2006

A finite family of q-orthogonal polynomials and resultants of Chebyshev polynomials

Jemal Emina Gishe
University of South Florida

A Finite Family of q-Orthogonal Polynomials
and
Resultants of Chebyshev Polynomials

by

Jemal Emina Gishe

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
Department of Mathematics
College of Arts and Sciences
University of South Florida

Major Professor: Mourad E. H. Ismail, Ph.D.
Boris Shekhtman, Ph.D.
Masahiko Saito, Ph.D.
Brian Curtin, Ph.D.

Date of Approval:
July 13, 2006

Keywords: continuous q-Jacobi polynomials, lowering operator, generating function, weight function, Rodrigues formula, discriminant.

Dedication

To Ahmed and his friends currently in Tanzania

Acknowledgments

I express my sincere and deepest gratitude to my research supervisor and dissertation advisor Professor Mourad Ismail. This work could never have been completed without his constant guidance and support. Prof. Ismail has supported me not only by providing research assistantship, but also academically and emotionally through the rough road over five years to complete the program.

I want to thank Dr. Masahiko Saito, Dr. Boris Shekhtman and Dr. Brian Curtin my supervising committee and Dr. David Rabson Chairperson of my defense, who read the manuscript with valuable comments. I am grateful to all assistance I obtained from the Mathematics department at University of South Florida; financially, academically and emotionally.

There are ample wonderful friends in the course of my life who influenced and believed in me. Jemal Dubie, Abduro Kelu, A. Hebo, Hussien Hamda, Jim Tremmel and Murat Thuran are among the few to mention.

My whole life is highly indebted to the support and love of my family. My dream is realized with the encouragement and great support of my parents Emina Gishe and Sinba Chawicha. Their courage, sacrifice and prayer to brought me up, influenced me and are source of my inspirations. It is also a blessing to have an intelligent, full of wonders and caring brother like Ahmed. His determination and strength to cope up with difficulties, fearless to fight injustice against himself and others, gifted nature to make fun are few of his qualities to mention.

Finally, love and emotional support I obtain from my wonderful wife Zebenay M. Kedir is so valuable to reach this level. Her courage and strength to properly care for our precious son Anatoli in my absence for years and her visions of life which helped me stay the course are a few of her blessed work that I can not afford not to mention.

Table of Contents

A Finite Family of q-Orthogonal Polynomials
and
Resultants of Chebyshev Polynomials

Jemal Emina Gishe

Abstract

Two problems related to orthogonal polynomials and special functions are considered. For $q > 1$ it is known that continuous q-Jacobi polynomials are orthogonal on the imaginary axis. The first problem is to find proper normalization to form a system of polynomials that are orthogonal on $\mathbb{R}$. By introducing a degree reducing operator and a scalar product one can show that the normalized continuous q-Jacobi polynomials satisfies an eigenvalue equation. This implies orthogonality of the normalized continuous q-Jacobi polynomials. As a byproduct, different results related to the normalized system of polynomials, such as its closed form, three-term recurrence relation, eigenvalue equation, Rodrigues formula and generating function will be computed. A discriminant related to the normalized system is also obtained.

The second problem is related to recent results of Dilcher and Stolarky [10] on resultants of Chebyshev polynomials. They used algebraic methods to evaluate the resultant of two combinations of Chebyshev polynomials of the second kind. This work provides an alternative method of computing the same resultant and also enables one to compute resultants of more general combinations of Chebyshev polynomials of the second kind. Resultants related to combinations of Chebyshev polynomials of the first kind are also considered.

Chapter 1

Introduction

1.1 Background and motivation

This dissertation deals with two separate problems which are related to orthogonal polynomials and special functions. In this section we will briefly explain the history, the core content and methodology of the dissertation and some general aspects of orthogonal polynomials and special functions.

The study of orthogonal polynomials and special functions is an old branch of mathematics. But the beginning of study of orthogonal polynomials as a discipline can be dated back to 1894 when Stieltjes published a paper about moment problem in relation to continued fraction. Stieltjes considered a bounded non-decreasing function $\phi(x)$ in the interval $[0,\infty)$ such that its moments given by $\int_0^\infty x^n d\phi(x)$, for $n = 0, 1, 2, \ldots$ has a prior given set of values $\{\mu_n\}$ as follows,

$$\int_0^\infty x^n d\phi(x) = \mu_n.$$

The values μ_n's are called the nth moments.

Similar results which preceded the work of Stelieltjes are those of Chebyshev in 1855, which discussed integrals of type $\int_{-\infty}^{\infty} \frac{p(y)}{x-y} dy$, where $p(x)$ is non-negative in $(-\infty, \infty)$, and the work of Heine in 1861. Heine considered continued fraction associated with the integral $\int_a^b \frac{p(y)}{x-y} dy$ for non-negative function $p(x)$ on (a, b).

A continued fraction is an expression of type,

$$\frac{A_0}{A_0 z + B_0 - \frac{C_1}{A_1 z + B_1 - \cdots}}.$$

The nth convergent of the continued fraction is a rational function of type $\frac{N_n(z)}{D_n(z)}$, for $n > 1$. If $A_n C_{n+1} \neq 0$ for $n = 0, 1, \ldots$ then the numerator and denominator polynomial solve the recurrence formula

$$y_{n+1}(z) = [A_n z + B_n] y_n(z) - C_n y_{n-1}(z), \qquad n > 0,$$

with initial conditions $D_0(z) := 1$, $D_1(z) := A_0 z + B_0$ and $N_0(z) := 0$, $N_1(z) := A_0$. If $A_{n-1} C_n > 0$, then D_n and N_n become orthogonal polynomials of first and second kind respectively. The polynomials D_n are usually denoted by P_n and N_n by P_n^*. Markov showed that if the true interval of orthogonality $[a, b]$ is bounded, then there is a measure μ such that

$$\lim_{n\to\infty} \frac{P_n^*(z)}{P_n(z)} = \int_a^b \frac{d\mu(t)}{z - t}, \quad z \notin [a, b].$$

Details on the above can be found in [14], [20] and [21]. From the above discussion it is straightforward to observe the connection among orthogonal polynomials, continued fractions and integrals of the type considered by Stelieltjes, Chebyshev and Heine.

Many mathematicians made important contributions to this field of mathematics. To mention a few, Euler's gamma and beta functions; Bessel functions; Polynomials of Legenderre, Jacobi, Laguerre and Hermite. Most of these functions were introduced to solve specific problems. For example, Euler's gamma and beta functions are discovered by Euler in the late 1720's, in the process of looking for a function of continuous variable x that equals $n!$ when $x = n$ for an integer n. Bessel functions

$$J_\alpha(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{\alpha+2n}}{\Gamma(n + \alpha + 1) n!},$$

are introduced to solve Bessel equation

$$\frac{d^2 y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + (1 - \frac{\alpha^2}{x^2}) y = 0,$$

which is obtained from Laplace's equation. Similarly for other polynomials mentioned above, their details and applications can be found in [1], [14], [21], [22].

In the 1970's the study of orthogonal polynomial was taken to a new level with Richard Askey's leadership in the area of special functions while at the same time George Andrew was advancing q-series and their applications to number theory and combinatorics. These advancement is due to strong team work of Mourad Ismail, M. Rahman, G. Gasper among the few to mention.

Orthogonal polynomials and special functions have a variety of applications in many areas. One area is in solving differential equations. The systems of classical orthogonal polynomials (such as Jacobi, Hermite, Laguerre) satisfy second order differential equations. For example, Poisson found that the theta function

$$u(x,t) = \sum_{-\infty}^{\infty} e^{-n^2 t + inx}$$

satisfies the heat equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t},$$

the Jacobi polynomials $y = P_n^{(\alpha,\beta)}(x)$ satisfy the second order differential equation

$$(1-x^2)y''(x) + [\beta - \alpha - x(\alpha+\beta+2)]y'(x) + n(n+\alpha+\beta+1)y(x) = 0,$$

and the Hermite polynomials solve

$$y'' - 2xy' + 2ny = 0, \quad y = H_n(x).$$

Another example is the following. Assuming $w(x) = \exp(-v(x)) > 0$, $x \in (a,b)$ and $v(x)$ and $v(x) + \ln A_n(x)$ be twice continuously differentiable functions whose second derivative is nonnegative on (a,b), then the equilibrium position of n movable unit charges in $[a,b]$ in the presence of the external potential $V(x) = v(x) + \ln A_n(x)$ is unique and is attained at the zeros of Jacobi polynomials, $p_n(x)$, provided that the particle interaction obeys a logarithmic potential that is $T(x) \to 0$ as x tends to any boundary point of $[a,b]^n$, where

$$T(x)=\prod_{j=1}^{n}\frac{\exp(-v(x_j))}{A_n(x_j)/a_n}\prod_{1\le l<k\le n}(x_l-x_k)^2.$$

It is also worth mentioning the minimal property of orthogonal polynomials. Suppose that we have a system of monic orthogonal polynomials $\{P_n(x)\}$ with respect to the weight function $w(x)$ over the interval Γ and $y(x)$ is an arbitrary monic polynomial of degree n. Then, since monic orthogonal polynomials form a basis we can write $y(x)=P_n(x)+\sum_{k=1}^{n-1}\gamma_k P_k(x)$. This implies that

$$\begin{aligned}\int_\Gamma y^2(x)w(x)dx &= \int_\Gamma P_n^2(x)w(x)dx+\int_\Gamma\sum_{k=1}^{n-1}\gamma_k^2P_k^2(x)w(x)dx\\ &\ge \int_\Gamma P_n^2(x)w(x)dx,\end{aligned}$$

which means that the monic orthogonal polynomials have minimal norm.

The beginning of q-polynomials is related to the work of Rogers and Ramanujan of the late 19th and early 20th century. In this regard the Rogers-Ramanujan identities (for notation refer to (1.2)),

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n}=\frac{1}{(q,q^4;q^5)_\infty}$$

and

$$\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q;q)_n}=\frac{1}{(q^2,q^3;q^5)_\infty}$$

can be mentioned. The Rogers-Ramanujan identities can be stated analytically and combinatorially. The analytic definition is due to independent work of Rogers, Ramanujan and Shur. This area of mathematics grown into an interesting level due to the connection work between Rogers-Ramanujan identities and certain families of orthogonal q-polynomials such as q-Hermite and q-Ultraspherical by Andrews, Askey, Ismail and Bressoud. For example, the generating function of q-Hermite polynomials is

$$\sum_{n=0}^{\infty}H_n(\cos\theta|q)\frac{t^n}{(q;q)_n}=\frac{1}{(te^{i\theta},te^{-i\theta};q)_\infty}.$$

For details and more examples one can refer to [14].

The combinatorial statement of the Rogers-Ramanujan identities was independently discovered by MacMahon and Shur during 1920's. The combinatorial interpretation of Rogers-Ramanujan follows in the same way as Euler interpretation of the identity

$$\sum_{n=0}^{\infty} \frac{q^n}{(q,q)_n} = \frac{1}{(q,q)_\infty}$$

in the form of

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q,q)_\infty},$$

where $p(n)$ is number of partitions of n. Here a partition is the number of ways of writing positive integers as a sum of positive integers. More precisely, suppose n is a positive integer then $p(n)$ is the number of ways of writing n as

$$n = \sum_j n_j,$$

where n_j are positive integers and $n_j \leq n_{j-1}$. For example, there are five partitions of 4, namely 4, $3+1$, $2+2$, $2+1+1$ and $1+1+1+1$.

There is a wide variety of work about q-series in relation to combinatorics and number theory, and their applications, especially in relation to partition theory.

The systems of q-orthogonal polynomials satisfy a second order difference equation. Here the Askey-Wilson and some other difference operators, usually called raising and lowering operators, play the role of differential operator. Definitions of these operators, are given in Chapter 2.

The second part of the dissertation is about resultants related to Chebyshev polynomials. The theory of resultant is an old and much studied topic in what used to be called the theory of equations [9]. Dickson introduced resultant in his book New First Course in the Theory of Equation [9] published in 1939, by considering two functions $f(x) = a_m x^m + \cdots + a_0$ and $g(x) = b_n x^n + \cdots + b_0$ where $a_m \neq 0$ and

$b_n \neq 0$. Suppose $\{x_j\}_{j=1}^m$ be zeros of $f(x)$. Then $g(x)$ and $f(x)$ have common roots if and only if $\prod_{j=1}^n g(x_j) = 0$. To avoid denominators while evaluating the product Dickson introduced resultant of $f(x)$ and $g(x)$ as $\mathrm{Res}(f(x), g(x)) = a_m^n \prod_{j=1}^m g(x_j)$. Dickson wrote in his book that resultant can be also represented in a matrix form which was later adopted as a base definition, and other equivalent working definitions and properties were derived.

From the above paragraph we observe that a resultant is a scalar function of two polynomials which is non zero if and only if the polynomials are relatively prime. The resultant of two polynomials is in general a complicated function of their coefficient. But there is an exceptionally elegant formula for resultant of two cyclotomic polynomials $\Phi_n(x)$ (the unique monic polynomial whose roots are the primitive n^{th} root of unity). It has degree $\phi(n)$ and written as

$$\Phi_n(x) = \prod_{k=1,(k,n)=1}^{n} (x - e^{\frac{2\pi ik}{n}}),$$

where $\phi(n)$ is the Euler function which represents number of positive integers less than or equal to n and relatively prime to n. Then for $m > n > 1$,

$$Res(\Phi_m(x), \Phi_n(x)) = \begin{cases} p^{\phi(n)} & \text{if m/n is power of prime p,} \\ 1 & \text{otherwise.} \end{cases}$$

The subject of resultants is an interesting topic for many reasons. For example, they can be used in matrix theory, they relate to problems on locations of roots of polynomials, they have applications in the theory of linear control systems, in robotics and computer aided geometric design, and they have extensions to polynomial matrices. There are many results on their theoretical properties especially in relation to algebraic geometry. For history and details of their application refer to [5], [6], [11] and [12].

Now we briefly introduce the core content of the dissertation and the methods used to solve the problems. In chapter 2, we follow the standard notation by Ismail as in [14]. In sections 2.1 and 2.2, we briefly review the construction of continuous q-Jacobi polynomials from Askey-Wilson polynomials, for proofs and more details one can refer to [16]. Toward the end of section 2.2, we consider continuous q-Jacobi polynomials for $q > 1$ which are orthogonal on the imaginary axis. Then, we normalize by choosing proper normalizing parameters to obtain a system of polynomials that are orthogonal on $\mathbb{R}$.

In section 2.3, we introduce the associated lowering operator and apply this operator to the normalized system of polynomials. Here, we define a related scalar product and show that the normalized system of polynomials are eigenfunctions under this scalar product. This will lead us to show the orthogonality of normalized system of polynomials. As a byproduct, we will compute a closed form, three-term recurrence relation, an eigenvalue equation, Rodrigues formula and a generating function of the normalized continuous q-Jacobi polynomials. As classical orthogonal polynomials satisfy second-order differential equations, this system of polynomials satisfies a second-order difference equations where the lowering and raising operators to be defined will play the role of differential operator in the later case. In the last section of this chapter we compute discriminant, using an elegant technique introduced by Ismail, related to the normalized polynomials.

In the last chapter, we compute resultants of combinations of different forms related to Chebyshev polynomials of first and second kind. The first section of this chapter deals with the preliminaries. In the second section we state and provide different proof for the resultants of two combinations of Chebyshev polynomials of second kind due to K. Dilcher and K. B. Stolarsky [10] and generalize their result. The last section of this chapter deals with the corresponding results for Chebyshev polynomials of first kind.

1.2 Basics, definitions and notation

In this section, we will give definitions, notation and results from orthogonal polynomials and hypergeometric series that we will be using later in the dissertation.

We will follow standard notations by [14].

The q-shifted factorials are defined as

$$(a;q)_0 := 1, \qquad (a;q)_n := \prod_{k=1}^{n}(1-aq^{k-1}) \tag{1.1}$$

for $n = 1, 2, \ldots,$ or ∞, and the multiple q-shifted factorials are defined by

$$(a_1, a_2, \ldots, a_k; q)_n := \prod_{j=1}^{k}(a_j; q)_n. \tag{1.2}$$

The basic hypergeometric series is defined as

$$\begin{aligned} {}_r\phi_s\left(\begin{array}{c} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{array}\middle| q; z\right) &= {}_r\phi_s(a_1, a_2, \ldots, a_r; b_1, b_2, \ldots, b_s; q, z) \\ &= \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_n}{(q, b_1, b_2, \ldots, b_s; q)_n} z^n (-q^{\frac{n-1}{2}})^{n(s+1-r)}. \end{aligned}$$

The above hypergeometric series is convergent for $r \leq s+1$. In this work we are considering hypergeometric series when $r = s+1$, and hence the above formula reduces to the form

$${}_r\phi_s\left(\begin{array}{c} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{array}\middle| q; z\right) = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_n}{(q, b_1, b_2, \ldots, b_s; q)_n} z^n. \tag{1.3}$$

The notion of q-shifted factorials and hypergeometric series introduced above are extensions of shifted and multishifted factorials because one can verify that

$$\lim_{q \to 1^-} \frac{(q^a; q)_n}{(1-q)^n} = (a)_n,$$

where the shifted and multishifted factorials are defined respectively as $(a)_n = a(a+1)\cdots(a+n-1)$ and $(a_1, a_2, \ldots, a_m)_n = \prod_{j=1}^{m}(a_j)_n$.

This implies that

$$\begin{aligned} &\lim_{q \to 1^-} {}_r\phi_s\left(\begin{array}{c} q^{a_1}, q^{a_2}, \cdots, q^{a_r} \\ q^{b_1}, q^{b_2}, \cdots, q^{b_s} \end{array}\middle| q; z(1-q)^{s+1-r}\right) \\ &= {}_r F_s\left(\begin{array}{c} a_1, a_2, \cdots, a_r \\ b_1, b_2, \cdots, b_s \end{array}\middle| (-1)^{s+1-r} z\right), \qquad r \leq s+1 \end{aligned}$$

where the right side of the above equation is hypergeometric series which is defined as

$$ {}_rF_s\left(\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{matrix} \,\middle|\, z\right) = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r)_n}{(b_1, b_2, \ldots, b_s)_n} \frac{z^n}{n!}. $$

There are ample identities that involves hypergeometric series, where we have sum on one side and product on the other side. I will mention few of them. The details, proofs and more identities can be found in [14].

Theorem 1.2.1

i) ***q-Pfaff-Saalschutz****: The sum of a terminating balanced* ${}_3\phi_2$ *is given by*

$$ {}_3\phi_2\left(\begin{matrix} q^{-n}, a, b \\ c, d \end{matrix} \,\middle|\, q, q\right) = \frac{(d/a, d/b; q)_n}{(d, d/ab; q)_n} \tag{1.4} $$

with $cd = abq^{1-n}$.

ii) The ***Euler*** *sums are given as*

$$ e_q(z) := \sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n} = \frac{1}{(z;q)_\infty}, \qquad |z| < 1, \tag{1.5} $$

and

$$ E_q(z) := \sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n} q^{n(n-1)/2} = (-z;q)_\infty. \tag{1.6} $$

iii) The ***Jacobi Triple Product Identity*** *is*

$$ \sum_{-\infty}^{\infty} q^{n^2} z^n = (q^2, -qz, -q/z; q^2)_\infty. \tag{1.7} $$

Below is a Lemma that gives important identities involving q-shifted factorials which will be used in the course of this dissertation.

Lemma 1.2.2 *The following identities holds:*

$$ (aq^{-n}; q)_k = \frac{(a;q)_k (q/a;q)_n}{(q^{1-k}/a;q)_n} q^{-nk}, \tag{1.8} $$

$$(a;q)_{n-k} = \frac{(a;q)_n (q^{1-n}/b;q)_k}{(b;q)_n (q^{1-n}/a;q)_k} \left(\frac{b}{a}\right)^k, \tag{1.9}$$

$$(a;q)_{n-k} = \frac{(a;q)_n}{(q^{1-n}/a;q)_k} (-a)^{-k} q^{\frac{1}{2}k(k+1)-nk}, \tag{1.10}$$

$$(a;q^{-1})_n = (1/a;q)_n (-a)^n q^{-n(n-1)/2}. \tag{1.11}$$

Proof. To prove (1.8) expand the left side as follows,

$$\begin{aligned}
(aq^{-n};q)_k &= (1-aq^{-n})(1-aq^{1-n})\cdots(1-aq^{k-1-n}) \\
&= (-1)^k q^{-nk+\frac{k(k-1)}{2}} \frac{(1-q^n/a)(1-q^{n-1}/a)\cdots(1-q/a)}{(1-q^{n-k}/a)(1-q^{n-1-k}/a)\cdots(1-q/a)} \\
&= (-1)^k q^{-nk+\frac{k(k-1)}{2}} \frac{(q/a;q)_n (1-1/a)(1-q/a)\cdots(1-q^{1-k}/a)}{(1-q^{n-k}/a)(1-q^{n-1-k}/a)\cdots(1-q^{1-k}/a)}.
\end{aligned}$$

The last equality indeed implies the right side of (1.8). The proofs of the remaining identities follow in the same way. □

1.3 General properties of orthogonal polynomials

In this section, we briefly explain basic notion, properties, and applications of general Orthogonal Polynomials.

Definition 1.3.1 We say $w(x) \geq 0$, for $x \in \mathbb{R}$, is a weight function if the integral

$$\mu_n := \int_{\mathbb{R}} x^n w(x) dx \tag{1.12}$$

exists for all $n \geq 0$. Existence here means that the resulting integrals are finite for all non-negative integers n.

The μ_n's are called the moments with respect to the weight function $w(x)$. In most cases the weight functions are supported on the finite interval. We mention few examples. It is known that $w(x) = (1-x)^\alpha(1+x)^\beta$ for $x \in (-1,1)$ is a weight function for Jacobi polynomials, $w(x) = x^\alpha e^{-x}$ for $x \in [0,\infty)$ is a weight function for Laguerre polynomials and $w(x) = e^{-x}$ for x on $\mathbb{R}$ is a weight function for Hermite polynomial .

Given a weight function $w(x)$, the n^{th} monic orthogonal polynomials (denoted by $Q_n(x)$) are constructed so that they satisfy

$$\int_{\mathbb{R}} Q_n(x)x^k w(x)dx = 0, \qquad k = 0,\ldots,n-1. \tag{1.13}$$

The above equation is equivalent to

$$\int_{\mathbb{R}} Q_n(x)p_k(x)w(x)dx = 0, \qquad k = 0,\ldots,n-1,$$

where $p_k(x)$ is any polynomial of degree k. Below is a more general theorem about existence and uniqueness of such polynomials. One can refer to [14] for the details and proof.

Theorem 1.3.2 *Given a positive Borel measure μ on $\mathbb{R}$ with infinite support and finite moments, there exists a unique sequence of monic polynomials $\{Q_n(x)\}_0^\infty$,*

$$Q_n(x) = x^n + \textit{lower order terms}, \qquad n = 0,1,\ldots,$$

and a sequence of positive numbers $\{\zeta_n\}_0^\infty$, with $\zeta_0 = 1$ such that

$$\int_{\mathbb{R}} Q_n(x)Q_m(x)d\mu(x) = \zeta_n\delta_{n,m}. \tag{1.14}$$

We introduce the determinant

$$\Delta_n = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ . & . & \cdots & . \\ . & . & \cdots & . \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{vmatrix}. \tag{1.15}$$

The existence of orthogonal system can be stated also as follows.

Theorem 1.3.3 *Suppose that we have a positive weight function $w(x)$ that satisfies the integral as in* (1.12) *with finite moments and Δ_n as defined by the matrix above. Then a necessary and sufficient condition for the existence of an orthogonal polynomial with respect the weight function is*

$$\Delta_n \neq 0, \qquad n = 0, 1, 2, \ldots.$$

Remark 1.3.4 The result of Theorem 1.3.3 can be extended and stated as follow. Let the moment integral in (1.12) be positive definite if it satisfies the condition that

$$\int_{\mathbb{R}} Q(x)w(x)dx > 0 \tag{1.16}$$

for any positive polynomial $Q(x)$. Then, the moment integral is positive definite if and only if all its moments are real and $\Delta_n > 0$.

Next is the Fundamental recurrence formula which states that all systems of orthogonal polynomials satisfies a three-term recurrence relation. For the proof refer to [14].

Theorem 1.3.5 *If $Q_n(x)$ is a simple set of real polynomials orthogonal with respect to $w(x) > 0$ on $a < x < b$, there exists sequence of numbers A_n, B_n, C_n such that for $n \geq 1$*

$$xQ_n(x) = A_nQ_{n+1}(x) + B_nQ_n(x) + C_nQ_{n-1}(x) \tag{1.17}$$

in which $A_n \neq 0$ and $C_n \neq 0$.

For the case of monic orthogonal polynomials that satisfy the recurrence relation

$$x\phi_n(x) = \phi_{n+1}(x) + B_n\phi_n(x) + C_n\phi_{n-1}(x), \tag{1.18}$$

with orthogonality

$$\int_{\mathbb{R}} \phi_n(x)\phi_m(x)d\mu(x) = \zeta_n\delta_{n,m}, \tag{1.19}$$

then $C_n > 0$ and $\zeta_n = C_1 \cdots C_n$. It is also possible to note that for the monic case $\zeta_n = \frac{\Delta_n}{\Delta_{n-1}}$.

Remark 1.3.6 It is also possible to verify using Remark 1.3.4 that the necessary and sufficient conditions for the system of polynomials $\{\phi_n(x)\}$ to be orthogonal on the real line with respect to positive measure is the three-term recurrence relation (1.18) holds with A_n, B_n, C_n real and $A_nC_{n+1} > 0$ for $n = 0, 1, 2, \ldots$.

Another important consideration in orthogonal polynomials is about properties of its zeros. The zeros of system of orthogonal polynomials are simple and interlace. Below is a theorem on Christoffel-Darboux identities which are used to verify about the simplicity and interlacing properties of zeros of orthogonal polynomials.

Theorem 1.3.7 *The Christoffel-Darboux identities hold for $N > 0$:*

$$\sum_{k=0}^{N-1} \frac{Q_k(x)Q_k(y)}{\zeta_k} = \frac{Q_N(x)Q_{N-1}(y) - Q_N(y)Q_{N-1}(x)}{\zeta_{N-1}(x-y)}, \tag{1.20}$$

$$\sum_{k=0}^{N-1} \frac{Q_k^2(x)}{\zeta_k} = \frac{Q_N'(x)Q_{N-1}(x) - Q_N(x)Q_{N-1}'(x)}{\zeta_{N-1}}, \tag{1.21}$$

where $Q_k(x)$ and ζ_k are as given in Theorem 1.3.2.

Theorem 1.3.8 *Let $\{Q_n(x)\}_{n=0}^{\infty}$ be a sequence of orthogonal polynomials. Then $Q_n(x)$ has n simple real zeros and zeros of $Q_n(x)$ and $Q_{n-1}(x)$ interlace.*

Finally it is also worth mentioning the expansion property of orthogonal polynomials. Given a weight function $w(x)$ over the interval $a < x < b$ we can introduce inner product as

$$(f, g) = \int_a^b f(x)g(x)w(x)dx$$

where $f(x)$, and $g(x)$ are functions where the integral above exists. With the above inner product one can verify the following theorem about expansion.

Theorem 1.3.9 *Let $\{Q_n(x)\}$ be a simple set of orthogonal polynomials with respect to the weight function $w(x) > 0$ over the interval $a < x < b$, and let $p(x)$ be a polynomial of degree m. Then*

$$P(x) = \sum_{k=0}^{m} C_k Q_k(x),$$

where

$$C_k = \frac{\int_a^b P(x)Q_k(x)w(x)dx}{\int_a^b Q_k^2(x)w(x)dx}.$$

From the above theorem it is immediate that a system of orthogonal polynomials forms a basis for polynomial functions.

Chapter 2

A Finite Family of q-Orthogonal Polynomials

For $q > 1$ the continuous q-Jacobi polynomials form a family of polynomials orthogonal on the imaginary axis. We renormalized to form a two parameter system of polynomials orthogonal on the real axis. This model leads to a four parameter finite family of orthogonal polynomials and its closed form expression, a three-term recurrence relation, an eigenvalue equation, Rodrigues formula and a generating function for this generalization are computed. The discriminant of the two parameter system of polynomials of arbitrary degree is also evaluated.

Lemma 2.0.10 *Routh[19] classified all orthogonal polynomials $\{p_n(x)\}$ satisfying the differential equation*

$$f(x)y'' + g(x)y' + h(x)y = \lambda_n y, \qquad n = 0, 1, \ldots, \tag{2.1}$$

with polynomial coefficients f, g and h independent of n, and λ_n a constant. He showed that in addition to the Jacobi, Hermite, and Laguerre polynomials (2.1) is satisfied by a finite family of Jacobi type polynomials $\{P_n^{(\alpha,\beta)}(ix) : n = 0, 1, \ldots, N\}$ with $\alpha = a + ib$, $\beta = a - ib$.

The system of polynomials constructed in the next section is one of such family of polynomials.

2.1 Continuous q-Jacobi polynomials

This section contains a brief introduction of the construction of continuous q-Jacobi polynomials from Askey-Wilson polynomials, some properties and related formulas. More detailed information can be obtained in [14] and [16].

The Askey-Wilson polynomials are built through the method of attachment, which involves generating functions and summation theorems to get new orthogonal or biorthogonal functions. We briefly explain the method of attachment using the construction of Al-Salam-Chihara polynomials.

The orthogonality relation of continuous q-Ultraspherical polynomials can be written in an equivalent integral form of

$$\int_0^\pi \frac{(t_1\beta e^{i\theta}, t_1\beta e^{-i\theta}, t_2\beta e^{i\theta}, t_2\beta e^{-i\theta}, e^{2i\theta}, e^{-2i\theta}; q)_\infty}{(t_1 e^{i\theta}, t_1 e^{-i\theta}, t_2 e^{i\theta}, t_2 e^{-i\theta}, \beta e^{2i\theta}, \beta e^{-2i\theta}; q)_\infty} d\theta$$

$$= \frac{(\beta, q\beta; q)_\infty}{(q, \beta^2; q)_\infty} {}_2\phi_1(\beta^2, \beta; q\beta; q, t_1 t_2), \qquad |t_1| < 1, \quad |t_2| < 1.$$

Taking $\beta = 0$, one obtains the following weight function whose mass is given by the integral

$$\int_0^\pi \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty}{(t_1 e^{i\theta}, t_1 e^{-i\theta}, t_2 e^{i\theta}, t_2 e^{-i\theta}; q)_\infty} d\theta = \frac{2\pi}{(q, t_1 t_2)_\infty}, \qquad |t_1| < 1, \quad |t_2| < 1. \tag{2.2}$$

To construct polynomials $p_n(x; t_1, t_2|q)$ orthogonal with respect to the weight function

$$w_1(x; t_1, t_2|q) = \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty}{(t_1 e^{i\theta}, t_1 e^{-i\theta}, t_2 e^{i\theta}, t_2 e^{-i\theta}; q)_\infty} \frac{1}{\sqrt{1-x^2}}, \qquad x = \cos\theta,$$

one can write $p_n(x; t_1, t_2|q)$ as

$$p_n(x; t_1, t_2|q) = \sum_{k=0}^{n} \frac{(q^{-n}, t_1 e^{i\theta}, t_1 e^{-i\theta}; q)_k}{(q; q)_k} a_{n,k},$$

because it can be easily attached to the weight function and the integral can be evaluated using (2.2). The polynomials are called Al Salam-Chihara polynomials and their closed form is

$$p_n(x; t_1, t_2|q) = {}_3\phi_2\left(\begin{matrix} q^{-n}, t_1 e^{i\theta}, t_1 e^{-i\theta} \\ t_1 t_2, 0 \end{matrix} \middle| q; q \right).$$

Details and proofs can be found in [14].

With the above motivation concerning the method of attachment, the polynomials orthogonal with respect to the weight function whose total mass is given by the Askey-Wilson q-beta integral,

$$\int_0^\pi \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty}{\prod_{j=1}^4 (t_j e^{i\theta}, t_j e^{-i\theta}; q)_\infty} d\theta = \frac{2\pi (t_1 t_2 t_3 t_4; q)_\infty}{(q;q)_\infty \prod_{1\le j<k\le 4} (t_j t_k; q)_\infty},$$

where $|t_j| < 1, \quad j = 1, \ldots, 4$ are called Askey-Wilson polynomials and have the basic hypergeometric representation

$$P_n(x; t_1, t_2, t_3, t_4 \mid q) = t_1^{-n} (t_1t_2, t_1t_3, t_1t_4; q)_n {}_4\phi_3 \left(\begin{matrix} q^{-n}, t_1t_2t_3t_4q^{n-1}, t_1e^{i\theta}, t_2e^{-i\theta} \\ t_1t_2, t_1t_3, t_1t_4 \end{matrix} \middle| q; q \right). \tag{2.3}$$

This system of polynomials satisfy the following orthogonality relation under the assumption that if t_1, t_2, t_3, t_4 are real, or occur in complex conjugate pairs if complex, then $\max\{| t_1 |, \;\; | t_2 |, \;\; | t_3 |, \;\; | t_4 |\} < 1$. It is known that

$$\frac{1}{2\pi} \int_{-1}^1 \frac{w(x)}{\sqrt{1-x^2}} P_m(x; t_1, t_2, t_3, t_4 \mid q) P_n(x; t_1, t_2, t_3, t_4 \mid q) dx = h_n \delta_{mn}, \tag{2.4}$$

where

$$w(x) := w(x; t_1, t_2, t_3, t_4 \mid q) = \left| \frac{(e^{2i\theta}; q)_\infty}{(t_1e^{i\theta}, t_2e^{i\theta}, t_3e^{i\theta}, t_4e^{i\theta}; q)_\infty} \right|^2$$

is the weight function for the Askey-Wilson polynomials and

$$h_n = \frac{(t_1t_2t_3t_4q^{n-1}; q)_n (t_1t_2t_3t_4q^{2n}; q)_\infty}{(q^{n+1}, t_1t_2q^n, t_1t_3q^n, t_1t_4q^n, t_2t_3q^n, t_2t_4q^n, t_3t_4q^n; q)_\infty}.$$

They satisfy the recurrence relation

$$2xp_n(x) = A_n p_{n+1}(x) + [t_1 + t_1^{-1} - (A_n + C_n)] p_n(x) + C_n p_{n-1}(x),$$

where

$$p_n(x) := p_n(x; t_1, t_2, t_3, t_4 \mid q) = \frac{t_1^n P_n(x; t_1, t_2, t_3, t_4 \mid q)}{(t_1t_2, t_1t_3, t_1t_4, ; q)_n},$$

$$A_n = \frac{(1-t_1t_2q^n)(1-t_1t_3q^n)(1-t_1t_4q^n)(1-t_1t_2t_3t_4q^{n-1})}{t_1(1-t_1t_2t_3t_4q^{2n-1})(1-t_1t_2t_3t_4q^{2n})},$$

and

$$C_n = \frac{t_1(1-q^n)(1-t_2t_3q^{n-1})(1-t_2t_4q^{n-1})(1-t_3t_4q^{n-1})}{(1-t_1t_2t_3t_4q^{2n-2})(1-t_1t_2t_3t_4q^{2n-1})}.$$

Askey-Wilson polynomials also satisfy the q-Difference equation

$$\begin{aligned}(1-q)^2 D_q \quad [\quad & \tilde{w}(x;t_1q^{1/2},t_2q^{1/2},t_3q^{1/2},t_4q^{1/2} \mid q)D_q y(x)] \\ & +\lambda_n \tilde{w}(x;t_1,t_2,t_3,t_4 \mid q)y(x) = 0, \qquad y(x) = p_n(x;t_1,t_2,t_3,t_4 \mid q),\end{aligned}$$

where

$$\tilde{w}(x;t_1,t_2,t_3,t_4 \mid q) := \frac{w(x;t_1,t_2,t_3,t_4 \mid q)}{\sqrt{1-x^2}},$$

and

$$\lambda_n = 4q^{-n+1}(1-q^n)(1-abcdq^{n-1}).$$

Here D_q is the Askey-Wilson operator which is defined as

$$(D_q f)(x) := \frac{\check{f}(q^{1/2}z) - \check{f}(q^{-1/2}z)}{(q^{1/2}-q^{-1/2})(z-1/z)/2}, \tag{2.5}$$

where $x = (z+1/z)/2$ and $\check{f}(z) = f((z+1/z)/2)$. Here $x = \cos\theta$ and one can observe that $z = e^{i\theta}$.

The expansion formula is an important notion related to the Askey-Wilson operator D_q. The Askey-Wilson operator D_q acts nicely on $(ae^{i\theta}, ae^{-i\theta}; q)_n$. The $(ae^{i\theta}, ae^{-i\theta}; q)_n$ are viewed as the basis for q-polynomials which is analogue to the basis x^n for classical polynomials. Indeed, for a polynomials of degree n the expansion formula is

$$f(x) = \sum_{k=0}^{n} f_k (ae^{i\theta}, ae^{-i\theta}; q)_k,$$

where

$$f_k = \frac{(q-1)^k}{(2a)^k (q;q)_k} q^{-k(k-1)/4} (D_q^k f)(x_k)$$

and

$$x_k = \frac{1}{2}\left(aq^{k/2} + \frac{q^{-k/2}}{a}\right).$$

The continuous q-Jacobi polynomials are special case of Askey-Wilson polynomials with parameter identification as follows. If we take $t_1 = q^{\frac{1}{2}\alpha+\frac{1}{4}}$, $t_2 = q^{\frac{1}{2}\alpha+\frac{3}{4}}$, $t_3 = -q^{\frac{1}{2}\beta+\frac{1}{4}}$, $t_4 = -q^{\frac{1}{2}\beta+\frac{3}{4}}$ in the definition of the Askey-Wilson polynomials after re-normalizing it follows that, for $n = 1, 2, \ldots,$

$$P_n^{(\alpha,\beta)}(x \mid q) = \frac{(q^{\alpha+1};q)_n}{(q;q)_n} {}_4\phi_3\left(\begin{matrix} q^{-n}, q^{n+\alpha+\beta+1}, q^{\frac{1}{2}\alpha+\frac{1}{4}}e^{i\theta}, q^{\frac{1}{2}\alpha+\frac{1}{4}}e^{-i\theta} \\ q^{\alpha+1}, -q^{\frac{1}{2}(\alpha+\beta+1)}, -q^{\frac{1}{2}(\alpha+\beta+2)} \end{matrix} \middle| q; q \right), \tag{2.6}$$

where $x = \cos\theta$.

The above polynomials are called continuous q-Jacobi polynomials and solve the normalized recurrence relation

$$xp_n(x) = p_{n+1}(x) + \frac{1}{2}[q^{\frac{1}{2}\alpha+\frac{1}{4}} + q^{-\frac{1}{2}\alpha-\frac{1}{4}} - (A_n + C_n)]p_n(x) + \frac{1}{4}A_{n-1}C_n p_{n-1}(x), \tag{2.7}$$

where

$$P_n^{(\alpha,\beta)}(x \mid q) = \frac{2^n q^{(\frac{1}{2}\alpha+\frac{1}{4})n}(q^{n+\alpha+\beta+1};q)_n}{(q, -q^{(\frac{1}{2}\alpha+\beta+1)}, -q^{(\frac{1}{2}\alpha+\beta+2)};q)_n} p_n(x), \tag{2.8}$$

$$A_n = \frac{(1-q^{n+\alpha+1})(1-q^{n+\alpha+\beta+1})(1+q^{n+\frac{1}{2}(\alpha+\beta+1)})(1+q^{n+\frac{1}{2}(\alpha+\beta+2)})}{q^{\frac{1}{2}\alpha+\frac{1}{4}}(1-q^{2n+\alpha+\beta+1})(1-q^{2n+\alpha+\beta+2})}, \tag{2.9}$$

and

$$C_n = \frac{q^{\frac{1}{2}\alpha+\frac{1}{4}}(1-q^n)(1-q^{n+\beta})(1+q^{n+\frac{1}{2}(\alpha+\beta)})(1+q^{n+\frac{1}{2}(\alpha+\beta+1)})}{(1-q^{2n+\alpha+\beta})(1-q^{2n+\alpha+\beta+1})}. \tag{2.10}$$

For $\alpha \geq -\frac{1}{2}$ and $\beta \geq -\frac{1}{2}$, the continuous q-Jacobi polynomials have the orthogonality formula

$$\frac{1}{2\pi}\int_{-1}^{1}\frac{w(x)}{\sqrt{1-x^2}}P_m^{(\alpha,\beta)}(x\mid q)P_n^{(\alpha,\beta)}(x\mid q)dx$$

$$=\frac{(q^{\frac{1}{2}(\alpha+\beta+2)},q^{\frac{1}{2}(\alpha+\beta+3)};q)_\infty}{(q,q^{\alpha+1},q^{\beta+1},-q^{\frac{1}{2}(\alpha+\beta+1)},-q^{\frac{1}{2}(\alpha+\beta+2)};q)_\infty}\times$$

$$\frac{(1-q^{\alpha+\beta+1})(q^{\alpha+1},q^{\beta+1},-q^{\frac{1}{2}(\alpha+\beta+3)};q)_n}{(1-q^{2n+\alpha+\beta+1})(q,q^{\alpha+\beta+1},-q^{\frac{1}{2}(\alpha+\beta+1)};q)_n}q^{(\alpha+\frac{1}{2})n}\delta_{mn},$$

where

$$w(x):=w(x;q^\alpha,q^\beta|q)=\left|\frac{(e^{2i\theta};q)_\infty}{(q^{\frac{1}{2}\alpha+\frac{1}{4}}e^{i\theta},q^{\frac{1}{2}\alpha+\frac{3}{4}}e^{i\theta},-q^{\frac{1}{2}\beta+\frac{1}{4}}e^{i\theta},-q^{\frac{1}{2}\beta+\frac{3}{4}}e^{i\theta};q)_\infty}\right|^2.$$

and δ_{mn} is the Kronecker delta, which is one if $m=n$ and zero otherwise.

For $q>1$, the continuous q-Jacobi polynomials $\{P_n^{(\alpha,\beta)}(x\mid q)\}$ are orthogonal on the imaginary axis. It is possible to renormalize in order to form a system of polynomials orthogonal on the real line.

First we will look at Jacobi and q-Hermite polynomials, for which similar results have already been considered by M. Ismail.

Example 2.1.1 The system of Jacobi polynomials has weight function $w(x;\alpha,\beta)$ $=(1-x)^\alpha(1+x)^\beta$, for $x\in(-1,1)$. This weight function has the integral value of

$$\int_{-1}^{1}w(x;\alpha,\beta)dx=2^{\alpha+\beta+1}\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}.$$

One can easily observe that the above integral follows from the definition of the Beta integral.

The closed form of Jacobi polynomials has hypergeometric representation

$$P_n^{(\alpha,\beta)}(x)=\frac{(\alpha+1)_n}{n!}{}_2F_1\left(\begin{matrix}-n,\alpha+\beta+n+1\\ \alpha+1,\end{matrix}\middle|\frac{1-x}{2}\right),$$

and this system of polynomials satisfies the orthogonality relation

$$\int_{-1}^{1}P_m^{(\alpha,\beta)}(x)P_n^{(\alpha,\beta)}(x)(1-x)^\alpha(1+x)^\beta=h_n^{(\alpha,\beta)}\delta_{m,n},$$

where

$$h_n^{(\alpha,\beta)} = \frac{2^{\alpha+\beta+1}\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{n!\Gamma(\alpha+\beta+n+1)\Gamma(\alpha+\beta+2n+1)}.$$

These polynomials solve the recurrence formula

$$\begin{aligned}
&2(n+1)(n+\alpha+\beta+1)(\alpha+\beta+2n)P_{n+1}^{(\alpha,\beta)}(x)\\
&=(\alpha+\beta+2n+1)[(\alpha^2-\beta^2)+x(\alpha+\beta+2n+2)(\alpha+\beta+2n)]\\
&\times P_n^{(\alpha,\beta)}(x)-2(\alpha+n)(\beta+n)(\alpha+\beta+2n+2)P_{n-1}^{(\alpha,\beta)}(x),
\end{aligned}$$

for $n \geq 0$, with initial conditions $p_{-1}^{(\alpha,\beta)}(x) = 0$ and $p_0^{(\alpha,\beta)}(x) = 1$.

Using the change of parameters $\alpha = a + bi$, $\beta = a - bi$, $x \to ix$, applied to the recurrence relation of Jacobi polynomials and redefining $\frac{p_n(ix)}{i^n} = Q_n(x)$ we obtain the recurrence formula

$$\begin{aligned}
(n+1)(n+2a+1)(n+a)Q_{n+1}(x) &= (2a+2n+1)[ab+x(a+n+1)(a+n)Q_n(x)\\
&- ((a+n)^2+b^2)(a+n+1)Q_{n-1}(x).
\end{aligned}$$

Because of the positivity condition on the coefficients of the recurrence relation, $a+n+1$ should be less than zero and we may redefine $a = -A$ and $n+1-A < 0$ for only finitely many values of n. Hence we observe that the normalization above gives finitely many system of orthogonal polynomials as suggested by Routh in Lemma 2.0.10.

Example 2.1.2 Another example is continuous q-Hermite polynomials. The continuous q-Hermite polynomials $\{H_n(x|q)\}$ are generated by the recursion relation

$$2xH_n(x|q) = H_{n+1}(x|q) + (1-q^n)H_{n-1}(x|q),$$

with initial conditions $H_0(x|q) = 1$, and $H_1(x|q) = 2x$. They have the closed form representation

$$H_n(cos\theta|q) = \sum_{k=0}^{n} \frac{(q;q)_n e^{i(n-2k)\theta}}{(q;q)_k (q;q)_{n-k}}$$

with the generating function

$$\sum_{n=0}^{\infty} H_n(x|q)\frac{t^n}{(q;q)_n} = \frac{1}{(te^{i\theta}, te^{-i\theta}; q)_\infty},$$

and satisfy the orthogonality relation

$$\int_{-1}^{1} H_m(x|q)H_n(x|q)w(x|q)dx = \frac{2\pi(q;q)_n}{(q;q)_\infty}\delta_{m,n},$$

where

$$w(x|q) = \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty}{\sqrt{(1-x^2)}}, \quad x = \cos\theta, \qquad 0 \le \theta \le \pi.$$

Continuous q-Hermite polynomials $H_n(x \mid q)$ are orthogonal on imaginary axis for $q > 1$. The normalization

$$h_n(x \mid q) = i^{-n} H_n(ix \mid 1/q)$$

gives

$$h_0(x \mid q) = 1, \qquad h_1(x \mid q) = 2x$$

and

$$h_{n+1}(x \mid q) = 2xh_n(x \mid q) - q^{-n}(1-q^n)h_{n-1}(x \mid q), \qquad n > 0,$$

and hence we can assume $0 < q < 1$. This implies the normalized system of polynomials are orthogonal on the real line. The weight of orthogonality, closed form, generating function of the normalized q-Hermite polynomials and more related formulas are given in [1] with details.

With the above motivation we now look back to continuous q-Jacobi polynomials for $q > 1$. For $q > 1$, the system of polynomials $\{P_n^{(\alpha,\beta)}(x)\}$ are orthogonal on the imaginary axis and we need to renormalize in order to make these polynomials orthogonal on the real line. After different trials and recalling Lemma 2.0.10, the following normalization is obtained that makes this system orthogonal on the real axis.

The proper normalization is to make the following change of parameters

$$i.\ q^{\frac{1}{2}\alpha+\frac{1}{4}} := A = a + bi, \tag{2.11}$$

$$ii.\ q^{\frac{1}{2}\beta+\frac{1}{4}} := B = a - bi \tag{2.12}$$

and replace x by ix. With this the three-term recurrence relation (2.7) becomes

$$ixp_n(ix) = p_{n+1}(ix) + iC_1p_n(ix) + C_2p_{n-1}(ix). \tag{2.13}$$

The *Maple* output shows that the values of C_1 and C_2 are real constants and is included below. Dividing (2.13) by i^{n+1} we have

$$\frac{xp_n(ix)}{i^n} = \frac{p_{n+1}(ix)}{i^{n+1}} + C_1\frac{p_n(ix)}{i^n} - C_2\frac{p_{n-1}(ix)}{i^{n-1}}.$$

With the normalization

$$Q_n(x;q) := \frac{p_n(ix)}{i^n},$$

we obtain the recursive relation

$$xQ_n(x;q) = Q_{n+1}(x;q) + C_1Q_n(x;q) - C_2Q_{n-1}(x;q). \tag{2.14}$$

C_1 and C_2 are computed by *Maple* to be as follows.

Set

$$
\begin{aligned}
e_1 &= (q^{n+2} - b^6 q^{4n+3/2} - b^6 q^{4n+1} + q^{2n+5/2}a^2 - q^{4n+3/2}a^6 + b^{10}q^{5n+1} \\
&+ b^{10}q^{5n+3/2} + b^8 q^{4n+2} - b^8 q^{5n+2} + b^8 q^{4n+1} + b^8 q^{4n+1/2} + b^8 q^{4n+3/2} \\
&- b^6 q^{4n+5/2} - b^4 q^{2n+1} - b^4 q^{2n+3/2} - b^4 q^{2n+2} - b^4 q^{2n+1/2} + q^{5n+1}a^{10} \\
&- q^{4n+5/2}a^6 + q^{4n+1}a^8 + q^{4n+1/2}a^8 + q^{4n+2}a^8 + q^{4n+3/2}a^8 - q^{5n+3/2}a^8 \\
&+ q^{5n+3/2}a^{10} - q^{4n+1}a^6 - q^{4n+2}a^6 - q^{n+1}b^2 + q^{n+3/2} - 2q^{2n+3/2}a^2b^2 \\
&- 6a^4b^4q^{5n+3/2} + 6a^4b^4q^{4n+1/2} - 6a^4b^4q^{5n+2} + 6a^4b^4q^{4n+3/2} \\
&- 3a^4b^2q^{4n+5/2} + 5a^2b^8q^{5n+1} + 5a^2b^8q^{5n+3/2} + 4a^2b^6q^{4n+3/2} \\
&- q^{5n+2}a^8 + q^{2n+1}a^2 + q^{2n+3/2}a^2 + q^{2n+2}a^2 - q^{2n+1/2}a^4 - q^{2n+1}a^4 \\
&- q^{2n+3/2}a^4 - q^{2n+2}a^4 + q^{2n+5/2}b^2 - q^{n+3/2}b^2 - q^{n+1}a^2 - q^{n+3/2}a^2 \\
&- 3q^{4n+1}b^4a^2 + b^2q^{2n+1} + b^2q^{2n+3/2} + b^2q^{2n+2} - b^8q^{5n+3/2} - b^6q^{4n+2} \\
&+ 4a^2b^6q^{4n+1} - 4a^2b^6q^{5n+3/2} - 4a^2b^6q^{5n+2} + 4a^2b^6q^{4n+1/2} \\
&- 3a^2b^4q^{4n+5/2} + 5a^8b^2q^{5n+1} + 5a^8b^2q^{5n+3/2} + 4a^2b^6q^{4n+2} \\
&+ 10a^6b^4q^{5n+1} - 2q^{2n+1/2}a^2b^2 + 10a^6b^4q^{5n+3/2} + 4a^6b^2q^{4n+2} \\
&+ 4a^6b^2q^{4n+3/2} + 4a^6b^2q^{4n+1} - 4a^6b^2q^{5n+3/2} - 4a^6b^2q^{5n+2} \\
&+ 4a^6b^2q^{4n+1/2} + 10a^4b^6q^{5n+1} + 10a^4b^6q^{5n+3/2} + 6a^4b^4q^{4n+1} \\
&+ 6a^4b^4q^{4n+2} - 3q^{4n+1}a^4b^2 - 3q^{4n+2}a^4b^2 - 3q^{4n+2}b^4a^2 \\
&- 3q^{4n+3/2}b^4a^2 - 3q^{4n+3/2}a^4b^2 - 2q^{2n+1}a^2b^2 - 2q^{2n+2}a^2b^2)b,
\end{aligned}
$$

and

$$
e_2 = \sqrt{q}A^2B^2[q - q^{2n}(1+q+q^2) + q^{4n}A^2B^2(1+q+q^2-q^{2n+1}A^2B^2)].
$$

Then $C_1 = e_1/e_2$ and

$$
\begin{aligned}
C_2 = {} & \frac{(1-q^{n-1}(a^2+b^2)^2)(1+q^{n-1}(a^2-b^2))(1-q^{n-1/2}(a^2-b^2))(1+q^{n-1/2}(a^2+b^2))}{(1-q^{2n-2}(a^2+b^2)^2)(1-q^{2n-1}(a^2+b^2)^2)^2(1-q^{2n}(a^2+b^2))} \\
& \times(1-q^n)(1+q^n(a^2+b^2)[(1-q^{n-1/2}a^2)^2+(1+q^{n-1/2}b^2)^2+2q^{2n-1}a^2b^2-1].
\end{aligned}
$$

Theorem 2.1.3 *The polynomial $Q_n(x;q)$ have closed form*

$$Q_n(x;q) = \frac{(AB, q^{1/2}AB, -q^{1/2}A^2;q)_n}{2^n A^n (q^n A^2 B^2;q)_n} {}_4\phi_3\left(\begin{matrix} q^{-n}, q^n A^2B^2, Ae^{\xi}, -Ae^{\xi} \\ AB, q^{1/2}AB, -q^{1/2}A^2 \end{matrix} \middle| q;q \right), \quad (2.15)$$

where $ix = \cos\theta$ and $\theta = \frac{\pi}{2} - i\xi$ (in particular $x = \sinh\xi$).

Proof. Making the substitution (2.11) and (2.12) in (2.6) and taking $\theta = \frac{\pi}{2} - i\xi$ we have

$$Q_n(x;q) = \frac{(-AB, -q^{1/2}AB, q^{1/2}A^2;q)_n}{2^n (iA)^n (q^n A^2 B^2;q)_n} {}_4\phi_3\left(\begin{matrix} q^{-n}, q^n A^2B^2, iAe^{\xi}, -iAe^{\xi} \\ -AB, -q^{1/2}AB, q^{1/2}A^2 \end{matrix} \middle| q;q \right).$$

Finally changing $iA \to A$ and $iB \to B$, the result of the theorem follows. □

Motivated by the form of the polynomials we shall study the following four parameter family of polynomials:

$$\begin{aligned} Q_n(\sinh\xi;\mathbf{t} \mid q) = Q_n(\sinh\xi; t_1,t_2,t_3,t_4 \mid q) &:= (-t_1t_2, -t_1t_3, -t_1t_4;q)_n t_1^{-n} \\ &\times {}_4\phi_3\left(\begin{matrix} q^{-n}, q^{n-1}t_1t_2t_3t_4, t_1e^{\xi}, -t_1e^{-\xi} \\ -t_1t_2, -t_1t_3, -t_1t_4 \end{matrix} \middle| q;q \right). \end{aligned}$$

We shall always use $\mathbf{t}$ to denote the quadruple (t_1,t_2,t_3,t_4), hence $c\,\mathbf{t}$ stands for $(c\,t_1, c\,t_2, c\,t_3, c\,t_4)$. Observe that these general four parameter polynomials are considered because of the parameter identification $A = t_1$, $q^{1/2}A = -t_2$, $B = t_3$, $q^{1/2}B = t_4$.

2.2 The polynomials Q_n

Here we construct three-term recurrence relation for the polynomials Q_n from Askey-Wilson polynomials three-term recurrence relation. The Askey-Wilson polynomials $P_n(x;\mathbf{s})$ are defined by [4],

$$P_n(\cos\theta;\mathbf{s}) := (s_1s_2, s_1s_3, s_1s_4;p)_n s_1^{-n} {}_4\phi_3\left(\begin{matrix} p^{-n}, p^{n-1}s_1s_2s_3s_4, s_1e^{i\theta}, s_1e^{-i\theta} \\ s_1s_2, s_1s_3, s_1s_4 \end{matrix} \middle| p;p \right).$$

These polynomials satisfy the three-term recurrence relation

$$xP_n(x;\mathbf{s}) = A_nP_{n+1}(x;\mathbf{s}) + B_nP_n(x;\mathbf{s}) + C_nP_{n-1}(x;\mathbf{s}), \quad n \ge 0, \quad (2.16)$$

with $P_{-1}(x;\mathbf{s}) = 0$, $P_0(x;\mathbf{s}) = 1$, where

$$\begin{aligned} A_n &= \frac{1 - s_1 s_2 s_3 s_4 p^{n-1}}{(1 - s_1 s_2 s_3 s_4 p^{2n-1})(1 - s_1 s_2 s_3 s_4 p^{2n})}, \\ C_n &= \frac{(1-p^n) \prod\limits_{1 \le j < k \le 4} (1 - s_j s_k p^{n-1})}{(1 - s_1 s_2 s_3 s_4 p^{2n-2})(1 - s_1 s_2 s_3 s_4 p^{2n-1})}, \end{aligned} \tag{2.17}$$

and

$$B_n = s_1 + s_1^{-1} - s_1^{-1} A_n \prod_{j=2}^{4} (1 - s_1 s_j p^n) - \frac{s_1 C_n}{\prod\limits_{2 \le k \le 4} (1 - s_1 s_k p^{n-1})}. \tag{2.18}$$

Now replace p by $1/q$ and s_j by i/t_j, $1 \le j \le 4$, respectively in (2.17) and (2.18) and denote the transformed A_n, B_n and C_n by A'_n, B'_n and C'_n. Thus,

$$\begin{aligned} A'_n &= -\frac{(t_1 t_2 t_3 t_4 q^{3n})(1 - t_1 t_2 t_3 t_4 q^{n-1})}{(1 - t_1 t_2 t_3 t_4 q^{2n-1})(1 - t_1 t_2 t_3 t_4 q^{2n})}, \\ C'_n &= -\frac{q^{3-3n}(1-q^n) \prod\limits_{1 \le j < k \le 4} (1 + t_j t_k q^{n-1})}{t_1 t_2 t_3 t_4 (1 - t_1 t_2 t_3 t_4 q^{2n-2})(1 - t_1 t_2 t_3 t_4 q^{2n-1})}, \end{aligned} \tag{2.19}$$

and

$$B'_n = i/t_1 - i t_1 + \frac{i q^{-3n}}{t_1^2 t_2 t_3 t_4} A'_n \prod_{j=2}^{4} (1 + t_1 t_j q^n) - \frac{i t_1^2 t_2 t_3 t_4 p^{3n-3} C'_n}{\prod\limits_{2 \le k \le 4} (1 + t_1 t_k q^{n-1})}. \tag{2.20}$$

Then replace x by ix in (2.16). This is done by writing $x = \cos\theta$ and replace θ by $\pi/2 - i\xi$. Under this replacement $P_n(x;\mathbf{s})$ is mapped to $Q_n(\sinh\xi;\mathbf{t})$, with

$$Q_n(\sinh\xi, \mathbf{t}) = (i t_1 t_2 t_3 t_4)^n q^{3n(n-1)/2} P_n(\cos\theta;\mathbf{s}). \tag{2.21}$$

Hence, the three-term recurrence relation for $Q_n(x)$ is

$$x Q_n(x;\mathbf{t}) = A''_n Q_{n+1}(x;\mathbf{t}) + B''_n Q_n(x;\mathbf{t}) + C''_n Q_{n-1}(x;\mathbf{t}), \quad n \ge 0, \tag{2.22}$$

where

$$A''_n = \frac{-A'_n}{t_1 t_2 t_3 t_4 q^{3n}}, \qquad B''_n = \frac{B'_n}{i}, \quad C''_n = C'_n t_1 t_2 t_3 t_4 q^{3n-1}, \tag{2.23}$$

with initial conditions $Q_{-1}(x;\mathbf{t}) = 0$, $Q_0(x;\mathbf{t}) = 1$.

Here the positivity condition of the recurrence relation is such that,

$$A''_{n-1}C''_n > 0. \tag{2.24}$$

It follows that from (2.19) and (2.23) the positivity condition holds for

$$1 - t_1t_2t_3t_4q^{n-2} < 0.$$

This in turn implies

$$t_1t_2t_3t_4q^{n-2} > 1.$$

Or equivalently that

$$\left|\frac{q^{2-n}}{t_1t_2t_3t_4}\right| < 1. \tag{2.25}$$

The above inequality holds only for finitely many n, and hence this normalization gives finitely many polynomials as suggested by Routh in Lemma 2.0.10. For purpose of the positivity condition of weight function considered later in this section, the pairs t_1 and t_3, t_2 and t_4 are assumed to be complex conjugates.

2.3 The Lowering operator

Systems of orthogonal polynomials are solutions of some differential equation or satisfy some difference equation. The polynomials under consideration satisfy a divided difference equation. The relevant operator here is the Lowering operator $\mathcal{D}_q$, which is an analogue of Askey-Wilson operator, and the averaging operator $\mathcal{A}_q$. They are defined respectively by

$$(\mathcal{D}_q f)(x) = \frac{\hat{f}(q^{1/2}z) - \hat{f}(q^{-1/2}z)}{(q^{1/2} - q^{-1/2})[(z + z^{-1})/2]}, \tag{2.26}$$

$$(\mathcal{A} f)(x) = \frac{1}{2}[\hat{f}(q^{1/2}z) + \hat{f}(q^{-1/2}z)], \tag{2.27}$$

with

$$\hat{f}(z) = f(\frac{z - z^{-1}}{2}), \quad x = (\frac{z - z^{-1}}{2}). \tag{2.28}$$

Here the parametrization is $x = \sinh \xi$ and therefore one may think z as e^{ξ}. The product rule for $\mathcal{A}_q$ is

$$\mathcal{D}_q fg = \mathcal{A}_q f \mathcal{D}_q g + \mathcal{A}_q g \mathcal{D}_q f. \tag{2.29}$$

We introduce inner product with respect to $w(x) = \frac{1}{\sqrt{1+x^2}}$. For f and g $\in \mathbf{L}^2(\mathbb{R}, (1+x^2)^{-1/2})$

$$\langle f, g \rangle = \int_{\mathbb{R}} f(x)\overline{g(x)} \frac{dx}{\sqrt{1+x^2}}. \tag{2.30}$$

Theorem 2.3.1 *Let f, $g \in \mathbf{L}^2(\mathbb{R}, (1+x^2)^{-1/2})$. Then the inner product defined above satisfies*

$$\langle \mathcal{D}_q f, g \rangle = -\langle f, \sqrt{1+x^2}\mathcal{D}_q g(x)(1+x^2)^{-1/2} \rangle. \tag{2.31}$$

Proof. We compute

$$\begin{aligned}
(q^{1/2} - q^{-1/2})\langle \mathcal{D}_q f, g \rangle &= \int_0^{\infty} \frac{\hat{f}(q^{1/2}u) - \hat{f}(q^{-1/2}u)}{(u^2+1)/2} \overline{\hat{g}(u)} du \\
&= \int_0^{\infty} \frac{\hat{f}(u)\overline{\hat{g}(q^{-1/2}u)}}{(q^{-1/2}u^2 + q^{1/2})/2} du \\
&\quad - \int_0^{\infty} \frac{\hat{f}(u)\overline{\hat{g}(q^{1/2}u)}}{(q^{1/2}u^2 + q^{-1/2})/2} du.
\end{aligned}$$

This implies the result. □

The polynomials $\{\phi_n(x \mid q)\}$ given by

$$\phi_n(x \mid q) := (ae^{\xi}, -ae^{-\xi}; q)_n \tag{2.32}$$

where $x = \sinh \xi$, form a basis in the vector space of all polynomials over $\mathbb{C}$. The result of applying the lowering operator to $\{\phi_n(x \mid q)\}$ is given by the following lemma.

Lemma 2.3.2 *Let $\{\phi_n(x \mid q)\}$ be as defined by* (2.32). *Then*

$$\mathcal{D}_q \phi_n(x \mid q) = \frac{-2a(1-q^n)}{1-q}(q^{1/2}ae^{\xi}, -q^{1/2}ae^{-\xi}; q)_{n-1}. \tag{2.33}$$

Proof. From the definition of $\mathcal{D}_q$ it follows that

$$
\begin{aligned}
&\mathcal{D}_q\phi_n(x \mid q) \\
&\quad = \frac{(q^{1/2}ae^{\xi}, -q^{-1/2}ae^{-\xi}; q)_n - (q^{-1/2}ae^{\xi}, -q^{1/2}ae^{-\xi}; q)_n}{(q^{1/2} - q^{-1/2})\cosh\xi} \\
&\quad = \frac{(q^{1/2}ae^{\xi}, -q^{1/2}ae^{-\xi}; q)_{n-1}[q^{-1/2}a(e^{-\xi}+e^{\xi}) - q^{n-1/2}a(e^{-\xi}+e^{\xi})]}{(q^{1/2} - q^{-1/2})\cosh\xi} \\
&\quad = \frac{-2a(1-q^n)}{1-q}(q^{1/2}ae^{\xi}, -q^{1/2}ae^{-\xi}; q)_{n-1}.
\end{aligned}
$$

□

Theorem 2.3.3 *The polynomials $\{Q_n(x; \mathbf{t} \mid q)\}$ have the property that*

$$
\mathcal{D}_q Q_n(x; \mathbf{t} \mid q) = \frac{2(1-q^n)(1-q^{n-1}t_1t_2t_3t_4)}{q^{(n-1)/2}(1-q)}\, Q_{n-1}(x; q^{1/2}\mathbf{t} \mid q). \tag{2.34}
$$

Proof.

$$
\begin{aligned}
&\mathcal{D}_q Q_n(x; \mathbf{t} \mid q) \\
&\quad = \frac{(-t_1t_2, -t_1t_3, -t_1t_4; q)_n}{t_1^n(q^{n-1}t_1t_2t_3t_4; q)_n} \sum_{k=0}^{n} \frac{(q^{-n}, q^{n-1}t_1t_2t_3t_4; q)_k}{(q, -t_1t_2, -t_1t_3, -t_1t_4; q)_k} q^k \\
&\qquad \times \frac{-2t_1(1-q^k)}{(1-q)}(q^{1/2}t_1e^{\xi}, -q^{1/2}t_1e^{-\xi}; q)_{k-1} \\
&\quad = \frac{-2q^{\frac{n+1}{2}}(1-q^{-n})(1-q^{n-1}t_1t_2t_3t_4)}{(1-q)} \frac{(-qt_1t_2, -qt_1t_3, -qt_1t_4; q)_{n-1}}{(q^{1/2}t_1)^{n-1}(q^n t_1t_2t_3t_4; q)_{n-1}} \\
&\qquad \times \sum_{k=0}^{n-1} \frac{(q^{-n+1}, q^n t_1t_2t_3t_4, q^{1/2}t_1e^{\xi}, -q^{1/2}t_1e^{-\xi}; q)_{k-1}}{(q, -qt_1t_2, -qt_1t_3, -qt_1t_4; q)_{k-1}} q^{k-1} \\
&\quad = \frac{2(1-q^n)(1-q^{n-1}t_1t_2t_3t_4)}{q^{(n-1)/2}(1-q)}\, Q_{n-1}(x; q^{1/2}\mathbf{t} \mid q),
\end{aligned}
$$

where we used (2.33) in the first equality and after simple manipulation and simplification we get the last equality which implies the result of the theorem. □

To construct weight function for the polynomials $Q_n(x|q)$ we begin with the weight function of continuous q-Jacobi polynomials. The weight function of continuous q-Jacobi polynomials is given as

$$w(x|q) := w(x; q^\alpha, q^\beta|q) = \left| \frac{(e^{2i\theta}; q)_\infty}{(q^{\frac{1}{2}\alpha+\frac{1}{4}}e^{i\theta}, q^{\frac{1}{2}\alpha+\frac{3}{4}}e^{i\theta}, -q^{\frac{1}{2}\beta+\frac{1}{4}}e^{i\theta}, -q^{\frac{1}{2}\beta+\frac{3}{4}}e^{i\theta}; q)_\infty} \right|^2, \tag{2.35}$$

where $\alpha \geq -1/2$ and $\beta \geq -1/2$. To introduce the weight function of the polynomials Q_n make the change of parameters (2.11) and (2.12) to obtain

$$w(x|q) = \left| \frac{(e^{2i\theta}; q)_\infty}{(Ae^{i\theta}, q^{1/2}Ae^{i\theta}, -Be^{i\theta}, -Bq^{1/2}e^{i\theta}; q)_\infty} \right|^2.$$

Now, taking $x = \sinh \xi$, i.e. $\theta = \frac{\pi}{2} - i\xi$, the above weight function becomes

$$\begin{aligned} w(x \mid q) = &\frac{(-e^{2\xi}, e^{-2\xi}; q)_\infty}{(iAe^{\xi}, q^{1/2}iAe^{\xi}, -iBe^{\xi}, -q^{1/2}iBe^{\xi}; q)_\infty} \\ &\times \frac{1}{(-iAe^{-\xi}, -q^{1/2}iAe^{-\xi}, iBe^{-\xi}, q^{1/2}iBe^{-\xi}; q)_\infty}. \end{aligned}$$

Finally, defining $iA := A$, $iB := B$ and defining $t_i's$ for $i = 1, \ldots, 4$ to be

$$A = t_1, \quad q^{1/2}A = -t_2, \quad B = t_3, \quad q^{1/2}B = t_4, \tag{2.36}$$

it follows that

$$w(x; t \mid q) = \frac{(-e^{2\xi}, e^{-2\xi}; q)_\infty}{\prod_{j=1}^{4}(t_j e^{\xi}, -t_j e^{-\xi}; q)_\infty}. \tag{2.37}$$

With this, we introduce the weight function

$$\tilde{w}(x; t \mid q) = \frac{w(x;\ t\ \mid\ q)}{\cosh \xi}.$$

In order for $\tilde{w}$ to be positive we require that the parameters t_1, t_2, t_3, t_4, to be nonreal and appear as two conjugate pairs. In addition to positivity we require that

$$\int x^n \tilde{w}(x, \mathbf{t}) = \int (\sinh \xi)^n \frac{(-e^{2\xi}, e^{-2\xi}; q)_\infty}{\prod_{j=1}^{4}(t_j e^{\xi}, -t_j e^{-\xi}; q)_\infty} d\xi < \infty$$

as $\xi \to +\infty$.

Indeed, setting $e^{\xi} = q^{-m}u$ for $q < u < 1$ it follows that the integrand above is

$$q^{-mn}\frac{(-q^{-2m}u^2;q)_{2m}}{\prod_{j=1}^4(t_jq^{-m}u;q)_m} = q^{-mn}\frac{(1+q^{-2m}u^2)\cdots(1+q^{-1}u^2)}{\prod_{j=1}^4(1-q^{-m}ut_j)\cdots(1-q^{-1}t_ju)},$$

and as $\xi \to +\infty$ the right side of the above equality becomes

$$\frac{q^{-mn+m}}{(t_1t_2t_3t_4)^m}.$$

Hence, for the above integral to be finite we require that

$$\left|\frac{q^{1-n}}{t_1t_2t_3t_4}\right| < 1.$$

One observes that this agrees with the positivity condition we obtained for three-term recurrence relation perviously in (2.25). The above inequality holds for finitely many values of n, which indeed is the reason why the system of polynomials obtained here is finite.

Theorem 2.3.4 *Let σ_j, $j = 1, \ldots, 4$, be the elementary symmetric function of t_j's, for $j = 1, \ldots, 4$. Then*

$$\mathcal{D}_q\tilde{w}(x;q^{1/2}t \mid q) = \frac{2\tilde{w}(x;t \mid q)}{(1-q)}[\sigma_1+\sigma_3+2x(1-\sigma_4)]. \tag{2.38}$$

Proof.

$$\begin{aligned}
&\mathcal{D}_q\tilde{w}(x;q^{1/2}t \mid q)\\
= \;& \frac{2(-qe^{2\xi},-1/qe^{-2\xi};q)_\infty}{(q^{1/2}-q^{-1/2})\cosh\xi\prod_{j=1}^4(qt_je^{\xi},-t_je^{-\xi};q)_\infty(q^{1/2}e^{\xi}+q^{-1/2}e^{-\xi})}\\
- \;& \frac{2(-1/qe^{2\xi},-qe^{-2\xi};q)_\infty}{(q^{1/2}-q^{-1/2})\cosh\xi\prod_{j=1}^4(t_je^{\xi},-qt_je^{-\xi};q)_\infty(q^{-1/2}e^{\xi}+q^{1/2}e^{-\xi})}\\
= \;& \frac{2\tilde{w}(x;t \mid q)}{(q^{1/2}-q^{-1/2})}[\frac{(1+q^{-1}e^{-2\xi})\prod_{j=1}^4(1-t_je^{\xi})}{(1+e^{2\xi})(q^{1/2}e^{\xi}+q^{-1/2}e^{-\xi}} -\\
& \frac{(1+q^{-1}e^{2\xi})\prod_{j=1}^4(1+t_je^{-\xi})}{(1+e^{-2\xi})(q^{-1/2}e^{\xi}+q^{1/2}e^{-\xi}}]\\
= \;& \frac{2\tilde{w}(x;t \mid q)}{(q^{1/2}-q^{-1/2})}[\frac{q^{-1/2}e^{-\xi}}{1+e^{2\xi}}\prod_{j=1}^4(1-t_je^{\xi}) - \frac{q^{-1/2}e^{\xi}}{1+e^{-2\xi}}\prod_{j=1}^4(1+t_je^{-\xi})]
\end{aligned}$$

$$= \frac{2\tilde{w}(x;t \mid q)e^{\xi}}{(q-1)(1+e^{2\xi})}[e^{-2\xi}(1-\sigma_1 e^{\xi}+\sigma_2 e^{2\xi}-\sigma_3 e^{3\xi}+\sigma_4 e^{4\xi}) -$$

$$e^{2\xi}(1+\sigma_1 e^{-\xi}+\sigma_2 e^{-2\xi}+\sigma_3 e^{-3\xi}+\sigma_4 e^{-4\xi})]$$

$$= \frac{2\tilde{w}(x;t \mid q)}{(q-1)}[2\sinh\xi(\sigma_4-1)-\sigma_1-\sigma_3]$$

which implies the result of the theorem. □

Theorem 2.3.5 *The polynomials* $\{Q_n(x;\mathbf{t} \mid q)\}$ *satisfy*

$$\frac{\mathcal{D}_q[\tilde{w}(x;q^{1/2}\mathbf{t} \mid q)Q_{n-1}(x;q^{1/2}\mathbf{t} \mid q)]}{\tilde{w}(x;\mathbf{t} \mid q)} = \frac{2q^{(1-n)/2}}{1-q}Q_n(x;\mathbf{t} \mid q). \tag{2.39}$$

Proof. Apply the definition of $\mathcal{D}_q$ and (2.38) to see that the left-hand side of (2.39) is

$$\begin{aligned}
&\frac{(-qt_1t_2,-qt_1t_3,-qt_1t_4;q)_{n-1}}{q^{(n-1)/2}t_1^{n-1}}\sum_{k=0}^{n-1}\frac{(q^{1-n},q^nt_1t_2t_3t_4;q)_k}{(q,-qt_1t_2,-qt_1t_3,-qt_1t_4;q)_k}\,q^k\\
&\times\frac{\mathcal{D}_q\tilde{w}(x;q^{k+1/2}t_1,q^{1/2}t_2,q^{1/2}t_3,q^{1/2}t_4 \mid q)}{\tilde{w}(x;\mathbf{t} \mid q)}\\
=\ &\frac{(-t_1t_2,-t_1t_3,-t_1t_4;q)_n}{q^{(n-1)/2}t_1^{n-1}(1-q)}\sum_{k=0}^{n-1}\frac{(q^{1-n},q^{n-1}t_1t_2t_3t_4;q)_k}{(q;q)_k(-t_1t_2,-t_1t_3,-t_1t_4;q)_{k+1}}\,q^k\\
&\times\frac{2\tilde{w}(x;t_1q^k,t_2,t_3,t_4 \mid q)}{\tilde{w}(x;\mathbf{t} \mid q)}\\
&\times[2\sinh\xi(1-q^k\sigma_4)+(q^kt_1+t_2+t_3+t_4)+q^kt_1(t_2t_3+t_2t_4+t_3t_4)+t_2t_3t_4].
\end{aligned}$$

It is easy to see that q^k times the term in the square bracket above is

$$(1+t_1t_2q^k)(1+t_1t_3q^k)(1+t_1t_4q^k)-(1-t_1q^ke^{\xi})(1+t_1q^ke^{-\xi})(1-q^k\sigma_4).$$

With this, the last equality above becomes

$$\frac{2(-t_1t_2,-t_1t_3,-t_1t_4;q)_n}{(1-q)q^{(n-1)/2}t_1^n}[\sum_{k=0}^{n-1}\frac{(q^{1-n},t_1e^{\xi},-t_1e^{-\xi},q^nt_1t_2t_3t_4;q)_k}{(q,-t_1t_2,-t_1t_3,-t_1t_4;q)_k}-$$

$$\sum_{k=1}^{n}\frac{(q^{1-n},q^nt_1t_2t_3t_4,q)_{k-1}(1-q^k)(t_1e^{\xi},-t_1e^{-\xi},;q)_k}{(q,-t_1t_2,-t_1t_3,-t_1t_4;q)_k}(1-q^{k-1}\sigma_4)]$$

$$= \frac{2(-t_1t_2,-t_1t_3,-t_1t_4;q)_n}{(1-q)q^{(n-1)/2}t_1^n}\sum_{k=1}^{n}\frac{(q^{1-n},q^nt_1t_2t_3t_4)_{k-1}(t_1e^{\xi},-t_1e^{-\xi};q)_k}{(q,-t_1t_2,-t_1t_3,-t_1t_4;q)_k}$$

$$\times[(1-q^{k-n})(1-q^{n+k-1}\sigma_4)-(1-q^k)(1-q^{k-1}\sigma_4)].$$

The term in the square bracket above, $[(1-q^{k-n})(1-q^{n+k-1}\sigma_4)-(1-q^k)(1-q^{k-1}\sigma_4)]$, simplifies to $q^k(1-q^{-n})(1-q^{n-1}\sigma_4)$. Substituting this into the last equality we have

$$\frac{2(-t_1t_2,-t_1t_3,-t_1t_4;q)_n}{(1-q)q^{(n-1)/2}t_1^n}\times {}_4\phi_3\left(\begin{matrix} q^{-n},q^{n-1}t_1t_2t_3t_4,t_1e^{\xi},t_1e^{-\xi} \\ -t_1t_2,-t_1t_3,-t_1t_4\end{matrix}\middle| q;q\right)$$

$$= \frac{2q^{(1-n)/2}}{1-q}Q_n(x;\mathbf{t}\mid q).$$

□

The above result can be rewritten as

$$\frac{2q^{(1-n)/2}}{1-q}Q_n(x;\mathbf{t}\mid q)\tilde{w}(x;\mathbf{t}\mid q)=\mathcal{D}_q[\tilde{w}(x;q^{1/2}\mathbf{t}\mid q)Q_{n-1}(x;q^{1/2}\mathbf{t}\mid q)], \tag{2.40}$$

which implies the following theorem.

Theorem 2.3.6 *The raising operator is given as*

$$\mathcal{D}_q[\tilde{w}(x;q^{1/2}t\mid q)Q_{n-1}(x;q^{1/2}t\mid q)]=\frac{2q^{(1-n)/2}}{1-q}Q_n(x;t\mid q)\tilde{w}(x;t\mid q). \tag{2.41}$$

Proof. It follows from (2.40). □

Theorem 2.3.7 *The system of polynomials $\{Q_n(x;q)\}$ is a solution of the following eigenvalue problem*

$$\frac{\mathcal{D}_q[\tilde{w}(x;q^{1/2}t\mid q)\mathcal{D}_qQ_n(x;t\mid q)]}{\tilde{w}(x;t\mid q)}=\frac{4(1-q^n)(1-q^{n-1}\sigma_4)}{q^{n-1}(1-q)^2}Q_n(x;t\mid q). \tag{2.42}$$

Proof. From (2.40) we have

$$\mathcal{D}_q[\tilde{w}(x;q^{1/2}t \mid q)Q_{n-1}(x;q^{1/2}t \mid q)] = \frac{2q^{(1-n)/2}}{1-q}Q_n(x;t \mid q)\tilde{w}(x;t \mid q).$$

Now replacing $Q_{n-1}(x;q^{1/2}t \mid q)$ using (2.34) the required result follows. $\square$

Given an eigenvalue problem one can characterize their eigenvalues and the orthogonality of the corresponding eigenfunctions. Below is a theorem and details of the proof from [14].

Theorem 2.3.8 *Assume that y, $p\mathcal{D}_q y \in L^2((1+x^2)^{-1/2})$, $p(x) \geq 0$ and $w(x) > 0$ for all $x \in \mathcal{R}$. Consider the eigenvalue problem:*

$$\frac{1}{w(x)}\mathcal{D}_q(p(x)\mathcal{D}_q y) = \lambda y.$$

Then the eigenvalues of this eigenvalue problem are real. The eigenfunctions corresponding to distinct eigenvalues are orthogonal with respect to w.

Proof. First we show that all eigenvalues are real. To the contrary consider that the above eigenvalue problem has complex eigenvalue λ and its corresponding eigenfunction is y. Then, $\bar{y}$ is an eigenfunction with eigenvalue $\bar{\lambda}$. With this,

$$\begin{aligned}
& (\lambda - \bar{\lambda}) \int_{\mathcal{R}} y(x)\overline{y(x)}w(x)dx \\
= \;& \langle w\lambda y, \sqrt{1+x^2}y\rangle - \langle \sqrt{1+x^2}y, w\lambda y\rangle \\
= \;& \langle \mathcal{D}_q(p\mathcal{D}_q y), \sqrt{1+x^2}y\rangle - \langle \sqrt{1+x^2}y, \mathcal{D}_q(p\mathcal{D}_q y)\rangle \\
= \;& -\langle p\mathcal{D}_q y, \sqrt{1+x^2}\mathcal{D}_q y\rangle + \langle \mathcal{D}_q y\sqrt{1+x^2}, p\mathcal{D}_q y)\rangle \\
= \;& 0.
\end{aligned}$$

This is a contradiction as the integrand above is strictly positive. Thus, the eigenvalue problem has only real eigenvalues. The next is to show that eigenfunctions corresponding to different eigenvalues are orthogonal. Let y_1 and y_2 be eigenfunctions corresponding to λ_1 and λ_2, respectively, where $\lambda_1 \neq \lambda_2$.

$$\begin{aligned}
& (\lambda_1 - \lambda_2) \int_{\mathcal{R}} y_1(x)\overline{y_2(x)}w(x)dx \\
= \;& \langle w\lambda_1 y_1, \sqrt{1+x^2}y_2\rangle - \langle \sqrt{1+x^2}y_1, w\lambda_2 y_2\rangle \\
= \;& \langle \mathcal{D}_q(p\mathcal{D}_q y_1), \sqrt{1+x^2}y_2\rangle - \langle \sqrt{1+x^2}y_1, \mathcal{D}_q(p\mathcal{D}_q y_2)\rangle \\
= \;& -\langle p\mathcal{D}_q y_1, \sqrt{1+x^2}\mathcal{D}_q y_2\rangle + \langle \mathcal{D}_q y_1\sqrt{1+x^2}, p\mathcal{D}_q y_2)\rangle \\
= \;& 0.
\end{aligned}$$

This implies the orthogonality of the eigenfunctions. □

With this we can now show the orthogonality of the system of polynomials $Q_n(x; \mathbf{t}|q)$.

Theorem 2.3.9 *The sequence of polynomials $\{Q_n(x; \mathbf{t}|q)\}$ are orthogonal with respect to the weight function*

$$\tilde{w}(x; t \mid q) = \frac{w(x;\ t\ \mid\ q)}{\cosh \xi}, \tag{2.43}$$

where

$$w(x; t \mid q) = \frac{(-e^{2\xi}, e^{-2\xi}; q)_\infty}{\prod_{j=1}^{4}(t_j e^{\xi}, -t_j e^{-\xi}; q)_\infty}. \tag{2.44}$$

Proof. From (2.42) the system of polynomials $Q_n(x; \mathbf{t}|q)$ satisfies the eigenvalue equation

$$\frac{\mathcal{D}_q[\tilde{w}(x; q^{1/2}t \mid q)Q_n(x; t \mid q)]}{\tilde{w}(x; t \mid q)} = \frac{4(1-q^n)(1-q^{n-1}\sigma_4)}{q^{n-1}(1-q)^2} Q_n(x; t \mid q).$$

Now, taking $p = \tilde{w}(x; q^{1/2}t)$ and using Theorem 2.3.8 the theorem follows. □

If we rewrite equation (2.41) and iterate we obtain,

$$\begin{aligned}
& Q_n(x; \mathbf{t} \mid q)\tilde{w}(x; \mathbf{t} \mid q) \\
= \;& \frac{(1-q)}{2q^{(1-n)/2}} \mathcal{D}_q[\tilde{w}(x; q^{1/2}\mathbf{t} \mid q)Q_{n-1}(x; q^{1/2}\mathbf{t} \mid q)] \\
= \;& \frac{(1-q)^2}{2^2 q^{(1-n+2-n)/2}} \mathcal{D}_q^2[\tilde{w}(x; q\mathbf{t} \mid q)Q_{n-2}(x; q\mathbf{t} \mid q)] \\
& \cdots \\
= \;& \frac{(1-q)^n}{2^n q^{\frac{n(1-n)}{2}}} \mathcal{D}_q^n \tilde{w}(x; q^{n/2}\mathbf{t} \mid q).
\end{aligned}$$

The above result is Rodrigues-type formula and we put in the following theorem.

Theorem 2.3.10 *The Rodrigues type formula for the polynomials $Q_n(x;\mathbf{t} \mid q)$ is*

$$Q_n(x;\mathbf{t} \mid q)w(x;\mathbf{t} \mid q) = \frac{(1-q)^n}{2^n q^{\frac{n(1-n)}{2}}}\mathcal{D}_q^n \tilde{w}(x;q^{n/2}\mathbf{t} \mid q). \tag{2.45}$$

A generating function for a sequence of polynomials $\{p_n(x)\}$ is a series of the form

$$\sum_{n=0}^{\infty} \lambda_n p_n(x) z^n = P(x,z) \tag{2.46}$$

for some suitable multipliers $\{\lambda_n\}$. If λ_n and $p_n(x)$ in (2.46) are assigned and we can determine the sum function $P(x,z)$ as a finite sum of products of a finite number of known special functions of one argument, we say the generating function is known. Generating functions play an important role in the study of orthogonal polynomials. For example, some orthogonal polynomials are defined using generating function. Below are few such polynomials.

The Legendre polynomials $P_n(x)$ are given as

$$(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)t^n,$$

and the Hermite polynomials $H_n(x)$ by

$$\exp(2xt-t^2) = \sum_{n=0}^{\infty} \frac{H_n(x)t^n}{n!}.$$

We observe that finding generating functions is among problems related to orthogonal polynomials. There is no general way of finding generating function and we apply manipulative technique to solve the problem.

To construct the generating function of the polynomials $Q_n(x;\mathbf{t} \mid q)$, it is important to mention the following hypergeometric identities or transformation formulas whose details are given in [14, Chapter 12].

Recall that

$$
\begin{aligned}
&{}_4\phi_3\left(\begin{array}{c} q^{-n}, a, b, c \\ d, e, f \end{array} \middle| q; q\right) \\
&= \left(\frac{bc}{d}\right)^n \frac{(de/bc, df/bc; q)_n}{(e, f : q)_n} {}_4\phi_3\left(\begin{array}{c} q^{-n}, a, d/b, d/c \\ d, de/bc, df/bc \end{array} \middle| q; q\right),
\end{aligned} \tag{2.47}
$$

where $abc = defq^{n-1}$, and

$$
\left(aq^{-n}; q\right)_n = (q/a; q)_n \, (-a)^n \, q^{-n(n+1)/2}, \tag{2.48}
$$

$$
\frac{(a; q)_{n-k}}{(b; q)_{n-k}} = \frac{(a; q)_n}{(b; q)_n} \frac{(q^{1-n}/b; q)_k}{(q^{1-n}/a; q)_k} \, (b/a)^k \,. \tag{2.49}
$$

Now taking $a = t_1 e^{\xi}$, $b = -t_1 e^{-\xi}$, $c = q^{n-1} t_1 t_2 t_3 t_4$, $d = -t_1 t_2$, $e = -t_1 t_3$, $f = -t_1 t_4$ and applying (2.47) we have

$$
\begin{aligned}
Q_n(\sinh \xi; \mathbf{t} \mid q) &= (-t_1 t_2, -q^{1-n} e^{\xi}/t_3, -q^{1-n} e^{\xi}/t_4; q)_n \left(q^{n-1} e^{-\xi} t_3 t_4\right)^n \\
&\times {}_4\phi_3\left(\begin{array}{c} q^{-n}, t_1 e^{\xi}, t_2 e^{\xi}, -q^{1-n}/t_3 t_4 \\ -t_1 t_2, -q^{1-n} e^{\xi}/t_3, -q^{1-n} e^{\xi}/t_4 \end{array} \middle| q; q\right).
\end{aligned} \tag{2.50}
$$

Applying (2.48) we have

$$
\begin{aligned}
&\left(-q^{1-n} e^{\xi}/t_3, -q^{1-n} e^{\xi}/t_4; q\right)_n \\
&= \left(-t_3 e^{\xi}, -t_4 e^{\xi}; q\right)_n (t_3 t_4)^{-n} \, q^{-n(n-1)} e^{2n\xi},
\end{aligned} \tag{2.51}
$$

and from (2.49) it follows that

$$
\begin{aligned}
&\frac{(q^{-n}, -q^{1-n}/t_3 t_4; q)_k}{(-q^{1-n} e^{\xi}/t_3, -q^{1-n} e^{\xi}/t_4; q)_k} \\
&= \frac{\left(-t_3 e^{-\xi}, -t_4 e^{-\xi}; q\right)_{n-k}}{(q, -t_3 t_4; q)_{n-k}} \frac{(q, -t_3 t_4; q)_n}{(-t_3 e^{-\xi}, -t_4 e^{-xi}; q)_n} \left(q e^{2\xi}\right)^{-k}.
\end{aligned} \tag{2.52}
$$

Applying (2.51) and (2.52), (2.50) becomes

$$
\begin{aligned}
\frac{Q_n(\sinh \xi; \mathbf{t} \mid q)}{(q, -t_1 t_2, -t_3 t_4; q)_n} &= \sum_{k=0}^{n} \frac{(t_1 e^{\xi}, t_2 e^{\xi}; q)_k}{(q, -t_1 t_2; q)_k} e^{-k\xi} \\
&\times \sum_{k=0}^{n} \frac{(-t_3 e^{-\xi}, -t_4 e^{-\xi}; q)_{n-k}}{(q, -t_3 t_4; q)_{n-k}} e^{(n-k)\xi}.
\end{aligned} \tag{2.53}
$$

The last equation above gives the following generating function for the polynomials $Q_n(x; \mathbf{t} \mid q)$.

Theorem 2.3.11 *The polynomials $Q_n(x;\mathbf{t}\mid q)$ have the generating function*

$$\sum_{n=0}^{\infty}\frac{Q_n(\sinh\xi;\mathbf{t}\mid q)}{(q,-t_1t_2,-t_3t_4;q)_n}t^n = {}_2\phi_1\left(\begin{array}{c|c} t_1e^{\xi},t_2e^{\xi} & \\ & q;te^{-\xi} \\ -t_1t_2 & \end{array}\right) \times {}_2\phi_1\left(\begin{array}{c|c} -t_3e^{-\xi},t_4e^{-\xi} & \\ & q;te^{\xi} \\ -t_3t_4 & \end{array}\right). \tag{2.54}$$

Proof. It follows from 2.53. □

2.4 Discriminants

Let

$$g(x)=\gamma\prod_{j=1}^{n}(x-x_j) \tag{2.55}$$

be a polynomial of degree n with leading coefficient γ. Then discriminant D of g, is defined by [Dickson, 1939]

$$D(g):=\gamma^{2n-2}\prod_{1\le j<k\le n}(x_j-x_k)^2. \tag{2.56}$$

An alternative definition is given as

$$D(g):=(-1)^{n(n-1)/2}\gamma^{n-2}\prod_{j=1}^{n}g'(x_j). \tag{2.57}$$

From the above definition one can observe that one of the importance of discriminant is to know if a polynomial has single or multiple zeros.

Definition 2.4.1 Ismail [14] introduced the concept of a generalized discriminant associated with a degree reducing operator T as

$$D(g;T)=(-1)^{n(n-1)/2}\gamma^{n-2}\prod_{j=1}^{n}(Tg)(x_j). \tag{2.58}$$

This is again can be stated as

$$D(g;T)=(-1)^{n(n-1)/2}\gamma^{-1}\mathrm{Res}(g,Tg), \tag{2.59}$$

where definition of $\mathrm{Res}(f,g)$ is given in the next chapter, refer to equation (3.2).

In order to compute the discriminant of the polynomials under consideration we first re-normalize to the monic case. We substitute (2.11) and (2.12) into (2.7), (2.9) and (2.10), then apply (2.8) and let $\theta = \frac{\pi}{2} - i\xi$. The result is that the polynomials $\{q_n(x;\mathbf{t}|q)\}$ satisfy

$$\begin{aligned} q_{n+1}(x;t|q) = {} & \frac{[2x + a_n + c_n - (t_1 + t_1^{-1})](1 - q^n t_1 t_2)}{a_n(1-q^{n+1})} q_n(x;t|q) - \\ & \frac{c_n}{a_n} \frac{(1-q^{n-1}t_1t_2)(1-q^n t_1 t_2)}{(1-q^n)(1-q^{n+1})} q_{n-1}(x;t|q), \end{aligned} \tag{2.60}$$

where

$$\begin{aligned} a_n &= \frac{(1-q^n t_1 t_2)(1-q^{n-1}t_1t_2t_3t_4)}{t_1(1+q^n t_1 t_3)(1+q^n t_2 t_3)}, \\ c_n &= \frac{t_1(1-q^n)(1-q^{n-1}t_3t_4)}{(1+q^{n-1}t_2t_3)(1+q^n t_1 t_3)}. \end{aligned} \tag{2.61}$$

The closed form of $q_n(x;t|q)$ is

$$q_n(x;t|q) = \frac{(-t_1t_2;q)_n}{(q;q)_n} {}_4\phi_3\left(\begin{matrix} q^{-n}, q^{n-1}t_1t_2t_3t_4, t_1e^{\xi}, -t_1e^{-\xi} \\ -t_1t_2, -t_1t_3, -t_1t_4 \end{matrix} \middle| q;q \right). \tag{2.62}$$

We can evaluate $q_n(x;t|q)$ at $x_1 = -(t_1 - t_1^{-1})/2$ and $x_2 = -(t_3 - t_3^{-1})/2$. Indeed,

$$q_n(x_1;t|q) = \frac{(-t_1t_2;q)_n}{(q;q)_n}, \quad q_n(x_2;t|q) = (\frac{t_1}{t_3})^n \frac{(-t_3t_4;q)_n}{(q;q)_n}. \tag{2.63}$$

Theorem 2.4.2 *The sequence of polynomials $\{q_n(x;t|q)\}$ has the property that*

$$\begin{aligned} & (1 - 2xt_1 - t_1^2)(1 - 2xt_3 - t_3^2)\mathcal{D}_q q_n(x;t|q) \\ & = A_n(x) q_{n-1}(x;t|q) + B_n(x) q_n(x;t|q), \end{aligned} \tag{2.64}$$

where

$$A_n(x) = \frac{2t_1(1+t_1t_3)(1+q^{n-1}t_1t_2)(1+q^{n-1}t_3t_4)}{q^{(n-1)/2}(1-q)(1-q^{n-1}t_2t_3)}, \tag{2.65}$$

and

$$B_n(x) = \frac{4t_1t_3(1-q^n)}{q^{(n-1)/2}(1-q)} x - \frac{(1-q^n)(t_1+t_3)(1+q^{n-1}t_1t_2t_3^2)}{q^{(n-1)/2}(1-q)(1-q^{n-1}t_2t_3)}. \tag{2.66}$$

Proof. Since

$$\frac{w(x;q^{1/2}t|q)}{w(x;t|q)} = (1-2xt_1-t_1^2)(1-2xt_3-t_3^2),$$

there are constants a, b, c such that

$$\begin{aligned}(1-2xt_1-t_1^2)(1-2xt_3-t_3^2)\mathcal{D}_q q_n(x;t|q) = \\ (ax+b)q_n(x;t|q)+cq_{n-1}(x;t|q).\end{aligned} \tag{2.67}$$

Equating the leading coefficient gives

$$a = \frac{4t_1t_3q^{\frac{1-n}{2}}(1-q^n)}{1-q}. \tag{2.68}$$

Applying (2.63) in (2.67) we have

$$ax_1 = b + \frac{c(1-q^n)}{(1+q^{n-1}t_1t_2)}, \qquad ax_2 = b + \frac{ct_3(1-q^n)}{t_1(1+q^{n-1}t_3t_4)}. \tag{2.69}$$

The theorem follows by solving the above system of equations. $\square$

Theorem 2.4.3 *The discriminant $D(f,\mathcal{D}_q)$ for the polynomials $\{q_n(x;t|q)\}$ is given by*

$$\begin{aligned}
& D(q_n(x;t|q),\mathcal{D}_q) \\
= \; & \frac{2^{n(n-1)}t_1^{2n(n-1)}q^{n(1-n)/2}(1+q^{n-1}t_1t_2)^n(1+q^{n-1}t_3t_4)^n}{(1-q)^n(1-q^{n-1}t_2t_3)^n(1+q^{n-1}t_1t_4)^n(1+t_1t_3)^{-n}} \\
\times \; & \prod_{k=1}^{n}\frac{(1+q^{k-1}t_1t_2)(1+q^{k-1}t_3t_4)(1+q^{k-1}t_1t_3)^{1-2k}(1-q^{k-2}t_1t_2)^{k-1}}{(1-q^k)^{2n-k-2}(1-q^{k-2}t_1t_2t_3t_4)^{n-k}(1-q^{k-2}t_3t_4)^{1-k}} \\
\times \; & \prod_{k=1}^{n-1}(1+q^{k-1}t_2t_3)^{1-3k}.
\end{aligned}$$

The result follows from definition of discriminant, Schur's theorem (Theorem 3.1.5) and Theorem 2.4.2.

Chapter 3

Resultants of Chebyshev Polynomials

3.1 Preliminaries

A resultant is a scalar function of two polynomials which is non-zero if and only if the polynomials are relatively prime. Relatively prime here means no common zero. The theory of resultant is an old and much studied topic in the theory of equations [9]. The subject of resultants is an interesting topic for many reasons. To mention few; it can be used in matrix theory (resultants are also defined by determinant of a matrix), to relate to problems on locations of roots of polynomials, it has relevant applications in the theory of linear control systems, in robotics and computer aided geometric design and it has extensions to polynomial matrices. There are many results on their theoretical properties specially in relation to algebraic geometry. For history and details of their application refer to [2], [21], [12] and [15]. Discriminants are special resultants and are useful in Ring Theory and electrostatic equilibrium problems, [13]. Two noteworthy results are Apostol's evaluation of the resultant of two cyclotomic polynomials [2] and Roberts' [18] recent evaluation of discriminants of certain polynomials which appear in Painlevé analysis.

In this chapter resultants of different forms of linear combinations of Chebyshev polynomials are considered. The resulting resultants are expressible in terms of Chebyshev polynomials whose coefficients and arguments are rational functions of the coefficients in the linear combinations. Resultant of two term linear combination of Chebyshev polynomials of second kind is given in the recent work of Dilcher and Stolarsky [10]. They used algebraic properties of resultants with respect to division, and Euclidean algorithms for polynomials. Here, simple method is used which also enabled to compute more general resultants related to combination of Chebyshev polynomials. In the next section resultants of different combinations of Chebyshev polynomials of second kind are considered and in the last section of this chapter different types of resultants related to Chebyshev polynomials of first kind are computed.

We follow classical definition of resultants. Alternative definitions are also given, such as the one by H. U. Gerber [12].

Definition 3.1.1 Given two polynomials

$$A(x) = \sum_{k=0}^{n} a_k x^k \quad \text{and} \quad B(x) = \sum_{j=0}^{m} b_j x^j,$$

their resultant which is usually denoted as $\mathrm{Res}(A, B)$ is defined to be the determinant of $m + n$ square matrix

$$\begin{vmatrix}
a_n & a_{n-1} & a_{n-2} & \dots & a_1 & a_0 & & & & & \\
 & a_n & a_{n-1} & \dots & a_2 & a_1 & a_0 & & & & \\
 & & & \dots & & & & & & & \\
 & & & & a_n & a_{n-1} & a_{n-2} & \dots & a_1 & a_0 & \\
b_m & b_{m-1} & b_{m-2} & \dots & b_1 & b_0 & & & & & \\
 & b_m & b_{m-1} & \dots & b_2 & b_1 & b_0 & & & & \\
 & & & \dots & & & & & & & \\
 & & & & b_m & b_{m-1} & b_{m-2} & \dots & b_1 & b_0 &
\end{vmatrix},$$

where entries of the first m rows are coefficients of $A(x)$, the last n rows are coefficients of $B(x)$ and the remaining entries are zeros.

Below is a theorem which is equivalent to the above definition and following are working corollaries. For details and proofs one can refer to [2].

Theorem 3.1.2 *If*

$$A(x) = a_n \prod_{k=1}^{n} (x - x_k) \quad \text{and} \quad B(x) = b_m \prod_{j=1}^{m} (x - y_j),$$

then their resultant is

$$\mathrm{Res}(A, B) = a_n^m b_m^n \prod_{k=1}^{n} \prod_{j=1}^{m} (x_k - y_j). \tag{3.1}$$

Corollary 3.1.3 *Suppose A and B are as defined in Theorem 3.1.2. Then*

$$\mathrm{Res}(A, B) = a_n^m \prod_{j=1}^{n} B(x_j). \tag{3.2}$$

Corollary 3.1.3 follows from Theorem 3.1.2 and usually called factorization formula. This is the formula we will be using in the course of the discussion.

Corollary 3.1.4 *Suppose A and B are again as defined in Theorem 3.1.2. Then*

$$\mathrm{Res}(A, B) = (-1)^{mn} \mathrm{Res}(B, A), \tag{3.3}$$

$$\mathrm{Res}(A, BC) = \mathrm{Res}(A, B)\mathrm{Res}(A, C). \tag{3.4}$$

Equations (3.3) and (3.4) easily follow from (3.1).

The following method is due to I. Schur, the sketch of the proof is included and if details are needed one can refer to [21] under discriminants of classical polynomials.

Theorem 3.1.5 *Let $p_n(x)$ be a sequence of polynomials satisfying the recurrence formula*

$$p_n(x) = (a_n x + b_n) p_{n-1}(x) - c_n p_{n-2}(x), \qquad n = 2, 3, 4, \ldots, \tag{3.5}$$

with initial conditions

$$p_0(x) = 1, \qquad p_1(x) = a_1 x + b_1.$$

Suppose that $a_1 a_n c_n \neq 0$ and let $\{x_{jn}\}$ be the zeros of $p_n(x)$. Then

$$\Delta_n = \prod_{j=1}^{n} p_{n-1}(x_{jn}) = (-1)^{n(n-1)/2} \prod_{j=1}^{n} \{a_j^{n-2j+1} c_j^{j-1}\}, \qquad n = 1, 2, 3, \ldots. \tag{3.6}$$

Proof.

$$\begin{aligned}
\Delta_n &= (a_1a_2\cdots a_{n-1})^n \prod_{j=1}^{n}(x_{j,n}-x_{1,n-1})(x_{j,n}-x_{2,n-1})\cdots(x_{j,n}-x_{n-1,n-1}) \\
&= \frac{(a_1a_2\cdots a_{n-1})^n}{(a_1a_2\cdots a_n)^{n-1}} p_n(x_{1,n-1})p_n(x_{2,n-1})\cdots p_n(x_{n-1,n-1}) \\
&= a_1a_2\cdots a_{n-1}a_n^{1-n}(-c_n)^{n-1}\prod_{j=1}^{n-1} p_{n-2}(x_{j,n-1}) \\
&= a_1a_2\cdots a_{n-1}a_n^{1-n}(-c_n)^{n-1}\Delta_{n-1}.
\end{aligned}$$

The theorem now follows by induction. □

Next is a Lemma that is important in the course of this discussion.

Lemma 3.1.6 *Let $[-a,a]$ be an interval symmetric with respect to the origin, and consider distribution of type $w(x)dx$ with an even weight function, that is $w(-x) = w(x)$. Let $p_n(x)$ be polynomials orthogonal with respect to $w(x)$. Then*

$$p_n(-x) = (-1)^n p_n(x). \tag{3.7}$$

Proof. For $\nu = 0, 1, 2, \ldots, n-1$,

$$\int_{-a}^{a} p_n(-x)x^\nu w(x)dx = (-1)^{\nu+1}\int_{-a}^{a} p_n(x)x^\nu w(x)dx = 0.$$

Consequently, $p_n(-x)$ possesses the same orthogonality relation as $p_n(x)$. Therefore, comparing the coefficient of x^n, we obtain $p_n(-x) = (-1)^n p_n(x)$. □

The method used is as follows. Assume $\{r_n(x)\}$ and $\{s_n(x)\}$ are sequences of polynomials such that $r_n(x)$ and $s_n(x)$ have exactly degree n for all n, $n > 0$. Construct functions $A_n(x)$ and $B_n(x)$ such that

$$s_{n-1}(x) = A_n(x)r_{n-1}(x) + B_n(x)r_n(x). \tag{3.8}$$

With $r_n(x) = \rho_n \prod_{j=1}^{n}(x-\zeta_j)$ then

$$\begin{aligned}
\mathrm{Res}(r_n(x), s_{n-1}(x)) &= \rho_n^{n-1}\prod_{j=1}^{n} s_{n-1}(\zeta_j) \\
&= \rho_n^{n-1}[\prod_{j=1}^{n} r_{n-1}(\zeta_j)][\prod_{j=1}^{n} A_n(\zeta_j)]. \qquad (3.9)
\end{aligned}$$

If $r_n(x)$ satisfies a three-term recurrence relation of the form (3.5) then Theorem 3.1.5 evaluates the first product on the last line of (3.9). The product $\prod_{j=1}^{n} A_n(\zeta_j)$ is evaluated on a case by case basis when $A_n(x)$ is a rational function.

The motivation for this approach came from the work of [15] on the discriminants of general orthogonal polynomials. In the next section, different derivation of the Dilcher-Stolarsky results is given, see Theorem 3.2.4. Then, general form of combinations of resultants of Chebyshev polynomials of three and more terms are given. In the last section similar results are established for Chebyshev polynomials of the first kind.

3.2 Chebyshev polynomials of second kind

We begin with the review of Chebyshev polynomials of second kind.

Definition 3.2.1 Chebyshev polynomials of second kind $\{U_n(x)\}$ are special ultraspherical polynomials that satisfies,

$$U_n(x) = \frac{(n+1)!}{(\frac{3}{2})_n} P_n^{(1/2,1/2)}(x)$$

and usually defined as

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta} \tag{3.10}$$

where, $x = \cos\theta$ and Ultraspherical polynomials are special Jacobi polynomials with $\alpha = \beta$.

Lemma 3.2.2 *These system of polynomials solve the three-term recurrence relation*

$$2xU_n(x) = U_{n+1}(x) + U_{n-1}(x), \tag{3.11}$$

with initial conditions

$$U_0(x) = 1, \qquad U_1(x) = 2x,$$

and satisfy the orthogonality relation

$$\int_{-1}^{1} U_n(x)U_m(x)(1-x^2)^{1/2}dx = \frac{\Pi}{2}\delta_{m,n}. \tag{3.12}$$

Lemma 3.2.3 *The closed form of Chebyshev polynomials of second kind is given by*

$$U_n(x) = \sum_{k=0}^{[n/2]} \frac{(n+1)!x^{n-2k}(x^2-1)^k}{(2k+1)!(n-2k)!},$$

and some of the generating functions satisfied by these polynomials are

$$\sum_{n=0}^{\infty} U_n(x)t^n = (1-2xt+t^2)^{-1},$$

$$\sum_{n=0}^{\infty} \frac{U_n(x)t^{n+1}}{(n+1)!} = \frac{e^{xt}\sinh(t\sqrt{x^2-1})}{\sqrt{x^2-1}}.$$

Details and proof of the above Lemmas can be found in [8], [14] and [17].

With the above motivation about Chebyshev polynomials of second kind, a result of Dilcher and Stolarsky [10] is stated and different short proof is given. The technique and approach used here enable to generalize the result of [10] and compute different forms of combinations of Chebyshev polynomials of both first and second kind.

Theorem 3.2.4 *([5]) Let $U_n(x)$ be the n^{th} Chebyshev polynomial of second kind. Then for $n \geq 2$,*

$$\mathrm{Res}(U_n(x)+kU_{n-1}(x), U_{n-1}(x)+hU_{n-2}(x)) = (-1)^{\frac{n(n-1)}{2}} 2^{n(n-1)} d_n(h,k), \tag{3.13}$$

where

$$d_n(x,y) = x^n \left[U_n(\frac{1+xy}{2x}) - yU_{n-1}(\frac{1+xy}{2x}) \right].$$

Proof. The proof is completed in several steps. For simplicity first define two sequences of polynomials,

$$r_n(x) = U_n(x) + kU_{n-1}(x), \tag{3.14}$$

$$s_n(x) = U_n(x) + hU_{n-1}(x). \tag{3.15}$$

Applying these two definitions the result of Theorem 3.2.4 reduces to show

$$\mathrm{Res}(r_n(x), s_{n-1}(x)) = (-1)^{\frac{n(n-1)}{2}} 2^{n(n-1)} d_n(h,k).$$

Comparing the degrees one can see that it is possible to express $s_{n-1}(x)$ as a combination of $r_{n-1}(x)$ and $r_n(x)$. Indeed,

$$s_{n-1}(x) = A_n(x) r_{n-1}(x) + B_n(x) r_n(x), \tag{3.16}$$

where the coefficients $A_n(x)$ and $B_n(x)$ will be evaluated below. Equation (3.16) together with definitions of $r_n(x)$, $s_n(x)$ and (3.11) implies that

$$\begin{aligned} U_{n-1}(x) + hU_{n-2}(x) &= A_n(x)(U_{n-1}(x) + kU_{n-2}(x)) \\ &\quad + B_n(x)(2xU_{n-1}(x) - U_{n-2}(x) + kU_{n-1}(x)). \end{aligned}$$

Solving it follows that

$$\begin{aligned} A_n(x) &= \frac{1 + h(k+2x)}{1 + k^2 + 2xk}, \\ B_n(x) &= \frac{k-h}{1 + 2xk + k^2}. \end{aligned} \tag{3.17}$$

For later use we rewrite the above result for $A_n(x)$ as

$$A_n(x) = \frac{h}{k}\left(\frac{-x - \frac{1+hk}{2h}}{-x - \frac{1+k^2}{2k}}\right). \tag{3.18}$$

The polynomial $r_n(x)$ has degree n with leading coefficient 2^n. Thus,

$$r_n(x) = 2^n \prod_{j=1}^{n} (x - x_j).$$

Hence applying (3.16) it follows that

$$s_{n-1}(x_j) = A_n(x_j) r_{n-1}(x_j).$$

This in turn implies

$$\begin{aligned} \prod_{j=1}^{n} s_{n-1}(x_j) &= \prod_{j=1}^{n} A_n(x_j) \prod_{j=1}^{n} r_{n-1}(x_j) \\ &= \Delta_n \frac{h^n}{k^n} \prod_{j=1}^{n} \left(\frac{-x_j - \frac{1+hk}{2h}}{-x_j - \frac{1+k^2}{2k}}\right) \\ &= \Delta_n \frac{h^n}{k^n} \frac{r_n\left(-\frac{1+hk}{2h}\right)}{r_n\left(-\frac{1+k^2}{2k}\right)} \end{aligned}$$

where

$$\Delta_n = \prod_{j=1}^{n} r_{n-1}(x_j). \tag{3.19}$$

Since $U_n(-x) = (-1)^n U_n(x)$ which follows from Lemma 3.1.6 and applying definition of $r_n(x)$ it follows that

$$\prod_{j=1}^{n} s_{n-1}(x_j) = \Delta_n \frac{h^n}{k^n} \frac{U_n\left(\frac{1+hk}{2h}\right) - kU_{n-1}\left(\frac{1+hk}{2h}\right)}{U_n\left(\frac{1+k^2}{2k}\right) - kU_{n-1}\left(\frac{1+k^2}{2k}\right)}.$$

The above equation together with the observation that $U_n\left(\frac{1+k^2}{2k}\right) = \frac{k^{n+1}- \ k^{-n-1}}{k- \ k^{-1}}$ implies

$$\prod_{j=1}^{n} s_{n-1}(x_j) = \Delta_n d_n(h,k). \tag{3.20}$$

To complete the proof it remains to compute Δ_n. Here we apply Schur's result, Theorem 3.1.5 about the discriminant of the classical polynomials which is mentioned in the introduction part.

From initial conditions and recurrence relation of Chebyshev polynomials of second kind it follows that $r_0(x) = 1$, $r_1(x) = 2x + k$, and

$$\begin{aligned} 2xr_1(x) - r_2(x) &= 2x(U_1(x) + ku_0(x)) - (U_2(x) + kU_1(x)) \\ &= U_0(x) + 2xkU_0(x) - kU_1(x) \\ &= r_0(x). \end{aligned}$$

Inductively one can show that the sequence of polynomials $r_n(x)$ satisfies the recurrence relation

$$2xr_n(x) = r_{n+1}(x) + r_{n-1}(x).$$

Applying 3.6 we have

$$\Delta_n = (-1)^{n(n-1)/2} \prod_{j=1}^{n} 2^{n-2j+1} = (-1)^{n(n-1)/2}. \tag{3.21}$$

Now Theorem 3.2.4 follows by putting together what we have from (3.2), (3.20), and (3.21). $\square$

Next we consider the more general combination of $\{U_n(x)\}$. Let

$$U_n(x;a,k) := U_n(x) + (ax+k)U_{n-1}(x), \tag{3.22}$$

and

$$\begin{aligned} f(x) &:= 1 + (bx+h)(2x+ax+k), \\ g(x) &:= 1 + (ax+k)(2x+ax+k). \end{aligned} \tag{3.23}$$

Theorem 3.2.5 *We have*

$$\begin{aligned} &\operatorname{Res}(U_n(x;a,k), U_{n-1}(x;b,h)) \\ &= \frac{(-1)^{\binom{n}{2}}}{(2+a)^2} 2^{(n-1)(n-2)} \operatorname{Res}(f(x), U_n(x;a,k)). \end{aligned}$$

Proof. Because of their corresponding degrees it is possible to express

$$U_{n-1}(x;b,h) = A_n(x)U_{n-1}(x;a,k) + B_n(x)U_n(x;a,k), \tag{3.24}$$

and it can be easily verified that

$$\begin{aligned} A_n(x) &= \frac{1 + (bx+h)(2x+ax+k)}{1 + (ax+k)(2x+ax+k)}, \\ B_n(x) &= \frac{(a-b)x + k - h}{1 + (ax+k)(2x+ax+k)}. \end{aligned} \tag{3.25}$$

For later use we rewrite the above result for $A_n(x)$ as

$$A_n(x) = \frac{b}{a}\,\frac{(x-c_1)(x-c_2)}{(x-d_1)(x-d_2)}, \tag{3.26}$$

where c_j and d_j for $j = 1, 2$ are respectively zeros of f and g defined in (3.23). From (3.22) we observe that $U_n(x;a,k)$ is polynomial of degree n with leading coefficient $2^{n-1}(2+a)$ and hence we can assume that,

$$U_n(x;a,k) = 2^{n-1}(2+a)\prod_{j=1}^{n}(x - x_{j,n}). \tag{3.27}$$

Applying (3.27) in (3.24) and using (3.26) we arrive at

$$
\begin{aligned}
\prod_{j=1}^{n} U_{n-1}(x_{j,n}; b, h) &= \prod_{j=1}^{n} A_n(x_{j,n}) \prod_{j=1}^{n} U_{n-1}(x_{j,n}; a, k) \\
&= \Delta_n \, \frac{b^n}{a^n} \frac{\prod_{j=1}^{n}(c_1 - x_{j,n})(c_2 - x_{j,n})}{\prod_{j=1}^{n}(d_1 - x_{j,n})(d_2 - x_{j,n})},
\end{aligned}
$$

where

$$
\Delta_n = \prod_{j=1}^{n} U_{n-1}(x_{j,n}; a, k).
$$

This implies

$$
\prod_{j=1}^{n} U_{n-1}(x_{j,n}; b, h) = \Delta_n \, \frac{b^n}{a^n} \frac{U_n(c_1; a, k) U_n(c_2; a, k)}{U_n(d_1; a, k) U_n(d_2; a, k)}. \tag{3.28}
$$

We now compute Δ_n. From initial conditions and three-term recurrence relation of Chebyshev polynomials of second kind and using representation (3.24) it is clear that $U_0(x; a, k) = 1$ and $U_1(x; a, k) = (2 + a)x + k$. Moreover $U_n(x; a, k)$ satisfies the three-term recurrence relation

$$
2xU_n(x; a, k) = U_{n+1}(x; a, k) + U_{n-1}(x; a, k). \tag{3.29}
$$

It follows that

$$
\begin{aligned}
\Delta_n &= \prod_{j=1}^{n} U_{n-1}(x_{j,n}; a, k) \\
&= 2^{n(n-2)}(2 + a)^n \prod_{j=1}^{n} (x_{j,n} - x_{1,n-1}) \cdots (x_{j,n} - x_{n-1,n-1}) \\
&= 2^{-1}(2 + a) U_n(x_{1,n-1}; a, k) \cdots U_n(x_{n-1,n-1}; a, k) \\
&= (-1)^{n-1} 2^{-1}(2 + a) \prod_{j=1}^{n} U_{n-2}(x_{j,n-1}; a, k) \\
&= (-1)^{n-1} 2^{-1}(2 + a) \Delta_{n-1}.
\end{aligned}
$$

In the second equality from the last we applied the recurrence formula in (3.29). Inductively it follows that

$$
\Delta_n = (-1)^{\frac{n(n-1)}{2}} 2^{1-n} (2 + a)^{n-1}. \tag{3.30}
$$

From the above result and (3.28) we have the following equality

$$\begin{aligned}\prod_{j=1}^{n} U_{n-1}(x_{j,n}; b, h) = (-1)^{n(n-1)/2} \\ \times(1+a/2)^{n-1}\frac{b^n}{a^n}\frac{U_n(c_1;a,k)U_n(c_2;a,k)}{U_n(d_1;a,k)U_n(d_2;a,k)}.\end{aligned} \tag{3.31}$$

Clearly,

$$(b(a+2))^n U_n(c_1;a,k)U_n(c_2;a,k) = \mathrm{Res}(f(x), U_n(x;a,k)),$$

and we only need to evaluate $U_n(d_1;a,k)U_n(d_2;a,k)$. Multiply (3.24) by g and let d be either d_1 or d_2. Thus,

$$f(d)U_{n-1}(d;a,k) + [(a-b)d + k - h]U_n(d;a,k) = 0,$$

which implies

$$U_n(d;a,k) = (2d + ad + k)U_{n-1}(d;a,k),$$

from which we conclude that $U_n(d;a,k) = [(a+2)d + k]^n$. Therefore,

$$U_n(d_1;a,k)U_n(d_2;a,k) = [(a+2)/a]^n. \tag{3.32}$$

The result of the theorem now follows from (3.31) and (3.32). □

Corollary 3.2.6 *Equation* (3.32) *implies that*

$$Res\,(g(x), U_n(x;a,k)) = (2+a)^{2n}.$$

Remark 3.2.7 Observe that

$$\begin{aligned}U_n(x;a,k) &= (1+a/2)U_n(x) + kU_{n-1} + (a/2)U_{n-2}(x), \\ U_n(x;b,h) &= (1+b/2)U_n(x) + hU_{n-1} + (b/2)U_{n-2}(x).\end{aligned} \tag{3.33}$$

Hence one can evaluate in closed form the resultants of polynomials of the form $\sum_{j=0}^{2} c_j U_{n-j}$ and $\sum_{j=0}^{2} d_j U_{n-j}$.

Remark 3.2.8 Let $\tilde{v}_n(x) = \sum_{j=0}^{m} c_j U_{n-j}(x)$ and $\tilde{w}_n(x) = \sum_{j=0}^{m} d_j U_{n-j}(x)$. Then, in general, there exist polynomials f, g and h of degrees m, m and $m-1$ respectively such that

$$f(x)\tilde{w}_{n-1}(x) = g(x)\tilde{v}_{n-1}(x) + h(x)\tilde{v}_n(x). \tag{3.34}$$

This is intuitively clear for the following reason. The left-hand side of (3.34) is a polynomial of degree $m+n-1$ and by repeated using of (3.11) it can be expressed as $\sum_{k=0}^{3m} \alpha_k U_{n+m-1-k}(x)$ and there is no loss of generality in assuming $\alpha_0 = 1$, that is $f(x) = 2^m x^m + \cdots$. By equating coefficients of various U_j's, we find $3m+1$ linear equations in the coefficients of f, g and h. The total number of coefficients in f, g, and h is $2(m+1)+m$ coefficients. Since one coefficient has already been specified we only have $3m+1$ unknowns and $3m+1$ equations, so the problem is tractable in general. The case of Chebyshev polynomials of first kind is more transparent, see Remark 3.3.6.

Let $x_1, x_2, \ldots, x_n$ be the zeros of $\tilde{v}_n(x)$, that is

$$\tilde{v}_n(x) = 2^n \prod_{j=1}^{n} (x - x_j). \tag{3.35}$$

Moreover we let

$$f(x) = 2^m \prod_{k=1}^{m} (x - f_k), \qquad g(x) = \gamma \prod_{k=1}^{m} (x - g_k). \tag{3.36}$$

The fact that

$$\prod_{j=1}^{n} \frac{g(x_j)}{f(x_j)} = \frac{\gamma^n}{2^{mn}} \prod_{j=1}^{n} \prod_{k=1}^{m} \frac{(x_j - g_k)}{(x_j - f_k)} = \frac{\gamma^n}{2^{mn}} \prod_{k=1}^{m} \frac{\tilde{v}_n(g_k)}{\tilde{v}_n(f_k)},$$

and (3.34) imply

$$\prod_{j=1}^{n} \tilde{w}_{n-1}(x_j) = \prod_{j=1}^{n} \frac{g(x_j)\tilde{v}_{n-1}(x_j)}{f(x_j)} = \frac{\gamma^n}{2^{mn}} \prod_{k=1}^{m} \frac{\tilde{v}_n(g_k)}{\tilde{v}_n(f_k)} \tilde{\Delta}_n, \tag{3.37}$$

where

$$\tilde{\Delta}_n := \prod_{j=1}^{n} \tilde{v}_{n-1}(x_j)$$

can be found from Theorem 3.1.5. This means one can compute resultant of type

$$\mathrm{Res}(\tilde{v}_n(x), \tilde{w}_{n-1}(x)),$$

where $\tilde{v}_n(x)$ and $\tilde{w}_n(x)$ are as given above in the remark.

3.3 Chebyshev polynomials of first kind

Below is the review of Chebyshev polynomials of first kind before considering the resultants related.

Definition 3.3.1 Chebyshev polynomials of first kind usually denoted as $T_n(x)$ are special type of Ultrasperical polynomials that satisfies

$$T_n(x) = \frac{n!}{(\frac{1}{2})_n} P_n^{(-1/2,-1/2)},$$

and usually defined as,

$$T_n(x) = \cos n\theta \tag{3.38}$$

where, $x = \cos\theta$ for $n = 0, 1, 2, \ldots$.

Lemma 3.3.2 *Chebyshev polynomials of first kind satisfy the recurrence relation*

$$2xT_n(x) = T_{n+1}(x) + T_{n-1}(x) \qquad n = 1, 2, 3, \ldots,$$

with initial conditions

$$T_0(x) = 1, \qquad T_1(x) = x,$$

and has the orthogonality relation

$$\int_{-1}^{1} T_m(x)T_n(x)(1-x^2)^{-1/2}dx = \begin{cases} \frac{\pi}{2}\delta_{mn} & if\ n > 0, \\ \pi & if\ n = m = 0. \end{cases}$$

Lemma 3.3.3 *Some of the generating functions satisfied by Chebyshev polynomials of first kind $T_n(x)$ are*

$$\sum_{n=0}^{\infty} T_n(x)t^n = (1-xt)(1-2xt+t^2)^{-1},$$

and

$$\sum_{n=0}^{\infty} \frac{T_n(x)t^n}{n!} = e^{xt}\cosh(t\sqrt{x^2-1}).$$

Moreover, the the closed form of $T_n(x)$ has the representation

$$T_n(x) = \sum_{k=0}^{[n/2]} \frac{n!x^{n-2k}(x^2-1)^k}{(2k)!(n-2k)!},$$

and can be also expressed as

$$T_n(x) = \frac{1}{2}\left[\left(x+\sqrt{x^2-1}\right)^n + \left(x-\sqrt{x^2-1}\right)^n\right].$$

□

With the brief review of $T_n(x)$ above, the first main result of this section is as follows.

Theorem 3.3.4 *Let $\{T_n(x)\}$ be the sequence of Chebyshev polynomials of first kind. Then*

$$\begin{aligned} &\mathrm{Res}(T_n(x)+kT_{n-1}(x), T_{n-1}(x)+hT_{n-2}(x)) \\ &= \frac{2^{n^2-3n+3}}{(-1)^{n(n-1)/2}} h^n \left[T_n\left((1+hk)/(2h)\right) - kT_{n-1}\left((1+hk)/(2h)\right)\right]. \end{aligned} \tag{3.39}$$

Proof. For simplicity we define

$$r_n(x) := T_n(x) + kT_{n-1}(x), \tag{3.40}$$

$$s_n(x) := T_n(x) + hT_{n-1}(x). \tag{3.41}$$

One can express $s_{n-1}(x)$ as a linear combination of $r_{n-1}(x)$ and $r_n(x)$. Indeed,

$$s_{n-1}(x) = A_n(x)r_{n-1}(x) + B_n(x)r_n(x). \tag{3.42}$$

and since $\{T_n(x)\}$ and $\{U_n(x)\}$ satisfy the same recurrence relation from (3.17) it follows that

$$A_n(x) = \frac{1+h(2x+k)}{1+k(2x+k)},$$
$$B_n(x) = \frac{k-h}{1+2xh+k^2}.$$

This implies

$$A_n(x) = \frac{h}{k}\left(\frac{-x - \frac{1+hk}{2h}}{-x - \frac{1+k^2}{2k}}\right). \tag{3.43}$$

From (3.40) we observe that $r_n(x)$ is a polynomial of degree n with leading coefficient 2^{n-1}. Let $\{y_j\}_{j=1}^n$ be zeros of $r_n(x)$. Therefore,

$$r_n(x) = 2^{n-1}\prod_{j=1}^{n}(x - y_j). \tag{3.44}$$

Applying this in 3.42 it follows that

$$s_{n-1}(y_j) = A_n(y_j)r_{n-1}(y_j).$$

This implies,

$$\begin{aligned}
\prod_{j=1}^{n} s_{n-1}(y_j) &= \frac{h^n}{k^n}\prod_{j=1}^{n}\frac{\left(\frac{-y_j-(1+hk)}{2h}\right)}{\left(\frac{-y_j-(1+k^2)}{2k}\right)}\prod_{j=1}^{n} r_{n-1}(y_j) \\
&= \Delta_n \frac{h^n}{k^n}\,\frac{r_n\left(\frac{-(1+hk)}{2h}\right)}{r_n\left(\frac{-(1+k^2)}{2k}\right)} \\
&= \Delta_n\,\frac{h^n}{k^n}\frac{T_n\left(\frac{-(1+hk)}{2h}\right) + kT_{n-1}\left(\frac{-(1+hk)}{2h}\right)}{T_n\left(\frac{-(1+k^2)}{2k}\right) + kT_{n-1}\left(\frac{-(1+k^2)}{2k}\right)},
\end{aligned}$$

where

$$\Delta_n = \prod_{j=1}^{n} r_{n-1}(y_j).$$

Since the weight function of Chebyshev polynomials of first kind is $w(x) = (1-x^2)^{1/2}$. It follows from Lemma (3.1.6) that these polynomials satisfy

$$T_n(-x) = (-1)^n T_n(x). \tag{3.45}$$

Chebyshev polynomials of first kind also have the property

$$T_n\left(\frac{z+z^{-1}}{2}\right) = \frac{z^n + z^{-n}}{2}, \tag{3.46}$$

which easily follows from the definition. Applying 3.45 and 3.46 in the last equality above it follows that

$$\prod_{j=1}^{n} s_{n-1}(x_j) = \frac{2\Delta_n h^n}{1-k^2}\left[T_n\left(\frac{1+hk}{2h}\right) - kT_{n-1}\left(\frac{1+hk}{2h}\right)\right]. \tag{3.47}$$

One can easily verify that the polynomials $\{r_n(x)\}$ satisfy the recurrence relation

$$2xr_n(x) = r_{n+1}(x) + r_{n-1}(x).$$

Applying (3.6) and observing that $\Delta_1 = 1-k^2$ it follows that

$$\Delta_n = (-1)^{n(n-1)/2} 2^{1-n}(1-k^2).$$

Using this in 3.47 we have

$$\begin{aligned}\prod_{j=1}^{n} s_{n-1}(y_j) &= (-1)^{n(n-1)/2}\, 2^{2-n}h^n \\ &\times \left[T_n\left(\frac{1+hk}{2h}\right) - kT_{n-1}\left(\frac{1+hk}{2h}\right)\right].\end{aligned} \tag{3.48}$$

The theorem now follows from (3.2) and (3.48). □

Next we consider resultant of the following combination of Chebyshev polynomials of first kind,

$$\operatorname{Res}\left(T_n(x) + (ax+k)T_{n-1}(x), T_{n-1}(x) + (bx+h)T_{n-1}(x)\right).$$

We let

$$T_n(x;a,k) := T_n(x) + (ax+k)T_{n-1}(x). \tag{3.49}$$

With similar argument before it is possible to express $T_{n-1}(x;b,h)$ as a linear combination of $T_{n-1}(x;a,k)$ and $T_n(x;a,k)$. Indeed,

$$T_{n-1}(x;b,h) = A_n(x)T_{n-1}(x;a,k) + B_n(x)T_n(x;b,k). \tag{3.50}$$

Applying (3.49) it follows that

$$\begin{aligned}T_{n-1}(x) + (bx+h)T_{n-2}(x) &= A_n(x)\{T_{n-1}(x) + (bx+h)T_{n-2}(x)\} \\ &+ B_n(x)\{2xT_{n-1}(x) - T_{n-2}(x) + (ax+k)T_{n-1}(x)\}.\end{aligned}$$

Equating the corresponding coefficients the following system of equations follows from above equality,

$$1 = A_n(x) + (2x + ax + k)B_n(x),$$
$$bx + h = (ax + k)A_n(x) - B_n(x).$$

A calculation leads to

$$A_n(x) = \frac{1 + (bx + h)(2x + ax + k)}{1 + (ax + k)(2x + ax + k)}, \tag{3.51}$$
$$B_n(x) = \frac{(a - b)x + k - h}{1 + (ax + k)(2x + ax + k)}. \tag{3.52}$$

This implies

$$A_n(x) \quad = \quad \frac{b}{a}\frac{(x - c_1)(x - c_2)}{(x - d_1)(x - d_2)}, \tag{3.53}$$

where c_i and d_i for $i = 1, 2$ are respectively zeros of quadratic functions of numerator and denominator of right side of (3.51) or respectively zeros of f and g defined in (3.23).

From (3.49) we observe that $T_n(x; a, k)$ is a polynomial of degree n with leading coefficient $2^{n-2}(2 + a)$. Hence we can assume that

$$T_n(x; a, k) = 2^{n-2}(2 + a)\prod_{j=1}^{n}(x - y_{j,n}). \tag{3.54}$$

The evaluation of $T_{n-1}(x; b, h)$ at the zeros of $T_n(x; a, k)$ is given by

$$T_{n-1}(y_{j,n}; b, h) = A_n(y_{j,n})T_{n-1}(y_{j,n}; a, k). \tag{3.55}$$

This together with (3.53) implies that

$$\prod_{j=1}^{n} T_{n-1}(y_{j,n}; b, h) = \frac{b^n}{a^n}\,\frac{\prod_{j=1}^{n}(c_1 - y_{j,n})(c_2 - y_{j,n})}{\prod_{j=1}^{n}(d_1 - y_{j,n})(d_2 - y_{j,n})}\prod_{j=1}^{n} T_{n-1}(y_{j,n}; a, k).$$

It follows that

$$\prod_{j=1}^{n} T_{n-1}(y_{j,n}; b, h) = \frac{b^n}{a^n}\Delta_n\frac{T_n(c_1; a, k)T_n(c_2; a, k)}{T_n(d_1; a, k)T_n(d_2; a, k)}, \tag{3.56}$$

where

$$\Delta_n = \prod_{j=1}^{n} T_{n-1}(y_{j,n}; a, k). \tag{3.57}$$

As in the proof of Theorem 3.2.5 we apply $T_0(x; a, k) = 1$ and $T_1(x; a, k) = (a+1)x+k$, three-term recurrence relation of Chebyshev polynomials of first kind and induction to show that the sequence of polynomials $\{T_n(x; a, k)\}$ satisfies the following three-term recurrence relation.

$$2xT_n(x; a, k) = T_{n+1}(x; a, k) + T_{n-1}(x; a, k). \tag{3.58}$$

Therefore,

$$\begin{aligned}
\Delta_n &= \prod_{j=1}^{n} T_{n-1}(y_{j,n}; a, k) \\
&= 2^{n-3}(2+a)\prod_{j=1}^{n}(y_{j,n} - y_{1,n-1}) \cdots (y_{j,n} - y_{n-1,n-1}) \\
&= 2^{-1}\prod_{j=1}^{n} T_n(y_{1,n-1}; a, k) \cdots T_n(y_{n-1,n-1}; a, k) \\
&= (-1)^{n-1}2^{-1}\Delta_{n-1}.
\end{aligned}$$

We used three-term recurrence relation given above in the last equality. This inductively implies that,

$$\Delta_n = (-1)^{n(n-1)/2}2^{1-n}. \tag{3.59}$$

The above discussion implies the following theorem.

Theorem 3.3.5 *Let $T_n(x; a, k)$ be defined by* (3.49)*. Then*

$$\begin{aligned}
&\operatorname{Res}(T_n(x; a, k), T_{n-1}(x; b, h)) \\
&= (-1)^{n(n-1)/2}\frac{2^{n^2-4n+3}}{(a+2)^{n-1}((a+1)^2-k^2)}\operatorname{Res}(f, T_n(x; a, k)).
\end{aligned} \tag{3.60}$$

Proof. It follow from (3.2), (3.56), and (3.59) that the left-hand side of Theorem 3.3.5 equals

$$(-1)^{n(n-1)/2}2^{n^2-4n+3}(a+2)^{n-1}\frac{b^n}{a^n}\frac{T_n(c_1; a, k)T_n(c_2; a, k)}{T_n(d_1; a, k)T_n(d_2; a, k)}.$$

A calculation gives

$$T_n(d_j; a, k) = [(a+2)d_j + k]T_{n-1}(d_j; a, k); \quad j = 1, 2,$$

$$\mathrm{Res}\,(g(x), T_1(x; a, k)) = (a+1+k)(a+1-k),$$

$$\mathrm{Res}\,(g(x), T_2(x; a, k)) = (a+2)^2(a+1+k)(a+1-k).$$

Hence,

$$\begin{aligned}\mathrm{Res}\,(g, T_n(.; a, k)) &= a^n(a+2)^n T_n(d_1; a, k)T_n(d_2; a, k)\\ &= (a+2)^{2n-2}(a+1+k)(a+1-k).\end{aligned}$$

Now the theorem follows by observing that

$$\mathrm{Res}\,(f, T_n(x; a, k)) = (b(2+a))^n\, T_n(c_1; a, k)T_n(c_2; a, k),$$

$$\mathrm{Res}\,(g, T_n(x; a, k)) = (a(2+a))^n\, T_n(d_1; a, k)T_n(d_2; a, k).$$

$\square$

Once again recall that Chebyshev polynomials are special Jacobi polynomials. Indeed,

$$T_n(x) = \frac{n!}{(1/2)_n} P_n^{(-1/2,-1/2)}(x), \tag{3.61}$$

and

$$U_n(x) = \frac{(n+1)!}{(3/2)_n} P_n^{(1/2,1/2)}(x). \tag{3.62}$$

It follows from the following orthogonality relations satisfied by the above mentioned polynomials that

$$\int_{-1}^{1} T_n(x)T_m(x)(1-x^2)^{-1/2}dx = 0,$$

$$\int_{-1}^{1} U_n(x)U_m(x)(1-x^2)^{1/2}dx = 0,$$

$$\int_{-1}^{1} P_n(x)P_m(x)(1-x)^\alpha(1+x)^\beta dx = 0,$$

for $m \neq n$, and taking $\alpha = \beta = -1/2$ for (3.61), $\alpha = \beta = 1/2$ for (3.62).

The expansion formula

$$\frac{(1-x)^n}{2^n(1+\alpha)_n} = \sum_{k=0}^{n} \frac{(-n)_k(1+\alpha+\beta+2k)(1+\alpha+\beta)_k}{(\alpha+1)_k(1+\alpha+\beta)_{n+k+1}} P_k^{(\alpha,\beta)}(x) \tag{3.63}$$

[[17], p.262] contains the expansions of powers of $1 \pm x$ in Chebyshev polynomials of the first and second kinds, since $P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\alpha,\beta)}(x)$ from Lemma 3.1.6. The term $k = 0$ in (3.63) when $\alpha = \beta = -1/2$ seems to be indeterminate but can be found by limiting procedure to be $1/n!$. Thus,

$$\frac{(1-x)^n}{2^n(1/2)_n} = \frac{1}{n!} + 2\sum_{k=1}^{n} \frac{(-n)_k k!}{(1/2)_k(n+k)!} T_k(x). \tag{3.64}$$

Remark 3.3.6 We now discuss the case when

$$\tilde{v}(x) = \sum_{j=0}^{m} c_j T_{n-j}(x) \tag{3.65}$$

and

$$\tilde{w}(x) = \sum_{j=0}^{m} d_j T_{n-j}(x). \tag{3.66}$$

As per Remark 3.2.8, in general, there exist polynomials f, g and h of degree m, m and $m-1$ respectively such that

$$f(x)\tilde{w}_{n-1} = g(x)\tilde{v}_{n-1}(x) + h(x)\tilde{v}_n(x). \tag{3.67}$$

In this case the analysis is made simpler by expanding f, g and h in powers of $1-x$, applying (3.63) and using

$$T_n(x)T_m(x) = \frac{1}{2}[T_{m+n}(x) + T_{m-n}(x)] \tag{3.68}$$

which follows from the trigonometric identity

$$\cos\theta\cos\beta = \frac{1}{2}[\cos(\theta+\beta) + \cos(\theta+\beta)]$$

to set up the linear system of equations satisfied by the coefficients of f, g and h. The difference between this case and the case of Chebyshev polynomials of second kind is that the linearization of product is more complicated.

Since $T_n(x) = 2^{n-1}x^n + \cdots$, we let

$$\tilde{v}_n(x) = 2^{n-1}\prod_{j=1}^{n}(x - x_j), \quad f(x) = 2^m \prod_{k=1}^{m}(x - f_k)$$

and

$$g(x) = \gamma \prod_{k=1}^{m}(x - g_k).$$

Then it follows that

$$\prod_{j=1}^{n} \frac{g(x_j)}{f(x_j)} = \frac{\gamma^n}{2^{mn}} \prod_{k=1}^{n} \frac{\tilde{v}_n(g_k)}{\tilde{v}_n(f_k)},$$

and (3.67) implies

$$\prod_{j=1}^{n} \tilde{w}_{n-1}(x_j) = \frac{\gamma^n}{2^{mn}} \prod_{k=1}^{n} \frac{\tilde{v}_n(g_k)}{\tilde{v}_n(f_k)} \tilde{\Delta}_n,$$

where $\tilde{\Delta}_n = \prod_{j=1}^{n} \tilde{v}_{n-1}(x_j)$.

Now $\tilde{\Delta}_n$ can be computed using Schur's Theorem. This means it is possible to consider resultants of the type

$$Res(\tilde{v}_n(x), \tilde{w}_{n-1}(x)),$$

where $\tilde{v}_n(x)$, $\tilde{w}_n(x)$ are as defined above.

Conclusion

There is more work that can be done related to both problems considered in this dissertation. The mass of the weight function of the polynomials $Q_n(x; \mathbf{t}|q)$ is not yet found. Computing resultants for more general combination is not an easy task, although one can argue that in general it is solvable. Similar problems can be also considered for other systems of orthogonal polynomials.

References

[1] G. E. Andrews, R. Askey and R. Roy, Special Functions. Cambridge, 1999.

[2] T. M. Apostol, The resulatants of the cyclotomic polynomials $F_m(ax)$ and $F_n(bx)$, Math. Comp. 29 (1975), $1-6$, MR0366801 (51:3047).

[3] R. Askey, Conitiuous q-Hermite polynomials when $q > 1$, In D. Stanton (Ed.), q-Series and Partitions, IMA volumes in Mathematics and Its Application (pp. $151-158$), New York, Springer-Verlag, 1989.

[4] R. Askey and J. Wilson, Some basic hypergeometric orthogonal polynomials that generalizes Jacobi polynomials. Amer.Math.Soc., Providence, 1985.

[5] S. Barnett, Matrices in control Theory, 2nd ed., Krieger, Malabar, Florida, 1984.

[6] S. Barnett, Polynomials and Linear Control Systems, Marcel Dekker, New York, 1983.

[7] Y. Chen and M. Ismail, Ladder Operators and Differetial Equations for Orthogonal Polynomials. J. Phys. A 30(1997),$7817-7829$.

[8] T.S. Chihara, An introduction to Orthogonal Polynomials, Gordon and Breach Science Publishers, Inc., New York, 1978.

[9] L. E. Dickson, New First Course on the Theory of Equations, Wiley, New York, 1939.

[10] K. Dilcher, and K.B. Stolarsky, Resulatnts and Discriminants of Chebyshev and related polynomials, Transaction of the Amer. Math. Soc. 357, no. 3 (2004), $965-981$. S 0002-9947(04)03687-6.

[11] I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky, Discriminants, Resulatants, and Multidimensional Determinants, Birkhuser Boston, Boston, 1994.

[12] H. U. Gerber, Wronski formula and resultant of two polynomilas, this MONTHLY, 91(1984) 644 – 646.

[13] M. E. H. Ismail, An electrostatic model for zeros of general orthogonal polynomials, Pacific J. Math. **193**(2000), 355 – 369.

[14] M. Ismail, Classical and Quantum Orthogonal Polynomials in one variable. Cambridge, 2005.

[15] M. E. H. Ismail, Discriminants and functions of the second kind of orthogonal polynomials, Results in Math. 34(1998), 132 – 149.

[16] R. Koekoek and R. Swarttouw, The Askey-Scheme of hypergeometric orthogonal polynomials and its q-analogues. reports of the Faculty of Technical Mathematics and Information 94-05, Delft University of Technology, Delft, 1999.

[17] E. D. Rainville, Special Functions. New York, 1960.

[18] D. P. Roberts, Discriminants of some Painlevé polynomials, to appear.

[19] E. Routh, On some properties of certain solutions of a differential equation of the second order, Proc. London Math. Soc. **16** (1884), 245 – 261.

[20] J. Shohat and J.D. Tamarkin, The problem of Moments. revised edition, Amer.Math.Soc., providence, 1950

[21] G. Szego, Orthogonal Polynomials, fourth edition, American Mathematical Society, Providence, Rhode Island, 1975.

[22] G. N. Watson, A treatise on the theory of Bessel functions, 2nd ed., The University Press, Cambridge, 1958.

About the Author

Jemal Emina Gishe was born in 1974, Goffore, Oromia (Ethiopia). He received B.Sc. (1995) and M.Sc. (2000) in pure Mathematics from Addis Ababa University and taught at Hara Maya University, Ethiopia as an Assistant Lecturer. He attended a diploma program at Abdus Salam International Center for Theoretical Physics (ICTP) in Trieste, Italy in the year 2000-2001.

In the Fall 2001 he was admitted to a graduate program in mathematics at University of South Florida, Tampa where he worked under supervisor Prof. Mourad Ismail. His scholarly interests are Analysis, Orthogonal Polynomials and Special Functions.