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Polynomial quandle cocycles, their knot invariants and applications

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Polynomial Quandle Cocycles, Their Knot Invariants and Applications

by

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Dedication

To my mother, father and my brothers and sisters. Their constant love and support has guided me to where I am. My gratitude to them could never be expressed through words.

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ABSTRACT

A quandle is a set with a binary operation that satisfies three axioms that correspond to the three Reidemeister moves on knot diagrams. Homology and cohomology theories of quandles were introduced in 1999 by Carter, Jelsovsky, Kamada, Langford, and Saito as a modification of the rack (co)homology theory defined by Fenn, Rourke, and Sanderson. Cocycles of the quandle cohomology, along with quandle colorings of knot diagrams, were used to define a new invariant called the quandle cocycle invariants, defined in a state-sum form. This invariant is constructed using a finite quandle and a cocyle, and it has the advantage that it can distinguish some knots from their mirror images, and orientations of knotted surfaces.

To compute the quandle cocycle invariant for a specific knot, we need to find a quandle that colors the given knot non-trivially, and find a cocycle of the quandle. It is not easy to find cocycles, since the cocycle conditions form a large, over-determined system of linear equations. At first the computations relied on cocycles found by computer calculations. We have seen significant progress in computations after Mochizuki discovered a family of 2- and 3-cocycles for dihedral and other linear Alexander quandles written by polynomial expressions.

In this dissertation, following the method of the construction by Mochizuki, a variety of n-cocycles for $n \geq 2$ are constructed for some Alexander quandles, given by polynomial expressions. As an application, these cocycles are used to compute the invariants for $(2, n)$-torus knots, twist knots and their r-twist spins. The calculations

in the case of $(2, n)$-torus knots resulted in formulas that involved the derivative of the Alexander polynomial. Non-triviality of some quandle homology groups is also proved using these cocycles.

Another application is given for tangle embeddings. The quandle cocycle invariants are used as obstructions to embedding tangles in links. The formulas for the cocycle invariants of tangles are obtained using polynomial cocycles, and by comparing the invariant values, information is obtained on which tangles do not embed in which knots. Tangles and knots in the tables are examined, and concrete examples are listed.

Chapter 1

Introduction

In this chapter, we first give an overview of the history of using quandles to define knot invariants. Next we give an outline of the results contained in each chapter of this dissertation. Then we review standard definitions and theorems that are used throughout this dissertation; in particular, quandles and quandle homology theories are reviewed, and the quandle cocycle invariants are defined.

1.1 Overview

The notion of a knot quandle was introduced by Joyce [25] and Matveev [30], independently, in 1982. They associated a quandle called the fundamental quandle to a given knot, and proved that it is a complete knot invariant up to orientation.

The fundamental quandle is defined similarly for knotted surfaces in 4-space using diagrams [13]. Although fundamental knot quandles are strong invariants, it is not easy to use them to distinguish knots by direct calculations, since they are defined by generators and relations. One way to use them is to calculate the number of homeomorphisms to a given finite quandle. For example, Fox 3-colorings are homeomorphisms of the fundamental knot quandle to the dihedral quandle of order 3.

In the late 1999's and early 2000's, homology and cohomology theories for quandles appeared in [9], as a modification of the rack (co)homology theory defined in [19]. Cocycles of the quandle cohomology, along with quandle colorings of knot diagrams, were used to define a new invariant called the quandle cocycle invariant [9]. The invariant has the advantage that it can distinguish some knoted surfaces from their orientation reversed counterpart, which the fundamental knot quandles fail to do.

To compute the quandle cocycle invariant for a specific knot, we need to find a quandle that colors a given knot non-trivially, and to find a cocycle of the quandle. It is not easy to find cocycles, since the cocycle conditions form a large, over-determined system of linear equations. At first the computations relied on cocycles found by computer calculations. We have seen significant progress in computations after Mochizuki [31] discovered a family of 2- and 3-cocycles for dihedral and other linear Alexander quandles written by polynomial expressions. Formulas for important families of knots and knotted surfaces and their applications followed [1, 24].

In knot theory, whenever a new invariant is defined, computation of the invariant for knots in the knot table, or for some typical families of knots are performed, so that we can get some data to be used for distinguishing knots and for other applications. The quandle cocycle invariants define a large family of knot invariants. Computation of these invariants involve finding cocycles. One of the main goals of this dissertation is to develop new computations of the quandle cocycle invariants, and this is done by constructing a new family of cocycles. This dissertation contains four chapters. In the current chapter, Chapter 1, we recall standard definitions. In Chapter 2 we give a new construction of a large family of $n-$cocycles for some Alexander quandles. In Chapter 3 we compute colorings by Alexander quandles for two families of knots, $(2, n)$-torus knots and twist knots, then we compute the 2-and 3-cocycle invariants for these knots using the cocycles constructed in chapter 2. In the case of $(2, n)$-torus knots, the formulas obtained involve the derivative of the Alexander Polynomial. These formulas are, then, used to prove nontriviality of some quandle (co)homology groups. Chapter 4 contains two applications; first we compute the quandle cocycle invariant for the r-twist spin of torus knots and twist knots, then, cocycle invariants are used as obstructions to embedding tangles in links. The formulas for the cocycle invariants of tangles are obtained using polynomial cocycles, and by comparing the invariant values, some information is obtained on which tangles do not embed in which knots. Tangles and knots in the tables are examined, and concrete examples are listed.

The following sections consist of reviews of standard definitions that are used throughout this dissertation.

1.2 Knot Diagrams

A classical knot is a smooth embedding of $S^1 = \{(x, y) \in \mathbb{R} : x^2 + y^2 = 1\}$ into $\mathbb{R}^3$ or into $S^3 = \{(x, y, z, w) \in \mathbb{R}^4 : x^2 + y^2 + z^2 + w^2 = 1\}$. An $n-$component link is a smooth embedding of n disjoint copies of S^1 into $\mathbb{R}^3$ or into S^3, for a positive integer $n \geq 1$. Since knots and links are 1-manifolds, they are orientable, and we specify an orientation on each component. We say that two knots K_1 and K_2 are equivalent if there is an orientation-preserving homeomorphism of $\mathbb{R}^3$ that maps K_1 to K_2 preserving the orientations of K_1 and K_2. Let $K \subset \mathbb{R}^3$ be a knot, and $P \subset \mathbb{R}^3$ be a plane such that $K \cap P = \emptyset$. Let $p : \mathbb{R}^3 \rightarrow P$ be the orthogonal projection. Then we may assume without loss of generality that the restriction $p|_K : K \rightarrow P$ is a generic immersion. This implies that the image $p(K)$ is an immersed closed curve in the plane with only transverse double intersection points (called *crossings*), other than embedded points. At a crossing, two arcs intersect. The preimage in K of one of the arcs is further away from P is called *over-arc* than the other that is called *under-arc*. A *knot diagram* is such a projection image $p(K)$ with the lower arc broken to indicate the crossing information. Instead of dealing with a knot or a link as closed curves in S^3 or $\mathbb{R}^3$, we will deal with their regular diagrams. Using knot diagrams, equivalence of knot can be defined as follows: two knots K and K' with diagrams D, D' respectively are equivalent if and only if D can be transformed into D' by a sequence of three moves called Reidemeister moves of type I, II, and III, see Figure 1.1. See [32] for more details.

1.3 Quandles

The name *quandle* appeared first in [25]. Similar structures were considered much earlier, for example in [42], from an algebraic point of view.

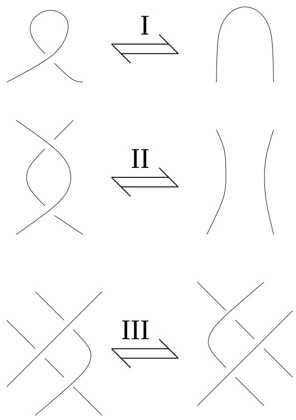

Figure 1.1: Reidmeister Moves

Definition 1.3.1 A *quandle*, X, is a set with a binary operation $(a, b) \mapsto a * b$ such that

(I) For any $a \in X$, $a * a = a$.

(II) For any $a, b \in X$, there is a unique $c \in X$ such that $a = c * b$.

(III) For any $a, b, c \in X$, we have $(a * b) * c = (a * c) * (b * c)$.

A *rack* is a set with a binary operation that satisfies (II) and (III). Racks and quandles have been studied in, for example, [3, 18, 25, 30].

The following typical examples are found in the literature mentioned above.

Example 1.3.2 Any non-empty set X with a quandle operation defined by $x * y = x$ for any x and y in X, is a quandle called *the trivial quandle*.

Example 1.3.3 A group G with conjugation as the quandle operation, $a * b = bab^{-1}$, denoted by X =Conj(G), is a quandle. Any subset of G that is closed under such conjugation is also a quandle.

Example 1.3.4 Let n be a positive integer, and for elements $i, j \in \{0, 1, \ldots, n-1\}$, define $i * j \equiv 2j - i \pmod{n}$. Then $*$ defines a quandle structure called the *dihedral quandle*, R_n.

Example 1.3.5 Let Λ be the Laurent polynomial ring $\mathbb{Z}[t, t^{-1}]$. Then, any Λ-module M is a quandle with $a*b = ta+(1-t)b$, $a, b \in M$, that is called an *Alexander quandle*.

In particular for any Laurent polynomial $h(t)$, where the coefficients of the highest and lowest degree are invertible in $\mathbb{Z}_n$, $\mathbb{Z}_n[t,t^{-1}]/(h(t))$ is a finite quandle. Then if $h(t) = 1+t$ the Alexander quandle operation $a*b$ becomes $a*b = 2b-a$; thus the quandle $\mathbb{Z}_n[t,t^{-1}]/1+t$ can be considered as the dihedral quandle R_n

1.4 Cohomology Theory of Quandles

Homology and cohomology theories for quandles appeared in [9], as a modification of the rack (co)homology theory defined in [19]. Using quandles, a new invariant was introduced by Carter, Jelsovsky, Kamada, Langford, and Saito [9]. In this section we review their definitions.

Let $C_n^R(X)$ be the free Abelian group generated by n-tuples $(x_1,\ldots,x_n)$ of elements of a quandle X. Define a homomorphism $\partial_n : C_n^R(X) \longrightarrow C_{n-1}^R(X)$ by:

$$\begin{aligned}\partial_n(x_1,\ldots,x_n) = & \sum_{i=2}^{n}(-1)^i[(x_1,x_2,\ldots,x_{i-1},x_{i+1},\ldots,x_n)\\ - & (x_1*x_i,x_2*x_i,\ldots,x_{i-1}*x_i,x_{i+1},\ldots,x_n)]\end{aligned}$$

for $n \geq 2$ and $\partial_n = 0$ for $n \leq 1$. Then $C_*^R(X) = \{C_n^R(X),\partial_n\}$ is a chain complex. Let $C_n^D(X)$ be the subset of $C_n^R(X)$ generated by n-tuples $(x_1,\ldots,x_n)$ with $x_i = x_{i+1}$ for some $i \in \{1,\ldots,n-1\}$ if $n \geq 2$; otherwise let $C_n^D(X) = 0$. If X is a quandle, then $\partial_n(C_n^D(x)) \subset C_{n-1}^D(X)$ and $C_*^D(X) = \{C_n^D(X),\partial_n\}$ is a subcomplex of $C_*^R(X)$. Put $C_n^Q = C_n^R(X)/C_n^D(X)$ and $C_*^Q(X) = \{C_n^Q(X),\partial'_n\}$ where, ∂'_n is the induced homomorphism. Henceforth, all boundary maps may be denoted by ∂_n. The superscripts R, Q and D, respectively, represent rack, quandle, and degenerate chain complexes. For an Abelian group G, define the chain and the cochain complexes by
$C_*^W(X;G) = C_*^W(X)\otimes G, \quad \partial = \partial\otimes id;$
$C^*_W(X;G) = \mathrm{Hom}(C_*^W(X),G), \ \delta = \mathrm{Hom}(\partial, id),$
in the usual way, where $W = D, R, Q$.

The coboundary map is written

$$(\delta_n f)(x_1,\ldots,x_{n+1}) = \sum_{i=2}^{n+1}(-1)^i[f(x_1,x_2,\ldots,x_{i-1},x_{i+1},\ldots,x_{n+1})$$
$$- \quad f(x_1 * x_i, x_2 * x_i, \ldots, x_{i-1} * x_i, x_{i+1}, \ldots, x_{n+1})]$$

where $f \in C^n_W(X;G)$. A map $f \in C^{n+1}_W(X;G)$ is called a *coboundary*, if there is a map $g \in C^n_W(X;G)$ such that $\delta_n g = f$.

The groups of cycles and boundaries are denoted respectively by $ker(\partial) = Z^W_n(X;G) \subset C^W_n(X;G)$ and $\mathrm{Im}(\partial) = B^n_W(X;G) \subset C^n_W(X;G)$ while the cocycles and coboundaries are denoted respectively by
$ker(\delta) = Z^n_W(X;G) \subset C^n_W(X;G)$ and $Im(\delta) = B^n_W(X;G) \subset C^n_W(X;G)$. The n-th quandle homology group with coefficient group G is defined by

$$H^Q_n(X;G) = H_n(C^Q_*(X;G)) = Z^Q_n(X;G)/B^Q_n(X;G).$$

The quandle cohomology group with coefficient group G is defined by

$$H^n_Q(X;G) = H^n(C^*_Q(X;G)) = Z^n_Q(X;G)/B^n_Q(X;G).$$

For the purpose of constructing knot invariants for applications to knots in later sections, we mainly use quandle 2- and 3-cocycles. The properties of these cocycles we need for the knot invariants are specifically formulated as follows. A 2-cocycle ϕ is regarded as a function $\phi : X \times X \to A$ with the 2-cocycle condition

$$\phi(x,y) + \phi(x*y,z) = \phi(x,z) + \phi(x*z,y*z)$$

for all $x,y,z \in X$, and $\phi(x,x) = 0$ for all $x \in X$. A 3-cocycle θ is regarded as a function $\theta : X \times X \times X \to A$ with the 3-cocycle condition

$$\theta(x,z,w)+\theta(x,y,z)+\theta(x*z,y*z,w) = \theta(x*z,y,w)+\theta(x,y,w)+\theta(x*w,y*w,z*w)$$

for any $x, y, z, w \in X$, and $\theta(x, x, y) = 0$, $\theta(x, y, y) = 0$ for all $x, y \in X$.

1.5 Colorings of Knot Diagrams by Quandles

Colorings of knot diagrams go back to Fox n-colorings [20], that correspond to homomorphisms of knot groups to the dihedral groups. Fox [20] gave conditions on the Alexander polynomial of a given knot for its diagram to have non-trivial n-colorings. Such colorings were generalized to racks and quandles [19]. We review the definitions.

Let X be a fixed quandle. Let K be a given oriented classical knot or link diagram, and let $\mathcal{R}$ be the set of (over-)arcs. The normals (normal vectors) are given in such a way that the ordered pair (tangent, normal) agrees with the orientation of the plane, see Fig. 1.2. A (quandle) *coloring* $\mathcal{C}$ is a map $\mathcal{C} : \mathcal{R} \to X$ such that at every crossing, the relation depicted in Fig. 1.2 holds. Specifically, let β be the over-arc at a crossing, and let α and γ be the under arcs, such that the normal of the over-arc points from α to γ, then $\mathcal{C}(\alpha) * \mathcal{C}(\beta) = \mathcal{C}(\gamma)$ holds.

The element $a = \mathcal{C}(\alpha) \in \mathcal{X}$ assigned to an arc α is called the *color* of α. The (ordered) colors $(\mathcal{C}(\alpha), \mathcal{C}(\beta))$ are called *source* colors. Let $\mathrm{Col}_X(D)$ denote the set of colorings of a knot diagram D of a knot K by a quandle X.

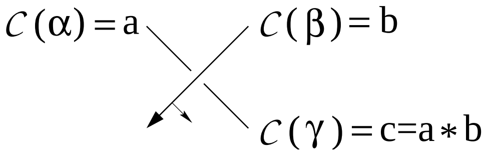

Figure 1.2: Quandle relation at a crossing

The following lemma is well-known [19], and is proved for each Reidemeister move.

Lemma 1.5.1 *Let D_2 be a diagram of a knot K obtained from D_1 by a Reidemeister move. Then there is a one-to-one correspondence between $\mathrm{Col}_X(D_1)$ and $\mathrm{Col}_X(D_2)$ induced from the move. In particular, the number of colorings $|\mathrm{Col}_X(D)|$ is a knot invariant.*

1.6 The Quandle 2-Cocycle Invariant

The cocycle invariant for classical knots [9] was defined as follows. Let $\phi \in Z^2_{\mathrm{Q}}(X; A)$ be a 2-cocycle of a finite quandle X with the coefficient group A. Recall that this ϕ is regarded as a function $X \times X \to A$ that satisfies the 2-cocycle condition

$$\phi(x,y) - \phi(x,z) + \phi(x*y,z) - \phi(x*z,y*z) = 0, \quad \forall x,y,z \in X$$

and $\phi(x,x) = 0$, for any $x \in X$. Let $\mathcal{C}$ be a coloring of a given knot diagram K by X. The *Boltzmann weight* $B(\mathcal{C},\tau)$ at a crossing τ of K is then defined by $B(\mathcal{C},\tau) = \phi(x_\tau, y_\tau)^{\epsilon(\tau)}$, where $(x_\tau,\, y_\tau)$ are source colors at τ and $\epsilon(\tau)$ is the sign (± 1) of τ. In Fig. 1.2, it is a positive crossing, and its sign is $+1$, if the under-arc is oriented downward. Here $B(\mathcal{C},\tau)$ is an element of A written multiplicatively. The formal sum (called a *state-sum*) in the group ring $\mathbb{Z}[A]$

$$\Phi_\phi(K) = \sum_{\mathcal{C} \in \mathrm{Col_X(K)}} \prod_\tau B(\mathcal{C},\tau)$$

is called the *quandle cocycle invariant.*

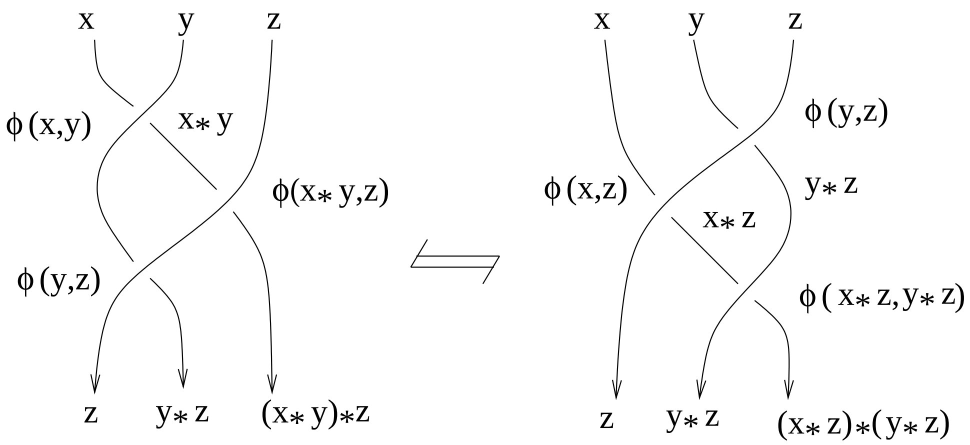

Figure 1.3: The 2-cocycle condition and type III move

Fig.1.3 shows how the cocycle condition corresponds to the type III Reidmeister move. The cocycle condition equates the sum of the cocycle values at the crossings before and after the move, which makes the state sum invariant under the type III Reidmeister move .The following was proved by checking Reidmeister moves.

Theorem 1.6.1 [9] *The state-sum $\Phi_\phi(K)$ does not depend on the choice of a diagram D of a given knot K, so that it is a well-defined knot invariant.*

The cocycle invariant can be also written as a family (multi-set, a set with repetition allowed) of weight sums

$$\left\{ \sum_{\tau} B(\mathcal{C}, \tau) \;\middle|\; \mathcal{C} \in \mathrm{Col}_{\mathrm{X}}(\mathrm{K}) \right\},$$

where the values of $B(\mathcal{C}, \tau)$ in A are denoted by additive notation.

1.7 The Quandle 3-Cocycle Invariant

The quandle cocycle invariant using region colorings (sometimes called "*shadow*" colorings) and 3-cocycles were considered in [19]. We review the definitions in this section.

For a coloring $\mathcal{C}$, there is a coloring of regions that extend $\mathcal{C}$ as depicted in Fig. 1.4. Let $(x, y, z) = (x_\tau, y_\tau, z_\tau)$ be the colors near a crossing τ such that x is the color of the region (called the source region) from which both orientation normals of over- and under-arcs point, y is the color of the under-arc (called the source under-arc) from which the normal of the over-arc points, and z is the color of the over-arc. See Fig. 1.4.

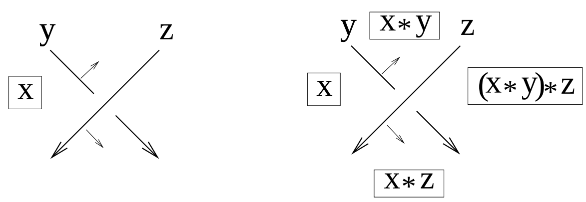

Figure 1.4: Quandle colorings of regions

Let $\phi \in Z^3_{\mathrm{Q}}(X; A)$ be a 3-cocycle. Then the weight in this case is defined by

$B(\mathcal{C},\tau) = \phi(x_\tau, y_\tau, z_\tau)^{\epsilon(\tau)}$ where $\epsilon(\tau)$ is $+1$ or -1, for a positive or a negative crossing respectively. Then the 3-cocycle invariant is defined by

$\Phi_\phi(K) = \sum_{\mathcal{C}\in \mathrm{Col_X(K)}} \prod_\tau B(\mathcal{C},\tau)$ as a state-sum, and by $\{\sum_\tau B(\mathcal{C},\tau) \mid \mathcal{C} \in \mathrm{Col_X(K)}\}$. as a multiset.

Chapter 2

Polynomial Cocycles of Alexander Quandles

We consider Alexander quandles $X = \mathbb{Z}_p[t, t^{-1}]/h(t)$ and their quandle cohomology groups with the coefficient group $A = \mathbb{Z}_p[t, t^{-1}]/g(t)$, where $h(t), g(t) \in \mathbb{Z}_p[t, t^{-1}]$ and $g(t)$ divides $h(t)$, so that there is the quotient homomorphism $X \to A$.

Some polynomial cocycles have been considered by Mochizuki [31]. Our goal is to examine certain polynomials in full generality, finding which polynomials become cocycles for which Alexander quandles.

2.1 Polynomial n-Cocycles

First we show that polynomials of the following form are n-cocycles for a positive integer n in certain Alexander quandles.

Proposition 2.1.1 *Let $a_i = p^{m_i}$, for $i = 1, \ldots, n-1$, where p is a prime and m_i are non-negative integers. For a positive integer n, let $f : X^n \to A$ be defined by*

$$f(x_1, x_2, \ldots, x_n) = (x_1 - x_2)^{a_1}(x_2 - x_3)^{a_2} \cdots (x_{n-1} - x_n)^{a_{n-1}} x_n^{a_n},$$

where the x_i's in the right-hand-side are regarded as elements of A via the quotient map.

1. *If $a_n = 0$, Then f is an n-cocycle ($\in Z^n_{\mathrm{Q}}(X; A)$).*
2. *If $a_n = p^{m_n}$ (for a positive integer m_n), then f is an n-cocycle if $g(t)$ divides $1 - t^a$, where $a = a_1 + a_2 + \cdots + a_{n-1} + a_n$.*

Proof. From the definition of δ, for $i = 1, \ldots, n$, we compute

$$\begin{aligned}
&\delta f(x_1, \ldots, x_{n+1}) \\
&= \sum_{i=2}^{n+1} (-1)^i [f(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1}) \\
&- f(x_1 * x_i, \ldots, x_{i-1} * x_i, x_{i+1}, \ldots, x_{n+1}) \\
&= \sum_{i=2}^{n} (-1)^i (x_1 - x_2)^{a_1} \cdots (x_{i-1} - x_{i+1})^{a_{i-1}} (x_{i+1} - x_{i+2})^{a_i} \\
&\qquad \cdots (x_n - x_{n+1})^{a_{n-1}} x_{n+1}^{a_n} \\
&+ (-1)^{n+1} (x_1 - x_2)^{a_1} \cdots (x_{n-1} - x_n)^{a_n} x_n^{a_n} \\
&- \sum_{i=2}^{n} (-1)^i (x_1 * x_i - x_2 * x_i)^{a_1} \cdots (x_{i-1} * x_i - x_{i+1})^{a_i - 1} \\
&\qquad \cdots (x_n - x_{n+1})^{a_n - 1} x_{n+1}^{a}] \\
&- (-1)^{n+1} t^a (x_1 * x_{n+1} - x_2 * x_{n+1})^{a_1} \\
&\qquad \cdots (x_{n-1} * x_{n+1} - x_n * x_{n+1})^{a_{n-1}} (x_n * x_{n+1})^{a_n}
\end{aligned}$$

which is simplified using $y_i = x_i - x_{i+1}$ as

$$\begin{aligned}
&= \sum_{i=2}^{n} (-1)^i y_1^{a_1} y_2^{a_2} \cdots (y_{i-1} + y_i)^{a_{i-1}} y_{i+1}^{a_i} \cdots y_n^{a_{n-1}} x_{n+1}^{a_n} \\
&+ (-1)^{n+1} y_1^{a_1} y_2^{a_2} \cdots y_{n-1}^{a_{n-1}} x_n^{a_n} \\
&- \sum_{i=2}^{n} (-1)^i (ty_1)^{a_1} (ty_2)^{a_2} \cdots (ty_{i-1} + y_i)^{a_{i-1}} \cdots y_n^{a_{n-1}} x_{n+1}^{a} \\
&- (-1)^{n+1} t^a y_1^{a_1} y_2^{a_2} \cdots y_{n-1}^{a_{n-1}} (ty_n + x_{n+1})^{a_n}.
\end{aligned}$$

Since each a_i is a power of p, with the coefficients in $\mathbb{Z}_p$ we obtain

$$\begin{aligned}
&\delta f(x_1, \ldots, x_{n+1}) \\
&= \sum_{i=2}^{n} (-1)^i y_1^{a_1} y_2^{a_2} \cdots y_{i-1}^{a_{i-1}} y_{i+1}^{a_i} \cdots y_n^{a_{n-1}} x_{n+1}^{a_n} \\
&+ \sum_{i=2}^{n} (-1)^i y_1^{a_1} y_2^{a_2} \cdots y_{i-2}^{a_{i-2}} y_i^{a_{i-1}} \cdots y_n^{a_{n-1}} x_{n+1}^{a_n}
\end{aligned}$$

$$
\begin{aligned}
- & \sum_{i=2}^{n}(-1)^i t^{a_1+\cdots+a_{i-1}} y_1^{a_1}\cdots y_{i-1}^{a_{i-1}} y_{i+1}^{a_i}\cdots y_n^{a_{n-1}} x_{n+1}^{a_n} \\
- & \sum_{i=2}^{n}(-1)^i t^{a_1+\cdots+a_{i-2}} y_1^{a_1}\cdots y_{i-2}^{a_{i-2}} y_i^{a_{i-1}}\cdots y_n^{a_{n-1}} x_{n+1}^{a_n} \\
+ & (-1)^{n+1} y_1^{a_1} y_2^{a_2}\cdots y_{n-1}^{a_{n-1}} x_n^{a_n} \\
- & (-1)^{n+1} t^{a_1+\cdots+a_{n-1}} y_1^{a_1}\cdots y_{n-1}^{a_{n-1}} (ty_n + x_{n+1})^{a_n}.
\end{aligned}
$$

By changing the indices in the second and the fourth sum, we obtain

$$
\begin{aligned}
& \delta f(x_1,\ldots,x_{n+1}) \\
= & \sum_{i=2}^{n}(-1)^i y_1^{a_1} y_2^{a_2}\cdots y_{i-1}^{a_{i-1}} y_{i+1}^{a_i}\cdots y_n^{a_{n-1}} x_{n+1}^{a_n} \\
- & \sum_{i=1}^{n-1}(-1)^i y_1^{a_1} y_2^{a_2}\cdots y_{i-1}^{a_{i-1}} y_{i+1}^{a_i}\cdots y_n^{a_{n-1}} x_{n+1}^{a_n} \\
- & \sum_{i=2}^{n}(-1)^i t^{a_1+\cdots+a_{i-1}} y_1^{a_1}\cdots y_{i-1}^{a_{i-1}} y_{i+1}^{a_i}\cdots y_n^{a_{n-1}} x_{n+1}^{a_n} \\
+ & \sum_{i=1}^{n-1}(-1)^i t^{a_1+\cdots+a_{i-1}} y_1^{a_1}\cdots y_{i-1}^{a_{i-1}} y_{i+1}^{a_i}\cdots y_n^{a_{n-1}} x_{n+1}^{a_n} \\
+ & (-1)^{n+1} y_1^{a_1} y_2^{a_2}\cdots y_{n-1}^{a_{n-1}} x_n^{a_n} \\
- & (-1)^{n+1} t^{a_1+\cdots+a_{n-1}} y_1^{a_1}\cdots y_{n-1}^{a_{n-1}} (ty_n + x_{n+1})^{a_n}
\end{aligned}
$$

which simplifies to

$$
\begin{aligned}
& (-1)^n y_1^{a_1}\cdots y_{n-1}^{a_{n-1}} x_{n+1}^{a_n} \\
- & (-1)^n t^{a_1+\cdots+a_{n-1}} y_1^{a_1}\cdots y_{n-1}^{a_{n-1}} x_{n+1}^{a_n} \\
+ & (-1)^{n+1} y_1^{a_1} y_2^{a_2}\cdots y_{n-1}^{a_{n-1}} x_n^{a_n} \\
- & (-1)^{n+1} t^{a_1+\cdots a_{n-1}} y_1^{a_1}\cdots y_{n-1}^{a_{n-1}} (ty_n + x_{n+1})^{a_n}.
\end{aligned}
$$

If $a_n = 0$ then we see $\delta f(x_1,..,x_{n+1}) = 0$. If $a_n = p^{m_n}$ then we have

$$
\delta f(x_1,\ldots,x_{n+1})
$$

$$
\begin{aligned}
&= (-1)^n y_1^{a_1} \cdots y_{n-1}^{a_{n-1}} x_{n+1}^a + (-1)^{n+1} y_1^{a_1} y_2^{a_2} \cdots y_{n-1}^{a_{n-1}} x_n^{a_n} \\
&\quad -(-1)^{n+1} t^{a_1+\cdots+a_{n-1}+a_n} y_1^{a_1} \cdots y_n^{a_n} \\
&= (-1)^{n+1}(1-t^a) y_1^{a_1} \cdots y_n^{a_n} \quad = \quad 0 \in A
\end{aligned}
$$

by assumption. Hence f is an n-cocycle. $\square$

Corollary 2.1.2 *Let $X = A = \mathbb{Z}_p[t, t^{-1}]/h(t)$, where $h(t) \in \mathbb{Z}_p[t, t^{-1}]$, $a_i = p^{m_i}$, for $i = 0, 1, \ldots, n$, $n > 1$, where p is a prime and m_i are non-negative integers. Let f be a map from X^{n+1} to A defined by*

$$f(x_1, x_2, \ldots, x_n, x_{n+1}) = (x_1 - x_2)^{a_1}(x_2 - x_3)^{a_2} \ldots (x_n - x_{n+1})^{a_n}.$$

If $1 - t^{a_1+a_2+\ldots+a_n}$ is invertible in X, then f is a coboundary.

Proof. Let $l(t)$ be the inverse of $1 - t^{a_1+a_2+\ldots+a_n}$ in X, and let

$$f_1(x_1, x_2, \ldots, x_n) = (-1)^{n+1} l(t)(x_1 - x_2)^{a_1}(x_2 - x_3)^{a_2} \cdots (x_{n-1} - x_n)^{a_{n-1}} x_n^{a_n},$$

then from Proposition 2.1.1 we have

$$\delta f_1(x_1, x_2, \ldots, x_{n+1}) = (-1)^{n+1} l(t)[(-1)^{n+1}(1 - t^a) y_1^{a_1} \cdots y_n^{a_n}],$$

where $y_i = x_i - x_{i+1}$ and $a = a_1 + a_2 + \cdots + a_n$.
Since $l(t)(1 - t^a) = 1$, $\delta f_1(x_1, x_2, \ldots, x_{n+1}) = f(x_1, x_2, \ldots, x_{n+1})$. $\square$

Corollary 2.1.3 *Let $X = A = R_p$, such that p is an odd prime. If n is odd, then*

$$f(x_1, x_2, \ldots, x_{n+1}) = (x_1 - x_2)^{a_1}(x_2 - x_3)^{a_2} \ldots (x_n - x_{n+1})^{a_n}$$

is a coboundary.

Proof. Since $X = R_p$, and n is odd, we have $1 - t^a = 2$ in X. Then $1 - t^a$ is invertible in X. So by Corollary 2.1.2, f is a coboundary. $\square$

Corollary 2.1.4 *Let $X = A = \mathbb{Z}_p[t, t^{-1}]/(t^k + t^a - 1)$, k is any nonzero integer and $a = a_1 + a_2 + \cdots + a_n$, where $a_i = p^{m_i}$, for $i = 0, 1, \ldots, n$, $n > 1$, p is a prime and m_i are non-negative integers. Then $f(x_1, x_2, \ldots, x_{n+1}) = (x_1 - x_2)^{a_1}(x_2 - x_3)^{a_2} \ldots (x_n - x_{n+1})^{a_n}$ is a coboundary.*

Proof. In X, $1 - t^{a_1+a_2+\cdots+a_n} = -t^k$, then $1 - t^{a_1+a_2,+\cdots+a_n}$ is invertible, then f is a coboundary by Corollary 2.1.2. □

Example 2.1.5 For $p = 2$, we find the following examples.

- For $X = A = \mathbb{Z}_2[t, t^{-1}]/(t^3 + t + 1)$, $f(x_1, x_2, x_3) = (x_1 - x_2)^2(x_2 - x_3)$ is a coboundary of $f_1(x_1, x_2) = (1 + t^2)(x_1 - x_2)^2 x_2$.
- For $X = A = \mathbb{Z}_2[t, t^{-1}]/(t^3 + t^2 + 1)$, $f(x_1, x_2, x_3) = (x_1 - x_2)^2(x_2 - x_3)$ is a coboundary of $f_1(x_1, x_2) = (1 + t)(x_1 - x_2)^2 x_2$.

2.2 Examples of Polynomial Cocycles

Here we list cocycles we use for the rest of the dissertation. Let p be a prime number, and take $X = \mathbb{Z}_p[t, t^{-1}]/g(t) = A$.

- $f(x, y) = (x - y)^{p^n}$ is a 2-cocycle for any Alexander quandle mod p.
- $f(x, y) = (x - y)^{p^{m_1}} y^{p^{m_2}}$ is a 2-cocycle for an Alexander quandle $\mathbb{Z}_p[t, t^{-1}]/g(t)$ if $g(t)$ divides $(t^{p^{m_1}+p^{m_2}} - 1)$.
- $f(x, y, z) = (x - y)^{p^{m_1}}(y - z)^{p^{m_2}}$ is a 3-cocycle for any Alexander quandle mod p.
- $f(x, y, z) = (x - y)^{p^{m_1}}(y - z)^{p^{m_2}} z^{p^{m_3}}$ is a 3-cocycle for an Alexander quandle $\mathbb{Z}_p[t, t^{-1}]/g(t)$ if $g(t)$ divides $(t^{p^{m_1}+p^{m_2}+p^{m_3}} - 1)$.

More specific polynomials and Alexander quandles we consider are as follows.

- 2-cocycles:

  (a) $p = 2$

* $f(x, y) = (x - y)^{2^m} y$.

  Alexander quandle $\mathbb{Z}_2[t, t^{-1}]/(t^2 + t + 1)$ has a non-trivial 2-cocycle $f(x, y) = (x - y)^2 y$. This 4 element quandle is well-known. It is isomorphic to the quandle consisting of 120 degree rotations of a regular tetrahedron. It is known to have 2-dimensional cohomology group $\mathbb{Z}_2$ with $\mathbb{Z}_2$ coefficient. The invariant values are all of the form 16, or $k[4 + 12u^{(t+1)}]$, so we conjecture that it is always the case. It is also an interesting problem to characterize the values of this invariant.

* $f(x, y) = (x - y)^{2^2} y$.

  The quandle must be mod $g(t)$ where $g(t)$ divides $t^5 - 1$, but $t^5 - 1$ is factored into prime polynomials $(t + 1)(t^4 + t^3 + t^2 + t^2 + 1)$ mod 2, so we set $g(t) = t^4 + t^3 + t^2 + t^2 + 1$. The Alexander quandle we use in this case, thus, is $\mathbb{Z}_2[t, t^{-1}]/(t^4 + t^3 + t^2 + t + 1)$ with 2-cocycle: $f(x, y) = (x - y)^{2^2} y$, which gives non-trivial invariants by computer calculations [41].

* $f(x, y) = (x - y)^{2^3} y$.

  We factor $t^9 - 1$ mod 2 to $(t+1)(t^2+t+1)(t^6+t^3+1)$, so we try Alexander quandle $\mathbb{Z}_2[t, t^{-1}]/(t^6 + t^3 + 1)$, which gives non-trivial invariants by computer calculations [41].

(b) $p = 3$

* $f(x, y) = (x - y)^3 y$.

  We have $t^4 - 1 = (t + 1)(t + 2)(t^2 + 1)$ mod 3, so we try the quandle $\mathbb{Z}_3[t, t^{-1}]/(t^2 + 1)$ which gives non-trivial invariants by computer calculations [41].

- 3-cocycles:

  (a) $p = 2$, $f(x, y, z) = (x - y)(y - z)^2$. The cocycle is non-trivial in the quandle: $\mathbb{Z}_2[t, t^{-1}]/(t^2 + t + 1)$.

  (b) $p = 3$, $f(x, y, z) = (x - y)(y - z)^3$. The cocycle is non-trivial in the quandles:

$\mathbb{Z}_3[t,t^{-1}]/(t^2+1)$, $\mathbb{Z}_3[t,t^{-1}]/(t^2-t+1)$.

(c) $p=5$, $f(x,y,z)=(x-y)(y-z)^5$. The cocycle is non-trivial in the quandle: $\mathbb{Z}_5[t,t^{-1}]/(t^2-t+1)$.

(d) $p=7$, $f(x,y,z)=(x-y)(y-z)^7$. The cocycle is non-trivial in the quandle: $\mathbb{Z}_7[t,t^{-1}]/(t^2-t+1)$.

Many computer calculations in [41] are based on these cocycles.

Chapter 3

Colorings and Cocycle Invariants of Knots with Alexander Quandles

In this chapter we compute the colorings by Alexander quandles for some torus knots and twist knots. We then use the cocycles constructed in Chapter 2 to compute their 2- and 3-cocycles invariants.

3.1 Colorings of Knot Diagrams by Alexander Quandles

3.1.1 Coloring $(2, m)$-Torus Knots by Alexander Quandles

Let X be an Alexander quandle, and $K(2, m)$ be a $(2, m)$-torus knot or link represented by the closure of the braid σ_1^m (see Fig 3.1), where σ_1 is the standard generator of the 2-string braid group B_2, and m is a positive integer, see [32] for definitions regarding braid theory. The generator σ_1 is identified with a positive crossing in the diagrams. We consider colorings of $K(2, m)$ using the closed braid form. For a given coloring, the vector $(a, b) \in X \times X$ of colors assigned to the left and right, respectively, of the top two strings, is called the top color vector. The situation is depicted in Fig. 3.1. The vector formed by a pair of colors assigned to a pair of strings just below the k-th crossing ($1 \leq k \leq m$) is called the k-th color vector. Denote by ξ_k the polynomial

$$\xi_k = \sum_{i=0}^{k-1} (-t)^i,$$

and define $\xi_0 = 0$ as convention. Note that for $k > 1$, the polynomial ξ_k is the Alexander polynomial $\Delta_{K(2,m)}(t)$ of the knot $K(2, m)$ [32].

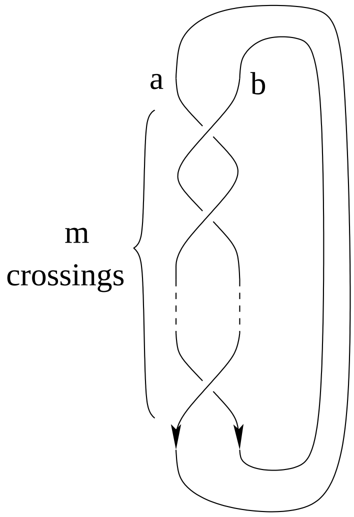

Figure 3.1: Torus knots $K(2,m)$

Lemma 3.1.1 *If* $(a,b) \in X \times X$ *is the top color vector of a coloring of* $K(2,m)$*, then the* k*-th color vector is* $(t\xi_{k-1}a + \xi_k b, t\xi_k a + \xi_{k+1}b)$*, where* $1 \leq k \leq m$*.*

Proof. Let m be a positive integer. Note that $1 - t\xi_{k-1} = \xi_k$. Then this lemma follows from the induction using the calculation

$$\begin{aligned} & t(t\xi_{k-1}a + \xi_k b) + (1-t)(t\xi_k a + \xi_{k+1}b) \\ = \quad & (t^2\xi_{k-1} + t(1-t)\xi_k)a + (t\xi_k + (1-t)\xi_{k+1})b \\ = \quad & t(1-t\xi_k) + t(1-t)\xi_k)a + ((1-\xi_{k+1}) + (1-t)\xi_{k+1})b \\ = \quad & t\xi_{k+1}a + \xi_{k+2}b. \end{aligned}$$

□

Corollary 3.1.2 *A top color vector* (a,b) *extends to a coloring of* $K(2,m)$ *if and only if* $(a-b)\xi_m = 0 \in X$*.*

Proof. In Lemma 3.1.1, the top and the bottom color vectors coincide if and only if

$$(a,b) = (t\xi_{m-1}a + \xi_m b, t\xi_m a + \xi_{m+1}b)$$

and the result follows. ☐

3.1.2 Coloring Twist Knots by Alexander Quandles

Let X be an Alexander quandle, and $k(2n)$ be the knot with $2n+2$ crossings, where n is a positive integer, as depicted in Fig. 3.2. Such a knot is commonly called a *twist knot*. The knot is oriented as indicated in the figure, and in particular, two strings with n full twists are oriented in opposite directions. We call such twisted strings with opposite orientations *antiparallel* strings.

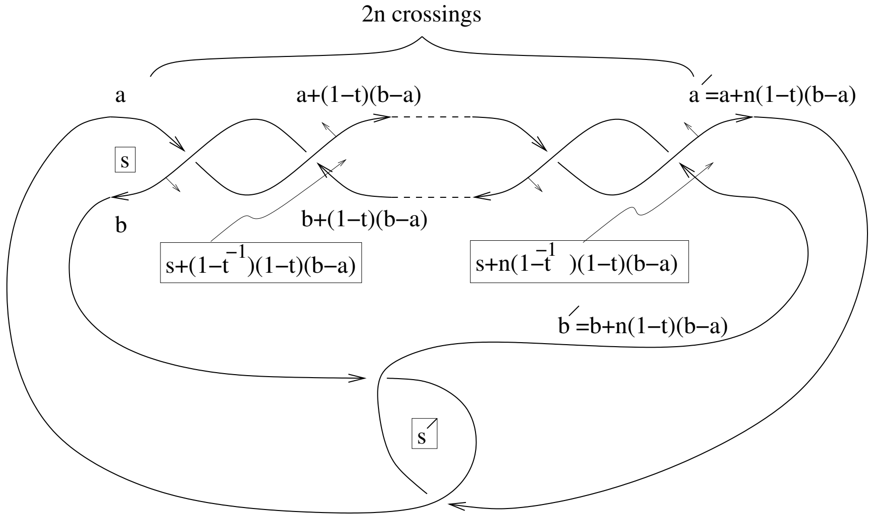

Figure 3.2: Twist knots

Lemma 3.1.3 *Let X be an Alexander quandle and $a, b \in X$ be the colors of the top (resp. bottom) left arc of the antiparallel strings with $2n$ crossings of a twist knot as in Fig. 3.2. Then the colors $a, b \in X$ extend to a coloring of the twist knot $k(2n)$ if and only if $(a-b)(t-n(1-t)^2) = 0$ in X.*

Proof. The end colors (the colors of the top and bottom right arcs) of the antiparallel strings are, respectively, $a' = a + n(1-t)(b-a)$ and $b' = b + n(1-t)(b-a)$ as depicted in the figure. The two crossings below antiparallel $2n$ crossings give the following conditions:

$$ta + (1-t)b' = b, \quad tb' + (1-t)a = a'.$$

Both conditions reduce to the same condition $(a-b)(t - n(1-t)^2) = 0$. □

3.2 Knot Invariants by 2-Cocycles

In this section we compute the quandle 2-cocycle invariants for $(2,m)$-torus knots $K(2,m)$ and twist knots $k(2n)$ considered in the preceding section, using the polynomial cocycle $f(x,y) = (x-y)^{a_1} y^{a_2}$ constructed in Chapter 2. Recall that such a polynomial is a 2-cocycle if $a_i = p^{m_i}$ for $i = 1, 2$, where p is a prime, for an Alexander quandle $X = \mathbb{Z}_p[t, t^{-1}]/g(t)$ if $g(t)$ divides $(t^{p^{m_1}+p^{m_2}} - 1)$, where we also take the coefficient group A to be the same as X. Throughout this section, we assume that $X = \mathbb{Z}_p[t, t^{-1}]/g(t) = A$ and $g(t)$ divides $(t^{p^{m_1}+p^{m_2}} - 1)$ in $\mathbb{Z}_p[t, t^{-1}]$. In addition to obtaining formulas of the invariant, our motivation includes proving nontriviality of cohomology groups.

3.2.1 The 2-cocycle Invariants for $(2, n)$-Torus Knots

Fix a coloring $\mathcal{C}$ by a quandle X of the torus knot $K(2,m)$ with a diagram in the closed braid form of a 2-string braid σ_1^m.

Lemma 3.2.1 *If (a, b) is the top color vector, then the contribution of the coloring $\mathcal{C}$ induced by (a, b) to the cocycle invariant $\Phi_f(K(2,m))$ is*

$$(a-b)^{a_1+a_2}(-t)^{a_2(2-m)}(\xi'_m)^{a_2},$$

where ξ'_m is the derivative of ξ_m.

Proof. Let (a, b) be the top color vector, then from Lemma 3.1.1, just below the k-th crossing we have the k-th color vector $(t\xi_{k-1} + \xi_k b, t\xi_k a + \xi_{k+1} b)$, so the contribution to the invariant is computed as

$$\begin{aligned}
&\sum_{k=1}^{m} f(t\xi_{k-1}a+\xi_k b, t\xi_k a+\xi_{k+1}b) \\
&= \sum_{k=1}^{m}(t\xi_{k-1}a+\xi_k b - t\xi_k a - \xi_{k+1}b)^{a_1}(t\xi_k a+\xi_{k+1}b)^{a_2} \\
&= \sum_{k=1}^{m}[(a-b)(\xi_{k+1}-\xi_k)]^{a_1}[b+t\xi_k(a-b)]^{a_2} \\
&= (a-b)^{a_1}b^{a_2}\sum_{k=1}^{m}(-t)^{ka_1} + (a-b)^{a_1+a_2}t^{a_2}\sum_{k=1}^{m}(-t)^{ka_1}\xi_k^{a_2}.
\end{aligned}$$

The first term is written as

$$\begin{aligned}
&(a-b)^{a_1}b^{a_2}\left(\sum_{k=1}^{m}(-t)^k\right)^{a_1} \\
&= (a-b)^{a_1}b^{a_2}\left[(-t)\sum_{k=1}^{m}(-t)^{k-1}\right]^{a_1} \\
&= (a-b)^{a_1}b^{a_2}[-t\xi_m]^{a_1}
\end{aligned}$$

which vanishes by Corollary 3.1.2. Note that by assumption $t^{a_1+a_2}=1$ in A, so that $t^{a_1}=t^{-a_2}$. Hence the second term is written as

$$(a-b)^{a_1+a_2}t^{a_2}\left(\sum_{k=1}^{m}(-1)^k t^{-k}\xi_k\right)^{a_2}.$$

Thus we compute $S_m=\sum_{k=1}^{m}(-t)^{-k}\xi_k$. We claim that

$$S_m=\sum_{k=0}^{m-1}(k+1)(-t)^{k-m},$$

and prove it by induction as follows.

$$\begin{aligned}
S_{m+1} &= \sum_{k=0}^{m+1}(-t)^{-k}\xi_k \\
&= S_m+(-t)^{-(m+1)}\xi_{m+1}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=0}^{m-1}(k+1)(-t)^{k-m} + \sum_{k=0}^{m}(-t)^{k-(m+1)} \\
&= \sum_{k=0}^{m-1}(k+1)(-t)^{k-m} + \sum_{k=0}^{m-1}(-t)^{k-m} + (-t)^{-(m+1)} \\
&= \sum_{k=0}^{m-1}(k+2)(-t)^{k-m} + (-t)^{-(m+1)} \\
&= \sum_{k=0}^{m+1}(k+1)(-t)^{k-(m+1)}.
\end{aligned}
$$

On the other hand, we compute

$$
\begin{aligned}
S_m &= (-t)^{-m}\sum_{k=0}^{m-1}(-t)^k + \sum_{k=0}^{m-1} k(-t)^k \\
&= (-t)^m(\xi_m + t\sum_{k=0}^{m-1}(-1)^k k t^{k-1}) \\
&= (-t)^m(\xi_m + t\xi'_m),
\end{aligned}
$$

where the last equality follows again from Corollary 3.1.2. □

Corollary 3.2.2 *The cohomology groups $H^2_{\mathrm{Q}}(X;X)$ are non-trivial for the following quandles:*

(i) $X = \mathbb{Z}_2[t,t^{-1}]/\xi_{2^n+1}$ *for any positive integer* n.

(ii) $X = \mathbb{Z}_p[t,t^{-1}]/\xi_{(p^n+1)/2}$ *for any odd prime* p *and for any positive integer* n.

Proof. It is known that if a cocycle is a coboundary, then the cocycle invariant is trivial (being trivial means that the invariant value is a positive integer corresponding to the number of colorings in a state-sum form, or that number of copies of zeros in a multi-set form) [9]. It is, therefore, sufficient to show that there is a coloring contributing a non-trivial value.

(i) For $X = \mathbb{Z}_2[t,t^{-1}]/\xi_{2^n+1}$, let $f(x_1,x_2) = (x_1-x_2)^{2^n}x_2$ in Lemma 2.1.1, so that $a_1 = 2^n$ and $a_2 = 1$. Note that $1 - t^{(a_1+a_2)} = (1-t)\xi_{2^n+1}$. Take $(1,0) \in X \times X$ as a top color vector, which extends to a coloring of $K(2,m)$, where $m = 2^n +$

1 by Corollary 3.1.2. Then by Lemma 3.2.1, the contribution to the invariant is $(-t)^{a_2(2-m)}\xi'_m$, which is non-trivial in X, as the degree of ξ'_m is less than that of ξ_m.

(ii) In this case take $a_1 = p^n$ and $a_2 = 1$ as before, then $1 - t^{(a_1+a_2)} = (1 - t^{(p^n+1)/2})(1 + t^{(p^n+1)/2})$. If $(p^n + 1)/2$ is odd, then $\xi_{(p^n+1)/2}$ divides $(1 + t^{(p^n+1)/2})$, and if even, it divides $(1 - t^{(p^n+1)/2})$, hence the result follows by the same argument. □

Proposition 3.2.3 *Let $X = A = \mathbb{Z}_p[t, t^{-1}]/\xi_m(t)$ for a prime p and a positive integer m, and let $f : X \times X \to A$ be defined by $f(x_1, x_2) = (x_1 - x_2)^{a_1} x_2^{a_2}$ for $a_i = p^{m_i}$, $i = 1, 2$, where m_i are non-negative integers. Suppose that ξ_m divides $1 - t^{(a_1+a_2)}$ in $\mathbb{Z}_p[t, t^{-1}]$. Then the cocycle invariant of the torus knot $K(2, m)$ is given by the multiset*

$$\Phi_f(K(2, m)) = \{\sqcup_{|X|}(-t)^{a_2(2-m)}(\xi'_m)^{a_2} s^{(a_1+a_2)} \mid s \in X\},$$

where $|X|$ denotes the number of elements of X and $\sqcup_r$, for a positive integer r, denotes the repeated r copies of the specified element.

If m is a negative integer, then the invariant consists of the negatives of the invariant for $K(2, |m|)$.

Proof. By Lemma 2.1.1, indeed $f \in Z^2_Q(X; A)$. By assumption any top color vector (a, b) extends to a coloring. Let $s = a - b$. Then by Lemma 3.2.1 we have

$$\{(a - b)^{(a_1+a_2)} \mid (a, b) \in X \times X\} = \{\sqcup_{|X|} s^{(a_1+a_2)} \mid s \in X\}$$

and the result follows.

If m is negative, then all crossings are negative. Then consider the diagram of $K(2, m)$ that is the mirror of the diagram used above of $K(2, |m|)$, with opposite orientation. Then also consider the colors of $K(2, m)$ at the bottom arcs (a, b). Then the contribution from the coloring induced by this bottom color vector coincides with the negative of the original. Hence the invariant $\Phi_f(K(2, m))$ is the multiset that consists of the negative of $\Phi_f(K(2, |m|))$. □

Example 3.2.4 From the proof of Corollary 3.2.2, we know that $K(2, m)$ has non-trivial cocycle invariants for the values $m = 2^n + 1$ and $m = (p^n + 1)/2$ for any positive integer n and any odd prime p, with $X = \mathbb{Z}_p[t, t^{-1}]/\xi_m(t)$. For example, with $f(x_1, x_2) = (x_1 - x_2)^4 x_2$, the above formula gives

$$\Phi_f(K(2,5)) = \{\sqcup_{16} 0, \sqcup_{80}(t+1), \sqcup_{80}(t^3), \sqcup_{80}(t^3+t+1)\}$$

where $m = 5 = 2^2 + 1$ and $X = \mathbb{Z}_2[t, t^{-1}]/\xi_5(t)$.

3.2.2 The 2-cocycle Invariants for Twist Knots

Let $k(2n)$ be the twist knot with n full twists as depicted in Fig. 3.2. In this section we compute the 2-cocycle invariants of $k(2n)$.

Proposition 3.2.5 *Let $X = A = \mathbb{Z}_p[t, t^{-1}]/(t - n(1-t)^2)$ for a prime p and a positive integer n, and let $f : X \times X \to A$ be defined by $f(x_1, x_2) = (x_1 - x_2)^{a_1} x_2^{a_2}$ for $a_i = p^{m_i}$, $i = 1, 2$, where m_i are non-negative integers. Suppose that $t - n(1-t)^2$ divides $1 - t^{(a_1+a_2)}$ in $\mathbb{Z}_p[t, t^{-1}]$. Then the cocycle invariant of the twist knot $k(2n)$ is given by the multiset*

$$\Phi_f(k(2n)) = \{\sqcup_{|X|}[-nt^{a_2} + (1 + n(1-t))^{(a_1+a_2)}]s^{(a_1+a_2)} \mid s \in X\}.$$

Proof. Consider the coloring as depicted in Fig. 3.2 and as considered in Lemma 3.1.3. In particular, the top and bottom left arcs of $2n$ twists have colors a and b assigned, respectively. The contribution of the antiparallel string with the $2n$ crossings is given by:

$$\begin{aligned}
& \sum_{k=1}^{n} f(a + (k-1)(1-t)(b-a), b + (k-1)(1-t)(b-a)) \\
+ & \sum_{k=1}^{n} f(b + k(1-t)(b-a), a + k(1-t)(b-a)) \\
= & \sum_{k=1}^{n} (a-b)^{a_1}(b + (k-1)(1-t)(b-a))^{a_2} \\
+ & \sum_{k=1}^{n} (b-a)^{a_1}(a + k(1-t)(b-a))^{a_2}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=1}^{n}(a-b)^{a_1}(b+(k-1)(1-t)(b-a)-a-k(1-t)(b-a))^{a_2} \\
&= \sum_{k=1}^{n}(a-b)^{a_1}(b-(1-t)(b-a)-a)^{a_2} \\
&= \sum_{k=1}^{n}(a-b)^{a_1}t^{a_2}(b-a)^{a_2} \\
&= -nt^{a_2}(a-b)^{a_1+a_2}.
\end{aligned}
$$

The contribution of the crossings at the twist is given by:

$$
\begin{aligned}
-f(a,b')-f(b',a) &= -(a-b')^{a_1}-b'^{a_1}-(b'-a)^{a_1}a^{a_2} \\
&= -(a-b')^{a_2}[b'^{a_2}-a^{a_2}] \\
&= (a-b')^{a_1+a_2} \\
&= (a-b-n(1-t)(b-a))^{a_1+a_2} \\
&= (a-b)^{a_1+a_2}[1+n(1-t)]^{a_1+a_2}.
\end{aligned}
$$

Hence the contribution of this coloring to the invariant is

$$
(a-b)^{a_1+a_2}[-nt^{a_2}+(1+n(1-t))^{a_1+a_2}],
$$

and as before, by taking all possible values of a, b, we obtain the result. □

3.3 Knot Invariants by 3-Cocycles

In this section we derive formulas of 3-cocycle invariants for the torus knots $K(2,m)$ and twist knots $k(2n)$ using polynomial cocycles. Again here non-triviality of 3-rd cohomology groups is part of our motivation. As before we consider the case $X = \mathbb{Z}_p[t,t^{-1}]/h(t)$ and $A = \mathbb{Z}_p[t,t^{-1}]/g(t)$, $h(t), g(t) \in \mathbb{Z}_p[t,t^{-1}]$ such that $g(t)$ divides $h(t)$ and p is a prime. Let $f : X^3 \to A$ be defined by $f(x_1,x_2,x_3) = (x_1 - x_2)^{a_1}(x_2-x_3)^{a_2}$ where $a_i = p^{m_i}$ for $i = 1, 2$, where m_i's are non-negative integers.

Then by Lemma 2.1.1, $f \in Z^3_{\mathrm{Q}}(X;A)$. We use these cocycles throughout this section.

3.3.1 The 3-cocycle Invariants for $(2,n)$-Torus Knots

Lemma 3.3.1 *Let $(a,b) \in X \times X$ be a top color vector for a coloring of $K(2,m)$ by X as in Fig. 3.1 and Lemma 3.1.1, with the region color $c \in X$ for the source region, where m is a positive integer. Then the contribution to the cocycle invariant of this coloring is given by*

$$(a-b)^{a_1+a_2}\sum_{k=1}^{m}\xi_k^{a_1}(-t)^{ka_2}.$$

In particular, if $t^{a_1+a_2}=1$ in A, then the contribution is given by

$$(a-b)^{a_1+a_2}(-t)^{a_1(1-m)}(\xi'_m)^{a_1}.$$

Proof. By Lemma 3.1.1, the contribution is computed as

$$\begin{aligned}
&\sum_{k=1}^{m} f(c, t\xi_{k-1}a+\xi_k b, t\xi_k a + t\xi_{k+1}b)\\
&= \sum_{k=1}^{m}[(c-a)+(a-b)\xi_k]^{a_1}[(a-b)(\xi_{k+1}-\xi_k)]^{a_2}\\
&= (c-a)^{a_1}(a-b)^{a_2}\sum_{k=1}^{m}(-t)^{ka_2} + (a-b)^{a_1+a_2}\sum_{k=1}^{m}\xi_k^{a_1}(-t)^{ka_2}.
\end{aligned}$$

The first term is $(c-a)^{a_1}(a-b)^{a_2}(-t\xi_k)=0$ by Corollary 3.1.2, and the second term is computed as $(a-b)^{a_1+a_2}\sum_{k=1}^{m}\xi_k^{a_1}(-t)^{ka_2}$. If $t^{a_1+a_2}=1$ in A, then $\sum_{k=1}^{m}\xi_k^{a_1}(-t)^{ka_2} = (S_m)^{a_1}$ where S_m is as in the proof of Lemma 3.2.1, and it was computed that $S_m = (-t)^{-m}(\xi_m + t\xi'_m)$, so that the contribution is

$$\begin{aligned}
(a-b)^{a_1+a_2}(S_m)^{a_1} &= (a-b)^{a_1+a_2}(-t)^{-ma_1}(\xi_m+t\xi'_m)^{a_1}\\
&= (a-b)^{a_1+a_2}(-t)^{a_1(1-m)}(\xi'_m)^{a_1},
\end{aligned}$$

where we used Corollary 3.1.2 again. □

Corollary 3.3.2 *For the quandles X stated in Corollary 3.2.2, we have $H^3_{\mathrm{Q}}(X;X) \neq$*

0.

Proof. Take $a_1 = 1$ and $a_2 = p^n$ for a positive integer n, then the top color vector $(1, 0)$ gives a non-trivial value in Lemma 3.3.1. The condition that ξ_m divides $1 - t^{a_1+a_2}$ is computed as in Corollary 3.2.2. □

Example 3.3.3 Using Lemma 3.3.1, we obtain the following list of calculations of 3-cocycle invariants for $K(2, m)$ torus knots carried out by a *Maple* program, for cocycles of the form $f(x, y, z) = (x - y)(y - z)^p$.

- $p = 2$, $f(x, y, z) = (x - y)(y - z)^2$.
  - $m = 3$, $16 + 48U^t$
  - $m = 5$, 4096
  - $m = 7$, 262144
  - $m = 9$, $4194304 + 12582912U^{(t^4+t^7+1)}$
  - $m = 11$, 1073741824
  - $m = 13$, 68719476736
  - $m = 15$, $1099511627776 + 3298534883328U^{(t^{13}+t^{10}+t^7+t^4+t)}$
- $p = 3$, $f(x, y, z) = (x - y)(y - z)^3$.
  - $m = 3$, $243 + 486U^{(2t+2)}$
  - $m = 5$, 531441
  - $m = 7$, 387420489
  - $m = 9$, $94143178827 + 188286357654U^{(2t^7+2t^6+t^4+t^3+2t+2)}$
- $p = 5$, $f(x, y, z) = (x - y)(y - z)^5$.
  - $m = 3$, $625 + 3750U^{(t+3)} + 3750U^{(4t+2)} + 3750U^{(3t+4)} + 3750U^{(2t+1)}$
  - $m = 5$, $48828125 + 97656250U^{(4t^3+2t^2+2t+4)} + 97656250U^{(t^3+3t^2+3t+1)}$
  - $m = 7$, 3814697265625

Proposition 3.3.4 *Let p be a prime and m be a positive integer. Let $X = \mathbb{Z}_p[t,t^{-1}]/\xi_m$, $A = \mathbb{Z}_p[t,t^{-1}]/g(t)$ and $f : X^3 \to A$ be defined by $f(x_1,x_2,x_3) = (x_1-x_2)^{a_1}(x_2-x_3)^{a_2}$, where $a_i = p^{m_i}$, for non-negative integers m_i, $i = 1,2$. Suppose that $g(t)$ divides both ξ_m and $(1-t^{a_1+a_2})$. Then the cocycle invariant is given by*

$$\Phi_f(K(2,m)) = \{\sqcup_{|X|^2}(-t)^{a_1(1-m)}(\xi'_m)^{a_1}s^{a_1+a_2} \mid s \in X\}.$$

Proof. Let (a,b) be a top color vector of the closed braid form of a torus knot $K(2,m)$ as before, and $c \in X$ be the color assigned to the region to the left of the diagram, where the braid closure is taken to the right of the diagram. Any such choice (c,a,b) uniquely extends to a coloring of $K(2,m)$ and the regions. Then the contribution of this coloring to the invariant is computed as in Lemma 3.3.1, and we obtain the result. □

Example 3.3.5 Let $p = 5$ and $X = \mathbb{Z}_5[t,t^{-1}]/\xi_3$. With

$$f(x_1,x_2,x_3) = (x_1-x_2)(x_2-x_3)^5,$$

we obtain the invariant for trefoil K

$$\Phi_f(K) = \{\ \sqcup_{625}(0),\ \sqcup_{3750}(t+3, 2t+1, 3t+4, 4t+2)\ \}.$$

If ξ_m does not divide $1-t^{a_1+a_2}$, then the contribution is likely to be zero, due to the following lemma.

Lemma 3.3.6 *Let $X = A = \mathbb{Z}_p[t,t^{-1}]/\xi_m$, where p is prime and m is a positive integer, and let $a_i = p^{m_i}$, $i = 1,2$. Then $(1-t^{a_1+a_2})\sum_{k=1}^{m}\xi_k^{a_1}(-t)^{ka_2} = 0$ in X. In particular, if X is an integral domain and $1-t^{a_1+a_2} \neq 0$ in X, then the invariant is trivial.*

Proof. We compute

$$\sum_{k=1}^{m}\xi_k^{a_1}(-t)^{ka_2} \quad = \quad \sum_{k=1}^{m}(1-t\xi_{k-1})^{a_1}(-t)^{ka_2}$$

$$
\begin{aligned}
&= \sum_{k=1}^{m}(-t)^{ka_2} - t^{a_1}\sum_{k=1}^{m}\xi_{k-1}^{a_1}(-t)^{ka_2} \\
&= (-t)^{a_2}\sum_{k=1}^{m}(-t)^{(k-1)a_2} + t^{a_1+a_2}\sum_{k=1}^{m}\xi_{k-1}^{a_1}(-t)^{(k-1)a_2} \\
&= -t^{a_2}\xi_m^{a_2} + t^{a_1+a_2}\sum_{k=0}^{m-1}\xi_k^{a_1}(-t)^{ka_2} \\
&= -t^{a_2}\xi_m^{a_2} + t^{a_1+a_2}\sum_{k=1}^{m-1}\xi_k^{a_1}(-t)^{ka_2}.
\end{aligned}
$$

Then we have

$$
\sum_{k=1}^{m-1}\xi_k^{a_1}(-t)^{ka_2} + \xi_k^{a_1}(-t)^{ma_2} = -t^{a_2}\xi_m^{a_2} + t^{a_1+a_2}\sum_{k=1}^{m-1}\xi_k^{a_1}(-t)^{ka_2}.
$$

Hence we obtain

$$
(1-t^{a_1+a_2})(\sum_{k=1}^{m-1}\xi_k^{a_1}(-t)^{ka_2}) = -(-t)^{ma_2}\xi_m^{a_1} - t^{a_2}\xi_m^{a_2}.
$$

Since $\xi_m = 0$ in X, the sum is written as

$$
\sum_{k=1}^{m-1}\xi_k^{a_1}(-t)^{ka_2} = \sum_{k=1}^{m}\xi_k^{a_1}(-t)^{ka_2}.
$$

Then the result follows. □

For $a_1 = a_2 = 1$, we have the following.

Proposition 3.3.7 *Let p be a prime and m be a positive even integer. Let $X = \mathbb{Z}_p[t,t^{-1}]/\xi_m$ and $f : X^3 \to A$ be defined by $f(x_1,x_2,x_3) = (x_1-x_2)(x_2-x_3)$. Then the cocycle invariant is given by*

$$
\Phi_f(K(2,m)) = \{\sqcup_{|X|^2}[(-t\xi_m)/(1-t)]s^2 \mid s \in X\}.
$$

In particular, the invariant is non-trivial in this case.

Proof. Take $a_1 = a_2 = 1$ in Lemma 3.3.1. It is computed by induction that

$$\sum_{k=1}^{m} \xi_k (-t)^k = -t\xi_m \xi_{m+1}/(1-t).$$

For even m, ξ_m is divisible by $1-t$, and $\xi_{m+1} = 1 - t\xi_m$. Hence we obtain

$$(-t\xi_m \xi_{m+1})/(1-t) = [(-t\xi_m)/(1-t)](1-t\xi_m).$$

The contribution, then, is $(a-b)^{a_1+a_2}[(-t\xi_m)/(1-t)]$ since $(a-b)\xi_m = 0$ by Lemma 3.1.2. Since m is even, ξ_m is divisible by $(1-t)$, and the quotient is of degree $m-1$, and therefore, the contribution is non-zero. □

3.3.2 The 3-cocycle Invariants for Twist Knots

Proposition 3.3.8 *The 3-cocycle invariant of the twist knot $k(2n)$ with $2n+2$ crossings as depicted in Fig. 3.2 is given by*

$$\Phi_f(K) = \{\sqcup_{|X|^2}[[-nt^{-a_1} + (1+n(1-t))^{a_1+a_2}]s^2 \mid s \in X\}.$$

Proof. We give the knot K a shadow coloring as described above, then the contribution of the antiparallel string with the $2n$ crossings is given by:

$$\begin{aligned}
\eta_n^+ &= \sum_{k=1}^{n} \{[(s+(k-1)(1-t^{-1})(1-t)(b-a)) - (a+(k-1)(1-t)(b-a))]^{a_1} \\
&\times [(a+(k-1)(1-t)(b-a)) - (b+(k-1)(1-t)(b-a)))]^{a_2} \\
&+ [(s+k(1-t^{-1})(1-t)(b-a)) - (b+k(1-t)(b-a))]^{a_1} \\
&\times [(b+k(1-t)(b-a)) - (a+k(1-t)(b-a)))]^{a_2}\} \\
&= \sum_{k=1}^{n} (a-b)^{a_2}(-(1-t^{-1})(1-t)(b-a) - (a-b) + (1-t)(b-a))^{a_1} \\
&= -nt^{-a_1}(a-b)^{(a_1+a_2)}.
\end{aligned}$$

The contribution from the bottom two crossings of the diagram is given by

$$\begin{aligned}
-f(s', a, b') - f(s', b', a) &= -(s'-a)^{a_1}(a-b')^{a_2} - (s'-b')^{a_1}(b'-a)^{a_2} \\
&= -(a-b')^{a_2}[(s'-a)^{a_1} - (s'-b')^{a_1}] \\
&= -(a-b')^{a_2}(b'-a)^{a_1} \\
&= (a-b')^{a_1+a_2} \\
&= (a-b-n(1-t)(b-a))^{a_1+a_2} \\
&= (a-b)^{a_1+a_2}[1+n(1-t)]^{a_1+a_2}.
\end{aligned}$$

Hence the total contribution is

$$(a-b)^{a_1+a_2}[-nt^{-a_1} + (1+n(1-t))^{a_1+a_2}],$$

and the result follows by taking all possible a, b as before. □

The program in Appendix A computes $[-nt^{-a_1} + (1+n(1-t))^{a_1+a_2}]$ that appear in the formula in Proposition 3.3.8, so that if the output is a non-zero polynomial, then the invariant is non-trivial. In the case of $p = 3$, for example below, the cocycle $f(x, y, z) = (x-y)(y-z)^3$ gives a non-trivial values $t+2$ and $t+1$ for $n = 1$ and $n = 2$, respectively, so that $k(2n)$ has non-trivial invariants for all $n \equiv 1, 2 \pmod 3$ for $p = 3$.

We obtained non-trivial values for the following p, $2 < p \leq 29$, and $0 < n < p$ with the cocycle $f(x, y, z) = (x-y)(y-z)^p$. In fact for all cases, the invariant is non-trivial for the cocycle $f(x, y, z) = (x-y)(y-z)^p$ if and only if it is non-trivial for $f(x, y, z) = (x-y)^p(y-z)$. Below we list the results for $2 < p \leq 13$

- $p = 3$, $n = 1, 2$.
- $p = 5$, $n = 1, 3, 4$.
- $p = 7$, all n except $n = 2, 6$.
- $p = 11$, all n except $n = 1, 2, 6, 9$.

- $p = 13$, all n except $n = 2, 4, 6, 7, 12$.

Hence we have the following corollary

Corollary 3.3.9 *Let $X = \mathbb{Z}_p[t, t^-1]/(t - n(1-t)^2)$, the cohomology groups $H^3_Q(X; X)$ for the followings p and n are nontrivial.*

(a) $p = 3$, $n \equiv 1, 2 \pmod 3$.

(b) $p = 5$, $n \equiv 1, 3$, *and* $4 \pmod 5$.

(c) $p = 7$, *all* n *except* $n \equiv 2, 6 \pmod 7$

(d) $p = 11$, *all* n *except* $n \equiv 1, 2, 6$, *and* $9 \pmod{11}$.

(e) $p = 13$, *all* n *except* $n \equiv 2, 4, 6, 7$, *and* $12 \pmod{13}$.

To compute the invariants, we used the program in Appendix B, which computes the term $(a-b)^{a_1+a_2}$, all possible values for this term, and the invariant values. In the outputs, we kept the outputs for $p = 2$, and listed other results below the program.

Below we give a summary of computational results for other primes. First, we used cocycles of the form $f(x, y, z) = (x-y)^{a_1}(y-z)^{a_2}$ where a_1 and a_2 are 1 or p. We always obtained trivial invariants for the cocycles $f(x, y, z) = (x-y)(y-z)$ and $f(x, y, z) = (x-y)^p(y-z)^p$, so we list the other cocycles, for which the invariant is non-trivial.

- $p = 3$
  - $n = 1$, $a_1 = 1$, $a_2 = 3$: $81 + 324U^{(t+2)} + 324U^{(1+2t)}$
  - $n = 1$, $a_1 = 3$, $a_2 = 1$: $81 + 324U^{(2t+2)} + 324U^{(t+1)}$
  - $n = 2$, $a_1 = 1$, $a_2 = 3$: $243 + 486U^{(t+1)}$
  - $n = 2$, $a_1 = 1$, $a_2 = 3$: $243 + 486U^{(2t+2)}$
- $p = 5$
  - $n = 1$, $a_1 = 1$, $a_2 = 5$: $3125 + 6250U^{(3t+3)} + 6250U^{(2t+2)}$

* $n = 1,\ a_1 = 5,\ a_2 = 1:\ 3125 + 6250U^{(3t+3)} + 6250U^{(2t+2)}$
* $n = 2,\ a_1 = 1,\ a_2 = 5$: 15625
* $n = 2,\ a_1 = 5,\ a_2 = 1$: 15625
* $n = 3,\ a_1 = 1,\ a_2 = 5$:
  $625 + 3750U^{(t)} + 3750U^{(2t)} + 3750U^{(3t)} + 3750U^{(4t)}$
* $n = 3,\ a_1 = 5,\ a_2 = 1$:
  $625 + 3750U^{(t+1)} + 3750U^{(2t+2)} + 3750U^{(3t+3)} + 3750U^{(4t+4)}$
* $n = 4,\ a_1 = 1,\ a_2 = 5$:
  $625 + 3750U^{(t+3)} + 3750U^{(2t+1)} + 3750U^{(3t+4)} + 3750U^{(4t+2)}$

- $p = 7$

* $n = 1,\ a_1 = 1,\ a_2 = 7$:

$$2401 + 19208U^{(t+3)} + 19208U^{(4t+5)} + 19208U^{(2t+6)} \\ +19208U^{(5t+1)} + 19208U^{(6t+4)} + 19208U^{(3t+2)}$$

* $n = 1,\ a_1 = 7,\ a_2 = 1:\ 2401 + 19208U^{(t+1)} + 19208U^{(2t+2)} + 19208U^{(3t+3)} + 19208U^{(4t+4)} + 19208U^{(5t+5)} + 19208U^{(6t+6)}$
* $n = 2,\ a_1 = 1,\ a_2 = 7$: 117649
* $n = 2,\ a_1 = 7,\ a_2 = 1:$ 117649
* $n = 3,\ a_1 = 1,\ a_2 = 7$: $2401 + 19208U^{(3t+4)} + 19208U^{(5t+2)} + 19208U^{(6t+1)} + 19208U^{(t+6)} + 19208U^{(2t+5)} + 19208U^{(4t+3)}$
* $n = 3,\ a_1 = 7,\ a_2 = 1$: $2401 + 19208U^{(t+1)} + 19208U^{(2t+2)} + 19208U^{(3t+3)} + 19208U^{(4t+4)} + 19208U^{(5t+5)} + 19208U^{(6t+6)}$
* $n = 4,\ a_1 = 1,\ a_2 = 7$: $2401 + 19208U^{(t+2)} + 19208U^{(2t+4)} + 19208U^{(3t+6)} + 19208U^{(4t+1)} + 19208U^{(5t+3)} + 19208U^{6t+5)}$
* $n = 4,\ a_1 = 7,\ a_2 = 1$: $2401 + 19208U^{(t+1)} + 19208U^{(2t+2)} + 19208U^{(3t+3)} + 19208U^{(4t+4)} + 19208U^{(5t+5)} + 19208U^{(6t+6)}$

* $n = 5$, $a_1 = 1$, $a_2 = 7$: $16807 + 33614U^{(3t+3)} + 33614U^{(5t+5)} + 33614U^{(6t+6)}$
* $n = 5$, $a_1 = 7$, $a_2 = 1$: $16807 + 33614U^{(t+1)} + 33614U^{(2t+2)} + 33614U^{(4t+4)}$
* $n = 6$, $a_1 = 1$, $a_2 = 7$: 117649
* $n = 6$, $a_1 = 7$, $a_2 = 1$: 117649

Chapter 4

Applications

In this chapter, two applications of quandle cocycle invariants are given: formulas for twist spun torus and twist knots, and invariants of tangles used as obstruction to tangle embedding.

Non-invertibility of the 2-twist-spun trefoil was the first application of the quandle cocycle invariants [6]. Two developments made it possible, since then, to extend this result to a large number of twist-spun knots; simpler diagrams of twist-spun knots given in [1] and an explicit formula of 3-cocycles of dihedral quandles R_p discovered by Mochizuki [31]. It is of interest, then, to obtain formulas of quandle cocycle invariants for twist-spun knots using our polynomial cocycles to see which twist-spuns have non-trivial invariants with some Alexander quandles, as well as to see whether they provide obstructions to non-invertibility. In the first section we investigate these problems.

In the second section of this chapter, we use cocycle invariants as obstructions to embedding tangles in knots and links. In application of knot theory to enzyme actions on DNA [17], tangle equations were formulated and solved, that were set up using experimental results. In tangle equations, which tangles can be embedded in a given knot is a part of the problem. The tangle embedding problem was later studied by Krebes [27] using evaluations of the Jones polynomial, that are determinants of the closures of a given tangle. This method was interpreted in terms of homology of double branched covers along knots [35], and then the results were further generalized in different directions [15, 33]. We use cocycle invariants as obstructions to tangle embeddings. First we define cocycle invariants for tangles by requiring that end points have the same color, and show that if a tangle embeds in a link, then the invariant of the tangle is a multi-subset of the invariant of the link. Then we investigate specific examples of tangles and knots, applying this obstruction.

4.1 Quandle Cocycle Invariants for Twist Spun Knots

In this section we derive formulas of the quandle cocycle invariants for twist-spun torus and twist knots by applying Asami-Satoh's formula [1] and our formulas of 3-cocycle invariants for these knots.

4.1.1 Twist Spinning

Twist spinning was introduced by E.C. Zeeman in 1965 and is a generalization of Artin's spinning. Spinning in general starts with an n-dimensional knot to produce an $(n+1)$-dimensional knot by a rotation of the upper-half space around the hyperplane axis. In Artin's spinning, the knot is rotated around the plane with its position in the upper-half plane fixed during the rotation. In the r-twist spinning, where r is a positive integer, the knot diagram is twisted around the axis while the upper-half space is rotated once around the hyperplane axis of rotation. The twist spinning can be compared to the earth orbiting the sun, also rotating around its axis, where the knot is contained inside of the earth. If K is a classical knot, the $r-$twist spin of K is denoted by $\tau^r K$. A specific surface diagram of $\tau^r K$ was considered in [1]. More specifically, pick a point b on K and let B be a small open ball neighborhood of b in S^3. Then $(S^3 \setminus B,\, K \setminus (K \cap B))$ is a ball pair. Such a ball pair is also called a $(1,1)$-tangle of K, or a $(1,1)$-tangle whose closure is K. Denote this $(1,1)$-tangle by T. In [1], a diagram $D^r T$ of the r-twist spun knot $\tau^r K$ of a knot K was described in terms of a diagram of the $(1,1)$-tangle T of K, as follows. First, a twist in the twist-spinning is described as a diagram of T going around an arc of axis of the twist as depicted in Fig. 4.1, where the tangle is represented by the same letter K. A diagram of $\tau^r K$ is obtained from T as follows. Consider a motion picture of tangle diagrams in the upper half plane $\mathbb{R}^2_+$ as shown in Fig. 4.1, with the boundary points of T fixed.

We give $\mathbb{R}^3$ an open book structure $\mathbb{R}^2_+ \times S^1$, where its binding is the boundary of $\mathbb{R}^2_+$, and we regard each page $\mathbb{R}^2_+$ as a frame of the motion, repeated r-times. Since the motion represents a single full twist of T, the continuous trace of the motion repeated r-times describes a surface diagram of $\tau^r K$ in $\mathbb{R}^2_+ \times S^1$. We denote the diagram thus obtained by D^rT.

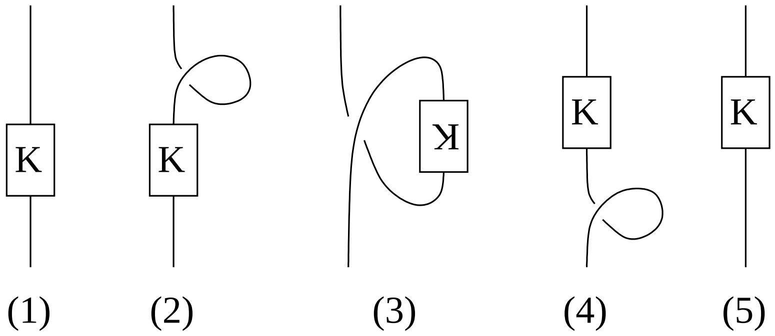

Figure 4.1: A movie of twist spinning

This description gives rise to a diagram as depicted in Fig. 4.2, where the tangle in the figure is a trefoil tangle for simplicity. These figures are used in [4] for their computations of cocycle invariants.

In Fig. 4.2, each vertical cross section of the broken surface diagram marked by (1) through (5) corresponds to the diagram in a twist depicted in Fig. 4.1 also marked by the corresponding numbers (1) through (5). Small black dots in Fig. 4.2 between (1) and (2), (4) and (5), respectively, are branch points of the projected surface corresponding to the type I Reidemeister move that occur between (1) and (2), (4) and (5) in Fig. 4.1.

In particular, the diagram of T can be regarded as a part (slice) of the diagram D^rT, and this situation is represented by the inclusion map $i : T \to D^rT$. Let $\mathrm{Col}_X(T)$ and $\mathrm{Col}_X(D^rT)$ be the set of colorings of T and D^rT, respectively, by a quandle X. Let $i : T \to D^rT$ be the inclusion map, and let $i^* : \mathrm{Col}_X(D^rT) \to \mathrm{Col}_X(T)$ e the induced map. Put $\mathrm{Col}^r_X(T) = Im(i^*) \in \mathrm{Col}_X(T)$. Asami and Satoh [1] proved that the map i^* is injective, and $C \in \mathrm{Col}_X(T)$ belongs to $\mathrm{Col}^r_X(T)$ if and only if $C * a^r_- = C$ where a_- is the color of the terminal arc of T. Here the notation $C*a$ for $C \in \mathrm{Col}_X(T)$

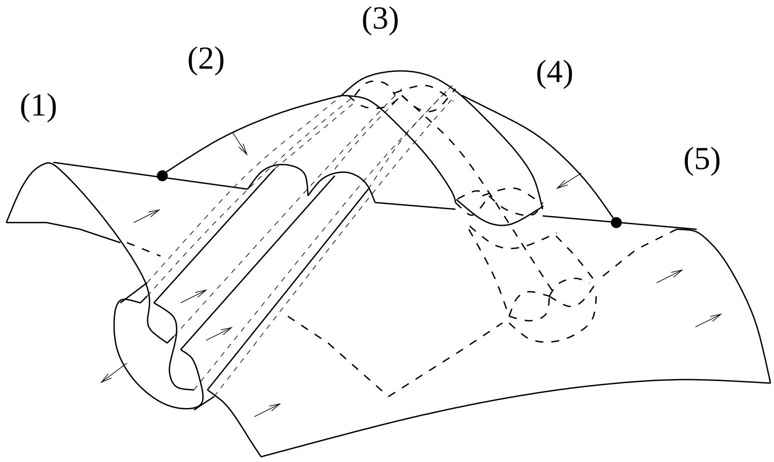

Figure 4.2: A part of a diagram of twist-spun trefoil

and $a \in X$ means that for any arc α in T, $(C * a)(\alpha) = C(\alpha) * a$, and $C * a^r$ denotes r-fold product $(\cdots(C * a) * a)*) * \cdots)*) * a$. Furthermore, they gave a formula for the quandle cocycle invariant as follows. Let $f \in Z^3_{\mathrm{Q}}(X; A)$ and denote by $\Phi(F) = \Phi_f(F)$ the cocycle invariant of a surface F with respect to a cocycle f. Then they proved that

$$\Phi(\tau^r K) = \left\{ r \left(\sum_{x \in \mathrm{Cr}(T)} W^*_f(x; C) \right) - \left(\sum_{\ell=0}^{r-1} \sum_{x \in \mathrm{Cr}(T)} W^\sharp_f(x; C * a^\ell_-) \right) \right\}_{C \in \mathrm{Col}^r_X(T)},$$

where $\mathrm{Cr}(T)$ denotes the set of crossings of the tangle diagram T, for a quandle triple (s, a, b) at each crossing x, $W^* f(x; C) = f(s, a, b)^{\epsilon(x)}$, a_- is the color of the terminal arc of T, and $W^\sharp_f(x; C) = f(a, b, a_-)^{\epsilon(x)}$.

As before we consider the case $X = \mathbb{Z}_p[t, t^{-1}]/h(t)$ and $A = \mathbb{Z}_p[t, t^{-1}]/g(t)$, $h(t), g(t) \in \mathbb{Z}_p[t, t^{-1}]$ such that $g(t)$ divides $h(t)$ and p is a prime. Let $f : X^3 \to A$ be defined by $f(x_1, x_2, x_3) = (x_1 - x_2)^{a_1}(x_2 - x_3)^{a_2}$ where $a_i = p^{m_i}$ for $i = 1, 2$, where m_is are non-negative integers. Then by Lemma 2.1.1, $f \in Z^3_{\mathrm{Q}}(X; A)$.

Lemma 4.1.1 *The quandle cocycle invariant of the r-twist spun $\tau^r K$ of K with the* *cocycle* $f(x_1, x_2, x_3) = (x_1 - x_2)^{a_1}(x_2 - x_3)^{a_2}$ *is given by :*

$$\Phi(\tau^r K) = \left\{ r \left(\sum_{x \in \mathrm{Cr}(T)} W_f^*(x; C) \right) - \sum_{k=0}^{r-1} t^{(a_1+a_2)k} \left(\sum_{x \in \mathrm{Cr}(T)} W_f^\sharp(x; C) \right) \right\}_{C \in \mathrm{Col}_X^r(T)}.$$

Proof. For any $a, b, c \in X$, the cocycle $f = (x_1 - x_2)^{a_1}(x_2 - x_3)^{a_2} x_3^{a_3}$ satisfies

$$f(a * c, b * c, c) = t^{a_1+a_2} f(a, b, c).$$

Hence

$$\sum_{k=0}^{r-1} \sum_{x \in \mathrm{Cr}(T)} W_f^\sharp(x; C * a_-^k) = \sum_{k=0}^{r-1} t^{(a_1+a_2)k} \sum_{x \in \mathrm{Cr}(T)} W_f^\sharp(x; C).$$

Then the result follows. □

4.1.2 Twist-spun $(2, m)$-Torus Knots

First we consider twist-spun $(2, m)$-torus knots.

Proposition 4.1.2 *Let* $X = A = \mathbb{Z}_p[t, t^{-1}]/h(t)$, $f(x_1, x_2, x_3) = (x_1 - x_2)^{a_1}(x_2 - x_3)^{a_2}$, *and suppose that* $1 - t^r = 0$ *in* X, *where* r *is a positive integer. Then the cocycle invariant* $\Phi_f(\tau^r K(2, m))$ *for the* r*-twist spin of* $K(2, m)$ *is given by*

$$\left\{ \left[r \left(\sum_{k=1}^{m} \xi_k^{a_1} (-t)^{k a_2} \right) - \left(\sum_{\ell=0}^{r-1} t^{(a_1+a_2)\ell} \right) \left(t^{a_2} \sum_{k=1}^{m} \xi_k^{a_2} (-t)^{k a_1} \right) \right] (a - b)^{a_1+a_2} \right\}_{a, b \in X}.$$

Proof. By Lemma 3.3.1, the contribution for the top color vector is computed as

$$\sum_{x \in \mathrm{Cr}(T)} W_f^*(x; C) = (a - b)^{a_1+a_2} \sum_{k=1}^{m} \xi_k^{a_1} (-t)^{k a_2},$$

and in particular, if $t^{a_1+a_2}=1$ in A, then one of the contribution is given by

$$\sum_{x\in \mathrm{Cr}(T)} W_f^*(x;C) = (a-b)^{a_1+a_2}(-t)^{a_1(1-m)}(\xi'_m)^{a_1}.$$

The other contribution is computed as follows.

$$\begin{aligned}
&\sum_{x\in \mathrm{Cr}(T)} W_f^\sharp(x;C)\\
&= \sum_{k=1}^{m} f(a+\xi_k(b-a), b+t\xi_k(a-b), a_-)\\
&= \sum_{k=1}^{m}(a-b)^{a_1}(-t)^{ka_1}[(b-a_-)+t\xi_k(a-b)]^{a_2}\\
&= (a-b)^{a_1}(b-a_-)(-t)\xi_m + \sum_{k=1}^{m}(a-b)^{a_1+a_2}(-t)^{ka_1}(t\xi_k)^{a_2}.
\end{aligned}$$

The first term vanishes by Corollary 3.1.2. The second term is written as

$$(a-b)^{a_1+a_2}t^{a_2}\sum_{k=1}^{m}\xi_k^{a_2}(-t)^{ka_1},$$

so that the invariant of the r-twist spun torus knot is given by

$$\left\{\left[r\left(\sum_{k=1}^{m}\xi_k^{a_1}(-t)^{ka_2}\right)\right.\right.$$
$$\left.\left. -\left(\sum_{\ell=0}^{r-1}t^{(a_1+a_2)\ell}\right)\left(t^{a_2}\sum_{k=1}^{m}\xi_k^{a_2}(-t)^{ka_1}\right)\right](a-b)^{a_1+a_2}\right\}_{a,b\in X}.$$

□

Proposition 4.1.3 *Suppose $h(t)$ divides $(1-t^{a_1+a_2})$ and ξ_m, and let r be a multiple of a_1+a_2. Then the cocycle invariant $\Phi_f(\tau^r K(2,m))$ for r-twist spin of $K(2,m)$ is given by*

$$\left\{\sqcup_{|X|}\left[(-t)^{a_1(1-m)}(\xi'_m)^{a_1}+(-t)^{a_2(2-m)}(\xi'_m)^{a_2}\right] rs^{a_1+a_2}\,\middle|\, s\in X\right\}.$$

If $1 - t^{a_1+a_2} = 0$, $t^{a_2} = t^{-a_1}$, then $\sum_{\ell=0}^{r-1} t^{(a_1+a_2)\ell} = r$ and

$$\sum_{k=1}^{m} \xi_k^{a_1}(-t)^{ka_2} = \sum_{k=1}^{m} \xi_k^{a_1}(-t)^{-ka_1} = (S_m)^{a_1} = ((-t)^{1-m}\xi'_m)^{a_1}$$

where S_m is as defined in the proof of Lemma 3.2.1. In the second term,

$$t^{a_2}\sum_{k=1}^{m} \xi_k^{a_2}(-t)^{ka_1} = t^{a_2}(S_m)^{a_2} = t^{a_2}((-t)^{1-m}\xi'_m)^{a_2} = -(-t)^{a_2(2-m)}(\xi'_m)^{a_2}$$

then using 4.1.2 the expression for the invariant simplifies as stated. □

4.1.3 Twist Spun Twist Knots

Next we consider formulas for twist-spun twist knots.

Proposition 4.1.4 *Let $X = A = \mathbb{Z}_p[t, t^{-1}]/h(t)$ such that $h(t)$ divides $t - n(1-t)^2$, and let $f : X \times X \times X \to A$ be a 3-cocycle defined by $f(x_1, x_2, x_3) = (x_1 - x_2)^{a_1}(x_2 - x_3)^{a_2}$, let r be a positive integer such that $1 - t^r = 0$. Then the cocycle invariant $\Phi(\tau^r K)$ of the r-twist spun of the the twist knot is given by*

$$\{\sqcup_{|X|}\ [r[-nt^{-a_1} + (1+n(1-t))^{a_1+a_2}] \\ -(-nt^{a_2} + [1+n(1-t)]^{a_1+a_2})\sum_{k=0}^{r-1} t^{(a_1+a_2)k}]s^{a_1+a_2} \mid s \in X\}.$$

Proof. First we compute the term $W_f^*(x; C)$. The contribution of the antiparallel strings with the $2n$ crossings to the term $W_f^*(x; C)$ is given by

$$\begin{aligned}
&\sum_{k=1}^{n} f(s + (k-1)(1-t^{-1})(1-t)(b-a), \\
&\qquad a + (k-1)(1-t)(b-a), b + (k-1)(1-t)(b-a)) \\
+ \quad & f(s + k(1-t^{-1})(1-t)(b-a), b + (k-1)(1-t)(b-a), a + k(1-t)(b-a)) \\
= \quad & \sum_{k=1}^{n} \{[(s + (k-1)(1-t^{-1})(1-t)(b-a)) - (a + (k-1)(1-t)(b-a))]^{a_1} \\
\times \quad & [(a + (k-1)(1-t)(b-a)) - (b + (k-1)(1-t)(b-a)))]^{a_2}
\end{aligned}$$

$$
\begin{aligned}
&+ \quad [(s+k(1-t^{-1})(1-t)(b-a)) - (b+k(1-t)(b-a))]^{a_1} \\
&\times \quad [(b+k(1-t)(b-a)) - (a+k(1-t)(b-a)))]^{a_2}\} \\
&= \quad \sum_{k=1}^{n}(a-b)^{a_2}(-(1-t^{-1})(1-t)(b-a) - (a-b) + (1-t)(b-a))^{a_1} \\
&= \quad -nt^{-a_1}(a-b)^{(a_1+a_2)}.
\end{aligned}
$$

The contribution of the two crossings at the bottom two crossing in Fig. 3.2 to the term $W_f^*(x;C)$ is computed as

$$
\begin{aligned}
&-(s'-a)^{a_1}(a-b')^{a_2} - (s'-b')^{a_1}(b'-a)^{a_2} \\
&\quad = \quad -(a-b')^{a_2}[(s'-a)^{a_1} - (s'-b')^{a_1}] \\
&\quad = \quad (a-b')^{a_1+a_2} \\
&\quad = \quad (a-b-n(1-t)(b-a))^{a_1+a_2} \\
&\quad = \quad (a-b)^{a_1+a_2}[1+n(1-t)]^{a_1+a_2}.
\end{aligned}
$$

Then we have

$$
W_f^*(x;C) = (a-b)^{a_1+a_2}[-nt^{-a_1} + (1+n(1-t))^{a_1+a_2}].
$$

Next we compute the term $W_f^\sharp(x;C)$, which is written as

$$
\begin{aligned}
&\sum_{k=1}^{n} f(b+(k-1)(1-t)(b-a), a+k(1-t)(b-a), a_-) \\
&\quad + \quad f(a+(k-1)(1-t)(b-a), b+(k-1)(1-t)(b-a), a_-) \\
&\quad - \quad f(a,b',s') - f(b',a,s').
\end{aligned}
$$

The contribution from the antiparallel $2n$ twists is

$$
\begin{aligned}
&\sum_{k=1}^{n} f(a+(k-1)(1-t)(b-a), b+(k-1)(1-t)(b-a), a_-) \\
+ \quad &\sum_{k=1}^{n} f(b+(k(1-t)(b-a), a+k(1-t)(b-a), a_-)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{k=1}^{n}(a-b)^{a_1}(b+(k-1)(1-t)(b-a)-a_-)^{a_2} \\
&+ (b-a)^{a_1}(a+k(1-t)(b-a)-a_-)^{a_2} \\
&= \sum_{k=1}^{n}(a-b)^{a_1}(b-a-(1-t)(b-a))^{a_2} \\
&= \sum_{k=1}^{n}(a-b)^{a_1}(t(b-a))^{a_2} \\
&= -nt^{a_2}(a-b)^{a_1+a_2}.
\end{aligned}
$$

The contribution to $W_f^\sharp(x;C)$ from the bottom two crossings is

$$
\begin{aligned}
&-f(a,b',s')-f(b',a,s') \\
&\quad = -(a-b')^{a_1}(b'-s')^{a_2}-(b'-a)^{a_1}(a-s')^{a_2} \\
&\quad = -(a-b')^{a_1}(b'-s'-a-s')^{a_2} \\
&\quad = (a-b')^{a_1+a_2} \\
&\quad = [a-b-n(1-t)(b-a)]^{a_1+a_2} \\
&\quad = (a-b)^{a_1+a_2}[1+n(1-t)]^{a_1+a_2}.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
W_f^\sharp(x;C) &= -n(a-b)^{a_1+a_2}+(a-b)^{a_1+a_2}[1+n(1-t)]^{a_1+a_2} \\
&= (a-b)^{a_1+a_2}(-nt^{a_2}+[1+n(1-t)]^{a_1+a_2}).
\end{aligned}
$$

Hence the contribution from this coloring to the invariant is

$$
\begin{aligned}
&r(a-b)^{a_1+a_2}[-nt^{-a_1}+(1+n(1-t))^{a_1+a_2}] \\
&- (a-b)^{a_1+a_2}\sum_{k=0}^{r-1}t^{(a_1+a_2)k}(-nt^{a_2}+[1+n(1-t)]^{a_1+a_2}) \\
&= (a-b)^{a_1+a_2}(r[-nt^{-a_1}+(1+n(1-t))^{a_1+a_2}] \\
&- (-nt^{a_2}+[1+n(1-t)]^{a_1+a_2})\sum_{k=0}^{r-1}t^{(a_1+a_2)k}),
\end{aligned}
$$

and the result follows by taking all possible $a, b \in X$. $\square$

Proposition 4.1.5 *Let p be an odd prime. Let $X = A = \mathbb{Z}_p[t, t^{-1}]/[t - n(1-t)^2]$, and let $f : X \times X \times X \to A$ be the 3-cocycle defined by $f(x_1, x_2, x_3) = (x_1 - x_2)^{a_1}(x_2 - x_3)^{a_2}$, $a_1 = p^{m_1}$, $a_2 = p^{m_2}$, for p prime and m_1, m_2 non-negative integers, then the cocycle invariant of the r-twist spin of the twist knot $k(2n)$ is trivial (contributions are all 0) if $2n + 1 \equiv 0$, $-n$ or $n \pmod p$.*

Proof. Recall that an r-twist spun knot is colored non-trivially by an Alexander quandle X, we have to have $1 - t^r = 0$ in X. Since trivial colorings give trivial contributions, we may assume now that $1 - t^r = 0$ in X. Note also that all three cases imply n is coprime with p.

Case (1) Suppose $2n + 1 \equiv 0 \pmod p$ (in particular p is an odd prime in this case), then $n(1-t)^2 - t = t^2 + 1$. Since $1 - t^r = 0$ and $1 + t^2 = 0$ in X, we have $1 - t^4 = 0$ in X, therefore r has to be a multiple of 4. In the formula of the invariant, $\sum_{k=0}^{r-1} t^{(a_1+a_2)k}$ appears, so that we compute powers of t. Let k be a positive integer, then in X we have $t^{2k} = (-1)^k$, $t^{2k+1} = (-1)^k t$, $t^{-2k} = (-1)^k$, $t^{-(2k+1)} = (-1)^{k+1} t$, hence $t^{a_1+a_2} = (-1)^{\frac{a_1+a_2}{2}}$.

Case (a) $t^{a_1+a_2} = (-1)^{\frac{a_1+a_2}{2}} = 1$.

The contribution to the state-sum of a coloring induced by a, b is given by

$$r[-nt^{-a_1} + (1 + n(1-t))^{a_1+a_2}](a-b)^{a_1+a_2}$$
$$-(-nt^{a_2} + [1 + n(1-t)]^{a_1+a_2}) \sum_{k=0}^{r-1} t^{(a_1+a_2)k}](a-b)^{a_1+a_2}.$$

We have

$$\sum_{k=0}^{r-1} t^{(a_1+a_2)k} = \sum_{k=0}^{r-1} (-1)^{\frac{a_1+a_2}{2}k} = \sum_{k=0}^{r-1} 1 = r.$$

Then the contribution is equal to

$$r(a-b)^{a_1+a_2}[-nt^{-a_1} + nt^{a_2}] = 0$$

since $t^{a_2} = t^{-a_1}$.

Case (b) $t^{a_1+a_2} = (-1)^{\frac{a_1+a_2}{2}} = -1$.

Since r is a multiple of 4, we have

$$\sum_{k=0}^{r-1} t^{(a_1+a_2)k} = \sum_{k=0}^{r-1} (-1)^{\frac{a_1+a_2}{2}k} = \sum_{k=0}^{r-1} (-1)^k = 0.$$

The contribution to the state-sum, then, is equal to

$$r(a-b)^{a_1+a_2}[-nt^{-a_1} + (1+n(1-t))^{a_1+a_2}].$$

We compute

$$\begin{aligned}
(1+n(1-t))^{a_1+a_2} &= (1+n(1-t))^{a_1}(1+n(1-t))^{a_2} \\
&= (1+n(1-t^{a_1}))(1+n(1-t^{a_2})) \\
&= 1+n(1-t^{a_1})+n(1-t^{a_2})+n^2(1-t^{a_1})(1-t^{a_2}) \\
&= 1+2n-(n^2+n)(t^{a_1}+t^{a_2}) \\
&= -(n^2+n)(t^{2a_1}-1)t^{-a_1} \\
&= -(n^2+n)(-2)t^{-a_1}
\end{aligned}$$

since $t^{a_2} = -t^{-a_1}$. Hence the contribution is equal to

$$r(a-b)^{a_1+a_2}[-nt^{-a_1} + 2(n^2+n)t^{-a_1}] = r(a-b)^{a_1+a_2}t^{-a_1}n(2n+1) = 0.$$

Case (2) Suppose now that $2n+1 \equiv -n \pmod{p}$. Then in X we have

$$\begin{aligned}
n(1-t)^2 - t &\equiv n-(2n+1)t+nt^2 \pmod{p} \\
&\equiv n+nt+nt^2 \pmod{p} \\
&\equiv n(1+t+t^2) \pmod{p}.
\end{aligned}$$

Since n is coprime with p, n is invertible in $\mathbb{Z}_p$, and $X = \mathbb{Z}_p[t, t^{-1}]/(t^2+t+1)$, hence $t^3 = 1$, so r is a multiple of 3. We then compute powers of t in X, and we have $t^2 = -t-1$, $t^3 = 1$. So if k is an integer, we have $t^{3k} = 1$, $t^{3k+1} = t$, $t^{3k+2} = -t-1$. From the condition that $1 - t^r = 0$ in X, we have that r is a multiple of 3. Suppose that $p = 3$, then the first part of the state-sum contribution with the factor r in front vanishes so that the contribution to the state-sum is equal to

$$-(a-b)^{a_1+a_2}(-nt^{a_2} + [1+n(1-t)]^{a_1+a_2})\sum_{k=0}^{r-1} t^{(a_1+a_2)k}.$$

First we compute the sum $\sum_{k=0}^{r-1} t^{(a_1+a_2)k}$. If $t^{a_1+a_2} = 1$ then $\sum_{k=0}^{r-1} t^{(a_1+a_2)k} = r = 0$, so the contribution to the state-sum is equal to zero, then the invariant is trivial. If $t^{a_1+a_2} = t$, then the sum is written as $\sum_{k=0}^{r-1} t^{(a_1+a_2)k} = \sum_{k=0}^{r-1} t^k = (1+t+t^2)(r/3) = 0$, so the invariant is also trivial. Suppose $t^{(a_1+a_2)} = t^2$. Then $\sum_{k=0}^{r-1} t^{(a_1+a_2)k} = \sum_{k=0}^{r-1} t^{2k} = (1+t^2+t^4)(r/3) = 0$, so the invariant is also trivial.

Now suppose that $p \neq 3$, then we have $t^p = t$, or $t^p = -t-1$. If $t^p = t$, then we have $t^{p^m} = t$, so $t^{a_1+a_2} = t^2$, and $\sum_{k=0}^{r-1} t^{(a_1+a_2)k} = \sum_{k=0}^{r-1} t^{2k} = (1+t^2+t^4)(r/3) = 0$. Then the contribution is equal to

$$r(a-b)^{a_1+a_2}[-nt^{-a_1} + (1+n(1-t))^{a_1+a_2}].$$

We have

$$\begin{aligned}
-nt^{-a_1} + (1+n(1-t))^{a_1+a_2} &= -nt^{-p^{m_1}} + (1+n(1-t))^{p^{m_1}+p^{m_2}} \\
&= -nt(-t-1) + (1+n(1-t^{p^{m_1}}))(1+n(1-t^{p^{m_2}})) \\
&= nt + n + (1+n(1-t))^2 \\
&= nt + n + 1 + 2n(1-t) + n^2(1-t)^2 \\
&= nt + n + 1 + 2n - 2nt + nt
\end{aligned}$$

$$= \quad n+1+2n \equiv 0 \pmod p.$$

So the invariant is trivial. If $t^p = -t-1$ (and we are still in the case $p \neq 3$), then $(t^p)^p = t^{p^2} = -t^p - 1 = -(-t-1)-1 = t$, and $t^{p^3} = t^p = -t-1$. So $t^{p^m} = t$ if m is even, and $t^{p^m} = -t-1$ if m is odd. If m_1, m_2 are both even, then $\sum_{k=0}^{r-1} t^{(a_1+a_2)k} = \sum_{k=0}^{r-1} t^{2k} = (1+t^2+t^4)(r/3) = 0$ as before. Then the contribution is given by $r(a-b)^{a_1+a_2}[-n(-t-1)+(1+n(1-t))^2]$, then by the same calculation as for the previous case, we get that this contribution is 0. If m_1 is even and m_2 is odd then, $t^{p^{m_1}} = t$ and $t^{p^{m_2}} = -t-1$, and the contribution is equal to

$$\begin{aligned}
& r(a-b)^{a_1+a_2}[-nt^{-a_1} + (1+n(1-t))^{a_1+a_2}] \\
& \qquad -r(a-b)^{a_1+a_2}[-nt^{a_2} + (1+n(1-t))^{a_1+a_2}] \\
&= r(a-b)^{a_1+a_2}[-nt^{-a_1} + nt^{a_2}] \\
&= r(a-b)^{a_1+a_2}[-n(-t-1) + n(-t-1)] = 0.
\end{aligned}$$

If m_1 is odd and m_2 is even, then $t^{a_1} = -t-1$ and $t^{a_2} = t$, and $t{-}a_1$=t, and the contribution is equal to

$$\begin{aligned}
& r(a-b)^{a_1+a_2}[-nt^{-a_1} \\
& \qquad +(1+n(1-t))^{a_1+a_2}] - r(a-b)^{a_1+a_2}[-nt^{a_2} + (1+n(1-t))^{a_1+a_2}] \\
&= r(a-b)^{a_1+a_2}[-nt^{-a_1} + nt^{a_2}] \\
&= (a-b)^{a_1+a_2}[-nt + nt] = 0.
\end{aligned}$$

If m_1 and m_2 are both odd, then $t^{a_1} = t^{a_2} = -t-1$ and we have $t^{a_1+a_2} = t^2+2t+1 = t$. So $\sum_{k=0}^{r-1} = t^{(a_1+a_2)k} = (1+t+t^2)(r/3) = 0$. Then the contribution is equal to

$$\begin{aligned}
& r(a-b)^{a_1+a_2}[-nt^{-a_1} + (1+n(1-t))^{a_1+a_2}] \\
&= r(a-b)^{a_1+a_2}[-nt + (1+n(1-(-t-1)))(1+n(1-(-t-1)))] \\
&= r(a-b)^{a_1+a_2}[-nt + (1+n(t+2))^2] \\
&= r(a-b)^{a_1+a_2}[-nt + 1 + 2n(t+2) + n^2(t^2+4t+4)]
\end{aligned}$$

$$
\begin{aligned}
&= r(a-b)^{a_1+a_2}[nt+1+4n+n^2(3t+3)] \\
&= r(a-b)^{a_1+a_2}[nt(3n+1)+n+3n^2] \\
&= n(3n+1)r(a-b)^{a_1+a_2}[t+1]=0.
\end{aligned}
$$

Case (3) Suppose that $2n+1 \equiv n \pmod p$. Then $X = \mathbb{Z}_p[t,t^{-1}]/(1-t+t^2)$, and then we have $t^2 = t-1$, $t^3 = -1$, $t^4 = -t$, $t^5 = -t^2 = 1-t$, $t^6 = 1$, so $t^{3k+1} = (-1)^k t$, $t^{3k+2} = (-1)^k(t-1)$ and $t^{3k} = (-1)^k$. Then $r = 6l$ for some l. If $t^p = t$, then $t^{p^m} = t$ for any positive integer m, so $t^{a_1+a_2} = t^2$ and the sum $\sum_{k=0}^{r-1} t^{2k} = 0$, and the contribution is given by

$$
\begin{aligned}
& r(a-b)^{a_1+a_2}[-nt^{-1}+(1+n(1-t))^2] \\
&\quad = r(a-b)^{a_1+a_2}[-nt^{-1}+1+2n(1-t)+n^2(1-t)^2] \\
&\quad = r(a-b)^{a_1+a_2}[-n+nt+1+2n-2nt+nt]=0.
\end{aligned}
$$

If $t^p = t-1$, then $p=2$, and the contribution is zero since $r=6l$.
If $t^p = 1-t$, then $t^{p^m} = t$ when m is even, and $t^{p^m} = 1-t$ when m is odd.
If m_1 is odd, and m_2 is even, $t^{a_1+a_2} = (1-t)t = 1$, then $\sum_{k=0}^{r-1} = t^{(a_1+a_2)k} = r$, and the contribution is equal to

$$
\begin{aligned}
& r(a-b)^{a_1+a_2}[-nt^{-a_1}+nt^{a_2}] \\
&\quad = r(a-b)^{a_1+a_2}[-nt^{-1}+n(1-t)] \\
&\quad = r(a-b)^{a_1+a_2}[-n(1-t)+n(1-t)]=0
\end{aligned}
$$

If m_1, m_2 are both even, then $t^{a_1+a_2} = t^2$ hence $\sum_{k=0}^{r-1} = t^{(a_1+a_2)k} = (1+t^2+t^4)(r/3) = 0$, and the contribution is equal to

$$
\begin{aligned}
& r(a-b)^{a_1+a_2}[-nt^{-a_1}+(1+n(1-t))^{a_1+a_2}] \\
&\quad = r(a-b)^{a_1+a_2}[-nt^{-1}+(1+n(1-t))^2] \\
&\quad = r(a-b)^{a_1+a_2}[-n(1-t)+(1+2n(1-t)+n^2(1-t)^2] \\
&\quad = r(a-b)^{a_1+a_2}[-n(1-t)+1+2n(1-t)+nt]
\end{aligned}
$$

$$= \quad r(a-b)^{a_1+a_2}[1+2n-n] = 0$$

If m_1, m_2 are both odd, then $t^{a_1+a_2} = (1-t)^2 = -t$, hence $\sum_{k=0}^{r-1} = t^{(a_1+a_2)k} = \sum_{k=0}^{r-1}(-t)^k = 0$, and the contribution is given by

$$\begin{aligned} & r(a-b)^{a_1+a_2}[-nt^{-a_1} + (1+n(1-t))^{a_1+a_2}] \\ = \quad & r(a-b)^{a_1+a_2}[-nt + 1 + 2n(1-t^{a_1}) + (n^2(1-t)^2)^{a_1}] \\ = \quad & r(a-b)^{a_1+a_2}[-nt + 1 + 2nt + n(1-t)] = 0 \end{aligned}$$

□

4.1.4 Remarks on Non-Invertibility of Twist Spun Knots

In this section we make some observations on non-invertibility of twist spun knots.

Let M be a multi-set of elements of an abelian group A. Let $\overline{M}$ be the multi-set defined by $\overline{M} = \{ -x \mid x \in M\}$. The following lemma was used in [9], but we state it here with a proof in a clearer manner.

Lemma 4.1.6 *Let K be an invertible classical knot, $\tau^r K$ its r-twist spun, and $-\tau^r K$ the same twist spun knot with the opposite orientation. Then for the cocycle invariants in the multi-set form, we have $\Phi_f(-\tau^r K) = \overline{\Phi_f(\tau^r K)}$.*

Proof. Let F be a knotted surface, $-F$ its orientation reversed counterpart, and let F^* be the mirror image of F. Then it is known [10] that for a knotted surface F, the cocycle invariant satisfies $\Phi_f(-F^*) = \overline{\Phi_f(F)}$.

On the other hand, Litherland [28] proved that if a classical knot K is invertible, then its twist-spun $\tau^r K$ is amphicheiral, so that F^* is equivalent to F. Hence we have $\Phi_f(-\tau^r K) = \Phi_f(-(\tau^r K)^*) = \overline{\Phi_f(\tau^r K)}$. □

Our remark here is that if we use cocycle invariants for detecting non-invertibility, then the lemma 4.1.6, together with our formulas for twist-spun torus and twist knots, the problem is reduced to a pure algebraic problem of determining whether the given multiset M that appear as the invariant of the twist knot is symmetric in the sense that $M = -M$.

Note that the cocycle invariants investigated in this section have the form

$$\Phi_f(D^r K) = \{g s^{a_1+a_2} \mid s \in X\},$$

for a fixed $g \in X$ Then $\Phi_f(D^r K) = -\Phi_f(D^r K)$ if and only if

$$\{s^{a_1+a_2} \mid s \in X\} = -\{s^{a_1+a_2} \mid s \in X\}.$$

This last condition is a purely algebraic question on polynomial rings. Thus we remark here that our formulas of twist-spun knots enabled us to reduce possibility of using the quandle cocycle invariants for the purpose of non-invertibility of twist-spun knots to an algebraic problem of determining whether powers of elements of polynomial rings are symmetric multisets.

For the case of torus knots, although it is proved in [21] that all twist spun torus knots are non-invertible, the argument applies only to the spherical case. If the non-invertibility is detected by quandle cocycle invariants, on the other hand, the non-invertibility remains valid for higher genus surfaces obtained by adding trivial 1-handles.

Using lemma 4.1.1 and a *Maple* program, we obtained the non-invertibilty of the r-twist $K(2,9)$ torus knot, where r is a multiple of 18, by confirming that the invariant values are not symmetric in the above sense. This case is not included in [1] in which they used dihedral quandles. Our *Maple* program could not finish computations for $K(2,m)$ when $m > 11$. We summarize the result as the following proposition.

Proposition 4.1.7 *The r-twist $K(2,9)$ torus knot, where r is a multiple of 18, is non-invertible.*

Proof. Take the quandle $X = A = \mathbb{Z}_p[t, t^{-1}]/\xi_9(t)$ and the cocycle $f(x_1, x_2, x_3) = (x_1 - x_2)(x_1 - x_2)^3$. Using our *Maple* program, we obtain that the invariant $\Phi_f(\tau^r K(2,9))$ is $94143178827 + 188286357654 U^{(2t^7+2t^6+t^4+t^3+2t+2)}$, where r is a multiple of 18. Then we see that it is not symmetric, hence the r-twist $K(2,9)$ is non-invertible, when r is a multiple of 18. $\square$

4.2 Quandle Cocycle Invariants and Tangle Embeddings

One of the first applications of knot theory is to molecular biology. The recombination of DNA caused by enzyme actions were mathematically determined by formulating tangle equations [17]. These equations were in the form of numerator of the addition of some tangles being equal to some knots that were found in experiments (see below for definitions). Solving tangle embedding problems are useful in checking if a tangle is an eventual solution to a tangle equation.

In this section we use quandle cocycle invariants as obstructions to embedding tangles in knots. We are interested in two types of tangles: tangles that are addition of two rational tangles with m and n vertical twists respectively, and tangles up to 7 crossings that were classified in [26]. The latter ones are parametrized by a pair of numbers in a symbol similar to those representing knots. The section is organized as follows. First we review definitions, then state and prove a theorem that is used as obstructions to tangle embeddings. In the third subsection, we compute and use invariants of tangles that are obtained by tangle additions of two parallel or antiparallel strings. In the last subsection, we investigate tangle embeddings for the table of tangles and the table of knots.

4.2.1 Tangles and Their Operations

In this section we recall definitions needed to discuss tangles. An (n, n)-tangle is a proper embedding of n disjoints arcs and some (possibly empty) circles into the 3-ball B^3 such that n end points lie in the upper hemisphere and n end points lie in the lower hemisphere.

An (n, n)-tangle is commonly abbreviated by an n-tangle. In this dissertation we are mostly interested in 2-tangles, and for simplicity we shall refer to them as just tangles. We will call the four points of the tangle that lie on the boundary of the ball B^3, NE, NW, SE, SW (where the abbreviations refer to, north east, north west et cetera). These points can be precisely described in $\mathbb{R}^3$ in terms of the following coordinates:

$$NE = (0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), \qquad NW = (0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$$
$$SE = (0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) \qquad SW = (0, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}).$$

A diagram of a tangle is defined in a manner similar to those for knots and links, where a diagram is drawn in a circle that represents a 3-ball that contain the tangle, and the end points of a diagram is placed on the circle.

The addition $T_1 + T_2$ of two tangles T_1, T_2 is another tangle defined from the original two as depicted in Fig. 4.3. The closures are two methods of obtaining a link from a tangle by closing the end points, and there are two ways called the *numerator* $N(T)$ and *denominator* $D(T)$ of a tangle T, defined as depicted in Fig. 4.4. These definitions can be found, for example, in [32].

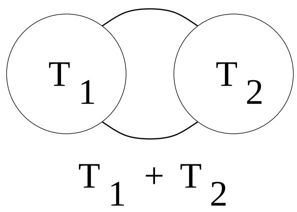

Figure 4.3: Addition of tangles

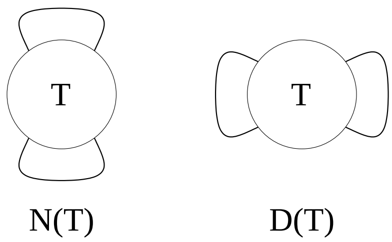

Figure 4.4: Closures (numerator $N(T)$, denominator $D(T)$) of tangles

There is a family of *trivial*, or *rational* tangles, some of which are depicted in Fig. 4.5. These are obtained from the trivial tangle of two vertical straight arcs by successively twisting end points vertically and horizontally. See again [32], for example, for more details.

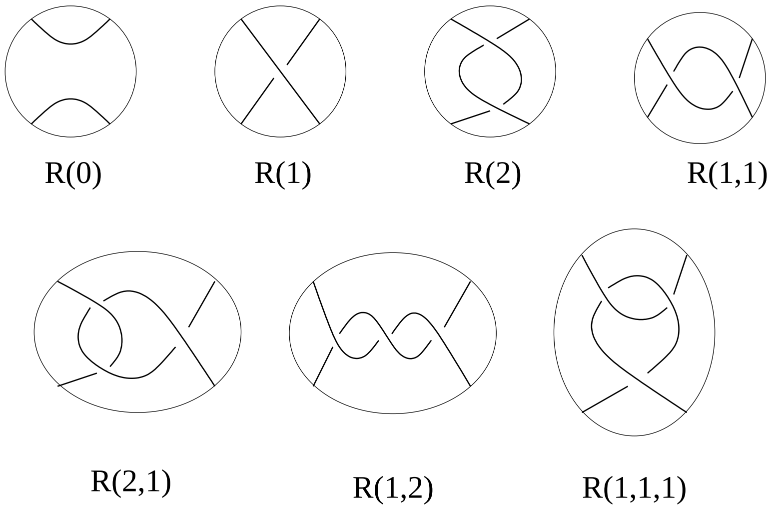

Figure 4.5: Some rational tangles

A straightforward way of identifying a tangle embedded in a knot is to construct a knot from a given tangle. Since we are interested in the knots in the table up to 9 crossings and tangles in the table up to 7 crossings, we try the following procedures: For a given tangle T from the table, add a rational tangle R to obtain $T + R$, then take closures $N(T + R)$ and $D(T + R)$, and see which knots in the table result.

4.2.2 Cocycle Invariants as Obstructions

In this section we prove a theorem that will be used as obstructions to tangle embeddings.

Definition 4.2.1 Let T be a tangle and X a quandle. A (boundary-monochromatic) *coloring* $\mathcal{C} : \mathcal{A} \to X$ is a map from the set of arcs in a diagram of T to X satisfying the same quandle coloring condition as for knot diagrams at each crossing, such that the (four) boundary points of the tangle diagram receives the same element of X.

For a coloring $\mathcal{C}$ of a tangle diagram T, a region colorings are defined in a similar manner as in the knot case. In this case, we allow region colors to change (not necessarily colored by the same element as the one assigned to the boundary points).

Denote by $\mathrm{Col}_X(T)$ the set of colorings of a diagram of T by X. Denote by $\mathrm{Col}_X(T, s)$ the set of colorings of a diagram of T by X with the color of the leftmost region (between the boundary arcs NW and SW) being $s \in X$. It is seen in a way similar to the knot case that the number of colorings $|\mathrm{Col}_X(T)|$ does not depend on a choice of a diagram of T, and that the set of colorings are in one-to-one correspondence between Reidemeister moves.

Let $\{T_i : i = 1, \ldots k\}$ be a finite family of disjoint tangle diagrams. Then the invariant $\Phi_\phi(\{T_i : i = 1, \ldots k\})$ is defined in a manner similar to the above. Specifically, for each coloring that is monochromatic on the boundary points of all tangles in the family, define the contribution to be the sum of contributions from each tangle with respect to this coloring. Then the cocycle invariant is defined as a multi-set with respect to all colorings.

The quandle 2- and 3-cocycle invariants are defined for tangles in a manner similar to the knot case, and denoted by $\Phi_\phi(T)$. Let ∂T denote the boundary points of a given tangle T. Let $\Phi(T, x) = \sum_{C \in \mathrm{Col}_x(T)} \prod_\tau B(C, \tau)$ where $\mathrm{Col}_x(T) = \{C \in \mathrm{Col}_X(T) \mid C(\partial T) = x\}$ for $x \in X$. Then $\Phi_\phi(T) = \sum_{x \in X} \Phi(T, x)$.

Let M, N be two multisets. The inclusion of multisets is denoted by $\subset_m$, and is defined as follows: if an element x is repeated n times in a multiset, call n the multiplicity of x. Then $M \subset_m N$ means that if $x \in M$, then $x \in N$, and the multiplicity of x in M is less than or equal to the multiplicity of x in N.

Theorem 4.2.2 *Let T be a tangle, and X a quandle. Suppose T embeds in a link L. Then we have the inclusion $\Phi_\phi(T) \subset_m \Phi_\phi(L)$ for the invariants in the multiset form.*

Suppose a finite set of tangles $T_1, \ldots, T_k$ embeds disjointly in a link L, for a positive integer k. Then we have $\Phi_\phi(\{T_i : i = 1, \ldots, k\}) \subset_m \Phi_\phi(L)$.

Proof. Suppose a diagram of T embeds in a diagram of L. We continue to use T and L for these diagrams. For a coloring $\mathcal{C}$ of T, let x be the color of the boundary points. Then there is a unique coloring $\mathcal{C}'$ of L such that the restriction of $\mathcal{C}'$ on T is $\mathcal{C}$ and all the arcs of L outside of T receive the color x. Then the contribution of

$\sum_{\tau \in T} B(\mathcal{C}, \tau)$ to $\Phi_\phi(T)$ is equal to the contribution $\sum_{\tau \in L} B(\mathcal{C}', \tau)$ to $\Phi_\phi(L)$, and the theorem follows. □

4.2.3 Tangles with Two Twists

In this section, let m, n be two positive integers and let $R_{(m)}$ denote the rational tangle with m vertical crossings. The tangle $R_{(m)}$ can be oriented by assigning direction at the NE and NW points. It is said to be parallel if both NE and NW are assigned the same direction, and it is said to be antiparallel if they are assigned opposite directions. Let $R_{(m)}^{(P,I)}$ be the parallel tangle $R_{(m)}$ with the NW arc is oriented in. The other cases are denoted in a similar way by using the letter A in the place of the letter P in the case where the tangle is antiparallel, and O instead of I if the NW arc is oriented out. Let X be an Alexander quandle, then the coloring of the top arcs of the tangle can be extended, in a unique way to the whole tangle.

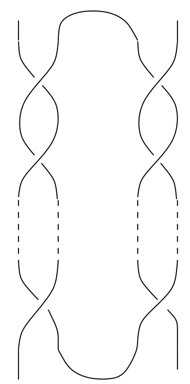

Figure 4.6: Tangle with two twists

We are interested in tangles with two twists as depicted in Fig. 4.6, and we establish notations and conventions. Let $T_{[m,n]}^{P,A}$ denote the oriented tangle obtained by adding the tangles $R_{(m)}$ and $R_{(n)}$ such that their orientations are parallel antiparallel, respectively, and the NW is oriented inward. The other cases are denoted in similar ways by putting in the letters P or A in superscript depending on the corresponding part (left and right twists) are parallel and antiparallel, respectively. Then without loss of generality we can consider the following three cases.

Case 1: $T^{P,A}_{[m,n]} = R^{(P,I)}_{(m)} + R^{(A,O)}_{(n)}$ (as depicted in Fig. 4.9), where m is an odd integer and n is an even integer.

Case 2: $T^{A,A}_{[m,n]} = R^{(A,I)}_{(m)} + R^{(A,O)}_{(n)}$ (as depicted in Fig. 4.10), where both m and n are odd integers.

Case 3: $T^{P,P}_{[m,n]} = R^{(P,I)}_{(m)} + R^{(P,O)}_{(n)}$ (as depicted in Fig. 4.11), where m, n are not both even.

We may consider only these three cases for the following reasons. First, we restrict ourselves here to tangles that do not have a separate component of a closed circle inside. This restriction excludes the case when m and n are both even. If both twists are parallel, then it is Case 3, and the orientations of middle arcs (the arcs that connect two twist parts) are determined by the requirement that both be parallel. Note that if the orientation of the NW is inward, then that of NE is outward. Hence the π rotation of this tangle about the horizontal axis make the orientation of NW outward, so that we can only consider the case where NW is inward, in this case.

The same applies when both are antiparallel, giving rise to the Case 2. The requirement that both antiparallel, in this case, determines the orientations of the middle arcs. Then the parities of twists are determined from these orientation to be both odd. This is the Case 2.

Then the case with one parallel, one antiparallel remains. If the left tangle is parallel as depicted in Fig. 4.9, and NW is inward, then the middle arc is to be oriented as depicted, and then, the right tangle must have even twists. The only other case is the same type with the opposite orientation. The invariant, in the case with opposite orientations is the negative of the first case, so we only consider the case of NW inward. This exhausts the above cases.

In this section we compute the 3-cocycle invariant for the tangles of Case 1, Case 2, and Case 3, with the cocycle $f(x_1, x_2, x_3) = (x_1 - x_2)^{a_1}(x_2 - x_3)^{a_2}$, where $a_1 = p^{m_1}$, $a_2 = p^{m_2}$. For this purpose we compute the contribution from antiparallel strings to the invariant.

First we compute the contribution from two antiparallel strings depicted in Fig. 4.7

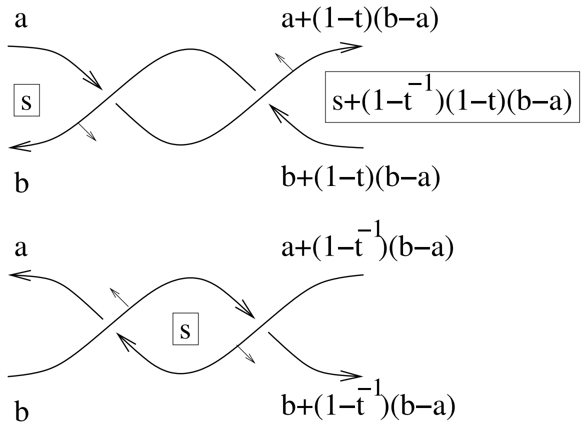

Figure 4.7: Two strings with an antiparallel orientation, case I

top. With the left most region colored by s, the top (resp. bottom) left arc with a (resp. b), the region left to the $(2k-1)$th crossing is colored by $s+(k-1)(1-t^{-1})(1-t)(b-a)$, and the arcs that are top (resp. bottom) arc with $a+k(1-t)(b-a)$ (resp. $b+k(1-t)(b-a)$) by induction. Then the contribution of the cocycle from these $2n$ crossings with this coloring is computed by

$$\begin{aligned}
\eta_n^+ &= \sum_{k=1}^{n} \{[(s+(k-1)(1-t^{-1})(1-t)(b-a)) - (a+(k-1)(1-t)(b-a))]^{a_1} \\
&\times \ [(a+(k-1)(1-t)(b-a)) - (b+(k-1)(1-t)(b-a)))]^{a_2} \\
&+ \ [(s+k(1-t^{-1})(1-t)(b-a)) - (b+k(1-t)(b-a))]^{a_1} \\
&\times \ [(b+k(1-t)(b-a)) - (a+k(1-t)(b-a)))]^{a_2}\} \\
&= \sum_{k=1}^{n} (a-b)^{a_2}(-(1-t^{-1})(1-t)(b-a) - (a-b) + (1-t)(b-a))^{a_1} \\
&= -nt^{-a_1}(a-b)^{(a_1+a_2)}.
\end{aligned}$$

For two antiparallel strings depicted in Fig. 4.7 bottom, we compute as follows similarly.

$$\begin{aligned}
\zeta_n^+ &= \sum_{k=1}^{n} \{[s_k - (a+k(1-t^{-1})(b-a)]^{a_1} \\
&\times \ [(a+k(1-t^{-1})(b-a) - (b+(k-1)(1-t^{-1})(b-a)))]^{a_2}
\end{aligned}$$

$$\begin{aligned}
&+ \quad [s_k - (b + (k-1)(1-t^{-1})(b-a)]^{a_1} \\
&\times \quad [(b + (k-1)(1-t^{-1})(b-a))) - (a + k(1-t^{-1})(b-a))]^{a_2}\} \\
&= \quad \sum_{k=1}^{n} (a - b + (1-t^{-1})(b-a))^{a_2}(-a + b - (1-t^{-1})(b-a))^{a_1} \\
&= \quad -nt^{-(a_1+a_2)}(a-b)^{(a_1+a_2)}
\end{aligned}$$

where s_k is the color of the middle region to the left of $(2k)$th crossing, which needs not be calculated as being cancelled.

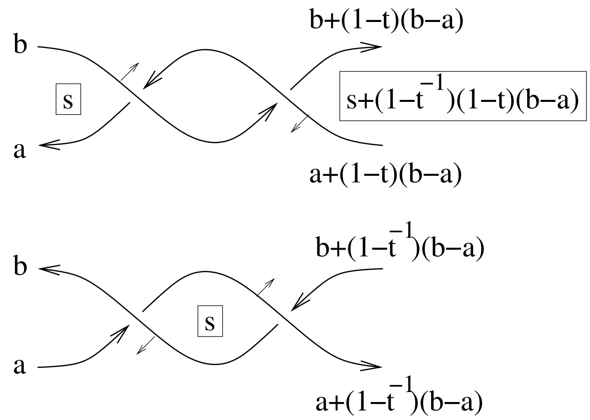

Figure 4.8: Two strings with an antiparallel orientation, case II

For crossings depicted in Fig. 4.8, since the figures are mirror images and only the sign of crossings change, we obtain

$$\begin{aligned}
\eta_n^- &= nt^{-a_1}(a-b)^{(a_1+a_2)} \\
\zeta_n^- &= nt^{-(a_1+a_2)}(a-b)^{(a_1+a_2)}
\end{aligned}$$

for the top and bottom figures, respectively, where the crossings are repeated $2n$ times as before for each figure, and the left-most colors are a and b as depicted.

In all what follows assume that a is the color of the NW arc, and b is the color of the arc between the NW arc and NE arc, unless otherwise specified.

Case 1: $T_{[m,n]}^{(P,A)}$ with m odd and n even.

Lemma 4.2.3 *The tangle* $T_{[m,n]}^{(P,A)}$ *with the orientation determined by NW inward, where* m *is odd and* n *is even, is colored nontrivially with an Alexander quandle* $X = \mathbb{Z}_p[t, t^{-1}]/h(t)$ *if* $(a - b)\xi_m = 0$ *for some* $a, b \in X$*, and* $n/2 \equiv 0 \pmod{p}$*.*

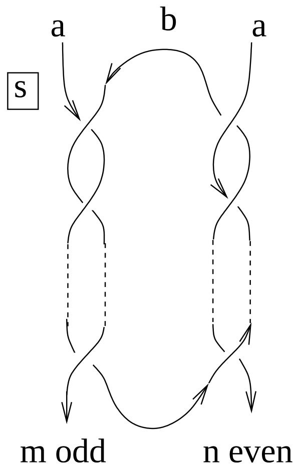

Figure 4.9: Case 1

Proof. By lemma 3.1.1, the colors at the bottom of the tangle $R_{(m)}$ are, respectively from left to right, $(a + \xi_m(b - a), b + t\xi_m(b - a))$, then we must have $a + \xi_m(b - a) = a$ and hence $(a - b)\xi_m = 0$. Suppose $(a - b)\xi_m = 0$, then the top arcs of the tangle $R_{(n)}$ are colored b, a, respectively from left to right, then by Lemma 3.1.3, the bottom arcs will get the colors $(b + n/2(a - b), a + n/2(a - b))$ from left to right. Therefore we must have $b + n/2(a - b) = b$ and $a + n/2(a - b)$, thus $n/2 \equiv 0 \pmod{p}$. □

Lemma 4.2.4 *Let* $X = A = \mathbb{Z}_p[t, t^{-1}]/\xi_m$*, and* $n/2 \equiv 0 \pmod{p}$*. Let* a *be the color of the NW arc, and* b *be the color of the middle arc between the NW arc and the NE arc. Suppose* ξ_m *divides* $1 - t^{a_1+a_2}$ *then the contribution to the state-sum of the coloring induced by* (a, b) *is given by*

$$(a - b)^{a_1+a_2}(-t)^{a_1(1-m)}(\xi'_m)^{a_1}.$$

Proof. From Lemma 3.3.1, the contribution of the left part of the tangle $R_{(m)}^{P,I}$ is given by $(a - b)^{a_1+a_2}(-t)^{a_1(1-m)}(\xi'_m)^{a_1}$. The contribution of the right tangle $R_{(n)}^{(A,O)}$ is given by $-(n/2)t^{-a_1}(a-b)^{a_1+a_2}$. The contribution to the state-sum is the sum of these two contributions. Since $n/2 \equiv 0 \pmod p$, the second contribution vanishes. Hence the contribution is only from the first one. □

Then we obtain

Proposition 4.2.5 *Let* $X = A = \mathbb{Z}_p[t,t^{-1}]/\xi_m$, *and* $n/2 \equiv 0 \pmod p$. *Then the cocycle invariant of the tangle* $T^{(P,A)}_{[m,n]}$ *is equal to that of the* $(2,m)$*-torus knot*

$$\Phi_f(T^{(P,A)}_{[m,n]}) = \Phi_f(K(2,m)) = \{\sqcup_{|X|}(-t)^{a_1(1-m)}(\xi'_m)^{a_1}s^{a_1+a_2} \mid s \in X\}.$$

Example 4.2.6 We consider the tangle $T^{(P,A)}_{[3,4k]}$. Using the the Alexander quandle $X = \mathbb{Z}_2[t,t^{-1}]/(1-t+t^2)$, and the cocycle $f(x_1,x_2,x_3) = (x_1-x_2)(x_2-x_3)^2$, the invariant $\Phi_f(T^{(P,A)}_{[3,4k]}) = 16 + 48U^t$. Comparing this invariant with the invariant of knots in the knot table we obtain that among 84 knots up the 9 crossings in the knot table the tangle $T^{(P,A)}_{[3,4k]}$ may embed only in 19 knots, and these knots are:

$$3_1, 4_1, 7_2, 7_3, 8_1, 8_4, 8_{11}, 8_{13}, 8_{18}, 9_1, 9_6, 9_{12}, 9_{13}, 9_{14}, 9_{21}, 9_{23}, 9_{35}, 9_{37}, 9_{40}.$$

Using the Alexander quandle $X = \mathbb{Z}_3[t,t^{-1}]/(1-t+t^2)$, and cocycle $f(x_1,x_2,x_3) = (x_1-x_2)(x_2-x_3)^3$, the invariant $\Phi_f(T^{(P,A)}_{[3,6k]}) = 243 + 486U^{(2t+2)}$. Comparing this invariant with the invariant of knots in the knot table we obtain that the tangle $T^{(P,A)}_{[3,6k]}$ may embed only in 16 knots, and these knots are:

$$3_1, 6_1, 7_4, 8_5, 8_{15}, 8_{18}, 8_{21}, 9_2, 9_4, 9_{16}, 9_{17}, 9_{28}, 9_{29}, 9_{34}, 9_{38}, 9_{40}.$$

Using the the Alexander quandle $X = \mathbb{Z}_5[t,t^{-1}]/1-t+t^2$, and the cocycle $f(x_1,x_2,x_3) = (x_1-x_2)(x_2-x_3)^5$, the invariant $\Phi_f(T^{(P,A)}_{[3,10k]}) = 625 + 3750U^{(t+3)} + 3750U^{(2t+2)} + 3750U^{(3t+4)} + 3750U^{(2t+1)}$. The tangle $T^{(P,A)}_{[3,10k]}$ may embed only in 18 knots, and these knots are:

$$3_1, 8_3, 8_5, 8_{11}, 8_{15}, 8_{18}, 8_{19}, 8_{21}, 9_1, 9_5, 9_6, 9_{16}, 9_{19}, 9_{23}, 9_{28}, 9_{29}, 9_{38}, 9_{40}.$$

Then combining all these results the tangle $T^{(P,A)}_{[3,60k]}$ may embedd only in $3_1, 8_{18}, 9_{40}$.

Case 2: $T^{A,A}_{[m,n]} = R^{(A,I)}_{(m)} + R^{(A,I)}_{(n)}$, with m and n both odd.

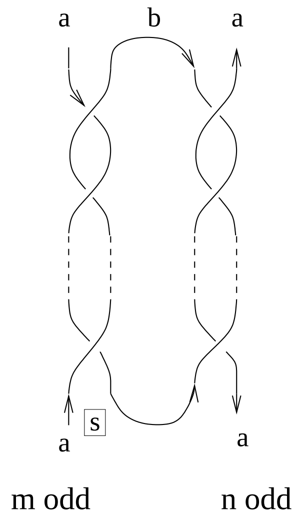

Figure 4.10: Case 2

Lemma 4.2.7 *The tangle* $T^{A,A}_{[m,n]} = R^{(A,I)}_{(m)} + R^{(A,I)}_{(n)}$*, with* m *and* n *both odd, with orientation NW inward, is colored non-trivially by the dihedral quandle* R_p*, if* $m \equiv n \equiv 0 \pmod{p}$.

Proof. Assign colors a, b for the NW arc and the arc between NW and NE arcs. For the left twists, if m is positive, then we have the twists depicted in the bottom of Fig. 4.8. In particular, the SW arc and the arc between SW and SE are colored by $b + m'(1 - t^{-1})(b - a)$ and $t^{-1}a + (1 - t^{-1})b + m'(1 - t^{-1})(b - a)$, respectively, where $m = 2m' + 1$. We must have $b + m'(1 - t^{-1})(b - a) = a$. To simplify the calculations, we take the condition $1 + m'(1 - t^{-1}) = 0$. Hence m' must be invertible and $t^{-1} = (1 + m')m'^{-1}$.

If $n > 0$ for the right part of twists, the top colors are (b, a), and the bottom are $a + n'(1 - t^{-1})(a - b)$ and $a + (n' + 1)(1 - t^{-1})(a - b)$, respectively, and $a + (n' + 1)(1 - t^{-1})(a - b) = a$, where $n = 2n' + 1$. The last condition gives $1 + n'$ is invertible and $t^{-1} = n'(1 + n')^{-1}$. Together with the conditions of m', we obtain $m' + n' + 1 \equiv 0$

(mod p). It is known [31] that a linear Alexander quandle has non-trivial 3-cocycle only when it is a dihedral quandle. Hence we restrict ur attention to dihedral quandles of prime order. Then the condition $t^{-1} = -1$ gives $m \equiv n \equiv 0 \pmod{p}$.

If n is negative, similar calculations imply that $t^{-1} = (1+m')m'^{-1} = (1+n')n'^{-1}$ and setting $t^{-1} = -1$ again, we obtain the same conclusion. The other cases follow from symmetry by these cases. □

Case 3: $T^{(p,p)}_{[m,n]} = R^{(P,I)}_{(m)} + R^{(P,O)}_{(n)}$, with m and n not both even.

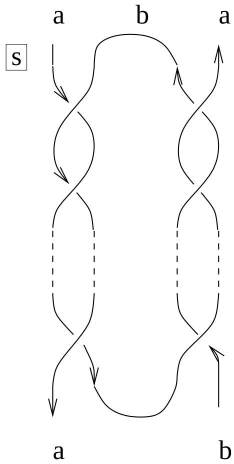

Figure 4.11: Case 3

Lemma 4.2.8 *The tangle* $T^{(P,P)}_{[m,n]} = R^{(P,I)}_{(m)} + R^{(P,O)}_{(n)}$ *is colored nontrivially by an Alexander quandle* X *if :* $(a-b)h(t) = 0$ *in* X *for* $a, b \in X$*, where* $h(t) = \gcd(\xi_m(t), \xi_n(t))$.

Proof. If the colors of the NW arc and the arc between the NW and NE arcs are colored by $a, b \in X$, then the colors at the SW arc and the arc between the SW arc and the SE arc are respectively $a + \xi_m(t)(b-a)$ and $b + t\xi_m(t)(a-b)$ by Lemma 3.1.1. Then we must have $a = a + \xi_m(t)(b-a)$ for the SW arc, and we obtain $\xi_m(t)(a-b) = 0$. Then the colors of the NE arc and the arc between the NE and NW arcs are, respectively, $a + \xi_n(t)(b-a)$ and $b + t\xi_n(t)(a-b)$ from the right n twists. From the color of the NE arc, we must have $a + \xi_n(t)(b-a) = a$, so that $(a-b)\xi_n = 0$. The both equalities are satisfied if $(a-b)h(t) = 0$ in X for $h(t) = gcd(\xi_m(t), \xi_n(t))$. □

Lemma 4.2.9 *Let* $X = A = \mathbb{Z}_p[t,t^{-1}]/\xi_k(t)$ *for a positive integer* k*, and suppose* $\xi_k(t)$ *divides* $1 - t^{a_1+a_2} = 0$*. Let* $T^{(P,P)}_{[m,n]} = R^{(P,I)}_{(m)} + R^{(P,O)}_{(n)}$*, and let* $m = uk$ *and* $n = vk$ *for a positive integers* u, v*. Then the contribution of the coloring induced by* a *and* b *is*

$$(u+v)(a-b)^{a_1+a_2}(-t)^{a_1(1-k)}(\xi_k')^{a_1}.$$

Proof. This follows from Lemma 3.3.1, since the source regions of the crossings of the left twists and right twists have the same color, so that the contribution to the invariant is the same as that of the torus knot of type $(2,(u+v)k)$. □

Hence we have the proposition

Proposition 4.2.10 *Let* $X = A = \mathbb{Z}_p[t,t^{-1}]/\xi_k(t)$ *for a positive integer* k*, and suppose* $\xi_k(t)$ *divides* $1 - t^{a_1+a_2} = 0$*. Let* $T^{(P,P)}_{[m,n]} = R^{(P,I)}_{(m)} + R^{(P,O)}_{(n)}$ *and let* $m = uk$ *and* $n = vk$ *for a positive integers* u, v*. Then the cocycle invariant of* $T^{(P,P)}_{[m,n]}$ *is given by*

$$\Phi_f(T^{(P,P)}_{[m,n]}) = \Phi_f(K(2,(u+v)k)) = \{\sqcup_{|X|}(u+v)(-t)^{a_1(1-k)}(\xi_k')^{a_1}s^{a_1+a_2} \mid s \in X\}.$$

Example 4.2.11 Let $X = \mathbb{Z}_p[t,t^{-1}]/\xi_3(t)$. For $p = 2$, and using the cocycle $f(x_1,x_2,x_3) = (x_1-x_2)(x_2-x_3)^2$, the invariant of the tangle $T^{(P,P)}_{[6,9]}$ is $16 + 48U^t$. Using [41] we obtain that among 84 knots in the knot table up to 9 crossings the tangle $T^{(P,P)}_{[6,9]}$ may embed only in 19 knots, and these knots are:

$$3_1, 4_1, 7_2, 7_3, 8_1, 8_4, 8_{11}, 8_{13}, 8_{18}, 9_1, 9_6, 9_{12}, 9_{13}, 9_{14}, 9_{21}, 9_{23}, 9_{35}, 9_{37}, 9_{40}.$$

For p=3, and $f(x_1,x_2,x_3) = (x_1-x_2)(x_2-x_3)^3$, the invariant of the tangle $T^{(P,P)}_{[6,9]}$ is $243 + 486U^{t+1}$, thus among 84 knots from the knot table up to 9 crossings, the tangle may only embed on 17 knots, and these knots are

$$3_1, 6_1, 7_4, 8_5, 8_{15}, 8_{18}, 8_{19}, 8_{21}, 9_2, 9_4, 9_{16}, 9_{17}, 9_{28}, 9_{29}, 9_{34}, 9_{38}, 9_{40}$$

. Combining these two lists, the tangle $T^{(P,P)}_{[6,9]}$ may embed only in $3_1, 8_{18}, 9_{40}$

4.2.4 Using Tangles and Knot Tables

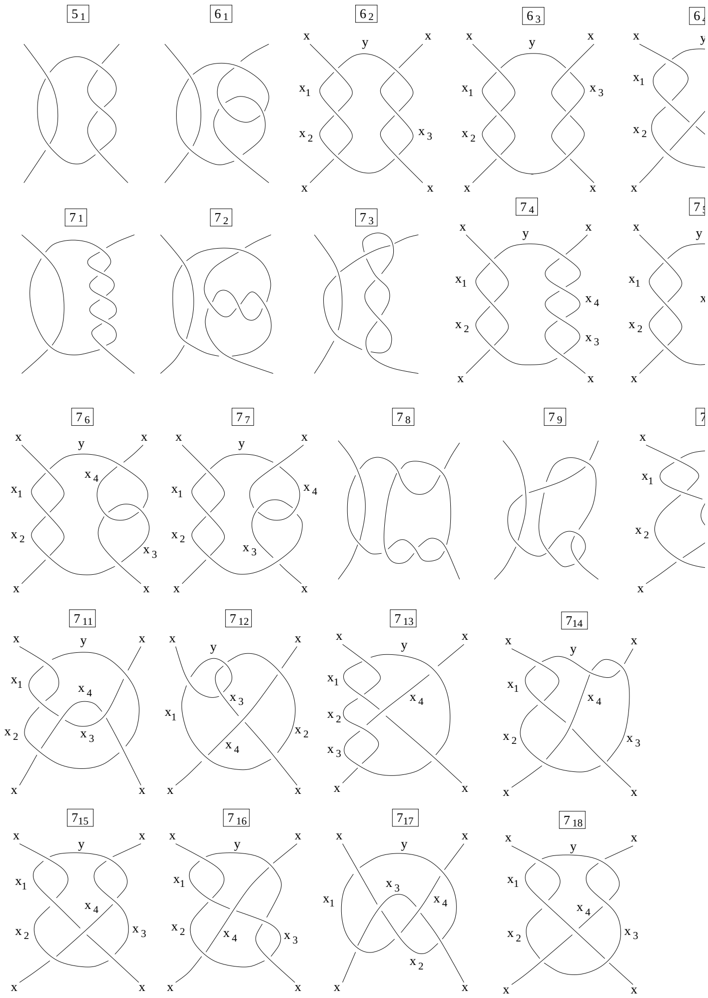

Figure 4.12: Table of tangles

In this section we investigate tangle embedding problem for tangles and knots in their tables. Knot theorists have tried to build a table since early days in history of knot theory. In recent years, the size of knots listed in the table has grown significantly, to millions [23]. A table of tangles was given for the first time in [26] only recently, which is depicted in Fig. 4.12. For each tangle in the table, we examine whether it is non-trivially colored by an Alexander quandle for which we know a polynomial cocycle, and determine the invariant value. Then we look up the table of invariant values posted at [41], and apply Theorem 4.2.2.

In the following we specify orientations of tangles when we consider the embedding problem. On the other hand, for the knots in the table, we use Livingston's [29] table with the braid presentations given there, and in particular, the orientations are specified by the downward orientation of such braids. Also, the knot table does not list distinct mirror images. Unless specifically mentioned, we tested whether a given tangle in the table (not its mirror) is embedded in a knot in the knot table of Livingston (and not their mirrors). It is still of interest to check all such possibilities of orientations and mirror images.

Proposition 4.2.12 *The tangle* 6_2 *with the orientation of the NW arc inward and the SW arc outward does not embed in the following knots up to* 9 *crossings in the table except for* 8_{18}, 9_{29}, 9_{38}.

Proof. With this orientation, the tangle is of the form of two copies of the mirror of the trefoil, and is colored non-trivially by the quandle $Z_p[t]/(t^2 - t + 1)$.

The invariant of the tangle is determined by Proposition 4.2.9, but here we exhibit a method to determine the invariant from the table in [41]. For $p = 2$, the table of quandle cocycle invariants in [41] gives $16 + 48u^t$ as the invariant for trefoil with the 3-cocycle $\phi(x, y, z) = (x - y)^2(y - z)$. This implies that any non-trivial coloring contributes t to the invariant. Its mirror has the same property. (Note that this case p=2 gives the same values of the invariant for mirror images.) With two copies, any non-trivial coloring of the tangle contributes $2t = 0$ when $p = 2$. Hence the invariant value of the tangle is 64. From the table this does not embed in knots up to 9 crossings except for the following possibilities : 8_5, 8_{10}, $8_{15}, 8_{18}, 8_{19}, 8_{20}, 8_{21}, 9_{16}, 9_{22}, 9_{24}, 9_{25}, 9_{28}, 9_{29}, 9_{30}, 9_{36}, 9_{38}, 9_{39}, 9_{40}, 9_{41}, 9_{42}$,
$9_{43}, 9_{44}$, $9_{45}, 9_{49}$.

For $p = 3$, the invariant table gives $243+486u^{(2t+2)}$ as the invariant for trefoil. This implies that 486 non-trivial colorings contributes $2t + 2$ to the invariant. Its mirror contributes $t + 1$. With two copies, 486 non-trivial colorings of the tangle contributes $2t + 2$. Hence the invariant value of the tangle is $243 + 486u^{(2t+2)}$. From the table this does not embed in knots up to 9 crossings except for: $3_1, 8_{18}, 9_2, 9_4, 9_{29}, 9_{34}, 9_{38}$.

For $p = 5$, the table gives

$$625 + 3750u^{(t+3)} + 3750u^{(4t+2)} + 3750u^{(3t+4)} + 3750u^{(2t+1)}$$

as the invariant for trefoil. As in the previous cases, the tangle has the invariant value

$$625 + 3750u^{(3t+4)} + 3750u^{(2t+1)} + 3750u^{(4t+2)} + 3750u^{(t+3)}$$

(for example, for the contribution $t+3$ of trefoil, the mirror contributes $4t+2$, its double contributes $3t+4$). This is the same as trefoil. From the table this does not embed in knots up to 9 crossings except for: $3_1, 8_3, 8_5, 8_{11}, 8_{15}, 8_{18}, 8_{19}, 8_{21}, 9_1, 9_5, 9_6, 9_{16}, 9_{19}, 9_{23}, 9_{28}, 9_{29}, 9_{38}, 9_{40}$. (by the symmetry of the invariant values).

For $p = 7$, the trefoil has 117649 as the invariant value, and so does the tangle. From the table this does not embed in knots up to 9 crossings except for: $3_1, 8_5, 8_{10}, 8_{11}, 8_{15}, 8_{18}, 8_{19}, 8_{20}, 8_{21}, 9_1, 9_6, 9_{16}, 9_{23}, 9_{28}, 9_{29}, 9_{38}, 9_{40}$.

All combined, this tangle does not embed in knots up to 9 crossings except for the only possibilities of $8_{18}, 9_{29}, 9_{38}$. $\square$

Remark 4.2.13 It is expected that these three knots contain the tangle, but we have not been able to confirm this guess.

Remark 4.2.14 For the same tangle 6_2 with the orientation of the both NW and SW arcs inward, the tangle is non-trivially colored by the dihedral quandle R_3 (and it is essentially this is the only Alexander quandle we could use). It is known that the invariant with the dihedral quandles does not depend on the orientation of knots, [38].) The 3-cocycle invariant for trefoil is $9+18u$ from the table. Its mirror has contribution 2 (u^2, multiplicatively) from any non-trivial coloring. The tangle contributes double of it, 1 and (u), so that its invariant is $9 + 18u$. From the table of invariants for dihedral quandles, this does not embed in knots up to 9 crossings except for: $3_1, 7_4, 7_7, 8_{18}, 9_{10}, 9_{29}, 9_{35}, 9_{37}, 9_{38}, 9_{46}, 9_{48}$, and mirrors of: $8_5, 8_{15}, 8_{18}, 8_{19}, 8_{21}, 9_2, 9_4, 9_{15}, 9_{28}, 9_{34}, 9_{37}, 9_{40}, 9_{46}, 9_{47}$. In this case the invariant was not able to exclude many.

Proposition 4.2.15 *The knots in the table up to 9 crossings in which the tangle 6_3 embeds are exactly 8_{10}, 8_{20}, 9_{24}. Here, the orientation of the tangle is such that the end point NW is oriented inward and the SW end point is oriented outward.*

Proof. The tangle 6_3 is written as the addition $R(3) + R(-3)$. Hence it is colored non-trivially by $Z_p[t]/(t^2 - t + 1)$ for any $p \in \mathbb{Z}$ (we use only primes), as well as the dihedral quandle R_3. For the quandle $Z_p[t]/(t^2 - t + 1)$ we used the 3-cocycle $f(x, y, z) = (x - y)(y - z)^p$. The colors of the source region for these two copies of the trefoil diagrams ($R(3)$ and $R(-3)$) coincide. The signs of the crossings are opposite. Hence the invariant is trivial, $(p^2)^3$ copies of 0, for $Z_p[t]/(t^2 - t + 1)$. Computer calculations are available at http://shell.cas.usf.edu/quandle. For $p = 5$, in particular, from the calculations of the invariant posted at the above web site, the Theorem implies that this tangle may embed, among knots in the table up to 9 crossings, only in: 8_{10}, 8_{12}, 8_{18}, 8_{20}, 9_{24}. Using calculations with R_3, the tangle does not embed in 8_{12} and 8_{18}. Therefore the tangle may embed only in 8_{10}, 8_{20}, and 9_{24}.

On the other hand, it is seen that

$$\begin{aligned} (8_{10}) &= N(T(6_3) + R(2,1))^*, \\ (8_{20}) &= N(T(6_3) + R(2))^*, \\ (9_{24}) &= N(T(6_3) + R(2,2))^*, \end{aligned}$$

where K^* denotes the mirror image of a knot K, and R denotes the rational tangles (we use the convention in [32]). Note that this tangle 6_3 is equivalent to its mirror. Therefore we have shown that the tangle 6_3 does indeed embed in these three knots. □

In general, for tangles for which we cannot use our formulas directly, we have to compute the set of colorings by Alexander quandles. For this purpose we assign symbols to arcs of a given tangle, and compute the coloring conditions at each crossing to determine which Alexander quandles color the tangle non-trivially. To illustrate our calculations, we exhibit examples that show how we computed the colorings, for the tangle 7_{17} for two different orientations. To simplify calculations, we assume that the quandle does not have zero divisors.

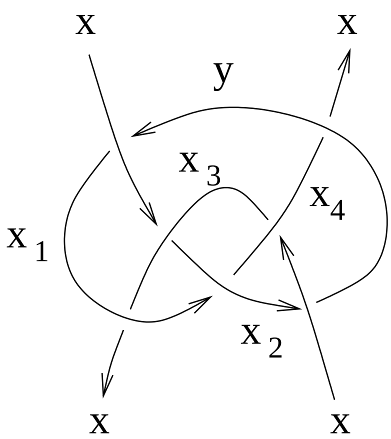

Figure 4.13: Tangle 7_{17}, NW in, SW out

- Tangle 7_{17}, with orientation NW in, SW out.

Let x be the color of the boundary arcs, and let y, x_1, x_2, x_3, x_4 be the colors of the arcs as depicted in Figure 4.13. We have $x_1 = t^{-1}y + (1-t^{-1})x = x + t^{-1}(y-x)$, $x_2 = ty+(1-t)x = x+t(y-x)$, $x_3 = tx+(1-t)x_1 = tx+(1-t)x+(t^{-1}-1)(y-x) = x+(t^{-1}-1)(y-x)$, $x_4 = tx+(1-t)y = y+t(x-y)$ from the top four crossings, with the following conditions that comes from the bottom three crossings

$$x * x_3 = x_2,\ x_1 * x_2 = x_4,\ x * x_4 = x_3.$$

Then it follows that

$$\begin{cases} x_2 = tx + (1-t)x_3 \\ x_4 = x_1 + (1-t)x_2 \\ x_3 = tx + (1-t)x_4 \end{cases}$$

which gives

$$\begin{cases} tx + (1-t)(x + (t^{-1}-1)(y-x)) = x + t(y-x) \\ tx + (y-x) + (1-t)(x + t(y-x)) = y + t(x-y) \\ tx + (1-t)(y + t(x-y)) = x + (t^{-1}-1)(y-x)\ . \end{cases}$$

Grouping together the term containing $(y - x)$, we obtain

$$\begin{cases} x + (1-t)(t^{-1}-1)(y-x) = x + t(y-x) \\ x - y + (y-x)(1+t(1-t)) = t(x-y) \\ (1-t)(y-x) + x + t(1-t)(x-y) = (t^{-1}-1)(y-x) + x \ . \end{cases}$$

We assumed that there is non-zero divisor, so that taking $x \neq y$, we obtain the following conditions.

$$\begin{cases} (1-t)(t^{-1}-1) = t \\ t(1-t) = -t \\ (1-t) - t(1-t) = 1 - t^{-1} \ , \end{cases}$$

which simplifies to

$$\begin{cases} (1-t)^2 - t^2 = 0 \\ t(2-t) = 0 \\ (1-t)(t(1-t)+1) = 0 \ . \end{cases}$$

The second equation of the above system gives $t = 2$, so that we consider the quandle of the form $\mathbb{Z}_p[t, t^{-1}]/(t-2)$. Then for the other two equations to be satisfied, we need to have $p = 3$, and conversely, $X = \mathbb{Z}_p[t, t^{-1}]/(t-2) = R_3$ colors the tangle non-trivially.

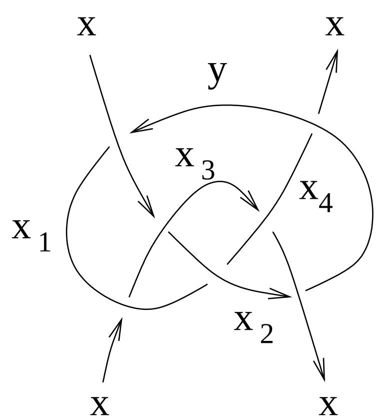

Figure 4.14: Tangle 7_{17}, NW in, SW in

- Tangle 7_{17}, with orientation NW in, SW In.

With the same notations as for the previous case and as depicted in Figure 4.14, we compute the colors using the top four crossings as follows.

$$\begin{cases} x_1 = x + t^{-1}(y - x) \\ x_2 = x + (t^{-1}(y - x)) \\ x_3 = x + (t^{-1} - 1)(y - x) \\ x_4 = x + t^{-1}(y - x)\ . \end{cases}$$

We have the following conditions from the bottom three crossings: $x = x_2 * x_3$, $x_3 = x * x_4$, and $x_4 = x * y$, which gives

$$\begin{cases} tx_2 + (1 - t)x_3 = x \\ tx + (1 - t)x_4 = x_3 \\ tx + (1 - t)y = x_4\ . \end{cases}$$

The last equality is plugged in to the second one to reduce it to

$$\begin{cases} tx + (y - x) + (1 - t)x + (1 - t)(t^{-1} - 1)(y - x) = x \\ tx + (1 - t)(tx + (1 - t)y) = x + (t^{-1} - 1)(y - x)\ , \end{cases}$$

which simplifies to

$$\begin{cases} (y - x)(1 + (1 - t)(t^{-1} - 1)) = 0 \\ (1 - t)^2(y - x) = (t^{-1} - 1)(y - x) \end{cases}$$

and we obtain $t + (1 - t)^2 = 0$ and $(1 - t)^2 = (t^{-1} - 1)$. Both equations reduce to $t^2 - t + 1 = 0$. Therefore the tangle is colored nontrivially by the Alexander quandle $X = \mathbb{Z}_p[t, t^{-1}]/(t^2 - t + 1)$. We conclude our demonstration of calculations of colorings of tangles by noting that the results above are different for the same tangle with different orientations. Thus we had to carefully inspect each possible orientations.

In a similar method we worked out all the remaining tangles in the tangle table in Figure 4.12. In the table below we list the Alexander quandles that colored some tangles in the tangle table nontrivially, and which tangles they color non-trivially.

Quandle	Tangle colored
$\mathbb{Z}_p[t, t^{-1}]/(t^2 - t + 1)$	6_2, 6_3, 7_{17}(NW In, SW In)
$\mathbb{Z}_2[t, t^{-1}]/(t^2 + t + 1)$	6_2, 6_3, 7_4(NW In, NE In), 7_5(NW In, NE In) 7_6(NW In, NE In), 7_7(NW In, NE In), 7_{17}(NW In, SW In)
R_3	6_2, 6_3, 7_{16}(NW In, NE Out), 7_{17}
R_5	7_{13}(NW In, NE Out), 7_{18}
R_7	7_{15}(NW In, SW In), 7_{15}(NW In, SW Out)

For the rest of the section, we give some results on other tangles in the table that have been obtained by cocycle invariants, to show how much information the cocycle invariants provide for other tangles.

To specify orientations, we use the notational convention, for example, "NW In, SE Out," to indicate that the northwest arc is oriented inward, and the southeast arc is oriented outward.

• **Tangle** 7_6

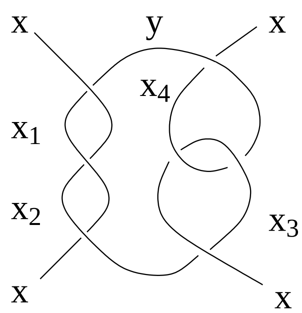

Figure 4.15: Tangle 7_6

(1) For Orientation (NW In, NE In):

The left half of the tangle is trefoil, so that we take the quandle $Z_p[t]/(t^2 - t + 1)$. Then the color x_2 in Fig. 4.12 must be y. Then the right half of the tangle can be closed to form the figure-eight knot with this coloring. Since the Alexander polynomial of the figure-eight is $t^2 - 3t + 1$, we take $p = 2$ to color the tangle non-trivially.

We use the 3-cocycle invariant with the 3-cocycle $f(x, y, z) = (x - y)(y - z)^2$. The knots 3_1 and 4_1 have the invariant $16 + 48U^t$. This implies that any non-trivial coloring gives non-trivial contribution for both 3_1 and 4_1. Hence any non-trivial coloring of this tangle contributes $u^{(t+t)} = 1$, so the invariant is 64. If this embeds in a knot, the knot must have at least 64 as the constant term. From the table of invariants, this tangle does not embed in knots in the table excluding: $8_5, 8_{10}, 8_{15}, 8_{18} - 8_{21}, 9_{16}, 9_{22}, 9_{24}, 9_{25}, 9_{28} - 9_{30}, 9_{36}, 9_{38}, 9_{39}, 9_{41} - 9_{45}, 9_{49}$.

(2) For Orientation (NW In, NE Out):

For this orientation the tangle 7_6 is colored trivially by Alexander quandles, thus we are unable to use the cocycle invariant.

- **Tangle** 7_7

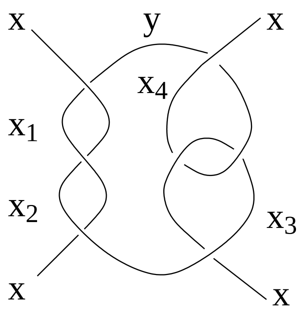

Figure 4.16: Tangle 7_7

(1) For Orientation (NW In, NE In):

By the same argument as for 7_6, we have to use the same quandle, $Z_p[t]/(t^2 - t + 1)$. Thus we obtain the same conclusions as 7_6.

(2) For Orientation (NW In, NE Out):

We noticed that for this orientation, the numerator of the tangle 7_7 is the unknot. Thus this tangle embeds in the unknot, and we are not able to use cocycle invariant since the tangle admits only trivial colorings.

- **Tangle** 7_{13}

(1) For Orientation (NW In, NE In):

The tangle has only the trivial colorings by any Alexander quandle, so that we are not able to use the quandle cocycle invariants with Alexander quandles. We also noticed that $(4_1) = N(T(7_{13}) + R(-1))$, so the tangle embeds in 4_1.

(2) For Orientation (NW In, NE Out):

The tangle is colored non-trivially by R_5. The tangle has the invariant value $25 + 50u^2 + 50u^3$. This is proved as follows. The invariant of the knot 7_4 is as above. Since we see that $N(T(7_{13})) = 7_4$, and the number of colorings are the same for the tangle $T(7_{13})$ and 7_4, the invariant value of $T(7_{13})$ must be the same as above. From the table for R_5, This tangle does not embed in knots in the table (up to 9 crossings) excluding: $4_1, 7_4, 8_{16}, 9_{24}, 9_{37}, 9_{39}, 9_{40}, 9_{49}$ and their mirrors. Those we do not know yet whether it embeds are: $4_1, 9_{24}, 9_{37}, 9_{40}$, (and possibly mirrors). We also noticed that $(8_{16}) = N(T(7_{13}) + R(1))$, $(9_{39}) = N(T(7_{13}) + R(1,1))$, $(9_{49}) = N(T(7_{13}) + R(-1,-1))$.

- **Tangle** 7_{15}

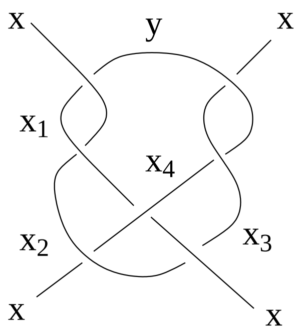

Figure 4.17: Tangle 7_{15}

(1) For Orientation (NW In, SW In):

For this orientation, we noticed that $(5_2) = N(T(7_{15}) + R(-1))$, and by computing the colorings, we found that the tangle is colored non-trivially by R_7. The tangle has the invariant value $49 + 98u^3 + 98u^5 + 98u^6$. This is proved as follows. The tangle can be non-trivially colored by R_7, and hence so does $(5_2) = N(T(7_{15}) + R(-1))$. By [41], the invariant for 5_2 is $49 + 98u^3 + 98u^5 + 98u^6$. The number of colorings is

343. The tangle has non-trivial colorings, hence there are at least $7^3 = 343$ colorings. Hence every coloring of 5_2 comes from a coloring of the tangle, so that they share the same invariant value. From the table for R_7, this tangle does not embed in knots in the table (up to 9 crossings) excluding: $5_2, 8_{16}, 9_{41}, 9_{42}$. Besides we noticed that $(5_2) = N(T(7_{15}) + R(-1))$. So the tangle does embed in the knot 5_2.

(2) For Orientation (NW In, SW Out):

The tangle is colored non-trivially by R_7. The tangle has the invariant value $49 + 98u + 98u^2 + 98u^4$. This is proved as follows. The tangle can be non-trivially colored by R_7, and hence so does its denominator (a knot K). Since K is a reduced alternating diagram, the crossing number of K is 7. By [41], there are only two 7 crossing knots that are colored non-trivially by R_7: 7_1 and 7_7. Both have the same (above) invariant value. Hence the tangle has this invariant value. (The knot K is presumably 7_7.)

From the table for R_7, This tangle does not embed in knots in the table (up to 9 crossings) excluding: $7_1, 7_7, 8_5, 9_4, 9_{12}, 9_{41}$.

- **Tangle** 7_{18}

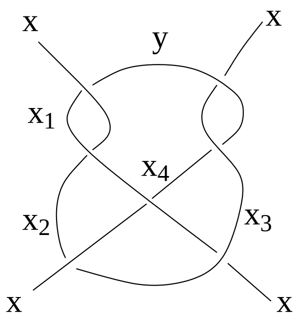

Figure 4.18: Tangle 7_{18}

(1) For Orientation (NW In, SW Out):

The tangle is colored non-trivially by R_5. The tangle has the invariant value $25 + 50u + 50u^4$. This is proved as follows. The invariant of the knot 5_1 is as above. Since we see that $D(T(7_{18})) = 5_1$, and the number of colorings are the same for the tangle $T(7_{18})$ and 5_1, the invariant value of $T(7_{18})$ must be the same as above. From the table for R_5, This tangle does not embed in knots in the table (up to 9 crossings) excluding: $5_1, 8_{18}, 8_{21}, 9_2, 9_{12}, 9_{23}, 9_{31}, 9_{40}, 9_{49}$, and their mirrors.

(2) For Orientation (NW In, SW In):

The tangle is colored non-trivially by R_5. The invariant does not depend on the choice of orientation for the dihedral quandle with the Mochizuki cocycle [37]. Hence the invariant is the same as the case above and $25 + 50u + 50u^4$. Then the same conclusion works for this case, this tangle does not embed in knots in the table (up to 9 crossings) excluding: $5_1, 8_{18}, 8_{21}, 9_2, 9_{12}, 9_{23}, 9_{31}, 9_{40}, 9_{49}$, and their mirrors.

4.2.5 Embedding Disjoint Tangles

In this section we discuss the embedding of a disjoint union of tangles in knots. We will prove a theorem that will be used as obstruction to embedding disjoint union of tangles and we give some examples. Let ∂T denote the boundary points of a given tangle T. Let $\Phi(T,x) = \sum_{C \in Col_x(T)} \prod_{\tau} B(C,\tau)$ where $Col_x(T) = \{C \in Col_X(T) \mid C(\partial T) = x\}$ for $x \in X$. Then $\Phi_\phi(T) = \sum_{x \in X} \Phi(T,x)$. For elements $a = \sum_{i=1}^{\ell} a_i u_i$, and $b = \sum_{i=1}^{\ell} b_i u_i \in \mathbb{Z}[A]$, denote by $a \subset_m b$ if $a_i \leq b_i$ for all $i = 1, \ldots, \ell$.

Theorem 4.2.16 *Let $T_1, \ldots, T_k$ be a disjoint union of tangles such that for all $i = 1, \ldots, k$, the condition $\Phi(T_i, x) = \Phi(T_i, y)$ holds for all $x, y \in X$. Then we have*

$$\Phi_\phi(T_1 \sqcup \cdots \sqcup T_k) = \frac{1}{|X|^{k-1}} \Phi_\phi(T_1) \times \ldots \times \Phi_\phi(T_k).$$

Furthermore if a disjoint union of $T_1, \ldots, T_k$ embed in a link L, then

$$\frac{1}{|X|^{k-1}} \Phi_\phi(T_1) \times \Phi_\phi(T_2) \times \cdots \times \Phi_\phi(T_k) \subset_m \Phi_\phi(L).$$

Proof. We compute

$$\Phi_\phi(T_1 \sqcup \cdots \sqcup T_k) = \sum_{x_j \in X} \prod_{i=1}^{k} \Phi(T_i, x_j) = |X| \prod_{i=1}^{k} \Phi(T_i, x)$$

for any fixed $x \in X$ since $\Phi(T_i, x) = \Phi(T_i, y)$ for all $x, y \in X$. The condition also implies that $\Phi(T_i, x) = \dfrac{1}{|X|}\Phi_\phi(T_i)$ for all $i = 1, \ldots, k$. Hence

$$\Phi_\phi(T_1 \sqcup \cdots \sqcup T_k) = \frac{1}{|X|^{k-1}}\Phi_\phi(T_1) \times \cdots \times \Phi_\phi(T_k).$$

Thus by theorem 4.2.2, if $T_1 \sqcup \cdots \sqcup T_k$ embeds in a link L, we have $\frac{1}{|X|^{k-1}}\Phi_\phi(T_1) \times \Phi_\phi(T_2) \times \cdots \times \Phi_\phi(T_k) \subset_m \Phi_\phi(L)$. □

Example 4.2.17 **(a)** For the dihedral quandle R_3, $\Phi_\phi(6_2) = 9(1 + 2U)$, then by theorem 4.2.16 $\Phi_\phi(6_2 \sqcup 6_2) = \frac{1}{3}81(1 + 4U + 4U^2) = 27 + 108U + 108U^2$. Using [41] we compare this invariant to the cocycle invariant of the knots in the knot table up to 12 crossings, we find that $6_2 \sqcup 6_2$ does not embedd in any knot in the knot table.

More specifically, from the invariant value, the number of colorings of $6_2 \sqcup 6_2$ is 243, and the knot in the table up to 12 crossings with this many number of 3-colorings are: 12_a0750, 12_n0553, to 12_n0556, 12_n0642, Hence the number of colorings alone can exclude all but only this many knots, but fails to exclude these, and the cocycle invariant is able to exclude them all. The situation is similar for the other examples below.

(b) In a similar way we can find that $\Phi_\phi(6_2 \sqcup 6_3) = \frac{1}{3}3^3 9(1 + 2U) = 81 + 162U$ with R_3, and $6_2 \sqcup 6_2$ does not embedd in any knot on the knot table up to 12 crossings.

(c) $\Phi_\phi(6_3 \sqcup 6_3) = \frac{1}{3}3^3 3^3 = 243$ with R_3, and $6_3 \sqcup 6_3$ does not embedd in any knot in the knot table up to 12 crossings.

(d) From the table we can consider the tangle $7_{16} \sqcup 7_{17}$ with orientation (NW In, NE out) on 7_{16} and orientation (NW In, SW In) on 7_{17} both colored by R_3. In this case the invariant of 7_{16} is $9 + 18U$ and the invariant of 7_{17} is $9 + 18U^2$. Thus the invariant of $7_{16} \sqcup 7_{17}$ is $27(5 + 2U + 2U^2)$. Then the tangle $7_{16} \sqcup 7_{17}$ does not embed in any knot in the knot table up to 12 crossings.

(e) For the quandle R_5 and tangles 7_{13} and 7_{18} both have invariant $25(1+2U+U^3)$, then $7_{13} \cup 7_{18}$ has invariant $125(5+U+4U^2+2U^3+4U^4)$. Thus it does not embed in any knot in the knot table.

Concluding Remarks

In this dissertation we constructed a family of n-cocycles using polynomial expressions for some Alexander quandles. Then we used the polynomial 2- and 3-cocycles to compute the quandle cocycle invariant for $(2, n)$-torus knots, twist knots and their twist spins. For the case of $(2, n)$-torus knots, formulas involving the derivative of the Alexander polynomial were obtained.

For the case of twist spun knots, the quandle cocycle invariants are used to detect non-invertibility. Our polynomial 3-cocycle provided a non-invertible twist-spun torus knot that has not been detected by dihedral quandles.

The 2 and 3-cocyles we constructed were also used to solve the tangle embedding problems. The tables of prime tangles up to 7 crossings and the table of knots up to 9 crossings are compared, and the cocycle invariants are used as obstructions to find knots that do not contain the tangles in the table.

The polynomial cocycles enable us to compute quandle cocycle invariants for a large class of Alexander quandles, and we expect further developments in calculations and applications. In particular, these are the only explicitly known higher dimensional cocycles, so that it is expected to be useful in the study of higher dimensional quandle cohomology groups and applications.

```
> restart;
>
> for ii from 2 to 2 do: #Picks the ii-th prime > 2.
> pp:= ithprime(ii) : #This is the ii-th prime > 2.
>
> for nn from 1 to pp-1 do # Loop for n (twists).
>
> printf("%s %s %a", n, is, nn);
>
> for y from 0 to 1 do
> for g from 0 to 1 do
>
> A1:=pp^y;
> A2:=pp^g;
>
> h(t):=nn*(1-2*t+t^2)-t mod pp;
>
> negpow:=proc(l) # This computes negative powers of t.
> local a,c;
> a:=tcoeff(h(t));
> c:=expand((a^(-1)mod pp)*(t^(-1)*(-h(t)+a*t^ldegree(h(t)))))^l;
> return(rem(c,h(t),t)mod pp);
> end proc:
>
> Fhi:=rem(
> -nn*negpow(A1)+(1+nn*(1-t))^(A1+A2), h(t),t) mod pp;
>
```

```
> print(A1, A2, Fhi);
>
> od; od;
>
> od; od;

                              pp := 3
n is 1
                              1, 1, 0
                            1, 3, t + 2
                           3, 1, 2 t + 2
                              3, 3, 0
n is 2
                              1, 1, 0
                            1, 3, 1 + t
                           3, 1, 2 t + 2
                              3, 3, 0
```

```
>restart;
>for ii from 1 to 10 do: #Picks ii-th prime.
>pp:=ithprime(ii) : #This is ii-th prime.
>for nn from 1 to pp-1 do # Loop for n (twists).
>
>h(t):=nn*(1-2*t+t^2)-t mod pp;
>negpow:=proc(l) # This computes negative powers of t.
>local a,c;
>a:=tcoeff(h(t));
>c:=expand((a^(-1)mod pp)*(t^(-1)*(-h(t)+a*t^ldegree(h(t)))))^l;
>return(rem(c,h(t),t)mod pp);
>end proc:
>
># This makes the list of elements of an Alexander quandle.
># polym is a polynomial to divide (like h(t)), mod m.
>rres:=proc(polym,m)
>local L,i,num,deg,C,prep,j;
>L:=[];
>if type(polym,polynom) then  deg:=degree(polym);
>else  printf("%s?ERROR); return;
>fi;
>#change from here
>if isprime(m)=true then continue;
>else printf (%s %s %s", "not", finished, yet);return;
>#need to solve for t^-1 so that we get a finite set of
>#equivalence classes. We will need to check
>#that certain coefficients are units in Z_m so
```

```
 that we can solve for t^-1.
>fi; #to here
>
>num:=m^deg;
>for i from 0 to num-1 do
> C:=convert(i,base,m);
> if C=[] then prep:=0;
> else
> prep:=0;
> for j from nops(C) to 1 by -1 do
 > prep:=prep+C[j]*t^(j-1);
> od;
>fi;
>L:=[op(L),prep];
>od;
>return(L):
>end:
>LX:=rres(h(t),pp);
>
> #Loop for different cocycles.
>for y from 0 to 1 do
>for g from 0 to 1 do
>A1:=pp^y;
>A2:=pp^g;
>f:=(x1-x2)^(A1)*(x2-x3)^(A2);
>printf("For n=%d, p= %d, f(x1,x2,x3)=(x1-x2)^(%d)(x2-x3)^(%d),
> The invariant value is:\n",nn,pp,A1,A2);
>
> #The following is the constant term that appear
in the cocycle invariant:
```

```
>Fhi:=rem( -nn*negpow(A1)+(1+nn*(1-t))^(A1+A2), h(t),t) mod pp;
>
>#The following computes the state-sum.
>SST:=0:
>
>for ss from 1 to nops(LX) do
>SSTContri:=rem(expand(Fhi*(LX[ss]^( A1 + A2 ) )), h(t), t) mod pp:
>
>SST := SST + U^SSTContri :
>od:
>print(nops(LX)^2*SST);
>
>od; od; od; od;

                              pp := 2

For n=1, p= 2, f(x1,x2,x3)=(x1-x2)^(1)(x2-x3)^(1),
 The invariant
value is:  64 For n=1, p= 2, f(x1,x2,x3)=(x1-x2)^(1)(x2-x3)^(2),
The invariant value is:  16 + 48 U^t

For n=1, p= 2, f(x1,x2,x3)=(x1-x2)^(2)(x2-x3)^(1), The invariant
value: 16 + 48 U^{(t+1)}

For n=1, p= 2, f(x1,x2,x3)=(x1-x2)^(2)(x2-x3)^(2), The invariant
value is: 64
```

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