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# On the theory of records and applications

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# On The Theory of Records and Applications

by

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## Dedication

To Doris, with all my love.

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On The Theory of Records and Applications

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**ABSTRACT**

The subject of the present study is to introduce the concept of "records" and "generalized order statistics" as applied to probability distribution functions. In addition to developing the analytical framework of the subject matter we shall illustrate its usefulness to various real world problems by performing statistical data analysis.

Let $X_1, X_2, ..., X_n$ be a sequence of independent and identically distributed random variables with cumulative distribution function $F(x)$. Denote $X_{L(n)} = min\{Y_1, X_2, \ldots, X_n\}$, $n = 2, 3, \ldots$. $X_{L(j)}$ is a lower **record** of $\{X_n\}$ if and only if $X_{L(j)} < X_{L(j-1)}$, $j = 2, 3, \ldots$ and $X_{L(1)} = X_1$. An analogous definition of **records** deals with upper **record** values. By definition, $X_1$ is an upper as well as lower **record** value.

We establish the analytic formulation of **records** as applied to the **Gumbel** probability distribution/**double exponential** probability distribution, inverse Weibull probability distribution, **half logistics** probability distribution and **power function** probability distribution. We shall illustrate the usefulness of the analytical results of the **Gumbel** probability distribution function in two real world applications, namely, the **Olympic records** of women's 100 meter free style swimming results from 1912 to 2004 and the mean concentration of sulphur dioxide from the city of Long Beach, Californian. Our analysis is compared with the initial published results of the subjects applications.

We will study the **lower generalized order statistics** (lgos) which was introduced by Kamp (1995) as a unified approach of **order statistics**, **sequential order statistics** and **lower record** values. We shall discuss some distributional properties of lgos, the problem of estimation of the parameters of some selected probability distribution function is also studied, and the results obtained are shown to include as special cases order statistics and lower **record** values. Coefficients of the best linear unbiased estimators of the location and scale parameters are given. We finally illustrate the applicability of our results by considering some simulation studies.

# 1 Introduction

In this thesis, we consider the analysis of **record** breaking data sets, where only observations that exceed, or only those that fall below, the current extreme value are recorded. Example of application areas include industrial stress testing, meteorological analysis, sporting and athletic events, and oil and mining surveys. A closely related area is that of threshold modeling, where the observations are those that cross a certain threshold value. Most of these events are well documented in the yearly "Guinness Book of World Records". If there exist information that is of interest to a lot of people, obviously, statisticians have to get involved as well. The interest in this research is not only in estimation but also in predicting future records. For example, will Australians be the number one bear drinkers in the world? What is the hottest year on record? The year that has the record increase in carbon dioxide release in the atmosphere, among others.

**Records** are very important when observations are difficult to obtain or when observations are being destroyed when subjected to an experimental test.

The easiest way to explain how to statistically define the theory of **records** is by examples.

**Example**

Suppose we have the following ten observations from a given experiment

$$10, 12, 6, 15, 20, 18, 17, 5, 22, 3$$

The lower **Record** values are

$$10, 6, 5, 3.$$

The upper record values are

$$10, 12, 15, 20, 22.$$

**An Example In Reliability**

Consider a sequence of product testing that may fail under stress. We are interested to determine the minimum failure stress of the products sequentially. For example let $X_1$ be the strength that a product fails (we stress $X_1$, then we record its failure stress) and in general, we record stress $X_m$ of the $m$th product failure, if $X_m < min(X_1, X_2, \ldots, X_{m-1}), m > 1$. The recorded failure stresses are the lower record values.

**An Example Of Multivariate Data**

Let $X_{ik}$ be the water level of a river at the $k$th day on the $i$th location. If one is interested to study at each location the local maximum values of $X_{ik}$, then the local maxima are the upper record values.

Chandler, [17] was first to introduce the concept of **record** values, **record** times and inter **record** times for analyzing the breaking strength data of certain material. He proved the result that for any given probability distribution function of a random variable, the expected value of the inter **record** time is infinite. Feller [19] gave some examples of **record** values with respect to gambling problems.

We proceed to finally introduce some basic definitions that play a central role in the present study.

**Definition 1.0.1** *Suppose that $X_1, X_2, \ldots, X_n$ are a sequence of independent and identically distributed random variables with cumulative probability distribution function $F(x)$. Let $X_n = min\{X_1, X_2, \ldots, X_n\}$ for $n \geq 1$. We say, $X_j$ is a lower record value of $\{X_n, n \geq 1\}$, if $X_j < X_{j-1}, j > 1$. An analogous definition exist for upper record values. By definition, $X_1$ is a lower as well as upper record value.*

The indices at which the lower record values occur are given by the record times $\{L(r), r > 0\}$, where $L(r) = min\{j|j > L(r-1), X_j < X_{L(r-1)}, r > 1\}$ and $L(1) = 1$.

The joint probability distribution function, pdf $f(x_1, x_2, \ldots, x_n)$ of $r$ lower record values $X_{L(1)}, X_{L(2)}, \ldots, X_{L(r)}$ from a continuous cumulative probability distribution function, cpdf

$F(x)$, is given by

$$f_{1,2,\ldots,r}(x_1, x_2, \ldots, x_r) = f(x_r) \prod_{i=1}^{r-1} \frac{f(x_i)}{F(x_i)} \tag{1.0.1}$$

for $-\infty < x_1 < x_2 < \ldots < x_{r-1} < x_r < \infty$.

The probability density function of $X_{L(r)}$ is given by

$$f_r(x) = \frac{1}{\Gamma(r)}(-\ln(F(x)))^{r-1} f(x), \quad -\infty < x < \infty, \tag{1.0.2}$$

and the cumulative probability distribution function of $X_{L(r)}$ is

$$\begin{aligned} F_r(x) &= \frac{1}{\Gamma(r)} \int_{-\infty}^{x} (-\ln(F(x)))^{r-1} f(x) dx \\ &= 1 - \Gamma_{-\ln(F(x))}(r). \end{aligned} \tag{1.0.3}$$

The joint probability density function of two lower record values $X_{L(r)}$ and $X_{L(s)}$ is given by

$$f(x_r, x_s) = \frac{1}{\Gamma(r)\Gamma(s-r)}(-\ln(F(x_r)))^{r-1}[\ln(F(x_r)) - \ln(F(x_s))]^{s-r-1} \frac{f(x_r)}{F(x_r)} f(x_s), \tag{1.0.4}$$

where $-\infty < x_s < x_r < \infty$.

## 1.1 Organization Of The Study

Mbah and Tsokos, [38], recently obtained the best linear unbiased estimates (BLUE) of the location and scale parameters of the power function distribution using lower **record** values. In Chapter 2, we discuss some important results of the power function distribution using **record** breaking data. Coefficients of the best linear unbiased estimators (BLUE) for location and scale parameters of the **Power Function Distribution** are also computed. Finally, the usefulness of our result is illustrated using a simulation study.

Keller and Kamath, [30], introduced the Inverse Weibull probability distribution as a suitable model to describe the degradation phenomena of mechanical components such as the dynamic components (pistons, crankshaft, etc.) of diesel engines. In Chapter 3, we discuss some important results of the Inverse Weibull probability distribution using **record** breaking data. Coefficients of the best linear unbiased estimators (BLUE) for location and scale parameters of this distribution are computed. We illustrate the usefulness of our result by using a simulation study.

In Chapter 4, we introduce the concept of "**records**" as applied to a probability distribution function that characterize real world phenomenon. **Records** are obtained by observing successive minimum or maximum values. The problem of parametric inference for record-breaking data was introduced by Samaniego and Whitaker, [47]. They studied the properties of maximum likelihood estimates of the mean of an underlying exponential probability distribution. Gulati and Padgett, [26], extended the work of Samaniego and Whitaker, [47], to the Weibul probability distribution. The Gumbel probability distribution plays a major role in analyzing and modeling the behavior of random phenomenon that occur in engineering, business, biology, finance, sports (Mbah and Tsokos, [38]), economics among others, additional references, see Luo and Zhu [34], Coles [18], Gumbel [25], Hosking et al. [27], Kotz [31]. We shall establish the analytic formulation of **records** as applied to the **Gumbel probability distribution/double exponential probability distribution**. In addition, we shall illustrate the usefulness of the subject results in two real world applications, namely, the Olympic record of women's 100 meter free style swimming results from 1912 to 2004 and the one hour mean concentration of sulphur dioxide from the city of Long Beach, Californian. Our analysis is compared with the initial modeling of these two applications with significant improvement.

In Chapter 5, we introduce the half logistics probability distribution and studied the maximum likelihood estimates of the location and scale parameters using **record** breaking data. The usefulness of the result is illustrated using the failure times of air conditioning equipment in a Boeing 720 airplane.

Kamps [29] introduced and gave detailed theory of the so-called generalized order statistics (gos) as a unified approach to order statistics, **record** values, and sequential **order** statistics. In Chapter 6, we introduce the concept of lower generalized order statistics (lgos). Burkschat et al. [13], created a connection between gos and lgos. Ahsanullah [3] presented several distributional properties of lgos. The problem of estimating the parameters of the power function probability distribution based on lower generalized order statistics. We also studied the coefficients of the best linear unbiased estimators (BLUE) for the location and scale parameters of the Power Function Probability Distribution. We also study some distributional properties of the generalized exponential distribution using lower generalized order statistics. The problem of estimation of the parameters is also discussed. Finally, we study the power function probability distribution using lower generalized order statistics.

# 2 Power Function Probability Distribution: Records and Application

## 2.1 Introduction

In the present study we investigate the problem of estimating the inherent parameters of the Power function probability distribution using record breaking data. Some characterizations of the power function distribution are also developed. We also presented the coefficients of the best linear unbiased estimators (BLUE) for location and scale parameters of the Power Function Distribution. Finally, the usefulness of our result is illustrated using a simulation study.

## 2.2 Properties of the Power Function Probability Distribution

Let $X$ be a complete random variable (rv) from the power function probability distribution with pdf given by

$$\begin{aligned} f(x) &= \delta((x-\mu)/\sigma)^{\delta-1}, \mu \leq x \leq \mu+\sigma, \sigma > 0, \mu \geq 0 \\ &= 0, \quad \text{otherwise,} \end{aligned} \tag{2.2.1}$$

where $\delta$ is the shape parameter, $\mu$ is the location parameter and $\sigma$ is the scale parameter.

Using equation (1.0.1) and letting $\mu = 0$ and $\sigma = 1$, the first moment of $X_{L(r)}$ from the power function probability distribution is given by

$$\begin{aligned} E(X_{L(r)}) &= \frac{\delta^r}{\Gamma(r)} \int_0^1 x^\delta (-\ln(x))^{r-1} dx \\ &= \left(\frac{\delta}{\delta+1}\right)^r = b_r. \end{aligned} \tag{2.2.2}$$

Similarly, the second moment is given by

$$\begin{aligned} E(X_{L(r)})^2 &= \frac{\delta^r}{\Gamma(r)} \int_0^1 x^{\delta+1}(-\ln(x))^{r-1}dx \\ &= \left(\frac{\delta}{\delta+2}\right)^r. \end{aligned} \tag{2.2.3}$$

Using expressions (2.2.2) and (2.2.3), we compute the variance of $X_{L(r)}$ to be

$$\begin{aligned} Var(X_{L(r)}) &= \left(\frac{\delta}{\delta+2}\right)^r - \left(\frac{\delta}{\delta+1}\right)^{2r} \\ &= \left(\frac{\delta}{\delta+1}\right)^r \left[\left(\frac{\delta+1}{\delta+2}\right)^r - \left(\frac{\delta}{\delta+1}\right)^r\right] \\ &= a_r b_r, \end{aligned} \tag{2.2.4}$$

where

$$a_r = \left(\frac{\delta+1}{\delta+2}\right)^r - \left(\frac{\delta}{\delta+1}\right)^r.$$

Using equation (1.0.4), for $s > r$, and $x_s < x_r$ we have

$$\begin{aligned} E(X_{L(r)}, X_{L(s)}) &= \frac{\delta^s}{\Gamma(r)\Gamma(s-r)} \int_0^1 \int_0^{x_r} x_s^{\delta}(-\ln(x_r))^{r-1} \\ &\times \ (\ln(x_r) - \ln(x_s))^{s-r-1} dx_s ds_r \\ &= \frac{\delta^s}{(\delta+1)^{s-r}(\delta+2)^r}. \end{aligned}$$

The covariance of $X_{L(r)}$ and $X_{L(s)}$ is given by

$$\begin{aligned} Cov(X_{L(r)}, X_{L(s)}) &= E(X_{L(r)}X_{L(s)}) - E(X_{L(r)})E(X_{L(s)}) \\ &= \left(\frac{\delta}{\delta+1}\right)^s \left[\left(\frac{\delta+1}{\delta+2}\right)^r - \left(\frac{\delta}{\delta+1}\right)^r\right] \\ &= b_s a_r. \end{aligned} \tag{2.2.5}$$

We shall proceed to introduce some interesting and useful properties of the subject pdf. We begin with the following characterization of the subject pdf, Mbah and Tsokos, [38].

**Theorem 2.2.1** *Let $X$ be a bounded non-negative absolute continuous (with respect to the Lebesque Measure) random variable. We assume Without loss of generality $F(0) = 0$ and $F(1) = 1$, then the following two statements are equivalent*

1. *$X$ has a power function distribution with $F(x) = x^{\delta}$, $0 < x < 1$, $\delta > 0$*

*2. for some $r$ and $s$, $1 \leq r < s \leq n$, $X_{L(s)}/X_{L(r)}$ and $X_{L(r)}$ are independent.*

*Proof.* (1) implies (2).

Using the transformation $U = X_{L(r)}$, $V = X_{L(s)}/X_{L(r)}$ and $F(x) = x^{\delta}$, we obtain from (1.0.4), the joint pdf of $U$ and $V$, for $m > -1$, that is,

$$\begin{aligned} f_{r,s}(u,v) &= \frac{1}{\Gamma(r)\Gamma(s-r)}[-\ln(F(u))]^{r-1} \\ &\times [\ln(F(u)) - \ln(F(uv))]^{s-r-1} \\ &\times \frac{uf(u)f(uv)}{F(u)} \\ &= \frac{1}{\Gamma(r)\Gamma(s-r)}(-\delta\ln(u))^{r-1} \\ &\times (-\delta\ln(v))^{s-r-1}\delta^2(uv)^{\delta-1}. \end{aligned}$$

Hence, if $X$ has cdf given by $F(x) = x^{\delta}$, then $U$ and $V$ are independent.

To prove (2) implies (1), using (1.0.4) we have that the joint pdf of $U$ and $V$,

$$\begin{aligned} f_{r,s}(u,v) &= \frac{1}{\Gamma(r)\Gamma(s-r)}[-\ln(F(u))]^{r-1} \\ &\times [\ln(F(u)) - \ln(F(uv))]^{s-r-1} \\ &\times \frac{uf(u)f(uv)}{F(u)}. \end{aligned} \tag{2.2.6}$$

Using (1.0.2) and (1.0.4), the conditional pdf of $V|U = u$, is given by

$$\begin{aligned} f_{V|U}(v|U = u) &= \frac{f(x_r, x_s)}{f(x_r)} \\ &= \frac{1}{\Gamma(s-r)}\left[-\ln\left(\frac{F(uv)}{F(u)}\right)\right]^{s-r-1}\frac{uf(uv)}{F(u)} \end{aligned} \tag{2.2.7}$$

for $0 < u \leq 1, 0 \leq v \leq 1$. Integrating (2.2.7) with respect to $v$ from $v_0$ to 1 we obtain

$$F_{V|U}(v_0|U = u) = \Gamma_{-\ln\left(\frac{F(uv_0)}{F(u)}\right)}(s-r). \tag{2.2.8}$$

Since $U$ and $V$ are independent, we have

$$\Gamma_{-\ln\left(\frac{F(uv_0)}{F(u)}\right)}(s-r) = G(v_0),$$

where $G(v_0)$ is a function of $v_0$ only.

Note that as $u \to 1$, we have

$$\Gamma_{-\ln\left(\frac{F(uv_0)}{F(u)}\right)}(s-r) = \Gamma_{-\ln F(v_0)}(s-r),$$

for all $u$, $0 < u \leq 1$ and almost all $v_0$, $0 \leq v_0 \leq 1$. Thus, we conclude that

$$F(uv_0) = F(u)F(v_0), \tag{2.2.9}$$

for all $u$, $0 < u \leq 1$, and almost all $v_0$, $0 \leq v_0 \leq 1$.
The non-zero solution of (2.2.9) with the condition that $F(x)$ is a probability distribution function with $F(0) = 0$ and $F(1) = 1$, is given by

$$F(x) = x^{\delta}, 0 \leq x \leq 1, \delta > 0. \tag{2.2.10}$$

■

The following theorem, Mbah and Tsokos, [38] gives further characterization of the power function probability distribution.

**Theorem 2.2.2** *Let $X$ be a bounded non-negative absolute continuous (with respect to the Lebesque Measure) random variable. We assume without loss of generality, that $F(0) = 0$ and $F(1) = 1$, then the following two statements are equivalent*

1. *$X$ has a power function distribution with $F(x) = x^{\delta}$, $0 < x < 1$, $\delta > 0$*

2. *for some $r$ and $s$, $1 \leq r < s \leq n$, $X_{L(s)}/X_{L(r)}$ and $X_{L(s-r)}$ are identically distributed and $F$ belong to the class of all continuous functions ($\mathcal{C}$).*

*Proof.* (1) implies (2).

Observe that

$$f(u, \nu) = f_{V/U=u}(\nu/u)f(u),$$

and if $F(x) = x^{\delta}$, $0 < x < 1$, $\delta > 0$, then we have

$$F(\nu) = \int_0^1 F_{V/U=u}(\nu/u)f(u)du$$

$$
\begin{aligned}
&= \int_0^1 \left[1 - \Gamma_{-\ln\left(\frac{F(u\nu)}{F(u)}\right)}(s-r)\right] f(u)du \\
&= 1 - \Gamma_{-\ln\left(\frac{F(u\nu)}{F(u)}\right)}(s-r). \qquad (2.2.11)
\end{aligned}
$$

We observe from (1.0.3) that the cdf of $X_{L(s-r)}$ is

$$
1 - \Gamma_{-\ln(F(x))}(s-r).
$$

Therefore, $X_{L(s)}/X_{L(r)}$ and $X_{L(s-r)}$ are identically distributed.

(2) implies (1).

If $V$ and $X_{L(s-r)}$ are identically distributed, then using equations (1.0.3) and (2.2.11), we have

$$
\begin{aligned}
&F_V(\nu) - F_{X_{L(s-r)}}(\nu) \\
&= \int_0^1 \left(\left[1 - \Gamma_{-\ln\left(\frac{F(u\nu)}{F(u)}\right)}(s-r)\right] - \left[1 - \Gamma_{-\ln\left(X_{L(s-r)}\right)}(s-r)\right]\right) f(\nu)d\nu \\
&= \int_0^1 \left(\Gamma_{-\ln\left(X_{L(s-r)}\right)}(s-r) - \Gamma_{-\ln\left(\frac{F(u\nu)}{F(u)}\right)}(s-r)\right) f(\nu)d\nu = 0 \qquad (2.2.12)
\end{aligned}
$$

Since $F$ belongs to class $C$, we obtain from the above expression, (2.2.12)

$$
F(uv) = F(u)F(v) \qquad (2.2.13)
$$

for all $u$, $0 < u \leq 1$, and almost all $v$, $0 \leq v \leq 1$. The non-zero solution of equation (2.2.13) with the conditions that F(x) is a probability distribution function with $F(0) = 0$, and $F(1) = 1$, is

$$
F(x) = x^{\delta}, 0 \leq x \leq 1, \delta > 0
$$

■

The following theorem, Mbah and Tsokos, [38], identifies the analytical estimator of the parameters for $\mu$ and $\sigma$ when $\delta$ is known.

**Theorem 2.2.3** *Let $x_1, x_2, \ldots, x_r$ be $r$ lower record values from the power function distribution given by equation (2.2.1). Then the best linear unbias estimates (BLUE), $\hat{\mu}$, $\hat{\sigma}$ for $\mu$*

*and $\sigma$ for known $\delta$ are respectively*

$$\begin{aligned}\hat{\mu} &= -\frac{1}{D_0}\Big[\frac{(\delta+2)^2 x_1}{\delta} + \left(\frac{\delta+2}{\delta}\right)^2 x_2 + \ldots \\ &+ \left(\frac{\delta+2}{\delta}\right)^{r-1} x_{r-1} - (\delta+1)\left(\frac{\delta+2}{\delta}\right)^r x_r\Big],\end{aligned}$$

*and*

$$\begin{aligned}\hat{\sigma} &= \frac{\delta+1}{\delta D_0}\Big[\left(D_0 + \frac{(\delta+2)^2}{\delta}\right) x_1 + \left(\frac{\delta+2}{\delta}\right)^2 x_2 + \ldots \\ &+ \left(\frac{\delta+2}{\delta}\right)^{r-1} x_{r-1} - (\delta+1)\left(\frac{\delta+2}{\delta}\right)^r x_r\Big],\end{aligned}$$

*where*

$$D_0 = \sum_{i=2}^{r}\left(\frac{\delta+2}{\delta}\right)^i.$$

*Proof.* Let

$$\mathbf{h}' = x_1, \ldots, x_r,$$

then

$$E(\mathbf{h}') = \mu\mathbf{1} + \sigma^2\alpha,$$

and

$$Var(\mathbf{h}') = \sigma^2\mathbf{V},$$

where, from equations (2.2.2) and (2.2.5),

$$\mathbf{1}' = (1, 1, 1, \ldots, 1),$$

$$\alpha' = (b_1, b_2, b_3, \ldots b_r),$$

and

$$\mathbf{V} = (v_{ij}),$$

$$v_{ij} = a_i b_j, 1 \le i, j \le r.$$

Let

$$\mathbf{V}^{-1} = (V^{ij}), 1 \le i < j \le r,$$

then the entries of $\mathbf{V}^{-1}$ are given by

$$\begin{aligned} V^{ii} &= \frac{a_{i+1}b_{i-1} - a_{i-1}b_{i+1}}{(a_i b_{i-1} - a_{i-1}b_i)(a_{i+1}b_i - a_i b_{i+1})} \\ &= (2\delta^2 + 4\delta + 1)\left(\frac{\delta+2}{\delta}\right)^i, i = 1, \ldots, r-1, \end{aligned}$$

$$\begin{aligned} V^{ij} &= V^{ji} \\ &= \frac{-1}{a_{i+1}b_i - a_i b_{i+1}} \\ &= -(\delta+1)\frac{(\delta+2)^{i+1}}{\delta^i}, j = i+1, i = 1, \ldots, r-1, \end{aligned}$$

$$V^{ij} = 0 \quad \text{for} \quad |i - j| > 1,$$

and

$$\begin{aligned} V^{rr} &= \frac{b_{r-1}}{b_r(a_r b_{r-1} - a_{r-1}b_r)} \\ &= (\delta\gamma_r + 1)^2 \frac{(\delta+2)^r}{\delta^r}. \end{aligned}$$

Using the method introduced by Lloyd, [33], we have that

$$\alpha'\mathbf{V}^{-1} = [(\delta+1)(\delta+2), 0, 0, \ldots, 0, 0],$$

and

$$\begin{aligned} \mathbf{1}'\mathbf{V}^{-1} &= \Big[ -(2\delta+3)\frac{\delta+2}{\delta}, -\left(\frac{\delta+2}{\delta}\right)^2, \ldots, \\ &- \left(\frac{\delta+2}{\delta}\right)^{r-1}, (\delta+1)\left(\frac{\delta+2}{\delta}\right)^r \delta^{r-2} \Big]. \end{aligned}$$

Therefore,

$$\alpha'\mathbf{V}^{-1}\alpha = \delta(\delta+2),$$

$$\alpha'\mathbf{V}^{-1}\mathbf{1} = (\delta+1)(\delta+2),$$

and

$$\mathbf{1}'\mathbf{V}^{-1}\mathbf{1} = \frac{(\delta+1)^2(\delta+2)}{\delta} + D_0.$$

Let

$$\begin{aligned}\Delta &= (\alpha'\mathbf{V}^{-1}\alpha)(\mathbf{1}'\mathbf{V}^{-1}\mathbf{1}) - (\alpha'\mathbf{V}^{-1}\mathbf{1})^2 \\ &= \delta(\delta+2)D_0,\end{aligned}$$

then

$$\begin{aligned}\alpha'\mathbf{V}^{-1}(\alpha\mathbf{1}' - \mathbf{1}\alpha')\mathbf{V}^{-1} &= \delta(\delta+2)^3\Big[-(\delta+2), -\frac{\delta+2}{\delta}, \dots, \\ &- \left(\frac{\delta+2}{\delta}\right)^{r-2}, (\delta+1)\left(\frac{\delta+2}{\delta}\right)^{r-1}\Big],\end{aligned}$$

and

$$\begin{aligned}\mathbf{1}'\mathbf{V}^{-1}(\mathbf{1}\alpha' - \alpha\mathbf{1}')\mathbf{V}^{-1} &= (\delta+1)(\delta+2)\Big[D_0 + \frac{(\delta+2)^2}{\delta}, \frac{(\delta+2)^2}{\delta^2}, \dots, \\ &\frac{(\delta+2)^{r-1}}{\delta^{r-1}}, -\frac{(\delta+1)(\delta+2)^r}{\delta^r}\Big].\end{aligned}$$

Thus, the BLUE, $\hat{\mu}$ and $\hat{\sigma}$ for $\mu$ and $\sigma$ based on $r$ lower record values from the power function probability distribution are given by

$$\hat{\mu} = \frac{\alpha' V^{-1}(\alpha\mathbf{1}' - \mathbf{1}\alpha')\mathbf{V}^{-1}\mathbf{h}}{\Delta},$$

and

$$\hat{\sigma} = \frac{\mathbf{1}'\mathbf{V}^{-1}(\mathbf{1}\alpha' - \alpha\mathbf{1}')\mathbf{V}^{-1}\mathbf{h}}{\Delta}.$$

The variance of $\hat{\mu}, \hat{\sigma}$ and covariance of $\hat{\mu}, \hat{\sigma}$ are given by

$$Var(\hat{\mu}) = \frac{\alpha\mathbf{V}^{-1}\alpha}{\Delta}\sigma^2 = \frac{\sigma^2}{D_0},$$

$$\begin{aligned}Var(\hat{\sigma}) &= \frac{\mathbf{1}'\mathbf{V}^{-1}\mathbf{1}}{\Delta}\sigma^2 \\ &= \left(\frac{(\delta+1)^2}{\delta^2 D_0} + \frac{1}{\delta(\delta+2)}\right)\sigma^2,\end{aligned}$$

and

$$\begin{aligned} Cov(\hat{\mu}, \hat{\sigma}) &= -\frac{\alpha' \mathbf{V}^{-1} \mathbf{1}}{\Delta} \sigma^2 \\ &= -\frac{(\delta+1)\sigma^2}{\delta D_0}. \end{aligned}$$

Coefficients of the BLUES for $\mu$, $\sigma$ and the variance covariance for $\mu$ and $\sigma$ are given in Tables 2.1, 2.2, and 2.3, respectively.

■

**Remark 2.2.4** . *If $\delta = 1$, then $D_0 = 9(3^{r-1} - 1)/2$, and BLUE $\hat{\mu}$, $\hat{\sigma}$ of $\mu$ and $\sigma$ from the uniform distribution based on lower record values $X_{L(1)}, X_{L(2)}, \cdots, X_{L(r)}$ are*

$$\hat{\mu} = \frac{1}{D_0} \left[ -3^2 X_{L(1)} + \sum_{i=2}^{r-1} 3^i X_{L(i)} + 2 \times 3^r X_{L(r)} \right],$$

*and*

$$\hat{\sigma} = 2(X_{L(1)} - \hat{\mu}),$$

*with*

$$Var(\hat{\mu}) = \frac{\sigma^2}{D_0}$$

$$Var(\hat{\sigma}) = \frac{3^r + 5}{9(3^{r-1} - 1)},$$

*and*

$$Cov(\hat{\mu}, \hat{\sigma}) = -\frac{4\sigma^2}{9(3^{r-2} - 1)}.$$

Given below, Mbah and Tsokos, [38], are further characterization of the estimates of the parameter inherent in the subject pdf

**Theorem 2.2.5** *The best linear invariant (in terms of minimum mean squared error and invariance with respect to the location parameter $\mu$) estimators (BLIE) $\tilde{\mu}, \tilde{\sigma}$ of $\mu$ and $\sigma$ are*

$$\tilde{\mu} = \hat{\mu} - \hat{\sigma} \left( \frac{E_{12}}{1 + E_{22}} \right),$$

*and*

$$\tilde{\sigma} = \frac{\hat{\sigma}}{1 + E_{22}},$$

*where $\hat{\mu}$ and $\hat{\sigma}$ are BLUE of $\mu$ and $\sigma$, and*

$$\begin{pmatrix} Var(\hat{\mu}) & Cov(\hat{\mu}, \hat{\sigma}) \\ Cov(\hat{\mu}, \hat{\sigma}) & Var(\hat{\sigma}) \end{pmatrix} = \sigma^2 \begin{pmatrix} E_{11} & E_{12} \\ E_{12} & E_{22} \end{pmatrix}.$$

*The mean square errors of these estimators are*

$$MSE(\tilde{\mu}) = \sigma^2 \left[ E_{11} - \frac{E_{12}^2}{1 + E_{22}} \right],$$

*and*

$$MSE(\tilde{\sigma}) = \sigma^2 \left[ \frac{E_{22}}{1 + E_{22}} \right].$$

*Proof.* From Theorem 2.2.3, we have

$$E_{11} = \frac{1}{D_0},$$

$$E_{22} = \frac{(\delta+1)^2}{(\delta)^2 D_0} + \frac{1}{\delta(\delta+2)}.$$

and

$$E_{12} = -\frac{\delta+1}{\delta D_0}.$$

Thus,

$$\tilde{\mu} = \hat{\mu} + \frac{\delta(\delta+1)(\delta+2)\hat{\sigma}}{\delta^2(\delta+2)D_0 + (\delta+1)^2(\delta+2) + \delta D_0},$$

and

$$\tilde{\sigma} = \frac{\delta^2(\delta+2)D_0\hat{\sigma}}{\delta^2(\delta+2)D_0 + (\delta+1)^2(\delta+2) + \delta D_0},$$

with

$$MSE(\tilde{\mu}) = \frac{\delta(\delta+1)\sigma^2}{(\delta+2)D_0 + (\delta+1)^2(\delta+2) + \delta D_0},$$

and

$$MSE(\tilde{\sigma}) = \frac{((\delta+1)^2(\delta+2) + \delta D_0)\sigma^2}{\delta^2(\delta+2)D_0 + (\delta+1)^2(\delta+2) + \delta D_0}.$$

∎

**Remark 2.2.6** . *If we let $\delta = 1$, then from Theorem 2.2.5, the BLIE $\tilde{\mu}$, $\tilde{\sigma}$, of $\mu$ and $\sigma$ from* *n lower record values $X_{L(1)}, X_{L(2)}, \ldots, X_{L(r)}$ from the uniform distribution are,*

$$\tilde{\mu} = \hat{\mu} + \frac{\hat{\sigma}}{3^r - 1},$$

*and*

$$\tilde{\sigma} = \frac{9(3^{r-1} - 1)\hat{\sigma}}{4(3^r - 1)},$$

*where $\hat{\mu}$ and $\hat{\sigma}$ are as in Theorem 2.2.3. The corresponding mean square errors are*

$$MSE(\tilde{\mu}) = \frac{8\sigma^2}{3(3^r - 1)},$$

*and*

$$MSE(\tilde{\sigma}) = \frac{(5 + 3^r)\sigma^2}{4(3^r - 1)}.$$

The following theorem, Mbah and Tsokos, [38], identifies the best linear least square estimates of the parameters $\mu$ and $\sigma$.

**Theorem 2.2.7** *The best linear least squares prediction (BLLSP), $\check{X}_{L(s+r)}$ of $X_{L(s+r)}$ based on $X_{L(1)} = x_1, X_{L(2)} = x_2, \ldots, X_{L(r)} = x_r$ is given by*

$$\check{X}_{L(s+r)} = \left(\frac{\delta}{\delta + 1}\right)^{s-1} x_r.$$

*Proof.* We can write

$$\begin{aligned}
\check{X}_{L(s+r)} &= E\Big(X_{L(s+r)} | X_{L(1)} = x_1, X_{L(2)} = x2, \ldots, X_{L(r)} = x_r\Big) \\
&= E\Big(X_{L(s+r)} | X_{L(r)} = x_r\Big) \\
&= E\Big(X_{L(r)} \prod_{j=r+1}^{s+r} W_j \Big| X_{L(r)} = x_r\Big) \\
&= x_r \prod_{j=r+1}^{s+r} E(W_j) \\
&= \left(\frac{\delta}{\delta + 1}\right)^s x_r.
\end{aligned}$$

∎

We proceed to develop the estimates of $\delta$ when $\mu$ and $\sigma$ are known

First we shall consider the estimation of $\delta$ when $\mu = 0$ and $\sigma = 1$. We shall use the method of moment and maximum likelihood to obtain the subject estimate.

**Method of moment**

Observe from (2.2.2) that

$$E(X_{L(r)}) = \left(\frac{\delta}{\delta+1}\right)^r,$$

and

$$E(X_{L(r+1)}) = \left(\frac{\delta}{\delta+1}\right)^{r+1}.$$

Therefore, a moment estimate $\hat{\delta}_{ME}$ of $\delta$ based on two consecutive lower record values $x_r$ and $x_{r+1}$ is given by

$$\hat{\delta}_{ME} = \frac{x_r}{x_r - x_{r+1}}.$$

**Method of Maximum likelihood**

Observe from equation (1.0.1) that the likelihood of a standard power function distribution is

$$L(\delta|x_1, x_2, \ldots, x_n) = x_r^{\delta} \prod_{i=1}^{r} \frac{\delta}{x_i}.$$

The loglikelihood function is given by

$$\log(L(\delta|x_1, x_2, \ldots, x_n)) = -\sum_{i=1}^{r} x_i + rlog(\delta) + \delta \log(x_r). \tag{2.2.14}$$

Differentiating (2.2.14) with respect to $\delta$ and equating to zero gives the MLE estimate of $\delta$, $\hat{\delta}_{MLE}$ to be

$$\hat{\delta}_{MLE} = -\frac{r}{\ln(x_r)}.$$

## 2.3 Simulation Study

We proceed with a simulation study to illustrate the performance of the estimators obtained in the previous section. We simulated a small random sample of size, $n = 15$ from the power function probability distribution with $\delta = 1$, $\mu = 5$ and $\sigma = 3$. The simulated values are

$$6.13, 6.34, 6.71, 6.59, 5.98, 6.18, 6.50, 5.20, 5.06, 5.03, 6.81, 6.46, 6.07, 5.93, 6.84.$$

Five Record values can be obtained from the above random samples, that is,

$$6.13, 5.98, 5.20, 5.06, 5.03.$$

Using the method of moments, we have that $\hat{\delta_{ME}} = 2.33$ while the the method of maximum likelihood gives $\hat{\delta_{MLE}} = 1.20$.

Observe that the method of maximum likelihood performs better than the method of moments estimate.

Using Table 2.1 for $\delta = 1$, $n = 5$, and $r = 1, \ldots, 5$ we obtain the BLUE for $\mu$:

$$\begin{aligned} \hat{\mu} &= -6.13 * 0.025 - 5.98 * 0.025 - 5.20 * 0.075 - 5.06 * 0.225 + 5.03 * 1.35 \\ &= 4.962, \end{aligned}$$

a very good estimate given the true value being 5.

Using Table 2.2 given below for $\delta = 1$, $n = 5$, and $r = 1, \ldots, 5$ we have that the BLUE for $\sigma$ is

$$\begin{aligned} \hat{\sigma} &= 6.13 * 2.05 + 5.98 * 0.05 + 5.20 * 0.15 + 5.06 * 0.45 - 5.03 * 2.7 \\ &= 2.335, \end{aligned}$$

also a very good estimate of the true value.

The standard error, S.E. are obtained from Table 2.3 to be

$$S.E.(\hat{\mu}) = \sqrt{0.00278\sigma^2} = 2.335\sqrt{0.00278} = 0.12,$$

$$S.E.(\hat{\sigma}) = \sqrt{0.34444\sigma^2} = 2.335\sqrt{0.34444} = 1.37.$$

The BLIE for $\mu$ and $\sigma$ are given by

$$\hat{\mu} = 4.966,$$

and

$$\tilde{\sigma} = 1.737,$$

with

$$S.E.(\tilde{\mu}) = \sqrt{0.0028\sigma^2} = 0.08,$$

$$S.E.(\tilde{\sigma}) = \sqrt{0.257\sigma^2} = 0.77.$$

The simulation results show that the BLUE and BLIE estimates $\hat{\mu}$ and $\hat{\sigma}$ for $\mu$ and $\sigma$ are close to the actual values. The S.E. of the BLIE are smaller than those of the BLUE.

## 2.4 Conclusion

In the present study, we have introduced the concepts of **records** for a given phenomenon that is probabilistically characterized by the power function probability distribution. We have developed some distributional properties of lower record values and have obtained some properties that are important to this distribution. We have developed the estimates of the location and scale parameters of this distribution given that the shape parameter is known and also the estimates of the shape parameter has been obtained given that the location and scale parameters are known. In addition, we have developed methods to predict future observations given the present. Coefficients of the best linear unbiased estimates have been obtained. In addition we have shown the importance of our results using simulation study.

Table 2.1: Coefficients for the BLUE of $\mu$

| n | r | $\delta = 0.5$ | $\delta = 1$ | $\delta = 2$ | $\delta = 2.5$ | $\delta = 3$ | $\delta = 3.5$ | $\delta = 4$ | $\delta = 5$ |
|---|---|---|---|---|---|---|---|---|---|
| 2 | 1 | -0.50000 | -1.00000 | -2.00000 | -2.50000 | -3.00000 | -3.50000 | -4.00000 | -5.00000 |
| 2 | 2 | 1.50000 | 2.00000 | 3.00000 | 3.50000 | 4.00000 | 4.50000 | 5.00000 | 6.00000 |
| 4 | 1 | -0.01613 | -0.07692 | -0.28571 | -0.41391 | -0.55102 | -0.69433 | -0.84211 | -1.14679 |
| 4 | 2 | -0.03226 | -0.07692 | -0.14286 | -0.16556 | -0.18367 | -0.19838 | -0.21053 | -0.22936 |
| 4 | 3 | -0.16129 | -0.23077 | -0.28571 | -0.29801 | -0.30612 | -0.31174 | -0.31579 | -0.3211 |
| 4 | 4 | 1.20968 | 1.38462 | 1.71429 | 1.87748 | 2.04082 | 2.20445 | 2.36842 | 2.69725 |
| 5 | 1 | -0.00321 | -0.02500 | -0.13333 | 1.69481 | -0.29779 | -0.39232 | -0.49231 | -0.70383 |
| 5 | 2 | -0.00641 | -0.02500 | -0.06667 | 0.11792 | -0.09926 | -0.11209 | -0.12308 | -0.14077 |
| 5 | 3 | -0.03205 | -0.07500 | -0.13333 | 0.21226 | -0.16544 | -0.17614 | -0.18462 | -0.19707 |
| 5 | 4 | -0.16026 | -0.22500 | -0.26667 | 0.38208 | -0.27574 | -0.2768 | -0.27692 | -0.2759 |
| 5 | 5 | 1.20192 | 1.35000 | 1.60000 | -2.40708 | 1.83824 | 1.95735 | 2.07692 | 2.31757 |
| 6 | 1 | -0.00064 | -0.00826 | -0.06452 | -0.11176 | -0.16863 | -0.23304 | -0.30332 | -0.4568 |
| 6 | 2 | -0.00128 | -0.00826 | -0.03226 | -0.0447 | -0.05621 | -0.06658 | -0.07583 | -0.09136 |
| 6 | 3 | -0.00640 | -0.02479 | -0.06452 | -0.08047 | -0.09368 | -0.10463 | -0.11374 | -0.12791 |
| 6 | 4 | -0.03201 | -0.07438 | -0.12903 | -0.14484 | -0.15614 | -0.16442 | -0.17062 | -0.17907 |
| 6 | 5 | -0.16005 | -0.22314 | -0.25806 | -0.26071 | -0.26024 | -0.25837 | -0.25592 | -0.25069 |
| 6 | 6 | 1.20038 | 1.33884 | 1.54839 | 1.64248 | 1.73491 | 1.82703 | 1.91943 | 2.10583 |
| 7 | 1 | -0.00013 | -0.00275 | -0.03175 | -0.06058 | -0.09788 | -0.14227 | -0.19248 | -0.3063 |
| 7 | 2 | -0.00026 | -0.00275 | -0.01587 | -0.02423 | -0.03263 | -0.04065 | -0.04812 | -0.06126 |
| 7 | 3 | -0.00128 | -0.00824 | -0.03175 | -0.04362 | -0.05438 | -0.06388 | -0.07218 | -0.08576 |
| 7 | 4 | -0.00640 | -0.02473 | -0.06349 | -0.07852 | -0.09063 | -0.10038 | -0.10827 | -0.12007 |
| 7 | 5 | -0.03200 | -0.07418 | -0.12698 | -0.14133 | -0.15105 | -0.15773 | -0.16241 | -0.1681 |
| 7 | 6 | -0.16001 | -0.22253 | -0.25397 | -0.25439 | -0.25175 | -0.24787 | -0.24361 | -0.23534 |
| 7 | 7 | 1.20008 | 1.33516 | 1.52381 | 1.60268 | 1.67830 | 1.75276 | 1.82707 | 1.97683 |
| 9 | 1 | -0.00915 | -0.125 | -0.73587 | -1.13927 | -1.57279 | -2.0248 | -2.48874 | -3.43852 |
| 9 | 2 | -0.00114 | 0.00000 | 0.04599 | 0.08544 | 0.13107 | 0.18079 | 0.23332 | 0.34385 |
| 9 | 3 | -0.0018 | 0.00000 | 0.05256 | 0.09521 | 0.14355 | 0.19554 | 0.24999 | 0.36350 |
| 9 | 4 | -0.00299 | 0.00000 | 0.06132 | 0.10790 | 0.15950 | 0.21417 | 0.27082 | 0.38773 |
| 9 | 5 | -0.00539 | 0.00000 | 0.07359 | 0.12517 | 0.18077 | 0.23864 | 0.29790 | 0.41875 |
| 9 | 6 | -0.01078 | 0.00000 | 0.09198 | 0.15020 | 0.21089 | 0.27273 | 0.33514 | 0.46063 |
| 9 | 7 | -0.02515 | 0.00000 | 0.12265 | 0.19026 | 0.25776 | 0.32468 | 0.39099 | 0.52204 |
| 9 | 8 | -0.07546 | 0.00000 | 0.18397 | 0.26636 | 0.34368 | 0.41745 | 0.48874 | 0.62645 |
| 9 | 9 | 1.13186 | 1.125 | 1.10381 | 1.11871 | 1.14559 | 1.18079 | 1.22185 | 1.31555 |
| 10 | 1 | -0.00685 | -0.11111 | -0.70697 | -1.10779 | -1.54012 | -1.99161 | -2.45541 | -3.40533 |
| 10 | 2 | -0.00076 | 0.00000 | 0.03928 | 0.07385 | 0.11408 | 0.15806 | 0.20462 | 0.30270 |
| 10 | 3 | -0.00114 | 0.00000 | 0.04419 | 0.08124 | 0.12359 | 0.16935 | 0.21741 | 0.31783 |
| 10 | 4 | -0.00179 | 0.00000 | 0.05050 | 0.09052 | 0.13536 | 0.18318 | 0.23293 | 0.33599 |
| 10 | 5 | -0.00299 | 0.00000 | 0.05891 | 0.10259 | 0.15040 | 0.20063 | 0.25235 | 0.35839 |
| 10 | 6 | -0.00538 | 0.00000 | 0.07070 | 0.11901 | 0.17045 | 0.22355 | 0.27758 | 0.38706 |
| 110 | 7 | -0.01076 | 0.00000 | 0.08837 | 0.14281 | 0.19886 | 0.25549 | 0.31228 | 0.42577 |
| 10 | 8 | -0.02511 | 0.00000 | 0.11783 | 0.18089 | 0.24305 | 0.30416 | 0.36432 | 0.48254 |
| 10 | 9 | -0.07534 | 0.00000 | 0.17674 | 0.25325 | 0.32407 | 0.39106 | 0.45541 | 0.57905 |
| 10 | 10 | 1.13014 | 1.11111 | 1.06046 | 1.06363 | 1.08024 | 1.10613 | 1.13851 | 1.21600 |

Table 2.2: Coefficients for the BLUE of $\sigma$

| n | r | $\delta = 0.5$ | $\delta = 1$ | $\delta = 2$ | $\delta = 2.5$ | $\delta = 3$ | $\delta = 3.5$ | $\delta = 4$ | $\delta = 5$ |
|---|---|---|---|---|---|---|---|---|---|
| 2 | 1 | 4.50000 | 4.00000 | 4.50000 | 4.90000 | 5.33333 | 5.78571 | 6.25000 | 7.20000 |
| 2 | 2 | -4.50000 | -4.00000 | -4.50000 | -4.90000 | -5.33333 | -5.78571 | -6.25000 | -7.20000 |
| 4 | 1 | 3.04839 | 2.15385 | 1.92857 | 1.97947 | 2.06803 | 2.17843 | 2.30263 | 2.57615 |
| 4 | 2 | 0.09677 | 0.15385 | 0.21429 | 0.23179 | 0.24490 | 0.25506 | 0.26316 | 0.27523 |
| 4 | 3 | 0.48387 | 0.46154 | 0.42857 | 0.41722 | 0.40816 | 0.40081 | 0.39474 | 0.38532 |
| 4 | 4 | -3.62903 | -2.76923 | -2.57143 | -2.62848 | -2.72109 | -2.8343 | -2.96053 | -3.2367 |
| 5 | 1 | 3.00962 | 2.05000 | 1.70000 | 1.69481 | 1.73039 | 1.79013 | 1.86538 | 2.04459 |
| 5 | 2 | 0.01923 | 0.05000 | 0.10000 | 0.11792 | 0.13235 | 0.14412 | 0.15385 | 0.16892 |
| 5 | 3 | 0.09615 | 0.15000 | 0.20000 | 0.21226 | 0.22059 | 0.22647 | 0.23077 | 0.23649 |
| 5 | 4 | 0.48077 | 0.45000 | 0.40000 | 0.38208 | 0.36765 | 0.35588 | 0.34615 | 0.33108 |
| 5 | 5 | -3.60577 | -2.70000 | -2.40000 | -2.40708 | -2.45098 | -2.5166 | -2.59615 | -2.78108 |
| 6 | 1 | 3.00192 | 2.01653 | 1.59677 | 1.55646 | 1.55818 | 1.58533 | 1.62915 | 1.74817 |
| 6 | 2 | 0.00384 | 0.01653 | 0.04839 | 0.06258 | 0.07495 | 0.08560 | 0.09479 | 0.10963 |
| 6 | 3 | 0.01921 | 0.04959 | 0.09677 | 0.11265 | 0.12491 | 0.13452 | 0.14218 | 0.15349 |
| 6 | 4 | 0.09603 | 0.14876 | 0.19355 | 0.20278 | 0.20819 | 0.21139 | 0.21327 | 0.21488 |
| 6 | 5 | 0.48015 | 0.44628 | 0.3871 | 0.36500 | 0.34698 | 0.33219 | 0.31991 | 0.30083 |
| 6 | 6 | -3.60115 | -2.67769 | -2.32258 | -2.29947 | -2.31321 | -2.34904 | -2.39929 | -2.52700 |
| 7 | 1 | 3.00038 | 2.00549 | 1.54762 | 1.48482 | 1.46384 | 1.46863 | 1.4906 | 1.56756 |
| 7 | 2 | 0.00077 | 0.00549 | 0.02381 | 0.03393 | 0.0435 | 0.05226 | 0.06015 | 0.07351 |
| 7 | 3 | 0.00384 | 0.01648 | 0.04762 | 0.06107 | 0.0725 | 0.08213 | 0.09023 | 0.10292 |
| 7 | 4 | 0.01920 | 0.04945 | 0.09524 | 0.10992 | 0.12084 | 0.12905 | 0.13534 | 0.14408 |
| 7 | 5 | 0.09601 | 0.14835 | 0.19048 | 0.19786 | 0.20140 | 0.20280 | 0.20301 | 0.20172 |
| 7 | 6 | 0.48003 | 0.44505 | 0.38095 | 0.35615 | 0.33566 | 0.31868 | 0.30451 | 0.2824 |
| 7 | 7 | -3.60023 | -2.67033 | -2.28571 | -2.24375 | -2.23774 | -2.25355 | -2.28383 | -2.37219 |
| 9 | 1 | 1.2334 | 1.25000 | 1.83231 | 2.23434 | 2.66808 | 3.12082 | 3.58565 | 4.53715 |
| 9 | 2 | 0.0014 | 0.00000 | -0.04855 | -0.08924 | -0.13592 | -0.18652 | -0.23980 | -0.35149 |
| 9 | 3 | 0.0022 | 0.00000 | -0.05548 | -0.09944 | -0.14887 | -0.20175 | -0.25693 | -0.37158 |
| 9 | 4 | 0.00366 | 0.00000 | -0.06473 | -0.11270 | -0.16541 | -0.22097 | -0.27834 | -0.39635 |
| 9 | 5 | 0.00659 | 0.00000 | -0.07768 | -0.13073 | -0.18746 | -0.24622 | -0.30617 | -0.42806 |
| 9 | 6 | 0.01318 | 0.00000 | -0.09709 | -0.15688 | -0.21870 | -0.28139 | -0.34445 | -0.47086 |
| 9 | 7 | 0.03074 | 0.00000 | -0.12946 | -0.19871 | -0.26730 | -0.33499 | -0.40185 | -0.53365 |
| 9 | 8 | 0.09223 | 0.00000 | -0.19419 | -0.27820 | -0.35641 | -0.43070 | -0.50232 | -0.64037 |
| 9 | 9 | -1.38338 | -1.25000 | -1.16513 | -1.16844 | -1.18802 | -1.21827 | -1.25579 | -1.34479 |
| 10 | 1 | 1.20822 | 1.22222 | 1.79232 | 2.1921 | 2.62479 | 3.07709 | 3.54179 | 4.49344 |
| 10 | 2 | 0.00091 | 0.00000 | -0.04124 | -0.07681 | -0.11789 | -0.16258 | -0.20973 | -0.30875 |
| 10 | 3 | 0.00137 | 0.00000 | -0.0464 | -0.08449 | -0.12771 | -0.17419 | -0.22284 | -0.32419 |
| 10 | 4 | 0.00215 | 0.00000 | -0.05302 | -0.09414 | -0.13987 | -0.18841 | -0.23876 | -0.34271 |
| 10 | 5 | 0.00359 | 0.00000 | -0.06186 | -0.1067 | -0.15541 | -0.20636 | -0.25865 | -0.36556 |
| 10 | 6 | 0.00646 | 0.00000 | -0.07423 | -0.12377 | -0.17614 | -0.22994 | -0.28452 | -0.39481 |
| 110 | 7 | 0.01292 | 0.00000 | -0.09279 | -0.14852 | -0.20549 | -0.26279 | -0.32008 | -0.43429 |
| 10 | 8 | 0.03014 | 0.00000 | -0.12372 | -0.18813 | -0.25116 | -0.31285 | -0.37343 | -0.49219 |
| 10 | 9 | 0.09041 | 0.00000 | -0.18558 | -0.26338 | -0.33488 | -0.40223 | -0.46679 | -0.59063 |
| 10 | 10 | -1.35616 | - 1.22222 | -1.11348 | -1.10618 | -1.11625 | -1.13774 | -1.16698 | -1.24032 |

Table 2.3: Variances Covariances of the BLUEs of $\mu$ and $\sigma$ in terms of $\sigma^2$.

| n | $\delta = 0.5$ | $\delta = 1$ | $\delta = 2$ | $\delta = 2.5$ | $\delta = 3$ | $\delta = 3.5$ | $\delta = 4$ | $\delta = 5$ |
|---|---|---|---|---|---|---|---|---|
| 2 | 0.04000 | 0.11111 | 0.25 | 0.30864 | 0.36000 | 0.40496 | 0.44444 | 0.51020 |
| | 1.16000 | 0.77778 | 0.6875 | 0.69383 | 0.70667 | 0.72137 | 0.73611 | 0.76327 |
| | -0.12000 | -0.22222 | -0.375 | -0.43210 | -0.48000 | -0.52066 | -0.55556 | -0.61224 |
| 3 | 0.00667 | 0.02778 | 0.08333 | 0.11023 | 0.13500 | 0.15748 | 0.17778 | 0.21259 |
| | 0.86000 | 0.44444 | 0.3125 | 0.30494 | 0.30667 | 0.31228 | 0.31944 | 0.33469 |
| | -0.02000 | -0.05556 | -0.125 | -0.15432 | -0.18000 | -0.20248 | -0.22222 | -0.2551 |
| 4 | 0.00129 | 0.00855 | 0.03571 | 0.05110 | 0.06612 | 0.08034 | 0.09357 | 0.11702 |
| | 0.81161 | 0.36752 | 0.20536 | 0.18904 | 0.18422 | 0.18475 | 0.18787 | 0.19708 |
| | -0.00387 | -0.01709 | -0.05357 | -0.07154 | -0.08816 | -0.10329 | -0.11696 | -0.14042 |
| 5 | 0.00026 | 0.00278 | 0.01667 | 0.02600 | 0.03574 | 0.04539 | 0.0547 | 0.07182 |
| | 0.80231 | 0.34444 | 0.1625 | 0.13984 | 0.13020 | 0.12698 | 0.12714 | 0.13199 |
| | -0.00077 | -0.00556 | -0.02500 | -0.03640 | -0.04765 | -0.05836 | -0.06838 | -0.08618 |
| 6 | 0.00005 | 0.00092 | 0.00806 | 0.0138 | 0.02024 | 0.02696 | 0.0337 | 0.04661 |
| | 0.80046 | 0.33701 | 0.14315 | 0.11593 | 0.10264 | 0.09652 | 0.09433 | 0.09569 |
| | -0.00015 | -0.00184 | -0.01210 | -0.01932 | -0.02698 | -0.03467 | -0.04213 | -0.05594 |
| 7 | 0.00001 | 0.00031 | 0.00397 | 0.00748 | 0.01175 | 0.01646 | 0.02139 | 0.03126 |
| | 0.80009 | 0.33455 | 0.13393 | 0.10355 | 0.08755 | 0.07916 | 0.07508 | 0.07358 |
| | -0.00003 | -0.00061 | -0.00595 | -0.01047 | -0.01566 | -0.02116 | -0.02673 | -0.03751 |
| 8 | 0.00000 | 0.00010 | 0.00197 | 0.00410 | 0.00691 | 0.01021 | 0.01381 | 0.02139 |
| | 0.80002 | 0.33374 | 0.12943 | 0.09692 | 0.07895 | 0.06883 | 0.06325 | 0.05937 |
| | -0.00001 | -0.00020 | -0.00295 | -0.00574 | -0.00922 | -0.01313 | -0.01727 | -0.02567 |
| 9 | 0.00106 | 0.01023 | 0.03679 | 0.04756 | 0.05632 | 0.06346 | 0.06935 | 0.07839 |
| | 0.03576 | 0.02273 | 0.04377 | 0.05369 | 0.06185 | 0.06850 | 0.07398 | 0.08238 |
| | -0.00129 | -0.01136 | -0.03884 | -0.04967 | -0.05841 | -0.06548 | -0.07127 | -0.08013 |
| 10 | 0.00075 | 0.00842 | 0.03214 | 0.04187 | 0.04979 | 0.05624 | 0.06154 | 0.06967 |
| | 0.02966 | 0.01852 | 0.03770 | 0.04676 | 0.05420 | 0.06027 | 0.06525 | 0.07287 |
| | -0.00090 | -0.00926 | -0.03374 | -0.04354 | -0.05145 | -0.05784 | -0.06308 | -0.07106 |

# 3 Record Values from the Inverse Weibull Probability Distribution

## 3.1 Introduction

The Weibull Probability distribution plays a very significant role in statistical analysis and modeling of real world problems. See, Weibull, [49], Lawless, [32], among others. This probability distribution is often used in the field of life data analysis due to its flexibility. It can mimic the behavior of other statistical distributions such as the normal and the exponential. The failure rate is either increasing, decreasing, or remains constant depending on the value of the shape parameter.

Keller and Kamath, [30], introduced the use of the Inverse Weibull probability distribution as a suitable model to describe the degradation phenomena of mechanical components such as the dynamic components (pistons, crankshaft, etc.) of diesel engines. The Inverse Weibull probability distribution distribution also provides a good fit to several data such as the times to breakdown of an insulating fluid, subject to the action of a constant tension, see Nelson, [41]. The Inverse Weibull probability distribution has initiated a large volume of research. For example, Carriere, [16], has used this distribution to model the mortality curve of a population, Mohamed et al., [40], have considered the single and product moments of order statistics from inverse Weibull probability distribution and doubly truncated inverse Weibull probability distributions, Calabria and Pulcini, [14], have discussed the maximum likelihood and least squares estimations of its parameters, and Calabria and Pulcini, [15], have considered Bayes 2-sample prediction of the distribution.

In this chapter we shall use the theory of **records** to obtain some distributional properties of the Inverse Weibull probability distribution. We shall obtain parameter of this distribution and we shall present coefficients of the BLUEs of the location and scale parameters of the Inverse Weibull Probability Distribution.

## 3.2 Distributional Properties of Inverse Weibull Probability Distribution.

Let $X$ be a complete random variable (rv) from the Inverse Weibull Probability Distribution with pdf given by

$$\begin{aligned} f(y) &= \frac{k}{x-\alpha}\left(\frac{b}{x-\alpha}\right)^k \exp\left\{-\left(\frac{x-\alpha}{b}\right)^{-k}\right\}, x>0, b>0, \alpha \geq 0, k>0 \quad (3.2.1) \\ &= 0, \quad \text{otherwise.} \end{aligned}$$

For the record sample $X_{L(1)}, X_{L(2)}, \ldots, X_{L(r)}$ from the inverse Weibull probability distribution, using equation (1.0.2) and letting $\alpha = 0$ and $b = 1$, the $n^{th}$ moment of $X_{L(r)}$ from the power function probability distribution is given by

$$\begin{aligned} E(X_{L(r)})^n &= \frac{1}{\Gamma(r)}\int_0^\infty x^n f(x_r)dx \\ &= \frac{k}{\Gamma(r)}\int_0^\infty x^{n-kr-1}e^{-x^{-k}}dx \\ &= \frac{\Gamma\left(r-\frac{n}{k}\right)}{\Gamma(r)}, k>n \end{aligned} \quad (3.2.2)$$

The first moment of $X_{L(r)}$ is obtained from equation 3.2.2 to be

$$E(X_{L(r)}) = \frac{\Gamma\left(r-\frac{1}{k}\right)}{\Gamma(r)}, k>1. \quad (3.2.3)$$

Observe also that

$$\begin{aligned} E(X_{L(1)}) &= \frac{\Gamma(1-1/k)}{\Gamma(1)} \\ E(X_{L(2)}) &= (1-1/k)E(X_{L(1)}), k>1. \end{aligned}$$

Recursively we have that

$$E(X_{L(r)}) = E(X_{L(1)})\prod_{i=1}^{r-2}\left[\frac{i-1/k}{i}\right], k>1$$

The moments for the inverse Weibull probability distribution have been computed and are presented in Table 3.1 for $k = 1.5, 2, 2.5, \ldots, 5$ and $r = 1, 2, \ldots, 10$.

The second moment of $X_{L(r)}$ is obtained from equation 3.2.2 to be

$$E(X_{L(r)})^2 = \frac{\Gamma\left(r-\frac{2}{k}\right)}{\Gamma(r)}, k > 2. \tag{3.2.4}$$

From equations 3.2.3 and 3.2.4, we have the variance of $X_{L(r)}$ to be

$$Var(X_{L(r)}) = \frac{\Gamma\left(r-\frac{1}{k}\right)}{\Gamma(r)}\left[\frac{\Gamma\left(r-\frac{2}{k}\right)}{\Gamma\left(r-\frac{1}{k}\right)} - \frac{\Gamma\left(r-\frac{1}{k}\right)}{\Gamma(r)}\right]. \tag{3.2.5}$$

Using equation 1.0.4, we have the $m^{th}$ and $n^{th}$ joint moments of $X_{L(r)}$ and $X_{L(s)}$, $s > r$ to be

$$\begin{aligned} E(X^m_{L(r)}, X^n_{L(s)}) &= \int_0^\infty \int_0^{x_s} x_r^m x_s^n f(x_r, x_s) dx_r dx_s \\ &= \frac{k^2}{\Gamma(r)\Gamma(s-r)} \int_0^\infty x_s^{n-k-1} e^{-x_s^{-k}} I(x_r) dx_s, \end{aligned} \tag{3.2.6}$$

where

$$\begin{aligned} I(x_r) &= \int_0^{x_s} \frac{1}{x_r^{rk-m+1}} \left(\frac{1}{x_s^k} - \frac{1}{x_r^k}\right)^{s-r-1} dx_r \\ &= \frac{1}{kx_s^{ks-m-k}} \int_0^{x_s} \frac{1}{x_r^{rk-m+1}} \left(1 - \left(\frac{x_s}{x_r}\right)^k\right)^{s-r-1} dx_r \\ &= \frac{1}{kx_s^{ks-m-k}} \frac{\Gamma\left(r-\frac{m}{k}\right)\Gamma(s-r)}{\Gamma\left(s-\frac{m}{k}\right)}. \end{aligned} \tag{3.2.7}$$

Substituting equation (3.2.7) in equation (3.2.6) gives

$$\begin{aligned} E(X^m_{L(r)}, X^n_{L(s)}) &= \int_0^\infty \frac{1}{kx^{ks-m-n+1}} e^{-x_s^{-k}} dx_s \\ &= \frac{k\Gamma\left(r-\frac{m}{k}\right)}{\Gamma\left(s-\frac{m}{k}\right)\Gamma(r)} \int_0^\infty \frac{1}{x_s^{ks-m-n+1}} e^{-x_s^{-k}} dx_s \\ &= \frac{\Gamma\left(r-\frac{m}{k}\right)\Gamma\left(s-\frac{m+n}{k}\right)}{\Gamma\left(s-\frac{m}{k}\right)\Gamma(r)}, k > m+n. \end{aligned} \tag{3.2.8}$$

The joint moment of $X_{L(r)}$ and $X_{L(s)}$ can be obtain from equation (3.2.8), by taking $m = n = 1$ to be

$$E(X_{L(r)}, X_{L(s)}) = \frac{\Gamma\left(r-\frac{1}{k}\right)\Gamma\left(s-\frac{2}{k}\right)}{\Gamma\left(s-\frac{1}{k}\right)\Gamma(r)}, s > r, k > 2. \tag{3.2.9}$$

The covariance of $X_{L(r)}$ and $X_{L(s)}$ is obtain from equations (3.2.3) and (3.2.9), to be

$$Cov(X_{L(r)}, X_{L(s)}) = \frac{\Gamma\left(r-\frac{1}{k}\right)}{\Gamma(r)}\left[\frac{\Gamma\left(s-\frac{2}{k}\right)}{\Gamma\left(s-\frac{1}{k}\right)} - \frac{\Gamma\left(s-\frac{1}{k}\right)}{\Gamma(s)}\right], s>r, k>2. \quad (3.2.10)$$

Table 3.2 presents computed values for the variance-covariance matrices of the inverse Weibull probability distribution function for $k = 2.5, 3, 3.5, 4, 4.5, 5$, $r = 1, \ldots, 10$, $s = 1, 2, 3, 4, 5, 6, 7, 8, 10$, for $s > r$.

| r | $k=1.5$ | $k=2$ | $k=2.5$ | $k=3$ | $k=3.5$ | $k=4$ | $k=4.5$ | $k=5$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 2.67894 | 1.77245 | 1.48919 | 1.35412 | 1.27599 | 1.22542 | 1.19015 | 1.16423 |
| 2 | 0.89298 | 0.88623 | 0.89352 | 0.90275 | 0.91142 | 0.91906 | 0.92567 | 0.93138 |
| 3 | 0.59532 | 0.66467 | 0.71481 | 0.75229 | 0.78122 | 0.80418 | 0.82282 | 0.83825 |
| 4 | 0.46303 | 0.55389 | 0.6195 | 0.6687 | 0.70682 | 0.73716 | 0.76187 | 0.78236 |
| 5 | 0.38586 | 0.48466 | 0.55755 | 0.61298 | 0.65633 | 0.69109 | 0.71954 | 0.74324 |
| 6 | 0.33441 | 0.43619 | 0.51295 | 0.57211 | 0.61883 | 0.65654 | 0.68756 | 0.71351 |
| 7 | 0.29725 | 0.39984 | 0.47875 | 0.54033 | 0.58936 | 0.62918 | 0.6621 | 0.68973 |
| 8 | 0.26894 | 0.37128 | 0.4514 | 0.5146 | 0.5653 | 0.60671 | 0.64108 | 0.67002 |
| 9 | 0.24653 | 0.34808 | 0.42883 | 0.49315 | 0.54511 | 0.58775 | 0.62327 | 0.65327 |
| 10 | 0.22827 | 0.32874 | 0.40977 | 0.47489 | 0.52781 | 0.57142 | 0.60788 | 0.63876 |

Table 3.1: Expected values of Standard Inverse Weibull pdf

## 3.3 Estimation of Parameter

In this section, we estimate the parameters of the inverse Weibull probability distribution using lower record values.

### 3.3.1 Estimating $\alpha$ and $b$ for known $k$

Let $x_1, x_2, \ldots, x_r$ be $r$ lower record values from the standard inverse Weibull probability distribution (3.2.1) with $\alpha = 0$ and $b = 1$. Further, let

$$\mathbf{h}' = x_1 + \ldots + x_r,$$

then

$$E(\mathbf{h}') = \mu\mathbf{1} + \sigma^2\alpha,$$

and

$$Var(\mathbf{h}') = \sigma^2\mathbf{V},$$

where,

$$\mathbf{1}' = (1, 1, 1, \ldots, 1),$$

$$\alpha' = (b_1, b_2, b_3, \ldots b_r),$$

and

$$\mathbf{V} = (v_{ij}), v_{ij} = a_i b_j, 1 \leq i, j \leq r.$$

Let

$$\mathbf{V}^{-1} = (V^{ij}), 1 \leq i < j \leq r,$$

then the entries of $\mathbf{V}^{-1}$ are

$$\begin{aligned} V^{ii} &= \frac{a_{i+1}b_{i-1} - a_{i-1}b_{i+1}}{(a_i b_{i-1} - a_{i-1}b_i)(a_{i+1}b_i - a_i b_{i+1})} \\ &= \frac{k^2 \Gamma(i)}{\Gamma\left(i - \frac{2}{k}\right)} \left[ \left(1 + \frac{1}{k}\right)^2 - 2i \left(1 + \frac{2}{k}\right) \right], i = 1, \ldots, r-1 \end{aligned}$$

$$\begin{aligned} V^{ij} &= V^{ji} \\ &= \frac{-1}{a_{i+1}b_i - a_i b_{i+1}} \\ &= -\frac{ik^2(i - \frac{1}{k})\Gamma(i)}{\Gamma(i - \frac{2}{k})}, j = i+1, i = 1, \ldots, r-1, \end{aligned}$$

and

$$V^{ij} = 0 \quad \text{for} \quad |i - j| > 1,$$

$$\begin{aligned} V^{rr} &= \frac{b_{r-1}}{b_r(a_r b_{r-1} - a_{r-1}b_r)} \\ &= \frac{k^2 b_{r-1}}{b_r} \frac{(r - 1 - \frac{1}{k})\Gamma(r)}{\Gamma(r - 1 - \frac{2}{k})}. \end{aligned}$$

Using the method of Lloyd, [33], we have that the BLUE, $\hat{\alpha}$ and $\hat{b}$ for $\alpha$ and $b$ based on $r$ lower record values from the inverse Weibull probability distribution are given by

$$\hat{\alpha} = \frac{\delta' V^{-1}(\delta \mathbf{1}' - \mathbf{1}\delta')\mathbf{V}^{-1}\mathbf{h}}{(\delta' \mathbf{V}^{-1}\delta)(\mathbf{1}'\mathbf{V}^{-1}\mathbf{1}) - (\delta' \mathbf{V}^{-1}\mathbf{1})^2},$$

and

$$\hat{b} = \frac{\mathbf{1}'\mathbf{V}^{-1}(\mathbf{1}\delta' - \delta\mathbf{1}')\mathbf{V}^{-1}\mathbf{h}}{(\delta'\mathbf{V}^{-1}\delta)(\mathbf{1}'\mathbf{V}^{-1}\mathbf{1}) - (\delta'\mathbf{V}^{-1}\mathbf{1})^2}.$$

The variance and covariance of the estimators are

$$Var(\hat{\alpha}) = \frac{(\delta\mathbf{V}^{-1}\delta)\sigma^2}{(\delta'\mathbf{V}^{-1}\delta)(\mathbf{1}'\mathbf{V}^{-1}\mathbf{1}) - (\delta'\mathbf{V}^{-1}\mathbf{1})^2},$$

$$Var(\hat{b}) = \frac{(\mathbf{1}'\mathbf{V}^{-1}\mathbf{1})\sigma^2}{(\delta'\mathbf{V}^{-1}\delta)(\mathbf{1}'\mathbf{V}^{-1}\mathbf{1}) - (\delta'\mathbf{V}^{-1}\mathbf{1})^2}$$

$$Cov(\hat{\alpha}, \hat{b}) = -\frac{(\delta'\mathbf{V}^{-1}\mathbf{1})\sigma^2}{(\delta'\mathbf{V}^{-1}\delta)(\mathbf{1}'\mathbf{V}^{-1}\mathbf{1}) - (\delta'\mathbf{V}^{-1}\mathbf{1})^2}.$$

Coefficients of the BLUES for $\alpha$, $b$ and the variance covariance for $\alpha$ and $b$ are given in Tables 2.1, 2.2, and 2.3.

## 3.4 Estimation of $k$ when $\alpha$ and $b$ are assumed known

**Method of Moments**

For simplicity, we let $\alpha = 0$ and $b = 1$ in (3.2.1). In the absence of this assumption, the the random variable $Y$ can be replaced by $(Y - \alpha)/b$ if $\alpha$ and $b$ are known. Observe from equation (3.2.3) that

$$E(X_{L(r)}) = \frac{\Gamma(r - 1/k)}{\Gamma(r)},$$

from which we have that

$$E(X_{L(1)}) = \frac{\Gamma(1 - 1/k)}{\Gamma(1)}.$$

Next,

$$\begin{aligned} E(X_{L(2)}) &= \frac{\Gamma(2 - 1/k)}{\Gamma(2)} \\ &= (1 - 1/k)E(X_{L(1)}), \end{aligned}$$

and

$$\frac{E(X_{L(1)})}{k} = E(X_{L(1)}) - E(X_{L(2)}),$$

similarly,

$$\frac{E(X_{L(2)})}{2k} = E(X_{L(2)}) - E(X_{L(3)}),$$

which gives

$$\frac{E(X_{L(1)})}{k} + \frac{E(X_{L(2)})}{2k} = E(X_{L(1)}) - E(X_{L(3)}) \tag{3.4.11}$$

The next term results in

$$\frac{E(X_{L(3)})}{3k} = E(X_{L(3)}) - E(X_{L(4)}). \tag{3.4.12}$$

Adding equations 3.4.11 and 3.4.12 together results in

$$\frac{E(X_{L(1)})}{k} + \frac{E(X_{L(2)})}{2k} + \frac{E(X_{L(3)})}{3k} = E(X_{L(1)}) - E(X_{L(4)})$$

Hence, continuing this procedure to the $r$th lower record value we have

$$\frac{E(X_{L(1)})}{k} + \frac{E(X_{L(2)})}{2k} + \ldots + \frac{E(X_{L(r-1)})}{k(r-1)} = E(X_{L(1)}) - E(X_{L(r)})$$

Dropping the Expectation and solving for $k$, we have a moment's estimate $\hat{k}_{ME}$ of $k$ to be

$$\hat{k}_{ME} = \frac{1}{X_1 - X_r} \sum_{i=1}^{r-1} \frac{X_i}{i} \tag{3.4.13}$$

**Method of Maximum Likelihood**

Using equations (1.0.1) and (3.2.1), we have that

$$f_{1,2,\ldots,r}(x_1, x_2, \ldots, x_r) = e^{-x_r^{-k}} \prod_{i=1}^{r} \frac{k}{x_i^{k+1}} \tag{3.4.14}$$

The loglikelihood of equation (3.4.14) is

$$\log f_{1,2,\ldots,r}(x_1, x_2, \ldots, x_r) = -x_r^{-k} + \sum_{i=1}^{r} \log(k) - (k+1)\log(x_i). \tag{3.4.15}$$

Differentiating equation (3.4.15) with respect to $k$ gives

$$\frac{\delta \log f_{1,2,\ldots,r}(x_1, x_2, \ldots, x_r)}{\delta k} = x_r^{-k} \log(x_r) + \sum_{i=1}^{r} \frac{1}{k} - \log(x_i). \qquad (3.4.16)$$

Solving equation (3.4.16) iteratively gives the maximum likelihood estimate $\hat{k}_{MLE}$ of $k$.

## 3.5 Simulation Study

To illustrate the performance of the estimators obtained in the previous section, we proceed with a simulation study. We simulated a small random sample of size, $n = 20$ from the power function probability distribution with $k = 3.5$, $\alpha = 3$ and $b = 5$. The simulated values are

$$10.87, 10.37, 10.03, 8.15, 9.14, 10.83, 10.02, 7.60, 10.47, 9.14,$$
$$8.72, 11.09, 8.27, 9.01, 10.40, 7.21, 8.09, 12.92, 10.55, 6.55.$$

From the above sample, we obtained six records, that is,

$$10.87, 10.37, 10.03, 8.15, 7.6, 6.55.$$

Using Table 3.3 for $k = 3.5$, $r = 6$, we have that the BLUE for $\alpha$ is:

$$\begin{aligned}\hat{\alpha} &= 10.87 \times 3.44 - 10.37 \times 2.02 - 10.03 \times 1.7 - 8.15 \times 2.10 - 7.6 \times 1.55 + 6.55 \times 5.03 \\ &= 3.45,\end{aligned}$$

Using Table 3.5, we obtained the estimated standard error for $\hat{\alpha}$ to be

$$S.E.(\hat{\mu}) = 3.45\sqrt{0.23929} = 1.69$$

Using Table 3.4 for $k = 3.5$, $r = 6$, we have that the BLUE for $b$ is

$$\begin{aligned}\hat{b} &= -10.87 \times 4.80 + 10.37 \times 2.04 + 10.03 \times 2.85 + 8.15 \times 3.52 + 7.6 \times 4.11 - 6.55 \times 7.37 \\ &= 6.86.\end{aligned}$$

Using Table 3.5, we obtained the estimated standard error for $\hat{b}$ to be

$$S.E.(\hat{b}) = 6.86\sqrt{0.69} = 5.70$$

## 3.6 Conclusion

We have introduced in this chapter the concepts of **records** for a given phenomenon that is probabilistically characterized by the inverse Weibull probability distribution function. We have developed some distributional properties of lower record values and have obtained some properties that are important to this distribution. We have developed the estimates of the location and scale parameters of this distribution given the shape parameter and also the estimates of the shape parameter has been obtained given that the location and scale parameters are known. In addition, we have tabulated the means and variance of lower record values from the inverse Weibull probability distribution function. Coefficients of the best linear unbiased estimates have been obtained. In addition we have shown the importance of our results using simulation study.

| s | r | k=2.5 | k=3 | k=3.5 | k=4 | k=4.5 | k=5 |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 2.37315 | 0.8453 | 0.43935 | 0.27081 | 0.18426 | 0.13376 |
| 2 | 1 | 0.19967 | 0.11705 | 0.07754 | 0.0554 | 0.04168 | 0.03255 |
| 2 | 2 | 0.1198 | 0.07803 | 0.05538 | 0.04155 | 0.03242 | 0.02604 |
| 3 | 1 | 0.08322 | 0.05289 | 0.03692 | 0.02738 | 0.02117 | 0.01688 |
| 3 | 2 | 0.04993 | 0.03526 | 0.02637 | 0.02053 | 0.01646 | 0.01351 |
| 3 | 3 | 0.03994 | 0.02938 | 0.02261 | 0.01797 | 0.01463 | 0.01216 |
| 4 | 1 | 0.04858 | 0.03213 | 0.02304 | 0.01742 | 0.01367 | 0.01103 |
| 4 | 2 | 0.02915 | 0.02142 | 0.01646 | 0.01307 | 0.01063 | 0.00883 |
| 4 | 3 | 0.02332 | 0.01785 | 0.01411 | 0.01143 | 0.00945 | 0.00794 |
| 4 | 4 | 0.02021 | 0.01587 | 0.01277 | 0.01048 | 0.00875 | 0.00741 |
| 5 | 1 | 0.03293 | 0.02235 | 0.01632 | 0.0125 | 0.0099 | 0.00805 |
| 5 | 2 | 0.01976 | 0.0149 | 0.01165 | 0.00937 | 0.0077 | 0.00644 |
| 5 | 3 | 0.01581 | 0.01242 | 0.00999 | 0.0082 | 0.00685 | 0.0058 |
| 5 | 4 | 0.0137 | 0.01104 | 0.00904 | 0.00752 | 0.00634 | 0.00541 |
| 5 | 5 | 0.01233 | 0.01012 | 0.00839 | 0.00705 | 0.00599 | 0.00514 |
| 6 | 1 | 0.02429 | 0.0168 | 0.01243 | 0.00961 | 0.00767 | 0.00628 |
| 6 | 2 | 0.01457 | 0.0112 | 0.00888 | 0.00721 | 0.00597 | 0.00502 |
| 6 | 3 | 0.01166 | 0.00933 | 0.00761 | 0.00631 | 0.0053 | 0.00452 |
| 6 | 4 | 0.01011 | 0.0083 | 0.00688 | 0.00578 | 0.00491 | 0.00422 |
| 6 | 5 | 0.00909 | 0.0076 | 0.00639 | 0.00542 | 0.00464 | 0.00401 |
| 6 | 6 | 0.00837 | 0.0071 | 0.00603 | 0.00515 | 0.00443 | 0.00385 |
| 7 | 1 | 0.01892 | 0.01328 | 0.00993 | 0.00773 | 0.00621 | 0.0051 |
| 7 | 2 | 0.01135 | 0.00885 | 0.00709 | 0.0058 | 0.00483 | 0.00408 |
| 7 | 3 | 0.00908 | 0.00738 | 0.00608 | 0.00508 | 0.00429 | 0.00368 |
| 7 | 4 | 0.00787 | 0.00656 | 0.0055 | 0.00465 | 0.00398 | 0.00343 |
| 7 | 5 | 0.00708 | 0.00601 | 0.00511 | 0.00436 | 0.00376 | 0.00326 |
| 7 | 6 | 0.00652 | 0.00561 | 0.00481 | 0.00414 | 0.00359 | 0.00313 |
| 7 | 7 | 0.00608 | 0.0053 | 0.00458 | 0.00397 | 0.00346 | 0.00302 |
| 8 | 1 | 0.0153 | 0.01087 | 0.0082 | 0.00643 | 0.00519 | 0.00428 |
| 8 | 2 | 0.00918 | 0.00725 | 0.00586 | 0.00482 | 0.00403 | 0.00342 |
| 8 | 3 | 0.00735 | 0.00604 | 0.00502 | 0.00422 | 0.00359 | 0.00308 |
| 8 | 4 | 0.00637 | 0.00537 | 0.00454 | 0.00387 | 0.00332 | 0.00288 |
| 8 | 5 | 0.00573 | 0.00492 | 0.00422 | 0.00363 | 0.00314 | 0.00273 |
| 8 | 6 | 0.00527 | 0.00459 | 0.00398 | 0.00344 | 0.003 | 0.00262 |
| 8 | 7 | 0.00492 | 0.00434 | 0.00379 | 0.0033 | 0.00289 | 0.00254 |
| 8 | 8 | 0.00464 | 0.00413 | 0.00363 | 0.00318 | 0.00279 | 0.00246 |
| 10 | 1 | 0.01082 | 0.00784 | 0.00599 | 0.00474 | 0.00386 | 0.0032 |
| 10 | 2 | 0.00649 | 0.00522 | 0.00428 | 0.00356 | 0.00300 | 0.00256 |
| 10 | 3 | 0.00519 | 0.00435 | 0.00367 | 0.00311 | 0.00267 | 0.00231 |
| 10 | 4 | 0.0045 | 0.00387 | 0.00332 | 0.00285 | 0.00247 | 0.00215 |
| 10 | 5 | 0.00405 | 0.00355 | 0.00308 | 0.00268 | 0.00233 | 0.00204 |
| 10 | 6 | 0.00373 | 0.00331 | 0.0029 | 0.00254 | 0.00223 | 0.00196 |
| 10 | 7 | 0.00348 | 0.00313 | 0.00277 | 0.00244 | 0.00215 | 0.0019 |
| 10 | 8 | 0.00328 | 0.00298 | 0.00265 | 0.00235 | 0.00208 | 0.00184 |
| 10 | 9 | 0.00311 | 0.00285 | 0.00256 | 0.00228 | 0.00202 | 0.0018 |
| 10 | 10 | 0.00298 | 0.00275 | 0.00248 | 0.00221 | 0.00197 | 0.00176 |

Table 3.2: Variance Covariance of Standard Inverse Weibull pdf

| r | $k = 2.5$ | $k = 3$ | $k = 3.5$ | $k = 4$ | $k = 4.5$ | $k = 5$ |
|---|---|---|---|---|---|---|
| 3 | -0.39628 | -0.67253 | -0.93153 | -1.18546 | -1.43740 | -1.68844 |
| | -2.28280 | -2.30989 | -2.46020 | -2.65333 | -2.86643 | -3.09047 |
| | 3.67907 | 3.98241 | 4.39172 | 4.83879 | 5.30383 | 5.77890 |
| 4 | 0.08390 | 0.36628 | 0.98284 | 2.23483 | 4.93390 | 12.28310 |
| | -1.59977 | -2.55895 | -4.10090 | -6.74035 | -11.87186 | -25.00456 |
| | -2.66628 | -3.83842 | -5.74126 | -8.98713 | -15.26382 | -31.25570 |
| | 5.18215 | 7.03109 | 9.85932 | 14.49265 | 23.20178 | 44.97716 |
| 5 | 0.06433 | 0.25227 | 0.61255 | 1.22095 | 2.20221 | 3.77892 |
| | -0.83573 | -1.31959 | -2.00272 | -2.96920 | -4.35933 | -6.42456 |
| | -1.39289 | -1.97938 | -2.80381 | -3.95894 | -5.60485 | -8.03070 |
| | -1.89939 | -2.54491 | -3.46353 | -4.75072 | -6.57961 | -9.26619 |
| | 5.06368 | 6.59161 | 8.65751 | 11.45791 | 15.34159 | 20.94252 |
| 6 | 0.05099 | 0.19033 | 0.44115 | 0.83247 | 1.40238 | 2.20284 |
| | -0.51822 | -0.81763 | -1.21523 | -1.73445 | -2.40705 | -3.27690 |
| | -0.86370 | -1.22645 | -1.70132 | -2.31259 | -3.09478 | -4.09612 |
| | -1.17777 | -1.57686 | -2.10163 | -2.77511 | -3.63300 | -4.72629 |
| | -1.47221 | -1.89224 | -2.45190 | -3.17156 | -4.08712 | -5.25144 |
| | 4.98091 | 6.32285 | 8.02892 | 10.16124 | 12.81957 | 16.14790 |
| 7 | -0.00409 | -0.01039 | -0.01766 | -0.02518 | -0.03255 | -0.03958 |
| | -0.02167 | -0.02261 | -0.02255 | -0.02201 | -0.02125 | -0.02041 |
| | -0.03612 | -0.03392 | -0.03157 | -0.02934 | -0.02732 | -0.02552 |
| | -0.04925 | -0.04361 | -0.03899 | -0.03521 | -0.03207 | -0.02944 |
| | -0.06156 | -0.05234 | -0.04549 | -0.04024 | -0.03608 | -0.03272 |
| | 0.03710 | 0.02174 | 0.01147 | 0.00436 | -0.00072 | -0.00445 |
| | 1.13560 | 1.14113 | 1.14479 | 1.14761 | 1.14999 | 1.15212 |
| 8 | -0.00262 | -0.00705 | -0.01251 | -0.01839 | -0.02431 | -0.03007 |
| | -0.01889 | -0.02008 | -0.02025 | -0.01992 | -0.01935 | -0.01867 |
| | -0.03149 | -0.03011 | -0.02835 | -0.02657 | -0.02488 | -0.02334 |
| | -0.04294 | -0.03872 | -0.03502 | -0.03188 | -0.02921 | -0.02693 |
| | -0.05367 | -0.04646 | -0.04086 | -0.03643 | -0.03286 | -0.02992 |
| | -0.50563 | -0.50568 | -0.50522 | -0.50468 | -0.50419 | -0.50381 |
| | 1.06035 | 1.08315 | 1.09823 | 1.10898 | 1.11714 | 1.12362 |
| | 0.59488 | 0.56495 | 0.54400 | 0.52889 | 0.51767 | 0.50912 |
| 10 | -0.00155 | -0.00440 | -0.00223 | -0.01223 | -0.01648 | -0.02069 |
| | -0.01246 | -0.01372 | 0.00410 | -0.01421 | -0.01399 | -0.01364 |
| | -0.02076 | -0.02059 | 0.00574 | -0.01895 | -0.01798 | -0.01704 |
| | -0.02831 | -0.02647 | 0.00709 | -0.02274 | -0.02111 | -0.01967 |
| | -0.03539 | -0.03176 | 0.00827 | -0.02598 | -0.02375 | -0.02185 |
| | -0.43398 | -0.44726 | 0.83452 | -0.46255 | -0.46735 | -0.47111 |
| | 0.95739 | 0.99739 | -2.05534 | 1.04585 | 1.06164 | 1.07420 |
| | -0.67657 | -0.67961 | 1.26150 | -0.68186 | -0.68231 | -0.68260 |
| | -0.06098 | -0.04971 | 0.01211 | -0.03618 | -0.03183 | -0.02841 |
| | 1.31261 | 1.27613 | -0.07575 | 1.22885 | 1.21316 | 1.20080 |

Table 3.3: Coefficients of the BLUE for $\alpha$ in terms of $b$.

| r | $k = 2.5$ | $k = 3$ | $k = 3.5$ | $k = 4$ | $k = 4.5$ | $k = 5$ |
|---|---|---|---|---|---|---|
| 3 | 0.57493 | 0.91662 | 1.21424 | 1.49452 | 1.76581 | 2.03173 |
| | 3.10453 | 2.97993 | 3.06617 | 3.22463 | 3.41620 | 3.62566 |
| | -3.67946 | -3.89654 | -4.28042 | -4.71915 | -5.18201 | -5.65739 |
| 4 | -0.16832 | -0.63722 | -1.56650 | -3.34699 | -7.04898 | -16.90941 |
| | 2.64841 | 3.99737 | 6.12633 | 9.71121 | 16.59580 | 34.07033 |
| | 4.41401 | 5.99605 | 8.57686 | 12.94827 | 21.33745 | 42.58792 |
| | -6.89410 | -9.35621 | -13.13669 | -19.31249 | -30.88427 | -59.74884 |
| 5 | -0.13966 | -0.47463 | -1.04811 | -1.94912 | -3.33327 | -5.48303 |
| | 1.52954 | 2.22987 | 3.18893 | 4.51182 | 6.37709 | 9.10590 |
| | 2.54923 | 3.34481 | 4.46450 | 6.01576 | 8.19911 | 11.38238 |
| | 3.47623 | 4.30047 | 5.51497 | 7.21891 | 9.62505 | 13.13351 |
| | -7.41534 | -9.40052 | -12.12030 | -15.79736 | -20.86797 | -28.13876 |
| 6 | -0.11857 | -0.38113 | -0.79766 | -1.39593 | -2.21862 | -3.32708 |
| | 1.02750 | 1.47211 | 2.03826 | 2.75352 | 3.65634 | 4.80015 |
| | 1.71249 | 2.20817 | 2.85357 | 3.67136 | 4.70101 | 6.00018 |
| | 2.33522 | 2.83908 | 3.52499 | 4.40564 | 5.51857 | 6.92329 |
| | 2.91902 | 3.40689 | 4.11249 | 5.03501 | 6.20839 | 7.69254 |
| | -7.87566 | -9.54513 | -11.73165 | -14.46961 | -17.86569 | -22.08908 |
| 7 | -0.00442 | -0.00922 | -0.01332 | -0.01665 | -0.01934 | -0.02154 |
| | -0.00138 | -0.00090 | -0.00061 | -0.00042 | -0.00030 | -0.00022 |
| | -0.00231 | -0.00136 | -0.00085 | -0.00056 | -0.00038 | -0.00027 |
| | -0.00314 | -0.00174 | -0.00105 | -0.00067 | -0.00045 | -0.00031 |
| | -0.00393 | -0.00209 | -0.00123 | -0.00077 | -0.00051 | -0.00035 |
| | 2.36822 | 2.12961 | 1.97407 | 1.86466 | 1.78354 | 1.72102 |
| | -2.35303 | -2.11430 | -1.95701 | -1.84559 | -1.76256 | -1.69833 |
| 8 | -0.00076 | -0.00149 | -0.00198 | -0.00228 | -0.00244 | -0.00252 |
| | 0.00554 | 0.00497 | 0.00444 | 0.00398 | 0.00359 | 0.00327 |
| | 0.00923 | 0.00746 | 0.00622 | 0.00531 | 0.00462 | 0.00409 |
| | 0.01258 | 0.00959 | 0.00768 | 0.00637 | 0.00543 | 0.00471 |
| | 0.01573 | 0.01151 | 0.00896 | 0.00728 | 0.00610 | 0.00524 |
| | 1.01666 | 0.90816 | 0.83729 | 0.78748 | 0.75062 | 0.72227 |
| | -2.54041 | -2.24859 | -2.05947 | -1.92732 | -1.82996 | -1.75534 |
| | 1.48145 | 1.30838 | 1.19686 | 1.11918 | 1.06204 | 1.01828 |
| 10 | -0.00086 | -0.00168 | -0.00223 | -0.00256 | -0.00274 | -0.00283 |
| | 0.00491 | 0.00452 | 0.00410 | 0.00372 | 0.00339 | 0.00310 |
| | 0.00818 | 0.00678 | 0.00574 | 0.00496 | 0.00436 | 0.00388 |
| | 0.01116 | 0.00871 | 0.00709 | 0.00595 | 0.00511 | 0.00448 |
| | 0.01395 | 0.01045 | 0.00827 | 0.00680 | 0.00575 | 0.00497 |
| | 1.00968 | 0.90396 | 0.83452 | 0.78554 | 0.74920 | 0.72119 |
| | -2.53035 | -2.24240 | -2.05534 | -1.92442 | -1.82782 | -1.75371 |
| | 1.59718 | 1.39219 | 1.26150 | 1.17124 | 1.10532 | 1.05514 |
| | 0.02404 | 0.01636 | 0.01211 | 0.00947 | 0.00771 | 0.00647 |
| | -0.13788 | -0.09889 | -0.07575 | -0.06071 | -0.05028 | -0.04269 |

Table 3.4: Coefficients of the BLUE for $b$ in tems of $b$.

| r | $k = 2.5$ | $k = 3$ | $k = 3.5$ | $k = 4$ | $k = 4.5$ | $k = 5$ |
|---|---|---|---|---|---|---|
| 3 | 0.5988546 | 0.5156701 | 0.4844271 | 0.4702911 | 0.46347 | 0.46023 |
| | 1.181577 | 0.9067008 | 0.7869827 | 0.7205605 | 0.67856 | 0.64971 |
| | -0.814424 | -0.6652533 | -0.6037476 | -0.5715508 | -0.55236 | -0.53993 |
| 4 | 0.41967 | 0.57127 | 0.80749 | 1.1947 | 1.91954 | 3.72366 |
| | 1.20515 | 1.43263 | 1.83086 | 2.50222 | 3.76888 | 6.92783 |
| | -0.69477 | -0.89239 | -1.20631 | -1.72127 | -2.68335 | -5.07373 |
| 5 | 0.21924 | 0.29459 | 0.39435 | 0.52628 | 0.70485 | 0.95674 |
| | 0.77532 | 0.8699 | 1.02113 | 1.23163 | 1.52146 | 1.93268 |
| | -0.40125 | -0.49781 | -0.62792 | -0.7997 | -1.0311 | -1.35604 |
| 6 | 0.13595 | 0.18253 | 0.23929 | 0.30742 | 0.38919 | 0.48799 |
| | 0.56707 | 0.61452 | 0.69006 | 0.78784 | 0.90838 | 1.05556 |
| | -0.2695 | -0.32864 | -0.40135 | -0.488 | -0.59119 | -0.71483 |
| 7 | 0.00568 | 0.00505 | 0.00444 | 0.0039 | 0.00344 | 0.00304 |
| | 0.0078 | 0.00524 | 0.00376 | 0.00284 | 0.00222 | 0.00178 |
| | 0.00036 | 2.00E-04 | 0.00012 | 7.00E-05 | 5.00E-05 | 3.00E-05 |
| 8 | 0.00496 | 0.00448 | 0.00399 | 0.00353 | 0.00313 | 0.00278 |
| | 0.00328 | 0.0022 | 0.00158 | 0.00118 | 0.00092 | 0.00074 |
| | -0.00145 | -0.00111 | -0.00087 | -0.00071 | -0.00058 | -0.00049 |
| 9 | 0.00397 | 0.00366 | 0.0033 | 0.00295 | 0.00263 | 0.00235 |
| | 0.00327 | 0.00219 | 0.00157 | 0.00118 | 0.00092 | 0.00074 |
| | -0.00136 | -0.00105 | -0.00084 | -0.00068 | -0.00056 | -0.00047 |
| 10 | 0.00327 | 0.00306 | 0.00279 | 0.00252 | 0.00226 | 0.00203 |
| | 0.00326 | 0.00219 | 0.00157 | 0.00118 | 0.00092 | 0.00074 |
| | -0.00129 | -0.00101 | -0.00081 | -0.00066 | -0.00055 | -0.00046 |

Table 3.5: Variance Covariance of $\alpha$ and $b$ in terms of $b$.

# 4 Theory Of Records For The Gumbel Probability Distribution

## 4.1 Introduction

The Gumbel probability distribution also known as the double exponential probability distribution plays a major role in analyzing and modeling the behavior of random phenomenon that occur in engineering, business, biology, finance, economics among others, see Luo and Zhu, [34], Coles, [18], Gumbel, [25], Hosking et al. [27], Kotz, [31]. Consider a random variable, $Y$, that characterizes a certain measurement of interest and it is said to follow the Gumbel probability distribution, if and only if its cumulative distribution function (cdf) is given by

$$F(y) = 1 - \exp\left\{-\exp\left(\frac{x-\theta}{\beta}\right)\right\}, \quad -\infty < x < \infty \tag{4.1.1}$$

where $\theta$ and $\beta$ are the location and scale parameters, respectively.

In the present study, we shall develop the theory of records for the subject cdf and illustrate their usefulness in two real world applications. Records are obtained by observing successive minimum or maximum values in a given phenomenon of interest. Chandler, [17], was first to introduce the concept of record for analyzing the breaking strength data of certain material. The problem of parametric inference for record-breaking data was introduced by Samaniego and Whitaker, [47]. They developed and studied the properties of maximum likelihood estimates of the mean of an underlying exponential probability distribution. Gulati and Padgett, [26], extended the work of Samaniego and Whitaker, [47] to the Weibul probability distribution.

In the present study, we shall utilize the concept of records to develop the key estimates of the parameters that are the key entities in the Gumbel probability distribution. In addition to developing the record estimates of the parameters, we shall illustrate their usefulness by applying the results to the Olympic records of women's 100 meter free style swimming from

1912 to 2004 and one hour mean concentration of sulphur dioxide ($SO_2$) from the city of Long Beach, Californian.

## 4.2 Analytic Formulation

Let $X_1, X_2, \ldots, X_n$ be a complete random sample from the Gumbel cumulative probability distribution function given by (4.1.1). To obtain the records as needed for the present study we proceed as follows

1. Given a complete random sample $X_1, X_2, \ldots, Y_n$.

2. The first record, $X_{L(1)}$, is $X_1$, the first observation, that is, $X_{L(1)} = X_1$. The second record value, $X_{L(2)}$, is obtained by observing the independent and identically distributed random variables $X_i$'s sequentially from $X_2, \ldots, X_n$. The next observation that is less than $X_{L(1)}$ is the second record observation, $X_{L(2)}$, and the number of trials to get $X_{L(2)}$ is $K_1$, for example, let the next observation that is less than $X_{L(1)}$ be $X_5$, then $X_{L(2)} = X_5$ and $K_1 = 4$. Now, $X_5$ is a standard for getting subsequent records.

3. The observe data will consist of $X_{L(1)} = x_1, K_1 = k_1, X_{L(2)} = x_2, K_2 = k_2, \ldots, X_{L(r)} = x_r, K_r = k_r$, where $\{X_{L(i)}, 1 \leq i \leq r\}$ is the record value sequence and $\{K_i, i > 0\}$ and $K_r = 1$ is the inter record time sequence. Note that by using this method, the number of records obtained $(r)$ will be less than $n$, the size of the complete random data sample.

Note that the record values without the inter record times form what is known as the lower record values.

For the record-breaking samples $x_1, k_1, x_2, k_2, \ldots, x_r, k_r$ defined above, we can write the likelihood function as

$$L(x) = \prod_{i=1}^{r} f(x_i)(1 - F(x_i))^{k_i - 1}, \tag{4.2.2}$$

where $f(x)$ and $F(x)$ are the pdf and cdf of the random variables from which the record observations are obtained.

Thus, using equations (4.1.1) and (4.2.2), the likelihood function for the Gumbel probability distribution is given by

$$L(z) = \prod_{i=1}^{r} \frac{1}{\beta} e^{z_i} [\exp(-e^{z_i})]^{k_i}, \tag{4.2.3}$$

where $z_i = (x_i - \theta)/\beta$.

The negative loglikelihood of expression (4.2.3) is given by

$$f(\psi) = -\log L(z) = \sum_{i=1}^{r} \log(\beta) - z_i + k_i e^{z_i}, \tag{4.2.4}$$

where $\psi = (\theta, \beta)$.

Taking the partial derivative of (4.2.4) with respect to $\theta$ and $\beta$ we have

$$\frac{\partial f(\psi)}{\partial \theta} = -\frac{1}{\beta} \sum_{i=1}^{r} [k_i e^{z_i} - 1] \tag{4.2.5}$$

and

$$\frac{\partial f(\psi)}{\partial \beta} = -\frac{1}{\beta} \sum_{i=1}^{r} [k_i z_i e^{z_i} - z_i - 1]. \tag{4.2.6}$$

Set equations (4.2.5) and (4.2.6) equal to zero we obtain the maximum likelihood estimates $\hat{\theta}$ for $\theta$ and $\hat{\beta}$ for $\beta$ for the record samples.

The second partial derivatives of equations (4.2.5) and (4.2.6) with respect to $\theta$, $\beta$ and $\theta\beta$ are given by

$$\frac{\partial^2 f(\psi)}{\partial \theta^2} = \frac{r}{\beta^2}, \tag{4.2.7}$$

$$\frac{\partial^2 f(\psi)}{\partial \beta^2} = \frac{1}{\beta^2} \sum_{i=1}^{r} (1 + k_i z_i^2 e^{z_i}) \tag{4.2.8}$$

and

$$\frac{\partial^2 f(\psi)}{\partial \theta \partial \beta} = \frac{1}{\beta^2} \sum_{i=1}^{r} k_i z_i e^{z_i}. \tag{4.2.9}$$

Equations, (4.2.7), (eqn41:7), and (eqn41:8) give the observed information matrix $I(\theta, \beta)$, at $(\hat{\theta}, \hat{\beta})$, for the Gumbel pdf model to be

$$I(\hat{\theta}, \hat{\beta}) = \frac{1}{\hat{\beta}^2} \begin{pmatrix} r & \sum_{i=1}^r k_i \hat{z}_i e^{\hat{z}_i} \\ \sum_{i=1}^r k_i \hat{z}_i e^{\hat{z}_i} & r + \sum_{i=1}^r k_i \hat{z}_i{}^2 e^{\hat{z}_i} \end{pmatrix}. \tag{4.2.10}$$

Approximate confidence intervals and hypothesis tests for $\theta$, and $\beta$ can be found by treating $(\hat{\theta}, \hat{\beta})$ as a bivariate normal for large samples with mean vector $(\theta, \beta)$ and covariance matrix the inverse of $I(\hat{\theta}, \hat{\beta})$, that is $I(\hat{\theta}, \hat{\beta})^{-1}$.

Next, we proceed to obtain the estimates of the parameters that are inherent in equations (4.2.5) and (4.2.6) as follows, for the complete sample $X_1, \ldots, X_n$, from the Gumbel probability density function given by equation (4.1.1), we can write the negative log-likelihood as

$$g(\psi) = \sum_{i=1}^{n} \log(\beta) - \xi_i + \exp(\xi_i), \tag{4.2.11}$$

where $\xi_i = (x_i - \theta)/\beta$.

Taking the partial derivatives of equation (4.2.11) with respect to $\theta$ and $\beta$, we have

$$\frac{\partial g(\psi)}{\partial \theta} = -\frac{1}{\beta} \sum_{i=1}^{n} [e^{\xi_i} - 1] \tag{4.2.12}$$

and

$$\frac{\partial g(\psi)}{\partial \beta} = -\frac{1}{\beta} \sum_{i=1}^{n} [\xi_i e^{\xi_i} - \xi_i - 1]. \tag{4.2.13}$$

Let expressions (4.2.12) and (4.2.13) equal to zero and taking the second partial derivatives with respect to $\theta$, $\beta$ and $\theta\beta$ we have

$$\frac{\partial^2 g(\psi)}{\partial \theta^2} = \frac{r}{\beta^2}, \tag{4.2.14}$$

$$\frac{\partial^2 g(\psi)}{\partial \beta^2} = \frac{1}{\beta^2} \sum_{i=1}^{n} (1 + \xi_i^2 e^{\xi_i}), \tag{4.2.15}$$

and

$$\frac{\partial^2 f(\psi)}{\partial\theta\partial\beta} \quad = \quad \frac{1}{\beta^2}\sum_{i=1}^{n}\xi_i e^{\xi_i}. \tag{4.2.16}$$

Using equations (4.2.14), (4.2.15), and (4.2.16) we obtain the observed information matrix at $(\hat{\theta}, \hat{\beta})$, that is, $I(\theta, \beta)$, for the Gumbel pdf model to be

$$I(\hat{\theta}, \hat{\beta}) = \frac{1}{\hat{\beta}^2}\begin{pmatrix} n & \sum_{i=1}^{n}\hat{\xi}_i e^{\hat{\xi}_i} \\ \sum_{i=1}^{n}\hat{\xi}_i e^{\hat{\xi}_i} & n + \sum_{i=1}^{n}\hat{\xi}_i{}^2 e^{\hat{\xi}_i} \end{pmatrix}. \tag{4.2.17}$$

Thus, we can use the estimates in equations (4.2.12) and (4.2.13) to predict future observations. We can accomplish this by using the return levels, that is,

$$F(x_s) = 1/s, \quad s > r$$

which gives

$$x_s = \hat{\theta} + \hat{\beta}\ln\ln\left(\frac{s}{s-1}\right). \tag{4.2.18}$$

## 4.3 Applications Of Records

In this section, we shall apply the results of the previous section to real interesting data sets. The first being the 100 meter women's free style swimming results of the Olympic games from 1912 to 2004. This data is given in Table 4.1. The second application is the one hour mean concentration of sulfur dioxide ($SO_2$) from the city of Long Beach, California. This information is presented in Table 4.4. Roberts, [45] fitted the Gumbel probability distribution given by (4.1.1) to the annual maxima of the hourly concentration of ($SO_2$). He used the least squares method to obtain estimates of the parameter $\theta$ and $\beta$, that resulted in $\hat{\theta} = 31.50$ and $\hat{\beta} = 12.34$ for the hourly concentration of $SO_2$. Ahsanullah, [5] also used the concept of records to analyze the hourly concentration of ($SO_2$).

### 4.3.1 Goodness-of-Fit

We shall begin by verifying that the data given in Table 4.1 and Table 4.4, that is, the 100 meter free style Olympic swimming data and concentration of $SO_2$, respectively, indeed follow the Gumbel pdf. Once that has been accomplished, we will obtain the estimated records. Our results are then compared with the estimates obtained by Ahsanullah, [5], which he initially analyzed the subject data.

We shall use the two sided Kolmogorov-Smirnov test statistic given by

$$D_n = \sup_x |F_n^*(x) - F(x)|, \tag{4.3.19}$$

where $F_n^*(x)$ is the empirical probability distribution function, $F(x)$ is the Gumbel probability distribution function that we are testing to see if the given data fits it well. Note that the statistic $(D_n)$ is distribution free.

For the Olympic data, using equations (4.2.12) and (4.2.13), we have the estimates $\hat{\theta}$ of $\theta$ and $\hat{\beta}$ of $\beta$ for the Gumbel pdf equal to 78.61 and 5.66, respectively.

Using these estimates for the Olympic data and those of Roberts, [45], for the hourly concentration of $SO_2$, we are able to determine if the data actually follows the Gumbel pdf. The results given below verify that indeed the Gumbel pdf fits the given data.

| | $D_n$ | p-value |
|---|---|---|
| Olympic | 0.1808 | $> 0.20$ |
| $SO_2$ | 0.0857 | $> 0.20$ |

Good-ness-of fit test

Thus, we can probabilistically characterize the behavior of the Olympic records using the Gumbel pdf.

We have also used the Akaike information criterion (AIC) and quantile plot to further verify the fit of our model with that of Ahsanullah, [5].

The AIC is defined by

$$-2 \log L + 2K,$$

where $L$ is the likelihood of the model of interest and $K$ is the number of parameters in the model. The smaller the value of the AIC, the better the model.

A quantile plot is fitted and plotted against the expected quantile defined by $F(x_i) = (i - 0.375)/(n + 0.25)$ (Royston [46]), where F is given by equation (4.1.1). For example, to check the quantile plot of our model, we would plot the expected quantile versus the fitted quantiles using the maximum likelihood estimates given by equations (4.2.5) and (4.2.6), and the cdf defined in equation (4.1.1).

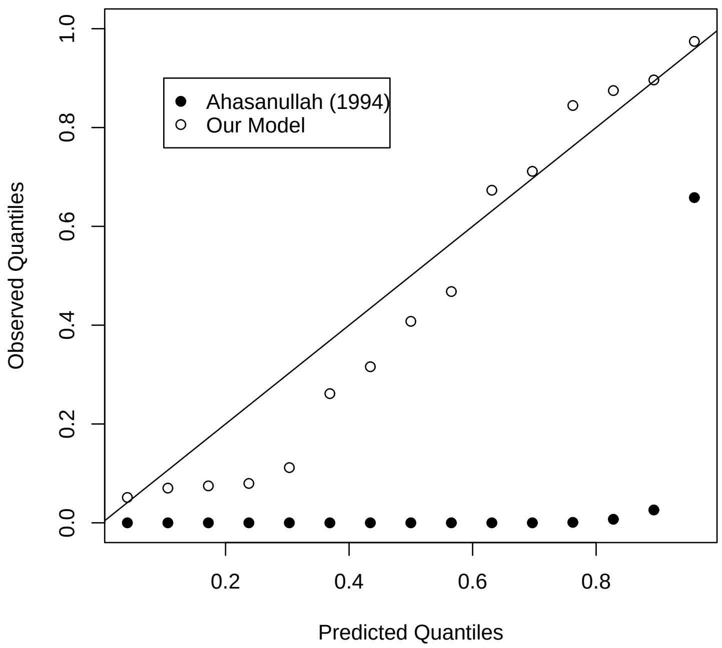


Figure 4.1: Q-Q Plot of the Olympic data; Our model versus Ahsanullah (1994)

Figure 4.1 shows the quantile plots of our model and that of Ahsanullah, [5]. As can be observed, the fit of our model is better than that of Ahsanullah, [5].

### 4.3.2 Estimation of Records

We shall now proceed to develop the records of the Olympic data along with the appropriate statistical estimates.

**Olympic Data**

First, using the entire data presented in Table 4.1, we have the following record values and inter record times

$$x_i = 82.2, 73.6, 72.4, 71, 66.8, 65.9, 62, 61, 59.5, 58.59, 55.65, 54.79, 54.64, 54.5, 53.83$$

and

$$k_i = 1, 1, 1, 1, 1, 3, 1, 1, 2, 1, 1, 3, 1, 1, 1$$

| Year | Time (in seconds) | Year | Time (in seconds) |
|---|---|---|---|
| 1912 | 82.20 | 1968 | 60.00 |
| 1920 | 73.60 | 1972 | 58.59 |
| 1924 | 72.40 | 1976 | 55.65 |
| 1928 | 71.00 | 1980 | 54.79 |
| 1932 | 66.80 | 1984 | 55.92 |
| 1936 | 65.90 | 1988 | 54.93 |
| 1948 | 66.30 | 1992 | 54.64 |
| 1952 | 66.80 | 1996 | 54.50 |
| 1956 | 62.00 | 2000 | 53.83 |
| 1960 | 61.20 | 2004 | 53.84 |
| 1964 | 59.50 | | |

Table 4.1: Olympic record of women's 100 meter free style swimming results.

The maximum likelihood estimates, mle, along with the standard error are $\hat{\theta} = 72.92(1.61)$, $\hat{\beta} = 7.40(1.35)$.

Table 4.2 gives the parameter estimates of our model, that is, using record values versus complete data analysis. As can be observed, the estimates are very good.

| | Our Model | Complete Data |
|---|---|---|
| $\hat{\theta}$ | 72.92 | 78.61 |
| $\hat{\beta}$ | 7.40 | 5.66 |

Table 4.2: Parameter Estimate For the Olympic data from 1912 to 2004

Using equation (4.2.18), the next record is expected to be 53.64 minutes.

Next, we compare our analysis with that of Ahsanullah, [5]. From the data presented in Table 4.1, Ahsanullah, [5] used the lower records, from 1912 to 1980, that is,

$$82.2, 73.6, 72.4, 71, 66.8, 65.9, 62, 61, 59.5, 58.59, 55.65, 54.79$$

to obtain $\hat{\theta} = 78.74$, $\hat{\beta} = 3.97$ as the best linear unbiased estimates (blue).

Using the method present described in Section 2, we obtain from Table 4.1 the record values and record times respectively for the period 1912 to 1980 as

$$x_i = 82.2, 73.6, 72.4, 71, 66.8, 65.9, 62, 61, 59.5, 58.59, 55.65, 54.79$$

and

$$k_i = 1, 1, 1, 1, 1, 3, 1, 1, 2, 1, 1, 1.$$

Using equations (4.2.5) and (4.2.6), we obtain the estimates (standard errors) for the 100 meter free style women's Olympic swimming data $\hat{\theta} = 71.95(1.75)$, $\hat{\beta} = 6.81(1.38)$.

Table 4.3 presents a summary of the record estimates obtained using the methods describe above versus those of Ahsanullah, [5].

| | Our Model | Ahsanullah (1994) |
|---|---|---|
| $\hat{\theta}$ | 71.95 | 78.74 |
| $\hat{\beta}$ | 6.81 | 3.97 |
| AIC | 94.5 | 103.61 |

Table 4.3: Parameter Estimate For the Olympic data from 1912 to 1980

Observe from Table 4.3 that based on the AIC, our results are better than those of Ahsanullah, [5].

| | | | | Difference | |
|---|---|---|---|---|---|
| s | Our model | Ahsanullah's model | Actual | Our model | Ahsanullah's model |
| 13 | 54.71 | 54.39 | 54.64 | +.07 | -.25 |
| 14 | 54.23 | 53.91 | 54.50 | -.27 | -.59 |
| 15 | 53.74 | 53.46 | 53.83 | -.09 | -.37 |
| 16 | 53.28 | 53.01 | – | – | – |
| 17 | 52.86 | 52.56 | – | – | – |

Table 4.4: Prediction of next record

Table 4.4 presents the next 5 swimming predictions using our model and are compared with those presented in Ahsanullah, [5], along with the actual record values. Our model performs better than that of the original conclusion.

Furthermore, we summarize what actually happened in the subject matter in the 1992, 2000, and 2004 Olympic games:

- China's Zhuang Yong, broke the record in the 1992 Olympic games in Barcelona, Spain in 54.64 minutes. Our model was off by 7 hundredth of a second while Ahasanullah, [5], was off by 25 hundredth of a second.

- China's Le Jingyi broke the 2000 Olympic games record in Atlanta, Georgia in 54.50 minutes. Our model was off by 27 hundredth of a second while Ahsanullah, [5], was off by 59 hundredth of a second.

- The next record was broken by Australia Jodie Henry in the 2004 Olympic games in Athens, Greece. She established a new world record of 53.83 minutes. Our model is off by 9 hundredth of a second while Ahasanullah, [5], is off by 37 hundredth of a second.

**Mean Concentration of $SO_2$**

The average monthly mean concentration of $SO_2$ and annual maximums from Long Beach, Californian from 1956 to 1974 are given in Table 4.5.

| Year | Jan | Feb | Mar | Apr | May | Jun | Jul | Aug | Sep | Oct | Nov | Dec | Max |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1956 | 47 | 31 | 44 | 12 | 13 | 3 | 14 | 21 | 33 | 26 | 40 | 32 | 47 |
| 1957 | 22 | 19 | 20 | 32 | 20 | 23 | 18 | 16 | 13 | 14 | 41 | 25 | 41 |
| 1958 | 15 | 13 | 20 | 12 | 24 | 13 | 37 | 20 | 32 | 27 | 27 | 68 | 68 |
| 1959 | 20 | 32 | 20 | 15 | 3 | 6 | 8 | 15 | 17 | 15 | 20 | 20 | 32 |
| 1960 | 22 | 18 | 23 | 20 | 8 | 13 | 14 | 9 | 13 | 16 | 27 | 20 | 27 |
| 1961 | 25 | 20 | 20 | 16 | 10 | 10 | 8 | 10 | 12 | 16 | 14 | 43 | 43 |
| 1962 | 20 | 13 | 15 | 18 | 10 | 1 | 10 | 10 | 11 | 11 | 14 | 7 | 20 |
| 1963 | 12 | 18 | 27 | 21 | 2 | 7 | 4 | 4 | 15 | 10 | 18 | 18 | 27 |
| 1964 | 16 | 10 | 3 | 3 | 19 | 9 | 16 | 25 | 4 | 14 | 18 | 21 | 25 |
| 1965 | 16 | 18 | 9 | 14 | 8 | 10 | 18 | 18 | 14 | 12 | 12 | 12 | 18 |
| 1966 | 27 | 33 | 25 | 10 | 17 | 30 | 13 | 18 | 22 | 15 | 25 | 23 | 33 |
| 1967 | 30 | 40 | 32 | 10 | 8 | 7 | 8 | 26 | 10 | 40 | 18 | 17 | 40 |
| 1968 | 51 | 30 | 18 | 22 | 10 | 19 | 22 | 25 | 26 | 29 | 50 | 40 | 51 |
| 1969 | 37 | 13 | 55 | 14 | 9 | 10 | 13 | 17 | 33 | 13 | 15 | 44 | 55 |
| 1970 | 23 | 19 | 10 | 11 | 15 | 12 | 25 | 40 | 25 | 20 | 12 | 8 | 40 |
| 1971 | 22 | 36 | 20 | 28 | 10 | 15 | 20 | 55 | 38 | 41 | 26 | 25 | 55 |
| 1972 | 30 | 32 | 18 | 27 | 37 | 13 | 23 | 19 | 21 | 31 | 25 | 13 | 37 |
| 1973 | 10 | 8 | 8 | 12 | 11 | 16 | 25 | 16 | 11 | 28 | 10 | 23 | 28 |
| 1974 | 8 | 9 | 9 | 13 | 8 | 14 | 9 | 9 | 25 | 11 | 19 | 15 | 34 |

Table 4.5: Average concentration (pphm) of $SO_2$ from Long Beach, California.

The following lower record values are obtained from the last column of Table 4.5,

$$47, 41, 32, 27, 20, 18.$$

Using the above data, Ahsanullah, [5], obtained the blue for the parameter $\theta$ and $\beta$ to be $\hat{\theta} = 44.3$ and $\hat{\beta} = 15.4$, respectively.

From the last column of Table 4.5, the following record values and inter record times are observed

$$x_i = 47, 41, 32, 27, 20, 18,$$

and

$$k_i = 1, 2, 1, 2, 3, 1.$$

Using equations (4.2.7) and (4.2.8), we obtain the maximum likelihood estimates (standard errors) of $\theta$ and $\beta$, to be $\hat{\theta} = 20.55(3.61)$ and $\hat{\beta} = 8.73(3.01)$, respectively.

The results of the estimates of $\theta$ and $\beta$ are summarized in Table 4.6.

| | Our Model | Ahsanullah (1994) | Roberts (1979) |
|---|---|---|---|
| $\hat{\theta}$ | 20.55 | 44.30 | 31.50 |
| $\hat{\beta}$ | 8.73 | 15.4 | 12.34 |
| AIC | 53.63 | 53.99 | |

Table 4.6: Parameter estimates of the $SO_2$ data sets

## 4.4 Conclusion

In the present study, we have introduced the concepts of Records for a given phenomenon that is probabilistically characterized by the Gumbel/double exponential pdf. we have developed the analytical structure of the records, along with their maximum likelihood estimates. We have illustrated the usefulness of our analytical developments in two interesting classical applications, namely, the 100 meter women's free style swimming results of the Olympic games from 1912 to 2004, given in Table 4.1 and the one hour mean concentration of Sulfur dioxide ($SO_2$) from the city of Long Beach, California, given in Table 4.5. In addition we have shown that the subject information fits very well the Gumbel distribution and we have compared our record estimates with those published in the literature. Finally, the estimates of our analysis and predictions are better than previous results.

# 5 Record values from the Half Logistic Probability Distribution

## 5.1 Introduction

The logistic probability distribution is very useful in many areas of human endeavor. Berkson, [10], [12], used the subject distribution extensively in analyzing bioassay and quantal response data. The works of Ojo, [42], McDonald and Xu, [39], are of interest among many publications on logistic probability distribution. The simplicity of the logistic probability distribution and its importance as a growth curve have made it even more important in statistical analysis and modeling. The shape of the logistic probability distribution which is similar to that of the Gaussian probability distribution makes it simpler and also profitable on suitable occasions to replace the Gaussian probability distribution by the logistic distribution with negligible errors in the respective theories and applications.

Balakrishnan and Chan, [9], have studied the best linear unbiased estimator of the scaled parameter half logistic probability distribution using double type II censored samples. In this Chapter, we shall study the theory of **records** for the half logistic probability distribution. In addition to developing the record estimates of the parameters, we shall illustrate their usefulness by applying the results to the failure time data of Boeing 720 airplane. This data has been initially analyzed by Balakrishnan and Chan, [9].

## 5.2 Analytical Formulation of the Record Model

Let $Y$ be a complete random variable having a half logistic probability density function given by

$$f(x) = \frac{1}{b} \frac{2e^{-\frac{x-\alpha}{b}}}{\left(1+e^{-\frac{x-\alpha}{b}}\right)^2}, \tag{5.2.1}$$

where $x \geq \alpha \geq 0$, $b > 0$.

The cumulative probability distribution function of equation 5.2.1 is given by

$$F(x) = \frac{1 - e^{-\frac{x-\alpha}{b}}}{1 + e^{-\frac{x-\alpha}{b}}}. \tag{5.2.2}$$

For the record-breaking samples, $x_1, k_1, \ldots, x_r, k_r$, using equations (4.2.2), (5.2.1) and (5.2.2), the likelihood function for the half logistic probability distribution function is given by

$$\begin{aligned} L(\omega) &= \prod_{i=1}^{r} \frac{1}{b} \frac{2e^{-\omega_i}}{(1+e^{-\omega_i})^2} \left( \frac{2e^{-\omega_i}}{1+e^{-\omega_i}} \right)^{k_i - 1} \\ &= \prod_{i=1}^{r} \frac{1}{b} \frac{1}{1+e^{-\omega_i}} \left( \frac{2e^{-\omega_i}}{1+e^{-\omega_i}} \right)^{k_i} \end{aligned} \tag{5.2.3}$$

where $\omega_i = (x_i - \alpha)/b$.

For convenience, the negative loglikelihood function of expression (5.2.3) is given by

$$-\log L(\omega) = \sum_{i=1}^{r} \log(b) + k_i \omega_i + (1 + k_i) \log(1 + e^{-\omega_i}). \tag{5.2.4}$$

Taking the partial derivatives of equation (5.2.4) with respect to $\alpha$ and $b$ we have

$$\frac{\partial(-\log L(\omega))}{\partial \alpha} = -\frac{1}{b} \sum_{i=1}^{r} \left[ k_i - (1 + k_i) \left( \frac{e^{-\omega_i}}{1 + e^{-\omega_i}} \right) \right] \tag{5.2.5}$$

and

$$\frac{\partial(-\log L(\omega))}{\partial b} = \frac{1}{b} \sum_{i=1}^{r} \left[ 1 - k_i \omega_i + (1 + k_i) \left( \frac{\omega_i e^{-\omega_i}}{1 + e^{-\omega_i}} \right) \right]. \tag{5.2.6}$$

The second partial derivative of equations (5.2.5) and (5.2.6) with respect to $\alpha$, $b$ and $\alpha b$ are given by

$$\frac{\partial^2(-\log L(\omega))}{\partial \alpha^2} = \frac{1}{b^2} \sum_{i=1}^{r} \frac{(1 + k_i) e^{-\omega_i}}{(1 + e^{-\omega_i})^2} \tag{5.2.7}$$

$$\frac{\partial^2(-\log L(\omega))}{\partial b^2} = \frac{1}{b^2}\sum_{i=1}^{r}\left[1-k_i\omega_i+(1+k_i)\left(\frac{\omega_i e^{-\omega_i}}{1+e^{-\omega_i}}\right)\right] + \frac{1}{b^2}\sum_{i=1}^{r} k_i\omega_i+(1+k_i)\frac{-\omega_i e^{-\omega_i}-\omega_i e^{-2\omega_i}+\omega_i^2 e^{-\omega_i}}{(1+\omega_i e^{-\omega_i})^2} \qquad (5.2.8)$$

and

$$\frac{\partial^2(-\log L(\omega))}{\partial\alpha\partial b} = \frac{1}{b^2}\sum_{i=1}^{r} k_i-(1+k_i)\frac{e^{-\omega_i}}{1+e^{-\omega_i}}+(1+k_i)\frac{\omega_i e^{-\omega_i}}{(1+e^{-\omega_i})^2}. \qquad (5.2.9)$$

Equating (5.2.5) and (5.2.6) to zero, we obtain the maximum likelihood estimates $\hat{\alpha}$ and $\hat{b}$ for $\alpha$ and $b$, respectively, and equations (5.2.8) and (5.2.9) become

$$\frac{\partial^2(-\log L(\omega))}{\partial b^2} = \frac{1}{b^2}\sum_{i=1}^{r} 1+\frac{(1+k_i)\omega_i^2 e^{-\omega_i}}{(1+\omega_i e^{-\omega_i})^2}, \qquad (5.2.10)$$

and

$$\frac{\partial^2(-\log L(\omega))}{\partial\alpha\partial b} = \frac{1}{b^2}\sum_{i=1}^{r}\frac{(1+k_i)\omega_i e^{-\omega_i}}{(1+e^{-\omega_i})^2}. \qquad (5.2.11)$$

Using equations, (5.2.7), (5.2.10), and (5.2.11) we obtain the observed information matrix $I(\alpha, b)$, at $(\hat{\alpha}, \hat{b})$, for the half logistic pdf model to be

$$I(\hat{\alpha},\hat{b}) = \frac{1}{\hat{b}^2}\begin{pmatrix} \sum_{i=1}^{r}\frac{(1+k_i)e^{-\omega_i}}{(1+e^{-\omega_i})^2} & \sum_{i=1}^{r}\frac{(1+k_i)\omega_i e^{-\omega_i}}{(1+e^{-\omega_i})^2} \\ \sum_{i=1}^{r}\frac{(1+k_i)\omega_i e^{-\omega_i}}{(1+e^{-\omega_i})^2} & \sum_{i=1}^{r}1+\frac{(1+k_i)\omega_i^2 e^{-\omega_i}}{(1+\omega_i e^{-\omega_i})^2} \end{pmatrix}. \qquad (5.2.12)$$

Note that the inverse of $I(\hat{\alpha}, \hat{b})$ from equation 5.2.12 gives the variance covariance matrix of $\hat{\alpha}, \hat{b}$.

Next, we proceed to obtain the estimates of the parameters that are inherent in (5.2.1) as follows: For the complete sample $X_1 = y_1, \ldots, X_n = y_n$, from the half logistic probability density function given by (5.2.1), we can write the negative log-likelihood function as

$$-\log L(\zeta) = -\sum_{i=1}^{n}\log(b)+\zeta+2\log(1+\exp(-\zeta), \qquad (5.2.13)$$

where $\zeta_i = (y_i - \mu/b$.

Taking the partial derivative of (5.2.13) with respect to $\mu$ and $b$, we have

$$\frac{\partial(-\log L(\zeta))}{\partial \mu} = \frac{1}{b}\sum_{i=1}^{n}\left[1 - \frac{2e^{-\zeta_i}}{1+e^{-\zeta_i}}\right] \tag{5.2.14}$$

and

$$\frac{\partial(-\log L(\zeta))}{\partial b} = -\frac{1}{b}\sum_{i=1}^{n}\left[1 - \zeta_i + \frac{2e^{-\zeta_i}}{1+e^{-\zeta_i}}\right]. \tag{5.2.15}$$

Let expressions (5.2.14) and (5.2.15) equal to zero and taking the second partial derivative with respect to $\mu$, $b$ and $\mu b$, we have

$$\frac{\partial^2(-\log L(\zeta))}{\partial \mu^2} = \frac{1}{b^2}\sum_{i=1}^{n}\frac{2e^{-\zeta_i}}{(1+e^{-\zeta_i})^2}, \tag{5.2.16}$$

$$\frac{\partial^2(-\log L(\zeta))}{\partial b^2} = \frac{1}{b^2}\sum_{i=1}^{n}\left[1 + \frac{2\hat{\zeta}_i^2 e^{-\zeta_i}}{(1+e^{-\hat{\zeta}_i})^2}\right], \tag{5.2.17}$$

and

$$\frac{\partial^2(-\log L)}{\partial \mu \partial b} = \frac{1}{b^2}\sum_{i=1}^{n}\frac{2\zeta_i e^{-\zeta_i}}{(1+e^{-\zeta_i})^2}. \tag{5.2.18}$$

Using equations (5.2.16), (5.2.17), and (5.2.18) we obtain the observed information matrix at $(\hat{\mu}, \hat{b})$, that is, $I(\mu, b)$, for the half logistic pdf model to be

$$I(\hat{\mu}, \hat{b}) = \frac{1}{\hat{b}^2}\begin{pmatrix} \sum_{i=1}^{n}\frac{2e^{-\hat{\zeta}_i}}{(1+e^{-\hat{\zeta}_i})^2} & \sum_{i=1}^{n}\left[1+\frac{2\hat{\zeta}_i^2 e^{-\hat{\zeta}_i}}{(1+e^{-\hat{\zeta}_i})^2}\right] \\ \sum_{i=1}^{n}\left[1+\frac{2\hat{\zeta}_i^2 e^{-\hat{\zeta}_i}}{(1+e^{-\hat{\zeta}_i})^2}\right] & \sum_{i=1}^{n}\frac{2\hat{\zeta}_i e^{-\hat{\zeta}_i}}{(1+e^{-\hat{\zeta}_i})^2} \end{pmatrix}.$$

Thus, we can use the estimates in equations (5.2.5) and (5.2.6) to predict future observations of the phenomenon of interest. We can accomplish this by using the return levels, that is,

$$F(x_s) = 1/s, \quad s > r$$

which gives

$$x_s = \hat{\alpha} - \hat{b}\log\left\{\frac{s-1}{s+1}\right\} \tag{5.2.19}$$

## 5.3 Application

We shall take $\alpha$ to be zero for comparison purpose consistent with published results,Balakrishnan and Chan, [9]. We consider two applications in this section and compare the results of our analysis with results of other statistical analysis.

**Application 1: Boeing 720 Airplane Data**

The following data which has been initially analyzed by Balakrishnan and Chan, [9], are the failure times of air conditioning equipment in a Boeing 720 airplane.

$$74, 57, 48, 29, 502, 12, 70, 21, 29, 386, 59, 27, 153, 26, 326.$$

Balakrishnan and Chan, [9], have shown that the above data fits the half logistic probability distribution function quite well. Using the scale half logistic probability distribution, that is, letting $\alpha = 0$ in equation (5.2.1), they obtained the best linear unbiased estimates for $b$ to be 90.92 with a standard error of 19.66.

The following record values and record times can be obtained from the above data, that is,

$$x_i = 74, 57, 47, 29, 12$$

with

$$k_i = 1, 1, 1, 2, 1.$$

Using equations (5.2.5) and (5.2.6) and letting $\mu = 0$ in (5.2.1) we have the maximum likelihood estimate and their standard error of $b$ is $\hat{b}_{mle} = 88.23$ with a standard error of 20.22.

Table 5.1 presents a comparison of the parameter estimate of our model with that of Balakrishnan and Chan, [9]. As can be observed from the table below, even though we have a reduced sample, our model performs equally well with the maximum likelihood estimate (MLE).

| Method | Estimate(Standard Error) |
|---|---|
| MLE using Records | 88.23(20.22) |
| BLUE using Complete data | 90.92(19.66) |
| MLE | 93.58 |

Table 5.1: Estimate of $b$ (the Boeing 720 Airplane Data)

The results of the analysis are displayed in Table 5.1. As can be observed, the results are not much different from each other despite the fact that we have reduced data.

**Application 2: Electrical Insulation Data**

The next application represent the failure times, in minutes, for a specific type of electrical insulation material that was subjected to a continuously increasing voltage stress (Lawless, [32], p.138):

$$12.3, 21.8, 24.4, 28.6, 43.2, 46.9, 70.7, 75.3, 95.5, 98.1, 138.6, 151.9.$$

A random sample of this data was obtained to be

$$138.6, 75.3, 95.5, 151.9, 46.9, 70.7, 24.4, 21.8, 28.6, 43.2, 12.3, 98.1.$$

We shall proceed to fit the scare-parameter half logistic distribution with pdf (5.2.1), that is, when $\alpha = 0$ to the data.

The best linear unbiased estimate for $b$ has been obtained using the complete data, Balakrishnan and Chan, [9]. Balakrishnan and Chan, [9] obtained the blue (standard error) for $b$ to be 50.50(12.68). Using records, we will obtain the blue for $b$ and compare our result with that of Balakrishnan and Chan, [9].

Using equation (5.2.15), we obtain the maximum likelihood estimate for the scale parameter of the half logistic pdf to be $\hat{b} = 47.42$.

The appropriateness of the assumption of the half logistic distribution for the above data is checked using the Q-Q plot given by Figure 5.1. In Figure 5.1, we plotted the quantile from the half logistic probability distribution versus the empirical quantiles. As can be observed from Figure 5.1, the half logistic distribution fits the data extremely well. The value of the correlation coefficient in the Q-Q plot is 0.98.

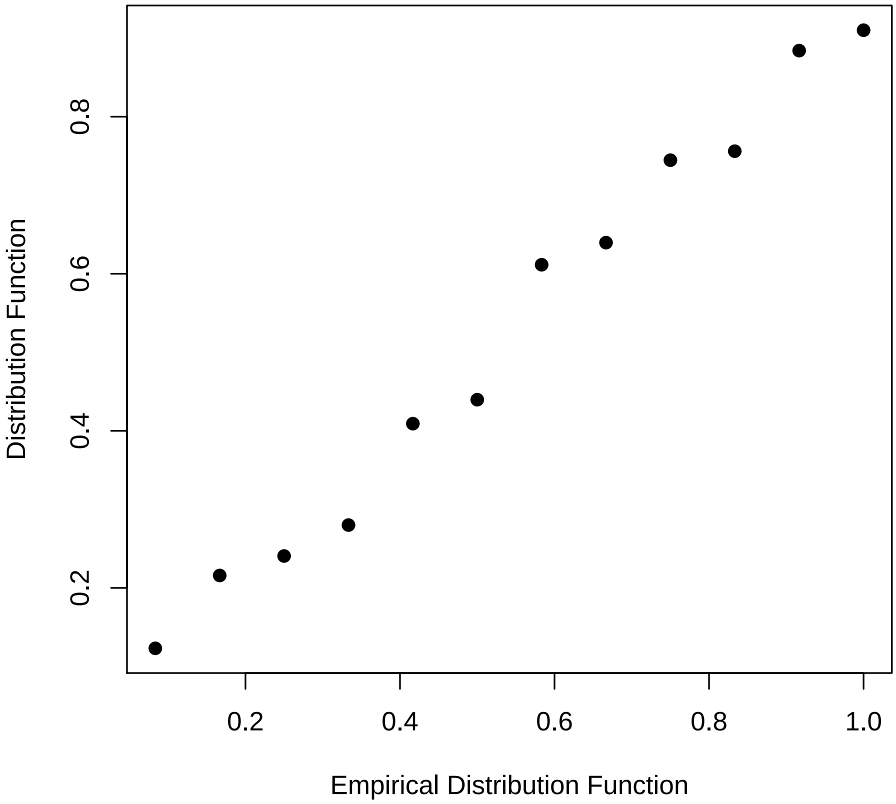


Figure 5.1: QQ plot of the data to verify goodness of fit.

Form the random data above, the following record values and record times have been obtained to be

$$x_i = 138.6, 75.3, 46.9, 24.4, 21.8, 12.3$$

and

$$k_i = 1, 3, 2, 1, 3, 1$$

Using equation (5.2.6) we calculated the maximum likelihood estimates and their standard errors of $b$ to be $\hat{b}_{mle} = 52.22$ with a standard error of 14.22. The results are summarized in Table 5.2 below.

Table 5.2 presents estimates of the data using our model, with those of Balakrishnan and Chan, [9]. As can be observed, we have successfully presented another method to estimate the parameter of the scale half logistic probability distribution. The subject method, records

| Method | Estimate(Standard Error) |
|---|---|
| MLE using Records | 52.23(14.22) |
| BLUE using Complete data | 50.50(12.68) |
| MLE | 47.42 |

Table 5.2: Estimate of $b$ (the Electrical Insulation)

and inter records, give good result with easier computational process and smaller samples.

## 5.4 Conclusion

In the present study, we have introduced the concepts of **records** for a given random phenomenon whose data is probabilistically characterized by the half logistic probability distribution function. We have developed the analytical methodology of this probability distribution using record breaking data. The usefulness of the theoretical results are applied to two real world problems. The air conditioning system of Boeing 720 and the Electrical Insulation Data that were initially analyzed by Balakrishnan and Chan, [9]. In addition to showing that both data sets fit the half logistic probability distribution, our estimate of the half logistic parameter are as good as the published estimates using classical approach. Thus, the present method offers an advantage in the computational aspect as well as reduced sample sizes.

# 6 Lower Generalized Order Statistics

## 6.1 Introduction

Kamps, [29], introduced and gave detailed theory of the so-called generalized order statistics (gos) as a unified approach to **order statistics**, that is, sorting the values of random variables in increasing order; **record values**: and **sequential order statistics**, that is, $(n-k+1)$th order statistics, for example a system will work as long as k components function.

In this chapter, we will consider **lower generalized order statistics** (lgos). Lgos is a generalization of **order statistics**, that is, sorting the values of random variables in decreasing order; **lower record values**. For a connection between generalized order statistics and lower generalized order statistics, see Burkschat et al., [13].

Suppose $\{X^*(1,n,m,k),\ldots,X^*(n,n,m,k)\}$, where $k \geq 1$, are real numbers, denote $n$ lgos from an absolute continuous cumulative distribution function with cdf $F(x)$ and corresponding probability density function (pdf) $f(x) = dF(x)/dx$, the joint pdf $f^*_{1,\ldots,n}(x_1,\ldots X_n)$ of $n$ lgos $X^*(1,n,m,k), X^*(2,n,m,k),\ldots,X^*(n,n,m,k)$ is

$$f^*_{12\ldots n}(x_1,x_2,\ldots,x_n) = k\prod_{j=1}^{n-1}\gamma_j\prod_{i=1}^{n-1}(F(x_i))^m(F(x_n))^{k-1}f(x_1)f(x_n) \qquad (6.1.1)$$

for

$$F^{-1}(1) \geq x_1 \geq x_2 \geq \ldots \geq x_n \geq F^{-1}(0),$$

$m > -1$,

$$\gamma_r = k + (n-r)(m+1), r = 1,2,\ldots,n-1, k \geq 1,$$

and $n$ is a positive integer.

Observe that equation (6.1.1) has as special case order statistics if we let $k = 1$ and $m = 0$, while lower record values are special cases of equation (6.1.1) if we let $k = 1$ and

$m = -1$.

Integrating out $x_1, \ldots, x_{r-1}, x_{r+1}, \ldots, x_n$ from (6.1.1), we obtain the pdf $f_{r,n,m,k}$ of $X^*(r,n,m,k)$ as

$$f^*_{r,n,m,k}(x) = \begin{cases} \frac{c_{r-1}}{\Gamma(r)}(F(x))^{\gamma_r-1}(g_m F(x))^{r-1} f(x), & F^{-1}(1) > x > F^{-1}(0), \\ 0, & \text{otherwise.} \end{cases} \tag{6.1.2}$$

where

$$c_r = \prod_{i=1}^{r} \gamma_i,$$

and

$$g_m(x) = \begin{cases} \frac{1}{m+1}(1 - x^{m+1}) & \text{for } m \neq -1, \\ -\ln x & \text{for } m = -1. \end{cases}$$

The joint pdf of $X(r,n,m,k)$ and $X(s,n,m,k)$, $r < s$, is expressed from equation (6.1.1) as

$$\begin{aligned} f^*_{r,s,n,m,k}(x,y) &= \frac{c_{s-1}}{(r-1)!(s-r-1)!}(F(x))^m (g_m(F(x)))^{r-1} \\ &\times [g_m(F(y)) - g_m(F(x))]^{s-r-1} F(y)^{\gamma_s-1} f(x) f(y), \end{aligned} \tag{6.1.3}$$

for $F^{-1}(0) < y < x < F^{-1}(1)$, and

$$f_{r,s,n,m,k}(x,y) = 0,$$

otherwise.

## 6.2 Distributional Properties Of Lower Generalized Order Statistics

The cumulative distribution function $F^*_{r,n,m,k}(x)$ of $X^*(r,n,m,k)$ can be expressed in terms of incomplete beta function if $m > -1$ and in terms of incomplete gamma function if $m = -1$, Mbah and Ahsanullah, [35].

**Lemma 6.2.1** *If $F(x)$ is absolutely continuous, then*

$$F^*_{r,n,m,k}(x) = \begin{cases} I_{\alpha_1^*(x)}\left(\frac{\gamma_r}{m+1}, r\right), & m > -1, \\ 1 - \Gamma_{\alpha_2^*(x)}(r), & m = -1. \end{cases}$$

*where*

$$\alpha_1^*(x) = (F(x))^m,$$

$$I_x(p,q) = \frac{1}{B(p,q)} \int_0^x u^{p-1}(1-u)^{q-1}du, 0 \le u \le \infty,$$

$$\alpha_2^*(x) = -k\ln(F(x)),$$

$$\Gamma_x(r) = \frac{1}{\Gamma(r)} \int_0^x u^{r-1}e^{-u}du,$$

*and*

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

*Proof.* Observe that

$$\begin{aligned} \Gamma(r+\frac{\gamma_r}{m+1}) &= (\frac{k}{m+1}+n-1)(\frac{k}{m+1}+n-2)\dots(\frac{k}{m+1}+n-r)\Gamma(\frac{\gamma_r}{m+1}) \\ &= \frac{c_{r-1}}{(m+1)^r}\Gamma(\frac{\gamma_r}{m+1}). \end{aligned} \tag{6.2.4}$$

and

$$\int_0^\infty w^{a-1}(1-w^b)^{r-1}dw = \frac{1}{b}B(a/b,r), \quad \gamma_r + i(m+1) = \gamma_{r-1} \tag{6.2.5}$$

Using equation (6.1.2), and for $m > -1$,

$$\begin{aligned} F^*_{r,n,m,k}(x) &= \frac{c_{r-1}}{\Gamma(r)} \int_{-\infty}^x (F(u))^{\gamma_r - 1}\left[\frac{1-(F(u))^m}{m+1}\right]^{r-1} f(u)du \\ &= \frac{c_{r-1}}{\Gamma(r)(m+1)^r} \int_0^{(F(x))^{m+1}} t^{\frac{\gamma_r}{m+1}-1}(1-t)^{r-1}dt \\ &= \frac{c_{r-1}B(r,\frac{\gamma_r}{m+1})}{\Gamma(r)(m+1)^r} I_{\alpha_1^*(x)}\left(\frac{\gamma_r}{m+1}, r\right) \\ &= \frac{c_{r-1}\Gamma(r)\Gamma(\frac{\gamma_r}{m+1})}{\Gamma(r)(m+1)^r\Gamma(r+\frac{\gamma_r}{m+1})} I_{\alpha_1^*(x)}\left(\frac{\gamma_r}{m+1}, r\right) \\ &= I_{\alpha_1^*(x)}\left(\frac{\gamma_r}{m+1}, r\right). \end{aligned}$$

For $m = -1$, we have that

$$F^*_{r,n,m,k}(x) = \frac{k^r}{\Gamma(r)} \int_{-\infty}^x (F(u))^{k-1}(-\ln(F(u))^{r-1})f(u)du$$

$$
\begin{aligned}
&= 1 - \frac{k^r}{\Gamma(r)} \int_x^{\infty} (F(u))^{k-1}(-\ln(F(u))^{r-1}) f(u) du \\
&= 1 - \frac{1}{\Gamma(r)} \int_x^{-k\ln(F(x))} t^{r-1} e^{-t} dt \\
&= 1 - \Gamma_{\alpha_2^*}(r).
\end{aligned}
$$

■

Using equations (6.1.2) and (6.1.3) we have that the conditional pdf $f^*_{s|r}(y|x)$ of $X^*(s,n,m,k)$ given $X^*(r,n,m,k) = x$, that is,

$$
\begin{aligned}
f^*_{s|r}(y|x) &= \frac{f^*_{s,m,n,k}(x,y)}{f^*_{r,m,n,k}(x)} \\
&= \frac{c_{s-1}[g_m(F(y)) - g_m(F(x))]^{s-r-1}(F(y))^{\gamma_s-1} f(y)}{c_{r-1}(s-r-1)!(F(x))^{\gamma_r+1}}.
\end{aligned}
\tag{6.2.6}
$$

The following lemma of Ahsanullah, [3], will be used in this chapter;

**Lemma 6.2.2** *For $m > -1$,*

$$
\gamma_{r+1}(F^*_{r,n,m,k}(x) - F^*_{r+1,n,m,k}(x)) = \frac{F(x)}{f(x)} f^*_{r+1,n,m,k}(x)
$$

*and for $m = -1$, we have*

$$
k(F^*_{r,n,m,k}(x) - F^*_{r+1,n,m,k}(x)) = \frac{F(x)}{f(x)} f^*_{r+1,n,m,k}(x)
$$

**Lemma 6.2.3** *The sequence of lgos $\{X^*(1,n,m,k), \ldots, X^*(n,n,m,k)\}$, which correspond to a continuous distribution function $F$, forms a continuous Markov chain.*

*Proof.* For convenience we shall give the proof for absolute continuous distributions. It can easily be shown from equation (6.1.1) that the joint pdf of $s$ lgos $\{X^*(1,n,m,k), X^*(2,n,m,k), \ldots, X^*(r,n,m,k), X^*(s,n,m,k)\}$ for $r < s$ can be written as

$$
\begin{aligned}
& f^*_{1,2,\ldots,r,s,m,n,k}(x_1, x_2, \ldots, x_r x_s) \\
&= \frac{c_{s-1}}{(s-r-1)!} \left[ \prod_{j=1}^{r} (F(x_j))^m f(x_j) \right] \\
&\quad [g_m(F(x_s)) - g_m(F(x_r))]^{s-r-1} (F(x_s))^{\gamma_s-1} f(x_s)
\end{aligned}
\tag{6.2.7}
$$

for $F^{-1}(0) < x_s < x_r < \ldots < x_2 < x_1 < F^{-1}(1)$.

Observe that the joint pdf of $r$ lgos $\{X^*(1,n,m,k), X^*(2,n,m,k), \ldots, X^*(r,n,m,k)\}$ is given by

$$f^*_{1,2,\ldots,r,m,n,k}(x_1, x_2, \ldots, x_r) = c_{r-1}\left[\prod_{j=1}^{r-1}(F(x_j))^m f(x_j)\right](F(x_r))^{\gamma_r-1}f(x_r). \tag{6.2.8}$$

Using equations (6.2.7) and (6.2.8), we obtain the conditional pdf of $X(s,n,m,k)$ given $X^*(1,n,m,k) = x_1, X^*(2,n,m,k) = x_2, \ldots, X^*(r,n,m,k) = x_r$ as

$$\begin{aligned} & f^*_{1,2,\ldots,r,m,n,k}(x_s|x_1, x_2, \ldots, x_r) \\ = & \frac{f^*_{1,2,\ldots,r,s,m,n,k}(x_1, x_2, \ldots, x_r, x_s)}{f^*_{1,2,\ldots,r,m,n,k}(x_1, x_2, \ldots, x_r)} \\ = & \frac{c_{s-1}[g_m(F(x_s)) - g_m(F(x_r))]^{s-r-1}(F(x_s))^{\gamma_s-1}f(x_s)}{c_{r-1}(s-r-1)!(F(x_r))^{\gamma_r+1}}. \end{aligned} \tag{6.2.9}$$

It follows that the right hand side, RHS of equation (6.2.9) coincides with equation (6.2.6).

■

For $F^{-1}(0) < x_s < x_r < F^{-1}(1)$, the conditional pdf $f_{r+1}(x_{r+1}|x_r)$ of $X(r+1,n,m,k)$ given $X^*(r,n,m,k) = x_r$ can be expressed as

$$f^*_{r+1|r}(x_{r+1}|x_r) = \gamma_{r+1}\left(\frac{F(x_{r+1})}{F(x_r)}\right)^{\gamma_{r+1}-1}\frac{f(x_{r+1})}{F(x_r)}.$$

The conditional probability $P(X(r+1,n,m,k) \geq y|X(r,n,m,k) = x)$ is

$$\begin{aligned} & P(X(r+1,n,m,k) \geq y|X(r,n,m,k) = x) \\ = & \int_y^\infty \gamma_{r+1}\left(\frac{F(x_{r+1})}{F(x)}\right)^{\gamma_{r+1}-1}\frac{f(x_{r+1})}{F(x)}dx_{r+1} \\ = & \gamma_{r+1}\int_{\frac{F(y)}{F(x)}}^\infty t^{\gamma_{r+1}-1}du \\ = & \left(\frac{F(y)}{F(x)}\right)^{\gamma_{r+1}}. \end{aligned}$$

Now in what follows, we shall study some important properties of the power function probability distribution.

### 6.3 Properties of the Power Function Probability Distribution

The following result by Ahsanullah, [3], will be used in this section;

**Theorem 6.3.1** *Let $X$ be an absolute continuous (with respect to the Lebesque Measure) bounded random variable with strictly increasing cdf $F(x)$ and pdf $f(x)$. Without any loss of generality we will take $F(0) = 0$ and $F(1) = 1$. Then the following two statements are equivalent*

1. *$X$ is uniformly distributed rv in* $(0, 1)$
2. $X^*(r+1, n, m, k) =^d X^*(r, n, m, k)W_{r+1}$, *where $W_{r+1}$ is independent of $X^*(r, n, m, k)$ and the pdf of $W_{r+1}$ is $f_{r+1}(w) = \gamma_{r+1}w^{\gamma_{r+1}-1}, 0 < w < 1$.*

Ahsanullah, [1], obtained the following results which will be used in this section;

**Theorem 6.3.2** *Let $X$ be an absolute continuous (with respect to the Lebesque Measure) bounded random variable with cdf $F(x)$ and pdf $f(x)$. Without any loss of generality we will take $F(0) = 0$ and $F(1) = 1$. Then the following two statements are equivalent*

1. *$X$ has a power function distribution with $F(x) = x^\delta$, $0 < x < 1$, $\delta > 0$*
2. $X^*(r+1, n, m, k) =^d X^*(r, n, m, k)W_{r+1}$,

*where $W_{r+1}$ is independent of $X^*(r, n, m, k)$ and the pdf of $W_{r+1}$ is*

$$f_{r+1}(w) = \delta\gamma_{r+1}w^{\delta\gamma_{r+1}-1}, 0 < w < 1, \delta > 0.$$

The following result, Mbah and Ahsanullah, [35], is a characterization of the power function probability distribution using lgos;

**Theorem 6.3.3** *Let $X$ be a bounded non-negative absolute continuous (with respect to the Lebesque Measure) random variable. We assume Without loss of generality $F(0) = 0$ and $F(1) = 1$, then the following two statements are equivalent*

1. *$X$ has a power function distribution with $F(x) = x^\delta$, $0 < x < 1$, $\delta > 0$*
2. *for some $r$ and $s$, $1 \leq r < s \leq n$, $X^*(s, n, m, k)/X^*(r, n, m, k)$ and $X^*(r, n, m, k)$ are independent.*

*Proof.* (1) implies (2)

Using the transformation $U = X^*(r,n,m,k)$ and $V = X^*(s,n,m,k)/X^*(r,n,m,k)$ and $F(x) = x^\alpha$, we obtain from equation (6.1.3), the joint pdf of $U$ and $V$, for $m > -1$, as

$$\begin{aligned} f^*_{r,s,n,m,k}(u,v) &= \frac{c_{s-1}\delta^2}{\Gamma(r)\Gamma(s-r)} u^{\delta m}\left[\frac{1-u^\delta}{m+1}\right]^{r-1} u \\ &\quad .\left[\frac{u^{\delta(m+1)}-(uv)^{\delta(m+1)}}{m+1}\right]^{s-r-1} \\ &\quad .(uv)^{\gamma_s-1}u^{\delta-1}(uv)^{\delta-1}. \end{aligned}$$

For $m = -1$, we have that

$$\begin{aligned} f^*_{r,s,n,m,k}(u,v) &= \frac{k^s}{\Gamma(r)\Gamma(s-r)} F(u)^{-1}[-\ln(F(u))]^{r-1} \\ &\quad .[\ln(F(u)) - \ln(F(uv))]^{s-r-1}F(uv)^{k-1}uf(u)f(uv) \\ &= \frac{k^s\delta^2}{\Gamma(r)\Gamma(s-r)} u^{-\delta}[-\delta\ln(u)]^{r-1}u[\delta\ln(u)-\delta\ln(uv)]^{s-r-1} \\ &\quad .(uv)^{\delta(k-1)}\delta^2u^{\delta-1}(uv)^{\delta-1} \\ &= \frac{\delta^2k^2}{\Gamma(r)\Gamma(s-r)}(-\delta\ln(u))^{r-1}(-\delta\ln(v))^{s-r-1}(uv)^{\delta k-1}. \end{aligned} \tag{6.3.10}$$

Thus, if $X$ has cdf $F(x) = x^\delta$, then $U$ and $V$ are independent.

To prove (2)*implies*(1), using equation (6.1.3) we have the joint pdf of $U$ and $V$, for $m > -1$, that is,

$$\begin{aligned} f^*_{r,s,n,m,k}(u,v) &= \frac{c_{s-1}}{\Gamma(r)\Gamma(s-r)}(F(u))^m g_m^{r-1}(F(u)) \\ &\quad .[g_mF(uv) - g_m(F(u))]^{s-r-1} \\ &\quad .(F(uv))^{\gamma_s-1}f(u)f(uv)u. \end{aligned} \tag{6.3.11}$$

Using equations (6.1.2) and (6.3.11), the conditional pdf of $V|U = u$, for $m > -1$, is given by

$$\begin{aligned} f^*_{V|U}(v|U=u) &= \frac{c_{s-1}}{c_{r-1}(m+1)^{s-r-1}\Gamma(s-r)}\left(\frac{F(uv)}{F(u)}\right)^{\gamma_s-1} \\ &\quad .\left[1-\left(\frac{F(uv)}{F(u)}\right)^{m+1}\right]^{s-r-1}\frac{f(uv)}{F(u)}u, \end{aligned} \tag{6.3.12}$$

for $0 < u \leq 1, 0 \leq v \leq 1$. Integrating equation (6.3.12) with respect to $v$ from 0 to $v_0$ we obtain

$$\begin{aligned} F^*_{V|U}(v_0|U=u) &= \frac{(m+1)^{r+1-s}c_{s-1}}{c_{r-1}\Gamma(s-r)} \int_0^{v_0} \Big\{ \left(\frac{F(uv)}{F(u)}\right)^{\gamma_s-1} \\ &\quad . \left[1-\left(\frac{F(uv)}{F(u)}\right)^{m+1}\right]^{s-r-1} \frac{f(uv)}{F(u)} u dv \Big\} \\ &= I_{\left(\frac{F(uv_0)}{F(u)}\right)^{m+1}} \left(\frac{\gamma_s}{m+1}, s-r\right). \end{aligned} \tag{6.3.13}$$

Since $U$ and $V$ are independent, we can write

$$I_{\left(\frac{F(uv_0)}{F(u)}\right)^{m+1}} \left(\frac{\gamma_s}{m+1}, s-r\right) = G(v_0),$$

where $G(v_0)$ is a function of $v_0$ only. Letting $u \to 1$, we have that

$$I_{\left(\frac{F(uv_0)}{F(u)}\right)^{m+1}} \left(\frac{\gamma_s}{m+1}, s-r\right) = I_{(F(v_0))^{m+1}} \left(\frac{\gamma_s}{m+1}, s-r\right),$$

for all $u$, $0 < u \leq 1$ and almost all $v_0$, $0 \leq v_0 \leq 1$. Thus,

$$F(uv_0) = F(u)F(v_0) \tag{6.3.14}$$

for all $u$, $0 < u \leq 1$, and almost all $v_0$, $0 \leq v_0 \leq 1$.

The non-zero solution of equation (6.3.14) with the condition that $F(x)$ is a probability distribution function with $F(0) = 0$ and $F(1) = 1$, is

$$F(x) = x^\delta, 0 \leq x \leq 1, \delta > 0. \tag{6.3.15}$$

For $m = -1$, the joint pdf of $X^*(r, n, m, k)$ and $X^*(s, n, m, k)$, $r < s$ is expressed from equation (6.1.3) to be

$$\begin{aligned} f^*_{r,s,n,m,k}(u,v) &= \frac{k^s}{\Gamma(r)\Gamma(s-r)} (-\ln(F(x)))^{r-1} \\ &\quad . [\ln(F(x)) - \ln(F(y))]^{s-r-1} \\ &\quad . h(x)(F(y))^{k-1} f(y), \end{aligned}$$

where $h(x) = f(x)/F(x)$.
The joint pdf of $U$ and $V$ is given by

$$\begin{aligned} f^*_{r,s,n,m,k}(u,v) &= \frac{k^s}{\Gamma(r)\Gamma(s-r)}[-\ln(F(u))]^{r-1} \\ &.[\ln(F(u)) - \ln(F(uv))]^{s-r-1} \\ &.h(u)(F(uv))^{k-1} f(uv)u. \end{aligned}$$

The conditional pdf of $V|U = u$ can be written as

$$\begin{aligned} f^*_{r,s,n,m,k}(u,v) &= \frac{k^{s-r}}{\Gamma(s-r)}[\ln(F(u)) - \ln(F(uv))]^{s-r-1} \\ &.\left[\frac{F(uv)}{F(u)}\right]^{k-1} \frac{f(uv)}{F(u)} u. \end{aligned} \tag{6.3.16}$$

Integrating equation (6.3.16) with respect to $v$ from 0 to $v_0$, we obtain

$$\begin{aligned} F^*_{V|U}(v|U=u) &= \frac{k^s}{\Gamma(s-r)} \int_0^{v_0} \Big\{[\ln(F(u)) - \ln(F(uv))]^{s-r-1} \\ &\left[\frac{F(uv)}{F(u)}\right]^{k-1} \frac{f(uv)}{F(u)} udv\Big\}, \\ &= 1 - \frac{1}{\Gamma(s-r)} \int_0^{-k\ln\left(\frac{F(uv_0)}{F(u)}\right)} t^{s-r-1} e^{-t} dt \\ &= 1 - \Gamma_{k\ln\left(\frac{F(uv)}{F(u)}\right)}(s-r), \end{aligned} \tag{6.3.17}$$

for all $u$, $0 < u \leq 1$ and almost all $v_0$, $0 \leq v_0 \leq 1$. Since $U$ and $V$ are independent, we must have

$$\Gamma_{\left(\frac{F(uv_0)}{F(u)}\right)^k}(s-r) = G(v_0),$$

for all $u$, $0 < u \leq 1$ and almost all $v_0$, $0 \leq v \leq 1$.

Letting $u \to 1$, we have that

$$\Gamma_{k\ln\left(\frac{F(uv_0)}{F(u)}\right)}(s-r) = \Gamma_{k\ln F(v_0)}(s-r)$$

for all $u$, $0 < u \leq 1$ and almost all $v_0$, $0 \leq v_0 \leq 1$.
Thus,

$$F(u\nu_0) = F(u)F(\nu_0). \tag{6.3.18}$$

The non-zero solution of equation (6.3.18) with the condition that $F(x)$ is a probability distribution function with $F(0) = 0$ and $F(1) = 1$, is

$$F(x) = x^{\delta}, 0 \leq x \leq 1, \delta > 0.$$

■

**Remark 6.3.4** *For $k = 1$ and $m = 0$, we obtain from Theorem 6.3.3 a characterization of the power function probability distribution based on the independence of the order random values $X_{n-s+1}/X_{n-r+1}$ and $X_{n+r-s,n}$.*
*For $k = 1$, $m = -1$, we obtain a characterization of the power function probability distribution based on the independence of the lower record values $X_{L(n)}/X_{L(r)}$ and $X_{L(s-r)}$*

The following theorem Mbah and Ahsanullah, [35], is an extension of Theorem 6.3.3.

**Theorem 6.3.5** *Let $X$ be a bounded non-negative absolute continuous (with respect to the Lebesque Measure) random variable. We assume Without loss of generality $F(0) = 0$ and $F(1) = 1$, then the following two statements are equivalent*

1. *$X$ has a power function distribution with $F(x) = x^{\delta}$, $0 < x < 1$, $\delta > 0$*
2. *for some $r$ and $s$, $1 \leq r < s \leq n$, $X^*(s, n, m, k)/X^*(r, n, m, k) and X^*(s - r, n - r, m, k)$ are identically distributed and and $F$ belongs to class of continuous function (C).*

*Proof.* (1) implies (2).
For $m > -1$, we have

$$f^*_{U,V}(u, v) = f^*_{V|U=u}(\nu|u) f^*(u)$$

using (6.3.13) we can write the probability distribution function as

$$\begin{aligned} F^*_V(\nu) &= \int_0^1 \int_0^{\nu} f^*_{V/U=u}(t/u) f^*(u) dt du \qquad (6.3.19) \\ &= \int_0^1 \Big\{ I_{\left(\frac{F(u\nu)}{F(u)}\right)^{m+1}} \left( \frac{\gamma_s}{m+1}, s - r \right) \frac{c_{r-1}}{\Gamma(r)} \\ &\quad .(F(u))^{\gamma_r - 1} \left[ \frac{1 - (F(u))^{m+1}}{m+1} \right]^{r-1} f(u) du \Big\}. \end{aligned}$$

If $F(x)=x^{\delta}$, and using equations (6.2.4) and (6.2.5), we obtain from equation (6.3.19)

$$
\begin{aligned}
F_V^*(\nu) &= I_{\nu^{\delta(m+1)}}\left(\frac{\gamma_s}{m+1}, s-r\right)\frac{\delta c_{r-1}}{\Gamma(r)(m+1)^{r-1}}\int_0^1 u^{\delta\gamma_r-1}\left(1-u^{\delta(m+1)}\right)^{r-1}du \\
&= I_{\nu^{\delta(m+1)}}\left(\frac{\gamma_s}{m+1}, s-r\right)\frac{\delta c_{r-1}}{\Gamma(r)(m+1)^{r-1}}B\left(\frac{\gamma_r}{m+1}, r\right) \\
&= I_{\nu^{\delta(m+1)}}\left(\frac{\gamma_s}{m+1}, s-r\right)
\end{aligned}
$$

By Lemma 6.2.1, it follows that the distribution function of $X(s-r, n-r, m, k)$ is

$$
F^*_{s-r,n-r,m,k}(x) = I_{x^{\delta(m+1)}}\left(\frac{\gamma^*_{s-r}}{m+1}, s-r\right),
$$

where $\gamma^*_{s-r}=k+(n-r-s+r)(m+1)=k+(n-s)(m+1)=\gamma_s$ .

Now, we shall prove that (2) implies (1).

If $V$ and $X^*(s-r, n-s, m, k)$ are identically distributed, then using equation (6.3.13) and Lemma 6.2.2, we can write

$$
\begin{aligned}
&\int_0^1 \left\{ I_{\left(\frac{F(uv)}{F(u)}\right)^{m+1}}\left(\frac{\gamma_s}{m+1}, s-r\right)\right. \\
&\left. \cdot\frac{c_{r-1}}{\Gamma(r)}(F(u))^{\gamma_r-1}\left[\frac{1-(F(u))^{m+1}}{m+1}\right]^{r-1} f(u)du\right\} \\
&= I_{(F(v))^{m+1}}\left(\frac{\gamma^*_{s-r}}{m+1}, s-r\right) \\
&= I_{(F(v))^{m+1}}\left(\frac{\gamma_s}{m+1}, s-r\right). \qquad (6.3.20)
\end{aligned}
$$

Upon simplification, we obtain from equation (6.3.20)

$$
\begin{aligned}
&\int_0^1 \left\{ \left[ I_{\left(\frac{F(uv)}{F(u)}\right)^{m+1}}\left(\frac{\gamma_s}{m+1}, s-r\right)\right.\right. \\
&- \left. I_{(F(v))^{m+1}}\left(\frac{\gamma_s}{m+1}, s-r\right)\right]\frac{c_{r-1}}{\Gamma(r)}(F(u))^{\gamma_r-1} \\
&\cdot \left.\left[\frac{1-(F(u))^{m+1}}{m+1}\right] f(u)du\right\} = 0. \qquad (6.3.21)
\end{aligned}
$$

Since $F$ belongs to class $C$, the class of all continuous functions, we obtain from equation (6.3.21)

$$
F(uv)=F(u)F(v), \qquad (6.3.22)
$$

for all $u$, $0 < u \leq 1$, and almost all $v$, $0 \leq v \leq 1$.

The non-zero solution of equation (6.3.22) with the conditions that F(x) is a probability distribution function with

$$F(0) = 0, \quad \text{and} \quad F(1) = 1,$$

is

$$F(x) = x^{\delta}, 0 \leq x \leq 1, \delta > 0$$

For $m = -1$,

(1) implies (2),we have that

$$\begin{aligned} F_V^*(\nu) &= \int_0^1 F_{V/U=u}^*(\nu/u) f(u) du \qquad (6.3.23) \\ &= \int_0^1 \left[ 1 - \Gamma_{-k \ln\left(\frac{F(u\nu)}{F(u)}\right)}(s-r) \right] \frac{\delta^r k^r}{\Gamma(r)} (F(u))^{k-1} \left[-\ln F(u)\right]^{r-1} f(u) du. \end{aligned}$$

If $F(x) = x^{\delta}$, we obtain from equation (6.3.23)

$$\begin{aligned} F_V^*(\nu) &= \left[1 - \Gamma_{-k\delta \ln(\nu)}(s-r)\right] \frac{\delta^r k^r}{\Gamma(r)} \int_0^1 u^{\delta k - 1} (-\ln u)^{r-1} du \\ &= 1 - \Gamma_{-k\delta \ln(\nu)}(s-r). \end{aligned}$$

Using Lemma 6.2.2, it follows that the probability distribution function of $X(s-r, n-r, m, k) = x$ is

$$F_{s-r,n-r,m,k}^*(x) = 1 - \Gamma_{-k\delta \ln(x)}(s-r),$$

which show that (1) implies (2).

Now, to show that (2) implies (1), If $V$ and $U$ are identically distributed, then using equation (6.3.17) and Lemma 6.2.2, we obtain

$$\begin{aligned} F_V^*(\nu) &= \int_0^1 \left[ 1 - \Gamma_{-k \ln\left(\frac{F(u\nu)}{F(u)}\right)}(s-r) \right] \frac{\delta^r k^r}{\Gamma(r)} (F(u))^{k-1} \left[-\ln F(u)\right]^{r-1} f(u) du \\ &= 1 - \Gamma_{-k \ln\left(\frac{F(u\nu)}{F(u)}\right)}(s-r) \\ &= 1 - \Gamma_{-k \ln(F(\nu))}(s-r). \qquad (6.3.24) \end{aligned}$$

Upon simplification, we obtain from equation (6.3.24);

$$\int_0^1 \left[\Gamma_{-k\ln(F(\nu))}(s-r) - \Gamma_{-k\ln\left(\frac{F(u\nu)}{F(u)}\right)}(s-r)\right] \frac{k^r}{\Gamma(r)}(-\ln F(u))^{r-1} f(u)du = 0. \tag{6.3.25}$$

Since $F$ belongs to class the $C$, we obtain from equation (6.3.25)

$$F(uv) = F(u)F(v), \tag{6.3.26}$$

for all $u$, $0 < u \leq 1$, and almost all $v$, $0 \leq v \leq 1$.

The non-zero solution of equation (6.3.26) with the conditions that F(x) is a probability distribution function with

$F(0) = 0$, and $F(1) = 1$, is

$$F(x) = x^\delta, 0 \leq x \leq 1, \delta > 0.$$

■

**Remark 6.3.6** *For $k = 1$ and $m = 0$ , we obtain a characterization of the power function probability distribution based on the identical distribution of order random values $X_{n-s+1,n}|X_{n-r+1,n}$ and $X_{n-s+1,n-r}$.*

*For $k = 1$ and $m = -1$, we obtained a characterization of the power function distribution based on the identical distribution of lower record values $X_{L(s)}|X_{L(r)}$ and $X_{L(s-r)}$.*

### 6.4 Moments Of Lower Generalized Order Statistics

Equations (6.1.2) and (6.1.3) can be used to obtain the single and product moments of lgos. The $j$th moment of $X^*(r, n, m, k)$ is given by

$$\begin{aligned} \mu^j_{r,n,m,k} &= E(X^*(r,n,m,k))^j \\ &= \frac{c_{r-1}}{\Gamma(r)} \int_{-\infty}^{\infty} x^j (F(x))^{\gamma_r - 1} g_m^{r-1}(F(x)) dF(x) \end{aligned} \tag{6.4.27}$$

Setting $j = 1$ and $j = 2$, in equation (6.4.27), we obtain the first moment ($\mu_{r,n,m,k}$) and second moments ($\mu^2_{r,n,m,k}$), respectively of $X^*(r, n, m, k)$. The variance $\sigma^2_{r,n,m,k}$ of $X^*(r, n, m, k)$ is given by

$$\sigma^2_{r,r,m,n} = \mu^2_{r,n,m,k} - \mu^2_{r,n,m,k}. \tag{6.4.28}$$

The joint $(i,j)$th moment of $X^*(r,n,m,k)$ and $X^*(s,n,m,k)$ is given by

$$\begin{aligned} \mu^{ij}_{r,n,m,k} &= E(X^*(r,n,m,k))^i(X^*(s,n,m,k))^j \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{x} u^i v^j f^*_{r,s,n,m,k}(u,v)dudv, \end{aligned} \tag{6.4.29}$$

with $f^*_{r,s,n,m,k}$ given by equation (6.1.3).
The covariance $\sigma^2_{r,s,n,m,k}$ of $X^*(r,n,m,k)$ and $X^*(s,n,m,k)$ is expressed as

$$\sigma^2_{r,s,n,m,k} = \mu^2_{r,s,n,m,k} - \mu_{r,n,m,k}\mu_{s,n,m,k} \tag{6.4.30}$$

We proceed to study some examples based on the developed theory.

### 6.4.1 Example: Uniform Probability Distribution

Suppose $X$ be a random variable (rv) from the uniform probability distribution with pdf given by

$$\begin{aligned} f(x) &= 1, 0 \leq x \leq 1 \\ &= 0, \quad \text{otherwise.} \end{aligned} \tag{6.4.31}$$

Using equation (6.4.27), with $j = 1$, we have that the first moment $\mu_{r,n,m,k}$ of $X^*(r,n,m,k)$ from the uniform probability distribution is given by

$$\begin{aligned} E(X^*(r,n,m,k)) &= \frac{c_r}{(m+1)^{r-1}\Gamma(r)}\int_0^1 x^{\gamma_r}(1-x^{m+1})^{r-1}dx \\ &= \frac{c_{r-1}}{(m+1)^r\Gamma(r)} B\left(\frac{\gamma_r+1}{m+1}, r\right) \\ &= \frac{c_{r-1}}{(\gamma_1+1)(\gamma_2+1)\ldots(\gamma_r+1)}. \end{aligned} \tag{6.4.32}$$

Similarly, the second moment is given by

$$E(X^*(r,n,m,k))^2 = \frac{c_r}{(m+1)^{r-1}\Gamma(r)}\int_0^1 x^{\gamma_r+1}(1-x^{m+1})^{r-1}dx \tag{6.4.33}$$

$$
\begin{aligned}
&= \frac{c_{r-1}}{(m+1)^r\Gamma(r)} B\left(\frac{\gamma_r+2}{m+1}, r\right) \\
&= \frac{c_r}{(\gamma_1+2)\ldots(\gamma_r+2)}
\end{aligned}
$$

**Remark 6.4.1** *If we take $m = 0$ and $k = 1$, $X^*(r, n, m, k)$ reduces to the $(n - r + 1)th$ order statistics from a sample of size $n$, (e.g. see Ahsanullah and Nevzorov, [6] p. 35) with $\gamma_r = n - r + 1$,*

$$E(X_{n-r+1,n}) = (n - r + 1)/(n + 1)$$

*and*

$$E(X_{n-r+1,n})^2 = \frac{n - r + 2}{(m + 1)(n + 2)}.$$

*If we take $m = -1$ and $k = 1$, $X^*(r, n, m, k)$ reduces to the $rth$ lower record statistics from a sample of size $n$, with $\gamma_r = k$, we have*

$$E(X_{L(r)}) = (1/2)^r$$

*and*

$$E(X_{L(r)})^2 = (1/3)^r.$$

### 6.4.2 Example: Power Function Probability Distribution

Let $X$ be a random variable (rv) from the power function probability distribution with pdf given by

$$
\begin{aligned}
f(x) &= \delta((x-\mu)/\sigma)^{\delta-1}, \mu \leq x \leq \mu+\sigma, \sigma > 0, \mu \geq 0 \qquad (6.4.34)\\
&= 0, \quad \text{otherwise.}
\end{aligned}
$$

Using equation (6.4.27) with $j = 1$, and letting $\mu = 0$ and $\sigma = 1$, in equation (6.4.34), we have the first moment $\mu_{r,n,m,k}$ of $X^*(r, n, mk)$ from the power function probability distribution to be

$$
\begin{aligned}
E(X^*(r,n,m,k)) &= \frac{\delta c_r}{(m+1)^{r-1}\Gamma(r)} \int_0^1 x^{\delta\gamma_r}(1-x^{\delta(m+1)})^{r-1}dx \qquad (6.4.35)\\
&= \frac{c_r\Gamma(r)\Gamma(\frac{\delta\gamma_r+1}{\delta(m+1)})}{(m+1)^r\Gamma(r)\Gamma(\frac{\delta\gamma_r+1}{\delta(m+1)}+r)}
\end{aligned}
$$

$$= \quad b_r,$$

where

$$b_r = \frac{c_r \delta^r}{(\delta\gamma_1 + 1)\dots(\delta\gamma_r + 1)}.$$

Similarly, the second moment is given by

$$\begin{aligned} E(X^*(r,n,m,k))^2 &= \frac{\delta c_r}{(m+1)^{r-1}\Gamma(r)} \int_0^1 x^{\delta\gamma_r+1}(1-x^{\delta(m+1)})^{r-1}dx \qquad (6.4.36) \\ &= \frac{c_r \Gamma(r)\Gamma(\frac{\delta\gamma_r+2}{\delta(m+1)})}{(m+1)^r\Gamma(r)\Gamma(\frac{\delta\gamma_r+2}{\delta(m+1)}+r)} \\ &= \frac{c_r\delta^r}{(\delta\gamma_1+2)\dots(\delta\gamma_r+2)}. \end{aligned}$$

Using equations (6.4.35) and (6.4.36) we compute the variance of $X^*(r,n,m,k)$, is given by

$$\begin{aligned} Var(X^*(r,n,m,k)) &= \frac{c_r\delta^r}{(\delta\gamma_1+2)\dots(\delta\gamma_r+2)} - \left(\frac{c_r\delta^r}{(\delta\gamma_1+1)\dots(\delta\gamma_r+1)}\right)^2 \\ &= b_r a_r, \end{aligned}$$

where

$$a_r = \frac{(\delta\gamma_1+1)\dots(\delta\gamma_r+1)}{(\delta\gamma_1+2)\dots(\delta\gamma_r+2)} - \frac{c_r\delta^r}{(\delta\gamma_1+1)\dots(\delta\gamma_r+1)}.$$

Using Theorem 6.3.2, we have that for $s > r$,

$$\begin{aligned} & Cov[X^*(r,n,m,k), X^*(s,n,m,k)] \qquad (6.4.37) \\ &= E(X^*(s,n,m,k)X^*(r,n,m,k)) - E(X^*(s,n,m,k))E(X^*(r,n,m,k)) \\ &= [E(X^*(r,n,m,k)X^*(r,n,m,k)) - E(X^*(s,n,m,k))E(X^*(r,n,m,k))] \prod_{i=r+1}^{s} EW_i \\ &= Var(X^*(r,n,m,k)) \prod_{i=r+1}^{s} \frac{\delta^{s-r}\gamma_i}{\delta\gamma_i+1} \\ &= a_r b_s. \end{aligned}$$

**Remark 6.4.2** *If we let $m = 0$ and $k = 1$, $X^*(r,n,m,k)$ reduces to the $(n-r+1)$th order statistics from a sample of size n, (e.g. see Ahsanullah and Nevzorov, [6] p. 35) with $\gamma_r = n-r+1$,*

$$E(X_{n-r+1,n}) = \frac{\Gamma(n+1)\Gamma(n-r+1+\frac{1}{\delta})}{\Gamma(n-r+1)\Gamma(n+1+\frac{1}{\delta})},$$

*and*

$$E(X_{n-r+1,n})^2 = \frac{\Gamma(n+1)\Gamma(n-r+1+\frac{2}{\delta})}{\Gamma(n-r+1)\Gamma(n+1+\frac{2}{\delta})}.$$

*If we let* $m = -1$ *and* $k = 1$, $X^*(r, n, m, k)$ *reduces to the* $r$*th lower record statistics from a sample of size* $n$, *with* $\gamma_r = k$,

$$E(X_{L(r)}) = \left(\frac{\delta}{\delta+1}\right)^r$$

*and*

$$E(X_{L(r)})^2 = (\frac{\delta}{2+\delta})^r.$$

### 6.4.3 Example: Generalized Exponential Distribution

Let $X$ be a rv from the Generalized exponential probability distribution (GED) with pdf given by

$$f(x) = \theta(1 - e^{-(x-\mu)/\sigma})^{\theta-1} e^{-(x-\mu)/\sigma}, \tag{6.4.38}$$

where $x > \mu, -\infty < \mu < \infty, \sigma > 0, \theta > 0$.

Without loss of generality, we will take $\mu = 0$, and $\sigma = 1$. Substituting equation (6.4.38) into equation (6.1.2), we obtain for $r = 1$,

$$f^*_{1,n,m,k}(x) = \theta\gamma_1(1 - e^{-x})^{\theta\gamma_1} e^{-x}. \tag{6.4.39}$$

Using equations (6.4.27), and (6.4.38) and the substitution $w = 1 - e^{-x}$, the first moment of $r$ lgos from the GED is given by

$$\begin{aligned}
& E(X^*(r,n,m,k)) \\
&= \frac{\theta c_{r-1}}{\Gamma(r)(m+1)^{r-1}} \int_0^\infty x(1-e^{-x})^{\theta\gamma_r - 1}[1-(1-e^{-x})^{\theta(m+1)}]^{r-1} e^{-x} dx \\
&= \frac{\theta c_{r-1}}{\Gamma(r)(m+1)^{r-1}} \int_0^1 -\ln(1-w) w^{\theta\gamma_r-1}[1-w^{\theta(m+1)}]^{r-1} dw \\
&= \frac{\theta c_{r-1}}{\Gamma(r)(m+1)^{r-1}} \sum_{i=1}^\infty \int_0^1 \frac{1}{i} w^{\theta\gamma_r+i-1}[1-w^{\theta(m+1)}]^{r-1} dw, \\
&= \frac{\theta c_{r-1}}{\Gamma(r)(m+1)^{r-1}} \sum_{i=1}^\infty \frac{1}{i} B\left(\frac{\theta\gamma_r+i-1}{\theta(m+1)}, r\right) \\
&= \sum_{i=1}^\infty \frac{\theta c_{r-1}}{i(\theta\gamma_{r-(r-1)}+\nu)(\theta\gamma_{r-(r-2)}+\nu)\ldots(\theta\gamma_r+\nu)} \\
&= \sum_{i=1}^\infty \frac{1}{i} \prod_{\nu=0}^{r-1} \frac{\theta\gamma_{r-\nu}}{\theta\gamma_{r-\nu}+i}.
\end{aligned} \tag{6.4.40}$$

The moment generation function (mgf) of $X^*(r,n,m,k)$ is

$$\begin{aligned}
M^*_{r,n,m,k}(t) &= \frac{\theta c_{r-1}}{\Gamma(r)} \int_0^\infty \left\{ \frac{e^{tx}(1-e^{-x})^{\theta\gamma_r-1}}{(m+1)^{r-1}} \right. \\
& \left. .[1-(1-e^{-x})^{\theta(m+1)}]^{r-1} e^{-x} dx \right\} \\
&= \frac{\theta c_{r-1}}{\Gamma(r)(m+1)^{r-1}} \int_0^1 \left\{ (1-w)^{-t} w^{\theta\gamma_r-1} \right.
\end{aligned}$$

$$.[1 - w^{\theta(m+1)}]^{r-1} dw\Big\}. \tag{6.4.41}$$

Using the Maclaurin series expansion, we can write

$$(1-w)^{-t} = \sum_{i=0}^{\infty} (t)_i w^i / i!,$$

where

$$(t)_i = \begin{cases} t(t+1)(t+2)(t+i-1) = \Gamma(t+i)/\Gamma(t), & i = 1, 2, 3, \ldots, \\ 1, & i = 0. \end{cases}$$

we have from equation (6.4.41) that

$$\begin{aligned} M^*_{r,n,m,k}(t) &= \frac{\theta c_{r-1}}{\Gamma(r)(m+1)^{r-1}} \sum_{i=0}^{\infty} \frac{(t)_i}{i!} \int_0^1 w^{\theta\gamma_r + i - 1} [1 - w^{\theta(m+1)}]^{r-1} dw, \\ &= \frac{\theta c_{r-1}}{\Gamma(r)(m+1)^{r-1}} \sum_{i=0}^{\infty} \frac{(t)_i}{i!} \frac{1}{\theta(m+1)} B\left(\frac{\theta\gamma_r + i}{\theta(m+1)}, r\right) \qquad (6.4.42) \\ &= \sum_{i=0}^{\infty} \frac{(t)_i}{i!} \frac{\theta^r c_r}{(\theta\gamma_{r-(r-1)} + i)\ldots(\theta\gamma_r + i)} \\ &= \sum_{i=0}^{\infty} \frac{(t)_i}{i!} \frac{\theta^r c_r}{\prod_{\nu=0}^{r-1}(\theta\gamma_{r-\nu} + i)} \\ &= \sum_{i=0}^{\infty} \frac{(t)_i}{i!} \prod_{\nu=0}^{r-1} \left(\frac{\theta\gamma_{r-\nu}}{\theta\gamma_{r-\nu} + i}\right). \end{aligned}$$

Observe that

$$(d/dt)(t)_i = [\psi(t+i) - \psi(t)](t)_i$$

and

$$\lim_{t\to 0} (\psi(t+i) - \psi(t))/\Gamma(t) = 1.$$

Differentiating $M^*_{r,n,m,k}(t)$ with respect to $t$ and evaluating at $t = 0$ we obtain the mean of $r$th lgos given in equation (6.4.40).

Using the substitution $w = 1 - e^{-x}$, the second moment of $r$th lgos from the GED is given by

$$\begin{aligned} &E(X^*(r,n,m,k))^2 \\ &= \frac{\theta c_{r-1}}{\Gamma(r)(m+1)^{r-1}} \int_0^{\infty} x^2 (1-e^{-x})^{\theta\gamma_r - 1} [1 - (1-e^{-x})^{\theta(m+1)}]^{r-1} e^{-x} dx \end{aligned}$$

$$
\begin{aligned}
&= \frac{\theta c_{r-1}}{\Gamma(r)(m+1)^{r-1}} \int_0^1 (\ln(1-w))^2 w^{\theta\gamma_r - 1}[1-w^{\theta(m+1)}]^{r-1} dw \qquad (6.4.43) \\
&= \frac{\theta c_{r-1}}{\Gamma(r)(m+1)^{r-1}} \int_0^1 \left[\sum_{i=1}^{\infty} \frac{w^i}{i}\right]^2 w^{\theta\gamma_r - 1}[1-w^{\theta(m+1)}]^{r-1} dw \\
&= \frac{\theta c_{r-1}}{\Gamma(r)(m+1)^{r-1}} \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{1}{ij} \int_0^1 w^{\theta\gamma_r + i + j - 1}[1-w^{\theta(m+1)}]^{r-1} dw \\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{1}{ij} \prod_{\nu=0}^{r-1} \frac{\theta\gamma_{r-\nu}}{(\theta\gamma_{r-\nu} + i + j)},
\end{aligned}
$$

The variance of the $r$th lgos is obtained from equations (6.4.40) and (6.4.43).

The joint pdf of two lgos $X^*(r,n,m,k)$ and $X^*(s,n,m,k)$ $(1 \le r < s \le n)$ from the GED can be written for $0 \le y < x < \infty$ as

$$
\begin{aligned}
f^*_{r,s,n,m,k}(x,y) &= \frac{\theta^2 c_{s-1}(1-e^{-x})^{(m+1)\theta-1}}{\Gamma(r)\Gamma(s-r)(m+1)^{s-2}}[1-(1-e^{-x})^{\theta(m+1)}]^{r-1} \\
&\quad .[(1-e^{-x})^{\theta(m+1)} - (1-e^{-y})^{\theta(m+1)}]^{s-r-1} \\
&\quad .e^{-x}e^{-y}(1-e^{-y})^{\theta\gamma_s - 1}. \qquad (6.4.44)
\end{aligned}
$$

The joint moment of two lgos is given by

$$
\begin{aligned}
&E(X^*(r,n,m,k)X^*(s,n,m,k)) \\
&= \int_0^{\infty} \Big\{ x \frac{\theta^2 c_{s-1}(1-e^{-x})^{(m+1)\theta-1}}{\Gamma(r)\Gamma(s-r)(m+1)^{s-2}} \\
&\quad .[1-(1-e^{-x})^{\theta(m+1)}]^{r-1} e^{-x} I(x) dx \Big\}, \qquad (6.4.45)
\end{aligned}
$$

where

$$
I(x) = \int_0^x y[(1-e^{-x})^{\theta(m+1)} - (1-e^{-y})^{\theta(m+1)}]^{s-r-1}(1-e^{-y})^{\theta\gamma_s - 1} e^{-y} dy,
$$

which upon making the transformation,

$$
(1-e^{-y})^{\theta(m+1)} = t(1-e^{-x})^{\theta(m+1)},
$$

$$
-\ln\left[1-(1-e^{-x})t^{\frac{1}{\theta(m+1)}}\right] = \sum_{i=1}^{\infty} \frac{(1-e^{-x})^i t^{\frac{i}{\theta(m+1)}}}{i},
$$

and integrating it with respect to $y$, we have

$$I(x) = \frac{1}{\theta(m+1)} \sum_{i=1}^{\infty} \frac{(1-e^{-x})^{\theta\gamma_{r+1}+i}}{i} B\left(\frac{\theta\gamma_s + i}{\theta(m+1)}, s-r\right).$$

Substituting $I(x)$ in equation (6.4.45) and simplifying gives

$$\begin{aligned} E(X^*(r,n,m,k)X^*(s,n,m,k)) &= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{1}{ij} \prod_{\nu=0}^{r-1} \frac{\theta\gamma_{r-\nu}}{\theta\gamma_{r-\nu}+i+j} \\ &\cdot \prod_{\omega=0}^{s-r-1} \frac{\theta\gamma_{s-\omega}}{\theta\gamma_{s-\omega}+i}, \end{aligned} \tag{6.4.46}$$

for $1 \le r < s$.

The covariance of any two lgos $X^*(r,n,m,k)$, $X^*(s,n,m,k)$, $s > r \ge 1$ is obtained from equations (6.4.40) and (6.4.46).

The moment generating function of two lgos is given by

$$\begin{aligned} & M^*_{r,s,n,m,k}(t_1,t_2) \\ &= \int_0^{\infty} \frac{\theta^2 c_{s-1} e^{t_1 x}(1-e^{-x})^{(m+1)\theta-1}}{\Gamma(r)\Gamma(s-r)(m+1)^{s-2}} \\ &.[1-(1-e^{-x})^{\theta(m+1)}]^{r-1} e^{-x} I(x)dx, \end{aligned} \tag{6.4.47}$$

where

$$I(x) = \int_0^x e^{t_2 y}[(1-e^{-x})^{\theta(m+1)} - (1-e^{-y})^{\theta(m+1)}]^{s-r-1}(1-e^{-y})^{\theta\gamma_s-1}e^{-y}dy.$$

Substituting

$$(1-e^{-y})^{\theta(m+1)} = w(1-e^{-x})^{\theta(m+1)},$$

and writing

$$[1-w^{\frac{1}{\theta(m+1)}}(1-e^{-x})]^{-t_2} = \sum_{i=0}^{\infty}(t_2)_i w^{\frac{i}{\theta(m+1)}}(1-e^{-x})^i,$$

we obtain

$$I(x) = \frac{1}{\theta(m+1)} \sum_{i=0}^{\infty} (t_2)_i B\left(\frac{\theta\gamma_s + i}{\theta(m+1)}, s-r\right) (1-e^{-x})^{\theta\gamma_{r+1}+i}.$$

Substituting $I(x)$ in equation (6.4.47) we have

$$\begin{aligned} & M^*_{r,s,n,m,k}(t_1,t_2) \\ = \ & \frac{c_s}{(m+1)^{s-1}\Gamma(r)\Gamma(s-r)} \sum_{i=0}^{\infty} (t_2)_i B\left(\frac{\theta\gamma_s + i}{\theta(m+1)}, s-r\right) \\ & .\int_0^{\infty} \Big\{ e^{t_1 x}(1-e^{-x})^{\theta\gamma_{r+1}\theta(m+1)+i-1} \\ & .[1-(1-e^{-x})^{\theta(m+1)}]^{r-1} e^{-x} dx \Big\}. \end{aligned} \tag{6.4.48}$$

Substituting $u = 1 - e^{-x}$ and using the expansion

$$(1-u)^{-t_1} = \sum_{j=0}^{\infty} (t_1)_j u^j$$

in equation (6.4.48) and integrating with respect to $t$ we have

$$\begin{aligned} & M^*_{r,s,n,m,k}(t_1,t_2) \\ = \ & \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{c_s (t_2)_i (t_1)_j}{(m+1)^s \Gamma(r)\Gamma(s-r)} B\left(\frac{\theta\gamma_s + i}{\theta(m+1)}, s-r\right) B\left(\frac{\theta\gamma_r + i + j}{\theta(m+1)}, r\right) \\ = \ & \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} (t_2)_i (t_1)_j \theta^s c_s \prod_{\nu=0}^{r-1} \frac{1}{\theta\gamma_{r-\nu} + i + j} \prod_{\omega=0}^{s-r-1} \frac{1}{\theta\gamma_{s-\omega} + i} \\ = \ & \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} (t_2)_i (t_1)_j \prod_{\nu=0}^{r-1} \frac{\theta\gamma_{r-\nu}}{\theta\gamma_{r-\nu} + i + j} \prod_{\omega=0}^{s-r-1} \frac{\theta\gamma_{s-\omega}}{\theta\gamma_{s-\omega} + i}. \end{aligned} \tag{6.4.49}$$

Differentiating equation (6.4.49) with respect to $t_1$ and then with respect to $t_2$ and evaluating at $t_1 = t_2 = 0$, we obtain the product moments of $X(r,n,m,k)$ and $X(s,n,m,k)$ $(1 \le r < s \le n)$ and the estimate is exactly equal to that obtained in equation (6.4.46).

**Remark 6.4.3** *If we let $r = 1$, then*

$$E(X^*(1,n,m,k)) = \sum_{i=1}^{\infty} \frac{\theta\gamma_1}{i(\theta\gamma_1 + i)},$$

*and*

$$E(X^*(1,n,m,k))^2 = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{\theta\gamma_1}{ij(\theta\gamma_1+i+j)}.$$

**Remark 6.4.4** *If we let $\theta = 1$, $X^*(r,n,m,k)$ reduces to the lgos statistics from the exponential distribution, that is, equations (6.4.40), (6.4.43) and (6.4.46) result in*

$$E(X^*(r,n,m,k)) = \sum_{i=1}^{\infty}\frac{1}{i}\prod_{\nu=0}^{r-1}\left(\frac{\gamma_{r-\nu}}{\gamma_{r-\nu}+i}\right),$$

$$E(X^*(r,n,m,k))^2 = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{ij}\prod_{\nu=0}^{r-1}\frac{\gamma_{r-\nu}}{(\gamma_{r-\nu}+i+j)},$$

*and*

$$E(X^*(r,n,m,k)X^*(s,n,m,k)) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{ij}\prod_{\nu=0}^{r-1}\frac{\gamma_{r-\nu}}{\gamma_{r-\nu}+i+j}\prod_{\omega=0}^{s-r-1}\frac{\gamma_{s-\omega}}{\gamma_{s-\omega}+i}.$$

**Remark 6.4.5** *If we let $m = 0$, $k = 1$, $\gamma_r = n-r+1$, tehn $X^*(r,n,m,k)$ reduces to the $(n-r+1)$th order statistics from a sample of size n, that is, equations (6.4.40), (6.4.43) and (6.4.46) give respectively*

$$E(X^*_{r,n}) = \sum_{i=1}^{\infty}\frac{1}{i}\prod_{\nu=0}^{r-1}\left(\frac{\theta(n-r+\nu+1)}{\theta(n-r+\nu+1)+i}\right)$$

$$E(X^*_{r,n})^2 = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{ij}\prod_{\nu=0}^{r-1}\left(\frac{\theta(n-r+\nu+1)}{\theta(n-r+\nu+1)+i+j}\right)$$

*and*

$$\begin{aligned} E(X^*(r,n,m,k)X^*(s,n,m,k)) &= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{ij}\prod_{\nu=0}^{r-1}\frac{\theta(n-r+\nu+1)}{\theta(n-r+\nu+1)+i+j} \\ &\times \prod_{\omega=0}^{s-r-1}\frac{\theta(n-s+\omega+1)}{\theta(n-s+\omega+1)+i}. \end{aligned}$$

*Tables for means, variances and covariances have been presented by Ahsanullah and Nevzorov, [7], for the case when $m = 0$, $k = 1$.*

**Remark 6.4.6** *If we take $m = -1$, $k = 1$, $\gamma_r = 1$, then $X^*(r,n,m,k)$ reduces to the rth lower record values and the estimates in equations (6.4.40), (6.4.43) and (6.4.46) coincide* *with those of Raqab, [43], that is,*

$$E(X_{L(r)}) = \theta^r \sum_{i=1}^{\infty} \frac{1}{i(\theta+i)^r},$$

$$E(X_{L(r)})^2 = \theta^r \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{1}{ij(\theta+i+j)^r},$$

*and*

$$E(X_{L(r)}X_{L(s)}) = \theta^s \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{1}{ij(\theta+i+j)^r(\theta+i)^{s-r}}.$$

*Tables for means, variances and covariances have been presented by Raqab, [43], for the case when* $m = -1$, $k = 1$.

## 6.5 Estimation Of Parameters

In this section, we shall obtain estimates for the parameters of the power function probability distribution and the GED.

### 6.5.1 Power Function Probability Distribution

We estimate the best linear unbiased estimate (BLUE) for $\mu$ and $\sigma$ when $\delta$ is known and also an estimate for $\delta$ when $\mu$ and $\sigma$ are known.

**Estimating $\mu$ and $\sigma$ when $\delta$ is known:**

The following results Mbah and Ahsanullah, [36], establishes the BLUE for $\mu$ and $\sigma$ when $\delta$ is known.

**Theorem 6.5.1** *Let* $X^*(1,n,m,k) = x_1, \ldots, X^*(n,n,m,k) = x_n$ *be* $n$ *lgos from the power function distribution given by (6.4.34). Then the best linear unbias estimates (BLUE)* $\hat{\mu}$, $\hat{\sigma}$ *for* $\mu$ *and* $\sigma$, *when* $\delta$ *is known, are*

$$\hat{\mu} = w_{11}x_1 + w_{12}x_2 + \ldots + w_{1n}x_n,$$

*and*

$$\hat{\sigma} = w_{21}x_1 + w_{22}x_2 + \ldots + w_{2n}x_n,$$

*where,*

$$w_{11} = -\frac{e_2}{D_0\delta\gamma_1},$$

$$w_{1i} = -\frac{(\delta\gamma_{i+1} - \delta\gamma_i + 1)e_i}{D_0\delta^i c_i}, i = 2, \ldots, n-1,$$

$$w_{1n} = \frac{(\delta\gamma_n + 1)e_n}{D_0\delta^n c_n},$$

$$w_{21} = \frac{(\delta\gamma_1 + 1)(D_0 + e_2/\delta\gamma_1)}{D_0\delta\gamma_1},$$

$$w_{2i} = \frac{(\delta\gamma_1 + 1)(\delta\gamma_{i+1} - \delta\gamma_i + 1)e_i}{\delta\gamma_1\delta^i c_i D_0}, i = 2, \ldots, n-1,$$

*and*

$$w_{2n} = -\frac{(\delta\gamma_n + 1)(\delta\gamma_1 + 1)e_n}{D_0\delta^{n+1}c_1 c_n}.$$

*The variance and the covariance of the estimates are given by*

$$Var(\hat{\mu}) = \frac{\sigma^2}{D_0},$$

$$Var(\hat{\sigma}) = \left(\frac{d_1^2}{(\delta\gamma_1)^2 D_0} + \frac{1}{\delta\gamma_1 e_1}\right)\sigma^2,$$

$$Cov(\hat{\mu}, \hat{\sigma}) = -\frac{d_1\sigma^2}{\delta\gamma_1 D_0},$$

*where*

$$D_0 = \sum_{i=2}^{n} \frac{e_i}{\delta^i c_i},$$

$$d_r = \prod_{j=1}^{r} \delta\gamma_j + 1$$

*and*

$$e_r = \prod_{j=1}^{r} \delta\gamma_j + 2.$$

*Proof.* Let $\mathbf{h}' = (x_1, x_2, \ldots, x_n)$, then

$$E(\mathbf{h}') = \mu\mathbf{1} + \sigma^2\alpha$$

and

$$Var(\mathbf{h}') = \sigma^2 \mathbf{V},$$

where,

$$\mathbf{1}' = (1, 1, 1, \ldots, 1),$$

$$\alpha' = (b_1, b_2, b_3, \ldots b_n),$$

$$\mathbf{V} = (v_{ij}),$$

$$v_{ij} = a_i b_j, 1 \leq i, j \leq n.$$

Let

$$\mathbf{V}^{-1} = (V^{ij}), 1 \leq i < j \leq n,$$

then the entries of $\mathbf{V}^{-1}$ are given by

$$\begin{aligned}
V^{ii} &= \frac{a_{i+1}b_{i-1} - a_{i-1}b_{i+1}}{(a_i b_{i-1} - a_{i-1}b_i)(a_{i+1}b_i - a_i b_{i+1})} \\
&= \big((\delta\gamma_i + 1)^2 + \delta\gamma_{i+1}(\delta\gamma_{i+1} + 2)\big)\frac{e_i}{\delta^i c_i}, i = 1, \ldots, n-1, \\
V^{nn} &= \frac{b_{n-1}}{b_n(a_n b_{n-1} - a_{n-1}b_n)} \\
&= (\delta\gamma_n + 1)^2 \frac{e_n}{\delta^n c_n}, \\
V^{ij} &= V^{ji} = \frac{-1}{a_{i+1}b_i - a_i b_{i+1}} \\
&= -(\delta\gamma_{i+1} + 1)\frac{e_{i+1}}{\delta^i c_i}, j = i+1, i = 1, \ldots, n-1, \\
V^{ij} &= 0, |i - j| > 1.
\end{aligned}$$

Using the method given by Lloyd, [33], we have that

$$\alpha' \mathbf{V}^{-1} = (d_1 e_1, 0, 0, \ldots, 0, 0),$$

and

$$\begin{aligned}
\mathbf{1}'\mathbf{V}^{-1} &= \Big(\frac{(d_1^2 - e_2)e_1}{\delta c_1}, -\frac{(\delta\gamma_3 - \delta\gamma_2 + 1)e_2}{\delta^2 c_2}, \ldots, -\frac{(\delta\gamma_{n-1} - \delta\gamma_{n-2} + 1)e_{n-2}}{\delta^{n-2} c_{n-2}}, \\
&- \frac{(\delta\gamma_n - \delta\gamma_{n-1} + 1)e_{n-1}}{\delta^{n-1} c_{n-1}}, \frac{(\delta\gamma_n + 1)e_n}{c_n \delta^n}\Big).
\end{aligned}$$

Therefore,

$$\alpha'\mathbf{V}^{-1}\alpha = \delta\gamma_1(\delta\gamma_1+2),$$

$$\alpha'\mathbf{V}^{-1}\mathbf{1} = (\delta\gamma_1+1)(\delta\gamma_1+2),$$

and

$$\mathbf{1}'\mathbf{V}^{-1}\mathbf{1} = \frac{(\delta\gamma_1+1)^2(\delta\gamma_1+2)}{\delta\gamma_1} + D_0.$$

Let

$$\begin{aligned}\Delta &= (\alpha'\mathbf{V}^{-1}\alpha)(\mathbf{1}'\mathbf{V}^{-1}\mathbf{1}) - (\alpha'\mathbf{V}^{-1}\mathbf{1})^2 \\ &= \delta\gamma_1(\delta\gamma_1+2)\left[\frac{(\delta\gamma_1+1)^2(\delta\gamma_1+2)}{\delta\gamma_1} + D_0\right] - [(\delta\gamma_1+1)(\delta\gamma_1+2)]^2 \\ &= \delta\gamma_1(\delta\gamma_1+2)D_0.\end{aligned}$$

Then we have that

$$\begin{aligned}\alpha'\mathbf{V}^{-1}(\alpha\mathbf{1'}-\mathbf{1}\alpha')\mathbf{V}^{-1} &= \delta\gamma_1(\delta\gamma_1+2)\Big(-\frac{e_2}{\delta c_1}, -\frac{(\delta\gamma_3-\delta\gamma_2+1)e_2}{\delta^2 c_2}, \ldots \\ &\qquad \ldots, -\frac{(\delta\gamma_n-\delta\gamma_{n-1}+1)e_{n-1}}{\delta^{n-1}c_{n-1}}, \ \frac{(\delta\gamma_n+1)e_n}{\delta^n c_n}\Big),\end{aligned}$$

and

$$\begin{aligned}\mathbf{1}'\mathbf{V}^{-1}(\mathbf{1}\alpha'-\alpha\mathbf{1}')\mathbf{V}^{-1} &= (\delta\gamma_1+1)(\delta\gamma_1+2)\Big(D_0+\frac{e_2}{\delta c_1}, \frac{(\delta\gamma_3-\delta\gamma_2+1)e_2}{\delta^2 c_2}, \ldots \\ &\qquad \ldots, \frac{(\delta\gamma_n-\delta\gamma_{n-1}+1)e_{n-1}}{\delta^{n-1}c_{n-1}}, -\frac{(\delta\gamma_n+1)e_n}{\delta^n c_n}\Big).\end{aligned}$$

The BLUE $\hat{\mu}$, $\hat{\sigma}$ of $\mu$ and $\sigma$ based on $n$ lgos are given by

$$\begin{aligned}\hat{\mu} &= \frac{\alpha' V^{-1}(\alpha\mathbf{1'}-\mathbf{1}\alpha')\mathbf{V}^{-1}\mathbf{h}}{\Delta} \\ &= w_{11}x_1 + \ldots + w_{1n}x_n,\end{aligned} \tag{6.5.50}$$

and

$$\begin{aligned}\hat{\sigma} &= \frac{\mathbf{1}'\mathbf{V}^{-1}(\mathbf{1}\alpha'-\alpha\mathbf{1}')\mathbf{V}^{-1}\mathbf{h}}{\Delta} \\ &= w_{21}x_1 + \ldots + w_{2n}x_n,\end{aligned} \tag{6.5.51}$$

where

$$w_{11} = -\frac{e_2}{D_0 \delta\gamma_1},$$

$$w_{1i} = -\frac{(\delta\gamma_{i+1} - \delta\gamma_i + 1)e_i}{D_0 \delta^i c_i}, i = 2, \ldots, n-1,$$

$$w_{1n} = \frac{(\delta\gamma_n + 1)e_n}{D_0 \delta^n c_n},$$

$$w_{21} = \frac{(\delta\gamma_1 + 1)(D_0 + e_2/\delta\gamma_1)}{D_0 \delta\gamma_1},$$

$$w_{2i} = \frac{(\delta\gamma_1 + 1)(\delta\gamma_{i+1} - \delta\gamma_i + 1)e_i}{\delta\gamma_1 \delta^i c_i D_0}, i = 2, \ldots, n-1,$$

and

$$w_{2n} = -\frac{(\delta\gamma_n + 1)(\delta\gamma_1 + 1)e_n}{D_0 \delta^{n+1} c_1 c_n}.$$

The variance and covariance of the estimators are given by

$$\begin{aligned} Var(\hat{\mu}) &= \frac{\alpha \mathbf{V}^{-1} \alpha}{\Delta} \sigma^2 \\ &= \frac{\delta\gamma_1(\delta\gamma_1 + 2)\sigma^2}{\delta\gamma_1(\delta\gamma_1 + 2) D_0} \\ &= \frac{\sigma^2}{D_0}, \end{aligned}$$

$$\begin{aligned} Var(\hat{\sigma}) &= \frac{\mathbf{1}' \mathbf{V}^{-1} \mathbf{1}}{\Delta} \sigma^2 \\ &= \frac{(\frac{(\delta\gamma_1+1)^2(\delta\gamma_1+2)}{\delta\gamma_1} + D_0)\sigma^2}{\delta\gamma_1(\delta\gamma_1 + 2) D_0} \\ &= \left( \frac{d_1^2}{(\delta\gamma_1)^2 D_0} + \frac{1}{\delta\gamma_1 e_1} \right) \sigma^2, \end{aligned}$$

and

$$\begin{aligned} Cov(\hat{\mu}, \hat{\sigma}) &= -\frac{\alpha' \mathbf{V}^{-1} \mathbf{1}}{\Delta} \sigma^2 \\ &= -\frac{d_1 \sigma^2}{\delta\gamma_1 D_0}. \end{aligned}$$

■

**Remark 6.5.2** *If $m = 0$, $k = 1$, $\gamma_j = n - j + 1, j = 1, 2, \ldots$, then the estimates in Theorem 6.5.1 reduces to $n$ order statistics, (see Ahsanullah and Nevzorov, [6], p. 178)).*

*Note that the coefficients of the BLUE for $\mu$, $\sigma$ and the variance covariance are given respectively in Table 6.1, Table 6.2 and Table 6.3 for $m = 0, k = 1$.*

*In particular, if $\delta = 1$, from Theorem 6.5.1, we have, $d_1 = n + 1, d_2 = (n + 1)n, d_3 = (n + 1)n(n - 1), \ldots$, $e_1 = n + 2, e_2 = (n + 2)(n + 1), e_3 = (n + 2)(n + 1)n, \ldots$, and,*

$$\begin{aligned} D_0 &= \frac{(n+2)(n+1)}{n(n-1)} + \frac{(n+2)(n+1)n}{n(n-1)(n-2)} + \ldots + \frac{(n+2)(n+1)n\ldots 3}{n(n-1)(n-2)\ldots 1} \\ &= \frac{(n-1)(n+1)(n+2)}{n}, \end{aligned}$$

$$\Delta = (n-1)(n+1)(n+2)^2,$$

*then,*

$$w_{11} = \frac{-1}{n-1}, w_{1i} = 0, i = 2 \ldots n-1, w_{1n} = \frac{n}{n-1},$$

*and the BLUE $\hat{\mu}$ and $\hat{\sigma}$ for $\mu$ and $\sigma$ based on $n$ order random variables $X_{1,n} > X_{2,n} > \ldots > X_{n,n}$ from the uniform distribution is*

$$\hat{\mu} = \frac{nX_{n,n} - X_{1,n}}{n-1},$$

*and*

$$\hat{\sigma} = \frac{n+1}{n-1}(X_{1,n} - X_{n,n}).$$

*The corresponding variances and covariance are given by*

$$Var(\hat{\mu}) = \frac{n\sigma^2}{(n+2)(n^2-1)}$$

$$Var(\hat{\sigma}) = \frac{(n+1)\sigma^2}{(n+1)(n+2)},$$

*and,*

$$Cov(\hat{\mu}, \hat{\sigma}) = -\frac{\sigma^2}{(n-1)(n+2)}.$$

*These results agree with those of Ahsanullah and Nevzorov, [6].*

*For $m = -1$, $k = 1$, $\gamma_j = 1, j = 1, 2, \ldots$, the BLUE $\hat{\mu}$, $\hat{\sigma}$ for $\mu$ and $\sigma$ based on $n$ lower* *record values* $X_{L(1)}, X_{L(2)}, \ldots, X_{L(n)}$ *are given by*

$$\hat{\mu} = -\frac{(\delta+2)^2}{D_0\delta} X_{L(1)} - \frac{1}{D_0}\sum_{i=2}^{n-1}\left(\frac{\delta+2}{\delta}\right)^i X_{L(i)} + \frac{\delta+1}{D_0}\left(\frac{\delta+2}{\delta}\right)^n X_{L(n)},$$

*and*

$$\hat{\sigma} = \frac{\delta+1}{\delta}\left(1+\frac{(\delta+2)^2}{\delta D_0}\right) X_{L(1)} - \frac{1}{D_0}\sum_{i=2}^{n-1}\frac{(\delta+1)(\delta+2)^i}{\delta^{i+1}D_0} X_{L(i)} + \frac{(\delta+1)^2(\delta+2)^n}{\delta^{n+1}D_0} X_{L(n)},$$

*where*

$$D_0 = \sum_{i=2}^{n}\left(\frac{\delta+2}{\delta}\right)^n.$$

*Note that the coefficients of the BLUE for* $\mu$, $\sigma$ *and the variance covariance are given in Chapter 3 for* $\delta = 0.5, 1, 2, 2.5, 3, 3.5, 4, 5$.

*In particular, if* $\delta = 1$, $m = -1$, $k = 1$, $D_0 = 9(3^{n-1} - 1)/2$, *the BLUE* $\hat{\mu}$, $\hat{\sigma}$ *of* $\mu$ *and* $\sigma$ *from the uniform distribution based on lower record values* $X_{L(1)}, X_{L(2)}, \ldots, X_{L(n)}$ *are given by*

$$\hat{\mu} = \frac{1}{D_0}\left[-3^2 X_{L(1)} + \sum_{i=2}^{n-1} 3^i X_{L(i)} + 2\times 3^n X_{L(n)}\right],$$

*and*

$$\hat{\sigma} = 2(X_{L(1)} - \hat{\mu}).$$

In the following theorem, Mbah and Ahsanullah, [36], we obtain the best linear invariant estimate, BLIE of $\mu$ and $\sigma$. Note that $\delta$ is assumed to be known.

**Theorem 6.5.3** *The best linear invariant (in terms of minimum mean squared error and invariance with respect to the location parameter* $\mu$*) estimators* $\tilde{\mu}, \tilde{\sigma}$ *for* $\mu$ *and* $\sigma$ *are*

$$\tilde{\mu} = \hat{\mu} - \hat{\sigma}\left(\frac{E_{12}}{1+E_{22}}\right)$$

*and*

$$\tilde{\sigma} = \frac{\hat{\sigma}}{1+E_{22}},$$

*where $\hat{\mu}$ and $\hat{\sigma}$ are BLUE of $\mu$ and $\sigma$, and*

$$\begin{pmatrix} Var(\hat{\mu}) & Cov(\hat{\mu}, \hat{\sigma}) \\ Cov(\hat{\mu}, \hat{\sigma}) & Var(\hat{\sigma}) \end{pmatrix} = \sigma^2 \begin{pmatrix} E_{11} & E_{12} \\ E_{12} & E_{22} \end{pmatrix}.$$

*The mean square errors of these estimators are given by*

$$MSE(\tilde{\mu}) = \sigma^2 \left[ E_{11} - \frac{E_{12}^2}{1 + E_{22}} \right],$$

*and*

$$MSE(\tilde{\sigma}) = \sigma^2 \left[ \frac{E_{22}}{1 + E_{22}} \right].$$

*Proof.* From Theorem 6.5.1, we can write

$$E_{11} = \frac{1}{D_0},$$

$$E_{22} = \frac{d_1^2}{(\delta\gamma_1)^2 D_0} + \frac{1}{\delta\gamma_1 e_1},$$

and

$$E_{12} = -\frac{d_1}{\delta\gamma_1 D_0},$$

therefore,

$$\tilde{\mu} = \hat{\mu} + \frac{\delta\gamma_1 d_1 e_1 \hat{\sigma}}{(\delta\gamma_1)^2 e_1 D_0 + e_1 d_1^2 + \delta\gamma_1 D_0},$$

and

$$\tilde{\sigma} = \frac{(\delta\gamma_1)^2 e_1 D_0 \hat{\sigma}}{(\delta\gamma_1)^2 e_1 D_0 + e_1 d_1^2 + \delta\gamma_1 D_0},$$

with

$$MSE(\tilde{\mu}) = \frac{\delta\gamma_1(\delta\gamma_1 + 1)\sigma^2}{e_1 D_0 + e_1 d_1^2 + \delta\gamma_1 D_0},$$

and

$$MSE(\tilde{\sigma}) = \frac{(e_1 d_1^2 + \delta\gamma_1 D_0)\sigma^2}{(\delta\gamma_1)^2 e_1 D_0 + e_1 d_1^2 + \delta\gamma_1 D_0}.$$

■

**Remark 6.5.4** *For $k = 1$, $m = 0$, $\gamma_i = n - i + 1, i = 1, 2, \ldots$, we obtain the BLIE $\hat{\mu}$, $\hat{\sigma}$ for $\mu$ and $\sigma$, from the order statistics $X_{1,n} > X_{2,n} > \ldots > X_{n,n}$. In particular, for $\delta = 1$, we* *obtain*

$$\tilde{\mu} = \hat{\mu} + \frac{\hat{\sigma}}{n(n+1)}$$
$$= \frac{(n+1)X_{n,n} - X_{1,n}}{n},$$

*and*

$$\tilde{\sigma} = \frac{(n+2)(n-1)}{n(n+1)}\hat{\sigma}$$
$$= \frac{(n+2)}{n}(X_{1,n} - X_{n,n}).$$

*The corresponding mean square errors are given by*

$$MSE(\tilde{\mu}) = \frac{\sigma^2}{n(n+2)},$$

*and*

$$MSE(\tilde{\sigma}) = \frac{2\sigma^2}{n(n+1)}.$$

*If we let* $\delta = 1$, $k = 1$, $m = -1$, $\gamma_i = 1, i = 1, 2, \ldots$, $d_1 = 2, d_2 = 2^2, \ldots$, $e_1 = 3, e_2 = 3^2, \ldots$ *then from Theorem 6.5.3, the BLIE* $\tilde{\mu}$, $\tilde{\sigma}$, *for* $\mu$ *and* $\sigma$ *from* $n$ *lower record values* $X_{L(1)}, X_{L(2)}, \ldots, X_{L(n)}$ *from the uniform distribution are given by*

$$\tilde{\mu} = \hat{\mu} + \frac{\hat{\sigma}}{3^n - 1},$$

*and*

$$\tilde{\sigma} = \frac{9(3^{n-1} - 1)\hat{\sigma}}{4(3^n - 1)},$$

*where* $\hat{\mu}$ *and* $\hat{\sigma}$ *are as in Theorem 6.5.1. The corresponding mean square errors are given by*

$$MSE(\tilde{\mu}) = \frac{8\sigma^2}{3(3^n - 1)},$$

*and*

$$MSE(\tilde{\sigma}) = \frac{(5 + 3^n)\sigma^2}{4(3^n - 1)}.$$

In the next result, Mbah and Ahsanullah, [36], we present the best linear least squares prediction (BLLSP) of $\mu$ and $\sigma$.

**Theorem 6.5.5** *The best linear least squares prediction (BLLSP),* $\ddot{X}^*(n+s,n,m,k)$ *of* $X^*(n+s,n,m,k)$ *based on* $X^*(1,n,m,k), X^*(2,n,m,k), \ldots, X^*(n,n,m,k)$ *is given by*

$$\ddot{X}^*(n+s,n,m,k) = \prod_{j=n+1}^{s+n} [\delta\gamma_j/(\delta\gamma_j+1)] X^*(n,n,m,k).$$

*Proof.* Using Theorem 6.3.2 and Lemma 6.2.3, we have

$$\begin{aligned}
&\ddot{X}^*(n+s,n,m,k) \\
&= E\Big(X^*(n+s,n,m,k)|X^*(1,n,m,k)=x_1,\ldots,X^*(n,n,m,k)=x_n\Big) \\
&= E\Big(X^*(n+s,n,m,k)|X^*(n,n,m,k)=x_n\Big) \\
&= E\Big(X^*(n,n,m,k)\prod_{j=n+1}^{s+n} W_j \Big| X^*(n,n,m,k)=x_n\Big) \\
&= x_n \prod_{j=n+1}^{s+n} E(W_j) \\
&= x_n \prod_{j=n+1}^{s+n} \frac{\delta\gamma_j}{\delta\gamma_j+1}.
\end{aligned}$$

■

**Remark 6.5.6** *For* $k=1$, $m=0$, $\gamma_j = n-j+1, j=1,2,\ldots$, *we obtain the BLLSP* $\ddot{X}_{n+s,n}$ *of* $X_{n+s,n}$ *from the order statistics* $X_{1,n}=x_1 > X_{2,n}=x_2 > \ldots > X_{n,n}=x_n$.

$$\ddot{X}_{n+s,n} = x_n \prod_{j=n+1}^{n+s} \frac{\delta(n-j+1)}{\delta(n-j+1)+1}.$$

*If we let* $\delta=1$, $k=1$, $m=-1$, $\gamma_i=1, i=1,2,\ldots$, *then from Theorem 6.5.5, the BLLSP* $\ddot{X}_{L(n+s)}$ *of* $X_{L(n+s)}$ *from* $n$ *lower record values* $X_{L(1)}=x_1, X_{L(2)}=x_2,\ldots,X_{L(n)}=x_n$ *from the power function distribution (see Chapter 3) is*

$$\ddot{X}_{L(n+s)} = x_n\left(\frac{\delta}{\delta+1}\right)^{s-1}.$$

**Estimation Of $\delta$ When $\mu$ And $\sigma$ Are Known**

We now consider the estimation of $\delta$ when $\mu = 0$ and $\sigma = 1$. We shall use the method of moment and maximum likelihood to obtain the estimate of $\delta$.

**Method of moment**

Using equation (6.4.32), the moment estimate $\tilde{\delta}$ for $\delta$ based on two consecutive lower generalized order statistics $X^*(r, n, m, k) = x_r$ and $X^*(r+1, n, m, k) = x_{r+1}$ for $\mu$ known is

$$\tilde{\delta} = \frac{1}{\gamma_{r+1}} \left[ \frac{x_{r+1} - \mu}{x_r - x_{r+1}} \right].$$

**Method of Maximum likelihood**

Using equations (6.1.1) and (6.3.15), we have the likelihood function of the power function probability density is given by

$$\begin{aligned} L(\theta|x_1, x_2, \ldots, x_n) &= \delta^r c_r x_r^{\delta\gamma_r} \prod_{i=1}^{r-1} x_i^{\delta(m+1)} \prod_{i=1}^{r} \frac{1}{x_i} \qquad (6.5.52) \\ &= c_r \left( \prod_{i=1}^{r} \frac{1}{x_i} \right) \delta^r \exp\left\{ -\delta \left[ (m+1) \sum_{i=1}^{r-1} \ln(1/x_i) + \gamma_r \ln(1/x_r) \right] \right\}. \end{aligned}$$

The log-likelihood of equation (6.5.52) is given by

$$\log(L(\delta|x_1, x_2, \ldots, x_n)) = \log(c) + r\log(\delta) - \delta \left[ (m+1) \sum_{i=1}^{r-1} \ln(1/x_i) + \gamma_r \ln(1/x_r) \right],$$

where $c = \log(c_r) - \sum_{i=1}^{r} x_i$.

Therefore, the maximum likelihood estimate for $\delta$, $\hat{\delta}_{MLE}$ is

$$\hat{\delta}_{MLE} = \frac{r}{(m+1)\sum_{i=1}^{r-1} \ln(1/x_i) + \gamma_r \ln(1/x_r)}. \qquad (6.5.53)$$

**Remark 6.5.7** *If we let $m = -1$ and $k = 1$, then $\gamma_r = 1$ and the maximum likelihood estimate $\hat{\delta}_{MLE}$ for $\delta$ is given by*

$$\hat{\delta}_{MLE} = -\frac{r}{\ln(x_r)}.$$

*If we let* $m = 0$ *and* $k = 1$*, then* $\gamma_r = n - r + 1$ *and (6.5.53) becomes*

$$\hat{\delta}_{MLE} = -\frac{r}{\sum_{i=1}^{r-1} \ln(x_i) + (n - r + 1)\ln(x_r)}.$$

## 6.6 Conclusion

In the present study, we have introduced the concepts of **Lower Generalized Order Statistics** for a given phenomenon that is probabilistically characterized by the power function probability distribution. We have developed some distributional properties of lower generalized order statistics. We have obtained some properties that are important to this distribution. We have developed the estimates of the location and scale parameters of this distribution given the shape parameter and also the estimates of the shape parameter given the location and scale parameters. In addition, we have developed methods to predict future observations given the present. Finally, coefficients of the best linear unbiased estimates are given.

| n | r | $\delta = 0.5$ | $\delta = 1$ | $\delta = 2$ | $\delta = 2.5$ | $\delta = 3$ | $\delta = 3.5$ | $\delta = 4$ | $\delta = 5$ |
|---|---|---|---|---|---|---|---|---|---|
| 2 | 1 | -0.50000 | -1.000000 | -2.00000 | -2.50000 | -3.00000 | -3.50000 | -4.00000 | -5.00000 |
| 2 | 2 | 1.50000 | 2.000000 | 3.00000 | 3.50000 | 4.00000 | 4.50000 | 5.00000 | 6.00000 |
| 4 | 1 | -0.07895 | -0.333333 | -1.09091 | -1.52439 | -1.97561 | -2.43839 | -2.90909 | -3.86598 |
| 4 | 2 | -0.02632 | 0.000000 | 0.18182 | 0.30488 | 0.43902 | 0.58057 | 0.72727 | 1.03093 |
| 4 | 3 | -0.07895 | 0.000000 | 0.27273 | 0.42683 | 0.58537 | 0.74645 | 0.90909 | 1.23711 |
| 4 | 4 | 1.18421 | 1.333333 | 1.63636 | 1.79268 | 1.95122 | 2.11137 | 2.27273 | 2.59794 |
| 5 | 1 | -0.04412 | -0.250000 | -0.96000 | -1.38274 | -1.82707 | -2.28522 | -2.75269 | -3.70553 |
| 5 | 2 | -0.01103 | 0.000000 | 0.12000 | 0.20741 | 0.30451 | 0.40807 | 0.51613 | 0.74111 |
| 5 | 3 | -0.02574 | 0.000000 | 0.16000 | 0.26272 | 0.37218 | 0.4858 | 0.60215 | 0.83992 |
| 5 | 4 | -0.07721 | 0.000000 | 0.24000 | 0.36781 | 0.49624 | 0.6246 | 0.75269 | 1.00791 |
| 5 | 5 | 1.15809 | 1.250000 | 1.44000 | 1.54480 | 1.65414 | 1.76674 | 1.88172 | 2.1166 |
| 6 | 1 | -0.02727 | -0.200000 | -0.87591 | -1.29154 | -1.73159 | -2.18708 | -2.65285 | -3.60383 |
| 6 | 2 | -0.00545 | 0.000000 | 0.08759 | 0.15498 | 0.23088 | 0.31244 | 0.39793 | 0.57661 |
| 6 | 3 | -0.01091 | 0.000000 | 0.10949 | 0.18598 | 0.26936 | 0.35707 | 0.44767 | 0.63427 |
| 6 | 4 | -0.02545 | 0.000000 | 0.14599 | 0.23558 | 0.32922 | 0.42509 | 0.52228 | 0.71884 |
| 6 | 5 | -0.07636 | 0.000000 | 0.21898 | 0.32981 | 0.43895 | 0.54654 | 0.65285 | 0.86261 |
| 6 | 6 | 1.14545 | 1.200000 | 1.31387 | 1.38519 | 1.46318 | 1.54594 | 1.63212 | 1.81149 |
| 7 | 1 | -0.01807 | -0.166667 | -0.81633 | -1.2268 | -1.66396 | -2.11779 | -2.58260 | -3.53272 |
| 7 | 2 | -0.00301 | 0.000000 | 0.06803 | 0.12268 | 0.18488 | 0.25212 | 0.32282 | 0.47103 |
| 7 | 3 | -0.00542 | 0.000000 | 0.08163 | 0.14231 | 0.20954 | 0.28093 | 0.35511 | 0.50871 |
| 7 | 4 | -0.01084 | 0.000000 | 0.10204 | 0.17077 | 0.24446 | 0.32106 | 0.39950 | 0.55958 |
| 7 | 5 | -0.0253 | 0.000000 | 0.13605 | 0.21631 | 0.29878 | 0.38222 | 0.46608 | 0.63419 |
| 7 | 6 | -0.0759 | 0.000000 | 0.20408 | 0.30283 | 0.39838 | 0.49143 | 0.58260 | 0.76103 |
| 7 | 7 | 1.13855 | 1.166667 | 1.22449 | 1.2719 | 1.32792 | 1.39003 | 1.45649 | 1.59817 |
| 9 | 1 | -0.00915 | -0.12500 | -0.73587 | -1.13927 | -1.57279 | -2.0248 | -2.48874 | -3.43852 |
| 9 | 2 | -0.00114 | 0.000000 | 0.04599 | 0.08544 | 0.13107 | 0.18079 | 0.23332 | 0.34385 |
| 9 | 3 | -0.00180 | 0.000000 | 0.05256 | 0.09521 | 0.14355 | 0.19554 | 0.24999 | 0.3635 |
| 9 | 4 | -0.00299 | 0.000000 | 0.06132 | 0.10790 | 0.1595 | 0.21417 | 0.27082 | 0.38773 |
| 9 | 5 | -0.00539 | 0.000000 | 0.07359 | 0.12517 | 0.18077 | 0.23864 | 0.29790 | 0.41875 |
| 9 | 6 | -0.01078 | 0.000000 | 0.09198 | 0.15020 | 0.21089 | 0.27273 | 0.33514 | 0.46063 |
| 9 | 7 | -0.02515 | 0.000000 | 0.12265 | 0.19026 | 0.25776 | 0.32468 | 0.39099 | 0.52204 |
| 9 | 8 | -0.07546 | 0.000000 | 0.18397 | 0.26636 | 0.34368 | 0.41745 | 0.48874 | 0.62645 |
| 9 | 9 | 1.13186 | 1.125000 | 1.10381 | 1.11871 | 1.14559 | 1.18079 | 1.22185 | 1.31555 |
| 10 | 1 | -0.00685 | -0.11111 | -0.70697 | -1.10779 | -1.54012 | -1.99161 | -2.45541 | -3.40533 |
| 10 | 2 | -0.00076 | 0.00000 | 0.03928 | 0.07385 | 0.11408 | 0.15806 | 0.20462 | 0.3027 |
| 10 | 3 | -0.00114 | 0.00000 | 0.04419 | 0.08124 | 0.12359 | 0.16935 | 0.21741 | 0.31783 |
| 10 | 4 | -0.00179 | 0.00000 | 0.0505 | 0.09052 | 0.13536 | 0.18318 | 0.23293 | 0.33599 |
| 10 | 5 | -0.00299 | 0.00000 | 0.05891 | 0.10259 | 0.1504 | 0.20063 | 0.25235 | 0.35839 |
| 10 | 6 | -0.00538 | 0.00000 | 0.070700 | 0.11901 | 0.17045 | 0.22355 | 0.27758 | 0.38706 |
| 10 | 7 | -0.01076 | 0.00000 | 0.08837 | 0.14281 | 0.19886 | 0.25549 | 0.31228 | 0.42577 |
| 10 | 8 | -0.02511 | 0.00000 | 0.11783 | 0.18089 | 0.24305 | 0.30416 | 0.36432 | 0.48254 |
| 10 | 9 | -0.07534 | 0.00000 | 0.17674 | 0.25325 | 0.32407 | 0.39106 | 0.45541 | 0.57905 |
| 10 | 10 | 1.13014 | 0.11111 | 1.06046 | 1.06363 | 1.08024 | 1.10613 | 1.13851 | 1.21600 |

Table 6.1: Coefficients for the BLUE of $\mu$

| n | r | $\delta = 0.5$ | $\delta = 1$ | $\delta = 2$ | $\delta = 2.5$ | $\delta = 3$ | $\delta = 3.5$ | $\delta = 4$ | $\delta = 5$ |
|---|---|---|---|---|---|---|---|---|---|
| 2 | 1 | 3.00000 | 3.00000 | 3.750000 | 4.20000 | 4.66667 | 5.14286 | 5.625 | 6.60000 |
| 2 | 2 | -3.00000 | -3.00000 | -3.750000 | -4.20000 | -4.66667 | -5.14286 | -5.625 | -6.60000 |
| 4 | 1 | 1.61842 | 1.66667 | 2.35227 | 2.77683 | 3.22358 | 3.68399 | 4.15341 | 5.10928 |
| 4 | 2 | 0.03947 | 0.00000 | -0.20455 | -0.33537 | -0.47561 | -0.62204 | -0.77273 | -1.08247 |
| 4 | 3 | 0.11842 | 0.00000 | -0.30682 | -0.46951 | -0.63415 | -0.79976 | -0.96591 | -1.29897 |
| 4 | 4 | -1.77632 | -1.66667 | -1.84091 | -1.97195 | -2.11382 | -2.26219 | -2.41477 | -2.72784 |
| 5 | 1 | 1.46176 | 1.50000 | 2.15600 | 2.57336 | 3.01554 | 3.47294 | 3.94032 | 4.89375 |
| 5 | 2 | 0.01544 | 0.00000 | -0.13200 | -0.224 | -0.32481 | -0.43139 | -0.54194 | -0.77075 |
| 5 | 3 | 0.03603 | 0.00000 | -0.17600 | -0.28374 | -0.39699 | -0.51356 | -0.63226 | -0.87352 |
| 5 | 4 | 0.10809 | 0.00000 | -0.26400 | -0.39723 | -0.52932 | -0.6603 | -0.79032 | -1.04822 |
| 5 | 5 | -1.62132 | -1.50000 | -1.58400 | -1.66838 | -1.76441 | -1.86769 | -1.97581 | -2.20126 |
| 6 | 1 | 1.36970 | 1.40000 | 2.03224 | 2.4443 | 2.88335 | 3.33885 | 3.80505 | 4.75729 |
| 6 | 2 | 0.00727 | 0.00000 | -0.09489 | -0.16532 | -0.24371 | -0.32732 | -0.41451 | -0.59583 |
| 6 | 3 | 0.01455 | 0.00000 | -0.11861 | -0.19838 | -0.28432 | -0.37408 | -0.46632 | -0.65542 |
| 6 | 4 | 0.03394 | 0.00000 | -0.15815 | -0.25128 | -0.34751 | -0.44533 | -0.54404 | -0.74281 |
| 6 | 5 | 0.10182 | 0.00000 | -0.23723 | -0.35179 | -0.46334 | -0.57257 | -0.68005 | -0.89137 |
| 6 | 6 | -1.52727 | -1.40000 | -1.42336 | -1.47753 | -1.54447 | -1.61955 | -1.70013 | -1.87187 |
| 7 | 1 | 1.30895 | 1.33333 | 1.94606 | 2.35404 | 2.79082 | 3.24505 | 3.71055 | 4.66223 |
| 7 | 2 | 0.00387 | 0.00000 | -0.07289 | -0.12969 | -0.19369 | -0.26241 | -0.33435 | -0.48449 |
| 7 | 3 | 0.00697 | 0.00000 | -0.08746 | -0.15044 | -0.21951 | -0.29240 | -0.36779 | -0.52325 |
| 7 | 4 | 0.01394 | 0.00000 | -0.10933 | -0.18053 | -0.2561 | -0.33417 | -0.41376 | -0.57557 |
| 7 | 5 | 0.03253 | 0.00000 | -0.14577 | -0.22867 | -0.31301 | -0.39782 | -0.48272 | -0.65231 |
| 7 | 6 | 0.09759 | 0.00000 | -0.21866 | -0.32014 | -0.41735 | -0.51148 | -0.6034 | -0.78278 |
| 7 | 7 | -1.46386 | -1.33333 | -1.31195 | -1.34458 | -1.39116 | -1.44677 | -1.50851 | -1.64383 |
| 9 | 1 | 1.23340 | 1.25000 | 1.83231 | 2.23434 | 2.66808 | 3.12082 | 3.58565 | 4.53715 |
| 9 | 2 | 0.00140 | 0.00000 | -0.04855 | -0.08924 | -0.13592 | -0.18652 | -0.2398 | -0.35149 |
| 9 | 3 | 0.00220 | 0.00000 | -0.05548 | -0.09944 | -0.14887 | -0.20175 | -0.25693 | -0.37158 |
| 9 | 4 | 0.00366 | 0.00000 | -0.06473 | -0.1127 | -0.16541 | -0.22097 | -0.27834 | -0.39635 |
| 9 | 5 | 0.00659 | 0.00000 | -0.07768 | -0.13073 | -0.18746 | -0.24622 | -0.30617 | -0.42806 |
| 9 | 6 | 0.01318 | 0.00000 | -0.09709 | -0.15688 | -0.2187 | -0.28139 | -0.34445 | -0.47086 |
| 9 | 7 | 0.030740 | 0.00000 | -0.12946 | -0.19871 | -0.2673 | -0.33499 | -0.40185 | -0.53365 |
| 9 | 8 | 0.09223 | 0.00000 | -0.19419 | -0.2782 | -0.35641 | -0.4307 | -0.50232 | -0.64037 |
| 9 | 9 | -1.38338 | -1.2500 | -1.16513 | -1.16844 | -1.18802 | -1.21827 | -1.25579 | -1.34479 |
| 10 | 1 | 1.20822 | 1.22222 | 1.79232 | 2.19210 | 2.62479 | 3.07709 | 3.54179 | 4.49344 |
| 10 | 2 | 0.00091 | 0.00000 | -0.04124 | -0.07681 | -0.11789 | -0.16258 | -0.20973 | -0.30875 |
| 10 | 3 | 0.00137 | 0.00000 | -0.04640 | -0.08449 | -0.12771 | -0.17419 | -0.22284 | -0.32419 |
| 10 | 4 | 0.00215 | 0.00000 | -0.05302 | -0.09414 | -0.13987 | -0.18841 | -0.23876 | -0.34271 |
| 10 | 5 | 0.00359 | 0.00000 | -0.06186 | -0.10670 | -0.15541 | -0.20636 | -0.25865 | -0.36556 |
| 10 | 6 | 0.00646 | 0.00000 | -0.07423 | -0.12377 | -0.17614 | -0.22994 | -0.28452 | -0.39481 |
| 10 | 7 | 0.01292 | 0.00000 | -0.09279 | -0.14852 | -0.20549 | -0.26279 | -0.32008 | -0.43429 |
| 10 | 8 | 0.03014 | 0.00000 | -0.12372 | -0.18813 | -0.25116 | -0.31285 | -0.37343 | -0.49219 |
| 10 | 9 | 0.09041 | 0.00000 | -0.18558 | -0.26338 | -0.33488 | -0.40223 | -0.46679 | -0.59063 |
| 10 | 10 | -1.35616 | -1.22222 | -1.11348 | -1.10618 | -1.11625 | -1.13774 | -1.16698 | -1.24032 |

Table 6.2: Coefficients for the BLUE of $\sigma$

| n | $\delta = 0.5$ | $\delta = 1$ | $\delta = 2$ | $\delta = 2.5$ | $\delta = 3$ | $\delta = 3.5$ | $\delta = 4$ | $\delta = 5$ |
|---|---|---|---|---|---|---|---|---|
| 2 | 0.06667 | 0.16667 | 0.33333 | 0.39683 | 0.45000 | 0.49495 | 0.53333 | 0.59524 |
| | 0.60000 | 0.50000 | 0.56250 | 0.60000 | 0.63333 | 0.66234 | 0.68750 | 0.72857 |
| | -0.13333 | -0.25000 | -0.41667 | -0.47619 | -0.52500 | -0.56566 | -0.60000 | -0.65476 |
| 3 | 0.02381 | 0.07500 | 0.16667 | 0.20140 | 0.23011 | 0.25407 | 0.27429 | 0.30637 |
| | 0.25661 | 0.20000 | 0.24769 | 0.27272 | 0.29419 | 0.31239 | 0.32786 | 0.35251 |
| | -0.03968 | -0.100000 | -0.19444 | -0.22825 | -0.25568 | -0.27827 | -0.29714 | -0.32680 |
| 4 | 0.01128 | 0.044444 | 0.10909 | 0.13372 | 0.15394 | 0.17069 | 0.1847 | 0.20674 |
| | 0.15038 | 0.111111 | 0.15057 | 0.17013 | 0.18662 | 0.20041 | 0.21199 | 0.23020 |
| | -0.01692 | -0.055556 | -0.12273 | -0.14709 | -0.16677 | -0.18288 | -0.19625 | -0.21707 |
| 5 | 0.00613 | 0.029762 | 0.08000 | 0.09934 | 0.11515 | 0.12818 | 0.13902 | 0.15596 |
| | 0.10090 | 0.071429 | 0.10513 | 0.12138 | 0.13494 | 0.14617 | 0.15555 | 0.17016 |
| | -0.00858 | -0.035714 | -0.08800 | -0.10728 | -0.12283 | -0.13550 | -0.14598 | -0.16220 |
| 6 | 0.00364 | 0.021429 | 0.06257 | 0.07859 | 0.09167 | 0.10241 | 0.11131 | 0.12513 |
| | 0.07313 | 0.050000 | 0.07938 | 0.09334 | 0.10492 | 0.11446 | 0.12238 | 0.13466 |
| | -0.00485 | -0.025000 | -0.06778 | -0.08383 | -0.10728 | -0.14598 | -0.11595 | -0.12930 |
| 7 | 0.00230 | 0.016204 | 0.05102 | 0.06476 | 0.07596 | 0.08513 | 0.09271 | 0.10443 |
| | 0.05575 | 0.037037 | 0.06303 | 0.07531 | 0.08544 | 0.09376 | 0.10064 | 0.11126 |
| | -0.00296 | -0.018519 | -0.05466 | -0.06846 | -0.07958 | -0.08860 | -0.09602 | -0.10741 |
| 8 | 0.00153 | 0.012698 | 0.04285 | 0.05491 | 0.06473 | 0.07275 | 0.07937 | 0.08957 |
| | 0.04405 | 0.028571 | 0.051850 | 0.06281 | 0.07184 | 0.07923 | 0.08533 | 0.0947 |
| | -0.00191 | -0.014286 | -0.04553 | -0.05766 | -0.06743 | -0.07535 | -0.08185 | -0.09181 |
| 9 | 0.00106 | 0.010227 | 0.03679 | 0.04756 | 0.05632 | 0.06346 | 0.06935 | 0.07839 |
| | 0.03576 | 0.022727 | 0.04377 | 0.05369 | 0.06185 | 0.06850 | 0.07398 | 0.08238 |
| | -0.00129 | -0.011364 | -0.03884 | -0.04967 | -0.05841 | -0.06548 | -0.07127 | -0.08013 |
| 10 | 0.00075 | 0.008418 | 0.03214 | 0.04187 | 0.04979 | 0.05624 | 0.06154 | 0.06967 |
| | 0.02966 | 0.018519 | 0.03770 | 0.04676 | 0.0542 | 0.06027 | 0.06525 | 0.07287 |
| | 0.00090 | -0.009259 | -0.03374 | -0.04354 | -0.05145 | -0.05784 | -0.06308 | -0.07106 |

Table 6.3: Variances Covariances of the BLUEs of $\mu$ and $\sigma$ in terms of $\sigma^2$

# 7 Future Research Studies

We have illustrated in the present study, the usefulness of records in terms of estimation, and prediction using a parametric model. In this chapter, we shall briefly discuss some possible extensions of what we have studied in the previous chapters.

The study of other distribution with respect to records is not very much exploited. This could be because of the complex form of the probability density function of record observation since close form solutions will not be obtained for the maximum likelihood estimates. However, The maximum likelihood estimates can be obtained by iteration and their properties studied using simulations. Therefore, probability density functions for example the Johnson probability density function can be considered with respect to record breaking data. This is because the Johnson probability density function is very useful in life science and engineering.

Investigating the concept of records with respect to using the kernel density approach to characterize the behavior of records is an interesting extension of the theory of records. This approach will be very useful in cases where a classical distribution cannot be identified to statistically fit the underlying data from which the record observations are obtained.

The inherent missing data structure present in these problems leads to likelihood functions that contain possibly high-dimensional integrals, rendering traditional maximum likelihood methods difficult or not feasible. An interesting extension will be to obtain arbitrarily accurate approximations to the likelihood function by iteratively applying Monte Carlo integration methods (Geyer and Thompson, [21]). Subiteration using the Gibbs sampler may help to evaluate any multivariate integrals encountered during this process.

Let $X_n = (X_n^{(1)}, X_n^{(2)}), n = 1, 2, \ldots$ be independent and identically distributed $R^2$ random vector with common distribution function $F$. To obtain the records of such a sequence, we employ the definition of the natural partial ordering of $R^2$, meaning $X_n$ is a record if there is a record simultaneously in both coordinates, that is, $X_n^{(1)} > X_j^{(1)}, j = 1, \ldots, n-1$ and $X_n^{(2)} > X_j^{(2)}, j = 1, \ldots, n-1$, see Goldie and Resnick, ([23],[24]) and Gnedin, [22]. An important extension of the results presented in the previous chapters is to develop the theory of records when we are dealing with multivariate data.

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