3-31-2009

Christoffel Function Asymptotics and Universality for Szegő Weights in the Complex Plane

Elliot M. Findley
University of South Florida

Christoffel Function Asymptotics and Universality for Szegő Weights in the Complex Plane

by

Elliot M. Findley

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
Department of Mathematics
College of Arts and Sciences
University of South Florida

Major Professor: Vilmos Totik, Ph.D.
Thomas Bieske, Ph.D.
Evguenii Rakhmanov, Ph.D.
Boris Schekhtman, Ph.D.

Date of Approval:
March 31, 2009

Keywords: Faber, Orthogonal Polynomials, Zero-Spacing, Ill-Posed Problems,
Potential Theory

Acknowledgements

I am inexpressibly grateful and deeply humbled to have studied under Vilmos Totik. Among the most insightful men I have ever known, he is certainly the kindest. His fatherly guidance and constant support have enriched my life more than I can articulate and I will forever treasure the time we have spent together. I sincerely thank the mathematics and physics faculty at the University of South Florida for their kind instruction and their many investments in the young minds who depend upon them. Among their rank, I have found at once the most gracious and brilliant people I have ever known. Finally, I pour out my gratitude to the Lord Jesus Christ, the architect of all wisdom and truth and the source of every fortuity. He has heard my prayer and answered.

Table of Contents

Christoffel Function Asymptotics and Universality for Szegő Weights in the Complex Plane

Elliot M. Findley

Abstract

In 1991, A. Máté precisely calculated the first-order asymptotic behavior of the sequence of Christoffel functions associated with Szegő measures on the unit circle. Our principal goal is the abstraction of his result in two directions: We compute the translated asymptotics, $\lim_n \lambda_n(\mu, x + a/n)$, and obtain, as a corollary, a universality limit for the fairly broad class of Szegő weights. Finally, we prove Máté's result for measures supported on smooth curves in the plane. Our proof of the latter derives, in part, from a precise estimate of certain weighted means of the Faber polynomials associated with the support of the measure. Finally, we investigate a variety of applications, including two novel applications to ill-posed problems in Hilbert space and the mean ergodic theorem.

1 Introduction

Let μ be a compactly supported, Borel measure in the complex plane. We are interested in the precise asymptotic behavior of the associated sequence of Christoffel functions, $\{\lambda_n(\mu, z)\}_{n=0}^{\infty}$. When μ is supported on the real line, the Christoffel functions are defined as follows:

$$\lambda_n(\mu, x) = \inf \frac{1}{|P(x)|^2} \int |P|^2 \, d\mu, \tag{1.0.1}$$

where the infimum is evaluated over all polynomials P of degree at most $n-1$ that do not vanish at x. For measures supported on the unit circle or any smooth homeomorphic image thereof, the infimum ranges over the complex polynomials and the definition is slightly modified:

$$\lambda_n(\mu, z) = \inf \frac{1}{|P(z)|^2} \frac{1}{2\pi} \int |P|^2 \, d\mu. \tag{1.0.2}$$

(In much of the literature, the latter is denoted $\omega_n(\mu, e^{i\theta})$ when μ is supported on the unit circle. We will adhere to this convention in chapter 2, but revert to the notation of (1.0.2) in the final chapter.)

From interpolation and quadrature to stochastic processes and statistical inference, this simple sequence has diverse utility. For the simplest example, we look to orthogonal polynomials. Let $\{p_n\}_n$ denote the sequence of orthonormal polynomials associated with μ. If we expand $P(x)$ in (1.0.1) in terms of this sequence, then elementary linear algebra provides the following representation of the Christoffel functions:

$$\frac{1}{\lambda_n(\mu, x)} = \sum_{k=0}^{n-1} |p_k(x)|^2. \tag{1.0.3}$$

(When μ is supported on a homeomorphic image of the unit circle, as in (1.0.2), we will take p_n orthonormal to $\mu/2\pi$, so that this identity will hold for all measures, regardless of the support.) The relationship between Christoffel functions and orthogonal polynomials is deep. Indeed, Paul Nevai, in his very helpful survey of the contributions of Géza Freud ([14]), easily derives Szegő's seminal theory of orthogonal polynomials on the unit circle entirely from a consideration of Christoffel-function asymptotics.

For a stellar exploration of further applications of Christoffel function asymptotics, we refer the reader to Grenander's and Szegő's treatise on Toeplitz matrices ([5]). Before proceeding to our main results, we will showcase the versatility of this subject with a few more examples, the first taken from classical approximation theory.

1.1 Classical Approximation

Consider some orthonormal basis of polynomials $\{p_k\}$ with respect to the measure μ. We may approximate any $f \in L_2(\mu)$ by the following polynomials:

$$S_n = \sum_{k=0}^{n-1} c_k p_k, \quad \text{where} \quad c_k := \int f(x)\overline{p_k(x)}\, d\mu(x).$$

S_n is a linear function of f, so we may define the linear operators $L_n(f) := S_n$ and associated norms

$$\Omega_n(x) := \sup_{\|f\|_{L^\infty(\mu)} \leq 1} |L_n(f)(x)|,$$

called the Lebesgue functions. $\{L_n\}$ are integral operators, with associated kernels

$$K_n(x, t) := \sum_{k=0}^{n-1} p_k(x)\overline{p_k(t)},$$

called the reproducing kernels associated with μ. The reproducing kernels are determined uniquely by their salient property:

$$\int K_n(x,t)P(t)\,d\mu(t) = P(x)$$

for any polynomial P of degree less than n. Let P_n denote the best uniform approximant of f by polynomials of degree at most $n-1$, with error $E_n(f)$. The error of approximation of f by its partial sums is determined by $|S_n - f|$. Since $L_n(P_k) = P_k$,

$$|S_n - f|(x) \leq |S_n - P_n|(x) + |P_n - f|(x) \leq (\Omega_n(x) + 1)E_n(f),$$

Now,

$$\begin{aligned}\Omega_n(x) = \int |K_n(x,t)|\,d\mu(t) &\leq \left(\mu(\mathbf{C})\int |K_n(x,t)|^2\,d\mu(t)\right)^{1/2} \\ &= \left(\mu(\mathbf{C})\sum_{k=0}^{n-1}|p_k(x)|^2\right)^{1/2} = \left(\frac{\mu(\mathbf{C})}{\lambda_n(\mu,x)}\right)^{1/2}, \quad (1.1.4)\end{aligned}$$

where $\mathbf{C}$ is the complex plane. Thus, lower bounds on the series of Christoffel functions imply upper bounds on the rate of convergence of a Fourier series. In fact, the Christoffel and Lebesgue functions sustain a more intimate relationship than the trivial inequality, (1.1.4), suggests: For a class of measures, μ, supported on the interval, $[-1,1]$, P. Nevai proves in [14, p. 12] that $\lambda_n(\mu,x)\,\Omega_n(x)^2 = o(1)$ as $n \to \infty$. The asymptotic behavior of the sequence of Christoffel functions has profound implications for approximation theory. For another (perhaps the oldest) example, let us consider integral quadrature ([4]).

1.2 Quadrature

Presume that the compact support of μ lies in the interval $[-1,1]$ so that the zeros of $p_n(x)$, denoted $x_1 < x_2 < \cdots < x_n$, are real. Let l_k denote the k-th Lagrange polynomial: the unique polynomial of degree at most n that vanishes at x_j when

$j \neq k$ and equals 1 at x_k. Any polynomial of degree at most n can be precisely interpolated at $\{x_k\}_{k=1}^n$ as follows:

$$p(x) = \sum_{k=1}^{n} p(x_k)\, l_k(x).$$

Since $\{l_k\}$ is an orthogonal set with respect to μ, this implies the following quadrature formula:

$$\int p(x)^2\, d\mu(x) = \sum_{k=1}^{n} p(x_k)^2 \lambda_{nk},$$

where $\lambda_{nk} = \int l_k^2\, d\mu$ are the Christoffel-Cotes numbers. The Christoffel-Darboux identity is fundamental for measures supported on the real line:

$$K_n(x,t) = c_n \frac{p_n(x)\, p_{n+1}(t) - p_n(t)\, p_{n+1}(x)}{x-t}, \tag{1.2.5}$$

where c_n is constant. It implies that $K_n(x_j, x_k) = 0$ whenever $j \neq k$. Therefore, $l_k(x) = \lambda_n(\mu, x_k) K_n(x, x_k)$. Squaring both sides and integrating against μ gives $\lambda_{nk} = \lambda_n(\mu, x_k)$. The asymptotic behavior of Christoffel functions is clearly central to the convergence of quadrature schemes.

1.3 Universality

The asymptotic behavior of the reproducing kernels, K_n, plays a significant role in the theory of random matrices and statistical mechanics. The so-called "universality" limit is especially important: For certain classes of measures,

$$\lim_{n\to\infty} \frac{K_n(x + a/n, x + b/n)}{K_n(x,x)} = \frac{\sin\left((a-b)/\sqrt{1-x^2}\right)}{(a-b)/\sqrt{1-x^2}}, \tag{1.3.6}$$

uniformly as a and b range over a compact interval. D.S. Lubinsky recently advanced the frontier of universality limits by means of a very simple relationship between K_n and the Christoffel functions, λ_n. (See [10].) For two measures $\mu \leq \mu^*$ and their associated Christoffel functions λ_n and λ_n^*,

$$\frac{|K_n(x,y) - K_n^*(x,y)|}{K_n(x,x)} \leq \left(\frac{\lambda_n(x)}{\lambda_n(y)}\right)^{1/2} \left(1 - \frac{\lambda_n(x)}{\lambda_n^*(x)}\right)^{1/2}.$$

Universality follows if the translated limit $\lim_{n\to\infty} n\,\lambda_n(\mu, x + a/n)$ can be calculated. Lubinsky does so, and proves (1.3.6) for a large class of measures. Among the primary contributions of this dissertation is the extension of Lubinsky's technique to a still broader class of measures–at present, the most general class–Szegő's class ([3]). Our final application concerns the asymptotic spacing of the zeros of p_n.

1.4 Zero-Spacing

The spacing of the zeros, $x_{nn} < x_{n-1,n} < \cdots < x_{1n}$, of $p_n(x)$ has received much attention in classical analysis. E. Levin and D. Lubinsky ([9]) were the first to compute the precise value of $\lim_{n\to\infty} n\,(x_{kn} - x_{k+1,n})$, from universality limits. The connection between universality and zero-spacing is fairly transparent: the Christoffel-Darboux identity, (1.2.5), implies that $K_n(x_{kn}, x_{k+1,n}) = 0$. Universality suggests that this is possible only if $n\,(x_{kn} - x_{k+1,n}) \sim \pi\sqrt{1-x^2}$ as $n \to \infty$. In 2.5, we follow Levin's and Lubinsky's technique to prove the following, the most general result presently known: for almost every $x \in [-1,1]$, if

$$|x_{kn} - x| = O\left(\frac{1}{n}\right), \tag{1.4.7}$$

for some sequence $k = k(n)$, then

$$\lim_{n\to\infty} (x_{kn} - x_{k+1,n}) \frac{n}{\pi\sqrt{1-x^2}} = 1. \tag{1.4.8}$$

For another application of Lubinsky's technique, see [22]. Such applications of our subject abound. Later, we will furnish a novel application of Christoffel-function asymptotics to ill-posed problems in operator theory. First, let us briefly review the history of this subject.

1.5 A Brief History

The asymptotic behavior of $\{\lambda_n(\mu, z)\}_n$ differs dramatically according to whether z is contained in the support of μ or not. If, for instance, μ is supported on the unit circle and $|z| < 1$, Szegő proved ([11, p. 434]) that

$$\lim_{n\to\infty} \lambda_n(\mu, z) = (1 - |z|^2)|D(\mu, z)|,$$

where $D(\mu, z)$ is the Szegő function associated with μ,

$$D(\mu, z) = \exp\left\{ \frac{1}{4\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log \mu'(\theta)\, d\theta \right\}, \qquad (|z| < 1).$$

On the support, when $|z| = 1$, the situation is much simpler: $\lambda_n(\mu, z) \to \mu(\{z\})$. What is the rate of this convergence? Szegő again found the correct formula: for absolutely continuous measures with sufficiently smooth, positive weights, μ',

$$\lim_{n\to\infty} n\, \lambda_n(\mu, e^{i\theta}) = \mu'(\theta). \tag{1.5.9}$$

As we have seen, this result has profound implications for a variety of subjects. Consequently, many have attempted to extend Szegő's calculation to ever-broader classes of measures.

The most precise asymptotic formulas for the sequence of Christoffel functions were given by Freud in his dissertation and reiterated in his treatise on orthogonal polynomials, [4, p. 271]. Let μ be an absolutely continuous measure supported on $[-1, 1]$ whose weight, μ', is in $Lip(\alpha)$ and satisfies $\mu'(x) \geq Q(x)^2$ for some polynomial $Q(x)$. He proves the the following representation [4, p. 254]:

$$\frac{1}{\lambda_n(\mu, x)} = \frac{n}{\pi} \frac{1}{\mu'(x)\sqrt{1 - x^2}} + O(n^{1-\alpha}) \qquad (\alpha < 1)$$
$$\frac{1}{\lambda_n(\mu, x)} = \frac{n}{\pi} \frac{1}{\mu'(x)\sqrt{1 - x^2}} + O(\log n) \qquad (\alpha = 1).$$

In particular,

$$\lim_{n\to\infty} n\,\lambda_n(\mu,x) = \pi\sqrt{1-x^2}\mu'(x), \tag{1.5.10}$$

the analogue of Szegő's result for measures supported on $[-1,1]$. (In fact, the two results are equivalent by virtue of a simple formula we will cite in the next chapter.) After Freud, the next substantial advancement was made by Máté and Nevai in [12].

Szegő's class consists of measures supported on the unit circle whose weights satisfy Szegő's condition,

$$\int_{-\pi}^{\pi} \log \mu'(\theta)\,d\theta > -\infty. \tag{1.5.11}$$

Máté and Nevai nearly obtained a full generalization of (1.5.9) for the Szegő class: they proved, for almost every $\theta \in [-\pi,\pi]$, that

$$\frac{e}{\pi}\,\mu'(\theta) \le \liminf_{n\to\infty} n\,\lambda_n(\mu,e^{i\theta}) \le \limsup_{n\to\infty} n\,\lambda_n(\mu,e^{i\theta}) \le \mu'(\theta). \tag{1.5.12}$$

Their result constitutes a dramatic improvement. Although they do not obtain the precise value of the limit (1.5.9), their hypothesis—Szegő's condition—is far less restrictive than Freud's: it makes no assumptions on the smoothness of the weight. In particular, (1.5.12) applies to every measure whose weight is bounded below by a positive constant.

The holy grail was finally recovered in 1991 by the circuitous and technical argument of Máté which establishes (1.5.9) for Szegő's class in his landmark paper authored jointly with Paul Nevai and Vilmos Totik ([11]). Nevai uses Máté's result to prove (1.5.10) via the simple formula to which we have already alluded. Totik further shows that Szegő's condition is unnecessarily restrictive: $\log \mu'$ need only be integrable over an open interval containing x. Under this condition, he proves (1.5.10) for the much larger class of regular measures, studied by Ullman ([23]). (This class is far broader than Szegő's: μ is regular if its Christoffel functions satisfy the very general condition $\liminf_{n\to\infty} \lambda_n(\mu,x)^{1/n} \ge 1$.) Since their paper, no further generalizations have been proved.

This dissertation comprises two papers by the author ([2, 3]) which substantially improve upon the result of Máté ([11]) in the following aspects: In [3], we combine Máté's approach with that of D. Lubinsky in [10] to prove the most general universality limit to date. In [2], we adapt Máté's technique to obtain (1.5.9) for the class of measures supported on smooth curves satisfying a modified Szegő condition. This is among the first calculation of the precise value of $\lim_{n\to\infty} n\,\lambda_n(\mu, z)$ for measures with general supports. Totik was the first to compute this value for measures supported on disjoint, real intervals ([21]). The only other result of this type was obtained by Golinskii for Chebyshev-type weights on circular arcs ([6]).

2 Universality

In this chapter, μ denotes a finite Borel measure supported on either the unit circle (equivalently on $(-\pi, \pi)$) or the interval $[-1, 1]$. μ' is the weight associated with its absolutely continuous part. If μ is supported on $[-1, 1]$, the Christoffel functions are defined by

$$\lambda_n(\mu, x) = \inf \int |P_{n-1}|^2 \, d\mu,$$

where the infimum is taken over all complex polynomials of degree at most $n-1$ which equal unity at x. For measures supported on the unit circle, the integral is evaluated with respect to $\mu/2\pi$, and the Christoffel functions are denoted by $\omega_n(\mu, z)$.

Let $\{p_n\}$ be the orthonormal polynomials associated with the measure μ. For μ supported on the interval, $[-1, 1]$, they are defined (up to a constant multiple of unit modulus) by the conditions

$$\int p_n(t) t^k d\mu(t) = 0 \quad \text{and} \quad \int |p_n|^2 \, d\mu = 1.$$

for all $0 \leq k < n$. As before, for measures on the unit circle, the integrals are evaluated with respect to $\mu/2\pi$. The reproducing kernels for μ are defined by

$$K_n(x, t) = \sum_{k=0}^{n-1} \overline{p_k(x)} p_k(t),$$

and are related to the Christoffel functions $\lambda_n(x) := \lambda_n(\mu, x)$ by

$$\frac{1}{\lambda_n(x)} = K_n(x, x). \tag{2.0.1}$$

The reproducing kernels are so named because of their salient feature: For all polynomials, P, with degree at most $n-1$,

$$\int P(x)K_n(x,t)\,d\mu(x) = P(t).$$

We will investigate measures whose weights satisfy Szegő's condition locally, that is,

$$\int_I \log\mu'(\theta)\,d\theta > -\infty,$$

for some interval I. Our results require that the measures also be *regular* in the sense of Ullman ([23]). For a comprehensive treatment of the theory of regular measures, see the book by Stahl and Totik ([19]). Regularity of a measure μ with compact support K is equivalent to the following condition: For every sequence, $\{P_n\}_{n=1}^{\infty}$, of polynomials whose degrees are not greater than their indices,

$$\limsup_{n\to\infty}\left(\frac{\|P_n\|_K}{\|P_n\|_\mu}\right)^{1/n} \le 1, \tag{2.0.2}$$

where

$$\|P\|_\mu^2 = \int |P|^2\,d\mu \qquad \text{and} \qquad \|P\|_K = \sup_{z\in K}|P(z)|.$$

The class of regular measures is far larger than Szegő's class.

D. S. Lubinsky, in [10], established the following inequality relating the reproducing kernels of two measures, $\mu \le \mu^*$, to their associated Christoffel functions, λ and λ^*:

$$\frac{|K_n(x,y)-K_n^*(x,y)|}{K_n(x,x)} \le \left(\frac{\lambda_n(x)}{\lambda_n(y)}\right)^{1/2}\left(1-\frac{\lambda_n(x)}{\lambda_n^*(x)}\right)^{1/2}.$$

He also proves the following asymptotic formula for translated Christoffel functions of regular measures whose weights are positive and continuous on some interval, I:

$$\lim_{n\to\infty} n\,\lambda_n(x+a/n) = \pi\sqrt{1-x^2}\mu'(x), \tag{2.0.3}$$

uniformly for $x \in I$ and a in a compact subset of $\mathbf{R}$. With these two formulae, he obtains a universality result for the aforementioned class of measures. The goal of this chapter is to prove Lubinsky's result for the broader class of regular measures which satisfy Szegő's condition locally. This is a substantial relaxation of Lubinsky's hypotheses: it obviates the requirement of continuity in favor of a less restrictive local Szegő condition.

Theorem 2.0.1 *Let μ be a regular Borel measure on $[-1,1]$ satisfying Szegő's condition,*

$$\int_I \log \mu'(t)\, dt > -\infty$$

on an open interval $I \subset [-1,1]$. Fix $A > 0$. Then, for almost every $x \in I$,

$$\lim_{n\to\infty} \frac{K_n(x+a/n, x+b/n)}{K_n(x,x)} = \frac{\sin\left((a-b)/\sqrt{1-x^2}\right)}{(a-b)/\sqrt{1-x^2}}, \tag{2.0.4}$$

uniformly for a, $b \in [-A, A]$.

The vehicle of this extension is the result of Máté, Nevai and Totik ([11]) which establishes the limit (2.0.3) (with $a = 0$) for regular measures satisfying Szegő's condition on an interval I at almost every $x \in I$. We will extend the techniques of these authors to obtain (2.0.3) for the broader class of regular, locally Szegő measures and then mimic Lubinsky's procedure to establish universality on I. Finally, we adapt the technique of Levin and Lubinsky ([9]) to prove a result on the distribution of the zeros of orthogonal polynomials associated with locally Szegő weights.

2.1 Translated Asymptotics

In this section we establish the asymptotics (2.0.3) of the translated Christoffel functions for measures on the unit circle which satisfy Szegő's condition.

Let $d\mu(x) = \mu'(x)dx + d\mu_s(x)$, where μ_s is the singular part of μ with respect to Lebesgue measure and $\mu'(x)dx$ is its absolutely continuous part. It is known (see [17, Theorem 8.6]) that

$$\mu'(t) = \lim_{\tau\to 0} \frac{\mu([t, t+\tau])}{\tau} \tag{2.1.5}$$

for almost all t (here, for $\tau < 0$, define $\mu([t, t+\tau])/\tau$ by $\mu([t+\tau, t])/|\tau|$). Recall that t is a Lebesgue point of μ' if

$$\lim_{\tau \to 0} \frac{1}{\tau} \int_0^\tau |\mu'(t+u) - \mu'(t)| du = 0. \tag{2.1.6}$$

We use the following terminology: t is a *Lebesgue point of* μ if the limit in (2.1.5) exists at t, and with this limit as $\mu'(t)$, (2.1.6) is true. Thus, almost all points are Lebesgue points of μ.

Theorem 2.1.1 *Let μ be a measure on $(-\pi, \pi)$ satisfying Szegő's condition. Fix $A > 0$. Then, for almost every $t \in (-\pi, \pi)$, we have*

$$\lim_{n \to \infty} n\, \omega_n\big(\mu, e^{i(t+a/n)}\big) = \mu'(t), \tag{2.1.7}$$

uniformly for $a \in [-A, A]$.

Furthermore, (2.1.7) *holds at every t which is a Lebesgue point of μ and for which e^{it} is a Lebesgue point of the Szegő function* (*see* (2.1.9)) *associated with μ.*

The upper limit actually holds for all finite Borel measures. This follows from the next lemma, an improvement of Lebesgue's result on the convergence of Fejér means (see [17, p. 244]). In what follows, $\sigma_n(\mu, z)$ is the n-th Fejér mean of the measure μ, given by

$$\sigma_n(\mu, z) = \int F_n(z - t)\, d\mu(t)$$

with normalized kernels

$$F_n(t) = \frac{1}{2\pi(n+1)} \frac{\sin^2\left((n+1)t/2\right)}{\sin^2(t/2)}.$$

Lemma 2.1.2 *Let μ be an absolutely continuous Borel measure on $(-\pi, \pi)$ such that*

$$\mu'(0) := \lim_{t \searrow 0} \frac{\mu([-t, t])}{2t},$$

exists. Then the translated Fejér means $\sigma_n(\mu, e^{ia/n}) \to \mu'(0)$ as $n \to \infty$ uniformly for

$a \in [-A, A]$.

The proof requires a result concerning Hardy's maximal function which is defined–for a measure μ supported on the real line–by

$$M\mu(x) = \sup_{t>0} \frac{\mu([x-t, x+t])}{2t}.$$

Lemma 2.1.3 *Let f be an even, positive function on $[-\pi, \pi]$, decreasing away from 0. Then, for any measure μ supported on an interval $I = [-t, t] \subseteq [-\pi, \pi]$, we have*

$$\int_I f \, d\mu \leq M\mu(0) \int_I f(t) \, dt.$$

Proof. Let $I = I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n$ be a nested sequence of symmetric intervals and choose positive numbers $a_1, \ldots, a_n$, such that $f(t) \leq s(t) := \sum_k a_k \chi_{I_k}(t)$. Then

$$\int f \, d\mu \leq \sum_k a_k \mu(I_k) \leq M\mu(0) \sum_k a_k |I_k| = M\mu(0) \int s(t) \, dt,$$

where $|\cdot|$ denotes Lebesgue measure. Since f decreases away from 0 and is even, there is a sequence of simple functions s_n of the form $s_n = \sum_k a_k \chi_{I_k}$ which dominate f and for which $\int s_n(t) dt \to \int f(t) dt$. This establishes the lemma.

■

Proof. (Lemma 2.1.2) Without loss of generality assume $\mu'(0) = 0$ (subtract a constant if necessary). We begin with the claim that for some constant $C > 0$,

$$F_n(t) \leq \min\left(n+1, \frac{\pi}{2(n+1)t^2}\right) \leq \frac{C(n+1)}{1+(n+1)^2 t^2} \tag{2.1.8}$$

for every $t \in [-\pi, \pi]$ and $n \geq 0$. To see the first inequality, observe that $\left|\sin \frac{t}{2}\right| \geq \frac{1}{\pi}|t|$ for all $t \in [-\pi, \pi]$. Thus,

$$F_n(t) \leq \frac{1}{2\pi(n+1)} \frac{\pi^2}{t^2}.$$

But the kernel F_n is the average of the first $n+1$ Dirichlet kernels, so its maximum is achieved at 0 and is equal to $(n+1)/2\pi$. This proves the first inequality. The second inequality is true for any $C > 1 + \pi/2$. Indeed,

$$n + 1 < \frac{\pi}{2(n+1)t^2} \ \Rightarrow\ \frac{C(n+1)}{1+(n+1)^2t^2} > \frac{C(n+1)}{1+\pi/2} > n+1.$$

On the other hand,

$$x \geq \frac{\pi}{2} \ \Rightarrow\ (2+\pi)x \geq (1+x)\pi \ \Rightarrow\ \frac{1}{1+x} \geq \frac{\pi/(2+\pi)}{x},$$

and so

$$\frac{\pi}{2(n+1)t^2} \leq n+1 \ \Rightarrow\ (n+1)^2t^2 \geq \frac{\pi}{2}$$
$$\Rightarrow\ \frac{C(n+1)}{1+(n+1)^2t^2} \geq \frac{C\pi/(2+\pi)}{(n+1)t^2} \geq \frac{1}{(n+1)t^2}\frac{\pi(1+\pi/2)}{(2+\pi)} = \frac{\pi}{2(n+1)t^2}.$$

This establishes (2.1.8).

Now, choose $\epsilon > 0$ and let $I_0 = [-a_0, a_0]$ be an interval centered at 0 such that $\mu(I) < \epsilon|I|$ for every symmetric interval $I \subseteq I_0$. Define $\mu_0 = \mu|_{I_0}$ and $\mu_1 = \mu - \mu_0$. We show that $\sigma_n(\mu_0, e^{ia/n})$ and $\sigma_n(\mu_1, e^{ia/n})$ converge to zero uniformly for $a \in [-A, A]$. Since $\mu = \mu_0 + \mu_1$, this will establish the lemma. Using (2.1.8), we find

$$\sigma_n(\mu_1, e^{ia/n}) = \int_{[-\pi,\pi]\setminus I_0} F_n(t - a/n)\, d\mu(t) \leq \int_{|t|>a_0} f_n^{(a)}(t)\, d\mu(t),$$

where the functions

$$f_n^{(a)}(t) = \frac{C(n+1)}{1+(n+1)^2(t-a/n)^2}$$

tend to zero uniformly for $a \in [-A, A]$ and $|t| > a_0$. So do their integrals, which establishes the convergence for μ_1.

To handle the integral with μ_0, define

$$L_n(t) := \sup_{a\in[-A,A]} f_n^{(a)}(t) = \begin{cases} C(n+1), & \text{if } |t| \leq A/n, \\ \frac{C(n+1)}{1+(n+1)^2(|t|-A/n)^2}, & \text{if } |t| > A/n. \end{cases}$$

The functions L_n are even and decreasing away from 0, so we may apply Lemma 2.1.3 together with the estimate $F_n(t - a/n) \leq L_n(t)$ to obtain

$$\sigma_n(\mu_0, e^{ia/n}) = \int_{-\pi}^{\pi} F_n(t - a/n)\, d\mu_0(t) \leq \int_{I_0} L_n(t)\, d\mu(t) \leq \epsilon |I_0| \int_{I_0} L_n(t)\, dt,$$

for all $a \in [-A, A]$. But,

$$\begin{aligned} \int_{I_0} L_n(t)\, dt \leq \int_{-\pi}^{\pi} L_n(t)\, dt &= AC\frac{n+1}{n} + 2\int_{A/n}^{\pi} \frac{C(n+1)}{1 + (n+1)^2(t - A/n)^2}\, dt \\ &\leq 2AC + 2\int_0^{\infty} \frac{C}{1+u^2}\, du \end{aligned}$$

which is finite and independent of n. Since ϵ is arbitrary, this completes the proof.

■

Lemma 2.1.4 *Let μ be a finite Borel measure on $[-\pi, \pi]$. Then, at every Lebesgue point $t \in [-\pi, \pi]$ of μ',*

$$\limsup_{n\to\infty} n\, \omega_n\big(\mu, e^{i(t-a/n)}\big) \leq \mu'(t),$$

uniformly for all $a \in [-A, A]$.

Proof. Without loss of generality, we may assume that μ is absolutely continuous. Fix a Lebesgue point of μ', $t \in [-\pi, \pi]$, and define the polynomial P by

$$P(\zeta) = \frac{1}{n} \sum_{j=0}^{n-1} e^{-ij(t-a/n)} \zeta^j.$$

$P(e^{i(t-a/n)}) = 1$ and

$$\left|P(e^{i\theta})\right|^2 = \left(\frac{\sin \frac{n(t-a/n-\theta)}{2}}{n \sin \frac{t-a/n-\theta}{2}} \right)^2 = \frac{2\pi}{n} F_{n-1}(t - a/n - \theta),$$

so that

$$n\,\omega_n\big(\mu, e^{i(t-a/n)}\big) \le \frac{n}{2\pi}\int \big|P\big(e^{i\theta}\big)\big|^2\,d\mu(t) = \sigma_{n-1}\big(\mu, e^{i(t-a/n)}\big)$$

By Lemma 2.1.2, the right hand side converges to $\mu'(t)$ (uniformly for $a \in [-A, A]$), which completes the proof.

■

The proof of the lower bound relies heavily on the Szegő function associated with the measure μ:

$$D(z) = D_\mu(z) = \exp\left\{\frac{1}{4\pi}\int_{-\pi}^{\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\log\mu'(\theta)\,d\theta\right\}, \quad (|z|<1) \tag{2.1.9}$$

For measures satisfying Szegő's condition, this function is in Hardy's class, H^2 (see e.g. [17, 242-244], where D^2 is called the outer function associated with μ'). The following properties will be implicitly invoked throughout the proof: $D(z)$ has nontangential limit $D(e^{it})$ at almost every point $z = e^{it}$, which satisfies $|D(e^{i\theta})|^2 = \mu'(\theta)$ at almost every $\theta \in [-\pi, \pi]$. In particular, the nontangential limit exists at every $z = e^{it}$ which is a Lebesgue point for $D(e^{i\theta})$ (see Fatou's theorem [7, p. 34] and apply it to the complex valued harmonic function D). Finally,

$$\int_{-\pi}^{\pi} D(e^{i\theta})\,d\theta = \lim_{r\to 1^-}\int_{-\pi}^{\pi} D(re^{i\theta})\,d\theta.$$

Proof. (Theorem 2.1.1)

We prove that for almost every $e^{it} \in \mathbf{T}$,

$$\liminf_{n\to\infty} n\,\omega_n(\mu, e^{i(t-a/n)}) \ge \mu'(t) \tag{2.1.10}$$

uniformly for $a \in [-A, A]$. This together with Lemma 2.1.4 proves Theorem 2.1.1. Without loss of generality we may assume that μ is absolutely continuous, since if not, the monotonicity of the Christoffel functions implies that $\omega_n(\mu, z) \ge \omega_n(\mu', z)$. In this case, (2.1.10) only increases. We shall show that (2.1.10) holds at every point t which is a Lebesgue point of μ' and for which e^{it} is a Lebesgue point of the Szegő function $D(e^{i\theta})$. Thus, let t be such a point. We may assume that $e^{it} = 1$ and hence that

$$\lim_{\tau\to 0} \frac{1}{\tau} \int_0^\tau |\mu'(u) - \mu'(0)| du = 0$$

and

$$\lim_{\tau\to 0} \frac{1}{\tau} \int_0^\tau |D(e^{iu}) - D(1)| du = 0.$$

These imply that there is a set S with 0 as a density point so that the limit at 0 of $\mu'(u)$ along S is $\mu'(0)$, while the limit of $D(e^{iu})$ along S is $D(1)$. This combined with $|D(e^{iu})|^2 = \mu'(u)$ a.e. implies that $|D(1)|^2 = \mu'(0)$.

Since we want to prove the lower estimate (2.1.10), we may also assume that for the particular $n \in \mathbf{N}$ appearing in the proof and $a \in [-A, A]$ the inequality $n\,\omega_n(\mu, e^{-ia/n}) \le |D(1)|^2 = \mu'(0)$ holds. For $a \in [-A, A]$ and $n \in \mathbf{N}$, let $q = q_{a,n} = e^{-ia/n}$ and choose polynomials, $P = P_{a,n}$, of degree at most $n-1$ for which

$$\omega_n(\mu, q) = \frac{1}{2\pi |P(q)|^2} \int \left|P\big(e^{i\theta}\big)\right|^2 d\mu(\theta).$$

Now fix a small $\epsilon > 0$ and $\alpha > 4/\epsilon^2$ with also $\alpha > 2A$ and define $K_1 = [-\alpha/n, \alpha/n]$ and $K_2 = [-\pi, \pi] \setminus K_1$.

We claim that for sufficiently large n, $|P(e^{i\theta})| \le 3|P(q)|$ for all $\theta \in K_1$ and $a \in [-A, A]$. If $|\zeta| = 1$, and $\rho = 1 - 1/n$, then for any n and a,

$$|P^2(\rho\zeta) D^2(\rho\zeta)| = \left| \frac{1}{2\pi i} \oint_{|z|=1} \frac{P^2(z) D^2(z)}{z - \rho\zeta} dz \right| \le \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|PD|^2(e^{i\theta})}{1-\rho} d\theta =$$

$$\frac{1}{2\pi(1-\rho)} \int_{-\pi}^{\pi} \left|P\big(e^{i\theta}\big)\right|^2 d\mu(\theta) = \frac{1}{1-\rho} |P(q)|^2 \omega_n(\mu, q) \le |P(q)|^2 |D(1)|^2.$$

It is known that the zeros of P lie on the unit circle, $\mathbf{T}$, so it is elementary (see [11, page 438]) that

$$|P(\rho\zeta)| \ge \left(\frac{1+\rho}{2}\right)^{n-1} |P(\zeta)| = \left(1 - \frac{1}{2n}\right)^{n-1} |P(\zeta)| \ge \frac{1}{2} |P(\zeta)| \qquad (2.1.11)$$

Thus,

$$|P^2(\zeta)D^2(\rho\zeta)| \leq 4|P(q)|^2|D(1)|^2$$

for all $n \in \mathbf{N}$, $a \in [-A, A]$, and $|\zeta| = 1$. Now $1 = e^{i0}$ is a Lebesgue point of $D(e^{i\theta})$, so $D(z)$ has a nontangential limit $D(1)$ at 1. Hence, for large n, and $\arg\zeta \in K_1$,

$$|D(\rho\zeta)|^2 \geq (1-\epsilon)|D(1)|^2$$

and, therefore,

$$|P(\zeta)| \leq \frac{2}{\sqrt{1-\epsilon}}|P(q)| \leq 3|P(q)|, \tag{2.1.12}$$

which proves the claim.

Now let $r = 1 + \epsilon/n$. We show that

$$|I_1/D(1) - \overline{\tilde{I}_1/D(1)}| < 4\epsilon|P(q)| \tag{2.1.13}$$

where

$$I_1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} P(e^{i\theta})\frac{r^{-n}q^n e^{-in\theta}}{r^{-1}qe^{-i\theta}-1}D(e^{i\theta})\,d\theta$$

$$\tilde{I}_1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{P(e^{i\theta})\frac{r^{-n}q^n e^{-in\theta}}{r^{-1}qe^{-i\theta}-1}}D(e^{i\theta})\,d\theta$$

.

To this end, define

$$I_{1j} = \frac{1}{2\pi}\int_{K_j} P(e^{i\theta})\frac{r^{-n}q^n e^{-in\theta}}{r^{-1}qe^{-i\theta}-1}D(e^{i\theta})\,d\theta$$

$$\tilde{I}_{1j} = \frac{1}{2\pi}\int_{K_j} \overline{P(e^{i\theta})\frac{r^{-n}q^n e^{-in\theta}}{r^{-1}qe^{-i\theta}-1}}D(e^{i\theta})\,d\theta$$

We establish (2.1.13) by analyzing the following decomposition:

$$I_1/D(1) - \overline{\tilde{I}_1/D(1)} = \left(\frac{I_{11}}{D(1)} - \frac{1}{2\pi}\int_{K_1} P(e^{i\theta})\frac{r^{-n}q^n e^{-in\theta}}{r^{-1}qe^{-i\theta}-1}\,d\theta\right)$$
$$+\frac{I_{12}}{D(1)} + \overline{\left(-\frac{\tilde{I}_{11}}{D(1)} + \frac{1}{2\pi}\int_{K_1} \overline{P(e^{i\theta})\frac{r^{-n}q^n e^{-in\theta}}{r^{-1}qe^{-i\theta}-1}}\,d\theta\right)} - \overline{\frac{\tilde{I}_{12}}{D(1)}}. \quad (2.1.14)$$

To establish an upper bound for the first term on the right hand side, observe that

$$\frac{1}{2\pi}\left|\int_{K_1} P(e^{i\theta})\frac{r^{-n}q^n e^{-in\theta}}{r^{-1}qe^{-i\theta}-1}\big(D(e^{i\theta}) - D(1)\big)\,d\theta\right|$$
$$\leq \max_{\theta\in K_1}\left|P(e^{i\theta})\frac{r^{-n}q^n e^{-in\theta}}{r^{-1}qe^{-i\theta}-1}\right| \times \frac{1}{2\pi}\int_{K_1}\big|D(e^{i\theta}) - D(1)\big|\,d\theta \quad (2.1.15)$$

By (2.1.12), the maximum is

$$\max_{\theta\in K_1}\left|P(e^{i\theta})\frac{r^{-n}q^n e^{-in\theta}}{r^{-1}qe^{-i\theta}-1}\right| \leq 3|P(q)|\frac{r^{-n}}{|r^{-1}-1|} \leq 3|P(q)|\frac{n}{\epsilon},$$

and for large n, by the Lebesgue point property,

$$\int_{K_1}\big|D(e^{i\theta}) - D(1)\big|\,d\theta < \frac{\epsilon^2}{6\alpha}\frac{2\alpha}{n} = \frac{\epsilon^2}{3n},$$

so that (2.1.15) is $\leq \epsilon|P(q)|$. But then,

$$\left|\frac{I_{11}}{D(1)} - \frac{1}{2\pi}\int_{K_1} P(e^{i\theta})\frac{r^{-n}q^n e^{-in\theta}}{r^{-1}qe^{-i\theta}-1}\,d\theta\right| \leq \epsilon\left|\frac{P(q)}{D(1)}\right| \quad (2.1.16)$$

An analogous argument yields

$$\left|-\frac{\tilde{I}_{11}}{D(1)} + \frac{1}{2\pi}\int_{K_1} \overline{P(e^{i\theta})\frac{r^{-n}q^n e^{-in\theta}}{r^{-1}qe^{-i\theta}-1}}\,d\theta\right| \leq \epsilon\left|\frac{P(q)}{D(1)}\right|, \quad (2.1.17)$$

for the third term of (2.1.14). For the second term of (2.1.14) the Cauchy inequality gives

$$|I_{12}|^2 \leq \frac{1}{2\pi}\int_{K_2}|P(e^{i\theta})|^2|D(e^{i\theta})|^2\,d\theta \times \frac{1}{2\pi}\int_{K_2}\left|\frac{r^{-n}q^n}{r^{-1}e^{-i\theta}q-1}\right|^2 d\theta.$$

The first integral is

$$\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |P(e^{i\theta})|^2 |D(e^{i\theta})|^2 \, d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} |P(e^{i\theta})|^2 \, d\mu(\theta) = \omega_n(q)|P(q)|^2$$

The second integral is

$$\frac{1}{2\pi}\int_{K_2} \left| \frac{r^{-n+1}}{e^{-i(\theta+a/n)} - r} \right|^2 d\theta \leq \frac{1}{2\pi}\int_{K_2} \frac{1}{|e^{-i(\theta+a/n)} - 1|^2} \, d\theta$$

$$= \frac{1}{2\pi}\int_{K_2} \frac{1}{2 - 2\cos(\theta + a/n)} \, d\theta = \frac{1}{2\pi}\int_{K_2} \frac{1}{(2\sin((\theta + a/n)/2))^2} \, d\theta$$

$$\leq \frac{1}{2\pi}\int_{K_2} \left(\frac{2}{\pi}(\theta + a/n) \right)^{-2} d\theta \leq \frac{1}{\pi}\int_{\alpha/n}^{\infty} \frac{\pi^2}{\theta^2} \, d\theta = \pi\frac{n}{\alpha},$$

since $2|\theta + a/n| \geq |\theta|$ on K_2. Thus,

$$|I_{12}|^2 \leq \pi\frac{n}{\alpha}\omega_n(q)|P(q)|^2 \leq \pi\frac{1}{\alpha}|D(1)|^2|P(q)|^2 \leq \epsilon^2|D(1)|^2|P(q)|^2,$$

so that

$$|I_{12}| \leq \epsilon|P(q)||D(1)|. \tag{2.1.18}$$

An analogous argument establishes that

$$|\tilde{I}_{12}| \leq \epsilon|P(q)||D(1)|. \tag{2.1.19}$$

Equations (2.1.16), (2.1.17), (2.1.18), and (2.1.19) prove (2.1.13).

If $P(\zeta) = \sum_{k=0}^{n-1} c_k\zeta^k$, let $\tilde{P}(\zeta) \equiv \sum_{k=0}^{n-1} \bar{c}_k\zeta^{n-1-k}$. For $\zeta = e^{i\theta}$, $\tilde{P}(\zeta) = \overline{P(\zeta)}\zeta^{n-1}$, so

$$\tilde{I}_1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \tilde{P}(e^{i\theta}) \frac{r^{-n}\overline{q}^n e^{i\theta}}{r^{-1}e^{i\theta}\overline{q} - 1} D(e^{i\theta}) \, d\theta = \frac{1}{2\pi i}\left(\frac{\overline{q}}{r}\right)^n \oint_{|\zeta|=1} \frac{\tilde{P}(\zeta)D(\zeta)}{r^{-1}\zeta\overline{q} - 1} \, d\zeta$$

But $r > 1$, so $F(\zeta) = \tilde{P}(\zeta)D(\zeta)/(r^{-1}\zeta\overline{q} - 1)$ is holomorphic in $\Delta = \{|z| < 1\}$ and has no singularities there. Also, $D \in H^2(\Delta)$, hence so is F, and therefore,

$$\oint_{|\zeta|=1} F(\zeta)\, d\zeta = \lim_{\tau \to 1^-} \oint_{|\zeta|=1} F(\tau\zeta)\, d\zeta = 0,$$

which, together with (2.1.13), implies that

$$|I_1| < 4\epsilon |P(q)||D(1)|.$$

Finally, let

$$H_n(z) = \frac{z^{-n} - 1}{z^{-1} - 1} = \sum_{k=0}^{n-1} z^{-k}$$

Then,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} P(e^{i\theta}) H_n(rq^{-1}e^{i\theta}) D(e^{i\theta})\, d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} P(e^{i\theta}) \frac{r^{-n}q^n e^{-in\theta} - 1}{r^{-1}qe^{-i\theta} - 1} D(e^{i\theta})\, d\theta$$

$$= I_1 - \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{P(e^{i\theta})D(e^{i\theta})e^{i\theta}}{r^{-1}q - e^{i\theta}}\, d\theta = I_1 - \frac{1}{2\pi i} \oint_{|\zeta|=1} \frac{P(\zeta)D(\zeta)}{r^{-1}q - \zeta}\, d\zeta$$

$$= I_1 + P(r^{-1}q)D(r^{-1}q),$$

so that for large n,

$$\left| P(r^{-1}q)D(r^{-1}q) - \frac{1}{2\pi} \int_{-\pi}^{\pi} P(e^{i\theta}) H_n(rq^{-1}e^{i\theta}) D(e^{i\theta})\, d\theta \right| \le 4\epsilon |P(q)D(1)|$$

But,

$$\left| \frac{1}{2\pi} \int_{-\pi}^{\pi} P(e^{i\theta}) H_n(rq^{-1}e^{i\theta}) D(e^{i\theta})\, d\theta \right|^2 \le \frac{1}{2\pi} \int_{-\pi}^{\pi} |H_n(rq^{-1}e^{i\theta})|^2\, d\theta \times$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |P(e^{i\theta})D(e^{i\theta})|^2\, d\theta = \left(\sum_{k=0}^{n-1} \left(1 + \frac{\epsilon}{n}\right)^{-2k} \right) \times \frac{1}{2\pi} \int_{-\pi}^{\pi} |P(e^{i\theta})|^2\, d\mu(\theta)$$

$$\le n|P(q)|^2 \omega_n(q)$$

Thus,

$$|P(r^{-1}q)D(r^{-1}q)| - 4\epsilon|P(q)||D(1)| \le |P(q)|\sqrt{n\omega_n(q)}$$

But, as in (2.1.11), we have

$$|P(r^{-1}q)| \ge \left(\frac{1+1/r}{2}\right)^{n-1}|P(q)| = \left(\frac{2+\epsilon/n}{2(1+\epsilon/n)}\right)^{n-1}|P(q)| =$$
$$\left(1-\frac{\epsilon/n}{2(1+\epsilon/n)}\right)^{n-1}|P(q)| \ge \left(e^{-\epsilon/n}\right)^{n-1}|P(q)| \ge e^{-\epsilon}|P(q)|$$

Also, as $n\to\infty$, $r^{-1}q\to 1$ non-tangentially, so that $D(r^{-1}q)\to D(1)$, and therefore

$$(e^{-\epsilon}-4\epsilon)|D(1)| \le \liminf_{n\to\infty}\sqrt{n\omega_n(q)}$$

This completes the proof of (2.1.10) since $\epsilon > 0$ is arbitrary.

■

2.2 Measures on $[-1,1]$

On $[-1,1]$ the Szegő class consists of all finite Borel measures μ with support on $[-1,1]$ for which

$$\int_{-1}^{1}\frac{\log\mu'(x)}{\sqrt{1-x^2}}dx > -\infty, \tag{2.2.20}$$

For measures in this class, the Szegő function is defined on $\mathbf{C}\setminus[-1,1]$ by

$$\tilde{D}(z) = \tilde{D}_\mu(z) = \exp\left(\sqrt{z^2-1}\frac{1}{2\pi}\int_{-1}^{1}\frac{\log\mu'(x)}{z-x}\frac{dx}{\sqrt{1-x^2}}\right) \tag{2.2.21}$$

with the branch of the square root that is positive for positive z. This has nontangential limit $\tilde{D}(x)$ at almost every $x\in[-1,1]$ with $|\tilde{D}(x)|^2 = \mu'(x)$.

Theorem 2.1.1 translates appropriately for measures on the interval $[-1,1]$.

Theorem 2.2.1 *Let μ be a measure on $[-1,1]$ satisfying Szegő's condition. Then, uniformly for $a\in[-A,A]$,*

$$\lim_{n\to\infty} n\,\lambda_n(\mu, x+a/n) = \pi\mu'(x)\sqrt{1-x^2} \tag{2.2.22}$$

for almost every $x \in [-1,1]$. Moreover, (2.2.22) *holds at every $x \in (-1,1)$ which is a Lebesgue point of μ and for $\tilde{D}$.*

The proof relies on orthogonal polynomials. Define the integral operators, G_n,by

$$G_n(f,x) = \lambda_n(\mu,x) \int f(t) K_n^2(\mu; x,t) d\mu(t),$$

where K_n are the reproducing kernels defined in the introduction. The next results are found in [13] as well as [15, p. 230, Corollary 4.3.1].

Lemma 2.2.2 *Let f be continuous on $[-1,1]$ and let μ be a measure on $[-1,1]$ which satisfies Szegő's condition. Then,*

$$\lim_{n\to\infty} \sup_{x\in[-1,1]} |G_n(f,x) - f(x)| = 0.$$

Lemma 2.2.3 *Let μ be a measure on $[-1,1]$ satisfying Szegő's condition. Then for any fixed $m \geq 1$ we have*

$$\lim_{n\to\infty} \sup_{x\in[-1,1]} \left| \frac{\lambda_{n+m}(\mu,x)}{\lambda_n(\mu,x)} - 1 \right| = 0.$$

We prove an extension of a corollary given in the same paper.

Corollary 2.2.4 *Let g be a nonnegative function on $[-1,1]$ and assume that there exists a polynomial P_m for which $P_m g$ and $P_m g^{-1}$ are continuous on $[-1,1]$. Let $d\mu_g(x) = g(x)d\mu(x)$, where μ is a measure on $[-1,1]$ satisfying Szegő's condition. Then*

$$\lim_{n\to\infty} \frac{\lambda_n(\mu_g, x+a/n)}{\lambda_n(\mu, x+a/n)} = g(x)$$

uniformly in every compact subset of $(-1,1)$ devoid of zeros of P_m and uniformly for all $a \in [-A,A]$.

Proof.

$$G_n(\mu; gP_m^2, x) = \lambda_n(\mu, x) \int P_m^2(t) g(t) K_n^2(\mu; x, t) d\,\mu(t)$$
$$= \lambda_n(\mu, x) \int P_m^2(t) K_n^2(\mu; x, t) d\,\mu_g(t) \geq \lambda_n(\mu, x)\lambda_{n+m}(\mu_g, x) P_m^2(x) K_n^2(\mu; x, x)$$
$$= \frac{\lambda_{n+m}(\mu_g, x)}{\lambda_n(\mu, x)} P_m^2(x).$$

The final equality follows from (2.0.1). Thus, uniformly in $a \in [-A, A]$

$$\limsup_{n\to\infty} \frac{\lambda_{n+m}(\mu_g, x + a/n)}{\lambda_n(\mu, x + a/n)} \leq \lim_{n\to\infty} \frac{G_n(\mu; gP_m^2, x + a/n)}{P_m^2(x + a/n)} = g(x) \tag{2.2.23}$$

uniformly on compact subsets of $(-1, 1)$ devoid of zeros of P_m. Also,

$$G_n(\mu_g, P_m^2/g, x) = \lambda_n(\mu_g, x) \int P_m^2(t) K_n^2(\mu_g; x, t) d\mu(t)$$
$$\geq \lambda_n(\mu_g, x)\lambda_{m+n}(\mu, x) P_m^2(x) K_n^2(\mu_g; x, x) = \frac{\lambda_{n+m}(\mu, x)}{\lambda_n(\mu_g, x)} P_m^2(x)$$

So, as above,

$$\limsup_{n\to\infty} \frac{\lambda_{n+m}(\mu, x + a/n)}{\lambda_n(\mu_g, x + a/n)} \leq \lim_{n\to\infty} \frac{G_n(\mu_g, P_m^2/g, x + a/n)}{P_m^2(x + a/n)} = \frac{1}{g(x)} \tag{2.2.24}$$

locally uniformly in $(-1, 1)$ (away from zeros of P_m) and uniformly in $a \in [-A, A]$, since $P_m/g \in L^\infty$ and μ_g also satisfies Szegő's condition. The result now follows from Lemma 2.2.3 and inequalities (2.2.23) and (2.2.24).

■

The proof of Theorem 2.2.1 relies on an equation which relates the Christoffel function of a measure, μ, supported on the interval $[-1, 1]$ to its 'projection' onto the unit circle. Defined ν by

$$\nu(E) := \mu(\{\cos\theta : \theta \in E\})$$

for $E \subset [-\pi, 0)$ or $E \subset [0, \pi)$. Note that $\nu'(t) = \mu'(\cos t)|\sin t|$. See [11, Lemma 6, p. 446] for a proof of the following:

Lemma 2.2.5 *Given an arbitrary positive finite Borel measure μ on $[-1,1]$, for every integer $n > 1$ and every $t \in [-\pi, \pi)$,*

$$\frac{1}{\omega_{2n-1}(\nu, e^{it})} = \frac{\pi}{\lambda_n(\mu, \cos t)} + \frac{\pi \sin^2 t}{\lambda_{n-1}(\mu_g, \cos t)},$$

where $g(x) = 1 - x^2$.

With these preliminaries, we can proceed with the proof of Theorem 2.2.1.

Proof. (Theorem 2.2.1) As $n \to \infty$,

$$\cos(t - a/n) = \cos t + \frac{a}{n} \sin t + O\left(\frac{1}{n^2}\right) = \cos t + \frac{a}{n} O(1).$$

From Lemma 2.2.3 and Corollary 2.2.4, it follows that

$$\lim_{n\to\infty} \frac{\lambda_{n-1}(\mu_g, \cos(t - a/n))}{\lambda_n(\mu, \cos(t - a/n))} = \sin^2 t$$

uniformly on compact subsets of $(-\pi, \pi)\backslash\{0\}$ and $a \in [-A, A]$. Consequently, Lemma 2.2.5 gives

$$\lim_{n\to\infty} \frac{\omega_{2n-1}(\nu, e^{i(t-a/n)})}{\lambda_n(\mu, \cos(t - a/n))} = \frac{1}{2\pi}, \tag{2.2.25}$$

uniformly on compact subsets of $(-\pi, \pi)\backslash\{0\}$ and $a \in [-A, A]$. We write $\cos(t - a/n) = \cos t + b/n$, and note that, while a runs through $[-A, A]$, the $b = b_{a,t}$ covers an interval $[-B_{A,t}, B_{A,t}]$ (depending on $t \in (-\pi, \pi) \setminus \{0\}$ and on A), and here for any $t \in (-\pi, \pi) \setminus \{0\}$ and any $B > 0$ there is an A such that $[-B, B] \subseteq [-B_{A,t}, B_{A,t}]$. The same is true uniformly if t runs through a compact subset of $(-\pi, \pi) \setminus \{0\}$. So, since the convergence in (2.2.25) is uniform over any interval $[-A, A]$, we get

$$\lim_{n\to\infty} \frac{\omega_{2n-1}(\nu, e^{i(t-a/n)})}{\lambda_n(\mu, \cos t + b/n)} = \frac{1}{2\pi},$$

uniformly for $a \in [-A, A]$. Therefore, by Theorem 2.1.1,

$$\lim_{n\to\infty} n\, \lambda_n(\mu, \cos t + b/n) = \pi \nu'(t),$$

for almost every t uniformly in $b \in [-B, B]$. Substituting $\cos t$ with x gives

$$\lim_{n\to\infty} n\,\lambda_n(\mu, x + b/n) = \pi\sqrt{1-x^2}\mu'(x),$$

for almost every x uniformly in $b \in [-B, B]$, which completes the proof of the first assertion of Theorem 2.2.1.

To prove the last statement, we only need to observe that under the conformal map $w = z - \sqrt{z^2-1}$, the complement of $[-1, 1]$ is mapped into the unit disk, and the Szegő function $\tilde{D}_\mu$ is mapped into the Szegő function D_ν (see (2.1.9)) associated with ν. Therefore, $x = \cos t$ is a Lebesgue point of μ and for $\tilde{D}_\mu(u)$ precisely when t is a Lebesgue point of ν and e^{it} is a Lebesgue point of $D_\nu(e^{iu})$. Taking these into account, the last statement follows from Theorem 2.1.1.

■

2.3 Asymptotics for Regular Measures

The assumptions of Theorem 2.2.1 are unnecessarily restrictive. Regular measures (defined in the introduction by (2.0.2)) that satisfy Szegő's condition locally–on I, an interval–generate Christoffel functions that exhibit the asymptotic behavior of (2.0.3) when $x \in I$. Since we shall now work with a local Szegő condition, we require a local Szegő function. Thus, let us suppose that μ is a finite Borel measure on $[-1, 1]$, and on some open interval $I \subset [-1, 1]$ it satisfies Szegő's condition, i.e.

$$\int_I \log \mu'(t)dt > -\infty.$$

We define

$$D^*(z) = D^*_\mu(z) = \exp\left(\frac{i}{2\pi}\int_I \frac{\log \mu'(x)}{z-x}dx\right). \tag{2.3.26}$$

$D^*(z)$ has a nontangential limit (from the upper half plane) $D^*(x)$ at almost every $x \in I$ and $|D^*(x)|^2 = \mu'(x)$ a.e. (see Lemma 2.3.2 below).

Theorem 2.3.1 *Let μ be a regular measure on $[-1, 1]$ and let I be an open interval* *in* $[-1,1]$ *such that*

$$\int_I \log \mu'(\theta)\, d\theta > -\infty.$$

Then, for almost every $x \in I$ *and for every* $A > 0$*,*

$$\lim_{n\to\infty} n\,\lambda_n(\mu, x + a/n) = \pi\sqrt{1-x^2}\mu'(x), \tag{2.3.27}$$

uniformly for $a \in [-A, A]$*.*

Moreover, (2.3.27) *holds at every* $x \in I$ *which is a Lebesgue point of* μ' *and of* D^**.*

Proof. For $\epsilon > 0$, let $d\nu_\epsilon(x) = d\mu(x) + \epsilon\chi_{[-1,1]}(x)dx$, where χ is the characteristic function. This measure clearly satisfies Szegő's condition globally and therefore, by Theorem 2.2.1,

$$\limsup_{n\to\infty} n\,\lambda_n(\nu_\epsilon, x + a/n) = \pi\sqrt{1-x^2}(\mu'(x) + \epsilon).$$

But $\mu < \nu_\epsilon$ for every ϵ and so, by virtue of the monotonicity of the Christoffel functions with respect to measures,

$$\limsup_{n\to\infty} n\,\lambda_n(\mu, x + a/n) \le \pi\sqrt{1-x^2}\mu'(x),$$

for almost every $x \in (-1,1)$ and uniformly for $a \in [-A, A]$.

To prove the lower bound,

$$\liminf_{n\to\infty} n\,\lambda_n(\mu, x + a/n) \ge \pi\sqrt{1-x^2}\mu'(x), \tag{2.3.28}$$

uniformly for $a \in [-A, A]$ and almost every $x \in I$, let

$$d\nu(x) = d\mu(x) + \chi_{[-1,1]\setminus I}(x)dx.$$

In Lemma 2.3.2 we will prove that $x \in I$ is a Lebesgue point of D^*_μ if and only it is a Lebesgue point of the Szegő function $\tilde{D}_\nu$ associated in (2.2.21) with ν. Therefore, it is enough to show (2.3.28) at every $x \in I$ which is a Lebesgue point of ν' and of

$\tilde{D}_\nu$. Let x be such a point. Assume to the contrary that there are sequences, $\mathcal{N} \subset \mathbf{N}$, $\{a_n \in [-A, A] : n \in \mathcal{N}\}$, and a real number $r < \pi\sqrt{1-x^2}$ for which

$$n\,\lambda_n(\mu, x + a_n/n) < r\mu'(x) \qquad \text{for } n \in \mathcal{N}. \tag{2.3.29}$$

Let P_n represent a polynomial of degree at most $n-1$ for which $P_n(x + a_n/n) = 1$ and

$$\lambda_n(\mu, x + a_n/n) = \int_{-1}^{1} |P(t)|^2\, d\mu(t). \tag{2.3.30}$$

Now, for some $\eta > 0$, define the polynomials

$$Q_n(t) = P_n(t)\left(1 - \left(\frac{x + a_n/n - t}{4}\right)^2\right)^{[\eta n]}.$$

(Here $[y]$ is the integral part of y.) Evidently, $Q_n(x+a_n/n) = 1$ and $|Q_n(t)| \le |P_n(t)|$, $t \in [-1, 1]$. Furthermore, since $a_n/n \to 0$ as $n \to \infty$, there is a $\rho < 1$ such that

$$\left|\frac{Q_n(t)}{P_n(t)}\right| = \left(1 - \left(\frac{x + a_n/n - t}{4}\right)^2\right)^{[\eta n]} < \rho^{2n} \qquad (n \to \infty) \tag{2.3.31}$$

on $[-1, 1] \backslash I$. Now, by (2.0.2), every regular measure σ on $[-1, 1]$ has the property that for any $s > 1$ and any sequence of polynomials, R_n,

$$\max_{t \in [-1,1]} |R_n(t)|^2 < s^n \int |R_n(t)|^2\, d\sigma(t),$$

for sufficiently large n (see [19, Theorem 3.2.3]). But,

$$\int |P_n(t)|^2\, d\mu(t) \le \mu([-1, 1])$$

since $\lambda_n(\mu, t) \le \mu([-1, 1])$. Therefore, for any $s > 1$ and sufficiently large $n \in \mathcal{N}$, we have

$$\max_{t \in [-1,1]} |P_n(t)|^2 < s^n,$$

which, together with (2.3.31) and with the assignment $s = 1/\rho$, implies that $|Q_n(t)| < \rho^n$ on $[-1,1]\backslash I$ for sufficiently large $n \in \mathcal{N}$. This, together with (2.3.29) and (2.3.30) implies that

$$\int |Q_n(t)|^2 \, d\nu(t) \leq \frac{r}{n}\mu'(x) + \rho^n \int_{[-1,1]\backslash I} dt = \frac{c}{n(1+\eta)}\nu'(x) + o(1/n),$$

where $c = r(1+\eta)$. Since this holds for arbitrary η, we can fix its value so that $c < \pi\sqrt{1-x^2}$. In this case

$$[n(1+\eta)] \int |Q_n(t)|^2 \, d\nu(t) \leq c\nu'(x) + o(1) \qquad \text{for } n \in \mathcal{N},$$

which implies that

$$\liminf_{n\to\infty} n\,\lambda(\nu, x + a_n/n) < \pi\sqrt{1-x^2}\nu'(x),$$

since $Q_n(x + a/n) = 1$ and Q_n has degree $[n(1+\eta)]$. This contradicts Theorem 2.2.1 (at x) and this contradiction proves the claim, pending the proof of the next lemma. ∎

Lemma 2.3.2 *With $d\nu(x) = d\mu(x) + \chi_{[-1,1]\backslash I}(x)dx$ the nontangential limit (from the upper half plane) at an $x \in I$ exists for $D_\mu^*(z)$ if and only if it exists for $\tilde{D}_\nu(z)$. Furthermore, x is a Lebesgue point of $D_\mu^*(u)$ precisely when it is a Lebesgue point of $\tilde{D}_\nu(u)$.*

Proof. Suppose first that the nontangential limit of $D_\mu^*(z)$ exists at $x \in I$. Consider the function

$$\tilde{D}_\mu^*(z) = \exp\left(\sqrt{z^2-1}\frac{1}{2\pi}\int_I \frac{\log \mu'(x)}{z-x}\frac{dx}{\sqrt{1-x^2}}\right). \tag{2.3.32}$$

Since this differs on $I \times \mathbf{R}$ from $\tilde{D}_\nu$ by an analytic multiplicative factor, it is enough to prove the existence of the nontangential limit for $\tilde{D}_\mu^*$ at x. But

$$\tilde{D}_\mu^*(z)/D_\mu^*(z) = \exp\left(\frac{i}{2\pi}\int_I \log \mu'(t) h(z,t) dt\right) \tag{2.3.33}$$

with

$$h(z,t) = \frac{1}{z-x}\left(\frac{\sqrt{1-z^2}}{\sqrt{1-t^2}} - 1\right) = -\frac{t+z}{\sqrt{1-t^2}(\sqrt{1-z^2}+\sqrt{1-t^2})},$$

which is an analytic function (in z) on I, so the nontangential limit

$$\tilde{D}^*_\mu(u)/D^*_\mu(u) = \exp\left(\frac{i}{2\pi}\int_I \log\mu'(t)h(u,t)dt\right) \tag{2.3.34}$$

of (2.3.33) certainly exists at any $u \in I$. This shows that, indeed, the nontangential limit of $\tilde{D}^*_\mu = D^*_\mu \times (\tilde{D}^*_\mu/D^*_\mu)$ also exists at x.

It is a simple exercise to show that if f is a nonzero C^1-function, then x is a Lebesgue point of $D^*_\mu(u)$ if and only if it is a Lebesgue point of $f(u)D^*_\mu(u)$. With $f(u) = \tilde{D}^*_\mu(u)/D^*_\mu(u)$ this is the same as x being a Lebesgue point of $\tilde{D}^*_\mu(u)$. Applying the same argument once more with $f(u) = \tilde{D}_\nu(u)/\tilde{D}^*_\mu(u)$ we find that x is a Lebesgue point of $D^*_\mu(u)$ if and only if it is a Lebesgue point of $\tilde{D}_\nu(u)$. The proof of the converse implication (i.e. going from $\tilde{D}^*_\mu(u)$ to $D^*_\mu(u)$) is very similar.

Since the real part of $i/(z-t)$ for $z = x+iy$ is the Poisson kernel $y/((x-t)^2 + y^2)$ of the upper half plane, it is a standard exercise to show that $|D^*_\mu(z)|^2$ tends nontangentially to $\mu'(x)$ at every Lebesgue point of μ.

■

2.4 Universality

We will now apply Lubinsky's technique ([10]) to prove the universality result, Theorem 2.0.1. The proof follows directly from the following:

Lemma 2.4.1 *Let μ and μ^* satisfy the conditions of the hypothesis in Theorem 2.0.1 and assume further that $\mu'(x_0) = (\mu^*)'(x_0) > 0$ for some $x_0 \in I$ which is a Lebesgue point of μ, μ^*, D^*_μ and $D^*_{\mu^*}$ (see (2.3.26)). Then*

$$\lim_{n\to\infty} \frac{1}{n}|K_n - K^*_n|(x_0 + a/n, x_0 + b/n) = 0,$$

uniformly for a, $b \in [-A, A]$, where K_n and K^*_n are the reproducing kernels associated respectively with μ and μ^*.

Proof. First assume that $\mu \le \mu^*$ on $[-1,1]$. It was proven in [10, (3.5)] that

$$\frac{|K_n(x,y) - K_n^*(x,y)|}{K_n(x,x)} \le \sqrt{\frac{\lambda_n(x)}{\lambda_n(y)} \left| 1 - \frac{\lambda_n(x)}{\lambda_n^*(x)} \right|}. \tag{2.4.35}$$

Here, the λ's are the associated Christoffel functions. By Theorem 2.3.1

$$\lim_{n\to\infty} n\, \lambda_n(x_0 + a/n) = \pi\sqrt{1 - x_0^2}\mu'(x_0) \tag{2.4.36}$$

and

$$\lim_{n\to\infty} n\, \lambda_n^*(x_0 + a/n) = \pi\sqrt{1 - x_0^2}(\mu^*)'(x_0), \tag{2.4.37}$$

uniformly for all $a \in [-A, A]$. Now replace x with $x_0 + a/n$ and y with $x_0 + b/n$ in (2.4.35). Then, since $\mu'(x_0) = (\mu^*)'(x_0)$, it follows that

$$\lim_{n\to\infty} \lambda_n(x_0 + a/n)|K_n - K_n^*|(x_0 + a/n, x_0 + b/n) = 0,$$

uniformly for $a, b \in [-A, A]$, which implies, again because of (2.4.36), that

$$\lim_{n\to\infty} \frac{1}{n}|K_n - K_n^*|(x_0 + a/n, x_0 + b/n) = 0 \quad \text{uniformly for } a, b \in [-A, A]. \tag{2.4.38}$$

For arbitrary μ and μ^* satisfying the conditions of the lemma, define the measure

$$d\nu(x) = \max(\operatorname{dist}(x, I), \mu'(x), (\mu^*)'(x))dx + d\mu_s(x) + d\mu_s^*(x),$$

where μ' and μ_s denote, respectively, the absolutely continuous and singular components of the measure μ. Clearly, $\nu \ge \mu, \mu^*$ so ν satisfies Szegő's condition locally on I and is a regular measure on $[-1, 1]$. Hence (2.4.38) holds for the pairs (ν, μ) and (ν, μ^*) and, consequently, for (μ, μ^*). This completes the proof.

■

Proof. (Theorem 2.0.1) Let x be a Lebesgue point of μ and of D^*_μ (see (2.3.26)) and assume $\mu'(x) > 0$. Define $d\mu^*(u) = \mu'(x)du$, $u \in [-1, 1]$. By Lemma 2.4.1

$$\lim_{n\to\infty} \frac{1}{n}|K_n - K^*_n|(x + a/n, x + b/n) = 0. \tag{2.4.39}$$

Applying Lubinsky's original theorem [10, Theorem 1.1] to K^*_n we find that, as $n \to \infty$,

$$\frac{K^*_n\,(x + \alpha/\mu'(x)K^*_n(x, x), x + \beta/\mu'(x)K^*_n(x, x))}{K^*_n(x, x)} \to \frac{\sin \pi(\alpha - \beta)}{\pi(\alpha - \beta)}, \tag{2.4.40}$$

uniformly for any fixed B and $\alpha, \beta \in [-B, B]$. Now, choose $\alpha = \alpha_n$ and $\beta = \beta_n$ so that $\alpha/\mu'(x)K^*_n(x, x) = a/n$ and $\beta/\mu'(x)K^*_n(x, x) = b/n$. Then, because of (2.4.37), as $n \to \infty$, $\alpha \to a/\pi\sqrt{1 - x^2}$ and $\beta \to b/\pi\sqrt{1 - x^2}$, hence the statement in Theorem 2.0.1 follows from (2.4.40) and (2.4.39) (see also (2.4.37)).

■

2.5 Zero Distribution of Orthogonal Polynomials

Finally, we apply the techniques of Levin and Lubinsky to extend their Theorem 1.1 in [9]. The application of universality to study zero spacing is also found in Freud's text, [4] as well as in [22]. In what follows, x_{kn} denotes the k-th zero of the orthogonal polynomial p_n associated with a given measure μ, defined on the interval $[-1, 1]$. Let the zeros be ordered according to

$$x_{nn} < x_{n-1,n} < x_{n-2,n} < \cdots < x_{1n}. \tag{2.5.41}$$

Theorem 2.5.1 *Let μ be a finite regular Borel measure on $[-1, 1]$ which satisfies Szegö's condition locally in some interval I. Fix an $x \in I$ for which* (2.0.4) *holds and for which $\mu'(x) > 0$. If for some sequence $k = k(n)$*

$$|x_{kn} - x| = O\left(\frac{1}{n}\right),$$

then

$$\lim_{n\to\infty}(x_{kn} - x_{k+1,n})\frac{n}{\pi\sqrt{1 - x^2}} = 1.$$

Proof. Let l_{kn} be the Lagrange interpolation polynomial associated with the point x_{kn} which vanishes at every other zero of p_n and satisfies $l_{kn}(x_{kn}) = 1$. l_{kn} has the representation

$$l_{kn}(z) = \frac{K_n(x_{kn}, z)}{K_n(x_{kn}, x_{kn})}.$$

There is a bounded sequence, a_n such that

$$x_{kn} = x + \frac{a_n}{n}.$$

Since $\mu'(x) > 0$, Theorem 2.3.1 implies that $K_n(x_{kn}, x_{kn})/K_n(x, x) \to 1$ as $n \to \infty$, so that Theorem 2.0.1 applied to the Lagrange polynomials gives

$$l_{kn}\left(x + \frac{b}{n}\right) = \frac{\sin((a_n - b)/\sqrt{1 - x^2})}{(a_n - b)/\sqrt{1 - x^2}} + o(1), \tag{2.5.42}$$

uniformly for bounded b. The first term on the right hand side (taken as a function of b), changes sign when $a_n - b = -\pi\sqrt{1 - x^2}$ and therefore, by (2.5.41), the zero $x_{k+1,n}$ has the representation,

$$x_{k+1,n} = x_{kn} + \frac{b_n}{n}, \tag{2.5.43}$$

for some bounded sequence $b_n < 0$ with

$$\liminf_{n\to\infty} b_n \geq -\pi\sqrt{1 - x^2}.$$

So, by (2.5.42),

$$0 = l_{kn}\left(x_{k+1,n}\right) = \frac{\sin(b_n/\sqrt{1 - x^2})}{b_n/\sqrt{1 - x^2}} + o(1). \tag{2.5.44}$$

We claim that $\lim_{n\to\infty} b_n = -\pi\sqrt{1 - x^2}$. To this end choose any subsequence of $\{b_n\}$ with limit point b. Equation (2.5.44) gives, upon passing through this subsequence,

$$\frac{\sin(b/\sqrt{1 - x^2})}{b/\sqrt{1 - x^2}} = 0.$$

Since $-\pi\sqrt{1-x^2} \leq b \leq 0$, we must have $b = -\pi\sqrt{1-x^2}$, which proves the claim. This together with equation (2.5.43) gives

$$(x_{k+1,n} - x_{kn})\, n = b_n \to -\pi\sqrt{1-x^2},$$

as $n \to \infty$, which completes the proof.

$\blacksquare$

3 Christoffel Function Asymptotics on General Curves

We now turn our discussion to measures with general supports in the plane. How does the asymptotic behavior of the sequence of Christoffel functions depends on the geometry of the support, $\Gamma := supp(\mu)$, when the point of evaluation, $z \in \Gamma$? It is easy to see, for instance, that $\lambda_n(\mu, z) \to \mu(\{z\})$ as $n \to \infty$ if Γ is sufficiently regular (e.g. smooth) and z lies on the outer boundary of Γ. What is the rate of this convergence for general measures, and how is this rate determined by Γ? For measures supported on a circle or a union of intervals, we already know the answer:

$$\lim_{n\to\infty} n\, \lambda_n(\mu, e^{i\theta}) = \frac{1}{2\pi}\mu'(e^{i\theta}) \quad \text{or} \quad \lim_{n\to\infty} n\, \lambda_n(\mu, x) = \pi\sqrt{1-x^2}\mu'(x) \tag{3.0.1}$$

(almost everywhere on the supports). These equations hint at the answer to our question, but to understand the hint we require some elementary potential theory.

3.1 Potential Theory

Every measure, μ, with compact support, Γ, has associated logarithmic energy defined as follows:

$$I(\mu) := \int\int \log\frac{1}{|z-w|}d\mu(z)d\mu(w).$$

If Γ is sufficiently dense, then $I(\nu) > \epsilon > 0$ for all probability measures ν supported on Γ. The fundamental theorem of potential theory guarantees a unique ν, denoted ν_Γ, whose energy is least among all such measures. It is known as the (logarithmic) equilibrium measure, supported on the outer boundary of Γ and determined entirely by the geometry of this boundary. Let Ω be the unbounded component of $\mathbf{C} \setminus \Gamma$

and $g_\Omega(z,\infty)$, its Green's function with pole at infinity. If Γ is smooth then we may exploit the following useful representation of equilibrium measure:

$$d\nu_\Gamma(z) = \frac{1}{2\pi}\frac{\partial g_\Omega(z,\infty)}{\partial \mathbf{n}}\,ds \quad (z \in \Gamma). \tag{3.1.2}$$

Here ds denotes arc length measure and $\mathbf{n}$ is the outward normal along Γ. Recall that $g_\Omega(z,\infty)$ is the unique harmonic function on Ω which vanishes on the boundary Γ and satisfies $g_\Omega(z,\infty) \sim \log z$ as $z \to \infty$. If Φ is a conformal mapping of Ω onto the exterior of the unit disk, then by the uniqueness of Green's function, $g_\Omega(z,\infty) = \log|\Phi(z)|$. Thus, by equation (3.1.2), $d\nu_\Gamma = (2\pi)^{-1}|\Phi'|\,ds$. For example, if $\Gamma = [-1,1]$ then $d\nu_\Gamma(x) = (\pi\sqrt{1-x^2})^{-1}dx$. (See [16, ch. 3] for a stellar introduction.)

Equations (3.0.1) can now be consolidated:

$$\lim_{n\to\infty} n\,\lambda_n(\mu,z) = \frac{d\mu}{d\nu_\Gamma}(z), \tag{3.1.3}$$

where Γ is the unit circle or interval. Since the Christoffel functions are infima, it is easy to show that the upper bound,

$$\limsup_{n\to\infty} n\,\lambda_n(\mu,z) \le \frac{d\mu}{d\nu_\Gamma}(z), \tag{3.1.4}$$

holds for all finite Borel measures (a.e. on the support). (See [11, p. 435].) Until fairly recently, however, (3.1.3) was known only for continuous weights bounded away from zero. That it holds almost everywhere (with respect to ν_Γ) for the more general class of measures satisfying Szegö's condition,

$$\int \log\left(\frac{d\mu}{d\nu_\Gamma}\right)\,d\nu_\Gamma > -\infty, \tag{3.1.5}$$

was a long-standing conjecture, finally proved by Máté, Nevai, and Totik in 1991 ([11]).

In this chapter, we aim to prove (3.1.3) for measures supported on smooth curves, $\Gamma \in C^{1,\alpha}$, whose weights satisfy Szegő's condition, (3.1.5). We accomplish this by an abstraction of Máté's technique in [11] and investigate some applications.

3.2 Main Results

In what follows, $\mathbf{C}$ denote the complex plane, U, a bounded, simply connected domain with boundary $\Gamma := \partial U$ in the class $C^{1,\alpha}$; and $\Omega := \overline{\mathbf{C}} \setminus \overline{U}$. Δ denotes the closed unit disk and $\Delta_R := \{z : |z| \leq R\}$ with boundaries γ and γ_R, respectively. Φ is the outer mapping function of Γ, a conformal mapping of Ω onto $\overline{\mathbf{C}} \setminus \Delta$ with $\Phi(\infty) = \infty$ and $\Psi := \Phi^{-1}$. Φ^* maps $U \to \Delta$ conformally and $\Psi^* := (\Phi^*)^{-1}$. $\Gamma_R := \Psi(\gamma_R)$ if $R \geq 1$ and $\Gamma_R := \Psi^*(\gamma_R)$ if $R < 1$. U_R is the interior of Γ_R and $\Omega_R := \overline{\mathbf{C}} \setminus \overline{U}_R$. Let ds denote arc length measure along Γ and let $d\mu = W|\Phi'|\,ds + d\mu_s$ be a measure supported on Γ with singular part μ_s. Note that $d\mu/d\nu_\Gamma = 2\pi\,W$ almost everywhere, so the following, our main result, is the correct abstraction of equation (3.1.3):

Theorem 3.2.1 *Let $d\mu = W|\Phi'|\,ds + d\mu_s$ be a positive, Borel measure supported on a closed curve $\Gamma \in C^{1,\alpha}$ for some $\alpha > 0$ and assume that W satisfies Szegő's condition, (3.1.5). If W is bounded and $d\mu_s \equiv 0$, or $\alpha > 1/2$, then*

$$\lim_{n\to\infty} n\,\lambda_n(\mu, \zeta_0) = W(\zeta_0), \tag{3.2.6}$$

for ν_Γ-almost every $\zeta_0 \in \Gamma$.

Theorem 3.2.1 is really the intersection of two broader results. Unfortunately, our proof of the upper bound, (3.1.4), requires a more stringent restriction on the smoothness of Γ than that of the lower bound. This inadequacy appears to be intrinsic to our method, as will become transparent in the proof.

Theorem 3.2.2 *Let $d\mu = W|\Phi'|\,ds + d\mu_s$ be a positive, Borel measure supported on a closed curve $\Gamma \in C^{1,\alpha}$. If W is bounded and $d\mu_s \equiv 0$, or $\alpha > 1/2$, then*

$$\limsup_{n\to\infty} n\,\lambda_n(\mu, \zeta_0) \leq W(\zeta_0), \tag{3.2.7}$$

for ν_Γ-almost every $\zeta_0 \in \Gamma$.

Theorem 3.2.3 *Let $d\mu = W|\Phi'|\, ds + d\mu_s$ be a positive, Borel measure supported on a closed curve $\Gamma \in C^{1,\alpha}$ where $\alpha > 0$. If W satisfies (3.1.5) then*

$$\liminf_{n\to\infty} n\,\lambda_n(\mu,\zeta_0) \geq W(\zeta_0), \tag{3.2.8}$$

for ν_Γ-almost every $\zeta_0 \in \Gamma$.

Theorem 3.2.2 admits the simpler proof, since it is an upper bound on the infimum, λ_n. We simply need a sequence of polynomials, $Q_n(z)$, whose $L_2(\mu)$-norms converge at the optimal rate dictated by Theorem 3.2.3. This is easy if Γ is the unit circle: We proved it already in Theorem 2.1.4 using the sequence

$$Q_n(z) = \frac{1}{n}\sum_{k=0}^{n-1}(\overline{\zeta_0}z)^k \tag{3.2.9}$$

For general Γ, the upper bound can be deduced from this case by substituting z^k in $Q_n(z)$ by the (k-th order) 1-Faber polynomial associated with Γ,

$$F_k(z) = \frac{1}{2\pi i}\int_{\Gamma_R}\frac{\Phi^k(\zeta)}{\zeta - z}\Phi'(\zeta)d\zeta, \quad (z \in U_R). \tag{3.2.10}$$

Let

$$Q_n(z) := \frac{1}{n}\sum_{k=0}^{n-1}F_k(z) \quad \text{and} \quad S_n(z) := \frac{1}{n}\sum_{k=0}^{n-1}\Phi'(z)\Phi^k(z). \tag{3.2.11}$$

With $d\tilde{\mu}(\theta) := d\mu(\Psi(e^{i\theta}))$, we have

$$\begin{aligned}\int |S_n|^2\, d\mu &= \int \Big|\frac{1}{n}\sum_{k=0}^{n-1}\Phi(\zeta)^k\Big|^2 |\Phi'(\zeta)|^2\, d\mu(\zeta)\\ &= \int_{-\pi}^{\pi}\Big|\frac{1}{n}\sum_{k=0}^{n-1}e^{ik\theta}\Big|^2|\Psi'(e^{i\theta})|^{-2}d\tilde{\mu}(\theta) = \frac{2\pi}{n}\sigma_n(d\tilde{\mu}/|\Psi'|^2, 1),\end{aligned} \tag{3.2.12}$$

and so the revised sequence, $\{Q_n\}$, will serve our purpose if we can show that $Q_n - S_n \to 0$ with sufficient rapidity. We can, at least if $\alpha > 1/2$ or W is bounded.

Lemma 3.2.4 *For any $\alpha > 0$ and $\Gamma \in C^{1,\alpha}$, $Q_n - S_n \to 0$ pointwise and*

$$\int_\Gamma |Q_n - S_n|^2 \, ds = o(n^{-1}). \tag{3.2.13}$$

If $\alpha > 1/2$ then the limit holds uniformly for $z \in \Gamma$:

$$\sup_{z\in\Gamma} |Q_n(z) - S_n(z)| = o(n^{-1/2}). \tag{3.2.14}$$

Our choice of Q_n in (3.2.11) is a natural generalization of (3.2.9): The Faber polynomials generalize the monomials, z^k, in a variety of contexts, e.g. locally uniform (taylor-series type) expansions of analytic functions in U. Let us quickly verify Theorem 3.2.2 using these estimates.

Proof. (Theorem 3.2.2) Minkowskii gives

$$\sqrt{n}\|Q_n\|_{L_2(\mu)} \leq \sqrt{n}\|Q_n - S_n\|_{L_2(\mu)} + \sqrt{n}\|S_n\|_{L_2(\mu)}. \tag{3.2.15}$$

Under the conditions of Theorem 3.2.2, Lemma 3.2.4 implies that $\|Q_n - S_n\|_{L_2(\mu)} = o(1/\sqrt{n})$ so the first term on the right-hand side vanishes as $n \to \infty$. Our choice of the particular conformal mapping in (3.2.11) is arbitrary, so we may presume that the Fejér means in (3.2.12) converge at $z = 1$. Thus, (3.2.15) implies that

$$\limsup_{n\to\infty} \frac{n}{2\pi} \int |Q_n|^2 \, d\mu \leq \lim_{n\to\infty} \frac{n}{2\pi} \int |S_n|^2 \, d\mu = W(\zeta_0)|\Phi'(\zeta_0)|^2,$$

where $\zeta_0 = \Psi(1)$. Finally, since $Q_n(\zeta_0) - S_n(\zeta_0) \to 0$ and $S_n(\zeta_0) = \Phi'(\zeta_0)$,

$$\limsup_{n\to\infty} \frac{1}{2\pi} \frac{n}{|Q_n(\zeta_0)|^2} \int |Q_n|^2 \, d\mu \leq W(\zeta_0). \tag{3.2.16}$$

■

The more difficult proof of the lower bound, (3.2.8), will employ Hardy space methods. $D(z)$ and $Q(z)$ represent, respectively, the outer functions associated with $\log \sqrt{W}$ in U and Ω. They are analytic and non-vanishing in their respective domains and have non-tangential limits almost everywhere at their common boundary, Γ, with $|D(\zeta)|^2 = |Q(\zeta)|^2 = W(\zeta)$, for almost every $\zeta \in \Gamma$. Fix $\zeta_0 \in \Gamma$, a Lebesgue point of D, Q and $\log W$; and choose a polynomial, P_n, of degree at most $n-1$ for which

$$\lambda_n(\mu, \zeta_0) = \frac{1}{|P_n(\zeta_0)|^2} \frac{1}{2\pi} \int |P_n|^2 \, d\mu.$$

P_n is defined only up to a multiplicative constant, so we may assume that $|P_n(\zeta_0)| = 1$ and, for the same reason, that $\Phi(\zeta_0) = 1$. If necessary, multiply Q by a constant of unit modulus so that $Q(\zeta_0) = D(\zeta_0)$.

The proof of the lower bound given in the last chapter for measures supported on the circle relies heavily on a knowledge of the locations of the zeros of P_n. Szegő proved that the zeros of P_n lie on the unit circle; for general supports, no analogous results exist. We can surmount this obstacle to our adaptation of Máté's method by means of a weighted Bernstein-Walsh inequality which permits an estimate of the sequence $\{|P_n|\}$ in vanishing neighborhoods of the point of evaluation, ζ_0. This is the approach conceived by V. Totik and the content of our next result.

Lemma 3.2.5 *Fix $c > 0$ and assume that $W \leq 1$ satisfies Szegő's condition. $|P_n(z)|$ is bounded uniformly for all $|z - \zeta_0| < c/n$.*

Before proceeding to prove the main theorem and supporting lemmata, we investigate some interesting applications.

3.3 Applications

Orthogonal Polynomials

First, we reiterate the trivial application to orthogonal polynomials mentioned in the introduction. $\{p_n(\mu, z)\}$ denotes the sequence of orthonormal polynomials associated with the measure μ. When W is smooth, positive and supported on a smooth curve Γ, Suetin ([20]) obtains precise norm estimates for this sequence using standard Fourier-analytic techniques. His methods fail for the more general Szegő class of measures since their weights may be highly erratic, not amenable to the methods of classical approximation theory. Although our results do not fully extend his, they do imply the following sup-norm estimate for Szegő weights which follows immediately from the Theorem and (1.0.3):

Corollary 3.3.1 *Let μ satisfy all the hypotheses of Theorem 3.2.1. For almost every $\zeta_0 \in \Gamma$,*

$$|p_n(\mu, \zeta_0)| = o(\sqrt{n}).$$

Operator Theoretic Formulation and Ill-Posed Problems

Let H be a Hilbert space with inner product $(\ ,\ )$. Let $y \in H$ and let A be a linear operator on H. The problem of solving the equation $Ax = y$ is called ill-posed if A is not invertible. Ill-posed problems obviously have no solution in general, although stable approximate solutions minimizing $\|Ax - y\|$ may be found by certain recursive algorithms. We will restrict our discussion to the case in which $A = \zeta_0 - T$, where T is a normal operator with spectrum $\sigma(T) = \Gamma \in C^{1,\alpha}$ and $\zeta_0 \in \Gamma$. A typical approach uses approximants of the form $x_n = Q_n(T)y$ where Q_n is a polynomial of degree at most n. (See, for example, [1].) Ideally, $\|(\zeta_0 - T)x_n - y\| \to 0$ as $n \to \infty$. What is the optimal rate of this convergence?

The answer follows immediately from Theorem 3.2.1: If $\{E_\zeta\}_{\zeta\in\Gamma}$ denotes the spectral family of projections associated with T, then

$$\|(\zeta_0 - T)x_n - y\|^2 = \int_\Gamma |(\zeta_0 - \zeta)Q(\zeta) - 1|^2 \, d\mu_y(\zeta),$$

where $d\mu_y(\zeta) = W_y(\zeta)|d\Phi(\zeta)| + d\mu_s(\zeta) := (y, dE_\zeta\, y)$. If $0 \notin \Gamma$ then W_y satisfies Szegö's condition precisely when y lies outside the closed span of $S_y := \{T^k y\}_{k\geq 1}$. Indeed, Szegö's classical result for measures $d\nu = W(\theta)\, d\theta$ on the unit circle is

$$\inf_{p\in A(\Delta), p(0)=0} \int |1 - p\,|^2 \, d\nu = \exp\left(\frac{1}{2\pi}\int \log W(\theta)\, d\theta\right),$$

where $A(\Delta)$ is the set of analytic functions in Δ. A conformal mapping generalizes this to any simply connected domain with boundary $\Gamma \in C^{1,\alpha}$ as long as $0 \notin \Gamma$. Applying the spectral theorem as above, it follows that y is isolated from the span of S_y precisely when $\log W_y$ is integrable. We now have an equivalent formulation of our main result.

Corollary 3.3.2 *Let T be a bounded normal operator on a Hilbert space, H, with spectrum $\Gamma \in C^{1,\alpha}$ and $0 \notin \Gamma$. If $y \in H \setminus S_y$ then*

$$\liminf_{n\to\infty} n \left\|(\zeta_0 - T)x_n - y\right\|^2 \geq W_y(\zeta_0),$$

for almost every $\zeta_0 \in \Gamma$. Equality holds if Q_n is given by (3.2.11) and $\alpha > 1/2$, or W_y is bounded and $d\mu_s \equiv 0$.

3.4 Proofs

We begin with the proof of Lemma 3.2.5. Let ω_D denote harmonic measure in the domain D. If ∂D is smooth, we have the representation $d\omega_D(z,\zeta) = K_D(z,\zeta)|d\zeta|$, for $\zeta \in \partial D$. Let Φ_r ($r<1$) denote a conformal map of Ω_r onto $\mathbf{C}\setminus\Delta$ with $\Phi_r(\zeta_0) > 0$ and $\Phi_r(z) \sim z$ ($z \to \infty$) and let Ψ_r denote its inverse. Since $\Gamma \in C^{1+\alpha}$, $\{\Phi_r'\} \subset Lip(\alpha)$ with uniformly bounded lipschitz constants for $1/2 < r < 1$. The same is true of Ψ_r', so Φ_r' are uniformly bounded away from 0 and ∞. This permits the following estimate:

Lemma 3.4.1 *There is a constant $C > 0$ such that for all $1/2 < r < 1$, $z \in \Omega_r$ and $\eta \in \Gamma$,*

$$\int_{\Gamma_r} K_U(\zeta,\eta) K_{\Omega_r}(z,\zeta)\,|d\zeta| \leq C \frac{|\Phi_r(\eta)|^2|\Phi_r(z)|^2 - 1}{|\Phi_r(\eta)\overline{\Phi_r(z)} - 1|^2}. \tag{3.4.17}$$

Proof. The conformal equivalence of harmonic measure implies that

$$K_{\Omega_r}(z,\zeta) = |\Phi_r'(\zeta)| K_{\mathbf{C}\setminus\Delta}(\Phi_r(z),\Phi_r(\zeta)) \leq C_1 \frac{|\Phi_r(z)|^2 - |\Phi_r(\zeta)|^2}{|\Phi_r(z) - \Phi_r(\zeta)|^2},$$

for all $\zeta \in \Gamma_r$ and $\zeta \in \Omega_r$. Similarly, for all $\eta \in \Gamma$ and $z \in \Omega_r$,

$$K_U(\zeta,\eta) \leq C_2 \frac{1-|\Phi^*(\zeta)|^2}{|\Phi^*(\zeta)-\Phi^*(\eta)|^2}.$$

Note that $1-|\Phi^*(\zeta)|^2 \sim dist(\Gamma,\Gamma_r) \sim |\Phi_r(\eta)|^2 - 1$, uniformly for $\zeta \in \Gamma_r$ and $\eta \in \Gamma$, and for all $1/2 < r < 1$. This implies, since $|(\Phi^*)'|$ is also bounded above and below, that

$$K_U(\zeta,\eta) \leq C_3 \frac{|\Phi_r(\eta)|^2-1}{|\Phi_r(\zeta)-\Phi_r(\eta)|^2}.$$

Thus, with $\Phi_r(\eta) = Re^{i\theta}$ and $\Phi_r(z) = \rho e^{i\phi}$,

$$\begin{aligned}\int_{\Gamma_r} K_U(\zeta,\eta) K_{\Omega_r}(z,\zeta)\,|d\zeta| &\leq C_4 \int_{\Gamma_r} \frac{|\Phi_r(\eta)|^2-1}{|\Phi_r(\zeta)-\Phi_r(\eta)|^2} \cdot \frac{|\Phi_r(z)|^2-|\Phi_r(\zeta)|^2}{|\Phi_r(z)-\Phi_r(\zeta)|^2}\,|d\zeta| \\ &\leq C_5 \int_{-\pi}^{\pi} \frac{R^2-1}{|Re^{i\theta}-e^{it}|^2} \frac{\rho^2-1}{|\rho e^{i\phi}-e^{it}|^2}\,dt\end{aligned}$$

The last inequality follows again from the uniform boundedness of Φ_r' away from 0. To evaluate, we use the series development of the Poisson kernel:

$$\frac{1-x^2}{|1-xe^{it}|^2} = \sum_{n=-\infty}^{\infty} x^{|n|} e^{int} \qquad (0 < x < 1).$$

Since $R, \rho > 1$,

$$\begin{aligned}&\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{R^2-1}{|Re^{i\theta}-e^{it}|^2} \frac{\rho^2-1}{|\rho e^{i\phi}-e^{it}|^2}\,dt \\ &\qquad = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left\{ \sum_{n=-\infty}^{\infty} R^{-|n|} e^{in(t-\theta)} \times \sum_{k=-\infty}^{\infty} \rho^{-|k|} e^{ik(t-\phi)} \right\} dt \\ &\qquad = \sum_{n=-\infty}^{\infty} (\rho R)^{-|n|} e^{in(\phi-\theta)} = \frac{\rho^2 R^2 - 1}{|\rho R - e^{i(\phi-\theta)}|^2},\end{aligned}$$

which equals the fraction on the right hand side of (3.4.17).

■

Proof. (Lemma 3.2.5) Let $h = h_r$ represent the solution to the dirichlet problem in Ω_r with boundary data $h(\zeta) = \log|D(\zeta)|$, $(\zeta \in \Gamma_r)$. First we prove that if $r = r_n < 1$

and $1 - r_n \sim 1/n$ then

$$|P_n(z)| \le Ce^{-h(z)+ng_{\Omega_r}(z,\infty)}, \qquad (z \in \Omega_r), \tag{3.4.18}$$

for some $C > 0$ independent of r and n. For $z \in \Gamma_r$ and $r < \rho < 1$,

$$|P_n(z)D(z)|^2 = \left| \frac{1}{2\pi} \int_{\Gamma_\rho} \frac{P_n(\zeta)^2 D(\zeta)^2}{\zeta - z} \, d\zeta \right| \le \frac{1}{2\pi} \int_{\Gamma_\rho} |P_n D|^2 \, ds \cdot \max_{\zeta \in \Gamma_\rho} \frac{1}{|\zeta - z|}.$$

Letting $\rho \to 1^-$, we find that

$$\begin{aligned} |P_n(z)D(z)|^2 &\le \max_{\zeta \in \Gamma} |\Phi'(\zeta)|^{-1} \times \frac{1}{2\pi} \int_\Gamma |P_n|^2 W |\Phi'| \, ds \cdot \frac{C_1}{1-r} \\ &\le C_2 \frac{|P_n(\zeta_0)|^2 \lambda_n(\mu, \zeta_0)}{1-r} \le C_3 \frac{|P_n(\zeta_0) D(\zeta_0)|^2}{n(1-r)}, \end{aligned}$$

by (3.2.16). The constant C_3 is independent of n and r. So, $|P_n(z)D(z)|$ is bounded uniformly for $z \in \Gamma_r$ as $r = r_n \to 1$ as long as $1 - r \sim 1/n$. This proves, under the stated assumptions, that

$$u(z) = \log |P_n(z)| + h(z) - ng_{\Omega_r}(z, \infty)$$

is bounded on Γ_r. It is also clearly subharmonic in Ω_r and bounded at ∞, so it must be true that $u(z) \le \log C$ in Ω_r, which establishes (3.4.18).

Let's examine the exponent on the right side of (3.4.18). Since Φ_r' are uniformly bounded, we may choose $\delta > 0$ so that $S_r := \{w : |w - \zeta_0| < \delta \cdot dist(\zeta_0, \Gamma_r)\}$ has $diam(\Phi_r(S_r)) < \frac{1}{2} dist(\zeta_0, \Gamma_r)$ for all $1/2 < r < 1$. Since $\Phi_r(\zeta_0)$ is real, this implies that the sets $\Phi_r(S_r)$ are contained in a fixed sector, Σ, of $\mathbf{C} \setminus \Delta$ emanating from 1 and symmetric about the real axis. We claim that the functions $-h_r$ are uniformly bounded on S_r. To see this, let $z \in S_r$ and apply (3.4.17) to obtain

$$
\begin{aligned}
-h(z) = & \int_{\Gamma_r} \log |D(\zeta)|^{-1} K_{\Omega_r}(z,\zeta)\,|d\zeta| \\
&= \int_{\Gamma_r} \left(\int_{\Gamma} \log |D(\eta)|^{-1} K_U(\zeta,\eta)\,|d\eta| \right) K_{\Omega_r}(z,\zeta)\,|d\zeta| \\
&= \int_{\Gamma} \log |D(\eta)|^{-1} \left(\int_{\Gamma_r} K_U(\zeta,\eta) K_{\Omega_r}(z,\zeta)\,|d\zeta| \right) |d\eta| \\
&\leq C_1 \int_{\Gamma} \log |D(\eta)|^{-1} \frac{|\Phi_r(\eta)|^2 |\Phi_r(z)|^2 - 1}{|\Phi_r(\eta)\overline{\Phi_r(z)} - 1|^2}\,|d\eta| \qquad (3.4.19)
\end{aligned}
$$

Now let $R_r(\theta)e^{i\theta} = \Phi_r(\eta)$ and set $\Phi_r(z) = \rho e^{i\phi}$ and $F_r(\theta) = \log |D \circ \Psi_r(R_r(\theta)e^{i\theta})|^{-1}$. With a change of variables, we find that

$$
\begin{aligned}
-h(z) \leq & C_2 \int_{\pi}^{\pi} F_r(\theta) \frac{1 - 1/(R_r(\theta)\rho)^2}{|e^{i\theta} - e^{i\phi}/(R_r(\theta)\rho)|^2}\,d\theta \\
&\leq C_3 \frac{1}{2\pi} \int_{\pi}^{\pi} F_r(\theta) \frac{1 - 1/(R'_r\rho)^2}{|e^{i\theta} - e^{i\phi}/(R'_r\rho)|^2}\,d\theta,
\end{aligned}
$$

where $R'_r > 1$ are constants. The second inequality follows from the fact that $R_r(\theta)\rho$ is bounded away from 1 uniformly for $z \in S_r$. But the last term is the Poisson integral, $[PF_r](e^{i\phi}/(R'_r\rho))$ of F_r evaluated at the point $e^{i\phi}/(R'_r\rho) = 1/R'_r\overline{\Phi_r(z)}$. Since $\Phi_r(S_r) \subset \Sigma$, this point is contained in the reciprocal sector, $\Sigma^{-1} \subset \Delta$. Σ^{-1} is based at 1 and non-tangential to $\partial\Delta$, so we may apply a fundamental inequality for Poisson integrals of finite measures (see [17, p. 242]) to conclude that

$$
-h(z) \leq C_4 (MF_r)(1) \qquad (z \in S_r).
$$

MF_r is the Hardy maximal function of F_r:

$$
\begin{aligned}
MF_r\,(1) = \sup_I \frac{1}{|I|} \int_I F_r(\theta)\,d\theta \leq & C_5 \sup_I \frac{1}{|I|} \int_{\Psi_r(I')} \log |D(\eta)|^{-1}\,|d\eta| \\
&= C_6 \sup_I \frac{|\Psi_r(I')|}{|I|} \frac{1}{|\Psi_r(I')|} \int_{\Psi_r(I')} \log W(\eta)^{-1}\,|d\eta|
\end{aligned}
$$

(Here the intervals I are centered at $\theta = 0$, $I' = \{e^{i\theta} : \theta \in I\}$ and $|\cdot|$ denotes arc length measure.) The last integral is bounded independently of r since the arcs $\Psi_r(I)$ shrink to ζ_0, which is a lebesgue point of $\log W$. This proves our claim.

Finally, since $z \in S_r$ implies that $|\Phi_r(z)| < 1 + c|1-r|$, $g_{\Omega_r}(z,\infty) = \log|\Phi_r(z)| \leq c|1-r|$ and, therefore, $n\, g_{\Omega_r}(z,\infty)$ is bounded on S_r as $n \to \infty$ since $1 - r_n \sim 1/n$. So, (3.4.18) ensures that $P_n(z)$ is bounded uniformly on the sets S_r. For sufficiently small $\epsilon > 0$ and large n, $r_n = 1 - 1/n\epsilon$ makes $\{z : |z - \zeta_0| < c/n\} \subset S_{r_n}$. This completes the proof.

■

Proof. (Theorem 3.2.3) Since the Christoffel functions are clearly monotonic in the measure, we may presume that $W \leq 1$. Otherwise, the left hand side of (3.2.8) is only increased. Choose a sequence $z_n \to 1$ non-tangentially in $\{z : |z| > 1\}$ in such a way that $|z_n - 1| \sim 1/n$. We will prove that

$$\limsup_{n\to\infty} \left\{ |P_n(\Psi(z_n))Q(\Psi(z_n))| - |P_n(\zeta_0)|\sqrt{\lambda_n(\zeta_0)} \left(\sum_{k=0}^{n} |z_n|^{2k}\right)^{1/2} \right\} \leq 0. \qquad (3.4.20)$$

Let derive (3.2.8) from (3.4.20). It suffices to show, for arbitrarily small $\epsilon > \epsilon' > 0$ and $\delta > 0$, the existence of a sequence $z_n \to 1$ non-tangentially in $\{z : |z| > 1\}$ and with $\epsilon'/n < |z_n - 1| < \epsilon/n$ such that

$$\liminf_{n\to\infty} |P_n(\Psi(z_n))| \geq (1-\delta)|P_n(\zeta_0)|. \qquad (3.4.21)$$

For then $\Psi(z_n) \to \zeta_0$ non tangentially in Ω, so $|Q(\Psi(z_n))|^2 \to W(\zeta_0)$ which implies by virtue of (3.4.20) that

$$\liminf_{n\to\infty} \lambda_n(\mu,\zeta_0) \sum_{k=0}^{n} |z_n|^{2k} \geq (1-\delta)^2 W(\zeta_0).$$

This, together with

$$\sum_{k=0}^{n} |z_n|^{2k} \leq n\left(1 + \frac{\epsilon}{n}\right)^{2n} < ne^{2\epsilon},$$

proves (3.2.8). Now, Lemma 3.2.5 provides constants M and c such that $|P_n(z)| \leq M$ whenever $|z - \zeta_0| \leq c/n$. Thus,

$$|z-\zeta_0| \leq \frac{c}{2n} \implies |P_n'(z)| = \Big| \frac{1}{2\pi i} \int_{|\zeta - z| = c/2n} \frac{P_n(\zeta)}{(z-\zeta)^2}\, d\zeta \Big| \leq \frac{2M}{c} n,$$

so if ϵ is sufficiently small and $|z - \zeta_0| < \epsilon/n$ then

$$|P_n(z) - P_n(\zeta_0)| = \Big| \int_{\zeta_0}^{z} P_n'(\zeta)\, d\zeta \Big| \leq \delta$$

and, consequently, $|P_n(z)| \geq (1-\delta)|P_n(\zeta_0)|$. Since $|\Psi'|$ is uniformly bounded the required sequence exists.

To establish (3.4.20), define the kernels

$$H_n(z) := \sum_{k=0}^{n} z_n^k \overline{\Phi(z)}^k.$$

With $\widetilde{F} := F \circ \Psi$, we have

$$\begin{aligned} \frac{1}{2\pi i} \int_\Gamma P_n(\zeta) Q(\zeta) H_n(\zeta) \Phi'(\zeta)\, d\zeta &= \frac{1}{2\pi i} \int_\gamma \widetilde{P}_n(w) \widetilde{Q}(w) \frac{1 - z_n^{n+1} \overline{w}^{n+1}}{1 - z_n \overline{w}}\, dw \\ &= \frac{1}{2\pi i} \int_\gamma \frac{\widetilde{P}_n(w)\widetilde{Q}(w)}{1 - z_n \overline{w}}\, dw - \frac{z_n^{n+1}}{2\pi i} \int_\gamma \frac{\widetilde{P}_n(w)\widetilde{Q}(w)}{w^n (w - z_n)}\, dw. \end{aligned}$$

$\widetilde{P}_n \sim w^{n-1}$ as $w \to \infty$, so the second integral gives, with the substitution $z = 1/w$,

$$\frac{z_n^n}{2\pi i} \int_\gamma \frac{\widetilde{P}_n(1/z)\widetilde{Q}(1/z) z^{n-1}}{(z_n^{-1} - z)}\, dz = -z_n \widetilde{P}_n(z_n) \widetilde{Q}(z_n).$$

Subtracting, we obtain

$$\begin{aligned} &- z_n \widetilde{P}_n(z_n) \widetilde{Q}(z_n) + \frac{1}{2\pi i} \int_\Gamma P_n(\zeta) Q(\zeta) H_n(\zeta) \Phi'(\zeta)\, d\zeta \\ &\qquad = \frac{1}{2\pi i} \int_\gamma \frac{\widetilde{P}_n(w)\widetilde{Q}(w)}{1 - z_n \overline{w}}\, dw =: I. \quad (3.4.22) \end{aligned}$$

We show that $I \to 0$ as $n \to \infty$. To this end, choose a large $a > 0$ and let $K_1 = [-a/n, a/n]$ and $K_2 = [-\pi, \pi] \setminus K_1$. With

$$I_j := \frac{1}{2\pi i} \int_{K_j} \frac{\widetilde{P}_n(w)\widetilde{Q}(w)}{1 - z_n \overline{w}}\, dw, \qquad (w = e^{i\theta})$$

$I = I_1 + I_2$. Since $z_n \to 1$ non-tangentially, $|e^{i\theta} - z_n| \geq r\theta$, for some $r > 0$. Therefore,

$$\begin{aligned} |I_2|^2 \leq \frac{1}{2\pi} \int_{K_2} |\widetilde{P}_n \widetilde{Q}|^2 \, d\theta \times \frac{1}{2\pi} \int_{K_2} \frac{d\theta}{|e^{i\theta} - z_n|^2} \leq \frac{1}{2\pi} \int_{\Gamma} |P_n Q|^2 \, |d\Phi| \\ \times \frac{1}{r^2\pi} \int_{a/n}^{\infty} \frac{d\theta}{\theta^2} = \frac{1}{2\pi} \int_{\Gamma} |P_n|^2 W |\Phi'| \, ds \times \frac{n}{r^2 \pi a} = |P_n(\zeta_0)|^2 \lambda_n(\zeta_0) \frac{n}{\pi a r^2}, \quad (3.4.23) \end{aligned}$$

which can be made arbitrarily small for sufficiently large a, since $n\lambda_n(\zeta_0)$ is bounded for all n, by (3.2.16). To estimate I_1, decompose it as $I_1 = I_{11} + I_{12}$, where

$$\begin{aligned} I_{11} &:= \frac{1}{2\pi i} \int_{K_1} \frac{\widetilde{P}_n(w)\widetilde{D}(w)}{1 - z_n \overline{w}}\, dw \quad \text{and} \\ I_{12} &:= \frac{1}{2\pi i} \int_{K_1} \frac{\widetilde{P}_n(w)}{1 - z_n \overline{w}} \left[\widetilde{Q}(w) - \widetilde{D}(w) \right] dw. \end{aligned}$$

Lemma 3.2.5 implies that

$$\begin{aligned} |I_{12}| &\leq \frac{1}{2\pi} \max_{w \in K_1} \left| \frac{\widetilde{P}_n(w)}{1 - z_n \overline{w}} \right| \times \int_{K_1} |\widetilde{Q}(e^{i\theta}) - \widetilde{D}(e^{i\theta})| \, d\theta \\ &\leq \frac{C}{|z_n| - 1} \int_{K_1} |\widetilde{Q}(e^{i\theta}) - \widetilde{D}(e^{i\theta})| \, d\theta, \end{aligned}$$

which tends to 0 as $n \to \infty$ since $n(|z_n| - 1) > \epsilon' > 0$; 1 is a Lebesgue point of $\widetilde{Q}$ and $\widetilde{D}$; and $\widetilde{Q}(1) = \widetilde{D}(1)$. With $J_1 := \Psi(K_1)$, we have

$$I_{11} = \frac{1}{2\pi i} \int_{K_1} \frac{\widetilde{P}_n(w)\widetilde{D}(w) w}{w - z_n}\, dw = \frac{1}{2\pi i} \int_{J_1} \frac{P_n(\zeta) D(\zeta) \Phi(\zeta)}{\Phi(\zeta) - z_n} \Phi'(\zeta)\, d\zeta.$$

Consider $G(z) := \Phi(\zeta_0) + \Phi'(\zeta_0)(z - \zeta_0)$, the linearization of Φ about ζ_0, and define

$$I_{11}^* := \frac{1}{2\pi i} \int_{J_1} \frac{P_n(\zeta) D(\zeta) \Phi^*(\zeta)}{G(\zeta) - z_n} (\Phi^*)'(\zeta)\, d\zeta.$$

Since $\Phi \in C^{1,\alpha}$, $|G(\zeta) - \Phi(\zeta)| \leq C_1|\zeta - \zeta_0|^{1+\alpha}$. Consequently, the curve $L := \{G(\zeta) : \zeta \in \Gamma\}$ is tangent to γ at 1 so, since $z_n \to 1$ non-tangentially to γ, $|G(\zeta) - z_n| \geq c|\Phi(\zeta) - z_n| \geq c'/n$ for all $\zeta \in \Gamma$ and sufficiently large n. Thus,

$$\left|\frac{1}{\Phi(\zeta) - z_n} - \frac{1}{G(\zeta) - z_n}\right| \leq C_2 \frac{|\zeta - \zeta_0|^{1+\alpha}}{n^{-2}} \leq C_3\, n^{1-\alpha},$$

for $\zeta \in J_1$. By applying Cauchy's inequality, we obtain the following:

$$\begin{aligned}
|I_{11} - I_{11}^*| \leq \frac{1}{2\pi} &\left|\int_{J_1} P_n(\zeta)D(\zeta)\Phi(\zeta)\left(\frac{1}{\Phi(\zeta) - z_n} - \frac{1}{G(\zeta) - z_n}\right)\Phi'(\zeta)\, d\zeta\right| \\
&+ \frac{1}{2\pi}\left|\int_{J_1} \frac{P_n(\zeta)D(\zeta)}{G(\zeta) - z_n}\left[\Phi(\zeta)\Phi'(\zeta) - \Phi^*(\zeta)(\Phi^*)'(\zeta)\right]\, d\zeta\right| \\
\leq \frac{1}{2\pi}\left(\int_{J_1} |P_n D|^2|\Phi'||d\zeta|\right)^{1/2} &\times \left(\int_{J_1}\left|\frac{1}{\Phi(\zeta) - z_n} - \frac{1}{G(\zeta) - z_n}\right|^2 |\Phi'(\zeta)d\zeta|\right)^{1/2} \\
+ C_4 n \frac{1}{2\pi}\left(\int_{J_1} |P_n D|^2|\Phi'||d\zeta|\right)^{1/2} &\times \left(\int_{J_1} \frac{|\Phi(\zeta)\Phi'(\zeta) - \Phi^*(\zeta)(\Phi^*)'(\zeta)|^2}{|\Phi'(\zeta)|}\,|d\zeta|\right)^{1/2}
\end{aligned} \tag{3.4.24}$$

The first term on the right side of (3.4.24) is bounded above by

$$C_5\sqrt{|P_n(\zeta_0)|\lambda_n(\mu, \zeta_0)}\, n^{1-\alpha}|J_1|^{1/2},$$

which, by Lemma 3.2.5, converges to 0 as $n \to \infty$. ($|\cdot|$ denotes arc-length measure.) The final term does as well, since its last integrand is continuous. Now, the integrand of I_{11}^* is holomorphic in U, so we may deform the contour, J_1, to the homologous contour, $-J_2 := -\Psi(K_2)$. This gives

$$\begin{aligned}
|I_{11}^*| = &\left|\frac{1}{2\pi i}\int_{J_2} \frac{P_n(\zeta)D(\zeta)\Phi^*(\zeta)}{G(\zeta) - z_n}(\Phi^*)'(\zeta)\, d\zeta\right| \leq \\
&\frac{1}{2\pi}\left(\int_\Gamma |P_n D|^2|(\Phi^*)'||d\zeta|\right)^{1/2} \times \left(\int_{J_2}\frac{|(\Phi^*)'(\zeta)|}{|G(\zeta) - z_n|^2}|d\zeta|\right)^{1/2} \\
&\leq C_6\left(\int_\Gamma |P_n D|^2|\Phi'||d\zeta|\right)^{1/2}\left(\int_{K_2}\frac{d\theta}{|e^{i\theta} - z_n|^2}\right)^{1/2}.
\end{aligned}$$

As in (3.4.23), this can be made arbitrarily small by choosing sufficiently large a. We conclude that I_1 and I_2 are arbitrarily small if n is sufficiently large, which establishes that $I \to 0$ as $n \to \infty$. Finally,

$$\left|\frac{1}{2\pi i}\int_\Gamma P_n(\zeta)Q(\zeta)H_n(\zeta)\,d\Phi(\zeta)\right|^2 \le \frac{1}{2\pi}\int_\Gamma |P_n(\zeta)Q(\zeta)|^2\,|d\Phi(\zeta)| \times \frac{1}{2\pi}\int_\Gamma |H_n(\zeta)|^2\,|d\Phi(\zeta)|$$

$$= |P_n(\zeta_0)|^2\lambda_n(\zeta_0)\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\sum_{k=0}^{n} z_n^k e^{-ik\theta}\right|^2 d\theta = |P_n(\zeta_0)|^2\lambda_n(\zeta_0)\sum_{k=0}^{n}|z_n|^{2k}.$$

This, together with (3.4.22), completes the proof.

■

Proof. (Lemma 3.2.4)

The Laurent series $\Phi(\zeta) \sim c_{-1}\zeta + c_0 + c_1/\zeta + \cdots$ converges uniformly on Γ_R for sufficiently large R, so

$$\frac{1}{2\pi i}\int_{\Gamma_R}\frac{\Phi'(\zeta)}{\zeta - z}d\zeta = c_{-1} \qquad (z \in \Gamma),$$

and $F_n(z) = I + c_{-1}\Phi^n(z)$, where

$$I := \frac{1}{2\pi i}\int_{\Gamma_R}\frac{\Phi^n(\zeta) - \Phi^n(z)}{\zeta - z}\Phi'(\zeta)d\zeta.$$

Make the substitutions $w = e^{it} = \Phi(\zeta)$ and $e^{i\theta} = \Phi(z)$ and factor the integrand to obtain

$$I = \frac{1}{2\pi i}\int_{\gamma_R}\frac{w - e^{i\theta}}{\Psi(w) - \Psi(e^{i\theta})}\sum_{k=0}^{n-1} w^k e^{i(n-1-k)\theta}\,dw$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{it} - e^{i\theta}}{\Psi(e^{it}) - \Psi(e^{i\theta})}\sum_{k=1}^{n} e^{ikt}e^{i(n-k)\theta}\,dt \quad (3.4.25)$$

(Since the later integrand is continuous and analytic in $|w| > 1$, we may deform the contour back to γ.) Now let

$$f(x,y) := \frac{e^{ix} - e^{iy}}{\Psi(e^{ix}) - \Psi(e^{iy})} \quad \text{and} \quad F_y(x) := f(x,y);$$

and let $S_n F$ denote the partial sums of the Fourier series of F and $D_n(t)$ the Dirichlet kernels. F_θ is the continuation to γ of a function analytic and bounded in $\mathbf{C} \setminus \Delta$, so its Fourier coefficients, $\hat{F}_\theta(k)$, with positive index vanish. Thus, from (3.4.25) we obtain

$$\begin{aligned} I = e^{in\theta} \frac{1}{2\pi} \int F_\theta(t) \sum_{k=1}^{n} e^{ik(t-\theta)}\, dt = e^{in\theta} \int F_\theta(t) \big[D_n(t-\theta) - 1 \big]\, dt \\ = e^{in\theta} \big(S_n F_\theta(\theta) - \hat{F}_\theta(0) \big) \end{aligned}$$

and, consequently, $F_n(z) = e^{in\theta}(S_n F_\theta(\theta) - \hat{F}_\theta(0) + c_{-1})$. In the following, the measures $dw/w|_{\gamma_R}$ are uniformly bounded and the integrand is analytic for $|w| > 1$ and continuous for $|w| \geq 1$ so we may deform the contour of integration and evaluate asymptotically: since $\Psi(w) \sim (c_{-1})^{-1} w$ as $w \to \infty$,

$$\hat{F}_\theta(0) = \frac{1}{2\pi i} \int_\gamma \frac{w - e^{i\theta}}{\Psi(w) - \Psi(e^{i\theta})} \frac{dw}{w} = \lim_{R \to \infty} \frac{1}{2\pi i} \int_{\gamma_R} \frac{w}{\Psi(w)} \frac{dw}{w} = c_{-1}.$$

Furthermore, since $F_\theta(\theta) = \Psi'(e^{i\theta})^{-1} = \Phi'(z)$, we have

$$F_n(z) - \Phi'(z)\Phi^n(z) = e^{in\theta}(S_n F_\theta(\theta) - F_\theta(\theta))$$

and so

$$\int \Big| \sum_{k=0}^{n} \big[F_k(z) - \Phi'(z)\Phi^k(z) \big] \Big|^2 ds \sim \int_{-\pi}^{\pi} \Big| \sum_{k=0}^{n} e^{ik\theta} \big(S_k F_\theta(\theta) - F_\theta(\theta) \big) \Big|^2 d\theta. \qquad (3.4.26)$$

This reduces the problem to an estimation the convergence $S_n F_\theta \to F_\theta$.

We show that the functions $F_\theta \in Lip\,(\alpha)$ and have uniformly bounded Lipschitz constants. Since F_θ are uniformly bounded, it suffices to prove this true of the family

$\{1/F_\theta\}$. Set $e^{i\theta_k} = u_k$ and $e^{i\theta} = w$. If $u_k \neq w$, we have

$$F_\theta(\theta_1)^{-1} - F_\theta(\theta_2)^{-1} = \frac{1}{u_1 - w}\int_w^{u_1} \Psi'(z)\,dz - \frac{1}{u_2 - w}\int_w^{u_2} \Psi'(z)\,dz$$
$$= \frac{1}{u_1 - w}\int_0^{u_1 - w} \big(\Psi'(z+w) - \Psi'(z/a + w)\big)\,dz,$$

where $a = (u_1 - w)/(u_2 - w)$. Thus,

$$|F_\theta(\theta_1)^{-1} - F_\theta(\theta_2)^{-1}| \leq \frac{C}{|u_1 - w|}\int_0^{u_1 - w} |z - z/a|^\alpha \,|dz|$$
$$\leq \frac{C'|1 - 1/a|^\alpha}{|u_1 - w|}|u_1 - w|^{1+\alpha} = C'|u_1 - u_2|^\alpha.$$

If $u_1 = w$, the result follows in a similar way. A standard result from approximation theory now applies: $|S_n F_\theta(x) - F_\theta(x)| \leq Cn^{-\alpha}\ln n$, where C is independent of x and θ. (See e.g. [8, pp. 180, 192-194].) Thus,

$$\frac{1}{n}\sum_{k=0}^{n-1} |S_k F_\theta(\theta) - F_\theta(\theta)| \leq \frac{C}{n}\sum_{k=0}^{n-1}\frac{\ln k}{k^\alpha} = O\left(\frac{\ln n}{n^\alpha}\right),$$

which proves (3.2.14) if $\alpha > 1/2$.

To prove (3.2.13), consider the partial sums, S_{nm}, of the double Fourier series of $f(x,y)$. We claim that they converge to f uniformly in x and y as $n \sim m \to \infty$. Let $D_n(t)$ denote the (normalized) Dirichlet kernels.

$$S_{nm}(x,y) - f(x,y) = \iint D_n(t)D_m(u)[f(x-t, y-u) - f(x,y)]\,dtdu$$
$$= \iint D_n(t)D_m(u)[f(x-t, y-u) - f(x-t,y)]\,dtdu$$
$$+ \int D_n(t)[f(x-t,y) - f(x,y)]\,dt =: I_1 + I_2.$$

We have already shown that $|I_2| = O(n^{-\alpha} \ln n)$. On the other hand, since $F_x(y) = F_y(x)$,

$$|I_1| \le \sup_t \Big| \int D_m(u) \big[f(x-t, y-u) - f(x-t, y) \big] \, du \Big| \times \int |D_n(\tau)| \, d\tau$$
$$= \sup_t |S_m F_{x-t}(y) - F_{x-t}(y)| \ln n \le C \frac{\ln n \, \ln m}{m^\alpha}.$$

This proves our claim.

We may now evaluate the right hand side of (3.4.26) in terms of the development

$$\frac{w-u}{\Psi(w) - \Psi(u)} \sim \sum_{m,l} \frac{c_{ml}}{w^m u^l} := \lim_{N\to\infty} \sum_{m,l=0}^{N} \frac{c_{ml}}{w^m u^l} \qquad (|w| \ge 1, |u| \ge 1),$$

which converges uniformly on the torus $\gamma \times \gamma$.

$$S_k F_\theta(\theta) = \sum_{|j|<k} \frac{1}{2\pi} \int_\pi^\pi e^{ij(\theta - t)} \sum_{m,l} c_{ml} \, e^{-i(m\theta + lt)} \, dt = \sum_{j=0}^{k} \sum_{m=0}^{\infty} c_{m,j} \, e^{-i(m+j)\theta}.$$

Also, $F_\theta(\theta) = \sum_{m,l} c_{ml} e^{-i(m+l)\theta}$, so

$$\frac{1}{2\pi} \int_\pi^\pi \Big| \sum_{k=0}^{n} e^{ik\theta} \big(S_k F_\theta(\theta) - F_\theta(\theta) \big) \Big|^2 d\theta = \frac{1}{2\pi} \int_\pi^\pi \Big| \sum_{k=0}^{n} \sum_{j>k} \sum_{m\ge 0} c_{mj} \, e^{-i(m+j-k)\theta} \Big|^2 d\theta$$
$$= \sum_{k=0}^{n} \sum_{r=j-k}^{\infty} \sum_{j>k} |c_{r+k-j,j}|^2 = \sum_{k=0}^{n} \sum_{r=l}^{\infty} \sum_{l=1}^{\infty} |c_{r-l,l+k}|^2 = \sum_{k=0}^{n} \sum_{r=0}^{\infty} \sum_{l=1}^{\infty} |c_{r,l+k}|^2.$$

These series converge since the continuity of f implies that

$$\sum_{r,l} |c_{r,l}|^2 = \frac{1}{4\pi^2} \iint |f(x,y)|^2 \, dx dy < \infty.$$

Therefore,

$$\frac{1}{n} \sum_{k=0}^{n} \sum_{r,l} |c_{r,l+k}|^2 \to 0 \quad (n \to \infty),$$

which, by virtue of (3.4.26), this completes the proof.

■

References

[1] A. Bakushinsky and A. Goncharsky. *Ill-Posed Problems: Theory and Applications*. Kluwer Academic Publishers, Boston, 1994.

[2] E. Findley. *Fine Asymptotics of Christoffel Functions for General Measures*, preprint.

[3] E. Findley. *Universality for Locally Szegő Measures.* J. Approx. Theory. 155 (2008), pp. 136-154.

[4] G. Freud. *Orthogonal Polynomials*. Pergamon Press, Oxford, 1971.

[5] U. Grenander and G. Szegő. *Toeplitz Forms and their Applications*. Unviersity of California Press, Los Angeles, 1958.

[6] L. Golinskii. The Christoffel Function for Orthogonal Polynomials on a Circular Arc. *J. Approx. Theory* **101**, pp. 165-174 (1999)

[7] K. Hoffman, *Banach Spaces of Analytic Functions*, Prentice Hall, London 1962.

[8] N. Korneichuk. *Exact Constants in Approximation Theory.* Cambridge University Press, New York, 1991.

[9] E. Levin, D. Lubinsky. Applications of Universality Limits to Zeros and Reproducing Kernels of Orthogonal Polynomials, to appear in *J. Approx. Theory.* http://www.math.gatech.edu/ lubinsky/SelectedPapers.html

[10] D. S. Lubinsky. A New Approach to Universality Limits Involving Orthogonal Polynomials, to appear in *Ann. of Math.* http://www.math.gatech.edu/ lubinsky/SelectedPapers.html

[11] A. Máté, P. Nevai and V. Totik. Szegő's Extremum Problem on the Unit Circle, *Ann. of Math.* **134** (1991), 433–453.

[12] A. Máté and P. Nevai. Bernstein's inequality in L_p for $0 < p < 1$ and $(C, 1)$ bounds for orthogonal polynomials, *Ann. of Math.* **111** (1980), 145-154.

[13] P. Nevai. *Orthogonal Polynomials*. Memoirs Amer. Math. Soc., Vol. 213. Amer. Math. Soc., Providence, RI, 1979.

[14] P. Nevai. Géza Freud: A Case Study, *J. Approx. Theory* **48** (1986).

[15] P. Nevai, V. Totik and J. Zhang. Orthogonal Polynomials, *J. Approx. Theory* **67** (1991).

[16] Ransford, Thomas. *Potential Theory in the Complex Plane*. Cambridge University Press. New York, NY. 1995.

[17] W. Rudin, *Real and Complex Analysis*, 3rd ed. Mc-Graw Hill, Singapore, 1987.

[18] B. Simon. Two Extensions of Lubinsky's Universality Theorem, to appear in *J. d'Analyse Mathematique.*

[19] H. Stahl and V. Totik. *General Orthogonal Polynomials*, Cambridge University Press, New York, 1992.

[20] P. K. Suetin. Fundamental Properties of Polynomials Orthogonal on a Contour. *Russian Math. Surveys* **21** (1968), 35-83

[21] V. Totik. Asymptotics for Christoffel functions for general measures on the real line. J. Anal. Math. 81 (2000), 283–303.

[22] V. Totik. Universality and Fine Zero Spacing on General Sets, to appear in *Arkiv. för Math.*

[23] J. L. Ullman. On the Regular Behavior of Orthogonal Polynomials. *Proc. London Math. Soc.* **24** (1972): 119-48.

[24] A. Zygmund, *Trigonometric Series*, Vols. I and II, 2nd ed., Cambridge Univ. Press, Cambridge, 1979.

About the Author

Elliot M. Findley completed a B. S. in Physics at the University of South Florida in 2003 and an M.A. in Applied Mathematics at USF in 2004. Raised in Bakersfield, California, he graduated from Bakersfield Christian High School in 1998. Later that year, he moved to Temple Terrace, Florida, to attend Florida College for two years before transferring to USF.

His scholarly interests include all areas of analysis, but especially approximation theory and general orthogonal polynomials.