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A Generalized Acceptance Urn Model

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A Generalized Acceptance Urn Model

by

Kevin P. Wagner

A dissertation submitted in partial fulfillment
of the requirements for the degree of
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A Generalized Acceptance Urn Model

Kevin P. Wagner

ABSTRACT

An urn contains two types of balls: \( p \) “+t” balls and \( m \) “−s” balls, where \( t \) and \( s \) are positive real numbers. The balls are drawn from the urn uniformly at random without replacement until the urn is empty. Before each ball is drawn, the player decides whether to accept the ball or not. If the player opts to accept the ball, then the payoff is the weight of the ball drawn, gaining \( t \) dollars if a “+t” ball is drawn, or losing \( s \) dollars if a “−s” ball is drawn. We wish to maximize the expected gain for the player.

We find that the optimal acceptance policies are similar to that of the original acceptance urn of Chen et al. [7] with \( s = t = 1 \). We show that the expected gain function also shares similar properties to those shown in that work, and note the important properties that have geometric interpretations. We then calculate the expected gain for the urns with \( t/s \) rational, using various methods, including rotation and reflection. For the case when \( t/s \) is irrational, we use rational approximation to calculate the expected gain. We then give the asymptotic value of the expected gain under various conditions. The problem of minimal gain is then considered, which is a version of the ballot problem.

We then consider a Bayesian approach for the general urn, for which the number of balls \( n \) is known while the number of “+t” balls, \( p \), is unknown. We find formulas for the expected gain for the random acceptance urn when the urns with \( n \) balls are distributed uniformly, and find the asymptotic value of the expected gain for any \( s \) and \( t \).

Finally, we discuss the probability of ruin when an optimal strategy is used for the \((m, p; s, t)\) urn, solving the problem with \( s = t = 1 \). We also show that in general, when the initial capital is large, ruin is unlikely. We then examine the same problem with the random version of the urn, solving the problem with \( s = t = 1 \) and an initial prior distribution of the urns containing \( n \) balls that is uniform.
1. The Acceptance Urn Model

1.1. Introduction and Main Results. The acceptance urn problem is as follows: An urn contains \( p \) balls of value \(+t\), and \( m \) balls of value \(-s\). The balls are drawn out from the urn uniformly at random, without replacement, until the urn is empty. Before each draw from the urn, a player is asked to decide whether he would like to accept the next ball, with the payout being the value of the next ball drawn from the urn. The problem is to find an optimal strategy that maximizes the player’s expected gain, and to calculate the expected gain. The original acceptance urn model with \( s = t = 1 \) was proposed in Chen et al. [7] (1998). In this work, we generalize the problem so that \( s \) and \( t \) may be arbitrary nonnegative real numbers, finding an optimal strategy and calculating the expected gain. We find that an optimal strategy depends only on the sign of the weight of the urn, that is, the combined value of the balls left in the urn after \( j \) balls have been drawn, where \( 0 \leq j < m + p \).

In Chapter 2, we examine the properties of the expected gain function. Using a graphical interpretation of the outcomes of the draws from the urn, we find that three geometric transformations are of great use for the acceptance urns. Using the vertical stretch, we are able to rescale \( s \) and \( t \) to suit our needs. Vertical reflection associates each acceptance urn with a “dual” urn, which we call the antiurn. Using reflection, we establish a fundamental equation relating the expected gain using an optimal strategy for the urn, the initial weight of the urn, and the expected gain for the corresponding antiurn, with the Antiurn Theorem (Theorem 2.3), which is a generalization of Theorem 2.1 of [7]. Locally, we find that reflection is a very useful tool for the urns with \( s = t (= 1) \). The third, and perhaps most important, geometric transformation that we note in this work is rotation. The rotation is accomplished by reversing the order that (some of) the balls are drawn from the urn. We translate the effect of the rotation on the outcomes of the draws from the urn with the Reversal Lemma (Lemma 2.5).

We show using combinatorial arguments that the expected gain will not decrease if “\(+t\)” balls are added, “\(-s\)” balls are removed, or a collection of balls with a combined value of zero is added. These results were shown for the case \( s = t = 1 \) in [7], though those results were shown via a
direct manipulation of the expected gain function (1.1). We also find that the expected gain will not decrease if \( t \) is increased, \( s \) is decreased, or the ball values are increased at the expense of the number of balls. In particular, we show that the expected gain function is continuous in both \( s \) and \( t \), which enables us to use rational approximation to calculate the expected gain when \( t/s \) is irrational.

We work toward finding formulas expressing the expected gain in Chapters 3 and 4. For the original model with \( s = t = 1 \), Chen [7, Theorem 2.2] showed that the expected gain with \( m \, “-1” \) balls and \( p \, “+1” \) balls using an optimal strategy equals

\[
\max\{0, p - m\} + \left(\frac{m + p}{p}\right)^{1 - \min\{m, p\}} \sum_{k=0}^{\min\{m, p\} - 1} \left(\frac{m + p}{p}\right)^{k}.
\]

We provide a more direct proof of this result, by showing that for \( \max\{0, p - m\} \leq k \leq p \), the probability the player gains at least \( k \) equals

\[
\left(\frac{m + p}{p}\right)^{1} \left(\frac{m + p}{p - k}\right).
\]

We use a reflection method to show this result.

Generalizing, we find that reflection is not a useful method for calculating the expected gain in general. However, we find that there is significant rotational symmetry when \( t \) is an integer multiple of \( s \), or vice versa. Applying the rotation method indicated by the Reversal Lemma, we show a fundamental result with the Crossing Lemma (Lemma 3.7). A version of this result appeared in Mohanty [15, eq. (19)] (1968) using the convolution of paths, but without the use of rotation. Mohanty’s motivation was to count the number of paths with a particular property, while in this work our interest is in the actual bijection that was indicated in [15], and using it as a tool to count paths with other properties. Using the result of the Crossing Lemma, we are able to give multiple expressions for the expected gain. When \( m \geq pt \), these include the zero count form,

\[
\left(\frac{m + p}{p}\right)^{1 - t} \sum_{k=1}^{p} \left(\frac{kt + k}{k}\right)^{1 - \min\{m, p\}} \left(\frac{m + p - k - kt}{p - k}\right),
\]

which is calculated by counting the number of times the urn is neutral, that is, the number of times the urn weight is zero, by location. This was the method originally used in [7] (along with the use of a combinatorial identity) to show (1.1). A different method of counting, again with the aid of the
Crossing Lemma, produces the negative binomial form for \( m \geq pt \),

\[
\left( \frac{m + p}{p} \right)^{-1} \frac{1}{2} \sum_{k=0}^{p-1} \left( \frac{m + k}{k} \right) (t + 1)^{p-k},
\]

and a modification of this method produces the binomial form, again for \( m \geq pt \):

\[
-\frac{t}{2} + \left( \frac{m + p}{p} \right)^{-1} \frac{1}{2} \sum_{k=0}^{p} \left( \frac{m + p + 1}{k} \right) t^{p-k}.
\]

We also indicate how to calculate the probability that the player will gain at least \( \ell \) using an optimal strategy, for any \( m, p, \) and \( \ell \), using a rotation argument. This results in another formula for the expected gain, the distribution form, which reduces to (1.1) when \( s = t = 1 \).

In general, the methods used to derive the formulas above do not work when \( s \) and \( t \) are arbitrary, as the result of the Crossing Lemma does not hold for a general \( s \) and \( t \). Using a more primitive approach in Chapter 4, we are able to give another formula, the crossings form, for the expected gain when \( s = 1 \) and \( t \) is a positive integer. This method of calculation can be generalized so that \( s \) may also be an integer, and the result is a rather complicated formula (when compared to the formulas above) for the expected gain, for the case when \( t/s \) is rational. For the urns with \( ms = pt \), we show that this formula can be simplified. When \( t/s \) is irrational, the continuity of the expected gain in \( s \) and \( t \) allows us to use a rational approximation of \( t/s \) to obtain a formula for the expected gain.

Finally, we show that finding the distribution of the gain is equivalent to finding the distribution of the maximum urn weight achieved during play. This result, like the earlier case with \( s = 1 \) and \( t \) a positive integer, is shown with the help of a rotation method.

For large \( m \) and \( p \), we show in Chapter 5 that if \( (ms - pt)/\sqrt{p} \) tends to zero, then the expected gain is asymptotically equal to

\[
\frac{\sqrt{2\pi pt(t+s)}}{4}.
\]

We also find the asymptotic value of the expected gain in the case \( (ms - pt)/\sqrt{p} \to \alpha \), generalizing the result of Chen et al. [7, Theorem 3.2]. In the case \( (ms - pt)/\sqrt{p} \to \infty \) and \( (ms - pt)/p \to 0 \), we show that the expected gain is asymptotically equal to

\[
\frac{t(t+s)}{2} \cdot \frac{p}{ms - pt}.
\]

When \( s = t = 1 \), it was shown in [7, Theorem 3.1] that if \( m/p \to \lambda > 1 \), then asymptotically the expected gain equals \( \frac{1}{\lambda - 1} = \frac{p}{m-p} \). We extend this result to the case with \( s = 1 \) and \( t \) a positive
integer, showing that if \( m/(pt) \to \lambda > 1 \), then the asymptotic value of the expected gain equals

\[
\frac{t + 1}{2(\lambda - 1)} = \frac{t(t + 1)}{2} \cdot \frac{p}{m - pt},
\]

using the binomial form of the expected gain. We then use this result to show that the value is bounded in the general case when \( ms/(pt) \to \lambda > 1 \), for arbitrary nonnegative \( s \) and \( t \). Finally, when \( (pt - ms)/\sqrt{p} \to \infty \), we show that the expected gain is asymptotically equal to \( pt - ms \).

We examine a related problem to the acceptance urn problem, the ballot problem, in Chapter 6. For the ballot problem, candidate \( A \) receives \( a \) votes in an election, while candidate \( B \) receives \( b \) votes, with \( a \geq b\mu \), with \( \mu \geq 0 \) a fixed real number. The question is to find the probability that, as the votes are counted, candidate \( A \) always has more than \( \mu \) times the number of votes for \( B \). The probability that the player makes the minimal gain possible using an optimal strategy is a variation on this ballot problem. The reflection method that we use in this work dates back to the solution of the ballot problem with \( \mu = 1 \). See, for example, Feller [9] (1968) and Renault [19] (2008). For the case with \( \mu \) a positive integer, the solution to the problem using a rotation method was shown in Goulden and Serrano [11] (2003). For arbitrary \( \mu \), the solution to the problem was given by Takács [24] (1962), though in general the solution is not explicit. We discuss generalized ballot numbers and generalized zero gain numbers, and in particular, find explicit solutions for both the ballot and zero gain problems for the case when \( \mu \) is the reciprocal of an integer. The generalization of the ballot problem of Irving and Rattan [12] (2009), which uses a rotation method, and its application to the acceptance urn model is also discussed.

In Chapter 7, we consider the Bayesian version of the urn, where the total number of balls \( n \) is known, but the number of \( +t \) balls, \( p \), is random and is determined by an initial prior distribution \( \theta \) on \( \{0, \ldots, n\} \). We develop a condition that, when met, indicates when the player shall accept the next ball drawn from the urn. In [7], it was determined that an optimal acceptance strategy for the case \( s = t = 1 \) and \( \theta \) uniform is “accept if and only if at least as many ‘+1’ balls as ‘−1’ balls have been drawn.” In this work, we find a larger family of initial prior distributions with \( s = t = 1 \) for which this rule is optimal. For this family of distributions, we give a formula for the expected gain, and find that the player attempts to gain the initial weight of the urn, if it is positive. We also find a second family of distributions for which the opposite rule is optimal: “bet if and only if at least as many ‘−1’ balls as ‘+1’ balls have been drawn.” With this second family of distributions, we find that the player aims to take advantage of fluctuations that will occur along
the away, instead of trying to capture the initial weight of the urn (when positive). We give a formula for the expected gain for these distributions as well. For other distributions, we develop an algorithm that gives an indication of when and when not to bet, that also calculates the expected gain. However, the algorithm is not very efficient, and in general the algorithm requires a significant amount of information to be known beforehand.

For arbitrary \( s \) and \( t \), we find that when \( \theta \) is a binomial distribution with \( n \) trials and probability of success \( s/(t+s) \), the random acceptance urn is a fair game. That is, any acceptance strategy is optimal, and the expected gain equals zero. When \( \theta \) is uniform, we find a relatively simple optimal acceptance policy, calculate the expected gain, and find that when \( n \) is large the expected gain is asymptotically equal to \( \frac{nt^2}{2(t+s)} \). We modify the algorithm used for the random urns with \( s = t = 1 \) so that it will also present an indication of when to bet and the expected gain, for the urns containing “\(-s\)” and “\(+t\)” balls, for any \( n \) and \( \theta \).

We introduce a ruin problem that can be associated with the acceptance urn model, in Chapter 8. We give the player a bank \( b \), and if at some point during play the player has lost more than \( b - s \), we say the player is ruined. For the case \( s = t = 1 \), we show that probability the player is ruined while using an optimal strategy that maximizes the expected gain depends on the choice of that strategy. We solve the problem with \( s = t = 1 \) with the zero-bet strategy (Definition 1.9) and zero-pass strategy (Definition 1.8) by finding the probability that the player is ruined and the bank attains a maximum value of \( b + k \) before the player is ruined, for each \( k \geq 0 \).

We also apply the ruin problem to the Bayesian version of the urn. For those urns, we solve the problem with \( s = t = 1 \) and any \( b \) for one family of initial distributions, as the solution is related to the calculation of the expected gain. In particular, we show that with \( \theta \) uniform, the probability of ruin tends to zero as \( b \to \infty \).

In addition to the ballot problem, many of the results in this work can be applied to the area of lattice paths. A particle starts at the point \((0,0)\), and at each stage moves either one unit up or one unit to the right, until it reaches the point \((p,m)\). The combined movements form a lattice path. For general information on lattice paths, see Mohanty [14] (1979) and Krattenthaler [13] (1997). The results for the acceptance urn model with \( m \) balls of value \(-s\) and \( p \) balls of value \(+t\) apply to the behavior of lattice paths from \((0,0)\) to \((p,m)\) relative to lines with slope \( t/s \). In particular, in Suen and Wagner [23], some of the applications to lattice paths with \( s = 1 \) and \( t \) a positive integer are shown.
1.2. Application of the Urn Model. Suppose that we expect the price of a bond or stock to go up \( p \) times and down \( m \) times, with the ratio of (the amount the stock goes) up to down equal to \( t/s \).

Before each up or down, the player can either buy (if the stock is not already held), sell (if the stock is held), or do nothing (wait or hold). (Short selling or multiple purchases are not considered.) After \( m + p \) time periods, the game finishes, and the player sells any stock held. We wish to maximize the player’s expected gain.

The translation of this problem to the acceptance urn model is as follows: Each “buy” (B) signifies a change from “pass” to “accept,” each “sell” (S) signifies a change from “accept” to “pass,” while “wait” (W) or “hold” (H) signifies no change from the previous decision of “pass” or “accept,” respectively. At the end of the game, if the player is holding stock, it is sold ([S]). Then, for example, the sequence

\[
P \ A \ A \ A \ P \ A \ P \ P \ P \ A \ A \ P \ A \ P,
\]

where where \( P \) indicates “pass” and \( A \) indicates “accept,” translates to the sequence

\[
W \ B \ H \ H \ S \ B \ S \ W \ W \ B \ H \ S \ B \ S \ [S],
\]

It will be shown with Theorem 1.6 that an optimal strategy is to buy or hold if \( s \) times the number of ups is at least (more than) \( t \) times the number of downs, and sell or wait otherwise.

If we insist that the actual price of the stock increases or decreases by the same quantity at each time period, then we may interpret the difference between \( s \) and \( t \) in terms of a “buy or hold fee.” The price of the stock will then go up or down \( \frac{t+s}{2} \) each time period, but the player will gain \( \frac{t+s}{2} + \frac{t-s}{2} = t \) or lose \( \frac{t+s}{2} - \frac{t-s}{2} = s \), if he chooses to buy or hold the stock. From the player’s point of view, the stock will go up \( t \), or down \( s \) at each stage, upon inclusion of this incentive (if \( t > s \)) or fee (if \( s < t \)).

More realistically, we will not know how many ups and downs will occur within a particular time frame. For that, we use the Bayesian version of the acceptance urn model. If our interest is over the next \( n \) periods, we assign an initial prior distribution \( \theta \) over the \( (n-j,j;s,t) \) urns (for which there will be \( j \) ups, \( 0 \leq j \leq n \)), basing \( \theta \) on, for example, previous market history.

1.3. A History of the Urn Model. In [20] (1969), L. A. Shepp proposed the following urn model: An urn contains \( m \) balls of value \(-1\) and \( p \) balls of value \(+1\). The balls are drawn from the urn
without replacement until the player chooses to stop, which includes the option of not drawing at all. The player’s gain will be the combined weight of the balls drawn. W. M. Boyce [4] (1970) offered that this urn model is a finite analogy to the following bond selling problem: An investor owns $10,000 of bonds which mature in three months. The current market value is $10,100. Should he sell now, sell later, or hold to maturity?

Shepp’s question was: which of these urns are favorable (i.e. have an expected gain, or value, \( V(m, p) > 0 \))? Shepp found that there exists a sequence \( \beta(1), \beta(2), \ldots \), such that the \((m, p)\) urns for which \( m \leq \beta(p) \) satisfy \( V(m, p) > 0 \), with \( V(m, p) = 0 \) for all other \( m \). In particular, the numbers \( \beta(p) \) satisfied the limit

\[
\lim_{p \to \infty} \frac{\beta(p) - p}{\sqrt{2p}} \to \alpha,
\]

where \( \alpha \approx 0.8399 \) is the unique real root to the equation

\[
(1.2) \quad \alpha = (1 - \alpha^2) \int_0^\infty \exp(\lambda\alpha - \lambda^2/2) \, d\lambda.
\]

The favorable urns are put into a collection \( \mathcal{C} \), and the optimal drawing policy is to draw the \( j^{th} \) ball as long as the urn, with the first \( j - 1 \) draws removed, remains in the collection \( \mathcal{C} \). As for the value \( V(m, p) \), defining

\[
E(m, p) = \frac{m}{m + p}(-1 + V(m - 1, p)) + \frac{p}{m + p}(1 + V(m, p - 1))
\]

we have \( V(m, p) = \max\{0, E(m, p)\} \). (See [4, Appendix 1].) It was shown in [20, Section 6] that adding a +1 into the urn never hurts, adding a −1 never helps, and swapping out a −1 for a +1 never hurts. Boyce [5, Theorem 3.8] (1973) showed that adding in both a +1 and a −1 never hurts, and also gave some tighter bounds. Defining the second-order differences

\[
\Delta^2 V_p(m) = V(m + 2, p) - 2V(m + 1, p) + V(m, p),
\]

\[
\Delta^2 V_m(p) = V(m, p + 2) - 2V(m, p + 1) + V(m, p),
\]

\[
\Delta^2 V(m, p) = V(m + 2, p) - 2V(m + 1, p + 1) + V(m, p + 2),
\]

it was shown in Chen and Hwang [6] (1984) that \( \Delta^2 V_p(m) \geq 0, \Delta^2 V_m(p) \geq 0, \) and \( \Delta^2 V(m, p) \geq 0. \)

Defining

\[
\Delta V(m, p) = V(m, p + 1) + V(m + 1, p) - V(m, p) - V(m + 1, p + 1),
\]

7
it was also shown that $\Delta V(m, p) \geq 0$, moreover, for nonnegative integers $a$ and $b$ the inequality

$$V(m + a, p + b) + V(m, p) \leq V(m + a, p) + V(m, p + b)$$

holds. As for what $V(m, p)$ looks like asymptotically:

1. $V(p, p) \sim (1 - \alpha^2)\sqrt{\frac{p}{\pi}}$ for large $p$, where $\alpha$ is given by (1.2). [20, eq. (6.7)]

2. For fixed $m(\leq \beta(p))$, $V(m, p) = (p - m) + \frac{m}{p+1} + O(p^{-3})$. [5, Section 6]

3. $V(\lambda p, p) \sim 0$ for $\lambda > 1$ and large $p$. [6, Theorem 4]

4. $V(\lambda p, p) \sim (1 - \lambda)p$ for $0 \leq \lambda < 1$ and large $p$. [6, Theorem 4]

Boyce [4] proposed the $(n; P)$ urn, a Bayesian version of the problem, where the total number of balls $n$ is fixed, and the urns with $n$ balls are distributed according to some probability distribution $P$ on $\{0, \ldots, n\}$. Boyce then described a series of computations [4, Theorem 4] that gave an indication of when the player should stop. The output of the algorithm was the value of the random urn $V(n; P)$.

**Theorem 1.1.** (Boyce) Given an $(n; P)$ urn, for $0 \leq j \leq n$ let $P^*(j) = P(j)\binom{n}{j}^{-1}$ and $a(n, j) = b(n, j) = 0$. For $0 \leq j \leq n - 1$, let $a(n - 1, j) = P^*(j) - P^*(j + 1)$, and for $0 \leq i \leq n - 2$ and each $j$ let

$$a(i, j) = a(i + 1, j) + a(i + 1, j + 1).$$

For $0 \leq i \leq n - 1$ and each $j$ let

$$b(i, j) = \max\{0, a(i, j) + b(i + 1, j) + b(i + 1, j + 1)\}.$$ 

Then an optimal stopping rule for maximizing the expected score at the stopping time is: if $k$ balls have been drawn, $\ell$ of them minus, draw again if and only if $b(k, \ell) > 0$. The value $V(n; P)$ of the $(n; P)$ urn under optimal play is $b(0, 0)$.

Chen et al. [7] offered a modification of Shepp’s urn model, the acceptance urn. It was shown that an optimal strategy maximizing the expected gain for this acceptance urn is to bet (i.e. accept) if and only if there are at least as many $+1$ balls as $-1$ balls remaining in the urn. Unlike Shepp’s urn, all but the trivial urns with $p = 0$ produce a value $\overline{V}(m, p) > 0$. In fact,

$$|\overline{V}(m, p) - \overline{V}(p, m)| = |m - p|$$ [7, Theorem 2.1]
follows by symmetry. The expected gain was given explicitly as

\[ V(m,p) = \max\{0, p - m\} + \left(\frac{m + p}{p}\right)^{-1} \sum_{i=0}^{\min\{m,p\}-1} \binom{m + p}{i}. \]

It was shown that \( V(m,p) \geq V(m,p) \), a fact intuitively clear since the player has the opportunity to resume betting after the first stop. Asymptotically,

1. \( V(m,p) \sim \frac{p}{m-p} \) if \( \frac{m}{p} \to \lambda > 1 \) as \( m \to \infty \). \[7, \text{ Theorem 3.1}\]

2. \( V(m,p) \sim \max\{0, 2\alpha\} + \exp(\alpha^2/2) \int_{|\alpha|}^{\infty} \exp(-t^2/2) \, dt \), if as \( m \to \infty \), \( \frac{p-m}{\sqrt{2}p} \to \alpha \). \[7, \text{ Theorem 3.2}\]

3. \( V(k + p, p) \to \sqrt{\pi p}/2 \) as \( p \to \infty \), for any fixed integer \( k \). \[7, \text{ Theorem 3.1}\]

Like the stopping urn, the acceptance urn has a similar structure: adding a +1 never hurts, adding a −1 never helps, and adding both a +1 and a −1 never hurts \[7, \text{ Theorems 2.5 and 2.6}\]. It was also shown that both \( V(km, m) \) and \( V(m, km) \) were strictly increasing in \( m \) for integer \( k \geq 1 \) \[7, \text{ Theorem 2.7}\]. Defining \( \Delta^2 V_{m}(m) \), etc., as was done with Shepp’s urn, it was shown \[7, \text{ Theorem 4.5}\] that \( \Delta^2 V_{p}(m) \geq 0 \), \( \Delta^2 V_{m}(p) \geq 0 \), and \( \Delta^2 V(m,p) \geq 0 \). The quadrangle inequality

\[ V(m,p) + V(m+1, p+1) \leq V(m+1, p) + V(m, p+1) \]

was also shown with \[7, \text{ Theorem 4.6}\], while the concavity of the function \( f(k) = V(m+k, p+k) \) was shown with \[7, \text{ Theorem 4.7}\]:

\[ V(m,p) + V(m+2, p+2) \leq 2V(m+1, p+1). \]

The Bayesian version of the acceptance model also was considered in \[7\]. It was shown that with an initial distribution that is uniform, an optimal policy maximizing the expected gain is to accept the next ball if and only if the player has seen at least as many +1 balls as −1 balls.

1.4. The Generalized Acceptance Model. We begin by defining the generalized acceptance urn.

**Definition 1.2.** The acceptance urn initially containing \( m \) “−s” balls and \( p \) “+t” balls is called the \( (m,p;s,t) \) urn.

The notation, while complicated, is a necessity as we will sometimes work with urns that have different sets of ball weights. Next, we define the outcomes of the draws from the urn, and the value of the \( (m,p;s,t) \) urn.
Definition 1.3. A realization is a sequence containing exactly \( m \) “−s”s and \( p \) “+t”s.

The \( j \)th member of the sequence reflects the \( j \)th ball drawn out of the urn. For the \((m, p; s, t)\) urn, there are \( \binom{m+p}{p} = \binom{m+p}{m} \) such realizations. It should be clear by the way the balls are drawn from the urn that each realization is equally likely. Graphically, we can regard each realization \( \omega \) as a path from \((0, pt - ms)\) to \((m+p, 0)\) using the steps \((1, s)\) and \((1, -t)\). Figure 1.1 shows a graphical representation of the realization \(-2, +3, -2, -2, +3, -2, +3\), from the \((4, 3; 2, 3)\) urn.

![Graphical representation of the realization \( \omega = -2, +3, -2, -2, +3, -2, +3 \) from the \((4, 3; 2, 3)\) urn.](image)

Remark. The balls in the \((m, p; s, t)\) urn with the same value are considered to be identical. However, in certain instances it is advantageous to consider some or all of the balls as being distinct. Whether the balls have numbers painted on them or not has no bearing on the process in any way.

Definition 1.4. For the \((m, p; s, t)\) urn, \( G(m, p; s, t) \) is the value of the urn, that is, the expected gain using an optimal acceptance policy.

An optimal acceptance policy maximizing the expected gain can be described as a function of the weight of the urn.

Definition 1.5. The weight of the urn \( X_j \) is the combined weight of the balls remaining in the \((m, p; s, t)\) urn after \( j \) balls have been drawn.
In particular, we will always have $X_0 = pt - ms$, and $X_{m+p} = 0$. Otherwise, $X_j$ is a random variable with

$$X_j = \begin{cases} 
X_{j-1} + s, & \text{if the } j^{th} \text{ ball drawn is } \text{“} -s \text{”} \\
X_{j-1} - t, & \text{if the } j^{th} \text{ ball drawn is } \text{“} +t \text{”} 
\end{cases}$$

when $1 \leq j \leq m + p$.

**Theorem 1.6.** An optimal strategy maximizing the expected gain $G(m, p; s, t)$ satisfies the following properties:

1. If $j - 1$ balls have been drawn, then the player shall accept the $j^{th}$ ball if $X_{j-1} > 0$.
2. If $j - 1$ balls have been drawn, then the player shall pass on the $j^{th}$ ball if $X_{j-1} < 0$.

**Proof.** We shall prove the result inductively. Clearly, if the urn contains only one “$-s$” ball, then we should not place a bet, otherwise we would lose $s$ dollars. If the urn contains only one “$+t$” ball, we should bet and receive the $t$ dollar payout. Given that we know on optimal strategy once there are less than $m + p$ balls remaining in the urn, we define $A(m, p; s, t)$ as the value if we bet on the first ball, then follow an optimal strategy thereafter; $B(m, p; s, t)$ as the value if we do not bet on the first ball, then follow an optimal strategy thereafter; and $G(m, p; s, t)$ as the value following an optimal strategy. Then

$$G(m, p; s, t) = \max\{A(m, p; s, t), B(m, p; s, t)\},$$

and

$$A(m, p; s, t) = \frac{p}{m + p} (t + G(m, p - 1; s, t)) + \frac{m}{m + p} (-s + G(m - 1, p; s, t)),$$

$$B(m, p; s, t) = \frac{p}{m + p} G(m, p - 1; s, t) + \frac{m}{m + p} G(m - 1, p; s, t).$$

Subtracting $B(m, p; s, t)$ from $A(m, p; s, t)$, we obtain

$$A(m, p; s, t) - B(m, p; s, t) = \frac{pt - ms}{m + p}.$$

An optimal strategy is thus to place a bet on the ball if $pt > ms$, that is, if $X_0 > 0$, and pass when $pt < ms$. When $pt = ms$, we have $A(m, p; s, t) = B(m, p; s, t)$, therefore we may use either option. The result now follows by induction. \qed

**Definition 1.7.** For any $j$, if $X_j > 0$, the urn is **positive**, if $X_j < 0$, the urn is **negative**, and if $X_j = 0$, the urn is **neutral**.
If we know that \( X_j = x \), we can figure out how many balls (\( a \) of \(+t\), \( b \) of \(-s\)) of each type remain by solving the system \( a + b = m + p - j \), \( at - bs = x \). Observe that Theorem 1.6 does not address what the player should do when \( X_{j-1} = 0 \). This is because the expected value of the next ball is zero, so the player will be neither at an advantage nor at a disadvantage if the player accepts the \( j^{th} \) ball. In this work, we shall employ two strategies that are optimal, which we will refer to as the zero-pass strategy and the zero-bet strategy.

**Definition 1.8.** The zero-pass strategy is the rule “bet on the \( j^{th} \) ball if and only if \( X_{j-1} > 0 \).”

**Definition 1.9.** The zero-bet strategy is the rule “bet on the \( j^{th} \) ball if and only if \( X_{j-1} \geq 0 \).”

In this work, unless specifically mentioned, we will be using the zero-pass strategy. For the realization depicted in Figure 1.1, a bet would be placed on every draw using the zero-bet strategy, while with the zero-pass strategy, a bet would be placed on every draw except the third. Generally, an optimal strategy is as follows: Bet when the urn is positive, pass when the urn is negative, and flip a coin (which need not be fair) to decide which option to use when the urn is neutral.

**Proposition 1.10.** Using any optimal strategy for the \((m, p; s, t)\) urn, if \( s > 0 \) then the last ball that a bet is placed on is a \( +t \).

**Proof.** If a bet is placed on the \( j^{th} \) ball and it is a \( -s \), then since \( X_{j-1} \geq 0 \), \( X_j = X_{j-1} + s > 0 \). As \( X_{m+p} = 0 \), there must be a \((j+1)^{th}\) ball, and a bet will be placed on it. Thus the \( j^{th} \) ball cannot be the last ball a bet is placed on. 

**Remark.** The minimum a player can gain using an optimal strategy is \( \max\{0, pt - ms\} \). If the urn is initially unfavorable, then the player will wait until the urn becomes favorable. Should that happen, the player will bet until the urn becomes unfavorable or is emptied, and will pick up a nonpositive gain. If the urn is initially favorable, then \( pt - ms \geq 0 \), and the player will bet until the urn becomes unfavorable or is emptied, picking up at least \( pt - ms \).
2. Properties of $G(m, p; s, t)$

We now show some of the properties possessed by the value function $G(m, p; s, t)$. We show that for any fixed $s$ and $t$, we increase $G(m, p; s, t)$ by adding more “$+\cdot t$” balls, removing “$-\cdot s$” balls, replacing “$-\cdot s$” balls with “$+\cdot t$” balls, and adding in balls while keeping the initial weight of the urn fixed, as was the case with the original acceptance urn. We show that for fixed $m$ and $p$, $G(m, p; s, t)$ is nondecreasing in $t$, nonincreasing in $s$, and continuous in both $s$ and $t$. We begin with three basic transformations that have geometric interpretations. The first is the vertical stretch.

**Lemma 2.1.** For any real $r \geq 0$, $G(m, p; rs, rt) = r G(m, p; s, t)$.

**Proof.** The result with $r = 0$ is trivial. Suppose $r > 0$. Denote the weight of the $(m, p; s, t)$ urn at stage $n$ to be $X_n$ and the weight of the $(m, p; rs, rt)$ at stage $n$ as $Y_n$. With the following correspondence:

\[
\text{draw a “$+\cdot rt$” ball at stage } n \text{ if and only if a “$+\cdot t$” ball was drawn,}
\]

we see that $Y_n = r X_n$ for all $n$ and $\omega$. Hence $X_n > 0$ if and only if $Y_n > 0$. Hence we bet on both urns, or we bet on neither. Thus, the gain from the $(m, p; rs, rt)$ urn will be $r$ times the gain from the $(m, p; s, t)$ urn. \hfill $\square$

Graphically speaking, we transition from the $(m, p; s, t)$ urn to the $(m, p; rs, rt)$ urn by a vertical stretch with factor $r$. If $t/s$ is rational, we can use Lemma 2.1 to rescale the urn to a $(m, p; s_1, t_1)$ urn with $s_1$ and $t_1$ positive integers satisfying $t/s = t_1/s_1$ and $\gcd(s_1, t_1) = 1$. For any $s$ and $t$, the lemma also allows us to use a standardized urn composed of “$-\cdot 1$” and “$+\cdot t/s$” balls.

Another useful transformation is vertical reflection, which is most powerful in the case $s = t = 1$.

**Definition 2.2.** For the $(m, p; s, t)$ urn, the $(p, m; t, s)$ urn is called its antiurn.

An example of the antiurn map is given in Figure 2.1. The next theorem is a generalization of Theorem 2.1 in Chen et al. [7]. This derivation, using a player trying to maximize gain and a second simulating the antiurn, allows us to eliminate the absolute value signs found in (1.3).
Theorem 2.3. (The Antiurn Theorem) For any nonnegative integers $m$ and $p$, and any nonnegative reals $s$ and $t$, we have

$$G(m, p; s, t) = (pt - ms) + G(p, m; t, s).$$

Remark. The method of proof is one which we will use to prove many of the inequalities to follow. We will sometimes ask our player to simulate one urn on another. What we mean by this is that, given a $(m, p; s, t)$ urn, our player will bet or pass as if the urn were actually a $(m', p'; s', t')$ urn, that is, use an optimal strategy associated with the $(m', p'; s', t')$ urn instead. Such a manipulation may include the addition of balls with value zero to the urn to accommodate the case $m + p \neq m' + p'$. The result is a simulation of the $(m', p'; s', t')$ urn on the $(m, p; s, t)$ urn.

Proof. To prove this assertion, we shall require two players, who we shall call Adam and Betty. Adam will follow the optimal zero-bet strategy associated with the $(m, p; s, t)$ urn. Betty will simulate the $(p, m; t, s)$ urn on the $(m, p; s, t)$ urn in the following manner: She will act as if each “$+t$” ball were a “$-t$” ball, and act as if each “$-s$” ball were a “$+s$” ball. She will follow the optimal zero-pass strategy on this simulated $(p, m; t, s)$ urn. If there are $p_1$ “$+t$” balls and $m_1$ “$-s$” balls left, Adam will place a bet on the next ball if and only if $p_1 t - m_1 s \geq 0$. Betty, meanwhile, will place a bet if and only if $p_1 (-t) - m_1 (-s) > 0$, i.e., if $p_1 t - m_1 s < 0$. Combined, one and only one player will place a bet on every draw from the urn. Thus Adam and Betty’s combined gain is the initial weight of the urn, $pt - ms$. Adam’s expected gain is $G(m, p; s, t)$, and, upon multiplication of the ball values
by $-1$, we see that Betty’s expected gain is $-G(p, m; t, s)$. Therefore, we must have 

$$G(m, p; s, t) = (pt - ms) + G(p, m; t, s).$$

The Antiurn Theorem will be a very useful tool for some of the inequalities that follow. For inequalities involving urns with different initial weights, this theorem produces a second, reversed inequality. The Antiurn Theorem also allows us to work exclusively with the urns with $pt - ms \leq 0$. Vertical reflection can be used locally (that is, on a subrealization of $\omega$) when $s = t$, but local reflections will not work for the $(m, p; s, t)$ urns in general.

The third, and perhaps most important, transformation is rotation. Unlike vertical reflection, rotation can be applied locally for all of the $(m, p; s, t)$ urns, and in particular it is very powerful when one of $s$ and $t$ is an integer multiple of the other. For an example, see Goulden and Serrano [11] (2003).

**Definition 2.4.** For a realization $\omega$, define $\omega(i, j]$ as the realization obtained by reversing the order the $(i + 1)$th through the $j$th balls appear in $\omega$. In particular, denote the reversal $\omega^R$ of $\omega$ as $\omega^R = \omega(0, m + p)$.

For the realization $\omega = -2, +3, -2, +3, -2, +3$ depicted in Figure 1.1, we have $\omega(2, 5] = -2, +3, +3, -2, -2, +3$, and $\omega(4, 7] = -2, +3, -2, -2, +3, +2, +3 = \omega$. Figure 2.2 provides another example using a realization from the $(5, 3; 1, 1)$ urn.

**Lemma 2.5.** (The Reversal Lemma) We have $X_k(\omega) = X_k(\omega(i, j])$ for $k \leq i$ and $k \geq j$, and $X_k(\omega(i, j]) + X_{j+i-k}(\omega) = X_j(\omega) + X_i(\omega)$ for $i < k < j$.

**Proof.** $\omega$ and $\omega(i, j]$ are identical up to and including the $i$th ball drawn, so it is obvious that $X_k(\omega) = X_k(\omega(i, j])$ for $k \leq i$. Also, when the $j$th ball has been drawn, identical sets of balls have been drawn for both $\omega$ and $\omega(i, j]$. Thus $X_j(\omega) = X_j(\omega(i, j])$, and since both draw identical balls from that point forward, we have $X_k(\omega) = X_k(\omega(i, j])$ for all $k \geq j$. This proves the first part. Now suppose $i < k < j$. Let $x_\ell$ denote the value of the $\ell$th ball of $\omega$. Then $X_k(\omega(i, j]) = X_i(\omega) + \sum_{\ell=i+j-k+1}^j x_\ell$, while $X_{i+j-k}(\omega) = X_i(\omega) + \sum_{\ell=1}^{i+j-k} x_\ell$. Therefore,

$$X_k(\omega(i, j]) + X_{i+j-k}(\omega) = 2X_i(\omega) + \sum_{\ell=i+1}^j x_\ell = X_i(\omega) + X_j(\omega),$$

as desired. \qed
Figure 2.2. For the realization $\omega$ from the $(5, 3; 1, 1)$ urn (solid), rotation about the point $P(4, -1)$ produces the reversed realization $\omega^R$ (dashed). To obtain the realization $\omega(3, 7]$ (dotted), only the part of the path $\omega$ over $[3, 7]$ is rotated, about the point $Q(5, -1)$.

The result of Lemma 2.5 implies that the graph of $\omega(i, j]$ can be obtained from the graph of $\omega$ by a half-turn rotation of the graph over $[i, j]$ about the point $P = \left( \frac{(i + j)}{2}, \frac{(X_i + X_j)}{2} \right)$, or equivalently, by a reflection through $P$. Thus, Lemma 2.5 can be considered as a rotation method or a midpoint reflection method, the latter being a more appropriate term when higher dimensions are considered. When $s = 1$ and $t$ is a positive integer, these rotations are primarily implemented via Lemma 3.7, the Crossing Lemma.

2.1. A Hierarchy of Inequalities. We now focus our attention to bounds for the urn families, by holding various parameters fixed, and letting other parameters vary. We find that the value $G(m, p; s, t)$ increases if we add “$+t$” balls, remove “$−s$” balls, add in a collection of balls with a combined weight of zero, or concentrate the gains and losses, that is, increasing $t$ or $s$ while $pt$ or $ms$ remains fixed. We show the first three statements hold with the next theorem.

**Lemma 2.6.** For any $m, p \geq 0$, we have

\begin{align*}
(2.1) & \quad \frac{t}{m + p + 1} \leq G(m, p + 1; s, t) - G(m, p; s, t) \leq t. \\
(2.2) & \quad 0 \leq G(m, p; s, t) - G(m + 1, p; s, t) \leq s \left( 1 - \frac{1}{m + p + 1} \right). \\
(2.3) & \quad \frac{t}{m + p + 1} \leq G(m, p + 1; s, t) - G(m + 1, p; s, t) \leq t + s - s \cdot \frac{s}{m + p + 1}.
\end{align*}
Remark. This is a generalization of Theorem 2.6 in Chen [7]. Unlike the proof there, which relies on the formula for \( G(m, p; 1, 1) \), we give a combinatorial proof.

**Proof.** We show the two inequalities:

\[
G(m, p + 1; s, t) - G(m, p; s, t) \geq \frac{t}{m + p + 1},
\]

(2.4)

\[
G(m, p; s, t) - G(m + 1, p; s, t) \geq 0.
\]

(2.5)

An application of the Antiurn Theorem produces the remaining inequalities for (2.1) and (2.2), while summing (2.1) and (2.2) gives (2.3).

To put the urns on equal footing, we add a ball with weight zero to the \((m, p; s, t)\) urn. The weight of this *magic ball* affects neither the player’s optimal strategy, the player’s gains, nor the probability of any realization (each realization appears \(m + p + 1\) times). Therefore, an optimal strategy will still result in an average gain of \(G(m, p; s, t)\). For the other urns, we shall distinguish a single ball (a “+t” for the \((m, p + 1; s, t)\) urn and a “−s” for the \((m + 1, p; s, t)\) urns) that shall correspond to the magic ball. Then for all three urns, there will be \(\binom{m+p+1}{m, p, 1}\) realizations. Each realization from the \((m, p; s, t)\) urn appears \(m + p + 1\) times, each realization from the \((m, p + 1; s, t)\) urn appears \(p + 1\) times, and each realization from the \((m + 1, p; s, t)\) urn appears \(m + 1\) times.

Figure 2.3. Identification of balls for the \((m, p + 1; s, t)\) urn (top), the modified \((m, p; s, t)\) urn with a blank ball (center), and the \((m + 1, p; s, t)\) urn (bottom).

To prove (2.4), we have the player simulate the \((m, p; s, t)\) urn on the \((m, p + 1; s, t)\) urn by regarding the magic “+t” ball as the blank “0” ball, and following an optimal \((m, p; s, t)\) zero-bet strategy based on that assumption. Since the player can certainly do this, this is a valid strategy, and therefore the expected gain cannot exceed the value \(G(m, p + 1; s, t)\). If the magic ball was in fact a “0” ball, the player’s expected gain would be \(G(m, p; s, t)\). Since the magic ball’s weight is “+t,” an error in the player’s favor, the player’s expected gain will be at least \(G(m, p; s, t)\).
particular, if the magic ball is the last drawn, an event occurring with probability \( \frac{1}{m + p + 1} \), the player will place a bet (since his strategy is zero-bet) and add \( t \) to his gain. Therefore, the player’s expected gain is at least
\[
G(m, p; s, t) + \frac{t}{m + p + 1}.
\]
This shows (2.4). We can show (2.5) by simulating the \((m + 1, p; s, t)\) urn on the \((m, p; s, t)\) urn.

An improvement cannot be made on a general lower bound for \( G(m, p; s, t) - G(m + 1, p; s, t) \), as \( G(m + 1, 0; s, t) = G(m, 0; s, t) = 0. \)

From (2.4), via the Antiurn Theorem, we have for \( m, p \geq 0 \):
\[
\frac{s}{m + p + 1} \leq G(p, m + 1; t, s) - G(p, m; t, s) \Rightarrow \frac{s}{m + p + 1} \leq G(m + 1, p; s, t) - G(m, p; s, t) + s
\]
\[
\Rightarrow G(m, p; s, t) - G(m + 1, p; s, t) \leq s \left( 1 - \frac{1}{m + p + 1} \right).
\]
From (2.5) we can similarly derive the remaining bound in (2.1). \(\square\)

The assumption that we must distinguish a magic ball, say, by painting it a different color as in Figure 2.3, is not a necessity. If we regard the balls as being identical, then from the \((m, p + 1; s, t)\) urn we can have the player select a positive integer \( i \) from \( \{1, \ldots, p + 1\} \) uniformly at random. Then, the player can treat the \( i \)th “+t” ball drawn as the magic ball.

If we add balls to the urn so that the initial weight remains the same, we would expect to gain more on average, simply because we have more time, hence it would seem that we have more opportunity. We show this is indeed the case with Lemma 2.7.

**Lemma 2.7.** For any nonnegative \( m, p \) and positive integers \( s, t \),
\[
G(m, p; s, t) \leq G(m + t, p + s; s, t).
\]

**Remark.** With the proper adjustments, this result can be extended to cover urns with \( t/s \) rational. (If \( t/s \) is irrational, we cannot add balls so that the initial weight remains constant.) If \( t/s \) is rational, the value \( G(m, p; s, t) \) may be rewritten as \( C \cdot G(m, p; s_1, t_1) \), where \( C = s/s_1 = t/t_1 \) is a nonnegative real constant and \( s_1 \) and \( t_1 \) are positive integers. Then the extended result is that
\[
G(m, p; s, t) \leq G(m + t_1, p + s_1; s, t).
\]
Proof. We wish to simulate the \((m, p; s, t)\) urn on the \((m + t, p + s; s, t)\) urn. In doing so, we show that the expected gain under the simulation is still \(G(m, p; s, t)\). This will require \(t + s\) magic balls.

We add \(t + s\) magic balls to the \((m, p; s, t)\) urn, with \(s\) of one color and \(t\) of another color. From the \((m + t, p + s; s, t)\) urn, we paint \(s\) of the “+t” balls and \(t\) of the “−s” balls accordingly, with these balls serving as the magic balls. Then, for both urns, there will be \((m + p + s + t)\) realizations by color. Our player will simulate the \((m, p; s, t)\) urn on the \((m + t, p + s; s, t)\) urn by assuming each magic ball has weight zero, and applying an optimal \((m, p; s, t)\) betting strategy. Since such a strategy is valid, the player’s expected gain cannot exceed \(G(m + t, p + s; s, t)\). On the other hand, the expected gain under the simulation is exactly \(G(m, p; s, t)\), as given that the player bets and a magic ball is drawn, the player’s expected gain is \(st − ts = 0\), the same as if all the magic balls had weight zero, as in the modified \((m, p; s, t)\) urn. (Of course, if the player does not bet no gain is made.) Therefore, \(G(m, p; s, t) \leq G(m + t, p + s; s, t)\).

It would seem conceivable to reverse the argument of Lemma 2.7. We can surely simulate the \((m + t, p + s; s, t)\) urn on the \((m, p; s, t)\) urn with \(t + s\) blanks, and the expected gain from the simulation \(G_{\text{sim}}(m + t, p + s; s, t)\) is at most \(G(m, p; s, t)\). However, because the blanks are located with the simulated urn, the realizations from the \((m + t, p + s; s, t)\) urn corresponding to realizations from the simulation do not have the same betting sequence. Thus, we cannot make the same conclusions as in the proof. We can, however, tighten the bound.

**Corollary 2.8.** For any nonnegative \(m, p,\) and positive integers \(s, t\),

\[
G(m, p; s, t) + \frac{st}{m + p + s + t} \leq G(m + t, p + s; s, t).
\]

Proof. We do this as in the tightened version of the Lemma 2.6. We allow our player to escape the simulated \((m, p; s, t)\) strategy on the last draw if the player knows it is a magic “+t” ball. This event occurs with probability \(s/(m + p + s + t)\). The player’s expected gain with this revised strategy is thus \(G(m, p; s, t) + st/(m + p + s + t)\), and as a valid strategy this cannot exceed \(G(m + t, p + s; s, t)\).

Finally, we show that an urn with fewer balls of larger value, plus or minus, is preferable to an urn with more balls of smaller value. That is, the player stands to gain more with a more volatile market in the short term when compared with a more steady market over the long term. We shall apply this tool to show that \(G(m, p; s, t)\) is bounded if \(ms/(pt) \to \lambda > 1\) as \(m \to \infty\).
Lemma 2.9. For any fixed $C \geq 0$ and $m, p > 0$ we have

$$G\left(m, p; s, \frac{C}{p}\right) \geq G\left(m, p + 1; s, \frac{C}{p + 1}\right), \quad \text{and} \quad G\left(m, p; \frac{C}{m}, t\right) \geq G\left(m + 1, p; \frac{C}{m + 1}, t\right).$$

Proof. For convenience, let $t = C/p$ and $\hat{t} = C/(p + 1)$. We show $G(m, p; s, t) \geq G(m, p + 1; s, \hat{t})$. The second result follows directly from the Antiurn Theorem, as we show below. We add a magic ball with value zero to the $(m, p; s, t)$ urn, and designate one of the “+$\hat{t}$” balls from the $(m, p + 1; s, \hat{t})$ urn to correspond with this blank ball.

We simulate the $(m, p + 1; s, \hat{t})$ urn on the $(m, p; s, t)$ urn by having the player assume that the “+$t$” balls and the blank ball are all “+$\hat{t}$” balls, using an optimal strategy under those assumptions. Then the player’s expected gain is at most $G(m, p; s, t)$, as this is a well-defined strategy for the $(m, p; s, t)$ urn. In fact, the player’s expected gain under the simulation is exactly $G(m, p + 1; s, \hat{t})$.

To show this, consider an arbitrary realization $\omega$ from the $(m, p + 1; s, \hat{t})$ urn. Since $\omega$ contains $p + 1$ “+$\hat{t}$” balls, and one of them is the magic ball, there are $p + 1$ realizations from the $(m, p; s, t)$ urn we may associate with $\omega$. Since the player’s strategy is an optimal $(m, p + 1; s, \hat{t})$ urn strategy, the betting sequences for all of these realizations are identical and exactly the same as that of $\omega$.

Suppose that $i$ of the “+$\hat{t}$” balls were bet on for $\omega$. Then the gain from “+$\hat{t}$” balls for $\omega$ is $i\hat{t}$. We show that the average gain over the corresponding realizations from the $(m, p; s, t)$ urn is also $i\hat{t}$. We have two cases:

Case 1. The magic ball was one of the $i$ balls bet on. Then the actual gains made from the “+$t$” balls equals $(i - 1)t$. There are $i$ such realizations.

Case 2. The magic ball was not one of the $i$ balls bet on. Then the actual gains made from the “+$t$” balls equals $it$. There are $p + 1 - i$ such realizations.

The average is thus

$$\frac{1}{p + 1}[i(i - 1)t + (p + 1 - i)\hat{t}] = \frac{i\hat{t}}{p + 1} = i\hat{t}.$$ 

Since this holds for any $i$ and each realization $\omega$, we conclude that the expected gain from the simulation is $G(m, p + 1; s, \hat{t})$. This completes the proof. \hfill \square

Example. We have $G(100, 100; 1, 1) \approx 8.3735$, while each of $G(100, 1; 1, 100)$, $G(1, 100; 100, 1)$, and $G(1, 1; 100, 100)$ equal 50. This also shows that we cannot make the inequalities strict.
2.2. **Continuity of** $G(m,p;s,t)$ **in** $s$ **and** $t$. Now we focus our attention on $s$ and $t$, and show that for fixed $m$ and $p$, $G(m,p;s,t)$ is a continuous function in both $s$ and $t$. Since $G(m,p;s,t)$ is continuous in $s$ and $t$, we have rational approximation at our disposal when we consider the urns for which $t/s$ is irrational.

**Lemma 2.10.** For any nonnegative reals $s$ and $t$, and any nonnegative integers $m$ and $p$, $G(m,p;s,t)$ is a continuous function in $s$ and $t$.

**Remark.** In proving this result, we show the monotonicity in $s$ and $t$ as well.

**Proof.** It suffices to show that $G(m,p;s,t)$ is continuous in $t$, as continuity in $s$ follows by the Antiurn Theorem. Let $\epsilon > 0$ be given. We prove:

\[(2.6) \quad G(m,p;s,t) \leq G(m,p;s,t+\epsilon)\]

\[(2.7) \quad G(p,m;t+\epsilon,s) \leq G(p,m;t,s).\]

To show (2.6), we simulate the $(m,p;s,t)$ urn on the $(m,p;s,t+\epsilon)$ urn by having our player assume the “$+(t+\epsilon)$” balls are “$+t$” balls, and following an optimal $(m,p;s,t)$ strategy. Then the player’s expected gain cannot exceed $G(m,p;s,t+\epsilon)$. On the other hand, the player’s expected gain will be at least $G(m,p;s,t)$, and strictly so if there is a “$+(t+\epsilon)$” ball present, as that ball could be the last one drawn. Thus, $G(m,p;s,t) \leq G(m,p;s,t+\epsilon)$.

We proceed similarly for (2.7), by simulating the $(p,m;t+\epsilon,s)$ urn on the $(p,m;t,s)$ urn. The simulation clearly gains at most $G(p,m;t,s)$, and since each loss on an accepted “$-t$” ball is less than what the player assumes $(-t-\epsilon)$, the simulation will fetch more than $G(p,m;t+\epsilon,s)$. Applying the Antiurn Theorem, we obtain the inequality

\[(2.8) \quad G(m,p;s,t+\epsilon) - p\epsilon \leq G(m,p;s,t).\]

Combining this inequality with (2.6), we find that $G(m,p;s,t)$ is continuous in $t$ from the right, by letting $\epsilon \to 0$. By replacing $t$ with $t-\epsilon$, we establish continuity from the left. \hfill \Box

For continuity in $s$, we apply the Antiurn Theorem to (2.8), yielding

\[G(m,p;s,t) - m\epsilon \leq G(m,p;s+\epsilon,t) \leq G(m,p;s,t).\]
Remark. It should be noted that the condition that \( t \) and \( s \) be non-negative is not a necessity. If \( s \leq 0 \) and \( t \geq 0 \), then every ball has a nonnegative value, and the player will accept every ball, gaining \( pt - ms \). Similarly, if \( s \geq 0 \) and \( t \leq 0 \), none of the balls have a positive value, and the player will not accept any balls, and thus will gain zero. If both \( s \) and \( t \) are at most zero, then the player will want to accept the “−\( s \)” balls instead of the “+\( t \)” balls. Thus, we multiply both \( s \) and \( t \) by −1 and swap their positions, so that the player’s expected gain \( G(m, p; s, t) \) will equal \( G(p, m; -t, -s) \).

Clearly, over these ranges \( G(m, p; s, t) \) is continuous, and since the forms above hold when \( s = 0 \) or \( t = 0 \), the extended function \( G(m, p; s, t) \) is continuous for all \( s \) and \( t \).

2.3. Miscellaneous Results. Here, we examine some other results related to the \((m, p; s, t)\) urn. Lemmas 2.11 and 2.12 relate to the maximum weight achieved during play, which we shall use when we examine the ruin problem in Chapter 8.

**Lemma 2.11.** Suppose \( m + p = \hat{m} + \hat{p} \), and \( \hat{m} > m \). Denote the weight of the \((m, p; s, t)\) and \((\hat{m}, \hat{p}; s, t)\) urns after \( n \) balls have been drawn as \( X_n \) and \( Y_n \), respectively. Then for any \( \ell \),

\[
P(X_n \geq pt - ms + \ell \text{ for some } n) \leq P(Y_n \geq \hat{p}t - \hat{m}s + \ell \text{ for some } n).
\]

**Proof.** First, observe that the result is trivial whenever \( \ell \leq 0 \), as both probabilities equal one. Therefore, suppose \( \ell > 0 \). From the \((m, p; s, t)\) urn, we distinguish \( p - \hat{p} = \hat{m} - m \) “+\( t \)” balls to match up with the distinguished “−\( s \)” balls from the \((\hat{m}, \hat{p}; s, t)\) urn. Thus, each urn contains \( m \) “−\( s \)” balls, \( \hat{p} \) “+\( t \)” balls, and \( \hat{m} - m \) magic balls.

Observe that \( X_0 = pt - ms = Y_0 - (\hat{m} - m)(t + s) \), and for \( n \geq 1 \)

\[
Y_n - X_n = \begin{cases} 
Y_{n-1} - X_{n-1} - (t + s), & \text{if a magic ball is drawn,} \\
Y_{n-1} - X_{n-1}, & \text{otherwise.}
\end{cases}
\]  

Thus, \( X_n \geq Y_n - (\hat{m} - m)(t + s) \) holds for all (matched) realizations and all \( n \). Therefore, if \( \hat{\omega} \) is a realization from the \((\hat{m}, \hat{p}; s, t)\) urn with \( Y_n(\hat{\omega}) \geq \hat{p}t - \hat{m}s + \ell \), we must have for the corresponding \( \omega \) that

\[
X_n(\omega) \geq Y_n(\hat{\omega}) - (\hat{m} - m)(t + s) \geq \hat{p}t - \hat{m}s + \ell - (\hat{m} - m)(t + s) = pt - ms + \ell.
\]

This proves the lemma. \( \square \)
The result of Lemma 2.11 implies that

\[ P(X_n < pt - ms + \ell \text{ for all } n) \geq P(Y_n < \hat{p}t - \hat{m}s + \ell \text{ for all } n). \]

Applying the Reversal Lemma, we also have that

\[ P(X_n \leq \ell \text{ for some } n) \leq P(Y_n \leq \ell \text{ for some } n), \quad \text{and} \quad P(X_n > \ell \text{ for all } n) \geq P(Y_n > \ell \text{ for all } n). \]

**Lemma 2.12.** Suppose \( m + p = \hat{m} + \hat{p} \), and \( \hat{m} > m \). Then for any \( \ell \)

\[ P(Y_n \geq \ell \text{ for some } n) \leq P(X_n \geq \ell \text{ for some } n). \]

**Proof.** The result with \( \ell \leq 0 \) is trivial since \( X_{m+p} = Y_{m+p} = 0 \). Assume \( \ell > 0 \). Note that (2.9) implies \( X_n \geq Y_n \). Then proceeding as in Lemma 2.11, if \( \hat{\omega} \) is a realization from the \((\hat{m}, \hat{p}; s, t)\) urn with \( Y_n(\hat{\omega}) \geq \ell \), then \( X_n(\omega) \geq Y_n(\hat{\omega}) \geq \ell \). □

Lemma 2.12 implies that

\[ P(X_n < \ell \text{ for all } n) \leq P(Y_n < \ell \text{ for all } n), \]
\[ P(X_n \leq pt - ms + \ell \text{ for some } n) \geq P(Y_n \leq \hat{p}t - \hat{m}s + \ell \text{ for some } n), \]
\[ P(X_n > pt - ms + \ell \text{ for all } n) \leq P(Y_n > \hat{p}t - \hat{m}s + \ell \text{ for all } n). \]

Similar results to Lemmas 2.11 and 2.12 hold upon exchanging “\( \leq \)” for “\( < \)” and “\( \geq \)” for “\( > \).”

Next, we take a look at some limits involving first-order differences.

**Proposition 2.13.** For fixed \( m \),

1. \( \lim_{p \to \infty} \left[ G(m, p + 1; s, t) - G(m, p; s, t) \right] = t. \)
2. \( \lim_{p \to \infty} \left[ G(m, p; s, t) - G(m + 1, p; s, t) \right] = s. \)

This is a generalization of Theorem 4.4 in Chen [7], and is proved in the same manner.

**Proof.** Since, for fixed \( p \), \( \lim_{m \to \infty} G(m, p; s, t) = 0 \), we have by the Antiurn Theorem:

\[ \lim_{p \to \infty} \left[ G(m, p + 1; s, t) - G(m, p; s, t) \right] = t + \lim_{p \to \infty} \left[ G(p + 1, m; t, s) - G(p, m; t, s) \right] = t. \]
Similarly,

$$\lim_{p \to \infty} \left[ G(m,p;s,t) - G(m+1,p;s,t) \right] = s + \lim_{p \to \infty} \left[ G(p,m;t,s) - G(p,m+1;t,s) \right] = s.$$ \hfill \Box

**Lemma 2.14.** For fixed $m > 0$ and $p > 0$, and any $\epsilon$:

1. For fixed $t$,
   $$\lim_{s \to \infty} \left[ G(m,p;s,t) - G(m,p;s+\epsilon,t) \right] = 0.$$

2. For fixed $t$,
   $$\lim_{s \to \infty} \left[ G(m,p;s,t+\epsilon) - G(m,p;s,t) \right] = \epsilon \begin{pmatrix} m + p \end{pmatrix}^{-1} \sum_{k=0}^{p} \binom{m+p-k-1}{m-1}.$$ 

3. For fixed $s$,
   $$\lim_{t \to \infty} \left[ G(m,p;s,t+\epsilon) - G(m,p;s,t) \right] = \epsilon \begin{pmatrix} m \end{pmatrix} \left( \begin{pmatrix} m + p \end{pmatrix}^{-1} \sum_{k=0}^{m} \binom{m+p-k-1}{p-1} \right).$$

4. For fixed $s$,
   $$\lim_{t \to \infty} \left[ G(m,p;s,t) - G(m,p;s+\epsilon,t) \right] = \epsilon \begin{pmatrix} m \end{pmatrix} \left( \begin{pmatrix} m + p \end{pmatrix}^{-1} \sum_{k=0}^{m} \binom{m+p-k-1}{m-1} \right).$$

**Proof.** We prove (1). Since we are letting $s$ go to infinity, assume $s$ and $s+\epsilon$ are both at least $pt$.

Then

$$G(m,p;s,t) = G(m,p;s+\epsilon,t) = t \begin{pmatrix} m + p \end{pmatrix}^{-1} \sum_{k=0}^{p} \binom{m+p-k-1}{m-1}. \tag{2.10}$$

We shall prove (2.10) with Theorem 4.13. Since equality holds for such $s$, the difference, and hence the limit, is zero. Statement (3) holds after applying the Antiurn Theorem to (1). To prove (2), if $s$ is greater than $pt$ and $p(t+\epsilon)$, from (2.10) (excluding the center expression) we have

$$G(m,p;s,t+\epsilon) - G(m,p;s,t) = (t + \epsilon - t) \begin{pmatrix} m + p \end{pmatrix}^{-1} \sum_{k=0}^{p} \binom{m+p-k-1}{m-1} = \epsilon \begin{pmatrix} m + p \end{pmatrix}^{-1} \sum_{k=0}^{p} \binom{m+p-k-1}{m-1}.$$ 

Statement (4) follows from (2) after applying the Antiurn Theorem. \hfill \Box

We close this section with some results related to other strategies. Denote by $G_{\geq n}(m,p;s,t)$ as the expected gain on the $(m,p;s,t)$ urn if a bet is placed if and only if the weight of the urn is at least $n$, $G_{\leq n}(m,p;s,t)$ as the expected gain if a bet is placed if and only if the weight of the urn is at least $n$. 

most $n$, and likewise define $G_{\geq n}(m, p; s, t)$ and $G_{< n}(m, p; s, t)$. Then $G_{\geq 0}(m, p; s, t)$ (zero-bet) and $G_{> 0}(m, p; s, t)$ (zero-pass) equal $G(m, p; s, t)$. Our motivation for defining these strategies is that some will be used for the Bayesian version of the acceptance urn.

**Lemma 2.15.** For any $m$, $p$, $s$, $t$, and $n$, the following hold:

1. $G_{\leq -n}(m, p; s, t) = -G_{\geq n}(p, m; t, s)$.
2. $G_{\geq -n}(m, p; s, t) = -G_{\leq n}(p, m; t, s)$.
3. $G_{< -n}(m, p; s, t) = -G_{\geq n}(p, m; t, s)$.
4. $G_{> -n}(m, p; s, t) = -G_{< n}(p, m; t, s)$.

**Proof.** Via the natural antiurn map, betting if and only if the urn weight is at most $-n$ for the $(m, p; s, t)$ urn corresponds with betting if and only if the urn weight is at least $n$ on the $(p, m; t, s)$ antiurn. Thus, $G_{\leq -n}(m, p; s, t)$ equals $G_{\geq n}(p, m; t, s)$ multiplied by $-1$. The other statements follow similarly. □

**Corollary 2.16.** (The Extended Antiurn Theorem) For any $m$, $p$, $s$, $t$, and $n$, we have

$$G_{\geq n}(m, p; s, t) + G_{< n}(m, p; s, t) = pt - ms.$$  

Therefore,

$$G_{\geq n}(m, p; s, t) - G_{\geq -n}(p, m; t, s) = pt - ms.$$  

**Remark.** The cases $n = 0$ and $n = 1$ reduce to the Antiurn Theorem.

**Proof.** If Adam uses the $n$-bet strategy, and Betty bets if and only if Adam does not bet, then Adam’s expected gain is $G_{\geq n}(m, p; s, t)$ and Betty’s expected gain is $G_{< n}(m, p; s, t)$. Since exactly one of the two place a bet on each ball, we have

$$G_{\geq n}(m, p; s, t) + G_{< n}(m, p; s, t) = pt - ms,$$

to which we apply Lemma 2.15 to $G_{< n}(m, p; s, t)$ and obtain the desired result. □

**Remark.** For any strategy, there is a complement- or anti-strategy, and for each urn there is an antiurn. Therefore, the results of Lemma 2.15 and Corollary 2.16 can be similarly be extended to any arbitrary strategy. If $S$ is a betting strategy for the $(m, p; s, t)$ urn, $S^C$ is its complement
strategy, and $G_S(m, p; s, t)$ denotes the expected gain using strategy $S$, then

$$G_S(m, p; s, t) + G_{S^C}(m, p; s, t) = pt - ms.$$ 

Furthermore, if $S^A$ is the strategy corresponding to $S$ via the antiurn map, then

$$G_S(m, p; s, t) - G_{(SC)^A}(p, m; t, s) = pt - ms.$$ 

This latter statement could be called the **Generalized Antiurn Theorem**.
3. The \((m, p; 1, t)\) Urns

We now look at the subcollection of acceptance urns, those with \(s = 1\) and \(t\) a positive integer. We shall give four precise formulas for \(G(m, p; 1, t)\) with \(pt \leq m\), which we call the zero count (Theorem 3.9), negative binomial (Theorem 3.12), binomial (Theorem 3.15), and distribution (Corollary 3.30) forms. A fifth form, the crossings form (Lemma 4.5), will be given in Chapter 4. For the urns with \(pt > m\), we give the zero count form (Theorem 3.25), and the distribution form (Corollary 3.30), with the crossings form (Lemma 4.6) to follow in Chapter 4. An application of the Antiurn Theorem extends the results of this chapter to the urns with \(t = 1\) and \(s\) a positive integer.

The acceptance urns with \(s = 1\) and \(t\) a positive integer represent a much larger collection of urns with the property that one of the ball weights is an integer multiple of the other. These urns exhibit the special property described by Lemma 3.7, the Crossing Lemma. This property follows by a rotation or midpoint reflection method, a mechanism described by Lemma 2.5, the Reversal Lemma. The original acceptance urns with \(s = t\) are special in their own right, since both ball weights are integer multiples each other. For those urns, we have the more powerful reflection method at our disposal.

We begin with the mathematical objects operating in the background, the generalized binomial series. We will then examine the \((m, p; 1, 1)\) urns, followed by the \((m, p; 1, t)\) urns, and conclude the chapter by finding the distribution of the gain for the \((m, p; 1, t)\) urns.

**Remark.** In this chapter, unless otherwise stated, we shall adopt the zero-pass strategy.

3.1. **Generalized Binomial Series.** The expected gain function \(G(m, p; 1, t)\) for the \((m, p; 1, t)\) urns, particularly when \(m \geq pt\), serves as a great example of the power of generalized binomial series. The generalized binomial series with parameter \(t\) (see Concrete Mathematics [10] (1994)) is defined as

\[
B_t(z) = \sum_{k \geq 0} \frac{1}{tk + 1} \binom{tk + 1}{k} z^k, \quad t \text{ real},
\]

and it is known to satisfy the equation

\[
B_t(z) = 1 + zB_t(z)^t.
\]
Lagrange’s Inversion formula gives that for any real number $r$,

$$
B_t(z)^r = \sum_{k \geq 0} \frac{r}{tk+r} \binom{tk+r}{k} z^k, \quad \frac{B_t(z)^r}{1 - ztB_t(z)^{t-1}} = \sum_{k \geq 0} \binom{tk+r}{k} z^k.
$$

A reference for the equations (3.1) can also be found in [10, eqs. (5.60), (5.61)]. When $t = 0$, $B_0(z) = 1 + z$ and equations (3.1) become

$$(1 + z)^r = \sum_{k \geq 0} \binom{r}{k} z^k,$$

which is the binomial theorem. In particular, $B_1(z)$ is the geometric series $1/(1 - z)$, while $B_2(z)$ is the generating function for the Catalan numbers.

For a power series $f$, we use $[z^n]f(z)$ to denote the coefficient of $z^n$ in $f(z)$. From (3.1) we can derive identities valid for $n \geq 0$:

$$
\sum_k \binom{tk+r}{k} \binom{t(n-k)+s}{n-k} \frac{r}{tk+r} \binom{t(n-k)+s}{n} \frac{r}{tn+r+s} = \binom{tn+r+s}{n},
$$

and

$$
\sum_k \binom{tk+r}{k} \binom{t(n-k)+s}{n-k} \frac{s}{tk+r} \cdot \frac{s}{t(n-k)+s} = \binom{tn+r+s}{n} \frac{r+s}{tn+r+s}.
$$

These two identities follow by the identities

$$
B_t(z)^r \cdot B_t(z)^s = \frac{B_t(z)^{r+s}}{1 - ztB_t(z)^{t-1}}, \quad B_t(z)^r \cdot B_t(z)^s = B_t(z)^{r+s},
$$

and examining the coefficient of $z^n$ for each expression. Write

$$
G^{(t,r)}(z) = \sum_{k \geq 0} \binom{tk+r}{k} z^k = \frac{B_t(z)^r}{1 - ztB_t(z)^{t-1}},
$$

and

$$
H^{(t,r)}(z) = G^{(t,r)}(z) G^{(t,0)}(z) = \frac{B_t(z)^r}{(1 - ztB_t(z)^{t-1})^2}.
$$

The function $H^{(t,r)}(z)$ generates the following identity, which shall be very useful to us.

**Lemma 3.1.** For any real $s$, $t$ and integer $m \geq 0$,

$$
\sum_{k=0}^m \binom{s-1-kt}{m-k} \binom{kt}{k} = \sum_{k=0}^m \binom{s}{k} (t-1)^{m-k}.
$$
Proof. Note that we may rewrite $G^{(t,0)}(z)$ as

\begin{equation}
G^{(t,0)}(z) = \frac{1}{1 - ztB(z)^{t-1}} = 1 + \frac{ztB(z)^{t-1}}{1 - ztB(z)^{t-1}}.
\end{equation}

Since

$$H^{(t,r)}(z) = \left( \sum_{k \geq 0} \binom{tk + r}{k} z^k \right) \left( \sum_{k \geq 0} \binom{tk}{k} z^k \right),$$

we have

$$[z^m]H^{(t,r)}(z) = \sum_{k=0}^m \binom{m+k(t-1)}{m} \binom{tk}{k}. \tag{3.5}$$

On the other hand, using (3.4), we have

$$H^{(t,r)}(z) = G^{(t,r)}(z)G^{(t,0)}(z) = G^{(t,r)}(z) + ztG^{(t,r)}(z) \frac{B(z)^{t-1}}{1 - ztB(z)^{t-1}}$$

$$= G^{(t,r)}(z) + zt \frac{B(z)^{r+t-1}}{(1 - ztB(z)^{t-1})^2} = G^{(t,r)}(z) + ztH^{(t,r+t-1)}(z).$$

By repeating the above argument iteratively for $H^{(t,r+t-1)}(z)$, we obtain that for any integer $m \geq 0$,

$$H^{(t,r)}(z) = \sum_{k=0}^m (zt)^k G^{(t,r+k(t-1))}(z) + (zt)^{m+1}H^{(t,r+(m+1)(t-1))}(z).$$

The above identity gives that

$$[z^m]H^{(t,r)}(z) = \sum_{k=0}^m [z^{m-k}] t^k G^{(t,r+k(t-1))}(z)$$

$$= \sum_{k=0}^m t^k \binom{tm+r-k}{m-k} = \sum_{k=0}^m t^k \binom{tm+r-k}{m-k}. \tag{3.6}$$

We have therefore proved that

$$\sum_{k=0}^m \binom{tm+r-k}{m-k} = \sum_{k=0}^m t^k \binom{tm+r-k}{m-k}. \tag{3.6}$$

Replacing $r$ with $s - mt - 1$, we have

\begin{equation}
\sum_{k=0}^m \binom{s-1-kt}{m-k} \binom{kt}{k} = \sum_{k=0}^m t^k \binom{s-1-k}{m-k} = \sum_{k=0}^m t^{m-k} \binom{s-m+k-1}{k}. \tag{3.5}
\end{equation}

It therefore remains to show that

\begin{equation}
\sum_{k=0}^m t^{m-k} \binom{s-m+k-1}{k} = \sum_{k=0}^m \binom{s}{k} (t-1)^{m-k}. \tag{3.6}
\end{equation}
We shall use the following identity (see [10, eq. (5.19)]):

\[
\sum_{k=0}^{m} \binom{r+k-1}{k} x^k (x+y)^{m-k} = \sum_{k=0}^{m} \binom{m+r}{k} x^k y^{m-k}, \quad \text{integer } m.
\]

By taking \(x = 1, y = t - 1\) and \(r = s - m\), equation (3.6) now follows and our proof of the theorem is complete.

Lemma 3.1 links the zero count, negative binomial, and binomial forms of \(G(m, p; 1, t)\). Of these, the binomial form is the easiest to analyze when \(m\) or \(p\) is large. This is because we can rewrite the right-hand side of (3.6) in terms of a binomial distribution.

3.2. The \((m, p; 1, 1)\) Urns. Before we examine the \((m, p; 1, t)\) urns, we start with the original \((m, p; 1, 1)\) urns.

**Theorem 3.2.** (Chen et al.) For any \(m\) and \(p\),

\[
G(m, p; 1, 1) = \max\{0, p - m\} + \left(m + p\right)^{-1} \min(m, p)^{-1} \sum_{k=0}^{m} \binom{m+p}{k}.
\]

Theorem 3.2 gives the distribution form of \(G(m, p; 1, t)\). We shall find the distribution of the gain of the \((m, p; 1, 1)\) urns with an optimal strategy via the reflection method. Recall that we reflect a realization \(\omega\) by swapping the signs of the balls, exchanging plus for minus and vice versa. This will reflect \(\omega\) through the \(n\)-axis, resulting in a realization \(\hat{\omega}\) from the antiurn. Here, we shall apply reflection locally, to subrealizations within \(\omega\). Graphically, after any reflections are performed, the various disjoint subpaths (reflected or not) are shifted vertically so that a continuous path \(\omega'\) ending at the point \((m + p, 0)\) is formed.

To obtain the distribution of the gain, we will reflect portions of the realizations that result in a net gain of one.

**Definition 3.3.** If \(X_n = a\) for some \(n\), \(X_q = b\) for some \(q > n\), and \(X_k \neq b\) for \(n < k < q\), then the urn makes a “\(+ (b - a)\)” trip.

Figure 3.1 shows a “\(-1\)” trip from zero in a realization from the \((10, 8; 1, 1)\) urn, and the realization from the \((11, 7; 1, 1)\) urn obtained after reflection of that “\(-1\)” trip.
Figure 3.1. For this realization from the (10, 8; 1, 1) urn (black), the dashed portion is a “−1” trip from neutral, and results in a gain of one when the zero-bet strategy is used. For the proof of Lemma 3.4 with $k = 1$, this “−1” trip is reflected, resulting in the path from the (11, 7; 1, 1) urn (pink over $[0, 11)$, black over $[11, 18]$). The black realization gains 2 using the zero-bet strategy, as there is a second “−1” trip from neutral over $[14, 17]$.

Lemma 3.4. For any integers $m$, $p$, integer $\min\{0, p - m\} \leq k \leq p$, and any optimal strategy,

$$P(\text{Player gains at least } k) = \binom{m + p}{p} \left(\frac{m + p}{p - k}\right)^{-1}.$$ 

Proof. We assume first that player uses the zero-bet strategy. Suppose that $m \geq p$. We shall count the number of realizations for which the player gains at least $k$. Note that when the urn first returns to neutral, the player starts betting until the urn reaches a weight of −1, with the player gaining one. The player then stops betting until the urn next becomes neutral. This must occur at least $k$ times in order for the player to gain at least $k$. Let $\omega$ be a realization gaining at least $k$. Each time the player gains one, the urn would have to make a “−1” trip from neutral, followed by a “+1” trip back to neutral.

We shall show that there is one-to-one correspondence between the realizations $\omega$ that gain at least $k$ from the $(m, p; 1, 1)$ urn and the realizations $\omega'$ from the $(m + k, p - k; 1, 1)$ urn. To turn $\omega$ into $\omega'$, we simply reflect the first $k$ “−1” trips of $\omega$ from neutral. Note that the number of “+1” balls involved in each “−1” trip is one more than the number of “−1” balls. Thus, each reflection of a “−1” trip has the net effect of increasing $m$ by one and decreasing $p$ by one. Hence, by reflecting the first $k$ “−1” trips from neutral, we have turned $\omega$ into $\omega'$. To get back $\omega$ from $\omega'$, we note first that the initial urn weight of $\omega'$ is $p - m - 2k \leq 0$, and thus $\omega'$ contains a “+(m − p + 2k)” trip, which we partition into a “+(m − p)” trip, and 2$k$ “+1” trips. Starting with the first of those “+1” trips, we reflect every other “+1” trip, creating $k$ segments composed of a “−1” trip followed by a “+1”
trip. Since they follow a “+\((m - p)\)” trip to neutral, the now-“−1” trips are from neutral, and the “+1” trips are back to neutral. Thus, we have shown that the map is a one-to-one correspondence.

Since the number of realizations of a \((m + k, p - k; 1, 1)\) urn equals \(\binom{m + p}{p - k}\), we have thus shown that for \(m \geq p\) and the zero-bet strategy,

\[
\binom{m + p}{p} P(\text{Player gains at least } k) = \binom{m + p}{p - k}.
\]

For the zero-pass strategy, in order for the player to gain one, the weight of the urn has to first reach +1, and the player will start betting until the urn next returns to neutral. The reflection will be of the first \(k - 1\) trips from 1. The proof is similar, and the details are omitted. Any other optimal strategy is a probabilistic hybrid of the zero-bet and zero-pass strategies. Therefore, the result holds true here as well.

Finally, the result with \(p > m\) follows by reflection of the “−(p − m)” trip from the start. Then, a gain of at least \(k \geq p - m\) from the \((m, p; 1, 1)\) urn corresponds to a gain of at least \(k - (p - m)\) from the \((p, m; 1, 1)\) urn. Thus,

\[
P(\text{Player gains at least } k) = \binom{m + p}{p}^{-1} \binom{m + p}{k - (p - m)} = \binom{m + p}{p}^{-1} \binom{m + p}{p - k}. \quad \square
\]

From Lemma 3.4, we have for \(m \geq p\)

\[
G(m, p; 1, 1) = \sum_{k=1}^{p} P(\text{player gains at least } k) = \binom{m + p}{p}^{-1} \sum_{k=0}^{p-1} \binom{m + p}{k},
\]

after an index shift. The case for \(p > m\) is obtained similarly.

The probability a player gains exactly \(k\) using an optimal strategy equals the probability that the maximum weight the urn takes during play equals \(k\). This is no coincidence; we will show that this also holds when \(s\) and \(t\) are arbitrary nonnegative reals in Chapter 4, with Theorem 4.18. For now, we show that this holds when \(s = t = 1\), but we shall give a different presentation of the result.

**Lemma 3.5.** If \(m \geq p\), then

\[
P(X_n \geq k \text{ for some } n) = P(X_n = k \text{ for some } n) = \binom{m + p}{p}^{-1} \binom{m + p}{p - k}.
\]

If \(p \geq m\) and \(k \geq 0\) is an integer, then

\[
P(X_n \geq p - m + k \text{ for some } n) = P(X_n = p - m + k \text{ for some } n) = \binom{m + p}{p}^{-1} \binom{m + p}{p + k}.
\]
Proof. Suppose \( m \geq p \). Let \( \omega \) be a realization from the \((m, p; 1, 1)\) urn with \( X_j(\omega) = k \) and \( j \) minimal. Observe that the first \( j \) balls form a \(+ (m - p + k)\) trip, and the remaining \( m + p - j \) balls contain \( k \) more \(+1\)’s than \( -1\)’s. Thus, by reflecting the last \( m + p - j \) balls we obtain a realization \( \hat{\omega} \) from the \((m + k, p - k; 1, 1)\) urn. This mapping is reversible, as each realization from the \((m + k, p - k; 1, 1)\) urn must start with a \(+ (m - p + k)\) trip, since \( m \geq p \). It follows that there are \( \binom{m+p}{p-k} \) realizations that reach the weight \( k \) during play.

The result with \( p \geq m \) is similar. This time, we reflect the \(+k\) trip from the initial weight \( p - m \) instead. We omit the remaining details. \( \square \)

When \( s = t \), reflection is a very useful tool, as we shall see when we examine the Bayesian version of the urn in Chapter 7 and the ruin problem in Chapter 8. When \( s \) and \( t \) are arbitrary, the only useful application of the reflection method is the antiurn map. However, when \( s = 1 \) and \( t \) is an integer (or the antiurn/vertical stretch equivalent), we have another geometric tool, rotation (reversal, or midpoint reflection), that we can use.

3.3. The Value of the \((m, p; 1, t)\) Urn. For the \((m, p; 1, t)\) urn, there are two ways to make what we shall call a permanent gain. The first is outlined below.

Suppose \( X_n = 0 \) and \( n \neq m + p \). Define

\[
\tau = \tau_n = \min\{h > n : X_h \leq 0\}, \quad G_n = \begin{cases} X_{n+1} - X_\tau, & \text{if } X_{n+1} = 1, \\ 0, & \text{otherwise}. \end{cases}
\]

\( G_n \) is thus the gain incurred from the series of bets on balls \( n + 2 \) through ball \( \tau_n \). Note that since we have \( -1 \) balls, if the player does not bet on the \( n^{th} \) ball and bets on the \((n + 1)^{th} \) ball, then we must have \( X_{n-1} = 0 \) and \( X_n = 1 \).

The only other way a gain can be made is if a bet is placed on the first ball. Then \( X_0 = pt - m > 0 \). When \( pt > m \), the player bets until the urn becomes nonpositive, picking up between \( pt - m \) and \( pt - m + t - 1 \). To deal with this ambiguity, we define for \( pt > m \)

\[(3.7) \quad \rho = \min\{n : X_n \leq 0\}, \quad \kappa = E[-X_\rho].\]

If \( pt \leq m \), we define \( \kappa \) to be zero. We call \( \kappa \) the crossing number. Clearly, we have that \( 0 \leq \kappa \leq t - 1 \).

In fact, we shall soon show that \( 0 < \kappa \leq (t - 1)/2 \) if \( pt > m > 0 \).
Theorem 3.6. For any $m$ and $p$,

$$G(m, p; 1, t) = \max\{0, pt - m\} + \kappa + \sum_{n \neq m + p} E(G_n \mid X_n = 0) P(X_n = 0).$$

Proof. In the case when the urn is initially positive, the player will bet until the first time the urn is nonpositive, averaging a gain of $pt - m + \kappa$. After that, the player, using the zero-pass strategy, will not place a bet until the first time, say time $n$, when the weight of the urn is positive. The player will gain $G_n$ until the next time $X_\tau \leq 0$. The process is then repeated until the urn is empty. Thus, if we let

$$\chi_n = \begin{cases} 
1, & \text{if } X_n = 0, \\
0, & \text{otherwise},
\end{cases}$$

the gain of the player is $\sum_{n \neq m + p} G_n \chi_n$, and hence the value is

$$G(m, p; 1, t) = \max\{0, pt - m\} + \kappa + \sum_{n \neq m + p} E(G_n \mid X_n = 0) P(X_n = 0).$$

We focus now on the two objects in the sum, $P(X_n = 0)$ and $E(G_n \mid X_n = 0)$. If $X_n = 0$, then for each “$+t$” ball present $t$ “$−1$” balls must be present. Hence there must be $kt + k$ balls left in the urn (with $k \leq p$, $kt \leq m$) after $n$ balls have been drawn. Since each realization is equally likely,

$$P(X_n = 0) = \binom{kt + k}{k} \left( \frac{m + p - k - kt}{p - k} \right) \left( \frac{m + p}{p} \right)^{-1}, \quad n = m + p - k - kt, \quad 0 \leq k \leq \min\{p, \lceil m/t \rceil\}.$$

As for $E(G_n \mid X_n = 0)$, we begin with a very important lemma that will allow us to give alternate forms of the expected gain when $m \geq pt$.

Lemma 3.7. (The Crossing Lemma) For the $(m, p; 1, t)$ urn, let

$$S := \{\omega: X_n = X_q = 0, X_k \neq 0 \text{ for } n < k < q\}.$$

Let $A_i$, $0 \leq i \leq t$, be the event that $X_k = t - i$, $X_{k+1} = -i$ for some $n \leq k < q$. Then the events

$$\{A_i\}_{i=0}^t$$

form a partition of $S$ and for each $i$

$$P(A_i \mid \omega \in S) = \frac{1}{t+1}.$$
Remark. We have set $X_n = 0 (= X_q)$ for convenience.

Proof. Let $\omega \in S$. The union of the events $\{A_i\}_{i=0}^t$ is

$$A := X_k \geq 0, X_{k+1} \leq 0$$

for some $n \leq k < q$.

If the $(n+1)^{\text{th}}$ ball is a “$+$,” then $A_t$ occurs, otherwise it is a “$-$,” in which case one of $A_0, \ldots, A_{t-1}$ occurs, since $X_q = 0$. Therefore, $S = \cup_{i=0}^t A_i$. If $A_i$ is the first event in $A$ occurring in $\omega$, then clearly another event in $A$ cannot occur, since the urn weight increases by ones. Therefore, the collection $\{A_i\}_{i=0}^t$ is a partition of $S$.

We now show that for each $i$, $|A_i| = |A_0|$. Given that $\omega$ is in $S$ and $A_i$ occurs, find $\nu$ so that $n \leq \nu \leq q$ and $X_\nu = t - i, X_{\nu+1} = -i$. Now consider the realization $\omega(\nu, q]$. The $q^{\text{th}}$ ball of $\omega(\nu, q]$ is a “$+$,” thus $A_0$ occurs. Also, since $\nu < k < q$ implies $X_k(\omega) < 0$, the Reversal Lemma implies that $\nu < k < q$ implies $X_n(\omega(\nu, q]) > t - i$, therefore $\omega(\nu, q]$ is also in $S$. Furthermore, the last time the urn takes the weight $t - i$ in $\omega(\nu, q]$ before time $q$ is after the $\nu^{\text{th}}$ ball has been drawn. Therefore, we can invert the map and recover $\omega$ from $\omega(\nu, q]$, so the map is injective. Given $\omega'$ in $S$ with the event $A_0$, since the urn weight increases by ones, the urn weight must be $t - i$ at some point between $n$ and $q$. Therefore, the map is surjective as well. Thus, $|A_i| = |A_0|$ for all $i$.

Since $S$ can be partitioned into the equal-size classes $A_i, 0 \leq i \leq t$, we conclude that

$$P(A_i \mid \omega \in S) = \frac{1}{t + 1},$$

as desired.

Corollary 3.8.

$$E(G_n \mid X_n = 0) = \frac{t}{2}.$$  

Proof. Suppose $X_n = 0$. Fix the next time the urn is neutral, after $q$ balls have been drawn. If $G_n = i$, where $1 \leq i \leq t$, then the event $A_{i-1}$ has occurred on $(n, q]$, and if $G_n = 0$, $A_t$ has occurred on $(n, q]$. Since each $A_i$ is equally likely,

$$E(G_n \mid X_n = X_q = 0) = \frac{1}{t + 1} \sum_{i=0}^t i = \frac{t}{2}.$$

Since this holds for any such $q > n$, we conclude that

$$E(G_n \mid X_n = 0) = \frac{t}{2}. \quad \square$$
We now give explicit formulas for $G(m, p; 1, t)$ when $m \geq pt$. We shall examine the urns with $m < pt$ later.

3.3.1. The $(m, p; 1, t)$ Urns with $m \geq pt$. With $P(X_n = 0)$ and $E(G_n | X_n = 0)$ now calculated, we can give our first form of $G(m, p; 1, t)$, the zero count form.

**Theorem 3.9.** (Zero Count Form) Suppose $m \geq pt$. Then

$$G(m, p; 1, t) = \left( \frac{m + p}{p} \right)^{-1} t \sum_{k=1}^{p} \left( kt + k \right) \left( m + p - k - kt \right).$$

We now explore writing $G(m, p; 1, t)$ in other ways. Define the random variables

$$N = |\{n: n \not= m + p, X_n = 0\}|,$$

$$N^+ = |\{n: X_n = 0, X_{n+1} = 1\}|,$$

$$N^- = |\{n: X_n = 0, X_{n+1} = -t\}|.$$

Note that $N = N^+ + N^-$. Furthermore, since

$$P(X_{n+1} = 1 \mid X_n = 0) = \frac{t}{t+1} = 1 - P(X_{n+1} = -t \mid X_n = 0),$$

we must have $E(N^+) = tE(N^-)$, thus

$$E(N) = \frac{t+1}{t} E(N^+).$$

Observe that we have calculated $E(N)$ for the zero count Form:

$$G(m, p; 1, t) = \frac{t}{t} E(N).$$

Our second derivation of $G(m, p; 1, t)$ will involve finding the distribution of $N$, while our third derivation will involve the distribution of $N^+$.

For the proofs that follow, we shall sometimes regard a realization $\omega$ from the $(m, p; 1, t)$ urn as a word containing $m$ “−1”s and $p$ “+t”s. For words $A$ and $B$, $AB$ denotes the concatenation of the words $A$ and $B$. If a word $W = AB$, then we say $A$ is a prefix of $W$, and $B$ is a suffix of $W$. The weight of a word is the sum of the weights of the letters of the word, and the size of a word $W$ is the number of letters in the word, and is denoted as $|W|$.
Lemma 3.10. Suppose $m \geq pt$. Then for integer $k \geq 0$ we have

$$P(N \geq k) = (t+1)^k \binom{m+p-k}{p-k} \binom{m+p}{p}^{-1}.$$ 

Proof. Assume $N \geq k$. We shall assume initially that the event $A_t$ follows for each of the final $k$ times the urn is nonempty and neutral. If there are $M$ realizations satisfying this property, then the Crossing Lemma implies that the number of realizations for which $N \geq k$ is $M(t+1)^k$, since we have $t+1$ equally likely options for the event $A_t$ to follow each time the urn is nonempty and neutral, independent of each other.

For a realization $\omega$ with $N \geq k$ and $A_t$ following the last $k$ times the urn is nonempty and neutral, we may write $\omega$ as

$$\omega = TM_kQ_k \cdots M_1Q_1,$$

where $M_iQ_i \cdots M_1Q_1$ is the $i$th smallest nonempty suffix of $\omega$ that has weight zero (that is, the urn is neutral for the $i$th to last time with $|M_iQ_i \cdots M_1Q_1|$ balls left), and $M_i$ is a solitary “$+t$” ball for each $i$. Therefore, $M_i$ is an indicator of the event $A_t$, and since $M_iQ_i$ has weight zero, $Q_i$ has weight $-t$ for each $i$. Furthermore, no proper prefix of $Q_i$ can have weight $-t$ by our choice of $M_iQ_i$. The identification of these segments within a realization from the $(10, 4; 1, 2)$ urn for the case $k = 2$ is shown in Figure 3.2.

![Figure 3.2](image.png)

**Figure 3.2.** Segment identification of a realization from the $(10, 4; 1, 2)$ urn with $N \geq 2$. Using the Crossing Lemma over $M_1Q_1$ and $M_2Q_2$, this realization is associated with $3^2 = 9$ realizations from the $(10, 4; 1, 2)$ urn with $N \geq 2$.

We map $\omega$ to the realization $\omega'$ from the $(m, p-k; 1, t)$ urn, where

$$\omega' = Q_kQ_{k-1} \cdots Q_1T.$$
From a realization $\omega'$ from the $(m, p - k; 1, t)$ urn, we recover a unique realization $\omega$ from the $(m, p; 1, t)$ urn with the desired properties as follows: Since the urn weight increases by ones and $m \geq pt$, we can find the smallest prefix of $\omega'$ with weight $-it$, $1 \leq i \leq k$. This shall be $Q_k \cdots Q_{k+1-i}$. Thus, we can recover $Q_i$ for each $i$. We then move the word $Q_k \cdots Q_1$ to the end of the realization, and insert a “+$t” ball before each $Q_i$. Therefore, the number of realizations $M$ equals $(m + p - k)_{p-k}$, the number of realizations for which $N \geq k$ equals $(m + p - k)_{p-k}(t + 1)^k$, thus

$$P(N \geq k) = \binom{m + p}{p}^{-1} \binom{m + p - k}{p - k} (t + 1)^k,$$

as desired. □

**Corollary 3.11.** For any integer $k \geq 0$,

$$P(N = k) = (t + 1)^k \frac{m - (p - k)t}{m + p - k} \binom{m + p - k}{p - k} \binom{m + p}{p}^{-1}.$$

**Remark.** The result is obtained by taking the difference $P(N = k) - P(N = k + 1)$. This result can also be shown combinatorially. The realizations with $N = 0$ are ballot permutations. In terms of our terminology thus far, an entire realization is a “+$ (m - pt)” trip if and only if $N = 0$. These will be discussed in greater detail in Chapter 6.

**Theorem 3.12.** (Negative Binomial Form) Suppose $m \geq pt$. Then

$$G(m, p; 1, t) = \binom{m + p}{p}^{-1} t \sum_{k=0}^{t-1} \binom{m + k}{k} (t + 1)^{p-k}.$$

**Proof.** We have

$$G(m, p; 1, t) = \frac{t}{2} E(N) = \frac{t}{2} \sum_{k \geq 1} P(N \geq k),$$

after which we reindex, summing over $p - k$. □

We call this form of $G(m, p; 1, t)$ the negative binomial form because of the presence of the binomial coefficient $\binom{m+k}{k}$. With a few adjustments, we can associate this form with a negative binomial distribution.

We now work toward the binomial form, using the random variable

$$N^+ = |\{n: X_n = 0, X_{n+1} = 1\}|.$$
Lemma 3.13. If \( m \geq pt \), then
\[
P(N^+ \geq k) = t^k \left( \frac{m + p}{p - k} \right) \left( \frac{m + p}{p} \right)^{-1}.
\]

Proof. We first assume that the last \( k \) occurrences of the event \( N^+ \) are followed by the event \( A_0 \). Then, if there are \( M \) realizations with this property, by the Crossing Lemma there are \( Mt^k \) realizations with \( N^+ \geq k \). (Note the event \( A_t \) is ineligible, as that is the contributor to \( N^- \).) We show that \( M = \binom{m+p}{p-k} \).

We may write a realization \( \omega \) with \( N^+ \geq k \) as
\[
\omega = TP_k M_k Q_k P_k \cdots Q_1 M_1 Q_1,
\]
where
1. \( M_i \) is a sequence consisting of a lone “+t” ball,
2. \( P_i \) has weight \( -t \) for each \( i \), and each proper prefix of \( P_i \) has negative weight, and
3. \( Q_i \) has weight zero for each \( i \), and each prefix of \( Q_i \) has nonpositive weight.

The identification of these segments in a realization from the \((10, 4; 1, 2)\) urn with \( N^+ \geq 2 \) is given in Figure 3.3. Note that \( |Q_i| \) may be zero, the initial “−1” ball at the beginning of \( P_i \) contributes one toward \( N^- \) (thus \( N^+ \geq k \)), while \( M_i \) serves as an indicator of the event \( A_0 \). We map \( \omega \) to the following realization from the \((m + k, p - k; 1, t)\) urn:
\[
\omega' = Q_1 M'_1 \cdots Q_{k-1} M'_{k-1} Q_k M'_k TP_k P_{k-1} \cdots P_1,
\]
where \( M'_i \) denotes a single “−1” ball.

We now show that the map is reversible. Let \( \omega' \) be a realization from the \((m + k, p - k; 1, t)\) urn. The smallest prefix of \( \omega' \) with weight \( -i \), \( 1 \leq i \leq k \), we set as \( Q_1 M'_1 \cdots Q_i M'_i \) (noting that the last member of this prefix must be a “−1” ball). The smallest suffix of \( \omega \) with weight \( -it \), \( 1 \leq i \leq k \), is the sequence \( P_i P_{i-1} \cdots P_1 \). Since \( m \geq pt \), we have \( pt - m - k(t+1) \leq -k(t+1) \), thus \( Q_1 M'_1 \cdots Q_k M'_k \) and \( P_i P_{i-1} \cdots P_1 \) do not intersect. Therefore, we can reverse the map, changing \( M'_i \) back to \( M_i \).

We have thus shown that \( M = \binom{m+p}{p-k} \). It follows from the Crossing Lemma that the number of realizations with \( N^+ \geq k \) equals \( t^k \binom{m+p}{p-k} \), and thus
\[
P(N^+ \geq k) = t^k \left( \frac{m + p}{p - k} \right) \left( \frac{m + p}{p} \right)^{-1}. \]
\( \square \)
Figure 3.3. Segment identification of a realization from the (10, 4; 1, 2) urn with \( N^+ \geq 2 \). Using the Crossing Lemma on the segments \( P_1M_1 \) and \( P_2M_2 \), this realization is associated with \( 2^2 = 4 \) realizations from the (10, 4; 1, 2) urn with \( N^+ \geq 2 \). Observe that \( Q_1 \) is empty.

Corollary 3.14. For any integer \( k \geq 0 \), we have

\[
P(N^+ = k) = t^k \frac{m + k + 1 - (p - k)t}{m + k + 1} \left( \frac{m + p}{p} \right)^{-1}.
\]

Remark. The realizations with \( N^+ = 0 \) are weak ballot permutations. We shall use an alternative name, and call the realizations with \( N^+ = 0 \) zero-gain realizations, as such realizations will give a player a gain of zero if the zero-pass strategy is used. If we adjoin a “−1” ball to the end of a realization \( \omega \) with \( N^+ = 0 \), then the extended realization \( \omega' \) would be a “+(m−pt+1)” trip. Again, see Chapter 6 for more details.

Theorem 3.15. (Binomial Form) Suppose \( m \geq pt \). Then

\[
G(m, p; 1, t) = -\frac{t}{2} + \left( \frac{m + p}{p} \right)^{-1} \frac{t}{2} \sum_{k=0}^{p} \left( \frac{m + p + 1}{k} \right) t^{p-k}.
\]

Proof. We have

\[
E(N) = \frac{t + 1}{t} E(N^+) = \frac{t + 1}{t} \left( \frac{m + p}{p} \right)^{-1} \sum_{k=0}^{p-1} \left( \frac{m + p}{k} \right) t^{p-k},
\]

and this sum equals

\[
\left( \frac{m + p}{p} \right)^{-1} \left[ \sum_{k=0}^{p-1} \left( \frac{m + p}{k} \right) t^{p-k} + \sum_{k=1}^{p} \left( \frac{m + p}{k - 1} \right) t^{p-k} \right] = -1 + \left( \frac{m + p}{p} \right)^{-1} \sum_{k=0}^{p} \left( \frac{m + p + 1}{k} \right) t^{p-k},
\]

after which we multiply through by \( t/2 \). □
The transformation of the summation of $G(m,p;1,t)$ from the zero count form, through the negative binomial form, and to the binomial form can also be done via Lemma 3.1, with (in that result) $m$ replaced by $p$, $t$ replaced by $t+1$, and $s$ replaced by $m+p+1$:

$$\sum_{k=0}^{p} \binom{kt+k}{k} \binom{m+p-k-kt}{p-k} = \sum_{k=0}^{p} \binom{m+k}{k} (t+1)^{p-k} = \sum_{k=0}^{p} \binom{m+p+1}{k} t^{p-k}. \tag{3.8}$$

For the third indicator, $N^-$, the method that produced the distribution of $N^+$ fails. This is because each occurrence of $N^+$ comes with a factor of $t$, and we will have no way to keep track of how many times $N^+$ occurs using the method that gave the result of Lemma 3.13. We can obtain the distribution of $N^- = \{|n: X_n = 0, X_{n+1} = -t|\}$ using another method.

**Lemma 3.16.** Suppose $m \geq pt$. Then, for any $0 \leq i \leq t$,

$$P(A_i \text{ occurs } k \text{ times}) = \left(\frac{m+p}{p}\right)^{-1} \sum_{j=k}^{p} \frac{m-(p-j)t}{m+p-j} \binom{m+p-j}{p-j} \binom{m+p}{p} t^{j-k}.$$

In particular, $P(A_t \text{ occurs } k \text{ times}) = P(N^- = k)$.

**Proof.** Corollary 3.11 implies that for $0 \leq j \leq p$

$$P(N = j) = (t+1)\frac{m-(p-j)t}{m+p-j} \binom{m+p-j}{p-j} \left(\frac{m+p}{p}\right)^{-1}.$$

Therefore, given a specific sequence of crossing events $A_{x_1}, \ldots, A_{x_j}$, there are

$$\frac{m-(p-j)t}{m+p-j} \binom{m+p-j}{p-j}$$

realizations with $N = j$ and that particular crossing sequence. Given $i$, our task is to count the number of sequences for which $A_i$ occurs $k$ times. Clearly, we must have $k \leq j \leq p$, and there are $\binom{j}{k} t^{j-k}$ sequences for which $A_i$ appears $k$ times. We then sum over $j$ and divide by $\left(\frac{m+p}{p}\right)^{-1}$ to complete the proof. \qed

**Lemma 3.17.** Suppose $m \geq pt$. Then, for any $0 \leq i \leq t$,

$$P(A_i \text{ happens at least } k \text{ times}) = \left(\frac{m+p}{p}\right)^{-1} \sum_{j=k}^{p} \binom{m+p-j}{p-j} \binom{j-1}{k-1} t^{j-k}.$$

**Proof.** The proof is similar to the proof of Lemma 3.16 in many respects. To avoid double-counting, we require that the first crossing event we actually count (that is, the $j^{th}$ from last crossing event) is indeed $A_i$. \qed
We can use these same procedures to obtain \( N^+ \) from \( N \). Combinatorial identities result when compared with the results in Lemmas 3.10 and 3.13. We state the following results here without proof.

**Lemma 3.18.** Suppose \( m \geq pt \). Then, for any \( 0 \leq i \leq t \),

\[
P(N^+ = k) = \binom{m + p}{p}^{-1} \sum_{j=k}^{p} \frac{m - (p - j)t}{m + p - j} \binom{m + p - j}{p - j} \binom{j}{k} t^k,
\]

and

\[
P(N^+ \geq k) = \binom{m + p}{p}^{-1} \sum_{j=k}^{p} \binom{m + p - j}{p - j} \binom{j - 1}{k - 1} t^k.
\]

We can also use our results about \( N^+ \) to obtain information on \( N^- \).

**Lemma 3.19.** Suppose \( m \geq pt \). Then, for any \( 0 \leq i \leq t \),

\[
P(A_i \text{ occurs } k \text{ times}) = \binom{m + p}{p}^{-1} \sum_{j=k}^{p} \frac{m - pt + j(t + 1) + 1}{m + j + 1} \binom{m + p}{p - j} \binom{j}{k} (t - 1)^{j-k},
\]

and

\[
P(A_i \text{ occurs at least } k \text{ times}) = \binom{m + p}{p}^{-1} \sum_{j=k}^{p} \binom{m + p}{p - j} \binom{j - 1}{k - 1} (t - 1)^{j-k}.
\]

**Proof.** The proof follows similarly to the proof of Lemma 3.16, when \( 0 \leq i \leq t - 1 \). Note that the event \( A_i \) is segregated from the other crossing events. For the case \( i = t \) (i.e. the distribution of \( N^- \)), we first take the (just proven) \( i = 0 \) case, then apply the Crossing Lemma by doing a circular shift on \( \{0, \ldots, t\} \), sending \( i \) to \( i - 1 \) modulo \( t + 1 \).

3.3.2. Other Strategies. Since the urn weight can only increase by ones, we can give explicit formulas for some of the alternate strategies given in section 2.3. Some will be used for the random version of the urn, which we discuss in Chapter 7.

**Theorem 3.20.** Suppose \( m \geq pt \). Then for \( pt - m \leq r \leq 0 \),

\[
G_{\geq r}(m, p; 1, t) = r + G(m, p; 1, t).
\]

**Proof.** We show that for each \( k \), the probability of gaining \( r + k \) with the \( r \)-bet strategy is the same as the probability of gaining \( k \) with the zero-bet strategy. Since \( m \geq pt \), each realization \( \omega \) begins with a “\((m - pt)\)” trip from the initial weight \( pt - m \). Since the urn weight can only increase by
ones, we can break this trip into a “−r” trip from pt − m and a “+ (m − pt + r)” trip from pt − m − r. Writing ω = QT, with Q the initial “−r” trip, if ω gains k with the zero-bet strategy, then the realization ω′ = TQ^R will gain k + r with the alternate strategy, as the betting sequences over T for both ω and ω′ are identical, while no betting occurs during Q in ω while no passing occurs over Q^R in ω′. Thus, the player will gain r more on ω′ compared with ω. The inverse map can be found by finding the last time the urn weight is r, reversing the balls that follow, and shifting them to the beginning of the realization. This result holds over all realizations ω, completing the proof. □

The Extended Antiurn Theorem then implies the following Corollary.

**Corollary 3.21.** Suppose m ≥ pt. Then for pt − m ≤ r ≤ 0,

\[ G_{< r}(m, p; 1, t) = pt - m - r - G(m, p; 1, t). \]

Since t is a positive integer, \( G_{> r}(m, p; 1, t) = G_{\geq r+1}(m, p; 1, t) \), and we have the following.

**Corollary 3.22.** Suppose m ≥ pt. Then for pt − m − 1 ≤ r ≤ −1,

\[ G_{> r}(m, p; 1, t) = r + 1 + G(m, p; 1, t), \quad \text{and} \quad G_{\leq r}(m, p; 1, t) = pt - m - r - 1 - G(m, p; 1, t). \]

3.3.3. The \((m, p; 1, t)\) Urns with \(m < pt\). If \(m < pt\) then the trek associated with the initial weight \(pt - m\) goes against the movements we used for the urns with \(m \geq pt\). The distributions of \(N\) and \(N^+\) are now on \(\{0, \ldots, \lfloor m/t \rfloor\}\), so things are a little more tricky here. So, we will stick with the zero count version of \(G(m, p; 1, t)\) for now. The problem with this form is that it does not address the value of the crossing number \(κ\). Exact, but more complicated, formulas for \(G(m, p; 1, t)\), the distribution form (Corollary 3.30) and the crossings form (Lemma 4.4), will come shortly.

Recall that when \(m < pt\), \(ρ = \min\{n: X_n \leq 0\}\), and the crossing number \(κ\) is defined as \(κ = -E[X_ρ]\). We begin by tightening the range of \(κ\).

**Lemma 3.23.** If \(pt > m\), then

\[ P(X_ρ = −t + 1) \leq P(X_ρ = −t + 2) \leq \cdots \leq P(X_ρ = 0). \]

**Proof.** Consider a realization \(ω\) with \(X_ρ(ω) = -i, i > 0\). We will provide an injective map of such realizations into realizations with \(X_ρ = −i + 1\). A sketch of the mapping we will use is depicted in Figure 3.4.
With $X_\rho(\omega) = -i$, we have $X_{\rho-1} = t - i$. Since we have \("-1\" balls and $X_{m+p} = 0$, the urn will have weight $-i + 1$ at some stage. Let $q$ denote the smallest value such that $X_q(\omega) = -i + 1$. We will examine the realization $\omega(\rho - 1, q]$. Clearly, $X_j(\omega(\rho - 1, q]) > 0$ while $0 \leq j \leq \rho - 1$. As for the range $\rho \leq j \leq q - 1$, we have by the Reversal Lemma that

$$X_{q+\rho-1-j}(\omega(\rho - 1, q]) + X_j(\omega) = X_{\rho-1}(\omega) + X_q(\omega) = t - 2i + 1.$$ 

For the range $\rho \leq j \leq q - 1$ we must have $X_j(\omega) \leq -i$, and thus we have $X_j(\omega(\rho - 1, q]) \geq t + 1 - i > 0$ in the same range. Therefore, $q$ is the first time that $X_n(\omega(\rho - 1, q])$ is negative. The map is injective as well - the fact that $X_j(\omega(\rho - 1, q]) \geq t + 1 - i > 0$ for $\rho \leq j \leq q - 1$ implies that if more than one realization is mapped into $\omega(\rho - 1, q]$, then both $\rho$ and $q$ are the same for both. Since nothing changes before stage $\rho - 1$ and after stage $q$, and the reversal is invertible given $\rho - 1$ and $q$, and the two realizations must be identical.

If $0 < pt - m \leq t$, we can explicitly find the probabilities $P(X_\rho = -i), 0 \leq i \leq t - 1$. Writing $m = pt - r$ (with $0 < r \leq t$), we neutralize the urn by adding $r$ \("-1\" balls to the urn. For the $(pt, p; 1, t)$ urn, by the Crossing Lemma the next crossing is equally likely to be to $0, \ldots, -t$. That is, there are $\frac{1}{t+1}(\frac{pt+p}{p})$ realizations for which $A_i$ is the first crossing event. If $i \leq t - r$, then the first $r$ balls must be \("-1\", so the number of realizations from the $(pt - r, p; 1, t)$ urn for which $X_\rho = -i$ equals $\frac{1}{t+1}(\frac{pt+p}{p})$ for these $i$. For $i > t - r$, we need to discard the realizations from the $(pt, p; 1, t)$ urn that do not start with $r$ \("-1\" balls. We can count these bad realizations easily, since

\begin{figure}[h]
\centering
\includegraphics[scale=0.5]{figure3.4}
\caption{A realization with crossing number $i > 0$, and the realization with crossing number $i - 1$ associated with it.}
\end{figure}
these realizations begin with $t - i \ (< r) \ "-1"$ balls, followed by a "+t" ball triggering the event $A_i$. Therefore, the number of realizations from the $(pt - r, p; 1, t)$ urn with $X_\rho = -i$ here equals

$$\frac{1}{t+1} \binom{pt+p}{p} - \frac{1}{t+1} \binom{pt+p-t+1+i}{p-1}.$$  

We can obtain the crossing probabilities, and thus calculate $\kappa$, by dividing by $\binom{pt+p-r}{p}$. In summary, if $pt - m = r$, with $0 < r \leq t$, we have

$$\binom{pt+p-r}{p} P(X_\rho = -i) = \begin{cases} 
\frac{1}{t+1} \binom{pt+p}{p}, & 0 \leq i \leq t - r, \\
\frac{1}{t+1} \binom{pt+p}{p} - \binom{(p-1)(t+1)+i}{p-1}, & t - r + 1 \leq i \leq t - 1.
\end{cases}$$

In particular, if $r = 1$, then each crossing is equally likely, and thus $\kappa = (t-1)/2$.

For other cases, we can at least use Lemma 3.23 to limit the range of $\kappa$.

**Lemma 3.24.** If $pt > m$ and $m > 0$, then $0 < \kappa \leq (t - 1)/2$.

**Proof.** Denote $P(X_\rho = -i)$ as $p_i$, for $0 \leq i \leq t - 1$. Lemma 3.23 tells us that $p_{t-1} \leq \cdots \leq p_1 \leq p_0$. For any $i$ in this range,

$$ip_i + (t - 1 - i)p_{t-1-i} = (p_i + p_{t-1-i}) \left(\frac{t-1}{2}\right) + (p_{t-1-i} - p_i) \left(\frac{t-1}{2} - i\right) \leq (p_i + p_{t-1-i}) \left(\frac{t-1}{2}\right).$$

Thus

$$\kappa = \sum_{i=0}^{t-1} ip_i \leq \frac{t-1}{2} \sum_{i=0}^{t-1} p_i = \frac{t-1}{2},$$

with equality holding if and only if $p_0 = p_1 = \cdots = p_{t-1}$. As for the removal of the equal sign on the lower bound, the realization $\omega_0$ consisting of drawing all but one of the "-1" balls, followed by all of the "+t" balls, followed by the last "-1" ball has $X_\rho(\omega_0) = -1$. Since there are finitely many realizations, we must have $\kappa > 0$. □

We can now present the zero count form for $m < pt$.

**Theorem 3.25.** (Zero Count Form) If $m < pt$, we have

$$G(m, p; 1, t) = pt - m + \kappa + \frac{t}{2} \binom{m+p}{p}^{-1} \sum_{k=1}^{\lfloor m/t \rfloor} \binom{kt+k}{k} \binom{m+p-k-kt}{p-k},$$

where $0 \leq \kappa \leq (t - 1)/2$.  

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Remark. Sums for which the upper limit is less than the lower limit (like Theorem 3.25 with $m < t$) are considered to be empty.

We shall give the distributions of $N$ and $N^+$ with Lemmas 3.27 and 3.28, noting that they are considerably more complicated when $m < pt$. To obtain those distributions, we shall need information on how far the urn weight drifts from the initial weight or zero. Given $m$, $p$, and $t$, let $Q_{m,p}(\ell)$ denote the number of realizations $\omega$ from the $(m,p; 1, t)$ urn for which $X_n(\omega) \geq \ell$ for some $n$, so that the number of realizations with a maximum weight of $\ell$ equals $Q_{m,p}(\ell) - Q_{m,p}(\ell + 1)$. For any $x$, let $x^+ = \max\{0, x\}$. Then, we have the following.

**Theorem 3.26.** For any integer $\ell$ and nonnegative integers $m$, $p$, and $t$:

$$Q_{m,p}(\ell) = \begin{cases} \binom{m + p}{p}, & \text{if } \ell \leq (pt - m)^+, \\ \sum_{k=0}^{m - pt + \ell} \binom{m - pt + kt + k + \ell}{k} \binom{pt + p - k - kt - \ell}{p - k}, & \text{otherwise.} \end{cases}$$

**Proof.** Since both the weights zero and $pt - m$ are always taken, the result when $\ell \leq (pt - m)^+$ is trivial. Suppose $\ell > (pt - m)^+$. Then a realization reaching $\ell$ begins with a “$+(m - pt + \ell)$” trip. From there, any method of returning to zero will do. There are $\frac{m - pt + \ell}{m - pt + kt + k + \ell} \binom{m - pt + kt + k + \ell}{k}$ such “$+(m - pt + \ell)$” trips containing $k$ “$+t$” balls, as a result of Corollary 3.11, and thus there are $\binom{pt + p - k - kt - \ell}{p - k}$ ways to finish the realization. We complete the proof by summing over $k$. \hfill $\square$

**Lemma 3.27.** If $pt > m$, then for $k > 0$ we have

$$P(N \geq k) = (t + 1)^k \left(\frac{m + p}{p}\right)^{-1} \sum_{j=0}^{k} \frac{kt}{jt + jkt} \binom{jt + j + kt}{j} \binom{m + p - k - kt - j - jt}{p - k - j}.$$ 

**Proof.** Again, we first assume that for the last $k$ times the urn is nonempty and neutral, the event $A_t$ follows. We then use the same mapping that proved Lemma 3.10. While the resulting realization is still from the $(m, p - k; 1, t)$ urn, this time the realization must take the weight $pt - m$, which is no longer between the initial weight $(p - k)t - m$ and zero. The total number of realizations taking the weight $pt - m$ equals $Q_{m,p-k}(pt - m)$, by Lemma 3.26. The mapping can be reversed, and we add the factor $(t + 1)^k$ similarly by Lemma 3.7, since the events $A_0, \ldots, A_t$ are equally likely. \hfill $\square$

While we could similarly arrive another form of $G_{m,p;1,t}$ for $pt > m$ via the result of Lemma 3.27, it is more complicated than the zero count form, and does not address the value of the crossing number $\kappa$. The same could be said for a formula involving $N^+$ as well.
Lemma 3.28. If $pt > m$, then for $k > 0$ we have

$$P(N^+ \geq k) = t^k \left( \frac{m + p}{p} \right)^{-1} \sum_{j=[p-m/t]}^{p-k} \frac{k}{p-j} \frac{(p-j)(t+1)}{p-k-j} \left( m - pt + jt + j \right).$$

Proof. We break up a realization $\omega$ with $N^+ \geq k$ again as in Lemma 3.27:

$$\omega = TP_k M_k Q_k P_k - 1 M_k - 1 Q_k - 1 \cdots P_1 M_1 Q_1.$$

Our map this time will be slightly different. We shall map $\omega$ to the $\omega'$ from the $(m + k, p - k; 1, t)$ urn, with

$$\omega' = Q_1 M'_1 \cdots Q_{k-1} M'_{k-1} Q_k M'_k (TP_k P_{k-1} \cdots P_1)^R = Q_1 M'_1 \cdots Q_{k-1} M'_{k-1} Q_k M'_k P^R \cdots P^R T^R.$$

Since $T^R$ has weight $pt - m$, $\omega'$ is a realization for which the weight $pt - m > 0$ is taken. This mapping is similarly invertible. We find the segments $Q_i M'_i$ as before, and we can find the segments $P^R_i$ similarly. This time, the smallest prefix of $\omega'$ with weight $-k - it$ is

$$Q_1 M'_1 \cdots Q_k M'_k P^R \cdots P^R_1.$$

Having identified each segment, we can undo the map and recover $\omega$ from $\omega'$. This can be done from any realization from the $(m + k, p - k; 1, t)$ urn that takes the weight $pt - m$, therefore the mapping is a one-to-one correspondence. Therefore, there are $Q_{m+k,p-k}(pt-m)$ such realizations. The result now follows from Lemma 3.26, after which we reindex with respect to $p - k - j$. \qed

Remark. If we set the indeterminate form $0/0$ to equal one, then the results of Theorems 3.27 and 3.28 also hold for $k = 0$.

3.4. Distribution of the Gain for the $(m, p; 1, t)$ Urns. For the original $(m, p; 1, 1)$ urns, we not only have the expected gain, but we also have the distribution for the gain via Lemma 3.4. We can also find the distribution of the gain for the $(m, p; 1, t)$ urns because there is a link between gain and maximum weight, and for the $(m, p; 1, t)$ urns counting the number of realizations with a certain maximum weight is not a difficult task.

Theorem 3.29. Let $m$, $p$, and $\ell \geq 0$ be given. Then

$$P(\text{player gains } \ell \text{ using an optimal strategy}) = \left( \frac{m + p}{p} \right)^{-1} \left( Q_{m,p}(\ell) - Q_{m,p}(\ell + 1) \right).$$
Theorem 3.29 implies that the probability the player gains at least $\ell$ equals $\binom{m+p}{p}^{-1}Q_{m,p}(\ell)$, and in particular the probability the player gains at least $\max\{0,pt-m\}$ equals one. The method of proof is to show a bijection between the realizations gaining exactly $\ell$ with the realizations with a maximum weight equal to $\ell$. Since this result holds under more general circumstances (Theorem 4.18), we will hold off on the proof until then. Our aim for the moment is the following result, the distribution form.

**Corollary 3.30.** (Distribution form) For any $m$ and $p$, we have

$$G(m,p;1,t) = \binom{m+p}{p}^{-1} \sum_{\ell=1}^{pt} Q_{m,p}(\ell).$$

Observe that Corollary 3.30 gives an exact form for $G(m,p;1,t)$ for any $m$ and $p$, including an accurate representation for the case $pt-m > 0$. When $pt-m > 0$, we have $Q_{m,p}(\ell) = \binom{m+p}{p}$ for $1 \leq \ell \leq pt-m$, and because of this we can hide the gain associated with the initial value of the urn inside the sum. In particular, when $t = 1$ we may rewrite the result of Theorem 3.2 in a way that matches the form of Corollary 3.30:

$$G(m,p) = \binom{m+p}{p}^{-1} \sum_{k=0}^{p-1} \binom{m+p}{\min\{k,m\}}.$$

The distributions of the gains of the urns with $t=2$ and $m+p=5$ balls are given in Table 3.1.

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<th>3</th>
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<tr>
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<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

48
4. The Value of the \((m, p; s, t)\) Urn

We now tackle the problem of finding the value of the \((m, p; s, t)\) urn, initially with \(t/s\) a positive rational. The Crossing Lemma is no longer at our disposal, so we lose the result of Corollary 3.8 with it. Thus, a zero count, negative binomial, or binomial form appears to be out of our reach. As for the distribution form, Theorem 3.29 does hold under these general circumstances (Theorem 4.18), but the objects involved are very difficult or impossible to obtain directly. Thus, we need another approach. To find that approach, we take another look at the \((m, p; 1, t)\) urns. Indeed, a rather primitive approach gives another form for \(G(m, p; 1, t)\), the crossings form (Lemmas 4.5, 4.6). Using the same approach, we can calculate \(G(m, p; s, t)\) (Theorems 4.10, 4.11) exactly for any \(s\) and \(t\), but not without some challenges along the way. For some special cases, we can simplify the crossings form. In particular, when \(pt = ms\) we can write \(G(m, p; s, t)\) in a form (Theorem 4.15) resembling the zero count form of \(G(m, p; 1, t)\).

When \(r = t/s\) is irrational, the crossings approach also works perfectly. The difficulty lies in actually writing a formula for the expected gain. Using a rational approximation and the continuity of \(G(m, p; s, t)\) in \(t\), we can write \(G(m, p; 1, r)\) in a form not involving a limit (Theorem 4.17) when \(r\) is irrational.

Finally, we show that the probability that \(k\) is gained equals the probability that the maximum weight of the urn equals \(k\). Therefore, the results for the expected gain using an optimal strategy also apply to the expected maximum weight (or minimum weight, by the Reversal Lemma) the urn achieves during play.

4.1. A More Primitive Formula for \(G(m, p; 1, t)\). The zero count, negative binomial, and binomial forms of \(G(m, p; 1, t)\) (with \(t\) a positive integer) all depend upon the Crossing Lemma, in particular Corollary 3.8, which reduced calculating the expected gain to counting the number of times the urn could be neutral. Since the result of the Crossing Lemma does not hold for the \((m, p; s, t)\) urns, and the distribution form will be exceedingly complicated, we need to find an approach that calculates \(G(m, p; 1, t)\) without the help of the Crossing Lemma. To do that, we need to figure out the event or events resulting in gains for the player.
4.1.1. *Crossings.* In our first derivation of $G(m, p; 1, t)$, we noted that if the urn weight was zero, the player gained an average of $t/2$ until the next time the urn was neutral. Now, we will indicate the precise events that explicitly give the value for the $(m, p; 1, t)$ urn. A *permanent gain* (that is, these gains combined will equal the total gain for the realization) can be made in one of three ways: First, if $pt - m > 0$, then since $X_{m+p} = 0$ it is a certainty that the player will gain (at least) $pt - m$. The remaining permanent gains are the result of two similar events, that we shall call down-crossings and up-crossings.

**Definition 4.1.** Suppose $X_n = t - j$ and $X_{n+1} = -j$. If the zero-pass strategy is used, then this event is a down-crossing to $-j$ if $0 \leq j \leq t - 1$. With the zero-bet strategy, this event is a down-crossing to $-j$ if $1 \leq j \leq t$.

When the zero-pass strategy is used, each down-crossing to $-j$ will result in the a permanent gain of $j$ for the player. This is because the player will wait until the urn weight is at least zero before placing another bet.

The notion of an up-crossing is defined similarly.

**Definition 4.2.** Suppose $X_n = i - s$, and $X_{n+1} = i$. If the zero-pass strategy is used, then this event is an up-crossing to $i$ if $1 \leq i \leq s$. With the zero-bet strategy, this event is an up-crossing to $i$ if $0 \leq i \leq s - 1$.

For the $(m, p; 1, t)$ urns with the zero-pass strategy, we have an up-crossing whenever $X_n = 0$ and the next ball drawn from the urn is a “$-1$.” Since the player will not bet on this ball, and will bet until the weight of the urn is nonpositive, the player will therefore gain one as a result of this event. Any remaining gain over the series of bets will be counted by the following down-crossing. Since zero is unfavorable with the zero-pass strategy, the last crossing will be always be a down-crossing.

Using these up- and down-crossings, we calculate yet another formula for $G(m, p; 1, t)$, the crossings form.

4.1.2. *The Crossings Form of $G(m, p; 1, t)$.* Using the zero-pass strategy, if $pt - m \leq 0$ and $p > 0$ (implying $m \geq t$), then the total number of up-crossings over all realizations is

$$
\frac{t}{t + 1} \sum_{k=1}^{p} \binom{kt + k}{k} \binom{m + p - k - kt}{p - k}.
$$
Figure 4.1. For this realization from the \((10, 4; 1, 2)\) urn, the third, sixth, and twelfth balls (black) result in up-crossings when the zero-pass strategy is used, with each resulting an a gain of one for the player. The fourth and thirteenth balls are down-crossings to \(-1\), with each resulting in a gain of one (shown in black) for the player. The eighth ball is a down-crossing to zero, but does not result in a gain for the player. This realization gains five for the player when the zero-pass strategy is used.

We associate a permanent gain of one with each occurrence of an up-crossing.

For each \(0 \leq i \leq t - 1\), the total number of down-crossings to \(-i\) is

\[
\sum_{k=0}^{p-1} \binom{kt + k + i}{k} \binom{m + p - k - kt - i - 1}{p - k - 1}.
\]

We associate the permanent gain of \(i\) with these. The up- and down-crossings of a realization from the \((10, 4; 1, 2)\) urn are shown in Figure 4.1.

**Lemma 4.3.** If \(pt - m \leq 0\), then

\[
\binom{m + p}{p} G(m, p; 1, t) = \frac{t}{t + 1} \sum_{k=1}^{p} \binom{kt + k}{k} \binom{m + p - k - kt}{p - k} + \sum_{i=0}^{t-1} \sum_{k=0}^{p-1} \binom{kt + k + i}{k} \binom{m + p - k - kt - i - 1}{p - k - 1}.
\]

If we have \(pt - m > 0\), then the total number of up-crossings is

\[
\frac{t}{t + 1} \sum_{k=1}^{\lfloor m/t \rfloor} \binom{kt + k}{k} \binom{m + p - k - kt}{p - k}.
\]

For down-crossings, a complication develops. A few down-crossings may appear before the urn is first neutral. However, which and how many are easily determined. The very first possible down-crossing can be found by drawing nothing but \(\text{“}+t\text{”}\) balls until the urn is nonpositive. Thus, if that first one crosses down to \(-K\), then \(pt - m\) (thus \(-m\)) and \(-K\) are congruent modulo \(t\). Then there
are extra down-crossings to $-K, -K + 1, \ldots, -1,$ and 0. Thus if we define $K$ as the residue of $m$ modulo $t$ and define, for $0 \leq i \leq t - 1$,

$$M(i) = \begin{cases} 
\lfloor m/t \rfloor, & \text{if } 0 \leq i \leq K, \\
\lfloor m/t \rfloor - 1, & \text{otherwise},
\end{cases}$$

(4.1)

we can say the total number of down-crossings to $-i$ equals

$$\sum_{k=0}^{M(i)} \binom{kt + k + i}{k} \binom{m + p - k - kt - i - 1}{p - k - 1}.$$ 

Like before, we associate a permanent gain of $i$ with each down-crossing to $-i$. Since each realization is equally likely, we have what we need to calculate $G(m, p; 1, t)$.

Lemma 4.4. Suppose $pt - m \geq 0$. With $K$ and $M(i)$ as defined in (4.1), we have

$$G(m, p; 1, t) = \binom{m + p}{p} (pt - m) + \frac{t}{t + 1} \sum_{k=1}^{\lfloor m/t \rfloor} \binom{kt + k}{k} \binom{m + p - k - kt}{p - k}$$

$$+ \sum_{i=0}^{t-1} \sum_{k=0}^{M(i)} i \binom{kt + k + i}{k} \binom{m + p - k - kt - i - 1}{p - k - 1}. $$

Note that we have included the case $pt = m$ as well. The difference in the argument is that there is a nonexistent down-crossing to zero added (since $K = 0$), coming before the first draw from the urn. However, it is associated with the vanishing $i = 0$ term.

If our player decides to take the zero-bet approach, the neutral urns are now deemed favorable. The up-crossings now run from $-1$ to 0, but since the urn is not guaranteed to return to back to the weight $-1$, we do not associate a positive gain with these events. Therefore, we can ignore them altogether. A down-crossing to $-i$, this time with $1 \leq i \leq t$, still results in a permanent gain of $i$ for the player. We now construct a more “official” set of formulas that turn out to be redundant to the formulas of Lemmas 4.3 and 4.4, though the next two results are in a better form since we can ignore the up-crossings.

Lemma 4.5. (Crossings Form for $m \geq pt$) Suppose $pt - m \leq 0$. Then we have

$$G(m, p; 1, t) = \binom{m + p}{p}^{-1} \sum_{i=1}^{t} \sum_{k=0}^{p-1} i \binom{kt + k + i}{k} \binom{m + p - k - kt - i - 1}{p - k - 1}. $$
Lemma 4.6. (Crossings Form for \(m \leq pt\)) Suppose \(pt - m \geq 0\). With \(K\) and \(M(i)\) as defined in (4.1), we have

\[
G(m, p; 1, t) = pt - m + \binom{m + p}{p}^{-1} \sum_{i=1}^{M(i)} \sum_{k=0}^{i} \binom{kt + k + i}{k} \binom{m + p - k - kt - i - 1}{p - k - 1}.
\]

Lemmas 4.4 and 4.6 are identical results, since \(K < t\) implies \(M(t) = \lfloor m/t \rfloor - 1\), and

\[
\frac{1}{t + 1} \sum_{k=0}^{\lfloor m/t \rfloor - 1} \binom{kt + k}{k} \binom{m + p - k - kt}{p - k} = \frac{1}{t + 1} \sum_{k=1}^{\lfloor m/t \rfloor} \binom{kt + k}{k} \binom{m + p - k - kt}{p - k}.
\]

Lemmas 4.3 and 4.5 also are identical by a similar manipulation. The common result of Lemma 4.4/4.6 is an exact formula for \(G(m, p; 1, t)\) for the case \(pt - m > 0\), unlike the zero count estimate.

4.2. The Expected Gain for the \((m, p; s, t)\) Urn with \(t/s\) Rational. With this more primitive approach, we can obtain formulas for the value of the \((m, p; s, t)\) urns, for any arbitrary positive rational \(s\) and \(t\). Without loss of generality, we will impose the following restrictions on \(s\) and \(t\): We require that \(s\) and \(t\) are positive integers with gcd\((s, t) = 1\), and we also require that \(ms - pt \geq 0\).

We can expand back to the cases with \(t/s\) rational via Lemma 2.1, while we can extend to the case \(ms - pt < 0\) via the Antiurn Theorem.

Two complications occur as a result of enlarging \(s\). First, we now will have multiple up-crossings to deal with. Another complexity arises in the “cycle” - the urn weight increases by \(s\) modulo \(t + s\) each time a ball is drawn, and the ordering only relates well with the regular integer order when \(s\) or \(t\) equals one. Formulas will be more complex, and be more difficult to simplify, as a result. So, our first task shall be to define an order on \([-t, -t + 1, \ldots, s - 1, s]\) to simplify notation.

Definition 4.7. The crossing order \(R\) on \([-t, -t + 1, \ldots, s]\) is defined as follows: For each \(i \in \{-t, \ldots, s\}\), there exists a unique \(c_i \in \{0, \ldots, s + t - 1\}\) such that \(i \equiv c_i t \mod (s + t)\). Then

\[
i R j \quad \text{if and only if} \quad c_i \leq c_j.
\]

Note that \(c_{-t} = c_s = t + s - 1\). This ambiguity shall not be a problem, as \(-t\) (for the zero-bet strategy) and \(s\) (for zero-pass) will not be used simultaneously.

The main reason for the introduction of the crossing order \(R\) is that the first possible up-crossing may come mid-cycle (that is, before the first time the urn could be neutral). If this first up-crossing
is to $K$, then we will have an extra crossing for each $i$ satisfying $i \not\sim R K$. The first possible up-crossing occurs when only “$-s$” balls are drawn until a favorable urn is reached. Therefore, $K$ and $pt$ (that is, $pt - ms$) will be congruent modulo $s$.

Next, we determine when the urn can possibly be anywhere in the range $\{-t, \ldots, s\}$.

**Definition 4.8.** For each integer $-t \leq i \leq s$, the unique non-negative integers $a_i$ and $b_i$ satisfying

$$a_i + b_i < s + t, \quad a_i + b_i = c_i, \quad tb_i - sa_i = i,$$

are the *crossing constants* for $i$.

The crossing constant $c_i$ for $i$ is defined in Definition 4.7. The crossing constants exist and are unique since $\{kt\}_{k=0}^{t+s-1}$ is a complete residue system modulo $t + s$. Also, observe that the crossing constants depend on neither $m$ nor $p$.

**Table 4.1.** Determination of the crossing constants $a_i$, $b_i$, and $c_i$ for $-3 \leq i \leq 4$, with $s = 4$ and $t = 3$.

<table>
<thead>
<tr>
<th>$c_i$</th>
<th>$3c_i$</th>
<th>$i$</th>
<th>$a_i$</th>
<th>$b_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>3</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>-1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>9</td>
<td>2</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>12</td>
<td>-2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>15</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>6</td>
<td>18</td>
<td>4</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>6</td>
<td>18</td>
<td>-3</td>
<td>3</td>
<td>3</td>
</tr>
</tbody>
</table>

*Example.* With $s = 4$ and $t = 3$, we have the transitive crossing order (see Table 4.1)

$$0 R 3 R -1 R 2 R -2 R 1 R (-3 \text{ or } 4).$$

The constants $a_i$, $b_i$, and $c_i$ have some noteworthy properties. We have a bijection between $i$ and $c_i$ once one of $-t$ and $s$ is thrown away. We also have a bijection between nonnegative $i$ and $b_i$ for those $i$, an assertion we shall prove momentarily and make good use of. We have a third bijection between nonpositive $i$ and $-a_i$ as well, which follows after an application of the Antiurn Theorem and our next result.

**Lemma 4.9.** The map $\theta$ with $\theta(i) = b_i$ is a permutation on $\{1, \ldots, s - 1\}$. 
Proof. Recall that \( b_i t - a_i s = i \). If \( b_i = b_j \), we have \( i - j = (a_j - a_i)s \), which means \( i = j \), since both \( i \) and \( j \) are between 1 and \( s - 1 \). Since the urn is crossing up from a negative value, we have \( b_i > 0 \). We also have \( b_i < s \). Otherwise, if \( b_i \geq s \), then \( i = b_i t - a_i s \geq s(t - a_i) \), but since \( a_i + b_i < s + t \), we have \( a_i < t \) which implies \( i \geq s \), which is not possible. Therefore, we have a one-to-one correspondence between \( \{b_1, \ldots, b_{s-1}\} \) and \( \{1, \ldots, s - 1\} \). \( \square \)

Gains are made in the \((m, p; s, t)\) urn in three ways. If \( pt - ms > 0 \), then there is a gain of \( pt - ms \) obtained through the first series of bets. Otherwise, gains are made via up- and down-crossings. For an up-crossing to \( i \), there must be \( kt + a_i \) “−s” balls and \( ks + b_i \) “+t” balls left in the urn at its completion, for some valid \( k \). The up-crossing consists of one “−s” ball, thus there have been \( m - kt - a_i - 1 \) “−s” balls and \( p - ks - b_i \) “+t” balls drawn from the urn before the crossing. A similar argument applies for the down-crossings, the only difference being that the down-crossing consists of a lone “+t” ball instead. We can now calculate \( G(m, p; s, t) \).

**Theorem 4.10.** Suppose \( pt - ms \leq 0 \). Let \( K \) be the residue of \( pt \) modulo \( s \). For \(-t \leq i \leq s\) define

\[
M(i) = \begin{cases} 
\lfloor p/s \rfloor, & \text{if } i \equiv R K, \\
\lfloor p/s \rfloor - 1, & \text{otherwise},
\end{cases}
\]

where the crossing order \( R \) is defined by (4.2). Then

\[
\binom{m+p}{p} G(m, p; s, t) = \sum_{i=0}^{s-1} \sum_{k=0}^{M(i)} \binom{kt+s+c_i}{ks+b_i} \binom{m+p-kt-s-c_i-1}{p-ks-b_i} + \sum_{j=1}^{t} \sum_{k=0}^{M(-j)} \binom{kt+s+c_{-j}}{ks+b_{-j}} \binom{m+p-kt-s-c_{-j}-1}{p-ks-b_{-j}-1}.
\]

The result for Theorem 4.10 is via the zero-bet strategy. Thus, we show the \( i = 0 \) term, and hide the \( j = 0 \) term.

**Example.** Taking the neutral \((3, 4; 4, 3)\) urn, we have \( K = 0 \), and \( M(i) = 0 \) for all \( i \). Then using the entries from Table 4.1 we obtain

\[
35 \cdot G(3, 4; 4, 3) = \left[ \binom{5}{3} \binom{1}{1} + 2 \binom{3}{2} \binom{3}{2} + 3 \binom{1}{1} \binom{5}{3} \right] + \left[ \binom{2}{1} \binom{4}{2} + 2 \binom{4}{1} \binom{2}{1} + 3 \binom{6}{1} \binom{0}{0} \right] = 154.
\]

Thus \( G(3, 4; 4, 3) = 154/35 = 4.4 \).

Using the zero-pass strategy we obtain another form.
Theorem 4.11. Suppose $pt - ms \leq 0$. With $K$ and $M(i)$ as defined in (4.4), we have
\[
\binom{m+p}{p} G(m,p; s,t) = \sum_{i=1}^{s} \sum_{k=0}^{M(i)} i \binom{k(t+s) + c_i}{ks + b_i} \binom{m + p - k(t+s) - c_i - 1}{p - ks - b_i} \\
+ \sum_{j=0}^{t-1} \sum_{k=0}^{M(-j)} j \binom{k(t+s) + c_{-j}}{ks + b_{-j}} \binom{m + p - k(t+s) - c_{-j} - 1}{p - ks - b_{-j} - 1}.
\]

For the $(3,4;4,3)$ urn, the $3\binom{6}{3}0\binom{0}{0}$ term with $j = t$ is replaced by the $i = s$ term of $4\binom{6}{4}0\binom{0}{0}$. Both terms contribute 60 to the sum, so we still get $G(3,4;4,3) = 4.4$.

We close by giving explicit formulas for some special cases, beginning with the “one ball” formulas.

Theorem 4.12. Let $N_1 = \max\{n \in \mathbb{Z} : ns \leq t\}$, and $N_2 = \max\{n \in \mathbb{Z} : nt \leq s\}$. Then for $m > 0$
\[
G(m,1; s,t) = \frac{N_1 + 1}{m + 1} \left( t - \frac{sN_1}{2} \right),
\]
and for $p > 0$
\[
G(1,p; s,t) = pt - s + \frac{N_2 + 1}{p + 1} \left( s - \frac{tN_2}{2} \right).
\]

Proof. We shall use the zero-bet strategy to calculate $G(m,1; s,t)$. The urn will be nonnegative only when the “$+t$” ball is one of the last $N_1 + 1$ balls drawn from the urn. Working with these realizations, the player will start betting once this event occurs, and will draw $i$ “$-s$” balls, where $0 \leq i \leq N_1$, then the “$+t$” ball, after which the player will stop betting. Thus the player’s expected gain is
\[
\binom{m+1}{1}^{-1} \sum_{i=0}^{N_1} (t - is) = \frac{N_1 + 1}{m + 1} \left( t - \frac{sN_1}{2} \right).
\]
The second result follows from this by the Antiurn Theorem. □

Our final formula is for the case where there is quite a disparity between $s$ and $t$.

Theorem 4.13. Suppose that $s \geq pt$, and $m > 0$. Then
\[
G(m,p; s,t) = t \binom{m+p}{p}^{-1} \sum_{k=0}^{p} \binom{m + p - k - 1}{m - 1}.
\]

Proof. If $s \geq pt$, then there are no contributing down-crossings (with the zero-pass strategy). The only up-crossings occur with the draw of the last “$-s$” ball. If there are $k$ “$+t$” balls left after the last “$-s$” ball has been drawn, then the player will gain $kt$. For each $k$, there are $\binom{m+p-k-1}{p-1}$
realizations resulting in a gain of $kt$. Thus we sum up and divide by the number of realizations, \( \binom{m+p}{p} \). \(\Box\)

$G(m, p; s, t)$ is nonincreasing as $s$ increases, and the formula given by Theorem 4.13 does not depend on $s$ (outside of the assumption $s \geq pt$). Thus, if $pt > 0$ we have $\inf_s G(m, p; s, t) > 0$, with any $s \geq pt$ giving the infimum. This is because the player will simply wait until all of the “$-s$” balls are drawn, and collect any “$+t$” balls that may be left.

The Antiurn Theorem provides us with a formula at the opposite end of the spectrum.

**Corollary 4.14.** Suppose that $t \geq ms$, and $p > 0$. Then

$$G(m, p; s, t) = (pt - ms) + s \left( \frac{m+p}{p} \right)^{-1} \sum_{k=0}^{m} \binom{m+p-k-1}{p-1}.$$

The expected gains for some of the urns with $s = 4$ and $t = 3$ are given in Table 4.2.

<table>
<thead>
<tr>
<th>$p$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G(0, p; 4, 3)$</td>
<td>0</td>
<td>3</td>
<td>6</td>
<td>9</td>
<td>12</td>
<td>15</td>
<td>18</td>
<td>21</td>
<td>24</td>
</tr>
<tr>
<td>$G(1, p; 4, 3)$</td>
<td>0</td>
<td>1.5</td>
<td>3.67</td>
<td>6.25</td>
<td>9</td>
<td>11.83</td>
<td>14.71</td>
<td>17.63</td>
<td>20.56</td>
</tr>
<tr>
<td>$G(2, p; 4, 3)$</td>
<td>0</td>
<td>1</td>
<td>2.33</td>
<td>4.1</td>
<td>6.4</td>
<td>8.95</td>
<td>11.64</td>
<td>14.42</td>
<td>17.24</td>
</tr>
<tr>
<td>$G(3, p; 4, 3)$</td>
<td>0</td>
<td>0.75</td>
<td>1.7</td>
<td>2.9</td>
<td>4.4</td>
<td>6.48</td>
<td>8.87</td>
<td>11.43</td>
<td>14.11</td>
</tr>
<tr>
<td>$G(4, p; 4, 3)$</td>
<td>0</td>
<td>0.6</td>
<td>1.33</td>
<td>2.23</td>
<td>3.31</td>
<td>4.72</td>
<td>6.58</td>
<td>8.8</td>
<td>11.24</td>
</tr>
<tr>
<td>$G(5, p; 4, 3)$</td>
<td>0</td>
<td>0.5</td>
<td>1.1</td>
<td>1.80</td>
<td>2.64</td>
<td>3.68</td>
<td>5</td>
<td>6.67</td>
<td>8.73</td>
</tr>
<tr>
<td>$G(6, p; 4, 3)$</td>
<td>0</td>
<td>0.43</td>
<td>0.93</td>
<td>1.51</td>
<td>2.19</td>
<td>3.00</td>
<td>4.00</td>
<td>5.23</td>
<td>6.73</td>
</tr>
<tr>
<td>$G(7, p; 4, 3)$</td>
<td>0</td>
<td>0.38</td>
<td>0.81</td>
<td>1.3</td>
<td>1.87</td>
<td>2.53</td>
<td>3.32</td>
<td>4.28</td>
<td>5.42</td>
</tr>
</tbody>
</table>

### 4.3. The Special Case $pt = ms$

A simpler formula for the value of the general urn seems to be difficult to obtain. However, for the neutral urns with $pt = ms$, we are able to do some simplification.

**Theorem 4.15.** Suppose $pt = ms$. Then

$$G(m, p; s, t) = \left( \frac{m+p}{p} \right)^{-1} \frac{t+s}{2} \sum_{i=0}^{s-1} \sum_{k=0}^{p-1} \left( k(t+s) + c_i \right) \left( m + p - k(t+s) - c_i - 1 \right).$$

**Proof.** We use the zero-bet formula given by Theorem 4.10. We begin by reversing the roles the two binomial coefficients in the double sum represent, by literally reversing the realization. In this manner, the balls left in the urn become the balls drawn from it, and vice versa. Upon this reversal,
each up-crossing to $i$ becomes an up-crossing to $s - i$ instead. When we put the gains associated with the two up-crossings together, we end up with a gain, $s$, that no longer depends on $i$. In terms of the algebra, we will show that

\[(4.5) \quad \sum_{i=1}^{s-1} \sum_{k=0}^{p/s-1} j \left( k(t + s) + c_i \right) \left( m + p - k(t + s) - c_i - 1 \right) \frac{ks + b_i}{p - ks - b_i} \]

\[= \sum_{i=1}^{s-1} \sum_{k=0}^{p/s-1} (s - i) \left( k(t + s) + c_{s-i} \right) \left( m + p - k(t + s) - c_{s-i} - 1 \right) \frac{ks + b_{s-i}}{p - ks - b_{s-i}} \]

Starting with the left hand side of (4.5), reindexing the outer sum with respect to $s - i$ gives us the sum

\[\sum_{i=1}^{s-1} \sum_{k=0}^{p/s-1} (s - i) \left( k(t + s) + (t + s - c_{s-i} - 1) \right) \left( m + p - k(t + s) - (t + s - c_{s-i}) \right) \frac{ks + (s - b_{s-i})}{p - ks - (s - b_{s-i})} \]

since $ms = pt$. By the definition of the crossing constants, $b_{s-i}t - a_{s-i}s = s - i$. We therefore have that

\[(s - b_{s-i})t - (t - 1 - a_{s-i})s = i.\]

We have $0 < b_{s-i} < s$ (by Lemma 4.9) and $0 < a_{s-i} < t - 1$, the latter since $i \neq 0$ and a “-s” ball precedes the up-crossing. Thus $0 < s - b_{s-i} < s$ and $0 < t - 1 - a_{s-i} < t - 1$. Therefore, we must have

\[b_i = s - b_{s-i}, \quad a_i = t - 1 - a_{s-i}, \quad \text{and} \quad c_i = t + s - c_{s-i} - 1,\]

by the uniqueness of $a_i$ and $b_i$. Making these substitutions, we obtain the desired sum on the right-hand side of (4.5).

Application of the Antiurn Theorem and (4.5) imply that

\[(4.6) \quad \sum_{j=1}^{t-1} \sum_{k=0}^{p/s-1} j \left( k(t + s) + c_{-j} \right) \left( m + p - k(t + s) - c_{-j} - 1 \right) \frac{ks + b_{-j}}{p - ks - b_{-j} - 1} \]

\[= \sum_{j=1}^{t-1} \sum_{k=0}^{p/s-1} (t - j) \left( k(t + s) + c_{-j} \right) \left( m + p - k(t + s) - c_{-j} - 1 \right) \frac{ks + b_{-j}}{p - ks - b_{-j} - 1} ,\]

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while a similar manipulation gives

\[
(4.7) \quad t \sum_{k=0}^{p/s-1} \left( \frac{k(t + s) + c_{-t}}{k s + c_{-t}} \right) \left( \frac{m + p - k(t + s) - c_{-t} - 1}{p - k s - b_{-t} - 1} \right)
\]

\[
= t \sum_{k=0}^{p/s-1} \left( \frac{k(t + s)}{k s} \right) \left( \frac{m + p - k(t + s) - 1}{p - k s - 1} \right)
\]

\[
= s \sum_{k=0}^{p/s-1} \left( \frac{k(t + s)}{k s} \right) \left( \frac{m + p - k(t + s) - 1}{p - k s} \right).
\]

Together, (4.5), (4.6), and (4.7) imply that

\[
(4.8) \quad 2 \left( \frac{m + p}{p} \right) G(m, p; s, t) = s \sum_{i=0}^{s-1} \sum_{k=0}^{p/s-1} \left( \frac{k(t + s) + c_i}{k s + b_i} \right) \left( \frac{m + p - k(t + s) - c_i - 1}{p - k s - b_i} \right)
\]

\[
+ t \sum_{j=1}^{t} \sum_{k=0}^{p/s-1} \left( \frac{k(t + s) + c_{-j}}{k s + b_{-j}} \right) \left( \frac{m + p - k(t + s) - c_{-j} - 1}{p - k s - b_{-j} - 1} \right).
\]

We can combine the two sums rather easily upon noticing that

\[
(4.9) \quad \sum_{i=0}^{s-1} \sum_{k=0}^{p/s-1} \left( \frac{k(t + s) + c_i}{k s + b_i} \right) \left( \frac{m + p - k(t + s) - c_i - 1}{p - k s - b_i} \right)
\]

\[
= \sum_{j=1}^{t} \sum_{k=0}^{p/s-1} \left( \frac{k(t + s) + c_{-j}}{k s + b_{-j}} \right) \left( \frac{m + p - k(t + s) - c_{-j} - 1}{p - k s - b_{-j} - 1} \right).
\]

The former sum is the total number of (zero-bet) up-crossings, and the latter sum is the total number of down-crossings. Since zero is favorable with the zero-bet strategy, the first crossing (if there is one) is a down-crossing. The down- and up-crossings alternate throughout, but the last crossing must be an up-crossing, again, since zero is favorable. Thus, we conclude that (4.9) holds.

Combining the two sums, we then have

\[
2 \left( \frac{m + p}{p} \right) G(m, p; s, t) = (t + s) \sum_{i=0}^{s-1} \sum_{k=0}^{p/s-1} \left( \frac{k(t + s) + c_i}{k s + b_i} \right) \left( \frac{m + p - k(t + s) - c_i - 1}{p - k s - b_i} \right),
\]

and we complete the proof by dividing through by \(2 \left( \frac{m + p}{p} \right)\). \(\square\)

For the neutral (3, 4; 4, 3) urn, we have a new calculation, but the same result:

\[
G(3, 4; 4, 3) = \frac{1}{35} \cdot \frac{7}{2} \cdot \left[ \binom{0}{0} \left( \frac{1}{4} \right) + \binom{6}{3} \left( \frac{1}{3} \right) + \binom{5}{3} \left( \frac{1}{3} \right) + \binom{3}{3} \left( \frac{1}{2} \right) + \binom{1}{3} \left( \frac{5}{3} \right) \right] = 4.4.
\]

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Corollary 4.16. Suppose $pt = ms$. Then

$$G(m, p; s, t) = \left(\frac{m + p}{p}\right)^{t - s - 1} \frac{1}{2} \sum_{j=0}^{t-1} \sum_{k=0}^{p/s-1} \left( \frac{k(t + s) + c_{-j}}{ks + b_{-j}} \right) \left( \frac{m + p - k(t + s) - c_{-j} - 1}{p - ks - b_{-j} - 1} \right).$$

Proof. From (4.8), we use (4.9) the other way to combine the two double sums. Then, using the intermediate step of (4.7) we can move the $j = t$ term to the $j = 0$ term. Thus,

$$2\left(\frac{m + p}{p}\right)G(m, p; s, t) = (t + s) \sum_{j=0}^{t-1} \sum_{k=0}^{p/s-1} \left( \frac{k(t + s) + c_{-j}}{ks + b_{-j}} \right) \left( \frac{m + p - k(t + s) - c_{-j} - 1}{p - ks - b_{-j} - 1} \right),$$

and we again divide through by $2\left(\frac{m + p}{p}\right)$. □

This gives (predictably)

$$G(3, 4; 4, 3) = \frac{44}{35} \cdot \frac{7}{2} = 4.4.$$  

Remark. A combinatorial proof of this special case is as follows: For a realization $\omega$, we examine the gains made from $\omega$ using the zero-pass strategy, and the gains made from the reversed realization $\omega^R = \omega(0, m + p]$ using the zero-bet strategy. From the perspective of $\omega$, the first crossing is an up-crossing. Each up-crossing gives a permanent gain of $i$ from $\omega$ and $s - i$ from its associated up-crossing in $\omega^R$, for $1 \leq i \leq s$. The following down-crossing gives a permanent gain of $j$ from $\omega$ and $t - j$ from $\omega^R$ for $0 \leq j \leq t - 1$. Since we have the pairing of up- and down-crossings, we can associate the gains from the down-crossing with the preceding up-crossing. Thus, for each up-crossing we can associate a combined permanent gain of $t + s$ from $\omega$ and $\omega^R$. Summing up over all $\omega$, we divide by 2, since each realization is counted twice.

4.4. Using Rational Approximation to Calculate $G(m, p; s, t)$ When $t/s$ Is Irrational. The approach that yielded $G(m, p; s, t)$ when $t/s$ is rational will also work when $t/s$ is irrational. The difficulty arises in writing the value in a readable form. Hence, we need to find a different way to express $G(m, p; s, t)$. A logical strategy would be to use some sort of rational approximation, since we have the continuity of $G(m, p; s, t)$ in both $s$ and $t$. For a start, we could say that if $\{r_n\}_{n=1}^\infty$ is a sequence of rational numbers converging to $r = t/s$, then

$$G(m, p; s, t) = s \cdot G(m, p; 1, r) = s \cdot \lim_{n \to \infty} G(m, p; 1, r_n),$$
but given the complicated nature of the formulas for $G(m,p;1,r_n)$, this is not a satisfactory representation of $G(m,p;s,t)$. Instead, we truncate the sequence $r_1, r_2, \ldots$ at an appropriate spot $N$, and use the apparatus associated with the $(m,p;1,r_N)$ urn to calculate and express $G(m,p;1,r)$. Suppose we have an urn with $m$ “−1” balls and $p$ “+1” balls, this time with $r$ positive and irrational. We shall use a rational approximation $y/x$ to $r$ with $y/x > r$ (though we could just as well use $y/x < r$).

Like the urns with $t/s$ rational, the permanent gains are made in the same manner. For each up-crossing and each down-crossing there is an associated gain, with the sum of those gains (along, possibly, with the initial weight of the urn) equal to the gains made over all realizations. We then divide by $\binom{m+p}{p}$ to calculate $G(m,p;1,r)$.

Since $r$ is irrational, each element in the set 

$$\{kr - j : k, j \text{ nonnegative integers}\}$$

has a unique representation. (Otherwise, we deduce $r$ is rational.) As a result, the cycle of crossings is infinite for irrational $r$, whereas there is a regular repeating cycle of crossings (of length $t + s$) for the $(m,p;s,t)$ urn when $s$ and $t$ are positive integers. Each crossing here is unique, so the methods used with rational $r$ do not seem to be of any direct use. Upon closer inspection, though, we note that if $x + y$ is bigger than $m + p$, the $(m,p;x,y)$ urn (the $(m,p;1,y/x)$ urn) does not complete the full cycle, i.e. each crossing is unique.

With the cycle problem taken care of, we now focus on replication of the crossings, in the sense that the correct crossing (up or down) occurs at the right position. That is, we want our rational approximation $y/x$ to $r$ to satisfy

$$\frac{ky}{x} - j \geq 0 \text{ if and only if } kr - j \geq 0, \ 0 \leq j \leq m, \ \text{and} \ 0 \leq k \leq p. \ (4.10)$$

Note that if we have $y/x > r$, then we are halfway there: $k(y/x) - j > kr - j \geq 0$, so our concern is choosing $y/x > r$ so that $kr - j < k(y/x) - j < 0$. This means that $j - kr > (j - kr) - (j - k(y/x)) = k(y/x - r)$. If we replace, on the right-hand side, $k$ with $p$, we will still have the above equation satisfied. Setting

$$\epsilon = \min\{j - kr : 0 \leq j \leq m, \ 0 \leq k \leq p, \ j - kr > 0\}$$

and choosing $y/x$ so that the following conditions are satisfied:
Let \( K(A_1) - (A_3) \) satisfying \( b_i \) if there is an up-crossing to \( a_i \) with \( b_i y - a_i x = i \). Dividing through by \( x \), for the \( (m, p; 1, y/x) \) urn, an up-crossing to \( i/x \) gives \( b_i + y/x \) balls and \( a_i \) balls. This is in direct correspondence with an up-crossing in the \( (m, p; 1, r) \) urn with \( b_i \) balls and \( a_i \) balls left to draw. That is, the permanent gain associated with the \( (m, p; 1, r) \) urn is

\[
b_i r - a_i = b_i \left( r - \frac{y}{x} \right) + b_i \left( \frac{y}{x} - a_i \right) = b_i \left( r - \frac{y}{x} \right) + \frac{i}{x}.
\]

Similarly, with a down-crossing to \( -j \) in the \( (m, p; y, x) \) urn, we have the permanent gain of

\[
a_{-j} - b_{-j} r = a_{-j} - b_{-j} \left( \frac{y}{x} \right) + b_{-j} \left( \frac{y}{x} - r \right) = \frac{j}{x} + b_{-j} \left( \frac{y}{x} - r \right)
\]

associated with the \( (m, p; 1, r) \) urn. Therefore, we have the following:

**Theorem 4.17.** Suppose \( r > 0 \) is irrational, \( pr - m < 0 \), and that \( x \) and \( y \) are positive integers satisfying (A1) - (A3). Define \( a_i \), \( b_i \) and \( c_i \) so that \( a_i y - b_i x = i \), \( a_i + b_i = c_i < x + y \) for \(-y \leq i \leq x\). Let \( K \) be the least positive residue of \( py \) modulo \( q \). With the relation \( R \) as defined in (4.2), for \(-y \leq i \leq x \) set \( \chi(i) = 1 \) if \( i \not\in K \), and set \( \chi(i) = 0 \) otherwise. Then

\[
\begin{align*}
(m + p) G(m, p; 1, r) &= \sum_{i=1}^{y} \chi(i) \left( \frac{i}{x} + b_i \left( r - \frac{y}{x} \right) \right) \left( \frac{c_i}{b_i} \right) \left( \frac{m + p - c_i - 1}{p - b_i} \right) \\
&\quad + \sum_{j=0}^{x} \chi(-j) \left( \frac{j}{x} + b_{-j} \left( \frac{y}{x} - r \right) \right) \left( \frac{c_{-j}}{b_{-j}} \right) \left( \frac{m + p - c_{-j} - 1}{p - b_{-j} - 1} \right).
\end{align*}
\]

We can similarly construct a formula for \( G(p, m; r, 1) \) with \( pr - m > 0 \), via the Antiurn Theorem, by observing that \( G(p, m; r, 1) = r G(p, m; 1, 1/r) \). We omit the details.

### 4.5. The Relationship Between Gain and Maximum Weight

We use the now-expanded notation \( Q_{m,p}^{(n,t)}(\ell) \) to denote the number of realizations from the \( (m, p; s, t) \) urn for which \( X_n(\omega) \geq \ell \) for some \( n \). (In the context of earlier results like Theorem 3.26, \( Q_{m,p}(\ell) = Q_{m,p}^{(1,1)}(\ell) \).)
Theorem 4.18. For any nonnegative \(s, t\), and integers \(m, p, \ell \geq 0\), using any optimal strategy we have

\[
P(\text{player gains at least } \ell) = \binom{m + p}{p}^{-1} Q_{m,p}^{(s,t)}(\ell).
\]

Proof. We show that the number of realizations for which the player gains \(\ell\) equals \(Q_{m,p}^{(s,t)}(\ell) - Q_{m,p}^{(s,t)}(\ell + 1)\), the number of realizations for which \(\max_n X_n = \ell\). From this the result immediately follows.

We need only to show the result for the two main strategies. Suppose that the zero-bet strategy is used. We divide each realization \(\omega\) into two parts, \(\omega^+\) and \(\omega^-\). The part \(\omega^-\) shall consist of the balls accepted, and \(\omega^+\) shall consist of the balls not accepted. In both, the positions of the balls relative to each other are preserved. We then form the realization \(\hat{\omega} = \omega^+ (\omega^-)^R\). The breakdown of a realization from the \((8, 5; 2, 3)\) urn is depicted in Figure 4.2.

![Figure 4.2](image)

**Figure 4.2.** For this realization \(\omega\) from the \((8, 5; 2, 3)\) urn, the dashed portions form \(\omega^+\), and the solid portions form \(\omega^-\) when the zero-bet strategy is used. This realization gains six using the zero-bet strategy. The mapped realization \(\hat{\omega} = \omega^+ (\omega^-)^R\) (not pictured) reaches a maximum weight of six, after six balls have been drawn.

Suppose \(\omega\) results in a gain of \(\ell\) with the zero-bet strategy. We claim \(\max_n X_n(\hat{\omega}) = \ell\). Clearly, \(\hat{\omega}\) takes a weight at least \(\ell\), and actually takes the weight \(\ell\), since \((\omega^-)^R\) contains all of the accepted balls, has weight \(\ell\), and ends the realization \(\hat{\omega}\). Two simple observations show that the maximum weight is \(\ell\). Call a maximal string of consecutive bets a **betting session**, and similarly define a **passing session**. The part \(\omega^-\) is therefore the concatenation of all of the betting sessions of \(\omega\), while \(\omega^+\) is...
the concatenation of the passing sessions. For each betting session, the urn weight is nonnegative at the beginning (in $\omega$), and the urn weight is negative at the end, except for possibly the last session, for which the urn weight is nonnegative at the end. Viewing $\omega^-$ as its own realization, $\min_n X_n(\omega^-) = 0$, with this minimum achieved at the end of $\omega^-$. Reversing, $\max_n X_n((\omega^-)^R) = \ell$, with this weight attained at $n = 0$. Similarly, each passing session begins with a negative weight and ends with a nonnegative weight. Therefore, as its own realization, $\max_n X_n(\omega^+) = \ell - (pt - ms)$, with this maximum weight achieved only at the end. In the context of $\hat{\omega}$, the urn weight at the conclusion of $\omega^+$ is $\ell$, and furthermore this is the first time the weight $\ell$ is taken. Therefore, $\max_n X_n(\hat{\omega}) = \ell$.

To complete the proof, we show that the mapping is a one-to-one correspondence. Suppose $\hat{\omega}$ satisfies $\max_n X_n(\hat{\omega}) = \ell$. We find the first time the weight $\ell$ is taken, and split $\hat{\omega}$ into two parts. The balls preceding this point we shall call $\omega^+$. The balls following this point are first reversed, and then we call it $\omega^-$. We then construct the realization $\omega$ by the following deterministic construction:

Starting with $X_0 = pt - ms$, we take the $n^{th}$ ball from $\omega^+$ (without replacement) if $X_{n-1} < 0$, otherwise we take the $n^{th}$ ball from $\omega^-$, until all balls have been used. (Note that the properties of $\omega^+$ and $\omega^-$ prevent a scenario in which, for example, $X_{n-1} < 0$ and $\omega^+$ is empty.) By construction, all of the accepted balls from $\omega$ come from $\omega^-$, while all of the balls not accepted come from $\omega^+$. Therefore, $\omega$ will gain $\ell$ via the zero-bet strategy.

Note that the segments $\omega^-$ and $\omega^+$ switch roles, and the strategy switches to the zero-pass strategy, when the antiurn map is applied. Thus, the result under the zero-pass strategy now follows from the zero-bet result and the Antiurn Theorem. If we want a more direct way, we can use the same construction as before. Create $\omega^-$, $\omega^+$, and $\hat{\omega}$ in the same manner, only this time in association with the zero-pass strategy. The only major difference is to find the last time the weight $\ell$ is taken in $\hat{\omega}$ before splitting. We omit the remaining details. □

**Corollary 4.19.** The distribution of the gain for the $(m, p; s, t)$ urn does not depend on the choice of optimal strategy.

Unfortunately, $Q_{m,p}^{(s,t)}(\ell)$, under general circumstances, appears extremely difficult to calculate. $Q_{m,p}^{(s,t)}(\ell) - Q_{m,p}^{(s,t)}(\ell + 1)$ can be calculated using the generalized ballot numbers discussed in Chapter 6, but a formula for the expression would be very complicated, and in addition will not be explicit in general.
5. ASYMPTOTICS

We now examine what kind of gain can be expected for large values of \( m \) and \( p \) under various circumstances. Throughout this section, we shall assume that \( s \) and \( t \) are nonnegative integers with \( \gcd(s, t) = 1 \), and \( ms - pt = h(t + s) \geq 0 \), unless specifically stated otherwise.

Recall the binomial form for \( G(m, p; 1, t) \), from Theorem 3.15:

\[
G(m, p; 1, t) = \frac{t}{2} + \left( \frac{m + p}{p} \right)^{-1} \frac{t}{2} \sum_{k=0}^{p} \binom{m + p + 1}{k} \left( \frac{t}{s} \right)^{p-k}.
\]

When \( s > 1 \), we have the more unwieldy crossings form given with Theorem 4.10. Our main task will be to mold \( G(m, p; s, t) \) into a form more like (5.1), namely the sum

\[
\left( \frac{m + p}{p} \right)^{-1} \frac{t}{2} \sum_{k=0}^{p} \binom{m + p + 1}{k} \left( \frac{t}{s} \right)^{p-k},
\]

which we shall call the pseudo-binomial form, as it is not an accurate form for \( G(m, p; s, t) \). Most of the time, it will be close enough. Let

\[
b(n, q, k) = \binom{n}{k} q^k (1 - q)^{n-k}, \quad B(n, q, m) = \sum_{k=0}^{m} b(n, q, k).
\]

When \( 0 \leq q \leq 1 \), \( b(n, q, k) \) and \( B(n, q, m) \) are related to a binomial distribution. Here,

\[
\sum_{k=0}^{p} \binom{m + p + 1}{k} \left( \frac{t}{s} \right)^{p-k} = \frac{(t+s)^{m+p+1}}{s^p t^{m+1}} B \left( m + p + 1, \frac{s}{t+s}, p \right).
\]

In this form, we may use tools associated with binomial distributions. This includes the Central Limit Theorem, which we state now.

**Theorem 5.1.** (Central Limit Theorem) Let \( \{X_n\} \) be a sequence of independent identically distributed random variables with \( 0 < \text{Var}(X_n) = \sigma^2 < \infty \) and common mean \( \mu \). Let \( S_n = \sum_{i=1}^{n} X_i \), \( n = 1, 2, \ldots \) Then for any real \( x \),

\[
\lim_{n \to \infty} P \left( \frac{S_n - n\mu}{\sigma \sqrt{n}} \leq x \right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} \, dt.
\]
As a consequence, for a binomial distribution $B(n, q)$ with $n$ trials, probability of success $q$, mean $nq = \mu$, and variance $nq(1-q) = \sigma^2 < \infty$, we have for $n$ large enough that

\begin{equation}
P\left( \frac{B - \mu}{\sigma} \leq x \right) \sim \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} \, dt.
\end{equation}

We shall denote the distribution function for the standard normal as

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} \, dt.$$ 

We shall frequently use Stirling’s Formula as well:

$$z! = \left( \frac{z}{e} \right)^z \sqrt{2\pi z} \left(1 + o\left(\frac{1}{z}\right)\right).$$

In particular, we use it to analyze the total number of realizations, $\binom{m+p}{p}$.

**Lemma 5.2.** Suppose $ms - pt = h(t+s) = o(\max\{m, p\})$, and let $A$ be a large constant. Then for $p$ large enough,

$$\binom{m+p}{p} = \left( \frac{t+s}{s} \right)^p \left( \frac{t+s}{t} \right)^m \sqrt{\frac{t+s}{2\pi pt}} \left(1 + \frac{h}{p}\right)^p \left(1 - \frac{h}{m}\right)^m (1 + O(1/A)).$$

**Proof.**

$$\binom{m+p}{p} = \sqrt{\frac{m+p}{2\pi mp}} \left( \frac{m+p}{p} \right)^p \left( \frac{m+p}{m} \right)^m (1 + O(1/A)),$$

and observe that

$$\frac{m+p}{p} = \frac{t+s}{s} \left(1 + \frac{h}{p}\right), \quad \frac{m+p}{m} = \frac{t+s}{t} \left(1 - \frac{h}{m}\right).$$

\[\square\]

**5.1. Analyzing the Pseudo-Binomial Form.** We will now analyze the pseudo-binomial form

$$\binom{m+p}{p}^{-1} \frac{1}{2} \frac{(t+s)^{m+p+1}}{s^{t+m}} B\left( m + p + 1, \frac{s}{t+s}, p \right).$$

We start with the assumption that $ms - pt = O(\sqrt{p})$. The mean and variance of the binomial random variable $B(m + p + 1, s/(t+s))$ are, respectively,

$$\mu = \frac{s(m + p + 1)}{t + s}, \quad \text{and} \quad \sigma^2 = \frac{st(m + p + 1)}{(t + s)^2}.$$ 

Normalizing, we have

$$B\left( m + p + 1, \frac{s}{t+s}, p \right) = P\left( \frac{B - \mu}{\sigma} \leq \frac{p - \mu}{\sigma} \right).$$
If \((ms - pt)p^{-1/2} \to \alpha\), with \(\alpha < \infty\), then

\[ p - \mu = p - \left( p + h + \frac{s}{t + s} \right) = -h(1 + o(1)), \quad \text{and} \quad \sigma = \sqrt{\frac{t(p + h)}{t + s} + \frac{st}{(t + s)^2}} = \sqrt{\frac{pt}{t + s}} (1 + o(1)). \]

Therefore, \(\frac{p - \mu}{\sigma} = \frac{-\alpha}{\sqrt{t(t + s)}} (1 + o(1))\), and it follows from (5.3) that as \(p \to \infty\),

\[
\frac{1}{2} \left( \frac{t + s}{pt} \right)^{m + p + 1} B\left(m + p + 1, \frac{s}{t + s}, p\right) \sim \frac{1}{2} \left( \frac{t + s}{pt} \right)^{m + p + 1} \Phi\left(\frac{-\alpha}{\sqrt{t(t + s)}}\right).
\]

We now give the asymptotic form of (5.2) with \((ms - pt)p^{-1/2} \to \alpha < \infty\), which will also be the asymptotic form of \(G(m, p; s, t)\) with those same conditions.

**Lemma 5.3.** Suppose that as \(p \to \infty\), \((ms - pt)p^{-1/2} \to \alpha < \infty\). Then asymptotically (5.2) equals

\[
\frac{\sqrt{2\pi pt(t + s)}}{2} \exp\left(\frac{\alpha^2}{2t(t + s)}\right) \Phi\left(\frac{-\alpha}{\sqrt{t(t + s)}}\right).
\]

**Proof.** Combining (5.4) with the result of Lemma 5.2, (5.2) equals

\[
\frac{\sqrt{2\pi pt(t + s)}}{2} \Phi\left(\frac{-\alpha}{\sqrt{t(t + s)}}\right) \left(1 + \frac{h}{p}\right)^{-p} \left(1 - \frac{h}{m}\right)^{-m} \left(1 + o(1)\right).
\]

Thus we are left to show

\[
\left(1 + \frac{h}{p}\right)^{-p} \left(1 - \frac{h}{m}\right)^{-m} \sim \exp\left(\frac{\alpha^2}{2t(t + s)}\right).
\]

The following inequalities hold for all \(x \geq 0\):

\[ 1 + x \geq \exp\left(\frac{x - x^2}{2}\right), \quad 1 - x \leq \exp\left(-x - \frac{x^2}{2}\right). \]

Then

\[ \left(1 + \frac{h}{p}\right)^p \geq \exp\left(h - \frac{h^2}{2p}\right), \quad \left(1 - \frac{h}{m}\right)^m \leq \exp\left(-h - \frac{h^2}{2m}\right). \]

Also, given \(\epsilon > 0\) and sufficiently small \(x\) we have

\[ 1 + x \leq \exp\left(x - \frac{x^2}{2}(1 - \epsilon)\right), \quad 1 - x \geq \exp\left(-x - \frac{x^2}{2}(1 + \epsilon)\right). \]

Since \(h = o(p)\) and \(h = o(m)\), for any \(\epsilon > 0\) there is a \(M\) such that \(\min\{m, p\} > M\) implies

\[ \left(1 + \frac{h}{p}\right)^p \leq \exp\left(h - \frac{h^2}{2p}(1 - \epsilon)\right), \quad \left(1 - \frac{h}{m}\right)^m \geq \exp\left(-h - \frac{h^2}{2m}(1 + \epsilon)\right). \]
Then for sufficiently large $p$ we have
\[
\exp\left(-\frac{h^2}{2} \left( \frac{1}{m} + \frac{1}{p} \right) (1 + \epsilon) \right) \leq \left( 1 + \frac{h}{p} \right)^{-p} \left( 1 - \frac{h}{m} \right)^{-m} \leq \exp\left(-\frac{h^2}{2} \left( \frac{1}{m} + \frac{1}{p} \right) (1 + \epsilon) \right).
\]
Now (5.5) follows since
\[
\frac{h^2}{2} \left( \frac{1}{m} + \frac{1}{p} \right) = \frac{1}{2} \cdot \frac{h^2}{p} \cdot \frac{m + p}{m} \sim \frac{1}{2} \cdot \frac{\alpha^2}{(t + s)^2} \cdot \frac{t + s}{t} = \frac{\alpha^2}{2t(t + s)},
\]
and we can set $\epsilon$ to be as small as we wish. Therefore, we have completed the proof. \[\square\]

If $\sqrt{p} = o(ms - pt)$, we are in trouble, as we will have $\mu \to -\infty$. Normal approximation will thus fail us, so another approach is needed. The following theorems will take care of our remaining cases. Since we are unable to find proofs in the literature, we shall provide proofs here. The proof of Lemma 5.4 is provided by the anonymous referee to [21] (2006), and will be used for the case with $\sqrt{p} = o(ms - pt)$ and $ms - pt = o(p)$.

**Lemma 5.4.** Assume that $m \leq nq$ (where $q$ can depend on $n$) and let $r = 1 - q$ and $h = nq - m$.

Suppose that the following conditions hold as $n \to \infty$:

(C1) $nqr/h^2 \to 0$,

(C2) $nqr/h \to \infty$.

Then as $n \to \infty$ we have
\[
B(n, q, m) \sim \frac{m(n - m)}{nh} b(n, q, m).
\]
That is,
\[
B(n, q, m) = \sum_{0 \leq k \leq m} \binom{n}{k} q^k r^{n-k} \sim \frac{nqr}{h} \binom{n}{m} q^m r^{n-m}.
\]

**Proof.** Note that, for all $n, q, k$, we have
\[
\frac{b(n, q, k - 1)}{b(n, q, k)} = \frac{kr}{q(n - k + 1)}.
\]
This ratio is bounded above by $x = nr/q(n - m + 1)$ when $k \leq m$. Thus $b(n, q, m - j) \leq b(n, q, m) x^j$.

We have
\[
\frac{1 - h}{1 + \frac{h+1}{nr}} = 1 - \frac{h}{nqr} \frac{1}{1 + \frac{h+1}{nr}},
\]

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and thus $1 - x \sim h/nqr$. Therefore,

$$B(n, q, m) \leq b(n, q, m)(1 + x + x^2 + \cdots) = \frac{b(n, q, m)}{1 - x} \sim b(n, q, m)\frac{nqr}{h}.$$  

Note also that if $k \geq m - z$ then

$$\frac{kr}{q(n - k + 1)} \geq \frac{(m - t)r}{q(n - m + t + 1)},$$

the latter we shall call $y$. Then for $j \leq z$ we have $b(n, q, m - j) \geq b(n, q, m)y^j$. With

$$y = \frac{1 - (h + t)q}{1 + \frac{h + t + 1}{nr}} = 1 - \frac{h + t}{nqr} + \frac{1}{nr},$$

we choose $z$ so that $z = o(h)$ and $zt/nqr \to \infty$, then $1 - y \sim h/nqr$ and $y^t = o(1)$. Thus we conclude that

$$B(n, q, m) \geq b(n, q, m)(1 + y + \cdots + y^t) = b(n, q, m)\frac{1 - y^{t+1}}{1 - y} \sim b(n, q, m)\frac{nqr}{h}.$$  

\[\Box\]

**Lemma 5.5.** Suppose that as $p \to \infty$, $(ms - pt)p^{-1/2} \to \infty$ and $(ms - pt)p^{-1} \to 0$. Then asymptotically (5.2) equals

$$t(t + s)\frac{p}{ms - pt}.$$  

**Proof.** With the conditions $(ms - pt)p^{-1/2} \to \infty$ and $(ms - pt)p^{-1} \to 0$, we have

$$\frac{st(m + p + 1)}{ms - pt + s} \to 0, \quad \frac{st}{t + s} \frac{m + p + 1}{ms - pt + s} \to \infty,$$

so conditions (C1) and (C2) of Lemma 5.4 are satisfied. Therefore,

$$B\left(m + p + 1, \frac{s}{t + s}, p \right) \sim \frac{st(m + p + 1)}{(t + s)(ms - pt + s)} \left(\frac{m + p + 1}{p}\right) \left(\frac{s}{t + s}\right)^p \left(\frac{t}{t + s}\right)^{m+1},$$

and (5.2) thus asymptotically

$$\frac{t}{2} \frac{(m + p + 1)st}{(t + s)(ms - pt + s)} \frac{m + p + 1}{m + 1} \sim \frac{t(t + s)}{2} \frac{p}{ms - pt},$$

since $\frac{m + p}{m} \sim \frac{t + s}{t}$ and $m + p \sim p\frac{t + s}{s}$.  

\[\Box\]

If $(ms - pt)p^{-1}$ tends to a constant greater than zero, that is, $ms/pt \to \lambda > 1$, then we cannot transition from the crossings form of $G(m, p; s, t)$ to (5.2) without significant error, outside of the
case $s = 1$, where we have the binomial form. Thus, the result below we shall only apply when $s = 1$.

**Lemma 5.6.** Let $m = nq - h$ and $h = vnq$ (where $v$ can depend on $n$). Assume that the following conditions are satisfied. As $n \to \infty$,

1. $(C3)$ $m \to \infty$, $n - m \to \infty$,
2. $(C4)$ $v > \epsilon$ for some constant $\epsilon > 0$.

Then as $n \to \infty$,

$$B(n, q, m) \sim \frac{1}{v} b(n, q, m).$$

**Proof.** Let

$$t_k = b(n, p, k) = \binom{n}{k} p^k q^{n-k}, \quad r_k = \frac{t_k}{t_{k-1}}.$$ 

Then

$$r_k = \frac{(n-k+1)p}{kq}.$$

For $k \leq m$, we have

$$r_k \geq \frac{(n-m+1)p}{mq} \geq \frac{(n-m)p}{mq} = \frac{(nq-h)p}{(np-h)q} = \frac{1+pv}{1-qv}.$$

We write

$$r = 1 - \frac{v}{1+pv} \leq 1 - \frac{v}{1+v} = \frac{1}{1+v} < \frac{1}{1+\epsilon} < 1,$$

where the second to last inequality holds for all large $n$. Thus, for $k \leq m$, we have

$$r_k \geq 1/r > 1.$$

We therefore have for $k = 1, 2, \ldots, m$ that

$$t_{m-k} \leq r^k t_m.$$

Hence,

$$B(n, p, m) = \sum_{k=0}^{m} t_{m-k} \leq \sum_{k=0}^{m} r^k t_m \leq \frac{t_m}{1-r} = t_m \frac{1+pv}{v}.$$
To get the corresponding asymptotic lower bound, we let \( \eta \) be such that as \( n \to \infty \), \( \eta \) satisfies the following:

\[
\eta \to \infty, \quad \eta = o(n - m), \quad \text{and} \quad \eta = o(m).
\]

Since

\[
B(n,p,m) = \sum_{k=0}^{m} t_k \geq \sum_{m-\eta \leq k \leq m} t_k,
\]

It is sufficient to show that as \( n \to \infty \),

\[
(5.9) \quad \sum_{m-\eta \leq k \leq m} t_k \geq \frac{t_m}{1 - r}.
\]

For \( m - \eta \leq k \leq m \), we have

\[
r_k = \frac{(n-k+1)p}{kq} \leq \frac{(n-m+\eta+1)p}{(m-\eta)q} = \left( \frac{(n-m)p}{mq} \right) \left( \frac{1 + \frac{\eta+1}{n-m}}{1 - \frac{\eta}{m}} \right).
\]

We write

\[
R = r \times \left( \frac{1 - \frac{\eta}{m}}{1 + \frac{\eta+1}{n-m}} \right)
\]

We note that since \( \eta/m = o(1) \) and \( (\eta+1)/(n-m) = o(1) \), we have \( R \sim r \), as \( n \to \infty \). Also, \( R < 1 \) for all large \( n \) because of (5.6). Therefore, since

\[
t_{m-k} \geq R^k t_m, \quad 0 \leq k \leq \eta,
\]

we have as \( n \to \infty \) that

\[
\sum_{m-\eta \leq k \leq m} t_k \geq \sum_{0 \leq k \leq \eta} R^k t_m \geq \frac{(1-R^\eta)t_m}{1-R} \sim \frac{t_m}{1 - r}.
\]

This shows (5.9) and our proof is complete. \( \square \)

**Lemma 5.7.** Suppose that, as \( p \to \infty \), \( ms/pt \to \lambda > 1 \). Then (5.2) is asymptotically

\[
\frac{t}{2} + \frac{t+s}{2(\lambda - 1)}.
\]

**Proof.** With

\[
v = \frac{t+s}{st} \cdot \frac{ms - pt + s}{m + p + 1} \sim \frac{(t+s)(\lambda - 1)}{\lambda t + s} > 0,
\]

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conditions (C3) and (C4) of Lemma 5.6 are met, and therefore

\[ B\left( m + p + 1, \frac{1}{t + 1}, p \right) \sim \frac{\lambda t + s}{(t + s)(\lambda - 1)} \left( 1 + \frac{s(\lambda - 1)}{\lambda t + s} \right) \left( \frac{m + p + 1}{p} \right) \frac{t^{m+1}s^p}{(t + s)^{m+p+1}} \]

\[ = \frac{\lambda}{\lambda - 1} \left( \frac{m + p + 1}{p} \right) \frac{t^{m+1}s^p}{(t + s)^{m+p+1}}. \]

Then (5.2) becomes

\[ \left( \frac{m + p}{p} \right)^{-1} t \frac{\lambda}{\lambda - 1} \left( \frac{m + p + 1}{p} \right) = t \frac{\lambda}{\lambda - 1} \frac{m + p + 1}{m + 1} \sim \frac{\lambda t + s}{2(\lambda - 1)} = \frac{t + s}{2(\lambda - 1)}. \]

5.2. Transitioning to the Pseudo-Binomial Form. In this section, we give a general outline on how we shall re-mold the double-double (zero-bet) sum formula

\[ \left( \frac{m + p}{p} \right)^{-1} \sum_{i=0}^{s-1} \sum_{k=0}^{M(i)} \left( k(t + s) + c_i \right) \left( m + p - k(t + s) - c_i - 1 \right) \]

\[ + \sum_{j=1}^{t} \sum_{k=0}^{M(-j)} \left( k(t + s) + c_{-j} \right) \left( m + p - k(t + s) - c_{-j} - 1 \right) \]

toward (5.2). We shall do this via the intermediary

\[ \left( \frac{m + p}{p} \right)^{-1} t \frac{\lambda}{\lambda - 1} \left( \frac{m + p + 1}{p} \right) \left( \frac{m + p - k(t + s) - c_i - 1}{p - ks - b_i} \right) \]

\[ + \sum_{j=1}^{t} \sum_{k=0}^{M(-j)} \left( k(t + s) + c_{-j} \right) \left( \frac{m + p - k(t + s) - c_{-j} - 1}{p - ks - b_{-j} - 1} \right) \]

which equals the sum

\[ \left( \frac{m + p}{p} \right)^{-1} t \frac{\lambda}{\lambda - 1} \left( \frac{m + p + 1}{p} \right) \left( \frac{t}{s} \right)^{p-k} \]

by Theorem 3.1. Part of this transition involves a tail exchange. Let \( A \) be a large constant, and define

\[ T_A = \{ k: 0 \leq k \leq A \text{ or } p/s - A \leq k \leq p/s \}, \quad T_A = \{ k: 0 \leq k \leq As \text{ or } p - As \leq k \leq p \}. \]

We shall assume a bit of flexibility on the upper end - be the term “p/s” it should be understood to actually mean “\( M(i) \)”.

**Lemma 5.8.** Let \( A \) be a large constant. Then for each \( i \), \( 0 \leq i \leq s - 1 \) and \( 1 \leq j \leq t \) we have

\[ \left( \frac{m + p}{p} \right)^{-1} \sum_{k \in T_A} \left( k(t + s) + c_i \right) \left( m + p - k(t + s) - c_i - 1 \right) \left( \frac{t}{s} \right)^{p-k} = O(A) \]
\[
\left( \frac{m + p}{p} \right)^{-1} \sum_{k \in \mathcal{T}_A} \left( k(t + s) + c_{-j} \right) \left( \frac{m + p - k(t + s) - c_{-j} - 1}{p - ks - b_{-j} - 1} \right) = O(A).
\]

**Proof.** This follows since both
\[
\left( \frac{m + p}{p} \right)^{-1} \left( k(t + s) + c_i \right) \left( \frac{m + p - k(t + s) - c_i - 1}{p - ks - b_i} \right)
\]
and
\[
\left( \frac{m + p}{p} \right)^{-1} \left( k(t + s) + c_{-j} \right) \left( \frac{m + p - k(t + s) - c_{-j} - 1}{p - ks - b_{-j} - 1} \right)
\]
are probabilities, and we are removing at most 2A terms for each.

Since the number of such sums is dependent only on the constants \( s \) and \( t \), the sum of all the tails over \( i \) and \( j \) are still \( O(A) \). The replacement tail is
\[
\left( \frac{m + p}{p} \right)^{-1} t \sum_{k \in \mathcal{T}_A} \left( \frac{k + s}{k} \right) \left( \frac{m + p - k + s}{p - k} \right).
\]
We prove that this tail also is similarly bounded in terms of \( A \):

**Lemma 5.9.** Let \( A \) be a large constant. Then
\[
\left( \frac{m + p}{p} \right)^{-1} \sum_{k \in \mathcal{T}_A} \left( \frac{k + s}{k} \right) \left( \frac{m + p - k + s}{p - k} \right) = O(A).
\]

**Proof.** For this sum, let \( a \) and \( b \) be arbitrary nonnegative real numbers and consider the sequence
\[
\left( \frac{a + p - k}{p - k} \right) \left( \frac{b + k}{k} \right), \quad 0 \leq k \leq m.
\]
This sequence is either increasing or decreasing, depending on whether \( a > b \) or not. Thus, for any nonnegative real numbers \( a, b \), we have
\[
\left( \frac{a + p - k}{p - k} \right) \left( \frac{b + k}{k} \right) \leq \max \left\{ \left( \frac{a + p}{p} \right), \left( \frac{b + p}{p} \right) \right\} \leq \left( \frac{a + b + p}{p} \right).
\]
Taking \( a = m - kt/s \), \( b = kt/s \), we have that
\[
\left( \frac{m + p}{p} \right)^{-1} \left( \frac{k + s}{k} \right) \left( \frac{m + p - k + s}{p - k} \right) \leq 1
\]
for \( 0 \leq k \leq p \). Thus the sum over \( \mathcal{T}_A \) is \( O(A) \).
The dominant part of the double-double sums lies within the remaining range of $k$. Further trimming of the “secondary tail” shall be dealt with case by case.

Once we have identified the dominant range of the double-double sum formula (5.10), we begin work to reduce that sum to (5.2). We begin by “folding” the two double sums together. Then we shall be a slight adjustment away from being able to rewrite the resulting double sum in terms of the pseudo-binomial form (5.2).

We shall fold the two double sums together by asserting that for small constants $a$ and $b$ with $0 \leq a + b \leq s + t$, under appropriate conditions we have

\[
\binom{k(t + s) + a + b}{ks + b} \binom{m + p - k(t + s) - a - b}{p - ks - b} \sim \binom{k(t + s)}{ks} \binom{m + p - k(t + s)}{p - ks}.
\]

Our goal is not to rid ourselves of the constants $a_i$, $b_i$ and $c_i$, rather, we want to exchange those within the down-crossing sums ($j < 0$) with another set corresponding to an up-crossing sum ($i \geq 0$). In this way we shall “fold” the down-crossings sum into the up-crossings sum.

We have

\[
(5.12) \quad \binom{k(t + s) + a + b}{ks + b} = \binom{k(t + s)}{ks} \frac{(k(t + s) + a + b)_{a+b}}{(kt + a)_{a}(ks + b)_{b}} \\
\leq \binom{k(t + s)}{ks} \frac{(k+1)(t + s)_{a+b}}{(ks)^b (kt)^a} = \binom{k(t + s)}{ks} \left( \frac{t + s}{s} \right)^b \left( \frac{t + s}{t} \right)^a \left( 1 + \frac{1}{k} \right)^{a+b}.
\]

Similarly, recalling that $b \leq s$ and $a \leq t$,

\[
(5.13) \quad \binom{k(t + s) + a + b}{ks + b} \geq \binom{k(t + s)}{ks} \frac{(k(t + s))^{a+b}}{(kt + b)^a((k+1)t)^b} \\
= \binom{k(t + s)}{ks} \left( \frac{t + s}{s} \right)^b \left( \frac{t + s}{t} \right)^a \left( 1 - \frac{1}{k+1} \right)^{a+b}.
\]

We work similarly with the other binomial coefficients.

\[
\binom{m + p - k(t + s) - a - b}{p - ks - b} = \binom{m + p - k(t + s)}{p - ks} \frac{(p - ks)_{a+b}(m - kt)_{a+b}}{(m + p - k(t + s))_{a+b}} \\
\leq \binom{m + p - k(t + s)}{p - ks} \frac{(p - ks)_{b}(m - kt)^a}{(m + p - (k+1)(t + s))^{a+b}}.
\]

Writing $ms - pt = h(t + s)$, we have

\[
\frac{m + p - (k+1)(t + s)}{p - ks} = \frac{t + s}{s} \left( 1 + \frac{h - s}{p - ks} \right) \quad \text{and} \quad \frac{m + p - (k+1)(t + s)}{m - kt} = \frac{t + s}{t} \left( 1 - \frac{h + t}{m - kt} \right).
\]
Therefore

\[
\frac{(p - ks)^b(m - kt)^a}{(m + p - (k + 1)(t + s))^{a+b}} = \left(\frac{t + s}{s}\right)^{-b}\left(\frac{t + s}{t}\right)^{-a}\left(1 + \frac{h - s}{p - ks}\right)^{-b}\left(1 - \frac{h + t}{m - kt}\right)^{-a}.
\]

Thus

\[
(5.14)
(m + p - k(t + s) - a - b)\frac{p - ks - b}{p - ks - b} \leq \left(\frac{m + p - k(t + s)}{p - ks}\right)\left(\frac{t + s}{s}\right)^{-b}\left(\frac{t + s}{t}\right)^{-a}\left(1 + \frac{h - s}{p - ks}\right)^{-b}\left(1 - \frac{h + t}{m - kt}\right)^{-a}.
\]

Similarly, we can show that

\[
(5.15)
(m + p - k(t + s) - a - b)\frac{p - ks - b}{p - ks - b} \geq \left(\frac{m + p - k(t + s)}{p - ks}\right)\left(\frac{t + s}{s}\right)^{-b}\left(\frac{t + s}{t}\right)^{-a}\left(1 + \frac{h + s}{p - ks - s}\right)^{-b}\left(1 - \frac{h - t}{m - kt - t}\right)^{-a}.
\]

Define

\[
(5.16) \quad U_1(m, p, s, t) = \left(1 + \frac{1}{k}\right)^{a+b}\left(1 + \frac{h - s}{p - ks}\right)^{-b}\left(1 - \frac{h + t}{m - kt}\right)^{-a}
\]

and

\[
(5.17) \quad L_1(m, p, s, t) = \left(1 - \frac{1}{k + 1}\right)^{a+b}\left(1 + \frac{h + s}{p - ks - s}\right)^{-b}\left(1 - \frac{h - t}{m - kt - t}\right)^{-a},
\]

so that for any \(i\),

\[
(5.18) \quad \left(\frac{k(t + s) + c_i}{ks + b_i}\right)\left(\frac{m + p - k(t + s) - c_i}{p - ks - b_i}\right) \leq \left(\frac{k(t + s)}{ks}\right)\left(\frac{m + p - k(t + s)}{p - ks}\right)U_1(m, p, s, t)
\]

and

\[
(5.19) \quad \left(\frac{k(t + s) + c_i}{ks + b_i}\right)\left(\frac{m + p - k(t + s) - c_i}{p - ks - b_i}\right) \geq \left(\frac{k(t + s)}{ks}\right)\left(\frac{m + p - k(t + s)}{p - ks}\right)L_1(m, p, s, t).
\]
This implies for any \(-t \leq i, j \leq s - 1\):

\[
(5.20) \quad \frac{(k(t + s) + c_j)}{ks + b_j} \left( \frac{m + p - k(t + s) - c_j}{p - ks - b_j} \right) \left( \frac{L_1(m, p, s, t)}{U_1(m, p, s, t)} \right) \leq \frac{(k(t + s) + c_i)}{ks + b_i} \left( \frac{m + p - k(t + s) - c_i}{p - ks - b_i} \right) \leq \frac{(k(t + s) + c_j)}{ks + b_j} \left( \frac{m + p - k(t + s) - c_j}{p - ks - b_j} \right) \left( \frac{L_1(m, p, s, t)}{U_1(m, p, s, t)} \right).
\]

Under the right conditions, we can marginalize \(U_1\) and \(L_1\). In this way we shall be able to “fold” the two double sums into a single double sum. As an added bonus, we can do it in a way that will leave only the two binomial terms inside the sum.

With (5.10) now reduced to a double sum, our next step is to reindex the inner sum with respect to \(b_i\), using Lemma 4.9. With the map \(\theta\) as defined in Lemma 4.9, if \(\theta(i) = b_i = j\), then from \(i = b_it - a_its = jt - a_is\) we have \(a_i = (jt - i)/s\). Therefore,

\[
(5.21) \quad \sum_{i=0}^{s-1} \sum_{k=0}^{M(i)} \left( \frac{(k(t + s) + c_j)}{ks + b_j} \left( \frac{m + p - k(t + s) - c_j}{p - ks - b_j} \right) \right) \leq \sum_{j=0}^{s-1} \sum_{k=0}^{M(i)} \left( \frac{(k + \frac{j}{2})(t + s) - \frac{j}{2}}{ks + j} \left( \frac{m + p - (k + \frac{j}{2})(t + s) - \frac{j}{2}}{p - ks - j} \right) \right),
\]

where \(\theta(i) = b_i = j\). Now we want to remove the two \(i/s\) terms. While basically a “trade” between the two binomial coefficients like the previous exchange, here we have to change every factor instead of discarding a couple. Since we have bottom entries not dependent on \(i\), we can place them to the side and work with the remaining falling factorials. To draw a comparison, we shall use the harmonic numbers. The \(n^{th}\) harmonic number is

\[
(5.22) \quad H(n) = \sum_{k=1}^{n} \frac{1}{k}.
\]

Asymptotically, for large \(n\) we have

\[
H(n) = \ln n + \gamma + o(1/n),
\]

where \(\gamma \approx 0.5772\) is Euler’s constant. Thus for large \(a\) we have

\[
\sum_{k=a+1}^{b} \frac{1}{k} = H(b) - H(a) = \ln \frac{b}{a} + o(1/n).
\]

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We begin with

\[ \left( \left( k + \frac{j}{s} \right) (t + s) - \frac{i}{s} \right)_{ks+j} \quad \text{and} \quad \left( \left( k + \frac{j}{s} \right) (t + s) \right)_{ks+j}. \]

We have a natural pairing of the factors. Taking a generic factor pair, we have

\[
\frac{(k + j/s)(t + s) - i/s - l}{(k + j/s)(t + s) - l} = 1 - \frac{i/s}{(k + j/s)(t + s) - l} \\
\leq \exp \left( -\frac{i/s}{(k + j/s)(t + s) - l} \right) \\
\leq \exp \left( -\frac{i/s}{(k + j/s)(t + s) + (s-i)/s - l} \right),
\]

since for \( 1 \leq i \leq s - 1 \), \((k + j/s)(t + s)\) is not an integer \((k + j/s)(t + s) - i/s\) is). Taking the product over \( 0 \leq l \leq ks + j - 1 \), we have

\[
(5.23) \quad \frac{(k + j/s)(t + s) - i/s}{(k + j/s)(t + s)}_{ks+j} \\
\leq \exp \left[ -\frac{i}{s} \left( H \left( \left( k + \frac{j}{s} \right) (t + s) + \frac{s-i}{s} \right) - H \left( \left( k + \frac{j}{s} \right) t + \frac{s-i}{s} \right) \right) \right].
\]

As for the other set of falling factorials,

\[
(5.24) \quad \frac{(m + p - (k + j/s)(t + s) + i/s)_{p-ks-j}}{(m + p - (k + j/s)(t + s))_{p-ks-j}} \\
\leq \exp \left[ \frac{i}{s} \left( H \left[ m + p - \left( k + \frac{j}{s} \right) (t + s) - \frac{s-i}{s} \right] - H \left[ m - \left( k + \frac{j}{s} \right) t - \frac{s-i}{s} \right] \right) \right],
\]

We can develop corresponding lower bounds by using the same method on the reciprocals.

With favorable circumstances we will be able to erase the \( i/s \) terms without introducing any significant error. We complete the transition to a single sum by noting that

\[
(5.25) \quad \sum_{k=0}^{p} \binom{k + \frac{j+s}{s}}{k} \left( m + p - \frac{k+t+s}{s} \right)_{p-k} = \sum_{j=0}^{s-1} \sum_{k=0}^{M(j)} \binom{(k + j/s)(t + s)}{ks+j} \left( m + p - (k + j/s)(t + s) \right)_{p-ks-j}
\]

by splitting the sum according to the residue of \( k \) modulo \( s \). Now, from (5.11) we shift to (5.2).

This completes the transition to the pseudo-binomial form. Now, we shall go to a case-by-case analysis, where we shall fill in the gaps.
5.3. Urns with $pt - ms = O(\sqrt{p})$. We begin with the urns for which $pt - ms = O(\sqrt{p})$ as $p \to \infty$.

**Theorem 5.10.** Suppose that, as $p \to \infty$, $(ms - pt)/\sqrt{p} \to \alpha$, with $|\alpha| < \infty$. Then, as $p \to \infty$,

$$G(m, p; s, t) \sim \max\{-\alpha\sqrt{p}, 0\} + \frac{\sqrt{2\pi pt(t + s)}}{2} \exp\left(\frac{-\alpha^2}{2t(t + s)}\right) \Phi\left(\frac{-|\alpha|}{\sqrt{t(t + s)}}\right).$$

In particular, if $pt - ms = o(\sqrt{p})$, we have $G(m, p; s, t) \sim \frac{1}{4} \sqrt{2\pi pt(t + s)}$.

**Proof.** We shall prove the formula first for $pt - ms \leq 0$. Then, we shall use the Antiurn Theorem to show that the result also holds when $pt - ms > 0$. Define for constants $A > 0$ and $\delta \in (0, 1)$:

$$U_{A, \delta} = \left\{ k: (1 - \delta)\frac{p}{s} \leq k \leq \frac{p}{s} - A \right\}, \quad U_{A, \delta} = \left\{ k: (1 - \delta)p \leq k \leq p - As \right\}$$

$$D_{A, \delta} = \left\{ k: A \leq k \leq (1 - \delta)\frac{p}{s} \right\}, \quad D_{A, \delta} = \left\{ k: As \leq k \leq (1 - \delta)p \right\}$$

With $pt - ms \leq 0$, we have

$$\left(\frac{m + p}{p}\right)G(m, p; s, t) = \sum_{i=0}^{s-1} \sum_{k=0}^{M(i)} \left(\frac{k(t + s) + c_i}{ks + b_i}\right) \left(\frac{m + p - k(t + s) - c_i - 1}{p - ks - b_i}\right)$$

$$+ \sum_{j=1}^{t} \sum_{k=0}^{M(-j)} j \left(\frac{k(t + s) + c_{-j}}{ks + b_{-j}}\right) \left(\frac{m + p - k(t + s) - c_{-j} - 1}{p - ks - b_{-j} - 1}\right).$$

We assert that we can effectively replace the portion of the sum (5.10) for which $k \in D_{A, \delta}$ with the portion of (5.11) with $k \in D_{A, \delta}$. Indeed, letting $ms - pt = h(t + s)$, from (5.16) and (5.17) we have that for $k \in D_{A, \delta}$ that $U_1(m, p; s, t) = 1 + O(1/A)$ and $L_1(m, p; s, t) = 1 + O(1/A)$. Thus for any $1 \leq j \leq t$ and $0 \leq i \leq s - 1$ and $k \in D_{A, \delta}$:

$$\left(\frac{k(t + s) + c_{-j}}{ks + b_{-j}}\right) \left(\frac{m + p - k(t + s) - c_{-j}}{p - ks - b_{-j}}\right) = (1 + O(1/A)) \left(\frac{k(t + s) + c_i}{ks + b_i}\right) \left(\frac{m + p - k(t + s) - c_i}{p - ks - b_i}\right).$$

Removing the factors

$$\frac{m + p - k(t + s) - c_i}{m - k t - a_i} = \frac{t + s}{t} (1 + o(1)) \quad \text{and} \quad \frac{m + p - k(t + s) - c_{-j}}{p - ks - b_{-j}} = \frac{t + s}{s} (1 + o(1)),$$

we have

$$s \left(\frac{k(t + s) + c_{-j}}{ks + b_{-j}}\right) \left(\frac{m + p - k(t + s) - c_{-j} - 1}{p - ks - b_{-j} - 1}\right)$$

$$= (1 + O(1/A)) t \left(\frac{k(t + s) + c_i}{ks + b_i}\right) \left(\frac{m + p - k(t + s) - c_i - 1}{p - ks - b_i}\right).$$
Since
\[(5.26) \quad \frac{s}{t+s} \left( \sum_{j=1}^{t} j \right) + \frac{t}{t+s} \left( \sum_{i=0}^{s-1} i \right) = \frac{1}{t+s} \left[ \frac{s(t+1)}{2} + \frac{t(s-1)s}{2} \right] = \frac{st}{2}, \]
over \( D_{A,\delta} \) we have for each such \( k \) that
\[
\sum_{i=0}^{s-1} i \left( \frac{(k(t+s) + c_i)}{ks + b_i} \right) \left( \frac{m + p - k(t+s) - c_i - 1}{p - ks - b_i} \right) + \sum_{j=0}^{t} j \left( \frac{(k(t+s) + c_{-j})}{ks + b_{-j}} \right) \left( \frac{m + p - k(t+s) - c_{-j} - 1}{p - ks - b_{-j} - 1} \right) = (1 + O(1/A)) \frac{t}{2} \sum_{i=0}^{s-1} \left( \frac{(k(t+s) + c_i)}{ks + b_i} \right) \left( \frac{m + p - k(t+s) - c_i - 1}{p - ks - b_i} \right),
\]
since we are splitting \( st/2 \) into \( s \) parts. Reindexing with respect to \( b_i = j \) by Lemma 4.9, we rewrite the sum
\[
\sum_{i=0}^{s-1} \left( \frac{(k(t+s) + c_i)}{ks + b_i} \right) \left( \frac{m + p - k(t+s) - c_i - 1}{p - ks - b_i} \right)
\]
as the sum
\[
\sum_{j=0}^{s-1} \left( \frac{(k + \frac{j}{s})(t+s) - \frac{i}{s}}{ks + j} \right) \left( \frac{m + p - (k + \frac{j}{s})(t+s) + \frac{i}{s}}{p - ks - j} \right).
\]
Since \( k \geq A \), we have
\[
H\left( \left( k + \frac{j}{s} \right)(t+s) + \frac{s-i}{s} \right) - H\left( \left( k + \frac{j}{s} \right) t + \frac{s-i}{s} \right) = O(1/A) + \ln \left( \frac{\left( k + \frac{j}{s} \right)(t+s) + \frac{s-i}{s}}{\left( k + \frac{j}{s} \right) t + \frac{s-i}{s}} \right) = O(1/A) + (1 + o(1)) \ln \frac{t+s}{t}.
\]
Thus from (5.23) we have
\[(5.27) \quad \exp \left[ -\frac{i}{s} \left( H \left[ \left( k + \frac{j}{s} \right)(t+s) + \frac{s-i}{s} \right] - H \left[ \left( k + \frac{j}{s} \right) t + \frac{s-i}{s} \right] \right) \right] = \exp \left( -\frac{i}{s} \left( O(1/A) + (1 + o(1)) \ln \frac{t+s}{t} \right) \right) = \left( \frac{t+s}{t} \right)^{-ij/s} (1 + O(1/A)),\]
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with the same result from the lower bound. As for the other set, since \( k \leq (1 - \delta)p/s \) we have

\[
H\left(m + p - \left(k + \frac{j}{s}\right)(t + s) - \frac{s - i}{s}\right) - H\left(m - \left(k + \frac{j}{s}\right)t - \frac{s - i}{s}\right)
= \ln \frac{(t + s)(p + h - ks - j)}{t(p + h - ks - j + h)} + O(1/p)
= \ln \left(\frac{t + s}{t}\right) (1 + O(1/A)) + O(1/p).
\]

Exponentiating, from (5.24) we thus have

\[
\exp\left[\frac{i}{s}\left(H\left[m + p - \left(k + \frac{j}{s}\right)(t + s) - \frac{s - i}{s}\right) - H\left[m - \left(k + \frac{j}{s}\right)t - \frac{s - i}{s}\right]\right]\right]
= \exp\left[\frac{i}{s}\left(\ln \left(\frac{t + s}{t}\right) (1 + O(1/A)) + O(1/p)\right)\right]
= \left(\frac{t + s}{t}\right)^{i/s} (1 + O(1/A)).
\]

Equations (5.27) and (5.28) imply that for \( k \in D_{A,\delta} \), we have

\[
\left((k + \frac{j}{s})(t + s) - \frac{i}{s}\right) \left(m + p - (k + \frac{j}{s})(t + s) + \frac{i}{s}\right)
= \left((k + \frac{j}{s})(t + s)\right) \left(m + p - (k + \frac{j}{s})(t + s)\right) (1 + O(1/A)).
\]

We have thus shown that for \( k \in D_{A,\delta} \),

\[
\sum_{i=0}^{s-1} \left(k(t + s) + c_i\right) \left(m + p - k(t + s) - c_i - 1\right)
+ \sum_{j=1}^{t} j \left(k(t + s) + c_{-j}\right) \left(m + p - k(t + s) - c_{-j} - 1\right)
= (1 + O(1/A)) \frac{t}{2} \sum_{i=0}^{s-1} \left((k + \frac{j}{s})(t + s)\right) \left(m + p - (k + \frac{j}{s})(t + s)\right).
\]

The sum

\[
\left(\frac{m + p}{p}\right)^{-1} \sum_{j=0}^{s-1} \sum_{k \in D_{A,\delta}} \left((k + \frac{j}{s})(t + s)\right) \left(m + p - (k + \frac{j}{s})(t + s)\right)
\]
equals

\[
O(1) + \left(\frac{m + p}{p}\right)^{-1} \sum_{k \in D_{A,\delta}} \left(k + \frac{j}{s}\right) \left(m + p - k + \frac{j}{s}\right).
\]
upon splitting the latter sum according to the residue of \( k \) modulo \( s \). We shall hide the \( O(1) \) term in the \( O(A) \) term created by the tails. As for those tails, Lemma 5.8 takes care of the extreme ends \((k \in T_A, k \in T_A)\). The remaining range is the "secondary tail," the sum over

\[
U_{A,\delta} = \left\{ k : (1 - \delta)\frac{p}{s} \leq k \leq \frac{p}{s} - A \right\}.
\]

To deal with this, we shall bound from above the contributions of the terms with \( k \in U_{A,\delta} \) in terms of \( \delta \) and \( A \). As \( p \to \infty \), we shall be able to let \( \delta \to 0 \), which will make these bounds, hence the contributions, tend to zero. We shall replace this section with the portion of the sum (5.11) for which \( k \in D_{A,\delta} \). Then, we will have completed the transition from \( G(m, p; s, t) \) to the pseudo-binomial form (5.2).

**Lemma 5.11.** Suppose \((ms - pt) p^{-1/2} \to \alpha < \infty \), with \( \alpha > 0 \), as \( p \to \infty \). Let \( A \) be a large constant, and let \( \delta \in (0, 1) \). Then, as \( p \to \infty \) there is a constant \( c_1(\delta) \), with \( c_1(\delta) \to 0 \) as \( \delta \to 0 \), such that the sum of

\[
(m + p)^{-1} \sum_{k \in U_{A,\delta}} \binom{k(t + s) + c_i}{ks + b_i} \binom{m + p - k(t + s) - c_i - 1}{p - ks - b_i}
\]

over \( 1 \leq i \leq s - 1 \) and

\[
(m + p)^{-1} \sum_{k \in U_{A,\delta}} \binom{k(t + s) + c_j}{ks + b_j} \binom{m + p - k(t + s) - c_j - 1}{p - ks - b_j - 1}
\]

over \( 1 \leq j \leq t \) is bounded above by \( c_1(\delta) \sqrt{p} \).

**Proof.** We apply Stirling’s formula to the various binomial coefficients. Write \( ms - pt = h(t + s) \).

By Lemma 5.2 we have

\[
(m + p)^{-1} = \left(\frac{t + s}{s}\right)^{-m} \left(\frac{t + s}{t}\right)^{m} \sqrt{\frac{2\pi mp}{m + p}} \left(1 - \frac{h}{m}\right)^{-m} \left(1 + \frac{h}{p}\right)^{-p} (1 + O(1/A)).
\]

We have for any \(-t \leq i \leq s - 1\) that

\[
\binom{k(t + s) + c_i}{ks + b_i} = (1 + O(1/A)) \left(\frac{s + t}{s}\right)^{ks + b_i} \left(\frac{s + t}{t}\right)^{kt + a_i} \sqrt{\frac{t + s}{2\pi kts}}.
\]

Let \( d_i = (s - i)/(t + s) \) for \( 1 \leq i \leq s - 1 \). Then we have (recalling that \( tb_i - sa_i = i \))

\[
\frac{m + p - k(t + s) - c_i - 1}{p - ks - b_i} = \frac{t + s}{s} \left(1 + \frac{h - d_i}{p - ks - b_i}\right).
\]
and
\[
\frac{m + p - k(t + s) - c_i - 1}{m - kt - a_i - 1} = \frac{t + s}{t} \left( 1 - \frac{h - d_i}{m - kt - a_i - 1} \right).
\]

Let \(e_j = (j - t)/(t + s)\) for \(1 \leq j \leq t\). Then we also have
\[
\frac{m + p - k(t + s) - c_{-j} - 1}{p - ks - b_{-j} - 1} = \frac{t + s}{s} \left( 1 + \frac{h - e_j}{p - ks - b_{-j} - 1} \right)
\]
and
\[
\frac{m + p - k(t + s) - c_{-j} - 1}{m - kt - a_{-j}} = \frac{t + s}{t} \left( 1 - \frac{h - e_j}{m - kt - a_{-j}} \right).
\]

Then
\[
(5.32)
\left( \frac{m + p - k(t + s) - c_i - 1}{p - ks - b_i} \right) = (1 + O(1/A)) \sqrt{\frac{m + p - k(t + s) - c_i - 1}{2\pi(p - ks - b_i)(m - kt - a_i - 1)}}
\times \left( \frac{t + s}{s} \right)^{p - ks - b_i} \left( \frac{t + s}{t} \right)^{m - kt - a_i - 1} \left( 1 + \frac{h - d_i}{p - ks - b_i} \right)^{p - ks - b_i} \left( 1 - \frac{h - d_i}{m - kt - a_i - 1} \right)^{m - kt - a_i - 1},
\]
and
\[
(5.33)
\left( \frac{m + p - k(t + s) - c_{-j} - 1}{p - ks - b_{-j} - 1} \right) = (1 + O(1/A)) \left( \frac{t + s}{s} \right)^{p - ks - b_{-j} - 1} \left( \frac{t + s}{t} \right)^{m - kt - a_{-j}}
\times \left( 1 + \frac{h - e_j}{p - ks - b_{-j} - 1} \right)^{p - ks - b_{-j} - 1} \left( 1 - \frac{h - e_j}{m - kt - a_{-j}} \right)^{m - kt - a_{-j}}
\times \sqrt{\frac{m + p - k(t + s) - c_{-j} - 1}{2\pi(p - ks - b_{-j} - 1)(m - kt - a_{-j})}}.
\]

Notice that in the case \(1 \leq i \leq s - 1\), all powers of \((t + s)/s\) from (5.30), (5.31), and (5.32) cancel, with the \((t + s)/t\) factors reduced to a lone \(t/(t + s)\). When \(1 \leq j \leq t\), (5.30), (5.31), and (5.33) has all powers of \((t + s)/t\) canceling, and a lone factor of \(s/(t + s)\) remaining.

We now focus on the “1 plus” and “1 minus” terms. Since the function \((1 + 1/x)^x\) is increasing in \(x\), we have
\[
(5.34)
\left( 1 + \frac{h - d_i}{p - ks - b_i} \right)^{p - ks - b_i} \leq \left( 1 + \frac{h}{p - ks - b_i} \right)^{p - ks - b_i} \leq \left( 1 + \frac{h}{p} \right)^p,
\]
\[
(5.35)
\left( 1 + \frac{h - e_j}{p - ks - b_{-j} - 1} \right)^{p - ks - b_{-j} - 1} \leq \left( 1 + \frac{h}{p - ks - b_{-j} - 1} \right)^{p - ks - b_{-j} - 1} \leq \left( 1 + \frac{h}{p} \right)^p.
\]

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Thus for each $Hence (5.38) is bounded above by

$$\left(1 - \frac{h - d_i}{m - kt - a_i - 1}\right)^{m-kt-a_i-1} \leq \left(1 - \frac{h}{m}\right)^m = \left(1 - \frac{h - d_i}{m}\right)^m (1 + o(1)),$$

(5.37)  $$\left(1 - \frac{h - e_j}{m - kt - a_j}\right)^{m-kt-a_j} \leq \left(1 - \frac{h - e_j}{m}\right)^m = \left(1 - \frac{h}{m}\right)^m (1 + o(1)),$$

since both $d_i$ and $e_j$ are $o(m)$. From (5.30), (5.31), (5.32), (5.34), and (5.36) we conclude that

$$\begin{align*}
(5.38) & \quad \left(\frac{m+p}{p}\right)^{-1} \left(\frac{k(t+s)+c_i}{ks+b_i}\right) \left(\frac{m+p-k(t+s)-c_i-1}{p-ks-b_i}\right) \\
& \leq \sqrt{\frac{mp(t+s)(m+p-k(t+s)-c_i-1)}{2\pi kts(m+p)(p-ks-b_i)(m-kt-a_i-1)}} (1 + O(1/A))^{\frac{t}{t+s}},
\end{align*}$$

while (5.30), (5.31), (5.33), (5.35), and (5.37) give us

$$\begin{align*}
(5.39) & \quad \left(\frac{m+p}{p}\right)^{-1} \left(\frac{k(t+s)+c_j}{ks+b_j}\right) \left(\frac{m+p-k(t+s)-c_j-1}{p-ks-b_j-1}\right) \\
& \leq (1 + O(1/A))^{\frac{s}{t+s}} \sqrt{\frac{mp(t+s)(m+p-k(t+s)-c_j-1)}{2\pi kts(m+p)(p-ks-b_j-1)(m-kt-a_j)}},
\end{align*}$$

Observe that

$$\frac{m}{m+p} \leq \frac{t}{t+s}, \quad \frac{m+p-k(t+s)-c_i-1}{m-kt-a_i-1} \leq t + s \quad \text{and} \quad \frac{p}{k} \leq \frac{ps}{(1-\delta)p} = \frac{s}{1-\delta}.$$

Hence (5.38) is bounded above by

$$\begin{align*}
(1 + O(1/A))^{\frac{t}{t+s}} \sqrt{\frac{t+s}{2\pi t(1-\delta)(p-ks-b_i)}.}
\end{align*}$$

Now we sum over $k$. Concentrating on the term dependent on $k$:

$$\text{(5.40)} \quad \sum_{k=(1-\delta)p/s}^{[p/s]-A} \frac{1}{\sqrt{p-ks-b_i}} \leq \sum_{k=(1-\delta)p/s+1}^{[p/s]-A+1} \frac{1}{\sqrt{p-ks}}$$

$$\leq \frac{1}{\sqrt{s}} \sum_{k=1}^{\delta p + 1} \frac{1}{\sqrt{k}} \leq 2 \sqrt{\frac{\delta p + 1}{s}} = \frac{2\sqrt{\delta p}}{s} (1 + o(1)).$$

Thus for each $i$, the sum of (5.38) over $k$ is bounded above by

$$\begin{align*}
(1 + O(1/A))^{\frac{2t}{s(t+s)} \sqrt{\frac{\delta p(t+s)}{2\pi t(1-\delta)}}},
\end{align*}$$

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thus the contribution of the first double sum in the statement of the lemma is bounded above by

\[(1 + O(1/A)) t(s - 1) \sqrt{\frac{\delta p(t + s)}{2\pi t(1 - \delta)}}.\]  

(5.41)

Working similarly with (5.39), for each \(j\) the sum over \(k\) is bounded above by

\[(1 + O(1/A)) \frac{2}{t + s} \sqrt{\frac{\delta p(t + s)}{2\pi t(1 - \delta)}}\]

and the contribution of the second double sum is bounded above by

\[(5.42) (1 + O(1/A)) \frac{t(t + 1)}{s + t} \sqrt{\frac{\delta p(t + s)}{2\pi t(1 - \delta)}}.\]

Thus we take

\[c_1(\delta) = (1 + O(1/A)) \sqrt{\frac{\delta t(t + s)}{2\pi t(1 - \delta)}}.\]

We want to replace this portion of the sum with

\[\sum_{k \in D_{A,\delta}} \left(\frac{k t + s}{s} \right) \left( m + p - \frac{k t + s}{s} \right) \left( m + p - k \right) \]

Lemma 5.12. Suppose \((m s - p t)p^{-1/2} \to \alpha < \infty, \) with \(\alpha > 0, \) as \(p \to \infty.\) Let \(A\) be a large constant, and let \(\delta \in (0, 1)\). Then, as \(p \to \infty\) there is a constant \(c_2(\delta),\) with \(c_2(\delta) \to 0\) as \(\delta \to 0,\) such that

\[\left(\frac{m + p}{p}\right)^{-1} \sum_{k \in D_{A,\delta}} \left(\frac{k t + s}{s} \right) \left( m + p - \frac{k t + s}{s} \right) \leq c_2(\delta) \sqrt{p},\]

with \(c_2(\delta) \to 0\) as \(\delta \to 0.\)

Proof. We write (with \(i\) defined so that \(\theta(i) = j\))

\[\left(\frac{m + p}{p}\right)^{-1} \sum_{k \in D_{A,\delta}} \left(\frac{k t + s}{s} \right) \left( m + p - \frac{k t + s}{s} \right)\]

\[= O(1) + \left(\frac{m + p}{p}\right)^{-1} \sum_{j=0}^{s-1} \sum_{k \in U_{A,\delta}} \left(\frac{k + \frac{j}{s}}{s} \right) \left( m + p - \frac{k + \frac{j}{s}}{s} \right)\]

\[\leq O(1) + \left(\frac{m + p}{p}\right)^{-1} \sum_{j=0}^{s-1} \sum_{k \in U_{A,\delta}} \left(\frac{k + \frac{j}{s}}{s} \right) \left( m + p - \frac{k + \frac{j}{s}}{s} \right)\]

\[= O(1) + \sum_{i=0}^{s-1} \sum_{k \in U_{A,\delta}} \left(\frac{k(t + s) + c_i + 1}{s} \right) \left( m + p - k(t + s) - c_i \right) \left( m + p - k(t + s) - c_i \right).\]
and since \( k \) is large enough,
\[
\frac{k(t + s) + c_i + 1}{kt + a_i + 1} = \frac{t + s}{t} (1 + o(1))
\]

reduces back to Lemma 5.11 after division by \( \binom{m+p}{p} \).

Letting \( C(\delta) = c_1(\delta) - c_2(\delta) \), we have shown that for \( p \) large enough and \( ms - pt \geq 0 \),
\[
(5.43) \quad G(m, p; s, t) = O(A) + C(\delta) \sqrt{p} + (1 + O(1/A)) \left( \frac{m + p}{p} \right)^{-1} \frac{t}{2} \sum_{k=0}^{p} \left( \frac{k + s}{k} \right) \left( \frac{m + p - k + s}{p - k} \right)
\]

\[
= O(A) + C(\delta) \sqrt{p} + (1 + O(1/A)) \left( \frac{m + p + 1}{k} \right) \left( \frac{t}{s} \right)^{p-k},
\]

by Lemma 5.8. Applying Lemma 5.3, then letting \( A \to \infty \) and \( \delta \to 0 \), we conclude that
\[
G(m, p; s, t) \sim \frac{\sqrt{2\pi pt(t + s)}}{2} \exp \left( \frac{\alpha^2}{2t(t + s)} \right) \Phi \left( \frac{-\alpha}{\sqrt{t(t + s)}} \right),
\]

as desired.

If \( pt - ms \geq 0 \), then from the Antium Theorem we have \( G(m, p; s, t) = G(p, m; t, s) + (pt - ms) \).

Then, since \( ms \sim pt \),
\[
\frac{ms - pt}{\sqrt{p}} \to \alpha \leq 0 \quad \Rightarrow \quad \frac{pt - ms}{\sqrt{m}} \to -\alpha \sqrt{\frac{t}{s}} \geq 0.
\]

Thus by our previous result,
\[
G(p, m; t, s) \sim \frac{\sqrt{2\pi ms(t + s)}}{2} \exp \left( \frac{\alpha^2 s/t}{2s(t + s)} \right) \Phi \left( \frac{\alpha \sqrt{s/t}}{\sqrt{s(t + s)}} \right)
\]

\[
\sim \frac{\sqrt{2\pi pt(t + s)}}{2} \exp \left( \frac{\alpha^2}{2t(t + s)} \right) \Phi \left( \frac{\alpha}{\sqrt{t(t + s)}} \right).
\]

Therefore, when \( pt - ms \geq 0 \) we have
\[
G(m, p; s, t) \sim -\alpha \sqrt{p} + \frac{\sqrt{2\pi pt(t + s)}}{2} \exp \left( \frac{\alpha^2}{2t(t + s)} \right) \Phi \left( \frac{-|\alpha|}{\sqrt{t(t + s)}} \right).
\]

If \( pt - ms = o(\sqrt{p}) \), then we have \( \alpha = 0 \). Then, since \( \exp(0) = 1 \) and \( \Phi(0) = 1/2 \), the secondary result holds. □
Corollary 5.13. The result of Theorem 5.10 holds for any positive \( s \) and \( t \).

Proof. Suppose first that \( t/s \) is rational. We can write \( G(m, p; s, t) = nG(m, p; r, q) \), where \( q/r \) is \( t/s \) written in lowest terms. That is, \( nr = s \) and \( nt = q \). Then, if \( (pt - ms)/\sqrt{p} \to \alpha \) as \( p \to \infty \), then \( pt - ms \to 0 \). Then by Theorem 5.10 we have

\[
G(m, p; s, t) \sim n \frac{2\pi pq(q + r)}{2} \exp \left( \frac{(\alpha/n)^2}{2q(q + r)} \right) \Phi \left( \frac{-|\alpha/n|}{\sqrt{q(q + r)}} \right)
\]

\[
= \frac{2\pi p n q + nr}{2} \exp \left( \frac{\alpha^2}{2nq(nq + nr)} \right) \Phi \left( \frac{-|\alpha|}{\sqrt{nq(nq + nr)}} \right)
\]

\[
= \frac{2\pi pt(t + s)}{2} \exp \left( \frac{\alpha^2}{2t(t + s)} \right) \Phi \left( \frac{-|\alpha|}{\sqrt{t(t + s)}} \right).
\]

The extension to the case with \( t/s \) irrational follows by the continuity of \( G(m, p; s, t) \) in \( s \) and \( t \). □

5.4. Urns with \( pt - ms \leq 0 \), \( \sqrt{p} = o(pt - ms) \), and \( pt - ms = o(p) \). We next examine urns for which \( (ms - pt)/\sqrt{p} \to \infty \), but \( (ms - pt)/p \to 0 \). We find that for large \( p \), we should not expect to gain as much as an urn with an initial weight closer to zero. This is intuitively clear. The urn has a longer trip to neutral, and thus will take a longer amount of time to neutralize. Therefore there will be less time to collect gains.

Theorem 5.14. Suppose that as \( p \to \infty \), \( (ms - pt)p^{-1/2} \to \infty \) and \( (ms - pt)p^{-1} \to 0 \). Then for sufficiently large \( p \),

\[
G(m, p; s, t) \sim \frac{t(t + s)}{2} \cdot \frac{p}{ms - pt}.
\]

Proof. As in the proof of Theorem 5.10, we want to replace the double-double sum found in the crossings form,

\[
\sum_{i=0}^{s-1} \sum_{k=0}^{M(i)} i \binom{k(t + s) + c_i}{ks + b_i} \left( m + p - k(t + s) - c_i - 1 \right) \frac{1}{p - ks - b_i}
\]

\[
+ \sum_{j=1}^{t} \sum_{k=0}^{M(-j)} j \binom{k(t + s) + c_{-j}}{ks + b_{-j}} \left( m + p - k(t + s) - c_{-j} - 1 \right) \frac{1}{p - ks - b_{-j} - 1},
\]

with the single sum

\[
\frac{t}{2} \sum_{k=0}^{p} \binom{k + s}{k} \left( m + p - k \right) \frac{t + s}{p - k} = \frac{(t + s)^{m + p + 1}}{2sp^{m}} B \left( m + p + 1, \frac{s}{t + s}, p \right).
\]
The process is similar in many respects. Write \( ms - pt = h(t + s) \), define \( T_A \) and \( T_A^* \) as before, and for \( \omega > 0 \) let

\[
U_{A,\omega} = \{ k : \frac{p}{s} - \omega h \leq k \leq \frac{p}{s} - A \}, \quad \overline{U}_{A,\omega} = \{ k : p - \omega hs \leq k \leq p - As \},
\]

\[
D_{A,\omega} = \{ k : A \leq k \leq \frac{p}{s} - \omega h \}, \quad \overline{D}_{A,\omega} = \{ k : As \leq k \leq p - \omega hs \}.
\]

Lemmas 5.8 and 5.9 will allow us to swap the tails. The secondary tails, with \( k \) in (respectively) \( U_{A,\omega} \) or \( \overline{U}_{A,\omega} \), shall be dealt with later. The dominant portion of the double-double sum has \( k \) in \( D_{A,\omega} \). Applying the same process as in Theorem 5.10, from (5.20) we have for \( k \in D_{A,\omega} \) and \( 0 \leq i \leq s - 1, 1 \leq j \leq t \):

\[
\frac{U_1(m,p;s,t)}{L_1(m,p;s,t)} = (1 + O(1/A))(1 + O(1/\omega)).
\]

With

\[
\frac{m + p - k(t + s) - c_i}{m - kt - a_i} = \frac{t + s}{t}(1 + o(1)),
\]

and

\[
\frac{m + p - k(t + s) - c_{-j}}{p - ks - b_{-j}} = \frac{t + s}{s}(1 + o(1)),
\]

from (5.26) we can say that for \( k \in D_{A,\omega} \):

\[
\sum_{i=0}^{s-1} \left( k(t + s) + c_i \right) \left( m + p - k(t + s) - c_i - 1 \right) \left( m - p - k(t + s) - c_{-j} - 1 \right) = (1 + O(1/A))(1 + O(1/\omega)) \frac{t}{2} \sum_{i=0}^{s-1} \left( k(t + s) + c_i \right) \left( m - p - k(t + s) - c_i \right),
\]

where again we split \( st/2 \) into \( s \) parts. Reindexing with respect to \( b_i \), the sum becomes

\[
(5.44) \sum_{j=0}^{s-1} \left( k + \frac{j}{s} \right) \left( m + p - k + \frac{j}{s} \right) \left( t + s - \frac{i}{s} \right) \left( m - p - k + \frac{j}{s} \right) \left( t + s - \frac{i}{s} \right) \left( 1 + O(1/A) \right).
\]

We now remove the \( i/s \) terms. From (5.23) we have

\[
\exp \left[ -\frac{i}{s} \left( \left( k + \frac{j}{s} \right) (t + s) + \frac{s - i}{s} \right) - \left( k + \frac{j}{s} \right) (t + s) + \frac{s - i}{s} \right] \leq \left( \frac{t + s}{s} \right)^{-i/s} (1 + O(1/A)).
\]
From (5.24) we have
\[
\exp\left[\frac{i}{s} \left( H \left[ m + p - \left( k + \frac{j}{s} \right) (t + s) - \frac{s - i}{s} \right] - H \left[ m - \left( k + \frac{j}{s} \right) t - \frac{s - i}{s} \right] \right) \right] 
\leq \left( \frac{t + s}{t} \right)^{i/s} (1 + O(1/A))(1 + O(1/\omega)).
\]

The same results occur with corresponding lower bounds. Thus we have from (5.44) that
\[
\sum_{j=0}^{s-1} \left( \frac{k}{s} + s \right) (t + s) - \frac{1}{s} \left( m + p - \left( k + \frac{j}{s} \right) (t + s) + \frac{1}{s} \right) 
= (1 + O(1/A))(1 + O(1/\omega)) \sum_{j=0}^{s-1} \left( \frac{k}{s} + s \right) (t + s) \left( m + p - \left( k + \frac{j}{s} \right) (t + s) \right).
\]

We have thus shown that for \( k \in D_{A,\omega}, \)
\[
\sum_{i=0}^{s-1} \sum_{k \in D_{A,\omega}} i \left( \frac{k(t + s) + c_i}{ks + b_i} \right) \left( m + p - k(t + s) - c_i - 1 \right) 
+ \sum_{j=1}^{t} \sum_{k \in D_{A,\omega}} j \left( \frac{k(t + s) + c_{-j}}{ks + b_{-j}} \right) \left( m + p - k(t + s) - c_{-j} - 1 \right) 
= (1 + O(1/A))(1 + O(1/\omega)) \sum_{j=0}^{s-1} \sum_{k \in D_{A,\omega}} \left( \frac{k}{s} + s \right) (t + s) \left( m + p - \left( k + \frac{j}{s} \right) (t + s) \right).
\]

This sum over \( j \) in turn equals, upon division by \( \left( \frac{m+p}{p} \right) \),
\[
\left( \frac{m+p}{p} \right)^{-1} \sum_{j=0}^{s-1} \sum_{k \in D_{A,\omega}} \left( \frac{k}{s} + s \right) (t + s) \left( m + p - \left( k + \frac{j}{s} \right) (t + s) \right) 
= O(1) + \left( \frac{m+p}{p} \right)^{-1} \sum_{k \in D_{A,\omega}} \left( \frac{k}{s} + s \right) (t + s) \left( m + p - \left( k + \frac{j}{s} \right) \right) \left( m + p - \left( k + \frac{j}{s} \right) \frac{t+s}{s} \right).
\]

We now deal with the secondary tail, the terms with \( k \in U_{A,\omega}. \)

**Lemma 5.15.** Let \( \omega > 0 \) be a constant. For \( 1 \leq i \leq s - 1 \) and \( 1 \leq j \leq t, \) we have for large \( p \) that
\[
\left( \frac{m+p}{p} \right)^{-1} \sum_{k \in U_{A,\omega}} \left( \frac{k(t + s) + c_i}{ks + b_i} \right) \left( m + p - k(t + s) - c_i - 1 \right) = o(p/h),
\]
and
\[
\left( \frac{m+p}{p} \right)^{-1} \sum_{k \in U_{A,\omega}} \left( \frac{k(t + s) + c_{-j}}{ks + b_{-j}} \right) \left( m + p - k(t + s) - c_{-j} - 1 \right) = o(p/h).
\]
Proof. We will show the first result, as the second result can be proven similarly. We have by Stirling’s Formula that

\[(5.45) \quad \left( k(t + s) \right)_{ks} = \left( \frac{t + s}{s} \right)^{ks} \left( \frac{t + s}{t} \right)^{kt} \sqrt{\frac{t + s}{2\pi kst}} (1 + O(1/A)). \]

Thus

\[(5.46) \quad \left( \frac{k(t + s) + c_i}{ks + b_i} \right) = \left( \frac{t + s}{s} \right)^{ks + b_i} \left( \frac{t + s}{t} \right)^{kt + a_i} \sqrt{\frac{t + s}{2\pi kst}} (1 + O(1/A)). \]

For the other binomial term, we have that

\[(5.47) \quad \left( m + p - k(t + s) - c_i - 1 \right) = \left( 1 + O(1/A) \right) \left( \frac{t + s}{s} \right)^{p - ks - b_i} \left( \frac{t + s}{t} \right)^{m - kt - a_i - 1} \times \left( 1 + \frac{h(t + s) + (i - s)}{(t + s)(p - ks - b_i)} \right) \left( 1 - \frac{h(t + s) + (i - s)}{(t + s)(m - kt - a_i - 1)} \right) \times \sqrt{\frac{m + p - k(t + s) - c_i - 1}{2\pi(p - ks - b_i)(m - kt - a_i - 1)}}. \]

Note that the powers of \((t + s)/s\) from the result of Lemma 5.2, along with (5.46) and (5.47) cancel, and a lone factor of \(t/(t + s)\) remains from the other proportion. We now focus in on the “one plus” and “one minus” terms from the result of Lemma 5.2 and (5.47). Since both \((1 + (1/x))^x\) is increasing and \(1 - x \leq \exp(-x)\), for \(k \in U_{\lambda, \omega}\) we have

\[(5.48) \quad \left( 1 + \frac{h(t + s) + (i - s)}{(t + s)(p - ks - b_i)} \right)^{p - ks - b_i} \leq \left( 1 + \frac{1 + (i - s)/[h(t + s)]}{\omega s} \right)^{\omega hs} \leq \exp\left( \left( 1 - a \right) \left( h + \frac{i - s}{t + s} \right) \right), \]

for some \(a \in (0, 1)\), and

\[(5.49) \quad \left( 1 - \frac{h + (i - s)/(t + s)}{m - kt - a_i - 1} \right)^{m - kt - a_i - 1} \leq \exp\left( - \left( h + \frac{i - s}{t + s} \right) \right). \]

Thus the product of (5.48) and (5.49) is bounded above by

\[(5.50) \quad \exp\left( -ah \left( 1 - \frac{s - i}{h(t + s)} \right) \right) \leq \exp(-bh), \]

for some \(b \in (0, 1)\). Since \(1 + x \geq \exp(x - x^2/2)\) for all \(x \geq 0\), we have

\[(5.51) \quad \left( 1 + \frac{h}{p} \right)^{-p} \leq \exp\left( -h + \frac{h^2}{2p} \right). \]
For sufficiently small $x \geq 0$ we have $1 - x \geq \exp(-x - x^2/2)$. As $h = o(m)$, we have as $p \to \infty$ that

$$\left(1 - \frac{h}{m}\right)^{-m} \leq \exp \left(h + \frac{h^2}{m}\right).$$

(5.52)

Therefore, for $p$ large enough, we have from the result of (5.50), (5.51), (5.52) that

$$\left(1 + \frac{h}{p}\right)^{-p} \left(1 + \frac{h + (i - s)/(t + s)}{p - ks - bi}\right)^{p - ks - bi} \left(1 - \frac{h}{m}\right)^{-m} \left(1 - \frac{h + (i - s)/(t + s)}{m - kt - a_i - 1}\right)^{m - kt - a_i - 1} \leq \exp \left(h \left(-b + \frac{h}{2p} + \frac{h}{m}\right)\right).$$

(5.53)

As for the terms under the square roots in (5.46), (5.47), and the result of Lemma 5.2, we have

$$\frac{s + t}{kt} = O \left(\frac{t + s}{pt}\right)$$

and

$$\frac{mp}{m + p} \sim \frac{pt}{t + s}.$$  

We also have

$$\frac{m + p - k(t + s) - c_i - 1}{(p - ks - bi)(m - kt - a_i - 1)} \leq \frac{(m - kt - a_i - 1) + (As - bi)}{(As - bi)(m - kt - a_i - 1)}$$

$$\leq \frac{(m - pt/s) - (As - bi)}{(As - bi)(m - pt/s - a_i - 1)} = 1 + o(1).$$

(5.54)

Thus the product of the square roots (as well as the extra factor of $t/(t + s)$) is $O(1)$. Since $h = o(p)$ and $h = o(m)$ there exists a constant $c > 0$ such that for sufficiently large $p$ via (5.53):

$$\left(\frac{m + p}{p}\right)^{-1} \left(\frac{k(t + s) + c_i}{ks + bi}\right) \left(\frac{m + p - k(t + s) - c_i - 1}{p - ks - bi}\right) \leq O(1) \left(1 + O(1/A)\right) \exp \left(h \left(-b + \frac{h}{2p} + \frac{h}{m}\right)\right) \leq \exp(-ch),$$

(5.55)

where the last line follows since both $h/p$ and $h/m$ go to 0. Summing over this range of $k$, there are at most $\omega h$ terms. Thus

$$\left(\frac{m + p}{p}\right)^{-1} \sum_{k=p/s-\omega h}^{p/s-A} \left(\frac{k(t + s) + b_i}{ks + bi}\right) \left(\frac{m + p - k(t + s) - c_i - 1}{p - ks - bi}\right) \leq \omega h \exp(-ch) = (\omega p/h)(h^2/p) \exp(-ch)$$

$$= (\omega p/h) o(1) \exp(-ch) = o(p/h).$$

□
There are a constant number of each type of term (around \( (s^2 + t^2)/2 \) of them). Therefore, the same result holds for the entire sums

\[
\left( \frac{m + p}{p} \right)^{-1} \sum_{i=0}^{s-1} \sum_{k \in U_{A, \omega}} \left( k(t + s) + c_i \right) \left( \frac{m + p - k(t + s) - c_i}{p - ks} - b_i \right)
\]

and

\[
\left( \frac{m + p}{p} \right)^{-1} \sum_{j=1}^{t} \sum_{k \in U_{A, \omega}} \left( k(t + s) + c_{-j} \right) \left( \frac{m + p - k(t + s) - c_{-j}}{p - ks} - b_{-j} - 1 \right).
\]

We shall replace these two sums with the sum

\[
\frac{t}{2} \sum_{k \in U_{A, \omega}} \left( \frac{k + s}{p} \right) \left( \frac{m + p - k + s}{p - k} \right).
\]

**Lemma 5.16.** For sufficiently large \( p \),

\[
\sum_{k \in U_{A, \omega}} \left( \frac{k + s}{p} \right) \left( \frac{m + p - k + s}{p - k} \right) = o(p/h).
\]

We proceed similarly to Lemma 5.12, reducing to sums appearing in Lemma 5.15.

We have thus shown that

\[
G(m, p; s, t) = O(A) + o(p/h) + \left( 1 + O(1/A) \right) \left( 1 + O(1/\omega) \right) \left( \frac{m + p}{p} \right)^{-1} \sum_{k=0}^{p} \left( \frac{k + s}{k} \right) \left( \frac{m + p - k + s}{p - k} \right)
\]

\[
= O(A) + o(p/h) + \left( 1 + O(1/A) \right) \left( 1 + O(1/\omega) \right) \left( \frac{m + p + 1}{p} \right)^{-1} \sum_{k=0}^{p-1} \left( m + p + 1 \right) \left( \frac{t}{k} \right)^{p-k}.
\]

Letting \( p \to \infty \), we let \( A \) and \( \omega \) be as large as needed, and Lemma 5.5 gives

\[
G(m, p; s, t) \sim \frac{t(t + s)}{2} \cdot \frac{p}{ms - pt}.
\]

**Corollary 5.17.** The result of Theorem 5.14 holds for arbitrary positive \( s \) and \( t \).

**Proof.** As in the proof of Corollary 5.13, if \( t/s \) is rational, \( G(m, p; s, t) = n G(m, p; r, q) \) for some constant \( n \) and coprime positive integers \( q \) and \( r \), and

\[
n G(m, p; r, q) = n \frac{q(q + r)}{2} \cdot \frac{p}{mr - pq} (1 + o(1)) = \frac{t(t + s)}{2} \cdot \frac{p}{ms - pt} (1 + o(1)).
\]

The remaining cases follow by continuity. \( \square \)
5.5. **Urns with** $ms/pt \to \lambda > 1$. The remaining case, for which $ms/pt \to \lambda > 1$ as $p \to \infty$, seems to be difficult to analyze. Since $G(m, p; s, t)$ turns out to be $O(1)$ for large $p$, we cannot afford to cut off the tails a la Lemma 5.8. Folding the two double sums together also fails. Without a direct connection to the sum

$$
\sum_{k=0}^{p} \binom{k+t+s}{k} \binom{m+p-k+t+s}{p-k}
$$

outside of the case $s = 1$, we cannot use results dealing with the tail of a binomial distribution. So, we are forced to manipulate into a form with $s = 1$. With $s = 1$, we have the binomial form to work with:

$$
G(m, p; 1, t) = -\frac{t}{2} + \left(\frac{m+p}{p}\right)^{-1} t \sum_{k=0}^{p} \binom{m+p+1}{k} t^{p-k}.
$$

The following follows immediately from Lemma 5.7.

**Theorem 5.18.** If $m/(pt) \to \lambda > 1$ as $p \to \infty$, then

$$
G(m, p; 1, t) \sim \frac{t+1}{2(\lambda - 1)} = \frac{t(t+1)}{2} \cdot \frac{p}{m - pt}.
$$

We can show $G(m, p; s, t)$ is bounded when $ms/(pt) \to \lambda > 1$ using Theorem 5.18.

**Lemma 5.19.** Suppose that as $p \to \infty$, $ms/pt \to \lambda > 1$. Then, as $p \to \infty$,

$$
G(m, p; s, t) \leq \frac{s(t+1)}{2(\lambda - 1)}.
$$

**Proof.** For any $p$, there is a minimal integer $\hat{p}$ such that $\hat{p}s \geq p$. Then by Lemma 2.1, Lemma 2.6, and Lemma 2.9, we have that

$$
G(m, p; s, t) \leq G(m, \hat{p}s; s, t) = s G(m, \hat{p}s; 1, t/s) \leq s G(m, \hat{p}; 1, t).
$$

If $\lambda < \infty$, we have $p \sim \hat{p}s$ for large enough $p$. Therefore, $\frac{ms}{pt} \to \lambda$ implies $\frac{m}{\hat{p}t} \to \lambda$ as well. Then, by Theorem 5.18,

$$
G(m, \hat{p}; 1, t) \sim \frac{t+1}{2(\lambda - 1)}.
$$

We have thus shown that $G(m, p; s, t)$ is bounded from above. \qed

In some circumstances, we can use the monotonicity in $s$ and $t$ to produce a (possibly) better bound.
Lemma 5.20. Suppose that as \( p \to \infty \), \( ms/pt \to \lambda > 1 \). If in addition

\[
\frac{m}{p[t/s]} \to \lambda_1 > 1,
\]

then as \( p \to \infty \),

\[
G(m, p; s, t) \leq \frac{s}{2} \cdot \frac{[t/s] + 1}{\lambda_1 - 1} (1 + o(1)).
\]

Proof. We have that \( G(m, p; s, t) = s G(m, p; 1, t/s) \). (2.6) implies that

\[
G(m, p; 1, t/s) \leq G(m, p; 1, [t/s]).
\]

Since \([t/s]\) is an integer, under the given conditions we can use Theorem 5.18:

\[
G(m, p; 1, [t/s]) = \frac{1}{2} \cdot \frac{[t/s] + 1}{\lambda_1 - 1} (1 + o(1)), \text{ as } p \to \infty.
\]

Multiplying through by \( s \) gives

\[
G(m, p; s, t) \leq s \cdot G(m, p; 1, [t/s]) \leq \frac{s}{2} \cdot \frac{[t/s] + 1}{\lambda_1 - 1} (1 + o(1)),
\]

as desired. \( \Box \)

We can use the same method to produce a better lower bound as well.

Lemma 5.21. Suppose that as \( p \to \infty \), \( ms/pt \to \lambda > 1 \), and that \( t > s \). Define \( \lambda_0 \) so that

\[
\frac{m}{p[t/s]} \to \lambda_0 \text{ as } p \to \infty.
\]

Then

\[
G(m, p; s, t) \geq \frac{s}{2} \cdot \frac{[t/s] + 1}{\lambda_0 - 1} (1 + o(1)).
\]

The proof follows that of Lemma 5.20, and is omitted. If \( s > t \), we use zero as a lower bound.

Remark. If \( p \) is bounded as \( m \) goes to infinity (that is, \( \lambda = \infty \)), then \( G(m, p; s, t) \to 0 \) under these circumstances, as the probability of placing a bet also goes to zero.

5.6. Urns with \( pt - ms \geq 0 \) and \( \sqrt{p} = o(pt - ms) \). For the remaining urns with \( pt - ms \geq 0 \), observe that if \( \sqrt{p} = o(pt - ms) \), then \( pt - ms \) is much larger than \( G(p, m; t, s) \), from Theorems 5.14 and 5.20. Therefore, the following is a result of the Antiurn Theorem.

Theorem 5.22. If \( (pt - ms)/\sqrt{p} \to \infty \) as \( p \to \infty \), then \( G(m, p; s, t) \sim pt - ms \).
6. The Probability of Minimal Gain

Using an optimal strategy for the \((m, p; s, t)\) urn, the player is guaranteed to gain \(\max\{0, pt - ms\}\). In this chapter, we shall examine the probability the player gains \(\max\{0, pt - ms\}\), which is a variation on the ballot problem.

6.1. A Brief History of the Ballot Problem. The generalized ballot problem can be stated as follows: In an election, candidate \(A\) has \(a\) votes, and candidate \(B\) has \(b\) votes, with \(a \geq b\mu\) and \(\mu \geq 0\), a fixed real number. What is the probability that, as the votes are counted, that (i) the number of votes for \(A\) is always greater than \(\mu\) times the number of votes for \(B\), or (ii) the number of votes for \(A\) is always at least \(\mu\) times the number of votes for \(B\)? An equivalent formulation is as follows: \(A\) has \(a\) votes of weight one, and \(B\) has \(b\) votes of weight \(\mu\), and we wish to calculate the probability that \(A\) always has a (i) greater, or (ii) greater or equal, weighted total than \(B\) throughout the count.

The problem was originally proposed by M. Bertrand [3] in 1887. He indicated that the solution to (i) with \(\mu = 1\) is \(\frac{a - b}{a + b}\). That same year, D. André [2] gave a direct proof of Bertrand’s result. In 1924 A. Aeppli [1] showed that the solution for integer \(\mu \geq 0\) for (i) is

\[
(6.1) \quad \frac{a - \mu b}{a + b}.
\]

The solution to (ii) for integer \(\mu \geq 0\) follows from (i) by adding an additional vote for \(A\) to be counted first. The solution to (ii) is thus

\[
\frac{a + 1 - \mu b}{a + 1}.
\]

For other values of \(\mu\), these two formulas do not work. In 1962, Takács [24] gave the solution to (i) for arbitrary \(\mu \geq 0\):

**Theorem 6.1.** (Takács) If \(a \geq b\mu\), then the probability that the number of votes for \(A\) is always more than \(\mu\) times the number of votes for \(B\) is

\[
\frac{a}{a + b} \sum_{j=0}^{b} C_j \left(\frac{b}{j}\right) \left(a + b - 1\right)^{-1} = \left(\frac{a + b}{a}\right)^{-1} \sum_{j=0}^{b} C_j \left(\frac{a + b - 1 - j}{b - j}\right),
\]

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where $C_0 = 1$ and the constants $C_j, j = 1, 2, \ldots$ are given by the recurrence formula

$$
\sum_{j=0}^{n} C_j \binom{n}{j} \left( \left\lfloor \frac{n\mu}{j} \right\rfloor + n - 1 \right)^{-1} = 0, \quad n = 1, 2, \ldots
$$

Takács made the important observation that these ballot numbers are uniquely determined by the binomial recurrence relation, along with the boundary conditions. The general solution to (ii) will be given with Theorem 6.10.

6.2. Zero Gain for the $(m,p;1,t)$ Urns, $t \geq 0$ an Integer. When $pt - m \leq 0$, Theorem 3.29 implies that the probability the player gains zero using any optimal strategy equals

$$
1 - \left( \frac{m + p}{p} \right)^{-1} Q_{m,p}(1).
$$

The one-to-one correspondence spelled out by Theorem 4.18 is actually the identity map under the zero-pass strategy. That is, using the zero-pass strategy, the player will gain zero if and only if $X_n \leq 0$ for all $n$ (and thus, the player will never bet), or equivalently, $N^+ = 0$. Thus the probability of zero gain equals $m - pt + 1$ via Corollary 3.14. This is version (ii) of the ballot problem. We have discussed version (i) as well - a “$+ (b - a)$” trip is, in fact, a ballot permutation of type (i).

When $pt - m > 0$, the minimum gain is $pt - m$ and the probability of minimum gain equals

$$
1 - \left( \frac{m + p}{p} \right)^{-1} Q_{m,p}(pt - m + 1),
$$

and by Theorem 3.26,

$$
Q_{m,p}(pt - m + 1) = \sum_{k=0}^{\lfloor (m-1)/t \rfloor} \frac{1}{kt + k + 1} \binom{kt + k + 1}{k} \binom{m + p - k - kt - 1}{p - k}.
$$

Recall the definition of $N$, the number of times that the urn is neutral:

$$
N = |\{n: n \neq m + p, X_n = 0\}|.
$$

We now examine the distribution of $N$ over the zero-gain realizations with $m \geq pt$.

Lemma 6.2. Suppose $m \geq pt$, with $t$ a positive integer. Then using any optimal strategy,

$$
P(\text{Player gains zero and } N = k) = \frac{m - (p - k)t}{m + p - k} \left( \frac{m + p - k}{p - k} \right) \left( \frac{m + p}{p} \right)^{-1}.
$$
and

\[ P(\text{Player gains zero and } N \geq k) = \frac{m - pt + kt + 1}{m + p - k + 1} \left( \begin{array}{c} m + p - k + 1 \\ p - k \end{array} \right)^{-1}. \]

Remark. The first result with \( k = 0 \) is (6.1).

Proof. The first result follows from Theorem 3.16. Since we have the added restriction \( N = k \), we take only the \( n = k \) term from the sum. So, we concentrate on the second result.

With the zero-pass strategy, if \( \omega \) is a zero-gain realization, then \( N = N^- \). We remove the “+t”s causing the the final \( k \) events \( A_t \). This forms \( k “+t” \) trips. The initial portion, upon an addition of a “−1” ball to the end, forms a “+(\( m - pt + 1 \))” trip. Therefore, a realization gaining zero with \( N \geq k \) is in a one-to-one correspondence with “+(\( m + 1 - pt - kt \))” trips containing \( k \) fewer “+t”s and one more “−1.” The total number of the latter is

\[ \frac{m - pt + kt + 1}{m + p - k + 1} \left( \begin{array}{c} m + p - k + 1 \\ p - k \end{array} \right). \]

We complete by dividing by \( \left( \begin{array}{c} m + p \\ p \end{array} \right) \). □


6.3. Generalized Ballot Numbers and Zero Gain Numbers. Now we take a look at \((s,t)\)-ballot numbers and \((s,t)\)-zero-gain numbers, their similarities, and the interplay between the two collections of numbers. We begin with the \((s,t)\)-ballot numbers.

Definition 6.3. The \((s,t)\)-ballot number \( \beta_{s,t}(a,b) \) is the number of ways that, in an election, a candidate receiving \( a \) votes of weight \( s \) remains ahead in weight over a candidate receiving \( b \) votes of weight \( t \) throughout the counting of the votes.

Remark. We shall take the convention that \( \beta_{s,t}(0,0) = 1 \).

Clearly, if \( bt \geq as \), we have \( \beta_{s,t}(a,b) = 0 \), as candidate \( B \) will win. The \((s,t)\)-ballot numbers satisfy the recurrence most often associated with the binomial coefficients, by considering the last vote counted:

\[ \beta_{s,t}(a,b) = \beta_{s,t}(a - 1,b) + \beta_{s,t}(a,b - 1), \quad as - bt > 0 \]

or equivalently, for \( a > 0 \),

\[ \beta_{s,t}(a,b) = \sum_{k=0}^{b} \beta_{s,t}(a - 1,k). \]
Some (2,3)-ballot numbers are given in Table 6.1.

The \((s,t)\)-zero-gain numbers are defined similarly.

**Definition 6.4.** The \((s,t)\)-zero-gain number \(\zeta_{s,t}(a,b)\) is the number of realizations from the \((a,b;s,t)\) urn that gain zero when the zero-pass strategy is used.

**Remark.** Again, we set \(\zeta_{s,t}(0,0) = 1\) by convention.

If \(bt > as\), we have \(\zeta_{s,t}(a,b) = 0\), as the player will definitely gain at least \(bt - as > 0\). We have the same binomial recurrence:

\[
\zeta_{s,t}(a,b) = \zeta_{s,t}(a-1,b) + \zeta_{s,t}(a,b-1), \quad \text{as } -bt \geq 0,
\]

and

\[
\zeta_{s,t}(a,b) = \sum_{k=0}^{b} \zeta_{s,t}(a-1,k), \quad a > 0.
\]

Some (2,3)-zero-gain numbers are given in Table 6.2.

**Table 6.1.** (2,3)-ballot numbers for \(0 \leq a \leq 11, 0 \leq b \leq 7\).

<table>
<thead>
<tr>
<th>(a)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\beta_{2,3}(a,0))</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>(\beta_{2,3}(a,1))</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\beta_{2,3}(a,2))</td>
<td>0</td>
<td>3</td>
<td>7</td>
<td>12</td>
<td>25</td>
<td>33</td>
<td>42</td>
<td>52</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\beta_{2,3}(a,3))</td>
<td>7</td>
<td>19</td>
<td>37</td>
<td>62</td>
<td>95</td>
<td>137</td>
<td>189</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\beta_{2,3}(a,4))</td>
<td>0</td>
<td>37</td>
<td>99</td>
<td>194</td>
<td>331</td>
<td>520</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\beta_{2,3}(a,5))</td>
<td>99</td>
<td>293</td>
<td>624</td>
<td>1144</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\beta_{2,3}(a,6))</td>
<td>0</td>
<td>624</td>
<td>1768</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\beta_{2,3}(a,7))</td>
<td>1768</td>
<td></td>
<td></td>
<td></td>
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</tr>
</tbody>
</table>

**Lemma 6.5.** For nonnegative \(s,t\), and \(r > 0\), we have \(\beta_{s,t}(a,b) = \beta_{rs,rt}(a,b)\) and \(\zeta_{s,t}(a,b) = \zeta_{rs,rt}(a,b)\).

This is clear because \(as - bt\) and \(ars - brt = r(as - bt)\) will always have the same sign. Thus \(\beta_{s,t}(a,b)\) is related to problem (i) and \(\zeta_{s,t}(a,b)\) is related to problem (ii), with \(\mu = t/s\). For the \((s,t)\)-zero-gain numbers, the boundary is included (in terms of the ballot problem, ties between \(A\) and \(B\) are allowed), while the boundary is not included for the \((s,t)\)-ballot numbers (in terms of the zero-gain problem, the weight of the urn stays negative while the urn is nonempty). Therefore, we
have

$$\beta_{s,t}(a,b) \leq \zeta_{s,t}(a,b), \text{ for all } a, b, s, t,$$

since each ballot permutation corresponds with a zero-gain realization. If \( t/s \) is irrational, problems (i) and (ii) are equivalent, as there cannot be any ties once the vote count starts. (That is, \( as - bt = 0 \) if and only if \( a = b = 0 \).) Therefore, we have

$$\beta_{s,t}(a,b) = \zeta_{s,t}(a,b), \text{ for all } a \text{ and } b, \text{ if } t/s \text{ is irrational.}$$

**Table 6.2.** \((2,3)\)-zero-gain numbers for \(0 \leq a \leq 11, 0 \leq b \leq 7\).

<table>
<thead>
<tr>
<th>( a )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \zeta_{2,3}(a,0) )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \zeta_{2,3}(a,1) )</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \zeta_{2,3}(a,2) )</td>
<td>2</td>
<td>5</td>
<td>9</td>
<td>14</td>
<td>20</td>
<td>27</td>
<td>35</td>
<td>44</td>
<td>54</td>
<td>23</td>
<td>66</td>
<td>136</td>
</tr>
<tr>
<td>( \zeta_{2,3}(a,3) )</td>
<td>9</td>
<td>23</td>
<td>43</td>
<td>70</td>
<td>105</td>
<td>149</td>
<td>203</td>
<td>23</td>
<td>66</td>
<td>136</td>
<td>377</td>
<td>1144</td>
</tr>
<tr>
<td>( \zeta_{2,3}(a,4) )</td>
<td>23</td>
<td>66</td>
<td>136</td>
<td>241</td>
<td>390</td>
<td>593</td>
<td>984</td>
<td>1512</td>
<td>2470</td>
<td>377</td>
<td>1144</td>
<td>2504</td>
</tr>
<tr>
<td>( \zeta_{2,3}(a,5) )</td>
<td>136</td>
<td>377</td>
<td>767</td>
<td>1360</td>
<td>2050</td>
<td>3100</td>
<td>4590</td>
<td>6540</td>
<td>9130</td>
<td>2504</td>
<td>2504</td>
<td>2504</td>
</tr>
<tr>
<td>( \zeta_{2,3}(a,6) )</td>
<td>377</td>
<td>1144</td>
<td>2504</td>
<td>377</td>
<td>1144</td>
<td>2504</td>
<td>2504</td>
<td>2504</td>
<td>2504</td>
<td>2504</td>
<td>2504</td>
<td>2504</td>
</tr>
</tbody>
</table>

Other results include the following.

**Lemma 6.6.** Suppose \( as = bt \). Then \( \zeta_{s,t}(a,b) = \zeta_{t,s}(b,a) \).

This is a direct consequence of the Antiurn Theorem and the Reversal Lemma.

**Lemma 6.7.** Suppose \( as - bt \leq 0 \), and \( s \) and \( t \) are positive integers with \( \gcd(s,t) = 1 \). Then we have

$$\zeta_{s,t}(a,b) - \beta_{s,t}(a,b) = \sum_{k=1}^{b/s} \beta_{s,t}(a-kt,b-ks)\zeta_{s,t}(kt,ks).$$

**Proof.** For each zero-gain realization, there is a first time the urn is neutral, since \( X_{m+p} = 0 \). (From the ballot perspective, there is always a last tie.) As \( \gcd(s,t) = 1 \), the urn can only be neutral when there are \( kt \) “−s” balls and \( ks \) “+t” balls left for \( 0 \leq k \leq \lfloor b/s \rfloor \). The balls left form a valid \((s,t)\)-zero-gain realization with \( kt \) “−s” balls and \( ks \) “+t” balls, while the balls that have been drawn form a valid \((s,t)\)-ballot permutation on the remaining \( a - kt \) votes for \( A \) and \( b - ks \) votes for \( B \), with the two events independent. \(\square\)
If \( \gcd(s, t) > 1 \), we refer to the result obtained after using Lemma 6.5. If \( as = bt \), then the only contributing term is \( k = b/s \) (which is an integer since \( \gcd(s, t) = 1 \)). If \( t/s \) is irrational, then the only contributing term is \( k = 0 \), making the \((s, t)\)-ballot and \((s, t)\)-zero-gain numbers equal, as already noted.

A second decomposition of \( \zeta_{s,t}(a, b) \) can be made as well:

**Lemma 6.8.** Suppose \( as - bt \leq 0 \), and \( s \) and \( t \) are positive integers with \( \gcd(s, t) = 1 \). Then we have

\[
\zeta_{s,t}(a, b) - \beta_{s,t}(a, b) = \left\lfloor \frac{b}{s} \right\rfloor \sum_{k=1}^{\left\lfloor \frac{b}{s} \right\rfloor} \zeta_{s,t}(a - kt, b - ks) \beta_{s,t}(kt, ks - 1).
\]

**Proof.** Each \((a, b; s, t)\)-zero-gain number that is not a \((a, b; s, t)\)-ballot number has, in terms of the ballot problem, a first tie after the vote count starts. The vote forcing that tie is for \( B \). If the first tie occurs when \( k(t + s) \) votes have been counted, then the first \( k(t + s) - 1 \) votes form a \((kt, ks - 1; s, t)\)-ballot number, with the remaining votes making a \((a - kt, b - ks; s, t)\)-zero-gain number. \( \square \)

A decomposition of \( \beta_{s,t}(a, b) \) is not so easy. When \( s = 1 \), we have the scenario that bridges the gap between (i) and (ii).

**Lemma 6.9.** \( \beta_{1,t}(a, b) = \zeta_{1,t}(a - 1, b) \) for any \( a > 0 \) and positive integer \( t \).

Generally, we have \( \beta_{s,t}(a, b) = \beta_{1,\mu}(a, b) \), with \( \mu = t/s \) and \( \beta_{1,\mu}(a, b) \left(\frac{a+b}{a}\right)^{-1} \) given by the formula in Theorem 6.1. The constants \( C_1, C_2, \ldots \) are determined by the boundary conditions

\[
\beta_{1,\mu}\left(\lfloor b\mu \rfloor, b\right) = 0.
\]

For the zero-gain numbers, we can obtain them similarly by adjusting the boundary conditions. We have in this case

\[
\zeta_{1,\mu}(\lfloor b\mu \rfloor, b) = 0, \quad \text{if } b\mu \text{ is not an integer.}
\]

If \( b\mu \) is an integer, then we have a tie, but this time ties are permitted. Thus, we take a vote away from \( A \) to fill out the boundary condition:

\[
\zeta_{1,\mu}(b\mu - 1, b) = 0, \quad \text{if } b\mu \text{ is an integer.}
\]
If $\mu$ is irrational, we will not have any ties, so we reduce back to the ballot numbers. Otherwise, we have the following.

**Theorem 6.10.** Suppose $\mu = t/s$, with $s$ and $t$ positive integers. If $a \geq b\mu$, then

$$\left(\frac{a+b}{a}\right)^{-1}\zeta_{1,\mu}(a,b) = \left(\frac{a+b}{a}\right)^{-1}\sum_{j=0}^{b} D_j \left(\frac{a+b-1-j}{b-j}\right),$$

where $D_0 = 1$ and the constants $D_j$, $j = 1, 2, \ldots$ are given by the recurrence formula

$$\sum_{j=0}^{n} D_j \left(\frac{\lfloor n\mu - 1/s \rfloor + n - 1 - j}{n - j}\right) = 0, \quad n = 1, 2, \ldots$$

**Proof.** Since $\mu = t/s$, $n\mu$ has the form $x/s$, with $x$ an integer, and

$$\lfloor n\mu - 1/s \rfloor = \begin{cases} 
\lfloor n\mu \rfloor, & \text{if } n\mu \text{ is not an integer,} \\
\lfloor n\mu \rfloor - 1, & \text{if } n\mu \text{ is an integer.}
\end{cases}$$

Thus, we have the proper boundary conditions. \hfill \Box

**Example.** With $\mu = 1$, we have $D_j = 1 - j$ for $j = 0, 1, \ldots$ and we have

$$\zeta_{1,1}(a,b) = \sum_{j=0}^{b} (1-j) \left(\frac{a+b-1-j}{b-j}\right) = \frac{a-b+1}{a+1} \left(\frac{a+b}{a}\right),$$

as expected.

We conclude with some special cases.

**Proposition 6.11.** Suppose $s$ and $t$ are positive integers with $as - bt = 1$. Then

$$\beta_{s,t}(a,b) = \frac{1}{a+b} \left(\frac{a+b}{a}\right).$$

This Lemma is a direct corollary of a theorem by Raney [16] (1960):

**Theorem 6.12.** (Raney) If $\langle x_1, x_2, \ldots, x_m \rangle$ is any sequence of integers whose sum is $+1$, exactly one of the cyclic shifts

$$\langle x_1, x_2, \ldots, x_m \rangle, \langle x_2, x_3, \ldots, x_1 \rangle, \ldots, \langle x_m, x_1, \ldots, x_{m-1} \rangle$$

has all of its partial sums positive.
Here, the sequence of integers is of length \( a + b \), and is composed entirely of “+”s and “−”s, after which we multiply through by \(-1\) to obtain corresponding realizations. An outline of the proof can be found in [10, pp. 359-360].

![Figure 6.1. A (zero-pass) zero-gain realization from the \((7, 4; 2, 3)\) urn and its corresponding \((5, 7; 3, 2)\) ballot permutation (reflected through the \(n\)-axis and shifted left one unit.](image)

We have a corollary since \( \beta_{s,t}(a, b) = \beta_{s,t}(a, b - 1) \) if \( as - bt = 1 \).

**Corollary 6.13.** If \( s \) and \( t \) are positive integers with \( as - bt = t + 1 \), then

\[
\beta_{s,t}(a, b) = \frac{1}{a + b + 1} \left( \frac{a + b + 1}{a} \right) = \frac{1}{b + 1} \left( \frac{a + b}{a} \right).
\]

Proposition 6.11 also has a consequence with some \((s, t)\)-zero-gain numbers.

**Proposition 6.14.** Suppose \( s \) and \( t \) are integers with \( as - bt = t - 1 \). Then

\[
\zeta_{s,t}(a, b) = \frac{1}{a + b + 1} \left( \frac{a + b + 1}{a} \right) = \frac{1}{b + 1} \left( \frac{a + b}{a} \right).
\]

**Proof.** Start with a zero-gain realization \( \omega \) from the \((a, b; s, t)\) urn. This realization is naturally associated with the realization \( \hat{\omega} \) from the \((b, a; t, s)\) antiurn by changing the signs on the balls. We now attach a “−” ball to the beginning of \( \hat{\omega} \), forming a realization \( \hat{\omega} \) from the \((b + 1, a; t, s)\) urn,
with \( X_n(\omega) = X_{n-1}(\omega) \geq 0 \) for \( 1 \leq n \leq a + b \), and \( X_0(\omega) = -1 \). An example with a zero-gain realization from the \((7, 4; 2, 3)\) urn is given with Figure 6.1. Then the reversed realization \( \omega^R \) satisfies \( X_n(\omega^R) \leq -1 < 0 \) for \( n < a + b + 1 \). These are precisely the valid \((b + 1, a; t, s)\) ballot permutations. We thus apply Proposition 6.11.

We have noted from (6.2) and (6.3) that the number of realizations gaining \( pt - m \) from the \((m, p; 1, t)\) urn with \( pt > m \) equals

\[
\binom{m + p}{p} - \sum_{k=0}^{\left\lfloor (m-1)/t \right\rfloor} \frac{1}{kt + k + 1} \binom{kt + k + 1}{k} \binom{m + p - k - kt - 1}{p - k}.
\]

When we apply the antiurn map to these realizations, they become zero-gain realizations (via a different strategy) from the \((p, m; 1, t)\) urn, and thus the \((p, m; 1, 1/t)\) urn. Therefore, we have:

**Theorem 6.15.** Suppose \( t \) is a positive integer. Then, if \( a \geq b/t \) we have

\[
\Zeta_{1,1/t}(a, b) = \binom{a + b}{b} - \sum_{k=0}^{\left\lfloor (b-1)/t \right\rfloor} \frac{1}{kt + k + 1} \binom{kt + k + 1}{k} \binom{a + b - k - kt - 1}{a - k}.
\]

A similar procedure can net the \((1, 1/t)\)-ballot numbers.

**Theorem 6.16.** Suppose \( t \) is a positive integer. Then, if \( a \geq b/t \) we have

\[
\Beta_{1,1/t}(a, b) = \binom{a + b - 1}{b} - \sum_{k=0}^{\left\lfloor b/t \right\rfloor - 1} \frac{t}{kt + k + t} \binom{kt + k + 1}{k} \binom{a + b - 1 - k - kt - t}{a - 1 - k}.
\]

**Proof.** Observe that a realization from the \((b, a; 1, t)\) urn with \( at > b \) and \( X_n > 0 \) for \( n \neq a + b \) is, in fact, a \((t, 1)\)-ballot \(((1, 1/t)\)-ballot\) permutation upon the antiurn reflection. We calculate their number. Obviously, the last ball must be a \("+t\)," so we remove it. Upon reversing the remaining part, we see that this is a realization from the \((b, a - 1; 1, t)\) urn with \( X_n < at - b \) for all \( n \). The number of such realizations equals

\[
\binom{a + b - 1}{b} - Q_{b,a-1}(at - b),
\]

where \( Q_{b,a-1}(at - b) \) is given by Theorem 3.26. \( \square \)

The proofs of Theorems 6.15 and 6.16 are more in line with those of the traditional ballot problems, as we count the bad ballot permutations instead of the good ones. Furthermore, in the proof of Theorem 6.16, we deleted the last (or first, depending on perspective) vote (a vote for \( A \)), as was
done with the traditional ballot problem (Lemma 6.9). With some manipulation, we can arrive at the following alternative forms for $\beta_{1,t}(a,b)$ and $\zeta_{1,t}(a,b)$:

$$\beta_{1,t}(a,b) = \left\lceil \frac{(a-1)t-b-1}{t+1} \right\rceil \sum_{j=0}^{\left\lceil \frac{at-b-1}{t+1} \right\rceil} \frac{(-1)^j t}{(a-j)(t+1)-1} \left( \frac{(a-j)(t+1)-1}{a-1-j} \right) \left( \frac{at-b-1-jt}{j} \right),$$

$$\zeta_{1,t}(a,b) = \left\lceil \frac{at-b-1}{t+1} \right\rceil \sum_{j=0}^{\left\lceil \frac{at-b-1}{t+1} \right\rceil} \frac{(-1)^j}{(a-j)(t+1)+1} \left( \frac{(a-j)(t+1)+1}{a-j} \right) \left( \frac{at-b-1-jt}{j} \right).$$

The application of the generalized ballot numbers to the problem of minimal gain is summarized with the next theorem.

**Theorem 6.17.** For the $(m,p;s,t)$ urn:

1) If $pt - ms \leq 0$, the probability the player will gain zero using any optimal strategy is

$$\left( \frac{m+p}{p} \right)^{-1} \zeta_{1,t/s}(m,p),$$

where $\zeta_{1,t/s}(m,p)$ is given by the result of Theorem 6.10 if $t/s$ is rational, and

$$\zeta_{1,t/s}(m,p) = \beta_{1,t/s}(m,p)$$

is given by the result of Theorem 6.1 if $t/s$ is irrational.

2) If $pt - ms \geq 0$, then the probability the player gains $pt - ms$ using any optimal strategy is the same as the probability of zero gain for the $(p,m;t,s)$ urn.

**Proof.** If $pt - ms \geq 0$, if the player uses the zero-bet strategy, then zero-gain realizations from the $(p,m;t,s)$ urn with the zero-pass strategy correspond to the minimal gain realizations from the $(m,p;s,t)$ urn via the natural antiurn map.

6.4. Another Extension of the Ballot Problem. We have noted that the solutions thus far to the two ballot problems are not pretty outside of a few nice cases. Now, we will extend the problem again, but this time we will expand upon the answer instead. Recall that when $\mu$ is a positive integer, the solutions to the ballot problem, versions (i) and (ii), are respectively

$$\frac{a-b\mu}{a+b} \left( \frac{a+b}{a} \right), \quad \frac{a-b\mu+1}{a+1} \left( \frac{a+b}{a} \right).$$

We will present a question with an answer for $\mu$ rational, reducing to (6.6) when $\mu$ is a positive integer. To do that, we shall need more notation.
Denote by $\zeta_{s,t}^{(i)}(a,b)$ as the number of ballot permutations for which candidate $A$ never trails by more than $i$ in weight once the vote count starts. Thus, $\zeta_{s,t}^{(0)}(a,b)$ is the $(s,t)$-zero-gain number and $\zeta_{s,t}^{(-1)}(a,b)$ is the $(s,t)$-ballot number. We now present the result.

**Theorem 6.18.** Suppose $s$ and $t$ are positive integers with $\gcd(s,t) = 1$. If $as \geq bt$, then

$$
(6.7) \quad \sum_{i=0}^{s-1} \zeta_{s,t}^{(i)}(a,b) = \frac{s(a+1) - bt}{a+1} \binom{a+b}{b}.
$$

If $as > bt$, then

$$
(6.8) \quad \sum_{i=1}^{s} \zeta_{s,t}^{(-i)}(a,b) = \frac{as - bt}{a+b} \binom{a+b}{b}.
$$

Observe that we have a reduction to (6.6) when $s = 1$. Theorem 6.18 is actually a special case of a more general result found by Irving and Rattan [12] (2009). We shall present their result shortly, but we shall need to adopt their notation.

A particle begins at the point $(0,0)$, and at each stage moves either one unit to the right or one unit up, eventually reaching the point $(n,m)$.

**Definition 6.19.** A weak $m$-part composition of $n$ is a list $f = (f_0, \ldots, f_{m-1})$ with $f_j$ a nonnegative integer for each $j$ and $\sum_{i=0}^{m-1} f_j = n$.

Each $m$-part composition of $n$ induces the piecewise linear boundary $\partial f$ defined by

$$
x = f_i(y - i) + \sum_{j=0}^{i-1} f_j, \quad y \in [i, i+1].
$$

Any point or path lying weakly below the boundary $\partial f$ is said to be dominated by $f$. Irving and Rattan’s result relates to the cyclic shifts of $f$. The $j$th cyclic shift of $f$ is

$$
f^{(j)} = (f_{m-1-j}, \ldots, f_{m-1}, f_0, \ldots, f_{m-2-j}).
$$

For any $f$, denote by $D(f)$ as the number of paths dominated by $\partial f$. The main result of [12] is the following:

**Theorem 6.20.** (Irving and Rattan) Let $f$ be a weak $m$-part composition of $n$ and let $T = (a,b)$, where $0 \leq a \leq n$ and $0 \leq b \leq m$. If the point $T' = (a+1,b)$ lies weakly to the right of $\partial f^{(j)}$ for all
Then

\[ \sum_{j=0}^{m-1} D(f^{(j)}) = \frac{m(a + 1) - nb}{a + 1} \left( \frac{a + b}{b} \right). \]

For the \((m, p; s, t)\) scenario, our boundary is the line \(y = sx/t\) and the urn empties at \((0, 0)\) so that realizations run in reverse, and our aim is to replace \(y = sx/t\) with a boundary more in line with Irving and Rattan’s result. Note that we only need to figure out what occurs up to the point \((t, s)\), as the pattern resets at that point. Lemma 6.21 gives the composition of the fundamental period of the piecewise linear boundary we shall use to mime the line \(y = sx/t\).

**Lemma 6.21.** Suppose \(\gcd(s, t) = 1\), and \(r\) is the integer satisfying \(r < t/s < r + 1\). Define the weak \(s\)-part decomposition \(f = (f_0, \ldots, f_{s-1})\) of \(t\) so that the following are satisfied:

1. \(f_i \in \{r, r + 1\}\) for each \(i\),
2. \(s \sum_{i=0}^{n-1} f_i \geq nt\) for each \(1 \leq n \leq t\),
3. \(\sum_{i=0}^{n-1} f_i\) is minimal.

Then for any realization from the \((t, s; s, t)\) urn

\[ \omega \text{ is dominated by } \partial f \iff X_n \leq 0 \text{ for all } n. \]

**Proof.** Since \(\gcd(s, t) = 1\), for any \(1 - s \leq i < 0\) there is a unique \(a_i\) and \(b_i\) with \(0 \leq a_i \leq t\) and \(0 \leq b_i \leq s\) with \(a_is - b_it = i\), while the only combinations that give zero are \((0, 0)\) and \((t, s)\), respectively. Thus, it suffices to show that

\[ \left\{ nt - \sum_{i=0}^{n-1} sf_i \right\}_{n=0}^{s-1} = \{1 - s, \ldots, 0\}. \]

By the definition of \(f\), \(nt - \sum_{i=0}^{n-1} sf_i \leq 0\) for each \(n\). Suppose \(n\) is such that \(nt - \sum_{i=0}^{n-1} sf_i \leq 0\) for each \(n\). Suppose \(n\) is so that \(nt - \sum_{i=0}^{n-1} sf_i \leq -s\). Then, if \(f_{n-1} = r\), we have \((n - 1)t - \sum_{i=0}^{n-2} sf_i \leq -s + rs - t \leq -s\), contradicting the minimality of \(n\). If \(f_{n-1} = r + 1\), then \(nt - rs - \sum_{i=0}^{n-2} sf_i = nt + s - \sum_{i=0}^{n-1} sf_i \leq 0\), contradicting the minimality requirement of the definition of \(f\). Therefore, \(nt - \sum_{i=0}^{n-1} sf_i\) is between \(1 - s\) and 0 for each \(n\). Since each \(r\) corresponds to \(r\) “–\(s\)” balls and one “+\(t\)” ball, and similarly for \(r + 1\), it follows from \(\gcd(s, t) = 1\) that the partial sums are distinct, and thus the collection of partial sums is in fact \(\{1 - s, \ldots, 0\}\). \(\square\)

Figure 6.2 shows the piecewise linear boundary obtained by Lemma 6.21 with \(s = 8\) and \(t = 3\).

Now, we observe the properties of the cyclic shifts of \(f\).
Figure 6.2. For $s = 8$, $t = 3$, we have $r = 0$, $f = (1, 0, 1, 0, 0, 1, 0, 0)$, and the piecewise boundary $\partial f$.

**Lemma 6.22.** Let $f$ be as defined in Lemma 6.21. If $kt - \sum_{i=0}^{k-1} sf_i = -j$, then the paths dominated by $\partial f^{(s-k)}$ satisfy $X_n \leq j$ for all $n$.

**Proof.** If $kt - \sum_{i=0}^{k-1} sf_i = -j$, then for $k < \ell \leq s$

\begin{equation}
1 - s \leq \ell t - \sum_{i=0}^{\ell-1} sf_i \leq 0
\end{equation}

implies that

\begin{equation}
1 - s + j \leq (\ell - k)t - \sum_{i=k}^{\ell-1} sf_i \leq j.
\end{equation}

Since

$$(s - k)t - \sum_{i=k}^{s-1} sf_i = j$$

and (6.9) holds for $\ell \leq k$, we have that (6.10) holds for $k < \ell \leq s - 1 + k$ (where $f_i$ is understood to be $f_{i \mod s}$). Therefore, we have that

$$\left\{ (\ell - k)t - \sum_{i=k}^{\ell-1} sf_i \right\}^{s+k}_{\ell=k+1} = \{1 - s + j, \ldots, j\}.$$ 

This shows that the boundary $\partial f^{(s-k)}$ dominates any path satisfying $X_n \leq j$ for all $n$. \qed

To extend this to the $(m, p; s, t)$ urns, we merely repeat the boundary $\partial f$ as many times as necessary to cover the initial urn point $(p, m)$, in which case $m$ times will certainly do the job. Let $f^m$ denote the generalized boundary (the terminus of $\partial f^m$ is $(ms, mt)$). The result of Theorem 6.20
is that for $as - bt \geq 0$,

$$
(6.11) \quad \sum_{j=0}^{ms-1} D \left( \partial (f^m)^{(j)} \right) = \frac{m(s(a+1)-bt)}{a+1} \binom{a+b}{a}.
$$

We then observe that only $s$ of the boundary shifts are unique due to the repetition of $f$, with each unique shift repeated $m$ times. Thus (6.11) implies that

$$
(6.12) \quad \sum_{j=0}^{s-1} D \left( \partial (f^m)^{(j)} \right) = \frac{s(a+1)-bt}{a+1} \binom{a+b}{a}.
$$

The result of Lemma 6.22 holds with $f^m$. As a result, for each $0 \leq j \leq s-1$ there is a unique $0 \leq i \leq s-1$ with

$$
D \left( (f^m)^{(j)} \right) = \zeta_{s,t}^{(i)}(a, b).
$$

The first result of Theorem 6.10 now follows. Adjoining a “−s” to the end (in the lattice path version, starting from the initial point $(0, -1)$ instead), gives the second result. That is,

$$
\sum_{i=1}^{s} \zeta_{s,t}^{(i-1)}(a, b) = \sum_{i=0}^{s-1} \zeta_{s,t}^{(i)}(a-1, b) = \frac{as - bt}{a} \binom{a+b-1}{a-1} = \frac{as - bt}{a+b} \binom{a+b}{a}.
$$

This completes the proof of Theorem 6.10. Observe that since $\zeta_{s,t}^{(i)}(a, b)$ is increasing in $i$, we can use the results of Theorem 6.10 to obtain upper and lower bounds on the generalized ballot numbers.

### 6.5. The First Crossing

Recall that when $t$ is a positive integer and $m < pt$, we have

$$
G(m, p; 1, t) = pt - m + \kappa + \binom{m+p}{p}^{-1} \sum_{k=1}^{[m/p]} \binom{kt+k}{k} \binom{m+p-k-kt}{p-k},
$$

where $0 \leq \kappa \leq \frac{t-1}{2}$. We now define $\kappa$, the crossing number, for the $(m, p; s, t)$ urns.

**Definition 6.23.** Suppose $ms - pt \geq 0$. Let $\rho = \min \{ j : X_j \geq 0 \}$. Then $\kappa = E[X_\rho]$ is called the crossing number. If $ms - pt \leq 0$, then we define $\kappa$ to equal the crossing number from the $(p, m; t, s)$ urn.

The crossing number is simply the average gain the player will achieve at the first crossing. When $s$ and $t$ are positive integers, we can use $(s, t)$-ballot and $(s, t)$-zero-gain numbers to calculate some probabilities associated with the crossing number $\kappa$. 

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Proposition 6.24. Suppose $s$ and $t$ are positive integers, and $ms - pt > 0$. Then

$$P(X_\rho = 0) = \left(\frac{m+p}{p}\right)^{-1} \sum_{k=0}^{\lfloor p/s \rfloor} \binom{k(t+s)}{ks} \beta_{s,t}(m-kt,p-ks).$$

Proof. Given a realization $\omega$ with $X_\rho = 0$, we will have $\rho = m + p - k(t+s)$ with $0 \leq k \leq \lfloor p/s \rfloor$. The first $\rho$ balls must form a valid $(m-kt,p-ks;s,t)$-ballot number. The last $k(t+s)$ balls can be drawn out in any fashion. \qed

Proposition 6.25. Suppose $s$ and $t$ are positive integers, and $ms - pt > 0$. Then

$$P(X_\rho = s - 1) = \left(\frac{m+p}{p}\right)^{-1} M(s-1) \sum_{k=0}^{M(s-1)} \binom{k(t+s)+c_{s-1}}{ks+b_{s-1}} \zeta_{s,t}(m-kt-a_{s-1}-1,p-ks-b_{s-1}),$$

where $M(s-1)$ is defined according to the conditions set in (4.4).

Proof. Let $\omega$ be a realization with $X_\rho = s - 1$. $\omega$ can be divided into three parts: The first $\rho - 1$ balls, for which $X_j \leq X_{\rho-1} = -1$ for $j < \rho$, the $\rho$th ball, a "$-s$" ball, and the remaining $m+p-\rho$ balls. The first $\rho$ balls form a $(s,t)$-zero-gain realization, upon removal of the remaining $m+p-\rho-1$ balls. The third part is composed of $ks+b_{s-1}$ "+t" balls and $kt+a_{s-1}$ "$-s$" balls with $0 \leq k \leq M(s-1)$, and they can be drawn in any order. Thus, the first $\rho$ balls consist of $m-kt-a_{s-1}-1$ "$-s$" balls (since another is used for the second part, the first crossing) and $p-ks-b_{s-1}$ "+t" balls. \qed

The remaining probabilities can be calculated, but become extremely complicated, with more and more cases to consider. We omit the details. Using Lemma 2.1, we can extend the previous results to cover the urns with $t/s$ rational. When $t/s$ is irrational, then since no two up-crossings are associated with the same gain, calculation of the various probabilities becomes very difficult in general.
7. A Bayesian Approach

In application of this model, the total number of balls is usually known, the number of "+t" balls is not, and is random. Thus, as in Chen [7] and Boyce [4], a Bayesian approach would be appropriate. We randomize the urn by assigning each urn with \( n \) balls a probability of occurring. We may interpret this initial prior distribution \( \theta \) as the player's outlook over the \( n \) draws that will follow. An optimal strategy can be determined based on this initial outlook \( \theta \), and in most cases it will be quite complicated. Thus, finding a formula for the expected gain will be very difficult in general, and we will focus most of our efforts on the original Bayesian urn with \( s = t = 1 \).

We will start with the general Bayesian urn, and develop the indicator (7.3) that determines whether a bet should be placed on the next ball based on the outcome of the previous draws. Then, working with the original random urn, we will find two families of distributions with a relatively simple betting rule. We will then calculate the expected gain for those distributions. For the remaining distributions, we present an algorithm similar to that of Boyce [4] that produces the expected gain. We then return back to the general random urn, taking a closer look at the cases where \( \theta \) is binomial and uniform, finding an optimal betting strategy and calculating the expected gain for both. Finally, we adapt the algorithm so that it produces the expected gain for any \( s, t, \) and \( \theta \).

Remark. In this chapter, the number of "+t" balls \( p \) is not known, and is a random variable.

7.1. Preliminaries. We begin by defining the random acceptance urn.

Definition 7.1. The random acceptance urn with \( n \) balls of value \(-s\) and \(+t\) and initial prior distribution \( \theta = \{q_j\}_{j=0}^n \) of the number of "+t" balls in the urn is called the \((n,\theta; s,t)\) urn.

We seek an optimal strategy maximizing the expected gain, which we shall denote as \( \mathcal{G}(n,\theta; s,t) \). As in [7], Let \( n = m + p \) be the total number of balls in the urn, and let \( \theta \) be the initial prior distribution of the random variable \( p \), and let \( Y_i \) denote the weight of the \( i^{th} \) ball drawn. Let \( \mathcal{A}(n,\theta; s,t) \) be the expected gain if a bet is placed on the first ball, and an optimal Bayesian betting policy is followed thereafter; Let \( \mathcal{B}(n,\theta; s,t) \) be the expected gain if no bet is placed, and an optimal
Bayesian betting policy is followed thereafter. Let

\[ G(n, \theta; s, t) = \max\{A(n, \theta; s, t), B(n, \theta; s, t)\} \]

denote the value of the urn with \( n \) balls and prior distribution \( \theta \).

Let \( y_1 \) be the weight of the first drawn ball. Then

\[ A(n, \theta; s, t) = \int (y_1 + G(n-1, \theta(y_1); s, t)) \theta(dy_1), \quad B(n, \theta; s, t) = \int G(n-1, \theta(y_1); s, t) \theta(dy_1), \]

where \( \theta(y_1) \) is the posterior distribution of the number of balls of weight \( +t \) after the first draw given that \( Y_1 = y_1 \). Then \( A(n, \theta; s, t) \geq B(n, \theta; s, t) \) if and only if

\[ \int y_1 \theta(dy_1) = t \cdot \theta(Y_1 = t) - s \cdot \theta(Y_1 = -s) \geq 0, \]

we would bet if \( t \cdot \theta(Y_1 = t) \geq s \cdot \theta(Y_1 = -s) \), i.e. \( \theta(Y_1 = t) \geq s/(s + t) \). Therefore, an optimal Bayesian betting policy can be stated as follows: For \( 0 \leq k \leq n - 1 \), bet on the \((k + 1)\)th draw if and only if

\[ t \cdot \theta(Y_{k+1} = t \mid y_1, y_2, \ldots, y_k) \geq s \cdot \theta(Y_{k+1} = -s \mid y_1, y_2, \ldots, y_k), \]

i.e. \( \theta(Y_{k+1} = t \mid y_1, y_2, \ldots, y_k) \geq s/(s + t) \), where \( \theta(\cdot \mid y_1, y_2, \ldots, y_k) \) is the posterior distribution of the number of \( +t \) balls given that \( Y_1 = y_1, Y_2 = y_2, \ldots, Y_k = y_k \).

**Remark.** If \( \theta(Y_{k+1} = t \mid y_1, y_2, \ldots, y_k) = s/(s + t) \), then the expected gain on the next drawn ball is zero. Hence, the policy of betting if and only if \( \theta(Y_{k+1} = t \mid y_1, y_2, \ldots, y_k) > s/(s + t) \) is also optimal. Since this strategy gives the same expected gain, we shall not use this strategy much in this chapter. We shall use this strategy when we discuss the ruin problem, in Chapter 8.

Since each nonrandom acceptance urn has an antiurn, the Bayesian version possesses its own version of the antiurn property.

**Theorem 7.2.** Let \( \theta = \{q_j\}_{j=0}^n \), where \( q_j \) is the probability the urn contains \( j \) \( +t \) balls initially, and let \( \mu = \sum_{j=0}^n j q_j \). Define \( \theta^R = \{w_j\}_{j=0}^n \) so that \( w_j = q_{n-j} \) for all \( j \). Then

\[ (7.1) \quad G(n, \theta; s, t) - G(n, \theta^R; t, s) = (t + s)\mu - ns. \]

**Proof.** We use a similar strategy to the proof of the Antiurn Theorem. Adam and Betty will play the \((n, \theta; s, t)\) urn simultaneously. Adam will use the (primary) optimal strategy, while Betty will
bet if and only if Adam does not bet. Then, exactly one of the two will bet on each ball drawn from the urn. Therefore, their combined expected gain will be \((t+s)\mu - ns\). Adam’s expected gain is \(\mathcal{G}(n, \theta; s, t)\), thus it remains to show that Betty’s expected gain is \(-\mathcal{G}(n, \theta^R; t, s)\).

Betty places a bet on the \((k + 1)\text{th}\) ball if and only if

\[ \theta(Y_{k+1} = t \mid y_1, \ldots, y_k) < \frac{s}{t + s}. \]

Since she bets if and only if Adam passes, her expected gain is minimal. Therefore, if all of the ball weights were multiplied by \(-1\), she would be maximizing her expected gain. Upon doing this, we find that \(\theta^R\) is the initial distribution of this (anti)urn. Letting \(Y'_i = -Y_i\) and \(y'_i = -y_i\), Betty bets if and only if

\[ \theta^R(Y'_{k+1} = -t \mid y'_1, \ldots, y'_k) < \frac{s}{t + s} \iff \theta^R(Y'_{k+1} = s \mid y'_1, \ldots, y'_k) > \frac{t}{t + s}. \]

This is an optimal strategy for the \((n, \theta^R; t, s)\) urn. Therefore, Betty’s expected gain is \(-\mathcal{G}(n, \theta^R; t, s)\), and (7.1) is shown. \(\square\)

**Remark.** If \(\theta\) is symmetric, that is, \(\theta = \theta^R\), then \(\mu = n/2\) and Theorem 7.2 implies that

\[ (7.2) \quad \mathcal{G}(n, \theta; s, t) - \mathcal{G}(n, \theta; t, s) = (t + s)\frac{n}{2} - ns = \frac{n(t - s)}{2}. \]

7.2. **Determining the Betting Rule.** Given an initial prior distribution \(\theta\), when is

\[ \theta(Y_{k+1} = t \mid y_1, y_2, \ldots, y_k) \geq \frac{s}{t + s}? \]

Let \(\theta = \{q_j\}_{j=0}^n\) be the distribution of the urns with \(n\) balls, i.e.

\[ q_j = \theta(\text{urn contains } j \text{ “}+t\text{” balls initially}), \quad 0 \leq j \leq n. \]

Suppose \(k\) balls have been drawn, with \(\ell\) of them “+t” balls. For \(0 \leq j \leq n\), let

\[ q_j(k, \ell) = q_j \binom{n}{j}^{-1} \binom{n-k}{j-\ell}. \]

For each \(p\), \(q_p(k, \ell)\) is the new “weight” associated with the urn initially containing \(p \text{ “}+t\text{” balls, as only } \binom{n-k}{p-\ell} \text{ of the } \binom{n}{p} \text{ realizations are still possible. (Given that the urn has } p \text{ “}+t\text{” balls, each realization must occur with equal probability, thus } q_p \text{ is decimated by this proportion.) Note that} \]

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if $\ell > p$ then the weight is zero. Thus the adjusted probabilities are

$$P(\text{urn has } p+ t \text{ balls} \mid \ell \text{ of the first } k \text{ balls drawn were } p+ t) = q_p(k, \ell) \left( \sum_{j=0}^{n} q_j(k, \ell) \right)^{-1}.$$

Given that the urn has $p+ t$ balls, the probability the next ball is a $p+ t$ equals $(p - \ell)/(n - k)$.

Thus, we should place a bet on the next ball if

$$\sum_{i=0}^{n} \frac{i - \ell}{n - k} q_i(k, l) \geq \frac{s}{t + s}.$$

We have the following result, using a different form of this equation.

**Theorem 7.3.** For the $(n, \theta; s, t)$ urn, if $k$ balls have been drawn, with $\ell$ of them $p+ t$, then an optimal acceptance policy is to accept the $(k + 1)$th ball if and only if

$$(7.3) \quad \sum_{j=0}^{n} q_j(k, \ell) [(j - \ell)(t + s) - s(n - k)] \geq 0.$$ 

**Remark.** If the player bets if and only if

$$\sum_{j=0}^{n} q_j(k, \ell) [(j - \ell)(t + s) - s(n - k)] > 0,$$

then his strategy is also optimal, since when equality holds in (7.3), the expected weight of the next ball equals zero.

7.3. **The Original Random Urn.** Using (7.3), we find an expanded set of initial distributions for which the policy stated in [7] is optimal. A second family of distributions, for which the opposite rule is optimal, will also be discussed.

**Lemma 7.4.** Suppose the probability distribution $\theta = \{q_j\}_{j=0}^{n}$ satisfies $q_j = q_{n-j}$. Then an optimal betting strategy for the $(n, \theta; 1, 1)$ urn is

$$(7.4) \quad \text{“bet on the } (k + 1)^{\text{th}} \text{ ball if and only if } \sum_{i=1}^{k} Y_i \geq 0”$$

if and only if $0 \leq j_1 \leq j_2 \leq n/2$ implies

$$(7.5) \quad \left( \begin{array}{c} n \\ j_1 \end{array} \right)^{-1} q_{j_1} \geq \left( \begin{array}{c} n \\ j_2 \end{array} \right)^{-1} q_{j_2}.$$
Proof. Suppose that the strategy indicated by (7.4) is optimal for the \((n, \theta; 1, 1)\) urn. If \(n - 1\) balls have been drawn, \(\ell\) of them “+1,” then the indicator (7.3) reduces to

\[
\sum_{j=\ell}^{\ell+1} \binom{n}{j}^{-1} q_j [2(j-\ell) - 1] = \left( \frac{n}{\ell + 1} \right)^{-1} q_{\ell+1} - \left( \frac{n}{\ell} \right)^{-1} q_{\ell}.
\]

If \(2\ell < n - 1\), then by (7.4) the player will not bet, thus (7.6) is negative. This holds for all \(0 \leq \ell \leq n/2 - 1\). Then, for \(0 \leq j_1 \leq j_2 \leq n/2\), we have

\[
\binom{n}{j_1}^{-1} q_{j_1} \leq \binom{n}{j_2}^{-1} q_{j_2}.
\]

for any \(\theta\), symmetric or not. Similarly, if \(2\ell \geq n - 1\), then \((n - 1)/2 \leq j_2 \leq j_1 \leq n\) implies

\[
\binom{n}{j_1}^{-1} q_{j_1} \leq \binom{n}{j_2}^{-1} q_{j_2}.
\]

As for the reverse implication, since \(\theta\) is symmetric and all the balls have absolute value 1, the \((n, \theta; 1, 1)\) urn is its own antiurn. Thus we need to show that if \(\sum_{i=1}^{k} Y_i \geq 0\), then (7.3) with \(s = t = 1\) holds. We potentially have a problem if equality holds in (7.3), as we could place a bet on the antiurn counterpart. Since the expected value of the next ball is zero under these circumstances, making either option optimal, we adopt the following convention: If \(\sum_{i=1}^{k} Y_i < 0\) and equality holds in (7.3), we pass instead of placing a bet.

If \(\sum_{i=1}^{k} Y_i \geq 0\), then \(2\ell \geq k\). We now look at (7.3), and consider the contributions of the terms with \(j = p\) and \(j = n - k - p + 2\ell\), with \(\ell \leq p \leq (n - k + 2\ell)/2\). Then

\[
\binom{n-k}{p-\ell} = \binom{n-k}{n-k-p+\ell} = \binom{n-k}{(n-k-p+2\ell)-\ell}
\]

and

\[
2(p-\ell) - (n-k) = -2 ((n-k-p+2\ell)-\ell) - (n-k),
\]

the latter showing that if \(p = (n-k+2\ell)/2\), the unpaired “center” term equals zero. The contribution of the two terms to (7.3) is thus

\[
\binom{n-k}{p-\ell} [(n-k) - 2(p-\ell)] \left[ \left( \frac{n}{n-k-p+2\ell} \right)^{-1} q_{n-k-p+2\ell} - \left( \frac{n}{p} \right)^{-1} q_{p} \right].
\]

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Since \( p \leq (n - k + 2\ell)/2 \), we have 
\[
(n - k) - 2(p - \ell) \geq 0.
\]
All that remains is determining the sign of the remaining part,
\[
\binom{n}{n - k - p + 2\ell}^{-1} q_{n-k-p+2\ell} - \binom{n}{p}^{-1} q_p.
\]

**Case 1.** If \( p \leq n/2 \), then
\[
\binom{n}{n - k - p + 2\ell}^{-1} q_{n-k-p+2\ell} = \binom{n}{p + k - 2\ell}^{-1} q_{p+k-2\ell},
\]
and since \( 2\ell \geq k \) we have by assumption that \( \binom{n}{p+k-2\ell}^{-1} q_{p+k-2\ell} \geq \binom{n}{p}^{-1} q_p \), thus the contributions of these two terms to (7.3) is nonnegative.

**Case 2.** If \( p > n/2 \), then \( p \leq (n - k + 2\ell)/2 \) implies \( n/2 \leq p \leq n - k - p + 2\ell \), and
\[
\binom{n}{n - k - p + 2\ell}^{-1} q_{n-k-p+2\ell} - \binom{n}{p}^{-1} q_p \geq 0
\]
follows by the symmetry of the distribution.

All other terms contribute zero. Therefore, we have shown that (7.3) will be satisfied, and the proof is complete. \( \square \)

**Remark.** Consider the distribution for which the urns containing \( n > 1 \) “+1” balls or \( n \) “−1” balls have probability zero, with all other urns equally likely. Then, if the first \( n - 1 \) draws result in all “+1” balls, \( \sum_{i=1}^{n-1} Y_i = n - 1 \geq 0 \), but the player would know the last ball is “−1” and not place a bet. Similarly, weighting the probabilities so that closer-to-neutral urns are more likely produce a point of diminished returns - if \( \sum_{i=1}^{k} Y_i \) becomes too large, the sum is more likely to decrease (a sell-off), and the player will stop betting, and if the sum becomes large on the negative side, the sum is more likely to increase (a rally), and the player will start to bet.

We have the tools to give an explicit formula for \( \overline{G}(n, \theta; 1, 1) \), where \( \theta \) satisfies the conditions of Lemma 7.4, based on properties of various \( (n - p, p; 1, 1) \) urns. Recall from Theorem 3.2 that for \( m \geq p \), 
\[
G(m, p; 1, 1) = \binom{m+p}{p}^{-1} \sum_{i=0}^{p-1} \binom{m+p}{i}.
\]
For \( 0 \leq p \leq n \), denote
\[
\overline{G}(n, p) = E\left(\overline{G}(n, \theta; 1, 1) \mid \text{there are } p \text{ “+1” balls initially}\right).
\]

For any initial prior distribution \( \theta \), the total probability theorem gives us that
\[
(7.7) \quad \overline{G}(n, \theta; 1, 1) = \sum_{j=0}^{n} q_j \overline{G}(n, j).
\]
If \( \theta \) satisfies the conditions of Lemma 7.4, then

\[
G(n,p) = G_{\leq 2p-n}(n-p,p;1,1), \quad 0 \leq p \leq n.
\]

**Theorem 7.5.** If \( \theta \) is a distribution satisfying the requirements of Lemma 7.4, then

\[
G(n,\theta;1,1) = \sum_{j=0}^{n} q_j G(n,j),
\]

where

\[
G(n,j) = \begin{cases} 
-\left(\begin{array}{c} n \\
 j 
\end{array}\right)^{-1} \sum_{i=0}^{j} \left(\begin{array}{c} n \\
 i 
\end{array}\right), & \text{if } 0 \leq j < n/2, \\
(2j-n) - \left(\begin{array}{c} n \\
 j 
\end{array}\right)^{-1} \sum_{i=0}^{n-j-1} \left(\begin{array}{c} n \\
 i 
\end{array}\right), & \text{if } n/2 \leq j \leq n.
\end{cases}
\]

**Proof.** It suffices from (7.7) and (7.8) to calculate \( G_{\leq 2j-n}(n-j,j;1,1) \) for \( 0 \leq j \leq n \). We have two cases to prove:

**Case 1.** If \( 0 \leq j \leq n/2 \), then from Corollary 3.22 and Theorem 3.2 we have

\[
G_{\leq 2j-n}(n-j,j;1,1) = -1 - G(n-j,j;1,1) = -\left(\begin{array}{c} n \\
 j 
\end{array}\right)^{-1} \sum_{i=0}^{j} \left(\begin{array}{c} n \\
 i 
\end{array}\right).
\]

**Case 2.** If \( n/2 < j \leq n \), then by the Extended Antiurn Theorem (twice), Theorem 3.20, and Theorem 3.2 we have

\[
G_{\leq 2j-n}(n-j,j;1,1) = (2j-n) - G_{> 2j-n}(n-j,j;1,1) = 2(2j-n) - G_{\geq n-2j}(j,n-j;1,1)
\]

\[
= (2j-n) - G(j,n-j;1,1) = (2j-n) - \left(\begin{array}{c} n \\
 j 
\end{array}\right)^{-1} \sum_{i=0}^{n-j-1} \left(\begin{array}{c} n \\
 i 
\end{array}\right).
\]

Having shown that (7.9) holds, we use the total probability rule to complete the proof.

Both the uniform and binomial distributions satisfy the requirements of Lemma 7.4. When \( \theta \) is uniform, we have the following result.

**Corollary 7.6.** Suppose \( \theta \) is the uniform distribution on \( \{0,1,\ldots,n\} \).

1) If \( n \) is odd, then

\[
(n+1)G(n,\theta;1,1) = \frac{n^2}{4} - 1 - 2 \sum_{j=1}^{[n/2]} \sum_{i=0}^{j-1} \left(\begin{array}{c} n \\
 i 
\end{array}\right) \left(\begin{array}{c} n \\
 j 
\end{array}\right)^{-1}.
\]

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2) If \( n \) is even, then

\[
(n + 1)G(n, \theta; 1, 1) = \frac{n^2 + 2}{4} - 2^{n-1} \left( \frac{n}{n/2} \right)^{-1} - 2 \sum_{j=1}^{n/2-1} \sum_{i=0}^{j-1} \left( \frac{n}{i} \right) \left( \frac{n}{j} \right)^{-1}.
\]

The formulas of Corollary 7.6 above give

\[
G(1, \theta; 1, 1) = 0, \quad G(2, \theta; 1, 1) = \frac{1}{6}, \quad G(3, \theta; 1, 1) = \frac{1}{3},
\]

\[
G(4, \theta; 1, 1) = \frac{8}{15}, \quad G(5, \theta; 1, 1) = \frac{11}{15}, \quad G(6, \theta; 1, 1) = \frac{199}{210},
\]

of which we note that the cases \( n = 1, 2, 3 \), give duplicate results to those of [7], with the result for \( n = 4 \) correcting a minor error. We shall give a “vertical” formula holding for general \( s \) and \( t \), to compare with the “horizontal” formula of Corollary 7.6, with Theorem 7.13. Since we have asymptotic knowledge at hand for the presentation of Corollary 7.6, we can quickly assess the asymptotic value of \( G(n, \theta; 1, 1) \) for large \( n \).

The sum \( \sum_{p=1}^{n-1} \left( \frac{n}{p} \right)^{-1} \) is the value for a \((n-p, p; 1, 1)\) urn with \( p \leq n/2 \) (using the original sum for the case \( p = n/2 \)), we know that as \( n \to \infty \) the sum is \( O(\sqrt{p}) = O(\sqrt{n}) \). Since we have \( n + 1 \) such sums, their total contribution to \((n + 1)G(n, \theta; 1, 1)\) is \( O(n^{3/2}) \). Therefore, as \( n \to \infty \),

\[
G(n, \theta; 1, 1) \sim \frac{n}{4}.
\]

We shall find a better bound for the big \( O \) term with Theorem 7.13.

**Remark.** In Chen et al. [8, Theorems 5 and 6] (2005), it was shown that the random stopping urn with an initial distribution that is uniform also has an asymptotic value of \( n/4 \) when an optimal strategy is used. Therefore, that strategy is designed to try to capture the initial weight of the urn if it is positive and stop before too long if the initial weight is negative. It should be noted, however, that if the player uses a “\( k \) in the hole strategy” with, say, \( k = An^{3/4} \), then the asymptotic value of the expected gain with this strategy will still be \( n/4 \), as the average of the positive initial weights is still far larger than \( k \), and for the major (positive) contributors it is extremely unlikely the player will get “\( k \) in the hole.”

As \( G(n, n; 1, 1) = \sqrt{n}/2 \ (1 + o(1)) = o(n) \) as \( n \to \infty \), \( G(2n, \theta; 1, 1) \sim n/2 \) is larger than \( G(n, n; 1, 1) \) for large \( n \), though this is not the case for \( n = 1, 2, 3 \). \( n = 4 \) is the smallest \( n \) for which
the Bayesian value exceeds the value of its corresponding neutral urn, as $G(8, \theta; 1, 1) = 436/315 > 93/70 = G(4, 4; 1, 1)$.

The Bayesian urn with $n$ balls generally gains more on the average than the corresponding neutral urn on $n$ balls, by a large margin. Why this happens is clear - there is roughly a $1/4$ chance that the initial weight of the urn is at least $n/2$, putting the gains from those urns to be larger that $n/8$, with the losses associated with those urns not enough to make a dent. A better comparison to $G(n, \theta; 1, 1)$ with $\theta$ uniform is the average of all of the values over urns with $n$ balls, i.e. $\frac{1}{n+1} \sum_{j=0}^{n} G(j, n-j; 1, 1)$. This sum is clearly bigger than $G(n, \theta; 1, 1)$, as each urn’s optimal strategy will gain the initial weight of the urn if it is positive, plus a bit more. This average is also asymptotically $n/4$, as the the extra gains from crossings are surely $O(\sqrt{n})$.

We seek a distribution $\hat{\theta}$ that gives a value $G(2n, \hat{\theta}; 1, 1)$ comparable to $G(n, n; 1, 1)$ for large $n$. Since we only have an explicit formula for distributions satisfying the conditions of Lemma 7.4, we shall assume that those conditions need to be met. Luckily enough, there is a distribution satisfying those conditions that gives an expected gain of zero.

**Theorem 7.7.** Suppose $\theta$ is the binomial distribution with $q_j = \binom{n}{j} 2^{-n}$ for $0 \leq j \leq n$. Then the $(n, \theta; 1, 1)$ urn is a fair game, that is, $G(n, \theta; 1, 1) = 0$.

We shall give a proof under more general circumstances with Theorem 7.11. It should be clear that $\theta$ satisfies (7.5). With $q_j = \binom{n}{j} 2^{-n}$, each realization on $n$ balls consisting of “$-1$”s and “$+1$”s is equally likely. Then, regardless of what has been drawn from the urn, the $k$th ball is equally likely to be “$-1$” or “$+1$,” and the expected gain if a bet is placed is zero. Therefore, any strategy is optimal, including the strategy indicated by (7.4).

With the uniform distribution giving a value too high, and the binomial distribution giving a value too low, a hybrid distribution of the two should give us what we want. For $\alpha \in [0, 1]$, let

$$q_j = \frac{1}{n+1}, \quad q'_j = \binom{n}{j} 2^{-n}, \quad \hat{q}_j = \alpha q_j + (1-\alpha) q'_j.$$  

Consider the initial distributions $\theta = \{q_j\}_{j=0}^{n}$, $\theta' = \{q'_j\}_{j=0}^{n}$, and $\hat{\theta} = \hat{\{q_j\}}_{j=0}^{n}$. It is clear from (7.7) that

$$\mathcal{G}(n, \hat{\theta}; 1, 1) = \alpha \mathcal{G}(n, \theta; 1, 1) + (1-\alpha) \mathcal{G}(n, \theta'; 1, 1) \sim \alpha n/4, \quad \text{as } n \to \infty,$$
using (7.10) and Theorem 7.7. Taking $\alpha = A n^{-1/2}$, with $A = \sqrt{2\pi}$ we have for large enough $k$ that 
\[ G(2k, \hat{\theta}; 1, 1) \sim G(k, k; 1, 1). \]

A reversal of the inequality of (7.5) gives a second family with a simplified betting rule.

**Lemma 7.8.** Suppose the probability distribution $\theta = \{q_j\}_{i=0}^n$ satisfies $q_j = q_{n-j}$. Then an optimal betting strategy for the $(n, \theta; 1, 1)$ urn is

\[ \text{"bet on the } (k+1)^{th} \text{ ball if and only if } \sum_{i=1}^k Y_i \leq 0" \]

if and only if $0 \leq j_1 \leq j_2 \leq n/2$ implies

\[ \binom{n}{j_1}^{-1} q_{j_1} \leq \binom{n}{j_2}^{-1} q_{j_2}. \]

The distributions covered by Lemma 7.8 might be considered more practical, as the weights are more concentrated at the center, whereas the distributions covered by Lemma 7.4 favored the extreme ends. With the center-favored distribution, capturing the initial weight becomes secondary to the gains made from the fluctuations along the way. Thus, the optimal strategy is more in line with optimal strategies associated with the nonrandom $(m, p; 1, 1)$ urns. Under the circumstances of Lemma 7.8, given the urn contains $j$ “+1” balls, using an optimal strategy the player will gain $G_{\geq p-m}(m, p; 1, 1)$. Therefore we have:

**Theorem 7.9.** If $\theta$ is a distribution satisfying the requirements of Lemma 7.8, then

\[ G(n, \theta; 1, 1) = \sum_{j=0}^n q_j G(n, j), \]

where

\[ G(n, j) = \begin{cases} 
(2j - n) + \binom{n}{j}^{-1} \sum_{i=0}^{j-1} \binom{n}{i}, & \text{if } 0 \leq j \leq n/2, \\
\binom{n}{j}^{-1} \sum_{i=0}^{n-j} \binom{n}{i}, & \text{if } n/2 < j \leq n.
\end{cases} \]

The proof is similar to that of Theorem 7.5, and is omitted.

7.4. **An Algorithm for the Value** $G(n, \theta; 1, 1)$. While it is unlikely to get a relatively simple formula for $G(n, \theta; 1, 1)$ with an arbitrary initial prior distribution, a slight modification to Boyce's
algorithm (Theorem 1.1) gives us an algorithm that calculates $G(n, 1, 1, 1)$. The weakness of this algorithm is in the number of calculations required to calculate the expected gain, as noted in [8].

**Theorem 7.10.** Given an $(n, 1, 1, 1)$ urn, for $0 \leq j \leq n$ let $P^*(j) = q_j \binom{n}{j}^{-1}$ and $a(n, j) = c(n, j) = 0$. For $0 \leq j \leq n - 1$, let $a(n - 1, j) = P^*(j + 1) - P^*(j)$, and for $0 \leq i \leq n - 2$ and each $j$ let

$$a(i, j) = a(i + 1, j) + a(i + 1, j + 1).$$

For $0 \leq i \leq n - 1$ and each $j$ let

$$c(i, j) = \max\{0, a(i, j)\} + c(i + 1, j) + c(i + 1, j + 1).$$

Then an optimal betting policy for maximizing the expected gain is: if $k$ balls have been drawn, $\ell$ of them plus, accept the next ball if and only if $a(k, \ell) > 0$. The value $G(n, 1, 1, 1)$ of the urn under optimal play is $c(0, 0)$.

![Diagram](image)

**Figure 7.1.** $a(i, j)$ and $P^*(j)$ for $n = 4$ and $\theta$ uniform.

The probabilistic structure of the algorithm is exactly the same as with the original stopping urn. See [4, Appendix 2] for the details. This time, since we can stop and start as many times as necessary, the decision to place a bet is based entirely on the next draw, whereas with the stopping urn, the player might be willing to take a (expected) loss on the next ball in the hopes of a later rebound. The results for this algorithm with $n = 4$ and $\theta$ uniform are given in Figures 7.1 and 7.2. The algorithm gives $G(4, 1, 1, 1) = 32/60 = 8/15$, matching the output of the formula given by Corollary 7.6. In this triangular table and the ones to follow, the top box in the column corresponds to all balls being “+1,” and decreasing on the way down.
Figure 7.2. $c(i, j)$ for $n = 4$ and $\theta$ uniform. The value of the $(4, \theta; 1, 1)$ urn is $c(0, 0) = 8/15$.

In Figure 7.3, we see that when the conditions of Lemma 7.8 are met, the optimal betting policy is the opposite - “bet if $\sum_{k=1}^{K} Y_k \leq 0$,” a policy indicating “expect a rally, anticipate a selloff.”

Figure 7.3. $P^*(j)$ and $a(i, j)$ for the initial distribution $\theta$ with $p_0 = p_4 = 1/20$, $p_1 = p_3 = 1/4$, and $p_2 = 2/5$.

7.5. A Fair $(n, \theta; s, t)$ Urn. We have noted that if $s = t = 1$ and $\theta$ is the binomial distribution, then $G(n, \theta; 1, 1) = 0$. This is also the case with the $(n, \theta; s, t)$ urns.

Theorem 7.11. Suppose $\theta$ is the binomial distribution with parameter $n$ and probability of success $s/(t + s)$. That is, the probability $q_j$ the urn contains $j$ “+t” balls satisfies

$$q_j = \binom{n}{j} \left( \frac{s}{t+s} \right)^j \left( \frac{t}{t+s} \right)^{n-j}.$$  

Then the $(n, \theta; s, t)$ random acceptance urn is a fair game, that is,

$$G(n, \theta; s, t) = 0.$$
Proof. Suppose that \( k \) balls have been drawn, with \( \ell \) of them “+t.” We show that (7.3) always returns zero if \( 0 \leq \ell \leq k \), beginning by discarding terms that do not contribute to the sum.

\[
\sum_{j=0}^{n} \binom{n-k}{j} \left( \frac{s}{t+s} \right)^j \left( \frac{t}{t+s} \right)^{n-j} (j-\ell)(t+s) - (n-k)) \left( j-\ell \right) (t+s) - (n-k)).
\]

Replacing \( j-\ell \) with \( j \), and \( n-k \) with \( n \), we see the familiar forms of a binomial distribution, and the summation (neglecting the outside factor) becomes, using this new binomial distribution,

\[
\sum_{j=0}^{n} \binom{n}{j} \left( \frac{s}{t+s} \right)^j \left( \frac{t}{t+s} \right)^{n-j} \left( (t+s) - sn \right) = (t+s)E[j] - sn = 0.
\]

Therefore, the sum vanishes for nonnegative integers \( \ell \leq k \). This implies that either option, pass or bet, is optimal at every stage. Therefore, \( O(n, \theta; s, t) = 0. \)

\[\square\]

7.6. The \((n, \theta; s, t)\) Urns with \( \theta \) Uniform. Now suppose \( \theta \) is uniform, that is,

\[q_p = \theta(\text{urn has } p \text{ “+t” balls}) = \frac{1}{n+1}, \quad 0 \leq p \leq n.\]

We are to place a bet if and only if \( \theta(Y_{k+1} = t \mid y_1, y_2, \ldots, y_k) \geq s/(s+t) \). Suppose \( y_1, \ldots, y_k \) are given, and that \( \ell \) of them are “+t” balls. Given a particular sequence of \( k \) balls, \( \ell \) of which are “+t,” we have

\[
\theta(Y_{k+1} = t \mid y_1, \ldots, y_k) = \frac{\theta(Y_1 = y_1, \ldots, Y_k = y_k, Y_{k+1} = t)}{\theta(Y_1 = y_1, \ldots, Y_k = y_k)},
\]

and we can calculate the two probabilities on the right-hand side explicitly. For \( 0 \leq j \leq n \),

\[
\theta(Y_1 = y_1, \ldots, Y_k = y_k \mid j) = \frac{(j)\ell(n-j)_{k-\ell}}{(n)_{k}},
\]

thus

\[
\theta(Y_1 = y_1, \ldots, Y_k = y_k) = \sum_{j=0}^{n} \frac{(j)\ell(n-j)_{k-\ell}}{(n)_{k+1}} = \frac{\ell!(k-\ell)!}{(k+1)!} \sum_{j=0}^{n} \frac{(j)_{k-\ell}}{(n)_{k+1}} = \frac{\ell!(k-\ell)!}{(k+1)!},
\]

the last following from the convolution identity \([10, \text{eq. (5.26)}]\). Therefore, we have

(7.15) \[
\theta(Y_{k+1} = t \mid y_1, \ldots, y_k) = \frac{\ell!(k-\ell)!/(k+2)!}{\ell!(k-\ell)!/(k+1)!} = \frac{\ell+1}{k+2}.
\]
Therefore an optimal policy is: If the player has seen $k$ balls, $\ell$ of which are “$+t$,” then the player should accept the next ball if and only if

$$\frac{\ell + 1}{k + 2} \geq \frac{s}{t + s} \iff t(\ell + 1) \geq s(k + 1 - \ell)$$
$$\iff t\ell - s(k - \ell) \geq s - t.$$  

Thus an optimal strategy is to have the player bet if and only if $\sum_{i=1}^{k} Y_i \geq s - t$. This condition is consistent with that given in [7] for the case $s = t = 1$. Thus, if $t = 1$ and $s = 100$, a player would be hesitant to bet even if the first 98 balls were “$+1$”, while on the other hand, if $t = 100$ and $s = 1$, then our player would not be discouraged even if the first 98 balls were “$-1$.” Note that if $t \geq ns$, then the lowest value $t\ell - s(k - \ell)$ can take with $k < n$ is $s(1 - n) \geq s - t$. Therefore, the player will bet every time if $t \geq ns$. Similarly, if $nt \leq s$ the player will never bet. The betting tree and the betting line for $n = 3$, $s = 5$, and $t = 3$ are given in Figure 7.4.

Figure 7.4. Betting tree and betting line for a $(3, \theta; 5, 3)$ urn, with $\theta$ uniform. Solid lines indicate placement of a bet.

Examples. With $n = 1$ and $\theta$ uniform, the value is simple enough.

$$\mathcal{G}(1, \theta; s, t) = \begin{cases} 
  t - s, & \text{if } t \geq s, \\
  0, & \text{if } t < s.
\end{cases}$$
As $n$ becomes larger, a general formula for $G(n,\theta; s, t)$ becomes increasingly dependent on $s$ and $t$, as the betting line can appear in more places:

$$G(2, \theta; s, t) = \begin{cases} 
  t - s, & \text{if } t \geq 2s, \\
  \frac{5t - 4s}{6}, & \text{if } s \leq t < 2s, \\
  \frac{2t - s}{6}, & \text{if } \frac{s}{2} \leq t < s, \\
  0, & \text{if } t < \frac{s}{2}.
\end{cases}$$

$$G(3, \theta; s, t) = \begin{cases} 
  \frac{3(t - s)}{2}, & \text{if } t \geq 3s, \\
  \frac{17t - 15s}{12}, & \text{if } 2s \leq t < 3s, \\
  \frac{15t - 11s}{12}, & \text{if } s \leq t < 2s, \\
  \frac{7t - 3s}{12}, & \text{if } \frac{s}{2} \leq t < s, \\
  \frac{3t - s}{12}, & \text{if } \frac{s}{3} \leq t < \frac{s}{2}, \\
  0, & \text{if } t < \frac{s}{3}.
\end{cases}$$

Observe the antiurn property (7.2):

$$\frac{5t - 4s}{6} - \frac{2s - t}{6} = t - s, \quad \text{when } n = 2,$$

$$\frac{17t - 15s}{12} - \frac{3s - t}{12} = \frac{15t - 11s}{12} - \frac{7s - 3t}{12} = \frac{3(t - s)}{2}, \quad \text{when } n = 3.$$

The maximum possible expected gain, for any $n$, will be $n(t - s)/2$, occurring only when it is always to the player’s advantage to bet, which occurs only when $t \geq ns$. The minimum possible expected gain is 0, which will occur when $t < s/n$. As for some other cases, we have

$$G(n, \theta; s, t) = \frac{tn - s}{n(n + 1)}, \quad \text{if } (n - 1)t \leq s \leq nt,$$

and

$$G(n, \theta; s, t) = \frac{n(t - s)}{2} + \frac{sn - t}{n(n + 1)}, \quad \text{if } (n - 1)s \leq t \leq ns.$$

These are particular examples of our next theorem.

**Theorem 7.12.** Suppose that $\theta$ is uniform on $\{0, \ldots, n\}$. Then

$$G(n, \theta; s, t) = \sum_{k=0}^{n-1} \sum_{t=0}^{t} \frac{1}{(k+1)(k+2)} \max\{0, t(\ell + 1) - s(k + 1 - \ell)\}.$$

In particular, we have

$$G(n, \theta; 1, 1) = \frac{n + 1}{4} - \frac{1}{2}H(n) + \frac{1}{4}H(\lfloor n/2 \rfloor) - \frac{\chi_n}{4(n + 1)}.$$

where $H(x)$ is the $x^{th}$ harmonic number, and $\chi_n = 1$ if $n$ is even and $\chi_n = 0$ if $n$ is odd.
Proof. With an initial distribution that is uniform, given a specific ordering of \(k\) balls, \(\ell\) of them "\(+t\)," we saw that

\[
\theta(Y_1 = y_1, \ldots, Y_k = y_k) = \frac{\ell!(k - \ell)!}{(k + 1)!}.
\]

Since there are \(\binom{k}{\ell}\) such sequences, we have

\[
\theta\left(\{Y_1, \ldots, Y_k\} = \{+t, \ldots, +t, -s, \ldots, -s\}\right) = \frac{1}{k + 1}.
\]

Given \(\ell\) of the first \(k\) balls are "\(+t\)," the expected gain on the next ball is

\[
\max \left\{ 0, \frac{1}{k + 2} \left(t(\ell + 1) - s(k + 1 - \ell)\right) \right\}.
\]

Thus, we sum over all possible \(\ell\) and \(k\). When \(s = t = 1\), since we bet if and only if \(\ell \geq k/2\), we have that

\[
\mathcal{G}(n, \theta; 1, 1) = \sum_{k=0}^{n-1} \sum_{k/2 \leq \ell \leq k} \frac{2\ell - k}{(k + 1)(k + 2)}.
\]

To derive the formula in the theorem, we observe first that for \(k \geq 0\),

\[
\sum_{k/2 \leq \ell \leq k} \frac{2\ell - k}{(k + 1)(k + 2)} = \begin{cases} 
\frac{1}{4} - \frac{1}{4(k + 1)}, & \text{if } k \text{ is even}, \\
\frac{1}{4} - \frac{1}{4(k + 2)}, & \text{if } k \text{ is odd}.
\end{cases}
\]

It now follows that

\[
\sum_{k=0}^{n-1} \sum_{k/2 \leq \ell \leq k} \frac{2\ell - k}{(k + 1)(k + 2)} = \frac{n + 1}{4} - \frac{1}{2} \sum_{\ell=0}^{[n/2]} \frac{1}{2\ell + 1} + \frac{\chi_n}{4(n + 1)}
\]

\[
= \frac{n + 1}{4} - \frac{1}{2} H(n) + \frac{1}{4} H\left([n/2]\right) - \frac{\chi_n}{4(n + 1)}. \quad \blacksquare
\]

If we want a form for \(\mathcal{G}(n, \theta; s, t)\) similar to that of Corollary 7.6, we note that

\[
E \left[ \mathcal{G}(n, \theta; s, t) \mid \text{there are } p \text{ "}+t\text{" balls} \right] = G_{\leq X_0 + t - s}(n - p, p; s, t),
\]

where \(X_0(p) = p(t + s) - ns\). Thus

\[
\mathcal{G}(n, \theta; s, t) = \frac{1}{n + 1} \sum_{j=0}^{n} G_{\leq X_0(j) + t - s}(n - j, j; s, t).
\]

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The optimal strategy aims to capture \( X_0(j) + t - s \) if it is nonnegative. Some of it may be missed due to crossings. Like the \( s = t = 1 \) urn, the player will suffer losses along the way in an effort to secure this (possibly large) positive gain.

With \( s = t = 1 \), we saw that the asymptotic value of \( \mathcal{G}(n, \theta; 1, 1) \) was \( n/4 \) with the uniform distribution. We can also calculate the asymptotic value of the expected gain with \( \theta \) uniform, for the \((n, \theta; s, t)\) urns.

**Theorem 7.13.** Suppose that \( \theta \) is uniform on \( \{0, \ldots, n\} \). Then as \( n \to \infty \), we have

\[
\mathcal{G}(n, \theta; s, t) = \frac{n}{2} \cdot \frac{t^2}{t + s} + O(\ln n).
\]

**Proof.** For \( 0 \leq \ell \leq k \), let

\[
f(\ell, k) = \max\{0, t(\ell + 1) - s(k + 1 - \ell)\},
\]

and let

\[
L_0 = \frac{ks}{t + s} + \frac{s - t}{t + s}, \quad L = \lceil L_0 \rceil, \quad \epsilon = L - L_0.
\]

For \( \ell > L_0 \) we have \( f(\ell, k) > 0 \). Observe that for \( \ell \geq L_0 \), \( f(\ell + 1, k) = f(\ell, k) + (t + s) \). Then

\[
\sum_{k=0}^{n-1} \sum_{\ell=0}^{k} \frac{f(\ell, k)}{(k+1)(k+2)} = (t + s) \sum_{k=0}^{n-1} \frac{1}{(k+1)(k+2)} \left[ (k - L + 1)\epsilon + \frac{(k - L)(k - L + 1)}{2} \right]
\]

\[
= O(\ln n) + \frac{t + s}{2} \sum_{k=0}^{n-1} \frac{(k - L)(k - L + 1)}{(k + 1)(k + 2)}
\]

\[
= O(\ln n) + \frac{t + s}{2} \sum_{k=0}^{n-1} \frac{(k - L_0)^2}{(k + 1)(k + 2)}.
\]

Since \( k - L_0 = kt/(t + s) + O(1) \), we then have

\[
= O(\ln n) + \frac{t^2}{2(t + s)} \sum_{k=0}^{n-1} \frac{k^2}{(k + 1)(k + 2)}
\]

\[
= O(\ln n) + \frac{n}{2} \cdot \frac{t^2}{t + s},
\]

completing the proof. \( \Box \)

**Remark.** If our player uses the corresponding strategy to that of Lemma 7.4, betting if and only if \( \sum_{i=1}^{k} Y_i \geq 0 \), then our player would bet if and only if \( \ell \geq ks/(t + s) \). Therefore, (since \( s, t \geq 0 \)
the difference between this strategy and the optimal one is less than a ball’s difference. In fact, the optimal strategy will capture a bit less of those positive initial values (up to $2t+s$ due to crossings) if $s > t$, while it might pick a small negative initial value if $t > s$. The player using the optimal strategy will not lose as much along the way, which more than makes up the difference.

### 7.7. An Algorithm for the General Random Urn

As for an algorithm for calculating the value $G(n, \theta; s, t)$ in the manner of Theorems 1.1 and 7.10, the underlying probabilistic structure does not change when the weights of the balls change. The difference is in how the $b(i, j)$’s (Theorem 1.1) and $c(i, j)$’s (Theorem 7.10) are calculated. If $q(i, j)$ is the probability of a specific sequence of $i$ balls, with $j$ of them “+$t$,” then (see [4, Appendix 2])

$$q(i, j) = q(i + 1, j + 1) + q(i + 1, j),$$

and given that sequence, the probability the next ball is a “+$t$” is $q(i + 1, j + 1)/q(i, j)$ and the probability it is a “$-s$” is $q(i + 1, j)/q(i, j)$. Thus the expected gain for the next draw with $s = t = 1$ equals

$$\frac{q(i + 1, j + 1)}{q(i, j)} - \frac{q(i + 1, j)}{q(i, j)} = \frac{a(i, j)}{q(i, j)}.$$

The $q(i, j)$ in the denominator can be dispatched with, thus we do not need to go through the unappealing task of calculating all of the transition probabilities. Unfortunately, for the general (stopping or acceptance) $(n, \theta; s, t)$ urn, the expected gain on the next draw will be

$$t \cdot \frac{q(i + 1, j + 1)}{q(i, j)} - s \cdot \frac{q(i + 1, j)}{q(i, j)},$$

![Figure 7.5. $a(i, j)$ and $P^*(j)$ for $n = 3$ and $\theta$ uniform.](image-url)
thus it appears that some transition probabilities need to be calculated in order for such an algorithm to work. The only question is whether only some can be calculated. We have

\[ a(i, j) = q(i + 1, j + 1) - q(i + 1, j), \]

and

![Figure 7.6](image)

(7.17) \[ t \cdot q(i + 1, j + 1) - s \cdot q(i + 1, j) = s \left(q(i + 1, j + 1) - q(i + 1, j)\right) + (t - s)q(i + 1, j + 1), \]

\[ = s \cdot a(i, j) + (t - s)q(i + 1, j + 1), \]

and we can reduce \( q(i + 1, j + 1) \) to

\[ q(i + 1, j + 1) = q(i + 1, 0) + \sum_{k=0}^{j} a(i, k). \]

Therefore, only the collection \( \{q(i, 0)\}_{i=1}^{n} \) (the probabilities that the first \( i \) balls are all \("-s"," \) \( 1 \leq i \leq n \)) would need to be known. (Knowing \( \{q(i, i)\}_{i=1}^{n} \) also is good enough.) This is an improvement compared to the total number of transition probabilities, which is roughly \( n^2/2 \). Then, defining for \( 0 \leq i \leq n - 1 \) (again, \( c(n, j) = 0 \) for all \( j \))

\[ \hat{c}(i, j) = s \cdot a(i, j) + (t - s) \left[q(i + 1, 0) + \sum_{k=0}^{j} a(i, k)\right], \]

\[ c(i, j) = \max\{0, \hat{c}(i, j)\} + c(i + 1, j) + c(i + 1, j + 1), \]

one should bet if \( \hat{c}(i, j) > 0 \) and pass when \( \hat{c}(i, j) < 0 \). The value \( G(n, \theta; s, t) \) would equal \( c(0, 0) \). With \( \theta \) uniform, we have from (7.15) that \( q(i, 0) = 1/(i+1) \) for \( 0 \leq i \leq n \). (Since \( q(i, j) = 1/(i+1) \) for \( 0 \leq j \leq i \) we do not actually need to go through the reduction to the outside transition probabilities;
we can work directly from (7.17). Figure 7.5 gives the structure for the case \( n = 3 \) and \( \theta \) uniform. The result of the algorithm with \( s = 2, t = 3, n = 3, \) and \( \theta \) uniform, is given in Figure 7.6, and the result with \( s = 9, t = 4, n = 3 \) and \( \theta \) uniform is given in Figure 7.7. The latter example shows that \( \hat{c}(i, j) \neq \hat{c}(i + 1, j + 1) + \hat{c}(i + 1, j) \) as it appeared to be in the former example.

The weakness of such an algorithm, again, is the number of calculations needed to calculate the final output. Since some transition probabilities are needed (not to mention using various sums of the \( a(i, j) \)'s to shorten the list of transition probabilities needed), these algorithms are even more cumbersome. We present the algorithm formally with Theorem 7.14.

**Theorem 7.14.** Given a \((n, \theta; s, t)\) urn, for \( 0 \leq j \leq n \) let \( P^*(j) = q_j(n)_{j}^{-1} \) and \( a(n, j) = c(n, j) = 0. \)

For \( 0 \leq j \leq n - 1 \), let \( a(n - 1, j) = P^*(j + 1) - P^*(j) \), and for \( 0 \leq i \leq n - 2 \) and each \( j \) let

\[
a(i, j) = a(i + 1, j) + a(i + 1, j + 1).
\]

For \( 0 \leq i \leq n - 1 \) and each \( j \) let

\[
\hat{c}(i, j) = s \cdot a(i, j) + (t - s) \left[ q(i + 1, 0) + \sum_{k=0}^{j} a(i, k) \right],
\]

\[
c(i, j) = \max \{ 0, \hat{c}(i, j) \} + c(i + 1, j) + c(i + 1, j + 1),
\]

where \( q(i + 1, 0) \) is explicitly given by

\[
q(i + 1, 0) = \sum_{j=i+1}^{n} \frac{(j)_{i+1}}{(n)_{i+1}} q_{n-j}.
\]
Then an optimal betting policy for maximizing the expected gain is: if $k$ balls have been drawn, $\ell$ of them plus, accept the next ball if and only if $c(k, \ell) > 0$. The value of the urn under optimal play is $c(0, 0)$.

We also remark that, with a few adjustments, an algorithm giving the value of a random generalized stopping urn can be given as well. We omit the details.
8. A Ruin Problem

The family of optimal betting strategies for the \((m, p; s, t)\) urn not only guarantees a nonnegative value, but also a nonnegative gain regardless of how the balls end up being drawn from the urn. However, while the game is in progress, our player may find himself digging in his own pockets in order to continue. The question at hand here is this: How much capital might one need to play the game using any of the optimal strategies? We can answer this question in the case \(s = t = 1\), thanks to the reflection method. Answering this question under more general circumstances is an open problem. Instead, we attempt to give a general idea of what is needed when \(m\) and \(p\) are large.

We first state the problem for the nonrandom urn.

8.1. The Ruin Problem for the \((m, p; s, t)\) Urn. We give the player a bank \(b = B_0\), and monitor its size as the game progresses. That is, for \(1 \leq n \leq m + p:\)

\[
B_n = \begin{cases} 
B_{n-1} + t, & \text{if a bet is placed on the } n\text{th ball and it is a } "+t", \\
B_{n-1} - s, & \text{if a bet is placed on the } n\text{th ball and it is a } "-s", \\
B_{n-1}, & \text{if no bet is placed.}
\end{cases}
\]

We say that a player is \textit{ruined} if \(B_n < s\) for some \(n\). (Thus, we shall assume that \(b \geq s\).) Then, at some point during play, the player cannot afford to place any further bets, having (virtually) emptied his bank. The probability of ruin does depend on the choice of (optimal) strategy. With that in mind, we shall focus our efforts on a player that uses the zero-bet strategy.

If \(X_n = X_0 = pt - ms\), then \(B_n\) equals \(b\) plus any permanent gains made due to crossings (any permanent gain associated with \(X_0\) itself can be said to not have occurred under this circumstance). Because of this, a necessary, but not a sufficient, condition for ruin to occur is that for some \(n, X_n > \max\{0, pt - ms\} + b - s\). We use this condition to show that if \(b\) is large compared to \(\sqrt{p}\), the probability of ruin goes to zero as \(p\) goes to infinity. However, given that under most circumstances, \(G(m, p; s, t) = o(b)\), from a practical standpoint the game probably would not be played considering the capital brought to the table.
8.1.1. **Trends and Preliminaries for the \((m, p; 1, 1)\) and \((m, p; 1, t)\) Urns.** We have many more tools at our disposal when \(s = t = 1\). In fact, we have enough to give a solution to the ruin problem, though it is quite complicated. We solve the problem for the \((m, p; 1, 1)\) urn in the next section. Here, we set up preliminaries and give general trends. Some of the coming results hold for the \((m, p; 1, t)\) urns as well, so we present those results under the more general circumstances.

By Lemma 3.5, with bank \(b\) we have that

\[
P(X_n = b \text{ for some } n) = \left(\frac{m + p}{p}\right)^{b} \left(\frac{m + p}{p - b}\right), \quad \text{if } m \geq p,
\]

and

\[
P(X_n = p - m + b \text{ for some } n) = \left(\frac{m + p}{p}\right)^{b} \left(\frac{m + p}{p + b}\right), \quad \text{if } p \geq m.
\]

Using these results, we can now show some trends:

**Lemma 8.1.** Suppose the player uses an optimal betting strategy. If \(p/m \rightarrow \lambda > 1\), then, as \(p \rightarrow \infty\), the probability of ruin with bank \(b\) is bounded above by \(\lambda^{-b}\). If \(m/p \rightarrow \lambda > 1\), then, as \(m \rightarrow \infty\), the probability of ruin with bank \(b\) is bounded above by \(\lambda^{-b}\).

**Proof.** For \(p > m\) and \(p/m \rightarrow \lambda > 1\), the probability of ruin is bounded above by

\[
\left(\frac{m + p}{p}\right)^{b} \left(\frac{m + p}{p - b}\right) = \frac{(m)_b}{(p + b)_b} \leq \left(\frac{m}{p}\right)^b \sim \lambda^{-b}.
\]

The other case is similar, and the proof is omitted. \(\square\)

**Lemma 8.2.** If \(p \geq m\), and \(h = p - m\), then the probability of ruin with bank \(b\) is at most \(e^{-hb/p}\). If \(m \geq p\), and \(h = m - p\), then the probability of ruin with bank \(b\) is at most \(e^{-hb/m}\).

**Proof.** Suppose \(p \geq m\). We have

\[
\left(\frac{m + p}{p}\right)^{b} \left(\frac{m + p}{p + b}\right) \leq \left(\frac{m}{p}\right)^b = \left(1 - \frac{h}{p}\right)^b \leq \exp \left(-\frac{hb}{p}\right),
\]

as desired. The other case is similar, and the proof is omitted. \(\square\)

Lemma 8.2 shows that if \(\max\{m, p\} = o(b|p - m|)\), then the ruin probability tends to zero as \(m + p\) becomes large.

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Lemma 8.3. Suppose the player uses an optimal strategy, and assume that \( b = \kappa \sqrt{p} \), with \( \kappa > 0 \). If \(|p - m| = \eta \sqrt{p} \), then the probability of ruin is bounded above by \((1 + o(1)) \exp(-\kappa (\eta + \kappa))\), as \( p \to \infty \).

Proof. If \( p \geq m \), then the probability of ruin is bounded above by

\[
\left( \frac{m + p}{p + b} \right) \left( \frac{m + p}{p} \right)^{-1} = \left( \frac{m}{p} \right)^{b} \times \prod_{i=0}^{b-1} \left( 1 - \frac{i}{m} \right) \prod_{i=1}^{b} \left( 1 + \frac{i}{p} \right).
\]

Since \( m = p - \eta \sqrt{p} \) and \( b = \kappa \sqrt{p} \), we have

\[
\left( \frac{m}{p} \right)^{b} = \left( 1 - \frac{\eta}{\sqrt{p}} \right)^{\kappa \sqrt{p}} \leq e^{-\kappa \eta}.
\]

Also,

\[
\prod_{i=0}^{b-1} \left( 1 - \frac{i}{m} \right) \leq \exp \left( -\frac{1}{m} \sum_{i=0}^{b-1} i \right) = \exp \left( -\frac{b(b-1)}{2m} \right),
\]

and

\[
\prod_{i=1}^{b} \left( 1 + \frac{i}{p} \right) \geq \exp \left( \frac{1}{p} \sum_{i=1}^{b} i - \frac{1}{2p^2} \sum_{i=1}^{b} i^2 \right) = \exp \left( \frac{b(b+1)}{2p} - \frac{b(b+1)(2b+1)}{12p^2} \right).
\]

Substituting in for \( b \) and \( m \), an upper bound for the quotient of the two products is

\[
\exp \left( -\kappa^2 + O(p^{-1/2}) \right).
\]

The upper bound for the ruin probability is thus \( \exp(-\kappa (\eta + \kappa) + O(p^{-1/2})) \), as desired. The case with \( m \geq p \) is similarly shown, only this time the roles of \( m \) and \( p \) are reversed (since \( m \sim p \), we have \( \sqrt{m} \sim \sqrt{p} \)). We omit the proof. \( \square \)

Lemma 8.3 shows that if we let either \( \eta \) or \( \kappa \) tend to infinity, the probability of ruin tends to zero.

Up to this point, we have been talking only about ruin while using the zero-bet strategy. The necessary condition for ruin with the zero-bet strategy is also necessary using any optimal strategy. Therefore, the results above apply with any optimal strategy. As mentioned at the start of this chapter, the ruin probabilities do vary according to the choice of optimal strategy. In fact, we can show that the zero-pass strategy is safer than the zero-bet strategy for the \((m, p; 1, 1)\) urns. This can be shown for the \((m, p; 1, t)\) urns, so we present and prove the next result in that context.
Lemma 8.4. Let \(r_b(m,p;1,t)\) and \(r'_b(m,p;1,t)\) denote the probability of ruin on the \((m,p;1,t)\) urn using the zero-pass strategy and zero-bet strategy, respectively, with initial bank \(b\). Then

\[ r_b(m,p;1,t) \leq r'_b(m,p;1,t). \]

Proof. Let \(S (S')\) denote the collection of realizations from the \((m,p;1,t)\) urn for which ruin occurs using the zero-pass (zero-bet) strategy with initial bank \(b\). Since \(r'_b(m,p;1,t) = |S'| \left( \frac{m+p}{p} \right)^{-1}\)
\(r_b(m,p;1,t) = |S| \left( \frac{m+p}{p} \right)^{-1}\), it suffices to show \(|S| \leq |S'|\) for the first inequality. To show this, we shall provide an injective map from \(S\) into \(S'\). Observe that since \(X_n > 0\) is equivalent to \(X_n \geq 1\), we can re-state the zero-pass strategy as “bet if and only if the urn weight is at least one.”

First, suppose \(m \geq pt\). Let \(\omega \in S\). Then there is a minimal \(N\) for which \(B_N(\omega) = 0\). Also, since \(X_n \geq b > 0\) holds for some \(n\), there is a “+1” trip from the initial starting position. We map \(\omega\) to the realization \(\omega'\) as follows: Shift the “+1” trip from the starting position, and move it so that it follows the first point of ruin. That is, if \(\omega = P_1P_2P_3\), with \(P_1\) a “+1” trip and \(|P_1P_2| = N\), then \(\omega' = P_2P_1P_3\). Clearly, \(\omega'\) is a realization from the \((m,p;1,t)\) urn. Figure 8.1 shows the mapping with \(m = p = 7, t = 1,\) and \(b = 1\).

We next show \(\omega'\) is in \(S'\). If \(|P_1| = k\), then

\[ X_n(\omega') = X_{n+k}(\omega) - 1, \quad 0 \leq n \leq N - k. \]

Figure 8.1. Mapping of a zero-pass ruin realization (black) to a zero-bet ruin realization (shadow) with \(b = 1\) on the \((7,7;1,1)\) urn. The dashed line indicates the “+1” trip to be moved, while the solid line indicates the ruin sequence.

Therefore, in this range \(X_n(\omega') \geq 0\) if and only if \(X_{n+k}(\omega) \geq 1\). Thus the sequence of bets/passes over \(P_2\) does not change upon mapping \(\omega\) to \(\omega'\). Therefore, it follows that \(B_{N-k}(\omega) = 0\), since no betting occurs on \(P_1\) in \(\omega\). Thus \(\omega' \in S'\). Furthermore, it follows from (8.1) that ruin first occurs
for \( \omega' \) after \( N - k \) balls have been drawn. We use this fact to show the map is injective. If \( \omega' \in S' \), ruin first occurs after \( N \) balls have been drawn, and there is a \( \text{"}+1\text{"} \) trip from \( X_N \), then shifting that \( \text{"}+1\text{"} \) trip to the beginning of the realization produces a unique realization \( \omega \) in \( S \). Thus, we conclude that \(|S| \leq |S'|\).

If \( m < pt \), then we have two cases. If ruin occurs before the first time the urn is neutral, then ruin will occur regardless of which strategy we have used (the two strategies are identical when the urn is nonneutral), so we map \( \omega \) to itself. Otherwise, the urn is neutral at some point before ruin occurs, in which case we proceed as before, this time shifting the first \( \text{"}+1\text{"} \) trip from neutral. The map is reversible using the same conditions. Thus, \(|S| \leq |S'|\) holds here as well. \(\square\)

8.1.2. General Results. For the \((m, p; 1, t)\) urns with \( t \) a positive integer, things become more complicated. Note that by Lemma 3.26 we have for \( pt > m \)

\[
P(X_n \geq pt - m + b \text{ for some } n) = \left(\frac{m + p}{p}\right)^{-1} Q_{m,p}(pt - m + b) = \left(\frac{m + p}{p}\right)^{-1} \sum_{k=0}^{[(m-b)/t]} \frac{b}{kt+k+b} \binom{kt+k+b}{k} \left(\frac{m+p-kt-k-b}{p-k}\right),
\]

while for \( m \geq pt \) we have

\[
P(X_n \geq b \text{ for some } n) = \left(\frac{m + p}{p}\right)^{-1} Q_{m,p}(b) = \left(\frac{m + p}{p}\right)^{-1} \sum_{k=\lceil b/t \rceil}^{p} \frac{m-pt+b}{m+p-k-kt+b} \binom{m+p-k-kt+b}{p-k} \binom{kt+k-b}{k}.
\]

For the \((m, p; s, t)\) urns, calculating this probability seems to be difficult in general. It is possible to calculate \( P(\max_n X_n = b) \) (and thus, \( P(X_n \geq b \text{ for some } n) = \sum_{j \geq b} P(\max_n X_n = j) \)) via the \((s, t)\)-zero-gain and ballot numbers, but due to their complexity, we shall omit those results. As a result, we are forced to use bounds that are not tight in general. Nevertheless, when \( b \) is large enough when compared with \( \sqrt{p} \), we can show the probability of ruin tends to zero.

**Theorem 8.5.** If, as \( p \to \infty \),

\[
\frac{b}{p^{1/2+\epsilon}} \to \infty, \quad \text{for some } \epsilon > 0,
\]

then the probability of ruin for the \((m, p; s, t)\) urn with bank \( b \) tends to zero.
Proof. The probability of ruin is bounded above by

\[ P(X_n > \max\{0, pt - ms\} + b - s) \].

By Lemmas 2.11 and 2.12 we may assume \(-(t + s) < pt - ms < t + s\). For such urns, there is a sequence of less than \(t + s\) balls that will neutralize the urn. If this sequence contains \(j\) balls, then after \(j\) balls have been drawn we must have \(b - js \leq Bj \leq b + tj\). Since \(b \gg \sqrt{p}\), the difference in the bank does not amount to much, thus we may assume \(pt = ms\) (after applying Lemmas 2.11 and 2.12 again). We do this here for relative ease of calculation.

If \(pt = ms\), then

\[ P(X_n > b - s \text{ for some } n) \leq \sum_{i=0}^{s-1} \sum_{n} P(X_n = b - i). \]

Once again, for ease of calculation, we shall assume that \(b\) is divisible by \(s\), and show only the \(i = 0\) term is small. The other terms will follow similarly.

Write \(b = a's = ast(t + s)\) and \(p' = p/s\). Then

\[ \sum_n P(X_n = b) = \left(\frac{m + p}{p}ight)^{-1} \sum_k \left(\frac{k(t + s) + a'}{ks}ight) \left(\frac{m + p - k(t + s) - a'}{m - kt - a'}\right). \]

Now

\[ (8.2) \quad \left(\frac{k(t + s) + a'}{ks}\right) \leq \exp\left(\frac{1}{12(k(t + s) + a')} + \frac{1}{12ks + 1} + \frac{1}{12(kt + a') + 1}\right) \times \frac{(t + s)^{k(t+s)+a'/2}}{s^{ks+1/2} t^{kt+a'/2} \sqrt{\frac{k + at}{2\pi k(k + a(t + s))}} \left(1 + \frac{at}{k}\right)^{ks} \left(1 - \frac{as}{k + a(t + s)}\right)^{kt+a'}}, \]

and

\[ (8.3) \quad \left(\frac{m + p - k(t + s) - a'}{m - kt - a'}\right) \leq \frac{(t + s)^{m+p-k(t+s)-a'+1/2}}{s^{p-ks+1/2} t^{m-kt-a'+1/2} \sqrt{\frac{p' - k - at}{2\pi (p' - k)(p' - k - a(t + s))}}} \times \left(1 - \frac{at}{p' - k}\right)^{p-ks} \left(1 + \frac{as}{p' - k - a(t + s)}\right)^{m-kt-a'} \times \exp\left(\frac{1}{12(m + p - k(t + s) - a')} + \frac{1}{12(p - ks + 1)} + \frac{1}{12(m - kt - a') + 1}\right), \]

while we still have

\[ (8.4) \quad \left(\frac{m + p}{p}\right) \geq \frac{(t + s)^{m+p+1/2}}{s^{p+1/2} t^{m+1/2} \sqrt{2\pi p'}} \exp\left(\frac{1}{12(m + p) + 1} + \frac{1}{12m} + \frac{1}{12p}\right). \]

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Combining (8.2), (8.3), and (8.4) (reciprocals), the exponent on $t + s$ is 1/2, the exponent on $s$ is $-1/2$, and the exponent on $t$ is $-1/2$, while the terms within the exponential function are negligible.

We now focus on the two pair of “one plus” and “one minus” terms. We have

\[ (8.5) \quad \left( 1 + \frac{at}{k} \right)^{ks} \left( 1 - \frac{as}{k + a(t + s)} \right)^{kt + a'} \leq 1, \]

\[ (8.6) \quad \left( 1 + \frac{as}{p' - k - a(t + s)} \right)^{m - kt - a'} \leq \exp(ast), \]

\[ (8.7) \quad \left( 1 - \frac{at}{p' - k} \right)^{p' - ks} \leq \exp\left( -ast - \frac{(at)^2 s}{2(p' - k)} \right), \]

with the product of the right-hand sides of (8.5), (8.6), and (8.7) equaling

\[ \exp\left( -\frac{b^2}{2(t + s)^2(p - ks)} \right). \]

If $b \gg p^{1/2+\epsilon}$ for some $\epsilon > 0$ (noting $p - ks \geq b/t$), this term will go to zero faster than the square root terms of (8.2), (8.3), and (8.4) will increase. Thus, the summand tends to zero. Similarly, the sum over $k$ will contribute $O(p)$ terms, and the exponential is small enough that the total sum will also tend to zero.

The ruin probabilities of some urns with $s = 2$, $t = 3$, and $b \leq 6$ with the zero-bet strategy and the zero-pass strategy are respectively given in Tables 8.1 and 8.2.

| Table 8.1. Some $(\cdot, \cdot; 2, 3)$ urn ruin probabilities (in percent) with five total balls and the zero-bet strategy. |
|---|---|---|---|---|---|
| urn | $b = 2$ | $b = 3$ | $b = 4$ | $b = 5$ | $b = 6$ |
| $(5, 0; 2, 3)$ | 0 | 0 | 0 | 0 | 0 |
| $(4, 1; 2, 3)$ | 20 | 20 | 0 | 0 | 0 |
| $(3, 2; 2, 3)$ | 60 | 60 | 30 | 30 | 10 |
| $(2, 3; 2, 3)$ | 50 | 40 | 10 | 10 | 0 |
| $(1, 4; 2, 3)$ | 20 | 20 | 0 | 0 | 0 |
| $(0, 5; 2, 3)$ | 0 | 0 | 0 | 0 | 0 |

8.2. **Solution of the Ruin Problem for the $(m, p; 1, 1)$ Urns.** The main difficulty in solving the ruin problem is that the ceiling $B_n \leq s$ is random. As permanent gains are made, the player’s pockets will deepen. However, we have seen with Theorem 4.18 that there is a relationship between the distribution of the gain and the distribution of the maximum weight achieved by the urn. Using
Table 8.2. Some $(\cdot,\cdot; 2,3)$ urn ruin probabilities (in percent) with five total balls and the zero-pass strategy.

<table>
<thead>
<tr>
<th>urn</th>
<th>$b = 2$</th>
<th>$b = 3$</th>
<th>$b = 4$</th>
<th>$b = 5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(5,0; 2,3)$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$(4,1; 2,3)$</td>
<td>20</td>
<td>20</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$(3,2; 2,3)$</td>
<td>40</td>
<td>40</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>$(2,3; 2,3)$</td>
<td>50</td>
<td>40</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>$(1,4; 2,3)$</td>
<td>20</td>
<td>20</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$(0,5; 2,3)$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

the same basic procedure, but this time focused only on the gains made before the point of ruin, we can calculate the ruin probability.

8.2.1. Ruin with the Zero-Bet Strategy. Let us begin with the zero-bet strategy. Given $m$, $p$, $b$, and $k$, define $E_{m,p}(b,k)$ as the number of realizations from the $(m,p; 1,1)$ urn for which ruin will occur with bank $b$, and for which a permanent gain of $k$ (not including the initial weight $p - m$) is made before ruin occurs. That is, $E_{m,p}(b,k)$ is the number of realizations for which $B$ attains a maximum value of $b + k$ before hitting zero. Then, the probability of ruin will be

$$r'_b(m,p; 1,1) = \left(\frac{m + p}{p}\right)^{-1} \sum_k E_{m,p}(b,k).$$

We calculate $E_{m,p}(b,k)$ as a convolution of two objects. One of these objects may be counted by the following result, found in Feller [9, p. 96, #3]. For an informal proof via repeated reflections, see S.G. Mohanty’s *Lattice Path Counting and Applications* [14, pp. 6-7] (1979).

**Lemma 8.6.** (Feller) Let $a$ and $b$ be positive, and $-b < c < a$. Then the number of paths from $(0,0)$ to $(n,c)$ that touch neither the line $x = a$ nor $x = -b$ equals

$$S_{a,b}(n,c) = \sum_i \left[ \left( \frac{n+c}{2} + i(a+b) \right) - \left( \frac{n-c}{2} + a + i(a+b) \right) \right].$$

The objects we desire here are paths to $(n,a-1)$ avoiding $x = a$ and $x = -b - 1$, but touching $x = -b$. Using Lemma 8.6, the number of paths with this property equals

$$S_{a,b+1}(n,a-1) - S_{a,b}(n,a-1).$$

The second object is described by Lemma 3.5. Convoluting these two objects, we obtain:
**Theorem 8.7.** Suppose \( k \geq 0 \). Let \( k^* = k + (p - m)^+ \). Then

\[
E_{m,p}(b,k) = \sum_{j=k^*}^{p-k^*} \left( \frac{m + p - 2j - b}{m - j + k^*} \right) (S_{b,k^*+1}(2j + b - 1, b - 1) - \chi(k)S_{b,k^*}(2j + b - 1, b - 1)),
\]

where \( \chi(k) = 0 \) if \( k = 0 \) and equals 1 otherwise.

**Proof.** Suppose \( \omega \) is a realization for which the player is ruined, but makes a permanent gain of \( k \) before ruin occurs. We separate \( \omega \) into three parts: \( \omega_1 \) shall be composed of the balls bet upon up to and including the point of ruin, as in the proof of Theorem 4.18; \( \omega_2 \) is similar to \( \omega_1 \), but contains the balls passed on; and \( \omega_3 \) is the realization following the point of ruin. Then \( \omega_2 \) is a \(+k + (m-p)^+\) trip, \( \omega_3 \) is a \( -(b + k + (p-m)^+)\)” path, while \( \omega_1 \) is a path that takes the weight \(-k - (p-m)^+\), avoids the weights \(-k - 1 - (p-m)^+\) and \( b \), and ends at weight \( b - 1 \), followed by a \( -1 \)” ball (ruining the player). The number of such \( \omega_1 \), given that there are \( j+1 \)” balls (with \( j \geq k^* \)) in \( \omega_1 \), equals

\[
S_{b,k^*+1}(2j + b - 1, b - 1) - \chi(k)S_{b,k^*}(2j + b - 1, b - 1).
\]

(When \( k = 0 \), the conditions of Lemma 8.6 are not met.) Concatenation of \( \omega_2 \) and \( \omega_3 \) gives a path from \( b \) to 0 that reaches \( b + k + (m-p)^+ \), containing \( p - j \)” \(+1\)” balls. Via Lemma 3.4, we find that there are \( \binom{m+p-2j-b}{m-j+k^*} \) such paths (with \( j \leq p - b - k^* \)). We complete by summing over the appropriate \( j \).

To reverse the mapping, it suffices to show that \( \omega_1 \), \( \omega_2 \), and \( \omega_3 \) can be identified. Let \( \omega \) be a realization with the following properties:

1) \( X_n(\omega) = b + (p - m)^+ \) for the first time when \( n = n_1 \).

2) \( X_n(\omega) = -k \) for some \( n < n_1 \).

3) If \( X_n = -k - 1 \), then \( n > n_1 \).

4) \( X_n = b + k + (p - m)^+ \) for some \( n \geq n_1 \).

Then, the realization up to and including the \( n_1^{th} \) ball is \( \omega_1 \). The \(+k^\prime\)” trip that follows from this point is \( \omega_2 \), while the remainder of the realization forms \( \omega_3 \). We complete the inverse map by properly interleaving \( \omega_1 \) and \( \omega_2 \) as in the proof of Theorem 4.18. \( \square \)

**Remark.** Note that \( \omega_2 \), the \(+\left( k + (m-p)^+ \right)\)” path, is not restricted from below. Therefore, reflection of this segment and concatenation with \( \omega_1 \) instead of \( \omega_3 \) will not work.
8.2.2. *Ruin with the Zero-Pass Strategy.* Calculating the ruin probabilities associated with the zero-pass strategy is similar, and we shall be brief. Lemma 8.4 establishes a one-to-one correspondence between ruin realizations with the zero-pass strategy and ruin realizations with the zero-bet strategy for which a “+1” trip follows the point of ruin, if $k > 0$ or $m \geq p$. Let $F_{m,p}(b,k)$ denote the realizations from the $(m,p)$ urn for which a permanent gain of $k$ precedes the point of ruin with bank $b$, so that

$$r_b(m,p; 1,1) = \left(\frac{m+p}{p}\right)^{-1} \sum_{k} F_{m,p}(b,k).$$

Then we have:

**Theorem 8.8.** Suppose $m \geq p$ or $k > 0$, and let $k^* = k + \max\{0, p - m\}$. Then

$$F_{m,p}(b,k) = \sum_{j=k^*}^{p-b-k^*-1} \binom{m+p-2j-b}{m-j+k^*+1} (S_{b,k^*+1}(2j+b-1, b-1) - \chi(k)S_{b,k^*}(2j+b-1, b-1))$$

where $\chi(k) = 0$ if $k = 0$ and equals 1 otherwise.

**Remark.** If we proved Theorem 8.8 in the manner of the proof of Theorem 8.7, then $\omega_2$ would be a “$+k + 1 + (m - p)^+$” trip, with the extra “+1” trip associated with the first trip from neutral to one.

For the case $m < p$ and $k = 0$, we split $F_{m,p}(b,k)$ into two classes - those that touch $x = 0$ before ruin, and those that do not.

**Theorem 8.9.** Suppose $m < p$. Then

$$F_{b,0}(m,p) = \sum_{j=p-m}^{m-b} \binom{m+p-2j-b}{p-j} S_{b,p-m}(2j+b-1, b-1)$$

$$+ \sum_{j=p-m}^{m-b-1} \binom{m+p-2j-b}{p-j+1} (S_{b,p-m+1}(2j+b-1, b-1) - S_{b,p-m}(2j+b-1, b-1)).$$

**Remark.** The same mapping presented in the proof of Theorem 8.7 works for any of the $(m,p; s,t)$ urns. For the $(m,p; 1,t)$ urns, an explicit $(1,t)$ form of the expression $S_{a,b}(n,c)$ of Lemma 8.6 does not seem to exist. This would not be a surprise, as the proof of Lemma 8.6 uses the reflection method, while in the $(1,t)$ case we have the weaker rotation/midpoint reflection method. For the $(m,p; s,t)$ urns, we lose the rotation method altogether, and we have the added problem that ruin occurs when the bank first reaches the range $[0, s)$. 

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8.3. **The Ruin Problem for the Random Acceptance Urn.** The ruin problem can similarly be applied to the Bayesian version of the urn. We again give the player a bank $b = B_0$, and monitor the random variables $B_i$, $0 \leq i \leq n$. Since the optimal strategy is very dependent on the initial prior distribution $\theta$, we shall concentrate solely on the $(n, \theta; 1, 1)$ urns.

8.3.1. **The Ruin Probability with $s = t = 1$.** We assume that our initial prior distribution $\theta$ of the urns on $n$ balls meet the conditions specified by Lemma 7.4. We shall have our player use the betting rule of [7] (betting if and only if $\sum_{j=1}^{k} Y_j \geq 0$), which we shall again call the zero-bet strategy. Then the player shall be ruined if and only if the event

$$(8.8) \quad Z := \sum_{j=1}^{k} Y_j = 0, \text{ then a } \text{“} -1 \text{” ball is drawn}$$

occurs for at least $b$ values of $k$. Then, the probability that the player is ruined using the zero-bet strategy equals $P(Z \geq b)$. This is because the permanent gain (the initial weight of the urn, if it is positive) is made during the final trip from the initial weight to zero. (Previous trips are undone, as the player bets not only from the initial weight to zero, but also from zero back to the initial weight, losing what had been gained. This does not happen for the last trip, as the urn weight finishes at zero.)

**Theorem 8.10.** Suppose $\theta$ satisfies the conditions of Lemma 7.4 and the zero-bet strategy is used. Then for any integer $b \geq 1$, the probability of ruin with bank $b$ equals

$$\sum_{j<n/2} q_j \left( ^n_j \right)^{-1} \left( ^n_{j-b+1} \right) + \sum_{j\geq n/2} q_j \left( ^n_j \right)^{-1} \left( ^n_{j+b} \right).$$

**Proof.** We calculate $P(Z \geq b)$. Given that $j$ “+1” balls are initially in the urn, (8.8) occurs each time the urn weight passes from $X_0(j) = 2j - n$ to $2j - n + 1$. Let $\omega$ be a realization from the $(n-j, j; 1, 1)$ urn. If $Z(\omega) \geq b$, then the urn weight of $\omega^R$ passes from $-1$ to $0$ at least $b$ times. Then we have

**Case 1.** If $2j \geq n$, the urn weight of $\omega^R$ must pass from $0$ to $-1$ before each passage from $-1$ to $0$. Therefore, $\omega^R$ gains at least $b$ upon play of the (nonrandom) $(n-j, j; 1, 1)$ urn using the zero-bet strategy. Since realization reversal is a bijection, we have from Lemma 3.4 that

$$P(Z \geq b \mid j) = P(\text{player gains at least } 2j - n + b \text{ on the } (n-j, j; 1, 1) \text{ urn}) = \left( ^n_j \right)^{-1} \left( ^n_{j+b} \right).$$
Case 2. If $2j < n$, we have a similar relation between the passages from 0 to $-1$ and the passages from $-1$ to 0 of $\omega^R$. This time, a passage from 0 to $-1$ does not precede the first passage from $-1$ to 0. Therefore, Lemma 3.4 implies that

$$P(Z \geq b \mid j) = P(\text{player gains at least } b-1 \text{ on the } (n-j,j;1,1) \text{ urn}) = \binom{n}{j}^{-1} \binom{n}{j-b+1}.$$  

We obtain the desired result via the total probability rule. □

Using Maple 12™, we have calculated some ruin probabilities and present them with Table 8.3.

**Table 8.3.** Probabilities of ruin approximated to three decimal places, with $\theta$ uniform, using the zero-bet strategy.

<table>
<thead>
<tr>
<th>bank</th>
<th>$n = 5$</th>
<th>$n = 10$</th>
<th>$n = 100$</th>
<th>$n = 1000$</th>
<th>$n = 10000$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b = 1$</td>
<td>.617</td>
<td>.646</td>
<td>.688</td>
<td>.693</td>
<td>.693</td>
</tr>
<tr>
<td>$b = 2$</td>
<td>.133</td>
<td>.188</td>
<td>.292</td>
<td>.305</td>
<td>.307</td>
</tr>
<tr>
<td>$b = 3$</td>
<td>.017</td>
<td>.050</td>
<td>.170</td>
<td>.190</td>
<td>.193</td>
</tr>
<tr>
<td>$b = 4$</td>
<td>0</td>
<td>.009</td>
<td>.109</td>
<td>.137</td>
<td>.140</td>
</tr>
<tr>
<td>$b = 5$</td>
<td>0</td>
<td>.001</td>
<td>.073</td>
<td>.105</td>
<td>.109</td>
</tr>
<tr>
<td>$b = 6$</td>
<td>0</td>
<td>0</td>
<td>.049</td>
<td>.085</td>
<td>.090</td>
</tr>
<tr>
<td>$b = 7$</td>
<td>0</td>
<td>0</td>
<td>.033</td>
<td>.070</td>
<td>.076</td>
</tr>
<tr>
<td>$b = 8$</td>
<td>0</td>
<td>0</td>
<td>.021</td>
<td>.059</td>
<td>.066</td>
</tr>
</tbody>
</table>

Asymptotically,

$$\sum_{j \geq n/2} \binom{n}{j+b} \binom{n}{j} \sim n \int_{1/2}^{1} \binom{1-x}{x}^b \, dx,$$

so that the asymptotic value of the ruin probability, as $n \to \infty$ and with $\theta$ uniform, equals

$$I_b = \int_{1/2}^{1} \binom{1-x}{x}^b + \left( \frac{1-x}{x} \right)^{b-1} \, dx = \int_{1/2}^{1} \left( \frac{1-x}{x} \right)^{b-1} \, dx.$$

Integration by parts gives for $b \geq 2$

$$I_b = \frac{1}{b-1} - I_{b-1} \quad \Rightarrow \quad I_b = (-1)^{b-1} \ln 2 + \sum_{k=1}^{b-1} \frac{(-1)^{b-1-k}}{k},$$

since $I_1 = \ln 2$. Clearly, the probability of ruin tends to zero as $b \to \infty$.

8.3.2. A Safer Optimal Strategy. We have noted that the probability of ruin appears to be higher for the riskier zero-bet strategy compared with the zero-pass strategy for the original urn, and showed this is the case when $s = 1$ and $t$ is a positive integer. This also appears to be the case with the Bayesian version of the urn. Recall that betting if and only if $\theta(Y_{k+1} = t \mid y_1, y_2, \ldots, y_k) > s/(s+t)$
is also an optimal strategy. For the distributions satisfying the conditions of Lemma 7.4, this betting rule can be stated when \( s = t = 1 \) as “bet if and only if \( \sum_{j=1}^{k} Y_j > 0 \).” We shall call this policy the zero-pass strategy. This time, the player is ruined if the event

\[
Z' := \sum_{j=1}^{k} Y_j = 1 \quad \text{then a \("-1\) ball is drawn}
\]

happens at least \( b \) times. Then, the probability that the player is ruined while using the zero-pass strategy equals \( P(Z' \geq b) \).

**Theorem 8.11.** Suppose \( \theta \) satisfies the conditions of Lemma 7.4. Then for any integer \( b \geq 1 \), the probability of ruin using the zero-pass strategy equals

\[
- q_{n/2} \left( \frac{n}{n/2} \right)^{-1} \left( \frac{n}{n/2 + b} \right) + 2 \sum_{j \geq n/2} q_j \left( \frac{n}{j} \right)^{-1} \left( \frac{n}{j + b} \right).
\]

**Proof.** We proceed similarly to the proof of Theorem 8.10. This time, if there are \( j \) \( +1 \) balls initially, (8.9) occurs each time the urn weight passes from \( 2j - n - 1 \) to \( X_0(j) = 2j - n \). Given \( \omega \) with \( Z(\omega) \geq b \), \( \omega^R \) has at least \( b \) passages from 0 to 1. Therefore, \( \omega^R \) gains at least \( \max\{0, 2j - n\} + b \) from the nonrandom \((n - j, j; 1, 1)\) urn. Therefore,

\[
P(Z' \geq b \mid j) = P\{\text{player gains at least } \max\{0, 2j - n\} \text{ from the } (n - j, j; 1, 1) \text{ urn} \}
\]

\[
= \begin{cases} 
\binom{n}{j}^{-1} \binom{n}{j + b}, & \text{if } 2j < n, \\
\binom{n}{j}^{-1} \binom{n}{j - b}, & \text{if } 2j \geq n.
\end{cases}
\]

We then sum over \( j \) using the total probability rule, and after some adjustment we obtain the desired form. \( \square \)

Asymptotically, the probability of ruin, as \( n \to \infty \) and with \( \theta \) uniform, equals

\[
I'_b = 2 \int_{1/2}^{1} \left( \frac{1-x}{x} \right)^b dx.
\]

Since

\[
I_b = \frac{1}{b-1} - \frac{b}{b-1} I'_b, \quad I'_1 = 2 \ln 2 - 1,
\]

we have

\[
I'_b = 1 + (-1)^{b-1} 2b \left( \ln 2 + \sum_{k=1}^{b} \frac{(-1)^k}{k} \right).
\]
Using Maple 12\textsuperscript{TM}, we have calculated some ruin probabilities using the zero-pass strategy, and present them with Table 8.4 for comparison with the values of Table 8.3.

Table 8.4. Probabilities of ruin approximated to three decimal places, with $\theta$ uniform, using the zero-pass strategy.

<table>
<thead>
<tr>
<th>bank $b$</th>
<th>$n = 5$</th>
<th>$n = 10$</th>
<th>$n = 100$</th>
<th>$n = 1000$</th>
<th>$n = 10000$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b = 1$</td>
<td>.233</td>
<td>.306</td>
<td>.377</td>
<td>.385</td>
<td>.386</td>
</tr>
<tr>
<td>$b = 2$</td>
<td>.033</td>
<td>.101</td>
<td>.209</td>
<td>.225</td>
<td>.227</td>
</tr>
<tr>
<td>$b = 3$</td>
<td>0</td>
<td>.026</td>
<td>.132</td>
<td>.156</td>
<td>.159</td>
</tr>
<tr>
<td>$b = 4$</td>
<td>0</td>
<td>.004</td>
<td>.088</td>
<td>.118</td>
<td>.121</td>
</tr>
<tr>
<td>$b = 5$</td>
<td>0</td>
<td>.000</td>
<td>.059</td>
<td>.093</td>
<td>.098</td>
</tr>
<tr>
<td>$b = 6$</td>
<td>0</td>
<td>0</td>
<td>.040</td>
<td>.076</td>
<td>.082</td>
</tr>
<tr>
<td>$b = 7$</td>
<td>0</td>
<td>0</td>
<td>.026</td>
<td>.064</td>
<td>.070</td>
</tr>
<tr>
<td>$b = 8$</td>
<td>0</td>
<td>0</td>
<td>.017</td>
<td>.055</td>
<td>.061</td>
</tr>
</tbody>
</table>
9. Conclusion

We have found that acceptance urn model can be generalized to the urns containing balls of values $-s$ and $+t$. Like the original acceptance urn model of [7], an optimal strategy that maximizes the expected gain is to bet whenever the urn weight is positive (that is, when the probability the next ball has value $+t$ is at least $s/(t+s)$), and to pass whenever the urn weight is negative. It would not be a surprise, then, that this would also be the case when the urn contains balls of multiple positive and negative values. Moreover, the method of counting crossings can be applied in this case, though any formulas will be extremely complicated. It is also likely that the problem of finding the probability the player gains (at least) $k$ is equivalent to finding the probability that the maximum weight of the urn equals (is at least) $k$. In this way the acceptance urn model can be interpreted as an extension of the ballot problem, with the question “What is the probability that candidate $A$ never trails by more than $k$ in weight as the votes are counted?”.

The use of vertical stretching is natural, and proved to be very useful for scaling purposes. However, it only has a global use for the $(m, p; s, t)$ urns (Lemma 2.1), and cannot be used locally, due to the restrictions placed on the ball values. Vertical reflection is also a natural tool to use, and gave a global result, the Antiurn Theorem. The reflection method, for these acceptance urns, has a local application, but only in the case $s = t$. For the urns with $s = t = 1$, we can map realizations from the $(m, p; 1, 1)$ urns to realizations from any $(m', p'; 1, 1)$ urn containing the same number of balls ($m' + p' = m + p$). These mappings produced elementary proofs for both the distribution of the gain using an optimal strategy for the urn (Lemma 3.4), and the maximum weight problem (Lemma 3.5) that did not require additional shuffling of the paths. The ruin problem was also solved for the $(m, p; 1, 1)$ urns with bank $b$ (Theorems 8.7, 8.8, and 8.9), using inclusion/exclusion and repeated reflection, though some shuffling was required for the proofs of those results. Unfortunately, localized reflection cannot be used in general for the $(m, p; s, t)$ urns, and as a result solving these problems became more difficult and results were more complicated.

Rotation, unlike vertical stretching and reflection, is a universal transformation, and in particular it can be applied locally for all of the $(m, p; s, t)$ urns. However, its main limitation is that a rotation
does not change the composition of the path at all, as \( \omega \) and \( \omega(i,j) \) are both realizations from the same urn. As a result, the use of rotation globally is of limited use, when the expected gain \( G(m,p; s,t) \) is considered, as the only result in this work that uses global rotation is Theorem 4.15. There is significant symmetry with the \( (m,p; s,t) \) urns, as shown with the regular repeating “cycle” of the crossings, but rotation does not seem to be a good enough tool in general to take advantage of this symmetry.

The rotational symmetry found for the case \( s = 1 \) and \( t \) a positive integer is perhaps the most noteworthy result of this work. For those urns (and the ones indicated by Lemma 2.1 and the Antiurn Theorem), rotation turned out to be a sufficient tool to properly use the symmetry, and we obtained a fundamental result, the Crossing Lemma (Lemma 3.7). The Crossing Lemma changed the problem of counting crossings to the simpler problem of counting the number of times the urn is neutral, for \( m \geq pt \). Counting the number of times the urn was neutral could be done in many different ways, also as a result of the Crossing Lemma. This induced combinatorial identities, which could be shown in a larger context using generalized binomial series (Lemma 3.1). The counting methods used for the proofs of Lemmas 3.10 and 3.13 can also be used to count similar objects at any fixed weight of the urn, and thus the Crossing Lemma has more general applications in the area of lattice paths. See [23].

When \( m < pt \), we still could count the number of times the urn was neutral with the help of the Crossing Lemma. Unfortunately, the first down-crossing in general does not meet the conditions specified by the Crossing Lemma. Recall that we let \( \rho = \min\{j: X_j \leq 0\} \), and defined the crossing number \( \kappa = \kappa(m,p; 1,t) = -E[X_\rho] \) for the \( (m,p; 1,t) \) urns with \( m < pt \), so that

\[
G(m,p; 1,t) = pt - m + \kappa + \left(\frac{m + p}{p}\right)^{-1} \sum_{k=1}^{p} \binom{kt + k}{k} \binom{m + p - k - kt}{p - k}.
\]

Using rotation (Lemma 3.23), we were able to reduce the possible range of \( \kappa \) to \([0, (t - 1)/2]\), with zero only achieved when \( m = 0 \). We also derived exact formulas for \( G(m,p; 1,t) \), the distribution form (Corollary 3.30) and the crossings form (Lemma 4.6), so in a way we can calculate \( \kappa \). However, calculating \( \kappa \) using catastrophic cancellation does not seem to be a good approach. In [23], we do indicate how to calculate the probability that \( X_\rho = -i \), where \( 0 \leq i \leq t - 1 \) (using the function \( Q_{m,p}(\ell) \) of Theorem 3.26), however, those results are complicated, and from there we would still need to take the average to calculate \( \kappa \). What we seek is a simpler way to express \( \kappa \), and in particular
we wish to find how to express $\kappa$ when $m$ and $p$ are large. Some calculations with Maple 12™, along with results (using the Crossing Lemma) from the corresponding random walk lead us to make the following conjectures.

**Conjecture 9.1.** Suppose $t$ is a positive integer, and that $pt - m > 0$ is a fixed constant. Then the sequence $\{\kappa(m, p; 1, t)\}$ is monotonically increasing, and converges as $p \to \infty$.

**Conjecture 9.2.** If as $p \to \infty$, $pt - m \to \infty$ and $pt - m = o(p)$, then for $0 \leq i \leq t - 1$, $P(X_p = -i) \to (t - i)/T_t$, where $T_t = \sum_{i=1}^{t} i = t(t + 1)/2$ is the $t^{th}$ triangular number. Thus, as $p \to \infty$ we have

$$\kappa(m, p; 1, t) \to \frac{1}{T_t} \sum_{i=1}^{t-1} i(t - i) = \frac{t^2 - 1}{3}.$$

We expect the condition $pt - m = o(p)$ to be required, since for very large urn weights the expected drift is far from zero. It is unclear how to compute an “approximate” mapping to explain this asymptotic result.

The crossing number problem has a similar extension to the $(m, p; s, t)$ urns, and its significance is that it will result in the player’s first nontrivial permanent gain during play on the $(m, p; s, t)$ urn. For $t/s$ rational, the problem can be reduced to the case with $s$ and $t$ positive integers with $\gcd(s, t) = 1$, and $ms - pt \geq 0$, in which case the first (zero-bet) up-crossing is to $0, \ldots, s - 1$. However, because of the ordering of the crossings, we cannot reduce the range $\kappa(m, p; s, t)$ (as defined in Definition 6.23) in the manner of Lemma 3.23. We also do not have a form of $G(m, p; s, t)$ that involves the crossing number, so we are unable to produce even a “catastrophic” cancellation form of $\kappa(m, p; s, t)$.

For the $(m, p; s, t)$ urns, we were able to show the basic structure of $G(m, p; s, t)$ in Chapter 2. However, the method of urn simulation (for example, the proof of Lemma 2.6) does not appear to work upon consideration of the second-order differences. Let

$$\Delta^2 G_m(\cdot, p; s, t) = G(m, p + 2; s, t) + G(m, p; s, t) - 2G(m, p + 1; s, t),$$

$$\Delta^2 G_p(m, \cdot; s, t) = G(m + 2, p; s, t) + G(m, p; s, t) - 2G(m + 1, p; s, t),$$

$$\Delta^2 G(m, p; s, t) = G(m + 2; p; s, t) + G(m, p + 2; s, t) - 2G(m + 1, p + 1; s, t).$$

We conjecture that these second-order differences share the same properties as those with $s = t = 1$.

**Conjecture 9.3.** For positive $s$ and $t$, and positive integers $m$ and $p$, we have $\Delta^2 G_m(\cdot, p; s, t) > 0$, $\Delta^2 G_p(m, \cdot; s, t) > 0$, and $\Delta^2 G(m, p; s, t) > 0$.  

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It should be noted that \( \Delta^2 G_m(\cdot, p; s, t) = \Delta^2 G_{m+p}(\cdot; t, s) \) by the Antiurn Theorem.

Most of the asymptotic results of Chapter 5 were obtained using the pseudo-binomial form (5.2), which, when \( s = 1, t \) is a positive integer, and \( m \geq pt \), is very close to the binomial form (Theorem 3.15). Unfortunately, if \( ms/(pt) \to \lambda > 1 \), the transition of \( G(m, p; s, t) \) to the pseudo-binomial form introduces an error term that is too large, and instead we are left with the less satisfying result \( G(m, p; s, t) = O(1) \) (Lemma 5.19) for this case. Some calculations with Maple 12™ indicate that the natural candidate for the asymptotic value of \( G(m, p; s, t) \), \( \frac{t+s}{2(\lambda-1)} \), does not appear to be equal to the asymptotic value. In particular, with \( s = 3, t = 5, \) and \( \lambda = 2 \), the natural candidate gives 4, but \( G(1000, 300; 3, 5) > 4.0326 \). With \( s = 2, t = 3, \) and \( \lambda = 2.5, G(7500, 2000; 2, 3) > 1.709 > 5/3 \).

(We also believe that the expected gain will increase as \( m \) increases, as shown in [7, Theorem 2.7].) As a result, we cannot speculate on the asymptotic value of \( G(m, p; s, t) \) with \( ms/(pt) \to \lambda > 1 \) in this work.

The Bayesian version of the acceptance urn model has a broader real-world application, since the future price of a commodity is generally unknown. Generally, we have shown that if the market is more likely to go up, then the player should bet, and the player should not bet if the market is more likely to go down. For a select group of distributions with \( s = t = 1 \), simple rules were given based on the outcome of the game through the first \( k \) balls, but a nicer description of a betting rule seems to be difficult to obtain for a general initial prior distribution \( \theta \). As a result, finding the expected gain for other distributions is an open question for the \( (n, \theta; s, t) \) urns, as well as the \( (n, \theta; 1, t) \) urns (with \( t \) a positive integer) and the \( (n, \theta; s, t) \) urns. The algorithms given with Theorems 7.10 (for the case \( s = t = 1 \) ) and 7.14 indicates when the player should bet, and outputs the expected gain \( \mathcal{G}(n, \theta; s, t) \), for any \( n \) and \( \theta \), but unfortunately the algorithm itself requires many calculations, even when \( s = t \).

We solved the ruin problem in the case \( s = t = 1 \) for any \( m, p \), with an initial bank \( b \), and the use of an optimal strategy that maximizes the expected gain. Unfortunately, the formula obtained is quite complicated. Using the rather soft necessary condition for ruin (and the simple solution), we were only able to obtain a limited number of asymptotic results. The method of the proof of Theorem 8.7 is sound when applied to the \( (m, p; 1, t) \) urns with \( t \) a positive integer. However, we do not have the solution to a generalized version of the problem found in Feller [9] (Lemma 8.6), and certainly the method of the proof of Lemma 8.6 will no longer apply. At the very least, we can give a description of what a realization producing ruin looks like in terms of the mapping used in the
proof of Theorem 8.7. For the \((m, p; s, t)\) urns, the ruin problem is further complicated by the fact that the player can be ruined if \(B_n\) equals \(0, \ldots, s - 1\).

We have found a necessary condition for ruin. We can find a sufficient condition for ruin as well, which can be expressed in terms of the generalized ballot numbers of Chapter 6. Unfortunately, this sufficient condition is quite soft, and as a result we cannot prove that the probability of ruin is near one when the bank is small. However, we do believe this to be the case.

**Conjecture 9.4.** If \(pt - ms = O(\sqrt{p})\) and \(B_0 = o(\sqrt{p})\), then for the \((m, p; s, t)\) urn we have

\[
P(B_{\min} < s) \rightarrow 1, \quad \text{as } p \rightarrow \infty.
\]

Based on this conjecture, along with Theorem 8.5, we are led to make a conjecture on what we believe to be the critical case.

**Conjecture 9.5.** If \(pt - ms = O(\sqrt{p})\) and \(B_0 p^{-1/2} \rightarrow \alpha \neq 0\), then for the \((m, p; s, t)\) urn

\[
P(B_{\min} < s) \not\rightarrow 0 \quad \text{and} \quad P(B_{\min} < s) \not\rightarrow 1.
\]

The weakness of the ruin problem lies in the fact that we force the player to use an optimal strategy maximizing the expected gain, that is, the player is unconcerned about being ruined. To combat this, we introduce a modified version of the acceptance urn model, as follows: the game is played as before, but this time the player starts with a bank \(b\) and aims to maximize the expected gain \textit{with the prospect of ruin}. Given a realization \(\omega\) from the \((m, p; s, t)\) urn, if the player uses “strategy A” (which need not be optimal) and gains \(g(\omega)\) with an infinite bank, then with bank \(b\) the player will gain \(g(\omega)\) if \(B_n \geq s\) throughout, but if \(B_n < s\) at some point the player is ruined, and the player’s “gain” will be the difference of the bank at the point of ruin and \(b\), e.g. when \(s = 1\), the gain will be \(-b\) instead. In other words, the bank will have an effect on the optimal acceptance policy for this \((m, p; s, t; b)\) urn, the \textit{acceptance urn with ruin}.

In this work, we have only considered the problem of maximizing the expected gain \(G(m, p; s, t)\). By changing the object we aim to optimize, we will almost certainly change the optimal acceptance strategy. For instance, for the \((m, p; s, t)\) urn, we might instead look to find a strategy maximizing the probability that the player gains, say, at least \(C \cdot G(m, p; s, t)\) for some constant \(C\). In that case, the optimal strategy maximizing the expected gain may no longer be the player’s best strategy.
LIST OF REFERENCES


About the Author

Kevin P. Wagner received both a Bachelor’s and Master’s Degree in Mathematics in 2001 from the University of South Florida as a part of the 5-Year B.A./M.A. Program. He served as a teaching assistant in the Mathematics Department at the University of South Florida from 2000 to 2006, and was a regular attendee at the Discrete Mathematics Seminar during that time. He has coauthored a paper to be published in the SIAM Journal on Discrete Mathematics in late 2010 or early 2011.