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A Survey of the Development of Daubechies Scaling Functions

Amber E. Age
University of South Florida

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by

Amber E. Age

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Major Professor: Catherine Bénéteau, Ph.D.
Dmitry Khavinson, Ph.D.
Sherwin Kouchekian, Ph.D.

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Dedication

I would like to dedicate this to all my mathematics teachers and instructors who have taught me over the years. It is because of their guidance and enthusiasm that I not only started my quest for mathematical education, but also kept my path steadfast and true. To all of them, I give thanks.

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Table of Contents

List of Figures

A Survey of the Development of Daubechies Scaling Functions

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ABSTRACT

Wavelets are functions used to approximate data and can be traced back to several different areas, including seismic geology and quantum mechanics. Wavelets are applicable in many areas, including fingerprint and data compression, earthquake prediction, speech discrimination, and human vision. In this paper, we first give a brief history on the origins of wavelet theory. We will then discuss the work of Daubechies, whose construction of continuous, compactly supported scaling functions resulted in an explosion in the study of wavelets in the 1990's. These scaling functions allow for the construction of Daubechies' wavelets. Next, we shall use the algorithm to construct the Daubechies $D4$ scaling filters associated with the $D4$ scaling function. We then explore the Cascade Algorithm, which is a process that uses approximations to get possible representations for the $D2N$ scaling function of Daubechies. Lastly, we will use the Cascade Algorithm to get a visual representation of the $D4$ scaling function.

Chapter 1

Wavelets: A Brief History and Basic Definitions

The study of wavelets can be traced back to several different fields, including mathematics, quantum physics, and electrical engineering. While several areas are responsible for independently developing wavelets, perhaps the most important contribution came from seismic geology and the work of Jean Morlet. Morlet needed a way to analyze seismic signals which carried information about geological layers. Building off the work of Dennis Gabor, Morlet, along with Alex Grossman, explicitly defined and began using the word wavelet. Stéphane Mallat and Yves Meyer then used these *Grossman-Morlet* wavelets as building blocks for *multiresolution analysis*, a notion built off previous work done by Burt and Adelson [1] and that uses the concept of *orthonormal wavelet bases*. The orthonormal wavelet bases, as first defined by Meyer in [7], were not perfected for use in applications until the work of Ingrid Daubechies in 1988. It was at this time that Daubechies presented a construction that resulted in a set of orthonormal wavelet bases that were of compact support and continuously differentiable [2]. This development resulted in an explosion in the study of wavelets and their applications. Currently, wavelets are used in a variety of areas, including human vision, speech discrimination, fingerprint compression, earthquake prediction, and nuclear engineering. For further information, the reader should see [3] or [6].

Wavelets are functions that can be used to approximate data. While other types of functions, such as the Fourier transform, can also be used to approximate data, the wavelet transform has the additional ability to analyze data at different resolutions. Data analyzed at a larger resolution gives a rough approximation while data analyzed at smaller resolutions provides more detailed information. It is this ability of the wavelet transform to zoom in and out of the data that makes it superior to the Fourier transform for certain applications. As stated in [3], wavelet algorithms allow us to see "both the forest and the trees."

While several different approaches were taken in the development of the analysis of wavelets, for the purposes of this paper, we shall focus on the work done by Haar and the idea of a multiresolution analysis presented by Mallat [4] and Meyer [7]. We will then briefly mention the Shannon multiresolution analysis and its shortcomings in terms of applications. These shortcomings will be corrected by Daubechies' construction of continuous, compactly supported scaling functions. These scaling functions are the essential tools needed to build orthonormal wavelet bases. In this paper, we will focus on the development of these scaling functions. Much of the following discussion is motivated by the works of [6] and [8].

While the study of wavelets started in the 1930's, the roots of wavelets go back to 1807 and the ideas of Fourier regarding the convergence of functions. Fourier claimed that every $2\pi-$ periodic function $f(\omega)$ could be expressed as a sum of its Fourier Series. For the purposes of this paper, we shall only be considering functions $f(\omega)$ in the space $L^2(\mathbb{R})$.

Definition 1.1 *We define the space $L^2(\mathbb{R})$ as the set of all Lebesgue measurable, complex-valued functions f such that $||f||_2^2 := \int_{\mathbb{R}} |f(\omega)|^2 d\omega < \infty$. We define the space $L^2([-\pi, \pi])$ in an analogous way.*

Definition 1.2 *We define the Fourier Series for a function for $f(\omega) \in L^2([-\pi, \pi])$, by*

$$\sum_{k=-\infty}^{\infty} c_k e^{ik\omega},$$

where the coefficient c_k is given by

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(\omega) e^{-ik\omega} \, d\omega.$$

Note that if $f \in L^2([-\pi, \pi])$, the above-mentioned series exists a.e. and converges to f in the L^2 norm, and so we often write

$$f(\omega) = \sum_{k=-\infty}^{\infty} c_k e^{ik\omega}.$$

However, it must be pointed out that given a continuous function f, it is not always the case that its Fourier series will converge to f pointwise. In fact, problems with convergence of Fourier series came in the form of an example constructed by P. Du Bois-Reymond. In his example, Du Bois-Reymond found a $2\pi-$periodic function whose Fourier series diverged at a given point. It was then concluded that certain restrictions would need to be met in order for Fourier's original theory to hold.

Mathematicians began trying to justify Fourier's theory in one of three ways. The first way was to attempt to re-evaluate the definition of a function and try to modify it in such a way that it would coincide with the idea of Fourier series. The second idea was to re-evaluate the notion of convergence for Fourier series. It was thought that by putting certain conditions on $f(x)$, convergence of the Fourier series would follow. For example, it can be shown that if $f(x)$ is a twice-differentiable function on the circle, then the Fourier Series of f converges absolutely and uniformly to f (see [10]). The third idea was to use orthogonal systems in the place of trigonometric polynomials. In other words, using orthogonal systems, could one achieve convergence of the Fourier series to a function? The first significant result was given by Haar in 1801 [6].

Recall that the *inner product* of two functions $f(t), g(t) \in L^2(\mathbb{R})$, is defined as

$$\langle f(t), g(t) \rangle = \int_{\mathbb{R}} f(t)\overline{g(t)}dt.$$

For a set of functions $\{\phi_k(t)\}_{k\in\mathbb{Z}} \in \mathcal{W}$, where $\mathcal{W}$ is a subspace of $L^2(\mathbb{R})$, we say that $\{\phi_k(t)\}_{k\in\mathbb{Z}}$ is a *basis* for $\mathcal{W}$ if every function $f \in \mathcal{W}$ can be expressed as a limit of finite linear combinations of $\{\phi_k(t)\}_{k\in\mathbb{Z}}$ and if $\{\phi_k(t)\}_{k\in\mathbb{Z}}$ is linearly independent. A basis $\{\phi_k(t)\}_{k\in\mathbb{Z}}$ is an *orthonormal basis* for $\mathcal{W}$ if

$$\langle \phi_k(t), \phi_j(t) \rangle = \begin{cases} 1 & j = k \\ 0 & \text{otherwise.} \end{cases}$$

Haar began by looking for an orthonormal system of functions $\{\phi_j(t)\} \in \mathbb{R}$ such that for a continuous function $f(t) \in L^2(\mathbb{R})$, the series given by

$$\sum_{j\in\mathbb{Z}} \langle f(t), \phi_j(t) \rangle \phi_j(t),$$

would converge uniformly to $f(t)$.

Haar created a vector space that contained all step-functions that had breakpoints at the integers. He defined an initial function as

$$\phi(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & \text{otherwise} \end{cases}$$

This function is known as the *Haar function*. It should be noted that the Haar function is sometimes referred to as the *box function*. The Haar function and its integer translates were used to define the space V_0, where

$$V_0 = \overline{\text{span}\{\phi(t-k)\}_{k\in\mathbb{Z}}} \cap L^2(\mathbb{R}).$$

This means that every function $f \in V_0$ can be expressed as an L^2 limit of finite linear combinations of $\{\phi(t-k)\}_{k\in\mathbb{Z}}$. V_0 is known as the *Haar Space*. Note that the functions $\{\phi(t-k)\}_{k\in\mathbb{Z}}$ form an orthonormal basis for V_0. Haar projected functions $f \in L^2(\mathbb{R})$ onto V_0 in the standard way:

$$P_0\big(f(t)\big) = \sum_{k\in\mathbb{Z}} \langle f(t), \phi(t-k)\rangle \phi(t-k).$$

This projection of $f(t) \in L^2(\mathbb{R})$ into V_0 gives a rough approximation of f. To alter the approximation, for better or worse, Haar simply changed where the breakpoints of the function occurred. Haar defined these spaces V_j as

$$V_j = \overline{\text{span}\{\phi(2^j t-k)\}_{k\in\mathbb{Z}}} \cap L^2(\mathbb{R}).$$

As before, for $j \in \mathbb{Z}$, by defining the function $\phi_{j,k}(t)$ as

$$\phi_{j,k}(t) = 2^{\frac{j}{2}}\phi(2^j t - k),$$

one can show that $\{\phi_{j,k}(t)\}$ is an orthonormal basis for the space V_j. Once again, we can project functions $f \in L^2(\mathbb{R})$ onto V_j by defining our projection $P_j\big(f(t)\big)$ as

$$\begin{aligned} P_j\big(f(t)\big) &= \sum_{k\in\mathbb{Z}} \langle \phi_{j,k}(t), f(t)\rangle \phi_{j,k}(t) \\ &= 2^j \sum_{k\in\mathbb{Z}} \langle \phi(2^j t-k), f(t)\rangle \phi(2^j t - k). \end{aligned}$$

The Haar Spaces V_j obey some nice properties that will be needed later, so we shall present them here. First, the V_j spaces are *nested* spaces. That is, they satisfy

$$... \subseteq V_{-2} \subseteq V_{-1} \subseteq V_0 \subseteq V_1 \subseteq V_2 ....$$

We can also move from one space to another with relative ease:

$$f(t) \in V_j \Leftrightarrow f(2t) \in V_{j+1}.$$

The reason we can "zoom" between spaces is because our function $\phi(t)$ satisfies what is called a *dilation equation* given by

$$\phi(t) = \frac{\sqrt{2}}{2}\phi_{1,0}(t) + \frac{\sqrt{2}}{2}\phi_{1,1}(t) = \phi(2t) + \phi(2t-1).$$

Because $\phi(t)$ satisfies such an equation, $\phi(t)$ is usually called a *scaling function*. By considering the union and intersection of the Haar spaces, we obtain two more useful properties, namely that these spaces satisfy a separation condition and are dense (see [8]):

$$\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$$

and

$$\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}).$$

We shall later see that these properties are essential in our definition of a multiresolution analysis. Haar's approach has its limitations, as its scaling function $\phi(t)$ is not continuous and its first derivative is zero a.e.. For purposes of application, we desire a scaling function that is both continuous and has a number of derivatives. Daubechies was able to build off of Haar's original construction to obtain a scaling function that has both of these properties. Before we can discuss how Daubechies built such scaling functions, we must describe the idea of *multiresolution analysis*, a result of the work of Meyer [7] and Mallat [4] built upon the earlier work of Burt and Adelson [1].

Definition 1.3 *Let $\{V_j\}_{j\in\mathbb{Z}}$ be a sequence of subspaces of $L^2(\mathbb{R})$. We say that $\{V_j\}_{j\in\mathbb{Z}}$ is a* multiresolution analysis *of $L^2(\mathbb{R})$ if*

1. $V_j \subseteq V_{j+1}$,
2. $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$.
3. $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$
4. $f(t) \in V_0 \Leftrightarrow f(2^j t) \in V_j$

and there exists a function $\phi(t) \in V_0$, called a scaling function, with $\int_{\mathbb{R}} \phi(t)dt \neq 0$, such that the set $\{\phi(t-k)\}_{k\in\mathbb{Z}}$ is an orthonormal basis for V_0.

Note that since we are requiring that $\int_{\mathbb{R}} \phi(t)dt \neq 0$, we can normalize the scaling function so that

$$\int_{\mathbb{R}} \phi(t)dt = 1.$$

We shall assume that the scaling function satisfies this property for the remainder of this paper. It is also important to note that while Definition 1.3 holds for complex-valued scaling functions, for the purposes of this paper, we shall assume that all scaling functions are real-valued functions.

One should notice immediately that the previously discussed Haar spaces V_j satisfy the requirements of a multiresolution analysis. In fact, the strength of a multiresolution analysis is that it gives us a way to decompose the space $L^2(\mathbb{R})$ into nested subspaces that we can use to approximate functions, much like the Haar spaces. Several of the properties that hold for the Haar spaces will also hold for a more general set of subspaces V_j satisfying the properties of a multiresolution analysis.

Using the scaling function $\phi(t)$ of a multiresolution analysis $V_j \in L^2(\mathbb{R})$, along with its integer translates, we define the function $\phi_{j,k}(t)$ as

$$\phi_{j,k}(t) = 2^{\frac{j}{2}}\phi(2^j t - k).$$

One should first observe that $\phi_{j,k}(t) \in V_j$ due to the nested property. It can also be shown that $||\phi_{j,k}(t)||_2 = 1$. These two facts can be used to show that, just as in the case of the Haar Spaces, $\{\phi_{j,k}(t)\}_{k\in\mathbb{Z}}$ forms an orthonormal basis for the space V_j. As a direct result, we can represent any function $f \in V_j$ as

$$f(t) = \sum_{k\in\mathbb{Z}} \langle f(t), \phi_{j,k}(t)\rangle \phi_{j,k}(t).$$

The scaling function $\phi(t)$ of a multiresolution analysis V_j satisfies a general dilation equation given by

$$\phi(t) = \sqrt{2}\sum_{k\in\mathbb{Z}} h_k \phi(2t - k)$$

where

$$h_k = \langle \phi(t), \phi_{1,k}(t)\rangle.$$

The coeffcients $h_k, k \in \mathbb{Z}$ form what is called the scaling filter.

In many applications, it is easier to consider the scaling function $\phi(t)$ in the transform domain, as first developed by Fourier. This should come as no surprise when one considers that the roots of wavelet analysis lie in Harmonic Analysis! Before we can describe the process, it is necessary to define the Fourier transform.

Definition 1.4 *We define the* Fourier Transform, $\hat{f}(\omega)$ *of* $f(t)$ *as*

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(t)e^{-it\omega}dt.$$

One can show that if $f \in L^2(\mathbb{R})$ and satisfies some additional conditions (see [8] or [10]), then

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(\omega) e^{i\omega t} d\omega.$$

This integral is known as the *Inverse Fourier Transform*. For the remainder of this paper, we shall refer to the domain of functions $f(t) \in L^2(\mathbb{R})$ as the time domain, while the Fourier transforms of these functions, denoted by $\hat{f}(\omega)$ shall be considered to lie in the transform domain. One of the reasons the Fourier transform is useful in solving problems is that *convolution* in the time domain becomes multiplication in the Fourier domain:

Definition 1.5 *For $f, g \in L^2(\mathbb{R})$, we define the* convolution *of $f(t)$ and $g(t)$ as*

$$(f * g)(t) = \int_{\mathbb{R}} f(u) g(t-u) du.$$

Note that $f * g \in L^1(\mathbb{R})$. Using this definition, we have the following theorem, as stated in [10]:

Theorem 1.1 *For $f, g \in L^2(\mathbb{R})$,*

$$\widehat{(f * g)} = \hat{f} \cdot \hat{g}.$$

We can translate the dilation equation

$$\phi(t) = \sqrt{2} \sum_{k \in \mathbb{Z}} h_k \phi(2t - k)$$

into the transform domain by first observing that if we let $g(t) = \phi(2t - k)$, the Fourier transform of g is

$$\hat{g}(\omega) = \frac{1}{2} e^{-\frac{ik\omega}{2}} \hat{\phi}\left(\frac{\omega}{2}\right).$$

Using Theorem 1.1 and basic properties of the Fourier transform (see [10]), we see that the dilation equation satisfied by $\phi(t) \in L^2(\mathbb{R})$ becomes, in the transform domain:

$$\hat{\phi}(\omega) = \frac{1}{\sqrt{2}} \sum_{k \in \mathbb{Z}} h_k e^{-\frac{ik\omega}{2}} \hat{\phi}\left(\frac{\omega}{2}\right). \tag{1.1}$$

This gives us a very important property of the scaling function $\phi(t)$, as it states that $\phi(t) \in L^2(\mathbb{R})$ satisfies the dilation equation in the time domain if and only if its Fourier transform, $\hat{\phi}(\omega)$, satisfies (1.1). Let $H(\omega)$ be the trigonometric series $H(\omega) = \frac{1}{\sqrt{2}} \sum_{k \in \mathbb{Z}} h_k e^{-ik\omega}$ where h_k are the scaling filter coefficients. We shall refer to $H(\omega)$ as the *symbol* of $\phi(t)$ and we can use it to re-write (1.1) as

$$\hat{\phi}(\omega) = H\left(\frac{\omega}{2}\right) \hat{\phi}\left(\frac{\omega}{2}\right).$$

We shall refer to this equation as the dilation equation in the transform domain.

If we think of our symbol being of the form $H(z) = \frac{1}{\sqrt{2}} \sum_{k \in \mathbb{Z}} h_k z^k$, which can be obtained from our original definition of $H(\omega)$ by letting $z = e^{-i\omega}$, we can make additional remarks about the support of our scaling function $\phi(t)$. For our purposes, we shall define the *support* of f, denoted by supp(f), as the set of all values $t \in \mathbb{R}$ such that $f(t) \neq 0$. We say f is *compactly supported* if supp(f) is contained in a closed interval of finite length. In this case, we say that the *compact support of f* is the smallest closed interval $[a, b]$ such that supp$(f) \subseteq [a, b]$. This interval is denoted by $\overline{\text{supp}(f)}$. It should be noted that is a different notion of compact support, and more details can be found in [8].

It can be shown that for $\phi(t)$ generating a multiresolution analysis $V_j \in L^2(\mathbb{R})$, if $\phi(t)$ has compact support, then $\overline{supp(\phi)} = [0, N]$, provided that the symbol $H(z)$ is a polynomial of the form $H(z) = \frac{1}{\sqrt{2}} \sum_{k=0}^{N} h_k z^k$ for $N \in \mathbb{Z}$. It can also be shown that $H(0) = 1$ and that $H(\omega)$ is $2\pi-$ periodic. These two properties are in fact necessary for the results obtained using the Cascade Algorithm, which will be discussed in more detail in Chapter 3. Perhaps the most important property of the symbol $H(\omega)$ is that is satisfies what we shall hereafter refer to as the *orthonormality condition*:

$$|H(\omega)|^2 + |H(\omega + \pi)|^2 = 1. \tag{1.2}$$

If the symbol $H(\omega)$ satisfies this condition, the result is that the scaling function $\phi(t)$, and its translates, are orthonormal in the time domain. Conversely, if $H(\omega)$ satisfies certain conditions, we can guarantee the existence of a scaling function $\phi(t)$ that will generate a multitresolution analysis: In other words, if $H(\omega)$ has finite degree and satisfies the orthonormality condition, and if $H(0) = 1$ and $H(z)$ is of the form

$$H(z) = \left(\frac{1+z}{2}\right)^N S(z),$$

where $S(z)$ satisfies

$$\max_{|z|=1} |S(z)| \leq 2^{N-1},$$

with $z = e^{-i\omega}$, then there exists a scaling function $\phi(t)$ that generates a multiresolution analysis. As we shall see, this idea is crucial in the construction of Daubechies' scaling functions.

In order for a multiresolution analysis to be useful in applications, we would like for it to satisfy three properties. We would like the scaling function to have compact support, in order to simplify computations. We would also like our scaling function $\phi(t)$ to be sufficiently smooth. That is, we would like $\phi(t)$ to have a finite number of continuous derivatives. Finally, we would also like our scaling function $\phi(t)$ to have orthogonal translates. That is, we would like our scaling function to satisfy the orthonormality condition given by (1.2). While there are several examples of multiresolution analyses, up until the work of Daubechies', there was not a single multiresolution analysis that satisfied all three of the above properties. The Shannon multiresolution analysis is such an example. The Shannon multiresolution analysis has a scaling function, namely $\frac{\sin(t)}{t}$ $:=$ sinc(t), whose Fourier transform is the characteristic function of an interval, and can therefore be thought of as the Haar multiresolution analysis in the transform domain. The scaling function of the Shannon multiresolution analysis does not have compact support, thus making it hard to use in many applications. In the next chapter, we shall discuss the Daubechies construction of a set of orthonormal scaling functions that were both compactly supported and possessed a sufficient number of continuous derivatives.

Chapter 2

An Algorithm for the Construction of Daubechies Scaling Functions

2.1 The Daubechies Algorithm

We will now explore an algorithm presented by Ingrid Daubechies in 1988 to obtain a multiresolution analysis. The following construction follows from an outline which can be found in [8]. As mentioned in the previous chapter, we would like to work with a scaling function that has compact support and has orthogonal translates. From our discussion in the previous chapter, we know that this is equivalent to saying that we desire to have the dilation equation be of the form

$$\hat{\phi}(\omega) = H(\frac{\omega}{2})\hat{\phi}(\frac{\omega}{2}) \tag{2.1}$$

where $H(\omega) = \frac{1}{\sqrt{2}}\sum_{k=0}^{N} h_k e^{-ik\omega}$ and satisfies the orthonormality equation given by

$$|H(\omega)|^2 + |H(\omega+\pi)|^2 = 1 \ \ \forall \omega \in \mathbb{R}. \tag{2.2}$$

Finally, we would like our scaling functions to be sufficiently smooth. In other words, we would like our scaling function to have continuous $N-1$ derivatives. Daubechies' approach was to require the symbol $H(\omega)$ to be of the form

$$H(\omega) = \left(\frac{1+e^{-i\omega}}{2}\right)^N S(\omega), \tag{2.3}$$

where

$$S(\omega) = \sum_{k=0}^{A} a_k e^{-ik\omega},$$

with real coeffcients a_k. By requiring our symbol $H(\omega)$ to satisfy these conditions, we can guarantee the existence of a scaling function $\phi(t)$ that will generate a multiresolution analysis $\{V_j\}_{j\in\mathbb{Z}}$ of $L^2\mathbb{R}$ that will have $H(\omega)$ as its symbol.

We begin by re-writing the orthonormality condition:

Lemma 2.1 *If*

$$H(\omega) = \left(\frac{1+e^{-i\omega}}{2}\right)^N S(\omega),$$

with

$$S(\omega) = \sum_{k=0}^{A} a_k e^{-ik\omega},$$

then the orthonormality condition

$$|H(\omega)|^2 + |H(\omega+\pi)|^2 = 1$$

can be re-written as

$$\left(\cos^2\left(\frac{\omega}{2}\right)\right)^N |S(\omega)|^2 + \left(\sin^2\left(\frac{\omega}{2}\right)\right)^N |S(\omega+\pi)|^2 = 1 \tag{2.4}$$

Proof. By first considering the symbol

$$H(\omega) = \left(\frac{1+e^{-i\omega}}{2}\right)^N S(\omega)$$

for a fixed N and substituting it into (2.2) we obtain

$$\left|\left(\frac{1+e^{-i\omega}}{2}\right)^N S(\omega)\right|^2 + \left|\left(\frac{1+e^{-i(\omega+\pi)}}{2}\right)^N S(\omega+\pi)\right|^2 = 1.$$

By observing that

$$\left|\frac{1+e^{-i\omega}}{2}\right|^2 = \cos^2\left(\frac{\omega}{2}\right)$$

and that

$$\left|\frac{1+e^{-i(\omega+\pi)}}{2}\right|^2 = \sin^2\left(\frac{\omega}{2}\right),$$

we obtain

$$\left(\cos^2\left(\frac{\omega}{2}\right)\right)^N |S(\omega)|^2 + \left(\sin^2\left(\frac{\omega}{2}\right)\right)^N |S(\omega+\pi)|^2 = 1. \tag{2.5}$$

□

We now want to find $S(\omega)$. We begin by letting $L(\omega) = |S(\omega)|^2$. We would like to transform $L(\omega)$ into a polynomial $P(y)$, which will be easier to work with.

Lemma 2.2 *If* $S(\omega) = \sum_{k=0}^{A} a_k e^{-ik\omega}$, *where the* a_k *are real, then by defining* $y = sin^2\left(\frac{\omega}{2}\right)$, *there exists a polynomial* $P(y)$ *such that* $|S(\omega)|^2 = P(y)$.

Proof. By first observing that

$$L(\omega) = |S(\omega)|^2 = S(\omega)\overline{S(\omega)}$$

and by noting that the coefficients of $S(\omega)$ are real we obtain

$$L(\omega) = S(\omega)S(-\omega),$$

which then makes $L(\omega)$ a product of trigonometric polynomials. We can rewrite $L(\omega)$ as

$$L(\omega) = S(\omega)S(-\omega) = \left(\sum_{k=0}^{A} a_k e^{-ik\omega}\right)\left(\sum_{k=0}^{A} a_k e^{ik\omega}\right) = \sum_{j=-A}^{A} c_j e^{-ij\omega}$$

where $c_j = c-_j$, which can then be written as

$$L(\omega) = c_0 + 2\sum_{j=1}^{A} c_j \cos(j\omega). \tag{2.6}$$

since $e^{ij\omega} + e^{-ij\omega} = 2\cos(j\omega)$. We can re-write $\cos(j\omega)$ as

$$cos(j\omega) = \sum_{k=0}^{j} t_k cos^k(\omega),$$

by first noting that

$$\begin{aligned} (e^{i\omega})^j &= \big(\cos(\omega) + i\sin(\omega)\big)^j \\ &= \sum_{k=0}^{j} \begin{pmatrix} j \\ k \end{pmatrix} (\cos(\omega))^k (i\sin(\omega))^{j-k} \\ &= \cos(j\omega) + i\sin(j\omega). \end{aligned} \tag{2.7}$$

Note that the contribution to the real part of (2.7) occurs when $j - k = 2n$ is even, and then we can write

$$(i\sin(\omega))^{(j-k)} = (-1)^n(1 - \cos^2(\omega))^n.$$

Thus, the real part of (2.7) can be written as linear combinations of powers of $\cos(\omega)$ of degree at most j :

$$\cos(j\omega) = \sum_{k=0}^{j} t_k \cos^k(\omega).$$

Thus,we can re-write $L(\omega)$ as

$$L(\omega) = \sum_{k=0}^{A} d_k \cos^k(\omega).$$

Now because

$$\cos(\omega) = 1 - 2\sin^2\left(\frac{\omega}{2}\right),$$

we have

$$L(\omega) = \sum_{k=0}^{A} d_k \left(1 - 2\sin^2\left(\frac{\omega}{2}\right)\right)^k.$$

Let $y = \sin^2(\frac{\omega}{2})$ Then

$$|S(\omega)|^2 = L(\omega) = P(y) = \sum_{k=0}^{A} d_k (1-2y)^k. \tag{2.8}$$

□

So now

$$\left(\cos^2\left(\frac{\omega}{2}\right)\right)^N |S(\omega)|^2 + \left(\sin^2\left(\frac{\omega}{2}\right)\right)^N |S(\omega+\pi)|^2 = 1$$

has been translated into

$$(1-y)^N P(y) + y^N L(\omega+\pi) = 1$$

with

$$\begin{aligned} L(\omega+\pi) &= \sum_{k=0}^{A} d_k \left(1 - 2\sin^2\left(\frac{\omega}{2} + \frac{\pi}{2}\right)\right)^k \\ &= \sum_{k=0}^{A} d_k \left(1 - 2(1 - \sin^2\left(\frac{\omega}{2}\right)\right)^k \\ &= \sum_{k=0}^{A} d_k \left(1 - 2\left(1-y\right)\right)^k \\ &= P(1-y). \end{aligned}$$

Thus we want to find a polynomial P(y) satisfying

$$(1-y)^N P(y) + y^N P(1-y) = 1. \tag{2.9}$$

To find the explicit form of $P(y)$, we shall follow [11]. Note that for a fixed $N \in \mathbb{N}$, we can write

$$\begin{aligned} 1 &= \left((1-y)+y\right)^{2N-1} \\ &= \sum_{k=0}^{2N-1} \binom{2N-1}{k} (1-y)^k y^{2N-1-k} \\ &= \sum_{k=0}^{N-1} \binom{2N-1}{k} (1-y)^k y^{2N-1-k} + \sum_{k=N}^{2N-1} \binom{2N-1}{k} (1-y)^k y^{2N-1-k}. \end{aligned}$$

Observe that

$$\left(\begin{array}{c} 2N-1 \\ k \end{array}\right) = \frac{(2N-1)!}{k!\,(2N-1-k)!} = \left(\begin{array}{c} 2N-1 \\ 2N-1-k \end{array}\right).$$

Letting $m = 2N-1-k$, we can re-write the second sum as

$$\sum_{m=0}^{N-1} \left(\begin{array}{c} 2N-1 \\ m \end{array}\right)(1-y)^{2N-1-m} y^m,$$

giving

$$\begin{aligned} 1 = \sum_{k=0}^{N-1} \left(\begin{array}{c} 2N-1 \\ k \end{array}\right)(1-y)^k y^{2N-1-k} &+ \sum_{m=0}^{N-1} \left(\begin{array}{c} 2N-1 \\ m \end{array}\right)(1-y)^{2N-1-m} y^m \\ = \; & y^N \sum_{k=0}^{N-1} \left(\begin{array}{c} 2N-1 \\ k \end{array}\right)(1-y)^k y^{N-1-k} + (1-y)^N \sum_{m=0}^{N-1} \left(\begin{array}{c} 2N-1 \\ m \end{array}\right)(1-y)^{N-1-m} y^m \\ = \; & y^N P(1-y) + (1-y)^N P(y) \end{aligned}$$

where

$$P(y) = \sum_{k=0}^{N-1} \left(\begin{array}{c} 2N-1 \\ k \end{array}\right) y^k (1-y)^{N-1-k}. \tag{2.10}$$

Clearly, $P(y) \geq 0$ for $0 \leq y \leq 1$. Also, since $y = sin^2(\frac{\omega}{2})$, all values of y fall in the interval [0,1]. We shall state the above results as a Theorem:

Theorem 2.1 *There exist a polynomial P(y), of degree N-1, such that*

$$(1-y)^N P(y) + y^N P(1-y) = 1$$

where the polynomial

$$P(y) = \sum_{k=0}^{N-1} \left(\begin{array}{c} 2N-1 \\ k \end{array}\right) y^k (1-y)^{N-1-k}. \tag{2.11}$$

To summarize, we have accomplished the following: In order to create a sufficiently smooth scaling function of compact support whose integer translates are orthogonal, the associated symbol $H(\omega)$ can be written in terms of a trigonometric polynomial $S(\omega)$ that satisfies Lemma 2.1. We can express $|S(\omega)|^2$ as a polynomial $P(y)$ that satisfies (2.9). Theorem 2.1 guarantees the existence of this polynomial.

Having now found polynomial $P(y)$, we wish to work backwards to find the explicit form of $L(\omega)$. We can do this by simply substituting $y = \sin^2(\frac{\omega}{2}) = \frac{1-\cos^2(\omega)}{2}$ into (2.11), we obtain

$$\begin{aligned} P(y) &= \sum_{k=0}^{N-1} \begin{pmatrix} 2N-1 \\ k \end{pmatrix} \left(\frac{1-\cos(\omega)}{2}\right)^k \left(1 - \left(\frac{1-\cos(\omega)}{2}\right)\right)^{N-1-k} \\ &= \sum_{k=0}^{N=1} \begin{pmatrix} 2N-1 \\ k \end{pmatrix} \left(\frac{1-\cos\omega}{2}\right)^k \left(\frac{1+cos(\omega)}{2}\right)^{N-1-k} \end{aligned}$$

Thus we have found $L(\omega)$ using $P(y)$.

We now wish to find an explicit formula for $S(\omega)$ using $L(\omega)$. To do this, we need to factor $L(\omega)$ in such a way that by building $S(\omega)$ using half of the factors from $L(\omega)$, we obtain $|S(\omega)|^2 = L(\omega)$. Since $P(y) = L(\omega) \geq 0, \forall \omega \in [-\pi, \pi]$, and since $L(\omega)$ is a trigonometric polynomial of degree $N-1$, we can apply a well-known result from harmonic analysis in order to factor $L(\omega)$.

Theorem 2.2 (Fejér-Riesz Theorem, [9]) *A trigonometric polynomial* $L(\omega) = \sum_{j=-A}^{A} c_j e^{-ij\omega}$, *that satisfies* $L(\omega) \geq 0$ *for all* $\omega \in [-\pi, \pi]$ *is expressible in the form*

$$L(\omega) = |F(z)|^2$$

for some polynomial $F(z)$, *with* $z = e^{-i\omega}$, *which takes the form*

$$F(z) = c \prod_{j=1}^{A} (z - \alpha_j),$$

where α_j *satisfy* $|\alpha_j| \leq 1$.

A direct application of this theorem allows us to factor $L(\omega)$ as

$$L(\omega) = |F(z)|^2,$$

where

$$F(z) = c \prod_{j=1}^{N-1} (z - \alpha_j).$$

We can obtain $S(\omega)$ by simply substituting $z = e^{-i\omega}$ into $F(z)$:

$$S(\omega) = F(e^{-i\omega}) = \sum_{k=0}^{N-1} a_k e^{-ik\omega}.$$

Note that this is the desired form for $S(\omega)$. To obtain our symbol $H(\omega)$, we substitute $S(\omega)$ into (2.3):

$$H(\omega) = \left(\frac{1+e^{-i\omega}}{2}\right)^N S(\omega) = \left(\frac{1+e^{-i\omega}}{2}\right)^N \cdot \sum_{k=0}^{N-1} a_k e^{-ik\omega}.$$

Expanging $H(\omega)$ gives a trigonometric polynomial of degree $2N-1$ of the form

$$H(\omega) = \frac{1}{\sqrt{2}} \sum_{k=0}^{2N-1} h_k e^{-ik\omega},$$

where the coefficients h_k are real. By construction, $H(\omega)$ satisfies the desired orthonormality condition (2.2). Therefore we know that there is a scaling function $\phi(t)$ that generates a multiresolution analysis $\{V_j\}_{j\in\mathbb{Z}}$ of $L^2(\mathbb{R})$. A Daubechies scaling function $\phi(t)$ whose symbol $H(\omega)$ for a given N obtained through this construction is called the $D2N$ *scaling function*. The corresponding filter h_k has $2N$ nonzero components, and is referred to as the *D2N Daubechies scaling filter*. We shall later use Daubechies' algorithm to construct the $D4$ scaling filters corresponding to the $D4$ scaling function. First we would like to give a more detailed description of the roots of the polynomial $F(z)$.

2.2 The Roots of $F(z)$

In her 1988 paper, Daubechies presented an alternative proof of the Fejér-Reisz theorem [2]. Her proof was based on the structure of the roots of the polynomial $F(z)$. In this section, we discuss her construction of the polynomial $F(z)$ based on the classification of these roots. We shall follow her construction as described in [8].

Recall that

$$P(y) = \sum_{k=0}^{N-1} \begin{pmatrix} 2N-1 \\ k \end{pmatrix} y^k (1-y)^{N-1-k} \tag{2.12}$$

and $L(\omega) = P(y)$ where $y = \sin^2\left(\frac{\omega}{2}\right)$.

We first begin by writing $L(\omega)$ as $T(z)$. Writing $L(\omega)$ in this form allows us to re-group the roots based on their location in terms of the unit circle. Classifying the roots in this way will lead us to the construction of $F(z)$.

By letting $z = e^{-i\omega}$ and by noting that we can write

$$\frac{1+\cos\omega}{2} = \left(\frac{1+z}{2}\right)\left(\frac{1+\frac{1}{z}}{2}\right)$$

and

$$\frac{1-\cos\omega}{2} = \left(\frac{1-z}{2}\right)\left(\frac{1-\frac{1}{z}}{2}\right),$$

we can re-write $L(\omega)$ as:

$$\begin{aligned} L(\omega) &= \sum_{k=0}^{N-1} \left(\begin{array}{c} 2N-1 \\ k \end{array} \right) \left(\frac{1-\cos\omega}{2}\right)^k \left(\frac{1+cos(\omega)}{2}\right)^{N-1-k} \\ &= \sum_{k=0}^{N-1} \left(\left(\frac{1-z}{2}\right)\left(\frac{1-\frac{1}{z}}{2}\right)\right)^k \left(\left(\frac{1+z}{2}\right)\left(\frac{1+\frac{1}{z}}{2}\right)\right)^{N-1-k} \\ &=: T(z). \end{aligned}$$

From this form, it is easily seen that $T(z) = T(\frac{1}{z})$ and that $z = 1$ and $z = -1$ are not roots of T(z). By expanding $\left(\left(\frac{1-z}{2}\right)\left(\frac{1-\frac{1}{z}}{2}\right)\right)$ and $\left(\left(\frac{1+z}{2}\right)\left(\frac{1+\frac{1}{z}}{2}\right)\right)$, we obtain

$$\begin{aligned} T(z) &= \sum_{k=0}^{N-1} \left(\begin{array}{c} 2N-1 \\ k \end{array} \right) \frac{(z+2-z^{-1})^k}{4^k} \frac{(z+2+z^{-1})^{N-1-k}}{4^{N-1-k}} \\ &= \frac{1}{4^{N-1}} \sum_{k=0}^{N-1} \left(\begin{array}{c} 2N-1 \\ k \end{array} \right) (z+2-z^{-1})^k (z+2+z^{-1})^{N-1-k} \\ &= \alpha z^{-(N-1)} \prod_{k=1}^{2N-2} (z - z_k) \end{aligned}$$

where α is the leading coefficient and z_k are the roots of $T(z)$.

In order to find a polynomial $F(z)$ such that

$$T(z) = |F(z)|^2 = F(z) \cdot \overline{F(z)},$$

it would be ideal to group the roots of $T(z)$ using conjugate pairs. Note that the Fejér-Riesz theorem guarantees the existence of the polynomial $F(z)$.The roots of $T(z)$ can be real, on the unit circle, or not real and not on the unit circle. Let us consider each of these cases in turn.

For $z_k \in \mathbb{R}$, we have that $z_k = \overline{z_k}$. From earlier, we noted that $T(z) = T(\frac{1}{z})$. Thus, if $z_k \neq 0$ is a root of $T(z)$, then so is $\frac{1}{z_k}$. Note that since we know that ± 1 are not roots of $T(z)$, this puts each real root either inside or outside of the unit circle, i.e., $|z_k| < 1$ or $|z_k| > 1$. We shall denote the number of pairs of such roots by $\mathcal{K}$ and we shall denote the roots inside the unit circle by r_k.

For $z_k \in \mathbb{C} \setminus \mathbb{R}$, either z_k lies on the unit circle or z_k lies off the unit circle. Let us denote roots on the unit circle by z_j^u and denote roots off the unit circle by z_i^c.

For each root z_j^u on the unit circle, $\overline{z_j^u} = \frac{1}{z_j^u}$ is also a root. It can be shown that the multiplicity of these roots is even (see [9]), so we will group the roots in pairs of two. We shall denote the number of these pairs by $\mathcal{J}$.

For each root z_i^c off the unit circle, then because $T(z) = T(\frac{1}{z})$, we have that $\frac{1}{z_i^c}$ is also a root. Since $T(z)$ has real coeffcients, if z_i^c is a root, then so is $\overline{z_i^c}$. In other words, if z_i^c is root, then so are $\frac{1}{z_i^c}$, $\overline{z_i^c}$, and $\frac{1}{\overline{z_i^c}}$. Now again, since roots of this type are either inside or outside of the unit circle,we have that either $|z_i^c| < 1$ or $|\frac{1}{z_i^c}| < 1$ which results in either $|z_i^c| > 1$ or $|\frac{1}{z_i^c}| > 1$ respectively. We shall create groups of 4 and denote the number of such groups by $\mathcal{L}$.

Lemma 2.3 *Let $z = e^{-i\omega}$. If*

$$T(z) = \frac{1}{4^{N-1}} \sum_{k=0}^{N-1} \begin{pmatrix} 2N-1 \\ k \end{pmatrix} (z + 2 - z^{-1})^k (z + 2 + z^{-1})^{N-1-k},$$

then there exists a polynomial $F(z)$ such that $|F(z)|^2 = T(z)$ and $F(z)$ is of the form

$$F(z) = \sqrt{|\alpha|}\Big[\prod_{i=1}^{\mathcal{L}} |z_i^c|^{-1} \prod_{k=1}^{\mathcal{K}} |r_k|^{-\frac{1}{2}}\Big] \prod_{i=1}^{\mathcal{L}} (z - z_i^c)(z - \overline{z_i^c}) \cdot \prod_{j=1}^{\mathcal{J}} (z - z_j^u)(z - 1/z_j^u) \cdot \prod_{k=1}^{\mathcal{K}} (z - r_k),$$

where r_k are real with $|r_k| < 1$, z_i^c are in $\mathbb{C} \setminus \mathbb{R}$, with $|z_i^c| < 1$, and $|z_u^c| = 1$.

Proof. We shall first write $T(z)$ as a product of its factors, where each of its factors is grouped according to the location of the roots:

$$\begin{aligned} T(z) &= \alpha z^{-(N-1)} \prod_{k=1}^{2N-2} (z - z_k) \\ &= \alpha z^{-N-1} \prod_{i=1}^{\mathcal{L}} (z - z_i^c)(z - 1/z_i^c)(z - \overline{z_i^c})(z - 1/\overline{z_1^c}) \\ &\quad \cdot \prod_{j=1}^{\mathcal{J}} (z - z_i^u)^2 (z - 1/z_i^u)^2 \cdot \prod_{k=1}^{\mathcal{K}} (z - r_k)(z - 1/r_k). \end{aligned}$$

By observing that because $L(\omega) = |L(\omega)|$ we have that $T(z) = |T(z)|$, and by regrouping complex roots on and off the unit circle, we obtain

$$|T(z)| = |\alpha| \prod_{i=1}^{\mathcal{L}} |(z - z_i^c)(z - 1/\overline{z_i^c})| \cdot |(z - \overline{z_i^c})(z - 1/z_i^c)| \cdot \prod_{j=1}^{\mathcal{J}} |(z - z_j^u)(z - 1/z_j^u)|^2 \cdot \prod_{k=1}^{\mathcal{K}} |(z - r_k)(z - 1/r_k)|.$$

Now since $|z| = 1$, we can write

$$|(z - z_i^c)(z - 1/z_i^c)| = |z_i^c|^{-1} |z - z_i^c|^2.$$

Similarly,

$$|(z-\overline{z_i^c})(z-1/z_i^c)| = |\overline{z_i^c}|^{-1}|z-\overline{z_i^c}|^2$$

and

$$|(z-r_k)(z-1/r_k)| = |r_k|^{-1}|z-r_k|^2.$$

Making the above substitutions gives

$$\begin{aligned}
|T(z)| &= |\alpha|\prod_{i=1}^{\mathcal{L}}|z_i^c|^{-1}|z-z_i^c|^2|\overline{z_i^c}|^{-1}|z-\overline{z_i^c}|^2\cdot\prod_{j=1}^{\mathcal{J}}|(z-z_j^u)(z-1/z_j^u)|^2\cdot\prod_{k=1}^{\mathcal{K}}|r_k|^{-1}|z-r_k|^2\\
&= |\alpha|\prod_{i=1}^{\mathcal{L}}|z_i^c|^{-2}|z-z_i^c|^2|z-\overline{z_i^c}|^2\cdot\prod_{j=1}^{\mathcal{J}}|(z-z_j^u)(z-1/z_j^u)|^2\cdot\prod_{k=1}^{\mathcal{K}}|r_k|^{-1}|z-r_k|^2.
\end{aligned}$$

Create the square root of T(z), denoted F(z), by choosing all factors with roots inside the unit circle and one factor from each double root z_j^u on the unit circle. Thus $F(z)$ is defined as

$$F(z)=\sqrt{|\alpha|}\left[\prod_{i=1}^{\mathcal{L}}|z_i^C|^{-1}\prod_{k=1}^{\mathcal{K}}|r_k|^{-\frac{1}{2}}\right]\prod_{i=1}^{\mathcal{L}}(z-z_i^c)(z-\overline{z_i^C})\cdot\prod_{j=1}^{\mathcal{J}}(z-z_j^u)(z-1/z_j^u)\cdot\prod_{k=1}^{\mathcal{K}}(z-r_k).$$

Note that $|F(z)|^2 = |T(z)| = T(z)$, as desired, and that the degree of $F(z)$ is $N-1$ since the degree of T(z) is $2N-2$. Note also that F(z) has real coefficients. Thus we have the desired polynomial $F(z)$. □

2.3 The Construction of the $D4$ scaling filter

In this section, we will construct the D4 scaling filter associated with the D4 scaling function using the algorithm of Daubechies. For clarity, we will make use of the construction presented in the previous section. We shall begin by finding the polynomial $P(y)$. Recall that $P(y)$ has the form

$$P(y)=\sum_{k=0}^{N-1}\begin{pmatrix}2N-1\\ k\end{pmatrix}y^k(1-y)^{N-1-k}. \tag{2.13}$$

Since we know that $N=2$, we obtain

$$P(y)=\sum_{k=0}^{1}\begin{pmatrix}3\\ k\end{pmatrix}y^k(1-y)^{1-k}=1+2y. \tag{2.14}$$

We now need to re-write $P(y)$ as $L(\omega)$. Substituting $y = \frac{1-cos(\omega)}{2}$ gives

$$\begin{aligned} P(y) &= 1 + 2y \\ &= 1 + 2\Big(\frac{1 - cos(\omega)}{2}\Big) \\ &= L(\omega) \end{aligned}$$

By using the identity

$$\frac{1 - cos(\omega}{2} = \Big(\frac{1-z}{2}\Big) \cdot \Big(\frac{1-\frac{1}{z}}{2}\Big),$$

we can convert $L(\omega)$ into $T(z)$:

$$\begin{aligned} L(\omega) &= 1 + 2\Big(\big(\frac{1-z}{2}\big)\big(\frac{1-\frac{1}{z}}{2}\big)\Big) \\ &= \frac{1}{2}\big(4 - z - \frac{1}{z}\big) \\ &= -\frac{1}{2} \cdot \frac{1}{z}\big(z^2 - 4z + 1\big) \\ &= T(z) \end{aligned}$$

Direct calculation gives that the roots of $T(z)$ are $z = 2 + \sqrt{3}$ and $z = 2 - \sqrt{3}$. We shall choose the root inside the unit circle to build the polynomial $F(z)$. Using the formula for $F(z)$ as described in the previous section, we have

$$F(z) = \sqrt{\frac{1}{2(2-\sqrt{3})}} \cdot \big(z - (2 - \sqrt{3})\big).$$

Substituting $z = e^{-i\omega}$ into $F(z)$, we obtain $S(\omega)$:

$$\begin{aligned} F(e^{-i\omega}) &= \sqrt{\frac{1}{2(2-\sqrt{3})}} \cdot \big(e^{-i\omega} - (2 - \sqrt{3})\big) \\ &= S(\omega). \end{aligned}$$

We can find our symbol $H(\omega)$ by first recalling that we wanted our symbol to be of the form

$$H(\omega) = \big(\frac{1 + e^{-i\omega}}{2}\big)^N \cdot S(\omega).$$

Substitution of $S(\omega)$ gives:

$$H(\omega) = \Big(\frac{1 + e^{-i\omega}}{2}\Big)^2 \Big(\sqrt{\frac{1}{2}}\Big)\Big(\frac{1}{\sqrt{2-\sqrt{3}}}\Big)\big(e^{-i\omega} - (2 - \sqrt{3})\big). \tag{2.15}$$

Further simplification of $H(\omega)$ gives

$$H(\omega) = \frac{1-\sqrt{3}}{4\sqrt{2}} + \frac{3-\sqrt{3}}{4\sqrt{2}}e^{-i\omega} + \frac{3+3\sqrt{3}}{4\sqrt{2}}e^{-2i\omega} + \frac{1+\sqrt{3}}{4\sqrt{2}}e^{-3i\omega}. \tag{2.16}$$

Thus, the $D4$ scaling filter coefficients are

$$h_0 = \frac{1+\sqrt{3}}{4\sqrt{2}},$$

$$h_1 = \frac{3+3\sqrt{3}}{4\sqrt{2}},$$

$$h_2 = \frac{3-\sqrt{3}}{4\sqrt{2}},$$

and

$$h_3 = \frac{1-\sqrt{3}}{4\sqrt{2}}.$$

Chapter 3
The Cascade Algorithm

It turns out that for $N \geq 3$, it is not possible to come up with a closed-form for the Daubechies $2N$ scaling functions. The Cascade Algorithm was developed by Daubechies and Lagarias in an attempt to obtain good approximations of these scaling functions. It was proposed that by taking sequences of approximations given by iterates based on a first guess, say $\phi_0(t)$, where $\phi_0(t)$ is the characteristic function on the interval $[0, 1)$, and with each successive approximation given by:

$$\phi_{n+1} = \sqrt{2} \sum_{k=0}^{M} h_k \phi_n(2t - k) \tag{3.1}$$

that we would ultimately have convergence of the approximations to the actual scaling function $\phi(t)$ associated with the symbol $H(\omega)$ described in the previous section. In fact, this algorithm produces the scaling function given a number of other first guesses, but for the purposes of this paper, we shall only consider the first guess of $\phi_0(t)$.

In this chapter, we shall show that we have convergence of the iterates in the time domain, given $\phi_0(t)$. By first showing that, for each $\omega \in \mathbb{R}$, we have convergence of the iterates in the transform domain to a continuous function $g(\omega)$ and then showing that this $g(\omega)$ satisfies the transform domain dilation equation, we shall be able to show pointwise convergence of the iterates $\{\phi_n(t)\}$ to the scaling function $\phi(t)$. The following argument is based on the outline as presented in [8].

Proposition 3.1 *[see [8], p. 258] Suppose the symbol given by*

$$H(\omega) = \frac{1}{\sqrt{2}} \sum_{k=0}^{M} h_k e^{-ik\omega} \tag{3.2}$$

satisfies $H(0) = 1$. Then the cascade algorithm iterates $\{\widehat{\phi}_n(\omega)\}$ defined by

$$\widehat{\phi_n}(\omega) = \prod_{k=1}^{n} H\left(\frac{\omega}{2^k}\right) \cdot \frac{1}{\sqrt{2\pi}} e^{\frac{-i\omega}{2^{n+1}}} \operatorname{sinc}\left(\frac{\omega}{2^{n+1}}\right) \tag{3.3}$$

converges for each $\omega \in \mathbb{R}$ in the transform domain to a continuous function $g(\omega)$.

Before proving the above proposition, recall that sinc(t) $= \frac{\sin(t)}{t}$ and note that the function appearing on the right hand side of (3.3) is a natural choice, since the Fourier transform for $\phi_0(\omega)$ is

$$\widehat{\phi_0}(\omega) = \frac{1}{2\pi} e^{\frac{-i\omega}{2}} \operatorname{sinc}\big(\frac{\omega}{2}\big).$$

Proof. We shall prove the proposition via a series of two claims. The result of the first claim is needed to give an explicit form for $g(\omega)$. Once we have our $g(\omega)$, our second claim will show that we have convergence of the cascade algorithm iterates to this $g(\omega)$ in the transform domain.

Claim 3.1

$$\sum_{k=1}^{\infty} \left| H\left(\frac{\omega}{2^k}\right) - 1 \right|$$

converges to a finite value.

Proof. Let $H(\omega) = \frac{1}{\sqrt{2}} \sum_{k=0}^{M} h_k e^{-ik\omega}$. We shall re-write $H(\omega)$ as

$$\begin{aligned} H(\omega) &= \frac{1}{\sqrt{2}} \sum_{k=0}^{M} h_k e^{-ik\omega} - \frac{1}{\sqrt{2}} \sum_{k=0}^{M} h_k + \frac{1}{\sqrt{2}} \sum_{k=0}^{M} h_k \\ &= \frac{1}{\sqrt{2}} \sum_{k=0}^{M} \big(h_k e^{-ik\omega} - h_k\big) + \frac{1}{\sqrt{2}} \sum_{k=0}^{M} h_k \\ &= \frac{1}{\sqrt{2}} \sum_{k=0}^{M} h_k (e^{-ik\omega} - 1) + \frac{1}{\sqrt{2}} \sum_{k=0}^{M} h_k \end{aligned}$$

Now, since $H(0) = \frac{1}{\sqrt{2}} \sum_{k=0}^{M} h_k = 1$, by our hypothesis, we have

$$\begin{aligned} H(\omega) &= \frac{1}{\sqrt{2}} \sum_{k=0}^{M} h_k (e^{-ik\omega} - 1) + H(0) \\ &= \frac{1}{\sqrt{2}} \sum_{k=0}^{M} h_k (e^{-ik\omega} - 1) + 1 \end{aligned}$$

We know that $1 - e^{-i\omega} = 2ie^{\frac{-i\omega}{2}} \sin(\frac{\omega}{2})$. This gives us that

$$\begin{aligned}
\left|\frac{1}{\sqrt{2}}\sum_{k=0}^{M} h_k(e^{-ik\omega} - 1)\right| &= \left|\frac{1}{\sqrt{2}}\sum_{k=0}^{M} h_k 2ie^{\frac{-ik\omega}{2}} \sin(\frac{-k\omega}{2})\right| \\
&= \left|\frac{2i}{\sqrt{2}}\sum_{k=0}^{M} h_k e^{\frac{-ik\omega}{2}} \sin(\frac{-k\omega}{2})\right| \\
&\leq \sqrt{2}\sum_{k=0}^{M} |h_k| \cdot |e^{-ik\omega}| \cdot |\sin(\frac{-k\omega}{2})| \\
&= \sqrt{2}\sum_{k=0}^{M} |h_k| \cdot |\sin(\frac{-k\omega}{2})|
\end{aligned}$$

From the above, we now have that

$$\begin{aligned}
\left|H(\omega) - 1\right| &= \left|\frac{1}{\sqrt{2}}\sum_{k=0}^{M} h_k(e^{-ik\omega} - 1)\right| \\
&\leq \sqrt{2}\sum_{k=0}^{M} |h_k| \cdot \left|\sin(\frac{-k\omega}{2})\right| \\
&\leq \sqrt{2}\sum_{k=0}^{M} |h_k| \cdot \left|\frac{k\omega}{2}\right| \\
&= \frac{\sqrt{2}}{2}\sum_{k=0}^{M} |h_k| \cdot |k\omega| \\
&= \frac{\sqrt{2}}{2}\sum_{k=0}^{M} |h_k| \cdot k|\omega| \\
&= |\omega|\frac{\sqrt{2}}{2}\sum_{k=0}^{M} |h_k| \cdot k
\end{aligned}$$

Let $\beta = \frac{\sqrt{2}}{2}\sum_{k=0}^{M} |h_k| \cdot k$. Then the above gives

$$\left|H(\omega) - 1\right| \leq \beta|\omega|$$

By relabeling and using geometric series, we obtain

$$\begin{aligned}
\sum_{k=1}^{\infty}\left|H(\frac{\omega}{2^k}) - 1\right| &\leq \sum_{k=1}^{\infty}\beta\left|\frac{\omega}{2^k}\right| \\
&= \sum_{k=1}^{\infty}\beta|\omega| \cdot \left|\frac{1}{2^k}\right| \\
&= \beta|\omega|
\end{aligned}$$

Therefore, we have that $\sum_{k=1}^{\infty} |H(\frac{\omega}{2^k}) - 1|$ converges for each ω in $\mathbb{R}$, and, in fact, converges uniformly on compact subsets of $\mathbb{R}$. $\square$

Now since $\sum_{k=1}^{\infty} |H(\frac{\omega}{2^k}) - 1|$ converges absolutely, by a theorem in complex analysis (see [11], Theorem 8.33, p. 238), the product given by $\prod_{k=1}^{\infty} H(\frac{\omega}{2^k})$ converges absolutely, and therefore, there exists a $g(\omega)$ such that

$$g(\omega) = \frac{1}{\sqrt{2}} \prod_{k=1}^{\infty} H\left(\frac{\omega}{2^k}\right).$$

Now that we have given an explicit formula for $g(\omega)$, we can now show that we have convergence of the cascade algorithm iterates to $g(\omega)$.

Claim 3.2 *The sequence of cascade algorithm iterates $\widehat{\phi_n}(\omega)$ converges to $g(\omega)$.*

Proof. Define

$$\widehat{\phi_n}(\omega) = \left(\prod_{k=1}^{n} H\left(\frac{\omega}{2^k}\right)\right) \cdot \frac{1}{\sqrt{2\pi}} e^{\frac{-i\omega}{2^{n+1}}} \operatorname{sinc}\left(\frac{\omega}{2^{n+1}}\right).$$

Then

$$\begin{aligned} \lim_{n\to\infty} \widehat{\phi_n}(\omega) &= \lim_{n\to\infty} \left((\prod_{k=1}^{n} H\left(\frac{\omega}{2^k}\right)) \cdot \frac{1}{\sqrt{2\pi}} e^{\frac{-i\omega}{2^{n+1}}} \operatorname{sinc}(\frac{\omega}{2^{n+1}})\right) \\ &= \frac{1}{\sqrt{2\pi}} \lim_{n\to\infty} \left(\prod_{k=1}^{n} H(\frac{\omega}{2^k})\right) \\ &= g(\omega) \end{aligned}$$

since $e^{\frac{-i\omega}{2^{n+1}}} \operatorname{sinc}(\frac{\omega}{2^{n+1}}) \to 1$ as $n \to \infty$. $\square$

It remains to show that $g(\omega)$ is continuous. Note that each $\widehat{\phi_n}(\omega)$ is continuous by construction. since, from claims, $\widehat{\phi_n}(\omega) \to g(\omega)$ uniformly on compact sets, we have that $g(\omega)$ is continuous. This completes the proof of Proposition 3.1. $\square$

Next we shall show that $g(\omega)$, as found above, satisfies the transform domain dilation equation when the symbol $H(z)$, is obtained from a Daubechies scaling function.

Proposition 3.2 *[see [8], p. 258] Suppose the symbol*

$$H(z) = \frac{1}{\sqrt{2}} \sum_{k=0}^{M} h_k e^{-ik\omega},$$

where $z = e^{-i\omega}$, can be factored as

$$H(z) = \left(\frac{1+z}{2}\right)^N S(z)$$

where $S(z)$ satisfies $S(1) = 1$ and $\max_{|z|=1} |S(z)| \leq 2^{N-1}$. Then $g(\omega)$ satisfies the transform domain dilation equation

$$g(\omega) = H(\frac{\omega}{2})g(\frac{\omega}{2}) \tag{3.4}$$

and

$$|g(\omega)| \leq \frac{C}{1+|\omega|}$$

for some constant C.

Proof. First, note that $g(\omega)$ satisfies the transform domain dilation equation

$$g(\omega) = H(\frac{\omega}{2})g(\frac{\omega}{2})$$

since by Proposition 3.1, we have

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \lim_{n\to\infty} \prod_{k=1}^{n} H(\frac{\omega}{2^k}),$$

and therefore we can re-write the right hand side of (3.4) as

$$\begin{aligned} H(\frac{\omega}{2})g(\frac{\omega}{2}) &= H(\frac{\omega}{2}) \cdot \frac{1}{\sqrt{2\pi}} \lim_{n\to\infty} \prod_{k=2}^{n} H(\frac{\omega}{2^k}) \\ &= \frac{1}{\sqrt{2\pi}} \lim_{n\to\infty} \prod_{k=1}^{n} H\left(\frac{\omega}{2^k}\right) \\ &= g(\omega). \end{aligned}$$

Thus $g(\omega)$ satisfies the dilation equation in the transform domain. It remains to show that

$$g(\omega) \leq \frac{C}{1+|\omega|}.$$

Claim 3.3

$$\prod_{k=1}^{\infty} \left(\frac{1+e^{\frac{-i\omega}{2^k}}}{2}\right)^N = \left(\frac{1-e^{-i\omega}}{i\omega}\right)^N$$

Proof. Observe that

$$\prod_{k=1}^{\infty}\left(\frac{1+e^{\frac{-i\omega}{2^k}}}{2}\right)^N = \lim_{n\to\infty}\prod_{k=1}^{n}\left(\frac{1+e^{\frac{-i\omega}{2^k}}}{2}\right)^N.$$

Now

$$\begin{aligned}
\lim_{n\to\infty}\prod_{k=1}^{n}\left(\frac{1+e^{\frac{-i\omega}{2^k}}}{2}\right)^N &= \lim_{n\to\infty}\prod_{k=1}^{n}\left(\frac{1+e^{\frac{-i\omega}{2^k}}}{2}\cdot\frac{1-e^{\frac{-i\omega}{2^k}}}{1-e^{\frac{-i\omega}{2^k}}}\right)^N \\
&= \lim_{n\to\infty}\prod_{k=1}^{n}\left(\frac{1-e^{\frac{-i\omega}{2^{k-1}}}}{2(1-e^{\frac{-i\omega}{2^k}})}\right)^N \\
&= \lim_{n\to\infty}\left(\frac{1}{2^n}\cdot\frac{1-e^{-i\omega}}{1-e^{\frac{-i\omega}{2^n}}}\right)^N
\end{aligned}$$

Now $1-e^{\frac{-i\omega}{2^n}} = 2ie^{\frac{-i\omega}{2^{n+1}}}\cdot\sin(\frac{\omega}{2^{n+1}})$, so we have that

$$\begin{aligned}
\lim_{n\to\infty}\left(\frac{1}{2^n}\cdot\frac{1-e^{-i\omega}}{1-e^{\frac{-i\omega}{2^n}}}\right)^N &= \lim_{n\to\infty}\left(\frac{1}{2^n}\cdot\frac{1-e^{-i\omega}}{2ie^{\frac{-i\omega}{2^{n+1}}}\sin(\frac{\omega}{2^{n+1}})}\right)^N \\
&= \lim_{n\to\infty}\left(\frac{1}{2^n}\cdot\frac{1-e^{-i\omega}}{2ie^{\frac{-i\omega}{2^{n+1}}}\sin(\frac{\omega}{2^{n+1}})}\right)^N\cdot\left(\frac{\frac{\omega}{2^{n+1}}}{\frac{\omega}{2^{n+1}}}\right)^N \\
&= \lim_{n\to\infty}\frac{(\frac{\omega}{2^{n+1}})^N(1-e^{-i\omega})^N}{(\sin(\frac{\omega}{2^{n+1}}))^N(i\omega e^{\frac{-i\omega}{2^{n+1}}})^N}
\end{aligned}$$

Now we know that

$$\lim_{n\to\infty}\left(\frac{(\frac{\omega}{2^{n+1}})}{\sin(\frac{\omega}{2^{n+1}})}\right)^N = 1,$$

since $\lim_{x\to 0}\frac{x}{\sin x} = 1$. So we are left with

$$\lim_{n\to\infty}\frac{(1-e^{-i\omega})^N}{(i\omega e^{\frac{-i\omega}{2^{n+1}}})^N} = \left(\frac{1-e^{-i\omega}}{i\omega}\right)^N,$$

since $e^{\frac{-i\omega}{2^{n+1}}}\to 1$ as $n\to\infty$. Thus we have

$$\prod_{k=1}^{\infty}\left(\frac{1+e^{\frac{-i\omega}{2^k}}}{2}\right)^N = \left(\frac{1-e^{-i\omega}}{i\omega}\right)^N,$$

and Claim 3.3 is shown, as required. □

Now

$$g(\omega) = \frac{1}{\sqrt{2\pi}}\prod_{k=1}^{\infty}H\left(\frac{\omega}{2^k}\right) = \frac{1}{\sqrt{2\pi}}\prod_{k=1}^{\infty}\left(\frac{1+e^{\frac{-i\omega}{2^k}}}{2}\right)^N\cdot S\left(\frac{\omega}{2^k}\right),$$

by the factorization of $H(z)$ from our hypothesis and from our previous claim, so we have that

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \prod_{k=1}^{\infty} \left(\frac{1 + e^{\frac{-i\omega}{2^k}}}{2} \right)^N \cdot S\left(\frac{\omega}{2^k}\right) = \frac{1}{\sqrt{2\pi}} \left(\frac{1 - e^{-i\omega}}{i\omega} \right)^N \cdot \prod_{k=1}^{\infty} S\left(\frac{\omega}{2^k}\right).$$

Thus in order to show that $|g(\omega)| \leq \frac{C}{1+|\omega|}$, we need to obtain estimates for both $|\frac{1-e^{-i\omega}}{i\omega}|^N$ and $\left| \prod_{k=1}^{\infty} S(\frac{\omega}{2^k}) \right|$.

Claim 3.4 $|\frac{1-e^{-i\omega}}{i\omega}|^N \leq 2^N \min(1, |\omega|^{-N})$

Proof. Note that we really need to show that

$$\left| \frac{1 - e^{-i\omega}}{i\omega} \right|^N \leq 2^N$$

and

$$\left| \frac{1 - e^{-i\omega}}{i\omega} \right|^N \leq \frac{2^N}{|\omega|^N}.$$

By the triangle inequality, we have that

$$|1 + e^{-i\omega}| \leq 1 + |e^{-i\omega}| = 2.$$

Thus,

$$\frac{|1 - e^{-i\omega}|}{|i\omega|} = \frac{|1 - e^{-i\omega}|}{|\omega|} \leq \frac{2}{|\omega|}.$$

Therefore

$$\left| \frac{1 - e^{-i\omega}}{i\omega} \right|^N \leq \frac{2^N}{|\omega|^N}.$$

So if $\frac{1}{|\omega|^N} \leq 1$, then $\min(1, \frac{1}{|\omega|^N}) = \frac{1}{|\omega|^N}$, and from the argument above, we have

$$\left| \frac{1 - e^{-i\omega}}{i\omega} \right|^N \leq \frac{2^N}{|\omega|^N} = 2^N \min(1, |\omega|^{-N}).$$

If $\frac{1}{|\omega|^N} \geq 1$, then $|\omega|^N \leq 1$, which implies $|\omega| \leq 1$. So $\min(1, |\omega|^{-N}) = 1$.
Note that $|1 - e^{-i\omega}| = |2ie^{\frac{-i\omega}{2}} \sin(\frac{\omega}{2})| = 2|\sin(\frac{\omega}{2})|$. So

$$\left| \frac{1 - e^{-i\omega}}{i\omega} \right| = \frac{2|\sin(\frac{\omega}{2})|}{|\omega|} = 2\left| \frac{\sin(\frac{\omega}{2})}{\omega} \right|.$$

Note that $\left| \frac{\sin(\frac{\omega}{2})}{\omega} \right| \leq 1 \ \forall \omega \in \mathbb{R}$. Thus we have that

$$\left| \frac{1 - e^{-i\omega}}{i\omega} \right| = 2\left| \frac{\sin(\frac{\omega}{2})}{\omega} \right| \leq 2.$$

Therefore

$$\left|\frac{1-e^{-i\omega}}{i\omega}\right|^N \leq 2^N$$

and we have that

$$\left|\frac{1-e^{-i\omega}}{i\omega}\right|^N \leq 2^N \min{(1, \frac{1}{|\omega|^N})}$$

as required for Claim 3.5. □

Now let us give an estimate for $\prod_{k=1}^{\infty} S(\frac{\omega}{2^k})$. For simplicity, let $T(\omega) = \prod_{k=1}^{\infty} S(\frac{\omega}{2^k})$. Fix M and ω such that

$$2^{M-1} \leq |\omega| \leq 2^M.$$

Then

$$T(\omega) = \prod_{k=1}^{\infty} S(\frac{\omega}{2^k}) = \prod_{k=1}^{M} S(\frac{\omega}{2^k}) \cdot \prod_{M+1}^{\infty} S(\frac{\omega}{2^k}).$$

Let $l = k - M$. This gives

$$\begin{aligned}
T(\omega) &= \prod_{k=1}^{M} S\left(\frac{\omega}{2^k}\right) \cdot \prod_{M+1}^{\infty} S\left(\frac{\omega}{2^k}\right) \\
&= \prod_{k=1}^{M} S\left(\frac{\omega}{2^k}\right) \cdot \prod_{l=1}^{\infty} S\left(\frac{\omega}{2^{l+M}}\right) \\
&= \prod_{k=1}^{M} S\left(\frac{\omega}{2^k}\right) \cdot \prod_{l=1}^{\infty} S\left(\frac{2^{-M}\omega}{2^l}\right) \\
&= \prod_{k=1}^{M} S\left(\frac{\omega}{2^k}\right) \cdot T(2^{-M}\omega).
\end{aligned}$$

Let $U = \max_{|\omega|\leq 1} |T(\omega)|$ and note that since $\frac{|\omega|}{2^M} \leq 1$, we have that

$$|T(2^{-M}\omega)| \leq U.$$

So we have, by our hypothesis on $S(z)$,

$$\begin{aligned}
|T(\omega)| &= \left|\prod_{k=1}^{M} S\left(\frac{\omega}{2}\right)\right| \cdot |T(2^{-M}\omega)| \\
&\leq (2^{N-1})^M \cdot |T(2^{-M}\omega)| \\
&= (2^M)^{N-1} \cdot |T(2^{-M}\omega)| \\
&= (2 \cdot 2^{M-1})^{N-1} \cdot |T(2^{-M}\omega)| \\
&\leq (2 \cdot 2^{M-1})^{N-1} \cdot U.
\end{aligned}$$

Because we fixed ω and M, we know that $2^{M-1} \leq |\omega|$. Then the above gives that

$$|T(\omega)| \leq 2^{N-1} \cdot |\omega|^{N-1} \cdot U = (2|\omega|)^{N-1} \cdot U.$$

This results holds independent of our choice of M and thus will hold for all ω. Therefore

$$\Big| \prod_{k=1}^{\infty} S(\frac{\omega}{2^k}) \Big| \leq (2|\omega|)^{N-1} \cdot U$$

We have now shown that

$$\begin{aligned} |g(\omega)| &= \frac{1}{\sqrt{2\pi}} \Big| \Big(\frac{1 - e^{-i\omega}}{i\omega} \Big) \Big|^N \cdot \Big| \prod_{k=1}^{\infty} S\Big(\frac{\omega}{2^k} \Big) \Big| \\ &\leq \frac{1}{\sqrt{2\pi}} \cdot 2^N \min(1, |\omega|^{-N}) \cdot 2^{N-1} |\omega|^{N-1} \cdot U. \end{aligned}$$

Claim 3.5

$$\frac{1}{\sqrt{2\pi}} \cdot 2^N \min(1, |\omega|^{-N}) \cdot 2^{N-1} |\omega|^{N-1} \cdot U \leq \frac{2^{2N-1} \cdot U}{\sqrt{2\pi}} \cdot \frac{2}{1 + |\omega|}.$$

Proof. Note that we really only need to show that $\min(1, |\omega|^{-N}) |\omega|^{N-1} \leq \frac{2}{1+|\omega|}$, since the rest of the inequality comes from rearranging terms. First assume $|\omega| \leq 1$. Then $\min(1, |\omega|^{-N}) = 1$ and

$$\begin{aligned} |\omega|^{N-1} + |\omega|^N \leq 2 &\Rightarrow |\omega|^{N-1} + |\omega|^{N-1}|\omega| \leq 2 \\ &\Rightarrow |\omega|^{N-1}(1 + |\omega|) \leq 2 \\ &\Rightarrow |\omega|^{N-1} \leq \frac{2}{(1 + |\omega|)} \\ &\Rightarrow \min(1, |\omega|^{-N})|\omega|^{N-1} \leq \frac{2}{(1 + |\omega|)}. \end{aligned}$$

Next assume $|\omega| > 1$. Then $\min(1, |\omega|^{-N}) = |\omega|^{-N}$ and

$$\begin{aligned} |\omega|^{-1} \leq 2 &\Rightarrow |\omega|^{-1} + |\omega|^0 \leq 2 \\ &\Rightarrow |\omega|^{-1}(1 + |\omega|) \leq 2 \\ &\Rightarrow |\omega|^{-1} \leq \frac{2}{(1 + |\omega|)} \\ &\Rightarrow |\omega|^{-N} \cdot |\omega|^{N-1} \leq \frac{2}{(1 + |\omega|)} \\ &\Rightarrow \min(1, |\omega|^{-N}) \cdot |\omega|^{N-1} \leq \frac{2}{(1 + |\omega|)}. \end{aligned}$$

Claim 3.6 is thus shown. □

Now from our series of claims, we have that

$$
\begin{aligned}
|g(\omega)| &= \frac{1}{\sqrt{2\pi}}\left|\left(\frac{1-e^{-i\omega}}{i\omega}\right)\right|^N \cdot \left|\prod_{k=1}^{\infty} S\left(\frac{\omega}{2^k}\right)\right| \\
&\leq \frac{1}{\sqrt{2\pi}} \cdot 2^N \min\left(1, |\omega|^{-N}\right) \cdot 2^{N-1}|\omega|^{N-1} \cdot U \\
&\leq \frac{2^{2N-1} \cdot U}{\sqrt{2\pi}} \cdot \frac{2}{1+|\omega|}.
\end{aligned}
$$

Let $C = \frac{2^{2N-1} \cdot U}{\sqrt{2\pi}} \cdot 2$. Then

$$|g(\omega)| \leq \frac{C}{1+|\omega|}.$$

□

To summarize, we have now shown that by putting the conditions on our symbol $H(z)$, as listed in Proposition 3.2, we have convergence of the cascade algorithm iterates $\{\widehat{\phi}_n(\omega)\}$ to a continuous function $g(\omega)$ for each $\omega \in \mathbb{R}$ and that this $g(\omega)$ satisfies the dilation equation in the transform domain. Next we shall show convergence of the iterates $\{\phi_n(t)\}$ to a function $\{\phi(t)\}$ in the time domain.

Theorem 3.1 *[see [8], p. 261] Suppose the symbol*

$$H(z) = \frac{1}{\sqrt{2}} \sum_{k=0}^{M} h_k e^{-ik\omega}$$

can be factored as

$$H(z) = \left(\frac{1+z}{2}\right)^N S(z)$$

where $S(z)$ satisfies $S(1) = 1$ and $\max_{|z|=1} |S(z)| \leq 2^{N-1}$. Then the sequence of functions $\{\phi_n(t)\}$ defined by

$$\phi_{n+1} = \sqrt{2} \sum_{k=0}^{M} h_k \phi_n(2t-k)$$

with $\phi_0(t)$ as our initial guess, converges in $L^2(\mathbb{R})$ to a function $\phi(t)$ that satisfies the dilation equation given by

$$\phi(t) = \sqrt{2} \sum_{k=0}^{M} h_k \phi(2t-k).$$

Moreover, $\overline{supp(\phi_n)} = [0, M - 2^{-n}(M-1)]$ for $n \geq 1$ and $\overline{supp(\phi)} = [0, M]$.

Proof. In order to translate $g(\omega)$ into the time domain, we first need to show that $g(\omega) \in L^2(\mathbb{R})$. This can be seen easily, since by Proposition 3.2, we know

$$\int_{\mathbb{R}} |g(\omega)|^2 d\omega \leq \int_{\mathbb{R}} \left(\frac{C}{1+|\omega|} \right)^2 d\omega$$

and it is clear that

$$\int_{\mathbb{R}} \left(\frac{C}{1+|\omega|} \right)^2 d\omega < \infty.$$

Thus we have that $g(\omega) \in L^2(\mathbb{R})$. We can now define

$$\widehat{g(\omega)} = \widehat{\widehat{\phi(\omega)}}$$

as the Fourier transform of $g(\omega)$. By Proposition 3.2, we know that $g(\omega)$ satisfies the transform domain dilation equation given by

$$g(\omega) = H(\frac{\omega}{2})g(\frac{\omega}{2}) = \frac{1}{\sqrt{2}} \sum_{k=0}^{M} h_k e^{-ik\frac{\omega}{2}} g(\frac{\omega}{2}).$$

So from an earlier result from Chapter 1(see (1.1)), we have that $\phi(t)$ satisfies the dilation equation, given by

$$\phi(t) = \sqrt{2} \sum_{k \in \mathbb{Z}} h_k \phi(2t-k)$$

in the time domain. Now in Proposition 3.1, we showed that $\widehat{\phi}_n(\omega) \to \widehat{\phi}(\omega) = g(\omega)$. Using this fact, one can show (see [11], Theorem 8.36, p.243), that $\phi_n(t) \to \phi(t)$ in $L^2(\mathbb{R})$.

It remains to show that $\overline{\text{supp}(\phi_n)} = [0, M - 2^{-n}(M-1)]$ for $n \geq 1$ and hence $\overline{\text{supp}(\phi)} = [0, M]$. Recall that $\overline{\text{supp}(\phi)}$ is the compact support of $\phi(t)$, as previously defined in Chapter 1. We shall show that $\overline{\text{supp}(\phi_n)} = [0, M - 2^{-n}(M-1)]$ using induction. First observe that

$$\phi_1(t) = \sqrt{2} \sum_{k=0}^{M} h_k \phi_0(2t-k) = \sqrt{2} \sum_{k=0}^{M} h_k B_0(2t-k),$$

since, by our hypothesis, $\phi_0(t)$ is our initial guess. Recall that $\phi_0(t)$ is defined as

$$\phi_0(t) = \begin{cases} 1 & 0 \leq t < 1 \\ 0 & \text{otherwise} \end{cases}$$

We know that $\phi_0(t)$ has a support of $[0,1)$, by the definition of the function. Thus, we know that the support of $\phi_0(2t-k)$ has a support of $[\frac{k}{2}, \frac{k+1}{2})$. Now, these support intervals do not overlap, so we have that

$$\overline{\text{supp}(\phi_1(t))} = \bigcup_{k=0}^{M} \left[\frac{k}{2}, \frac{k+1}{2}\right] = \left[0, \frac{M+1}{2}\right] = [0, M - 2^{-1}(M-1)].$$

Now assume that for $\phi_n(t)$, $\overline{\text{supp}(\phi_n(t))} = [0, M - 2^{-n}(M-1)] = [0, A]$ and consider

$$\phi_{n+1}(t) = \sqrt{2}\sum_{k=0}^{M} h_k \phi_n(2t-k).$$

By our induction hypothesis, $\phi_n(t)$ has compact support $[0, A]$ so the compact support of $\phi_n(2t-k)$ is $[\frac{k}{2}, \frac{A+k}{2}]$. Taking $k = 0$ on the left side of the interval gives 0. Taking $k = M$ on the right side of the interval, we have that

$$\begin{aligned}
\frac{A+k}{2} &= \frac{M - 2^{-n}(M-1) + M}{2} \\
&= \frac{2M - 2^{-n}(M-1)}{2} \\
&= M - \frac{(M-1)}{2^{n+1}} \\
&= M - 2^{-(n+1)}(M-1).
\end{aligned}$$

Thus we have shown that $\overline{\text{supp}(\phi_{n+1})} = [0, M - 2^{-(n+1)}(M-1)]$.
Therefore, $\overline{\text{supp}(\phi_n)} = [0, M - 2^{-n}(M-1)]$ for $n \geq 1$
To show that $\overline{\text{supp}(\phi)} = [0, M]$, we simply need to take the limits of the supports of the iterates,

$$\lim_{n\to\infty} \overline{\text{supp}(\phi_n(t))} = \lim_{n\to\infty} [0, M - 2^{-n}(M-1)] = [0, M].$$

The proof of Theorem 3.1 is now complete. □

The above results hold for Daubechies scaling functions, as described in the previous section. In fact, as a direct result of Theorem 2.9, we have the following corollary:

Corollary 3.1 *Suppose $\phi(t)$ is the Daubechies $D2N$ scaling function where N is a positive integer. Then*

$$\overline{\textit{supp}\phi(t)} = [0, 2N-1].$$

Let $\phi(t)$ be the Daubechies $D4$ scaling function. We know that the $D4$ scaling filter coeffcients are

$$h_0 = \frac{1+\sqrt{3}}{4\sqrt{2}},$$

$$h_1 = \frac{3+3\sqrt{3}}{4\sqrt{2}},$$

$$h_2 = \frac{3-\sqrt{3}}{4\sqrt{2}},$$

and

$$h_3 = \frac{1-\sqrt{3}}{4\sqrt{2}}.$$

From Theorem 3.1, we know that the scaling function iterates are represented by

$$\phi_{n+1}(t) = \sqrt{2}\sum_{k=0}^{2N-1} h_k \phi_n(2t-k). \tag{3.5}$$

We know that the support of $B_0(t) = \phi_0(t) = [0, 1]$. Using (3.5), one can show that the compact support of $\phi_1(t)$ is $[0, 2]$. Furthermore, the compact support of $\phi_2(t)$ is $[0, 2.5]$, the compact support of $\phi_3(t)$ is $[0, 2.75]$, and the compact support of $\phi_4(t)$ is $[0, 2.875]$. Taking further iterates shows that

$$\overline{\text{supp}\phi(t)} = [0, 3],$$

verifying the above corollary for $N = 2$.

Plotted on the following pages are the first eight iterates of the cascade algorithm for the Daubechies $D4$ scaling function, with an initial guess of $\phi_0(t) = B_0(t)$. One can see from the graphs that the support for the scaling function is $[0, 3]$, and that within a very few number of iterations, the Cascade Algorithm gives a good estimate of the scaling function.

In conclusion, we have explored Daubechies' construction of scaling functions that are both compactly supported and smooth, which are desired properties for many applications. These scaling functions are associated with a multiresolution analysis, $\{V_j\}_{j\in\mathbb{Z}}$, which is a sequence of subspaces of $L^2(\mathbb{R})$ that satisfies a number of different properties. While no closed-form formula for a scaling function exists for values of $N \geq 3$, we can approximate the scaling functions using the Cascade Algorithm, as first presented by Daubechies and Lagarias. Daubechies results led to an explosion in the study of wavelets in the 1990's and resulted in many modern-day applications of wavelet theory, including speech discrimination and earthquake prediction, as well as fingerprint and data compression.

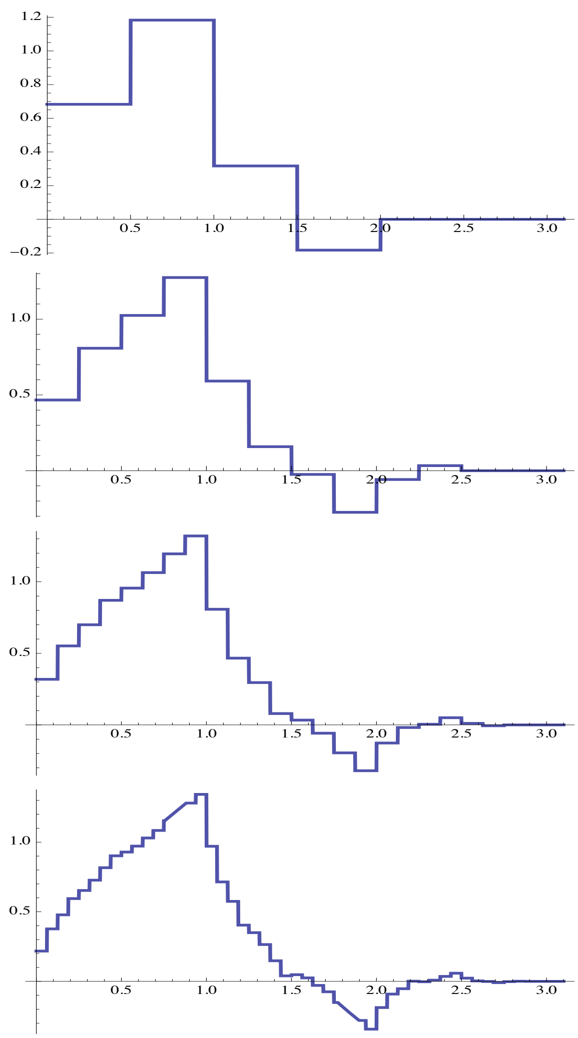

Figure 1.: Iterates of the Cascade Algorithm for D4 scaling function with n=1,2,3,4

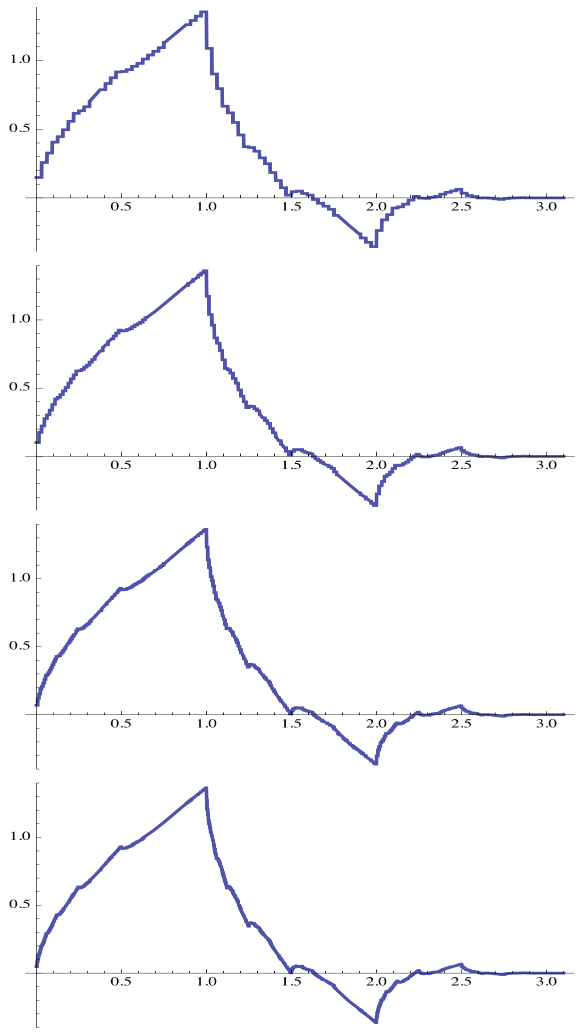

Figure 2.: Iterates of the Cascade Algorithm for D4 scaling function with n=5,6,7,8

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