12-6-2002

Extensions of Quandles and Cocycle Knot Invariants

Marina Appiou Nikiforou
University of South Florida

Office of Graduate Studies
University of South Florida
Tampa, Florida

CERTIFICATE OF APPROVAL

This is to certify that the dissertation of

MARINA APPIOU NIKIFOROU

in the graduate degree program of
Mathematics
was approved on December 6, 2002
for the Doctor of Philosophy degree

Examining Committee:

____
Major Professor: Masahiko Saito, Ph.D.

____
Member: W. Edwin Clark, Ph.D.

____
Member: Nataša Jonoska, Ph.D.

____
Member: David Rabson, Ph.D.

Committee Verifcation:

____
Program Director

EXTENSIONS OF QUANDLES AND COCYCLE KNOT INVARIANTS

by

MARINA APPIOU NIKIFOROU

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
Department of Mathematics
College of Arts and Sciences
University of South Florida

Date of Approval:
December 6, 2002

Major Professor: Masahiko Saito, Ph.D.

DEDICATION

To my parents for having faith in me

ACKNOWLEDGMENTS

I gratefully acknowledge the help of Masahiko Saito, who has supervised my dissertation and guided me through all steps of research and writing. I am grateful to Edwin Clark, Nataša Jonoska, and David Rabson for their valuable quidance and feedback. I also thank my co-authors J. Scott Carter, Mohamed Elhamdadi, and Angela Haris.

I have received invaluable support from my husband, Savvas Nikiforou, who has provided help and encouragement through my graduate studies. I am also thankful to Savvas for his feedback and help with LaTeX. Finally, I would like to thank to my parents, Andreas Appios and Paraskevi Andreou, for their support and understanding.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

EXTENSIONS OF QUANDLES AND COCYCLE KNOT INVARIANTS

by

MARINA APPIOU NIKIFOROU

An Abstract

Of a dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
Department of Mathematics
College of Arts and Sciences
University of South Florida

Date of Approval:
December 6, 2002

Major Professor: Masahiko Saito, Ph.D.

Knot theory has rapidly expanded in recent years. New representations of braid groups led to an extremely powerful polynomial invariant, the Jones polynomial. Combinatorics applied to knot and link diagrams led to generalizations. Knot theory also has connections with other fields such as statistical mechanics and quantum field theory, and has applications in determining how certain enzymes act on DNA molecules, for example.

The principal objective of this dissertation is to study the relations between knots and algebraic structures called quandles. A quandle is a set with a binary operation satisfying some properties related to the three Reidemeister moves. The study of quandles in relation to knot theory was intitiated by Joyce and Matveev. Later, racks and their (co)homology theory were defined by Fenn and Rourke. The rack (co)homology was also studied by Graña from the viewpoint of Hopf algebras. Furthermore, a modified definition of homology theory for quandles was introduced by Carter, Jelsovsky, Kamada, Langford, and Saito to define state-sum invariants for knots and knotted surfaces, called quandle cocycle invariants.

This dissertation studies the quandle cocycle invariants using extensions of quandles and knot colorings. We obtain a coloring of a knot by assigning elements of a quandle to the arcs of the knot diagram. Such colorings are used to define knot invariants by state-sum. For a given coloring, a 2-cocycle is assigned at each crossing as the Boltzmann weight. The product of the weights over all crossings is the contribution to the state-sum, which is the formal summation of the contributions over all possible colorings of the given knot diagram by a given quandle. Generalizing the cocycle invariant for knots to links, we define two kinds of invariants for links: a component-wise invariant, and an invariant defined as families of vectors.

Abelian extensions of quandles are also defined and studied. We give a formula for creating infinite families of abelian extensions of Alexander quandles. These extensions give rise to explicit formulas for computing 2-cocycles. The theory of quandle

extensions parallels that of groups. Moreover, we investigate the notion of extending colorings of knots using quandle extensions. In particular, we show how the obstruction to extending the coloring contributes to the non-trivial terms of the cocycle invariants for knots and links. Moreover, we demonstrate the relation between these new cocycle invariants and Alexander matrices.

Abstract Approved: ______________________________

Major Professor: Masahiko Saito, Ph.D.
Associate Professor, Department of Mathematics

Date Approved: ______________________________

CHAPTER 1
INTRODUCTION

1.1 History and organization

Knot theory is the mathematical study of knots. A knot is a closed, non-intersecting curve in 3-dimensional Euclidean space. More precisely, a knot is the image of an embedding of a unit circle in the Euclidean 3-space. Much of knot theory is concerned with telling which knots are the same and which are different. Knot theory has been used, for example, to determine how certain enzymes act on DNA molecules. One of the early and main achievements was the discovery in 1923 of the Alexander polynomial of a knot or link [1]. Homology theory applied to infinite cyclic covers of the complement of a knot led to the Alexander polynomial. Conway [13] gave a recursive formula of this polynomial. The study of knot invariants changed dramatically in 1984, when new representations of braid groups led to another extremely powerful polynomial invariant, the Jones polynomial [26]. Since then many generalizations were discovered. These caused interactions between knot theory and various other fields such as combinatorics, statistical mechanics, and quantum field theory.

This dissertation consists of two parts. The first part (Chapters 1–3) is an overview of invariants of knots and links defined by using quandles. The second part (Chapters 4–6), which is the main contribution of the author, deals with extensions of quandles and colorings, as well as relations to Alexander matrices.

In Chapter 1, we review background information needed to present this work. Constructions of quandles and some of their properties are discussed in Chapter 2. In particular, we consider colorings of knot diagrams by quandles.

Homology and cohomology theories for quandles are introduced in Chapter 3. Generalizing the quandle cocycle invariants for knots [8], we define a component-wise invariant, and an invariant for links defined by families of vectors.

In Chapter 4, we discuss abelian extensions of quandles. Formulas that produce infinite families of abelian extensions of Alexander quandles are given, and these families are shown to be non-trivial as extensions. For example, we show that $\mathbb{Z}_{q^{m+1}}[T, T^{-1}]/(T-1+q)$ is an abelian extension of the quandle $X = \mathbb{Z}_{q^m}[T, T^{-1}]/(T-1+q)$, for some cocycle $\phi \in Z^2_{\rm Q}(X; \mathbb{Z}_q)$. Moreover, these extensions give rise to explicit formulas for computing cocycles. In Chapter 5, we describe the notion of extending colorings of knot diagrams using the extension theory of quandles. In addition, we show how the previously defined invariants determine for both knots and links the number of colorings by a quandle that can be extended to colorings by an extension of the quandle. Finally, in Chapter 6 we relate the new cocycle invariants to Alexander matrices.

1.2 Knots and links

We often deal with knots by depicting them in a plane; in other words, we study their diagrams. Moreover, we describe the equivalence of knots by some moves among their diagrams, called the Reidemeister moves. Tait [38] attempted to classify knot types in the late 19th century.

One of the main topics of knot theory is the study of knot invariants. An invariant is a tool to distinguish knots. It is a well-defined algebraic object such as a number, a polynomial, or a group.

We denote the n-dimensional Euclidean space by $\mathbb{R}^n$ and the n-dimensional sphere by S^n.

Definition 1.2.1 [30] A *link L of m components* is a subset of S^3, or of $\mathbb{R}^3$, that consists of m disjoint, piecewise linear, simple closed curves. A link of one component is a *knot.*

Definition 1.2.2 [29] An *orientation* of an n-simplex is an equivalence class of orderings of the $n+1$ vertices modulo even permutations. By $[v_0, v_1, \ldots, v_n]$, we denote an oriented simplex with vertices ordered as $v_0, v_1, \ldots, v_n$ and by $-[v_0, v_1, \ldots, v_n]$, we denote the simplex with opposite orientation. By the *induced orientation* of the face of $[v_0, v_1, \ldots, v_n]$ opposite to v_i, we mean the orientation given by

$$(-1)^i[v_0, v_1, \ldots, \hat{v}_i, \ldots, v_n],$$

where $\hat{v}_i$ means that the vertex i is ommited. An *orientation* of a piecewise linear manifold M is an assignment of an orientation for each n-simplex of K_M, a triangulation of M, such that the orientation of A_0 induced from the orientation of A_1 is opposite to that of A_0 induced from the orientation of A_2 for any n-simplices A_1, A_2 in K_M, where $A_0 = A_1 \cap A_2$ is an $(n-1)$-simplex. According to whether or not such an orientation exists, we say that M is *orientable* or *non-orientable.* When M is orientable and an orientation is specified, M is said to be *oriented.*

Since S^1 is a 1-manifold, it is orientable, and so are knots and links.

Definition 1.2.3 [30] Links L_1 and L_2 in S^3 are *equivalent* if there is an orientation-preserving piecewise linear homeomorphism $h : S^3 \to S^3$ such that $h(L_1) = (L_2)$.

A simple way to study links is to work with their "diagrams", which are two-dimensional representations with respect to the standard projection $p : \mathbb{R}^3 \to \mathbb{R}^2$. This means that each line segment of L projects to a line segment in $\mathbb{R}^2$ satisfying the following conditions:

1. The projections of two segments intersect in at most one point (i.e., no projections of the two segments overlap in a subsegment as depicted in Figure 1(a)).

2. Any two segments do not intersect at an endpoint (see Figure 1(b)).

3. No intersection point belongs to the projections of three segments (so that the situation depicted in Figure 1(c) does not happen).

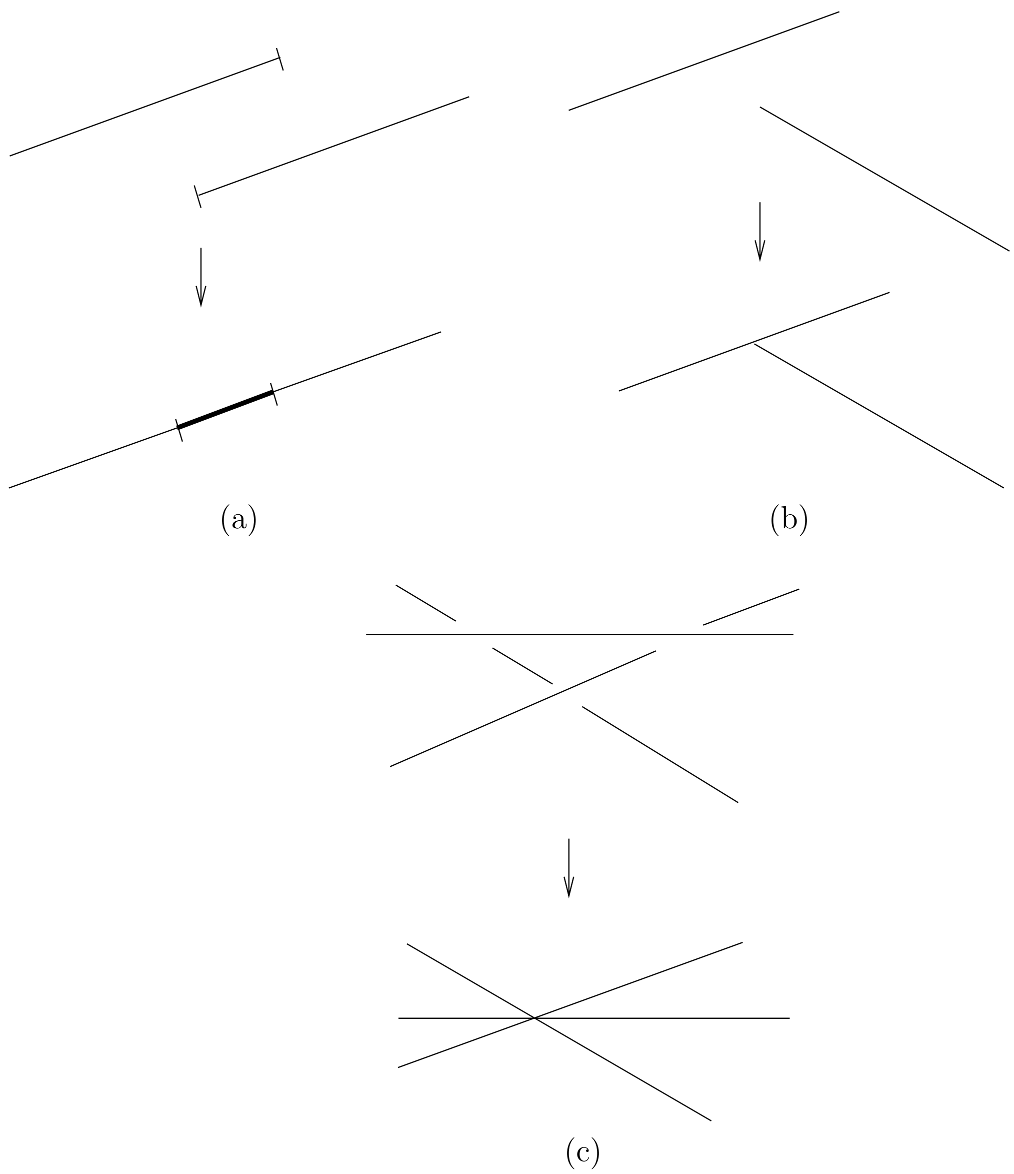

Figure 1. Restrictions of the projection on line segments

Given such a situation, the image of L in $\mathbb{R}^2$ together with "over and under" information at the crossings is called a *link diagram* of L and is denoted by D_L. A *crossing* is a point of intersection of the projections of two line segments of L. The "over and under" information refers to the relative heights above $\mathbb{R}^2$ of the two inverse images of a crossing. This information is indicated in pictures by breaking the under-passing segments as shown in Figure 3. After breaking under-passing segments, the projection of L becomes a disjoint union of *arcs*. Thus, an arc is a connected component of the projection after breaking under-passing segments. In practice, we draw well rounded curves for knot diagrams (see Figure 2) instead of polygonal segments.

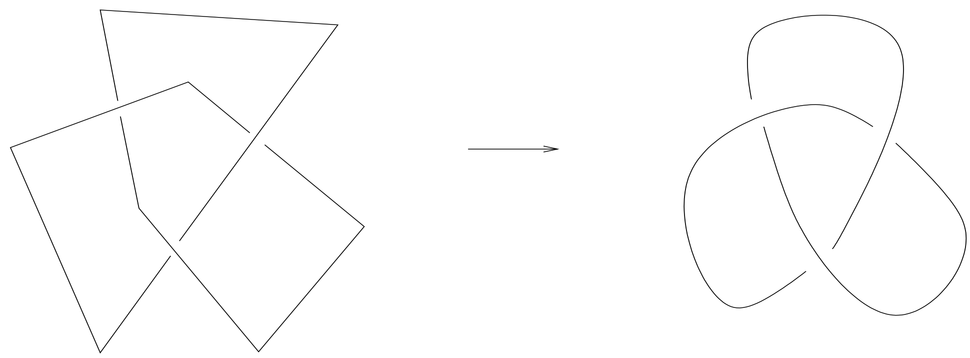

Figure 2. A well rounded trefoil

The projection map preserves the orientation of a link. The *co-orientation* is a family of vectors in $\mathbb{R}^2$ normal to the link diagram, such that the pair (orientation, co-orientation) matches the given orientation (right-handed, or counterclockwise) of the plane. At a crossing, if the pair of the orientation of the over-arc and that of the under-arc matches the (right-hand) orientation of the plane, then the crossing is called *positive*; otherwise it is *negative*. In Figure 3, the crossing depicted at the left is positive and the other on the right is negative.

Definition 1.2.4 [35] The *sign* of a crossing τ, denoted $\varepsilon(\tau)$, is taken to be 1 if the crossing is positive and -1 if the crossing is negative, as illustrated in Figure 3.

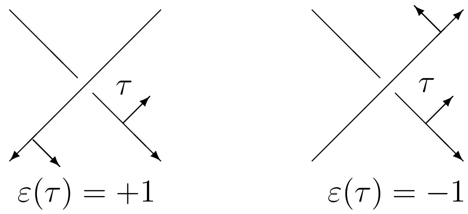

Figure 3. Positive and negative crossings

If two links L_1 and L_2 are equivalent, then their respective diagrams D_{L_1} and D_{L_2} are related by a sequence of *Reidemeister moves* and an orientation-preserving homeomorphism of the plane. In this case, the two diagrams D_{L_1} and D_{L_2} are *equivalent.* There are three such moves called the first (Type I), second (Type II), and third (Type III) Reidemeister move, respectively. The three Reidemeister moves are depicted in Figure 4 (see, for example, [30]). The Reidemeister Type I move allows to put in or take out a small twist in the string. The second move is used to either add two crossings or remove two crossings locally. Type III move allows to slide a strand from one side of a crossing to the other.

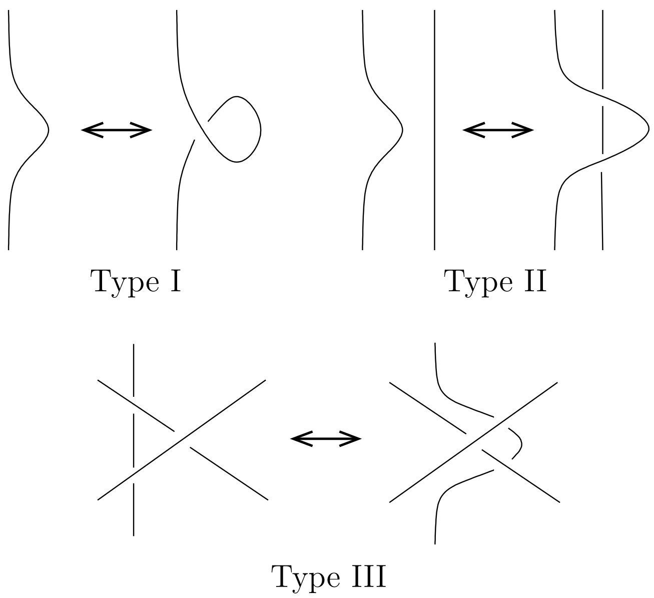

Figure 4. The Reidemeister moves

Definition 1.2.5 [35] A property of a link L, or a link diagram D_L respectively, is *invariant* if it remains the same for all links (resp. link diagrams) equivalent to L (resp. D_L).

1.3 Fox n-coloring of knot diagrams

The most elementary knot invariant is Fox's 3-colorability. Since our invariants are generalizations of this, we review its definition in this section.

Definition 1.3.1 [20] A link is *3-colorable* if it has a diagram, such that we can assign either 0, 1 or 2 (these numbers are called *colors*) to its arcs in such a way that the following conditions are satisfied:

1. each arc is assigned a single color,
2. at least two colors are used, and
3. at each crossing, either all arcs have the same color, or all of the three colors meet. See Figure 5. In the figure, $\{a, b, c\} = \{0, 1, 2\}$, i.e. a, b and c represent distinct numbers from the set $\{0, 1, 2\}$.

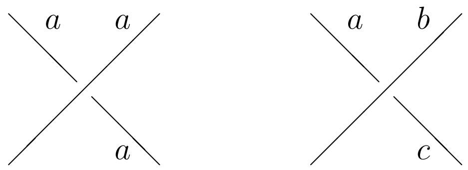

Figure 5. The 3-colorability condition

Theorem 1.3.2 [20] 3-colorability is a link invariant.

Proof. This is proved by checking that the 3-colorability remains unchanged by the Reidemeister moves. See section 1.3 of [20] for details.

Example 1.3.3 The trefoil, as shown in Figure 6, is 3-colorable.

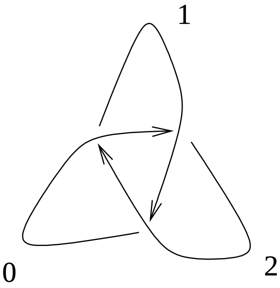

Figure 6. A colored trefoil

For any positive integer n, 3-colorability can be generalized to n-colorability as follows. Let n be a natural number greater than 2. Let $a_1, a_2, \ldots, a_k$ be the arcs of a link diagram. Assign to each arc an integer $\lambda_i \in \{0, 1, \ldots, n-1\}$ (called *color*). Let λ_q be the color assigned to the over-arc and λ_r, λ_s be the colors assigned to the two under-arcs at a crossing. Then, it is required that the condition $\lambda_r + \lambda_s \equiv 2\lambda_q \pmod{n}$ be satisfied at every crossing. See Figure 7.

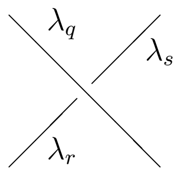

Figure 7. The n-colorability condition

Definition 1.3.4 [35] A link is said to be *Fox n-colorable* if for some diagram D_L of L, colors can be assigned to the arcs satisfying the above properties using at least two distinct colors.

Theorem 1.3.5 [19] *n-colorability is a link invariant.*

In the next chapter we introduce an algebraic structure called quandle that generalizes n-colorability.

CHAPTER 2
QUANDLES

The algebraic structure *kei* defined by Takasaki [39] appears to be the first occurence of a quandle in the literature. Since then, similar algebraic structures have been defined, but were often motivated from symmetric transformations, rather than from knot theory. A dramatic change in the study of quandles occured when Joyce [27], and at the same time Matveev [33], initiated the study of quandles in relation to knot theory. The term quandle was invented by Joyce. Then, Brieskorn [3] introduced the *automorphic sets*, dropping the idempotency ($x*x = x, \forall x$) condition, and pointed out many occurences of this structure. Later, Fenn and Rourke [15] called this structure *racks*. Kauffman [28] also gave a description of a similar structure called *crystal*. Furthermore, (co)homology theory for racks was defined in [17], and from the point of view of Hopf algebras in [22]. A modified version of (co)homology theory for quandles was described in [8] for defining state-sum invariants for knots and knotted surfaces.

We give definitions and examples of quandles in Sections 2.1 and 2.2. Then, in Section 2.3 we classify 4-element quandles. Colorings of knot diagrams by quandles are defined in Section 2.4.

2.1 Definitions

Definition 2.1.1 [8] A *quandle*, X, is a set with a binary operation $(a, b) \mapsto a * b$ such that

(I) For any $a \in X$, $a * a = a$.

(II) For any $a, b \in X$, there is a unique $c \in X$ such that $a = c * b$.

(III) For any $a, b, c \in X$, we have $(a * b) * c = (a * c) * (b * c)$.

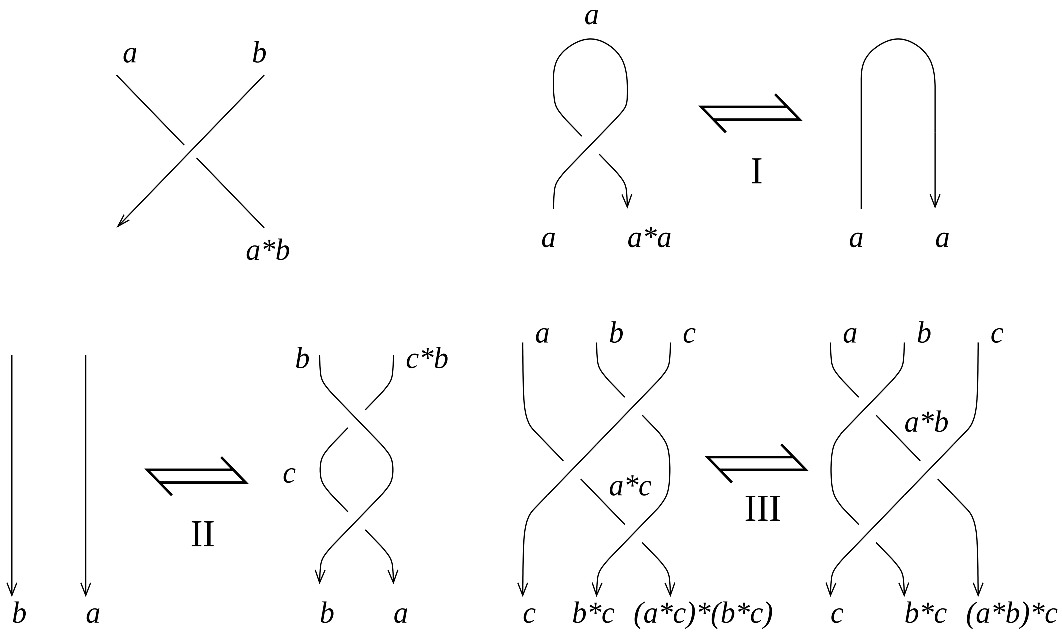

Figure 8. Reidemeister moves and quandle conditions

A *rack* is a set with a binary operation that satisfies (II) and (III). If a rack or a quandle is finite, then the number of elements in X is called the *order* of X.

We call Axiom (III) in the definition of a quandle the *rack identity.* Note that it is the right self-distributive law. Racks and quandles have been studied in, for example, [3, 15, 27, 28, 33].

The three axioms for a quandle arise from Reidemeister moves of type I, II, and III, respectively [15, 28]. At the top left of Figure 8, a coloring rule is depicted, which will be precisely defined in Section 2.4. Under this coloring rule we observe that the colors at the bottom segments of figures I, II, and III, match before and after Reidemeister moves I, II, and III. Thus, Figure 8 shows that Reidemeister moves of type I, II, and III correspond to the quandle axioms I, II, and III, respectively. Quandle structures have been found in areas other than knot theory, see [2, 3] for example.

Definition 2.1.2 A function $f : X \to Y$ between quandles or racks is a *homomorphism* if $f(a * b) = f(a) * f(b)$ for any $a, b \in X$. A homomorphism f is called an *isomorphism* if f is one-to-one and onto. An isomorphism $f : X \to X$ is called an *automorphism* [8, 27].

There are several immediate consequences of the quandle and rack axioms. Let X denote a quandle. From Axiom (II) of Definition 2.1.1, each element $b \in X$ defines a bijection $S(b) : X \to X$ with $aS(b) = a * b$ (the function is on the right). The bijection is a quandle automorphism by Axiom (III) of Definition 2.1.1. For a *word* $w = b_1^{\varepsilon_1} \cdots b_n^{\varepsilon_n}$ with $b_1^{\varepsilon_1}, \ldots, b_n^{\varepsilon_n} \in X; \varepsilon_1, \ldots, \varepsilon_n \in \{\pm 1\}$, we define $a * w = aS(w)$ by $aS(b_1)^{\varepsilon_1} \cdots S(b_n)^{\varepsilon_n}$, where $S(b)^{-1}$ denotes the inverse mapping of $S(b)$. The terminology $S(b)$ follows Joyce's paper [27] and $a * w$ $(= a^w)$ follows Fenn and Rourke [15].

Definition 2.1.3 [25] Let X denote a quandle. An automorphism of X is called *inner-automorphism* of X if it is $S(w)$ for a word w.

Definition 2.1.4 [10] Define a relation on X by $a \sim b$ if a is mapped to b by an inner-automorphism of X. The relation $\sim$ is an equivalence relation. The *orbit* of $a \in X$ is the equivalence class of a, which is denoted by $\mathrm{Orb}(a)$. The set of equivalence classes of X by $\sim$ is denoted by $\mathrm{Orb}(X)$. When we regard $\mathrm{Orb}(X)$ as a trivial quandle (see Section 2.2), the projection map $\pi : X \to \mathrm{Orb}(X)$ is a quandle homomorphism. In this case $\mathrm{Orb}(X)$ is called the *orbit quandle* of X.

If $c * b = a$, we write $c = a \,\overline{*}\, b$. Note that if $(X, *)$ is a quandle or a rack, then so is $(X, \overline{*})$. It is called the *dual quandle* of $(X, *)$ [8].

A subset of a quandle that forms a quandle by itself is called a *subquandle*.

2.2 Examples of quandles

- The Trivial Quandle [15]. Any set T with the operation $x * y = x$ for any $x, y \in X$ is a quandle called a *trivial* quandle. If T has n elements, the elements of T can be represented by the numbers $1, 2, \ldots, n$, and T is denoted by T_n.

- The Conjugation Quandle [15]. For any group G, define $g * h$ to be $h^{-1}gh$ for any $g, h \in G$. Then, $(G, *)$ defines the *conjugation quandle*, sometimes written by $conj(G)$.

- The Dihedral Quandle [15]. Let D_{2n} be the dihedral group of order $2n$, which may be taken to be the symmetry group of a regular n-gon. Then, D_{2n} has a presentation

$$D_{2n} = \langle x, y | x^2 = 1 = y^n, xyx = y^{-1} \rangle,$$

where x is a reflection through a fixed vertex and y is a rotation of a regular n-gon through an angle of $\frac{2\pi}{n}$ about its center. The set R_n of all possible reflections is written as $\{a_i = xy^i | i = 0, \ldots, n-1\}$, and it is closed under conjugation. We use the subscripts from $\mathbb{Z}_n$ in the following computations. The operation by conjugation is computed as

$$a_i * a_j = a_j^{-1} a_i a_j = xy^j xy^i xy^j = xy^j y^{-i} y^j = a_{2j-i}.$$

Therefore, we may consider the dihedral quandle as

$$R_n = \{0, 1, 2, \ldots, n-1\}$$

with the operation $a * b \equiv 2b - a \pmod{n}$.

- The Alexander Quandle [8]. Let Λ be the ring of Laurent polynomials $\mathbb{Z}[T, T^{-1}]$ in the variable T. Any Λ-module M has the structure of a quandle with the operation $a * b = Ta + (1 - T)b$ for any $a, b \in M$. We call such a quandle an *Alexander quandle*. In particular, for any Laurent polynomial $h(T)$ such that the highest and lowest terms are invertible in $\mathbb{Z}_n$, $\mathbb{Z}_n[T, T^{-1}]/(h(T))$ is a finite quandle.

If $T = -1$, then the quandle operation of a finite Alexander quandle becomes $a * b \equiv 2b - a \pmod{n}$, which is the operation for dihedral quandles. This value of T is achieved when $h(T) = T + 1$. If $T = 1$, then the quandle operation becomes $a * b = a$, which is the operation for trivial quandles (if a positive integer n is used instead of 0). This value occurs if $h(T) = T - 1$. Thus, both dihedral and trivial quandles may be considered as special cases of Alexander quandles.

2.3 Classification of 4-element quandles

In this section we use new constructions of quandles to describe 4-element quandles, in addition to extensions by cocycles (see Chapter 4).

Let $\mathcal{X} = \{(X_\alpha, *_\alpha) : \alpha \in \Lambda\}$ be a family of racks, where Λ is an index set. A rack W, called the *disjoint union* of $\mathcal{X}$, is defined as follows. As a set, $W = \sqcup_\Lambda \mathcal{X}$. For $x_1 \in X_\alpha$ and $x_2 \in X_\beta$, the rack operation is defined by

$$x_1 * x_2 = \begin{cases} x_1 *_\alpha x_2 & \text{if} \quad \alpha = \beta, \\ x_1 & \text{if} \quad \alpha \neq \beta. \end{cases}$$

It is checked that W is a rack (or quandle) if X_α is for every $\alpha \in \Lambda$. This construction is found in [3, 15].

Another construction is given as follows. Let X_0, X_1 be trivial quandles, and let $\tau : X_1 \to \mathcal{A} \subset S(X_0)$ be a map into an abelian subgroup $\mathcal{A}$ of the permutation group of elements of X_0. The image of τ need not be a subgroup of $\mathcal{A}$. Denote the image by $\tau(k) = \tau_k : X_0 \to X_0$. Let $X = X_0 \cup X_1$ and define a binary operation on X by

$$a * b = \begin{cases} a & \text{if } a, b \in X_0, \text{ or } a \in X_1 \\ \tau_b(a) & \text{if } a \in X_0, \ b \in X_1 \end{cases}$$

Lemma 2.3.1 *The above constructed X is a quandle with $*$ as its operation.*

Proof. The conditions (I) and (II) of Definition 2.1.1 are easily checked case by case. If $a \in X_1$, then the both sides of the self-distributive law (III) are a, and thus the law is satisfied. Axiom (III) is also satisfied, if all a, b, c are in X_0. If $a \in X_0$ and $b, c \in X_1$, then

$$(a * c) * (b * c) = (a * c) * b = (\tau_b \circ \tau_c)(a) = (\tau_c \circ \tau_b)(a) = (a * b) * c,$$

as $\mathcal{A}$ is an abelian group. If $a, b \in X_0$ and $c \in X_1$, then

$$(a * b) * c = \tau_c(a * b) = \tau_c(a) = a * c = (a * c) * (b * c)$$

as the element $b * c \in X_0$ acts trivially on X_0. If $a, c \in X_0$ and $b \in X_1$, then

$$(a * b) * c = \tau_b(a) * c = \tau_b(a) = \tau_b(a * c) = (a * c) * b = (a * c) * (b * c).$$

These exhaust all the cases. □

Example 2.3.2 Let $V_n = T_{n-1} \cup T_1$ for $X_0 = T_{n-1}, \ X_1 = T_1$ in the previous construction, where the right action of T_1 on T_{n-1} is a permutation and T_n denotes the trivial quandle of n elements. Then V_n is a quandle. In particular, there is a quandle V_4 of four elements up to isomorphism, where T_1 acts as a 3-cycle on T_3.

We use the above constructions of quandles as a tool to describe quandles, and as an application we classify 4-element quandles. Quandles with three elements are

R_4

$x * y$	$y = a$	b	c	d
$x = a$	a	a	b	b
b	b	b	a	a
c	d	d	c	c
d	c	c	d	d

$P_3 \sqcup T_1$

$x * y$	$y = a$	b	c	d
$x = a$				a
b		P_3		b
c				c
d				d

Y_4

$x * y$	$y = a$	b	c	d
$x = a$	a	a	a	a
b	b	b	b	b
c	d	d	c	c
d	c	c	d	d

Table 1. Multiplication tables for R_4, $P_3 \sqcup T_1$, and Y_4

classified in [15]: T_3, R_3, and P_3 where P_3 is the homomorphic image of R_4 with even numbers identified to a single element.

Let Y_4 be the quandle defined by the multiplication table to the right of Table 1. We remark here that using the extensions of quandles $E(X, A, \phi)$ are defined in Section 4.1, R_4 is described as $E(T_2, \mathbb{Z}_2, \chi_{0,1} + \chi_{1,0})$ and the Y_4 is described as $E(T_2, \mathbb{Z}_2, \chi_{0,1})$. See Section 4.1 for more details on notation.

Proposition 2.3.3 *Any 4-element quandle is isomorphic to exactly one of the quandles in the following list:* T_4, $\mathbb{Z}_2[T, T^{-1}]/(T^2 + T + 1)$, V_4, $R_3 \sqcup T_1$, $P_3 \sqcup T_1$, R_4, Y_4.

Thus extensions, their homomorphic images, and disjoint unions are expected to be effective ways of describing small quandles. The proof of Proposition 2.3.3 follows from the following sequence of lemmas.

Lemma 2.3.4 *Let X be a quandle with four elements. If there are $a, b \in X$ such that $b * a = b$, then there is a trivial subquandle $T_2 \subset X$.*

Proof. If $a * b = a$, then $T_2 = \{a, b\}$ is a trivial subquandle. Hence, assume that $a * b = c$, where c is an element of X distinct from a and b. Then, $c * a = (a * b) * a = a * (b * a) = a * b = c$.

(Case 1) $c * b = a$. Then, $a * c = (c * b) * (a * b) = (c * a) * b = c * b = a$, so that X would contain a subquandle $T_2 = \{a, c\}$ (if this information completes to form a quandle X).

(Case 2) $c * b = d$. Then, $a * c = (d * b) * (a * b) = (d * a) * b = d * b = a$, so we obtain $T_2 = \{a, c\}$. Here, $d * a = d$ since the action of a is a permutation, and we already have $b * a = b$ and $c * a = c$. Similarly, $d * b = a$ since we already have $a * b = c$ and $c * b = d$. □

Lemma 2.3.5 *If a 4-element quandle X does not have a trivial subquandle T_2 of two elements, then X is isomorphic to $\mathbb{Z}_2[T, T^{-1}]/(T^2 + T + 1)$.*

Proof. Let $a, b \in X$ be distinct elements. By Lemma 2.3.4, we may assume that $b * a = c$, where $c \in X$ is distinct from a and b. If $c * a = b$, then $d * a = d$, which implies $T_2 \subset X$ by Lemma 2.3.4. Hence we assume that $c * a = d$. Then, we have $d * a = b$ considering that the action by a from the right is a permutation.

(Case 1) Suppose $a * b = a$. By Lemma 2.3.4 there is $T_2 \subset X$.

(Case 2) Suppose $a * b = c$. We have $a * c = (a * a) * (b * a) = (a * b) * a = c * a = d$. Then, by considering the action by b, we have that $c * b$ can be either a or d. Similarly, by considering the action by c, we have that $b * c$ can be either a or b. Since $c * b = (b * a) * b = b * (a * b) = b * c$ then $c * b = b * c = a$, which contradicts $b * c = (d * a) * (b * a) = (d * b) * a = d * a = b$. Therefore, this choice does not yield a quandle.

(Case 3) Suppose $a * b = d$. By Lemma 2.3.4 and actions by elements, this condition uniquely determines a quandle isomorphic to $\mathbb{Z}[T, T^{-1}]/(T^2 + T + 1)$. Specifically, the isomorphism is given by $a \mapsto 0$, $b \mapsto 1$, $c \mapsto T$, and $d \mapsto 1 + T$. □

Lemma 2.3.6 *Suppose a 4-element quandle X has a subquandle isomorphic to R_3. Then X is isomorphic to $R_3 \sqcup T_1$.*

Proof. Let $X = \{a, b, c, d\}$ and $R_3 = \{a, b, c\}$. Since the right action is injective, we have $d * x = d$ for $x = a, b, c$. Also, for any $x = a, b, c$, we have $x * d = x * (d * x) = (x * d) * x$ and there is a unique solution y for $y * x = x$, namely, $y = x$, and we obtain $x * d = x$. The result follows. □

Lemma 2.3.7 *Suppose a non-trivial quandle $X = \{a, b, c, d\}$ has trivial subquandles $T_2 \cong \{a, b\}$ and $T_2 \cong \{c, d\}$. Then, X is isomorphic to either $P_3 \sqcup T_1$, R_4 or Y_4.*

Proof. If the right action of $\{a, b\}$ on $\{c, d\}$ is trivial, then $c*a = c, d*a = d, c*b = c, d*b = d$. Then, there are are two possibilities, either $a*c = a$ or $a*c = b$. When $a*c = a$ and $a*d = a$, then X is isomorphic to T_4. Otherwise, when $a*c = a$ and $a*d = b$ we have that X is isomorphic to $P_3 \sqcup T_1$. If $a*c = b$ and $a*d = a$, then X is isomorphic to $P_3 \sqcup T_1$. Otherwise, for $a*c = b$ and $a*d = b$, X is isomorphic to Y_4.

Suppose the right action of $\{a, b\}$ on $\{c, d\}$ is not trivial. There are three such cases: (Case 1) $c*a = c, d*a = d, c*b = d, d*b = c$, in which case, $a*d = (a*b)*(c*b) = (a*c)*b$. If $a*c = a$, then $a*d = a*b = a$, and $P_3 \sqcup T_1$ results. If $a*c = b$, then $a*d = b*b = b$, and the quandle X has the following multiplication table:

$x*y$	$y=$ a	b	c	d
$x=a$	a	a	b	b
b	b	b	a	a
c	c	d	c	c
d	d	c	d	d

However, $(c*a)*c = c$ and $c*(a*c) = d$, so that this is not a quandle.

(Case 2) $c*a = d, d*a = c, c*b = c, d*b = d$, which is similar to the case above.

(Case 3) $c*a = d, d*a = c, c*b = d, d*b = c$, in which case, $a*d = a*(c*a) = (a*c)*a$. If $a*c = a$, then $a*d = a*a = a$, and the quandle Y_4 appears. If $a*c = b$, then $a*d = b*a = b$, in which case $X \cong R_4$. □

Lemma 2.3.8 *Suppose $X = \{a, b, c, d\}$ has a trivial subquandle $T_2 = \{a, b\}$, and $\{c, d\}$ does not form a trivial quandle. Then X is isomorphic to $R_3 \sqcup T_1$ or V_4.*

Proof. There are three cases for the right actions of $\{a, b\}$ on c. (Case 1) $c * a = c, c * b = c$. It follows that $d * a = d, d * b = d$. If there is a subquandle T_3, then X is isomorphic to V_4. Therefore, assume that there is no subquandle T_3. Since a and b play symmetric roles, there are two cases: $d * c = a$, and $d * c = d$. If $d * c = a$, then $a * c = b$ and $b * c = d$ (otherwise it falls into the next case, by switching b and d). Then,

$$a * d = (d * c) * (b * c) = (d * b) * c = d * c = a$$

and

$$b * d = (a * c) * (b * c) = (a * b) * c = a * c = b,$$

thus X has a subquandle T_3. If $d * c = d$, then $a * c = b$ and $b * c = a$, since otherwise it has a subquandle T_3. Note that in this case $a * d = (b * c) * d = (b * d) * (c * d)$, and $b * d = (a * c) * d = (a * d) * (c * d)$. Hence, $c * d$ transposes $a * d$ and $b * d$ by right action. This implies that $c * d = c$, then $\{c, d\}$ is trivial.

(Case 2) $c * a = c$ and $c * b = d$. If $d * c = a$, then $c * d = (d * c) * b = a * b = a$, and $a * c = (c * d) * c = c * (d * c) = c * a = c$, which contradicts the action of c on the remaining elements by permutation. Hence, this choice does not yield a quandle. If $d * c = b$, then $c * d = (d * c) * b = b$, $b * d = (d * c) * d = d * (c * d) = d * b = c$, and $b * c = (c * d) * c = c * (d * c) = c * b = d$, and $\{b, c, d\}$ form a subquandle isomorphic to R_3.

Suppose $d * c = d$. Then, $c * d = (d * c) * b = d * b = c$, and $\{c, d\}$ form T_2. We observe that the case $c * a = d$ and $c * b = c$ is similar.

(Case 3) $c * a = d, c * b = d$. Assume $d * c = d$. Then, $c * d = (d * c) * a = d * a = c$, thus $\{c, d\}$ is a trivial quandle. Hence, either $d * c = a$ or b. One case is similar to the other, so assume $d * c = a$. It follows that $c * d = (d * c) * a = a$, as well as $a * c = (c * d) * c = c * (d * c) = c * a = d$, and $a * d = (d * c) * d = d * (c * d) = d * a = c$.

Hence, there is a subquandle $R_3 = \{a, c, d\}$. In fact, since $c * b = d$, this case contradicts Lemma 2.3.6 and does not form a quandle. □

It is easy to see that the quandles listed in Proposition 2.3.3 are not isomorphic to each other, by looking at subquandles or orbits. This is proved in the following lemma.

Lemma 2.3.9 *The quandles* T_4, $\mathbb{Z}_2[T, T^{-1}]/(T^2 + T + 1)$, V_4, $R_3 \sqcup T_1$, $P_3 \sqcup T_1$, R_4, Y_4 *are pairwise non-isomorphic.*

Proof. The quandles listed above, except for T_4, are not trivial quandles. The only proper subquandle of $\mathbb{Z}_2[T, T^{-1}]/(T^2+T+1)$ is T_1, while the remaining quandles have proper subquandles of higher order. The quandle V_4 is the only quandle from the list that has a proper subquandle isomorphic to a trivial quandle T_3. By Lemma 2.3.8, if a quandle contains a subquandle isomorphic to T_3, then that quandle must be V_4. The largest subquandle that is isomorphic to a trivial quandle is T_2 for the remaining quandles, $R_3 \sqcup T_1$, $P_3 \sqcup T_1$, Y_4, and R_4. By Lemma 2.3.6, $R_3 \sqcup T_1$ is the only quandle that contains a subquandle isomorphic to R_3.

The quandle $P_3 \sqcup T_1$ has one element that acts (from the right) trivially on the remaining elements. This can be seen from a multiplication table for $P_3 \sqcup T_1$, shown in the middle of Table 1, where d acts trivially on a, b, c and d. However, R_4 has no such element, as shown in the multiplicaton table corresponding to R_4 in Table 1.

A multiplication table for the quandle Y_4 is given in Table 1. There are two elements, c and d, that act trivially on the other three elements. These result the three element subquandles shown in Table 2, which differ from the quandle P_3, also shown in Table 2. Hence, Y_4 is not isomorphic to $P_3 \sqcup T_1$.

By Lemma 2.3.7, R_4 and Y_4 appear when we have two trivial subquandles of order two. The quandle Y_4 has an element a such that all the elements act on a trivially. The quandle R_4, however, does not have such an element. Therefore, the two extensions are not isomorphic. □

$T_1 = \{c\}$	$y =$		
$x * y$	a	b	d
$x = a$	a	a	a
b	b	b	b
d	c	c	d

$T_1 = \{d\}$	$y =$		
$x * y$	a	b	c
$x = a$	a	a	a
b	b	b	b
c	d	d	c

P_3	$y =$		
$x * y$	0	1	3
$x = 0$	0	0	0
1	3	1	1
3	1	3	3

Table 2. Comparison of 3-element quandles of Y_4 with P_3

Recently, Graña [21] classified indecomposable racks (an extension of a rack with no proper quotients) of order p^2, where p is prime. Nelson [36] obtained a procedure for classifying finite Alexander quandles in terms of their submodules.

An important problem in knot theory is to determine which knots are equivalent and which are not. One way of approaching this problem is by coloring knot diagrams. In the next section we see that Fox's n-coloring generalizes to colorings by quandles.

2.4 Colorings of knot diagrams by quandles

The motivation for studying quandles partly arises from knot and link diagrams. Consider an oriented knot diagram, with co-orientaion given by the right hand rule. Let X be a quandle. It is possible to label each arc of the knot diagram by a quandle element as follows.

Definition 2.4.1 [28, 35] A *coloring* of an oriented classical link diagram is a function $\mathcal{C} : R \to X$, where X is a fixed quandle and R is the set of over-arcs in the diagram, satisfying the condition depicted in Figure 9. In the figure, a crossing with over-arc, β, has color $\mathcal{C}(\beta) = b \in X$. The under-arcs are called α and γ from top to bottom; the normal (co-orientation) of the over-arc β points from α to γ. Then, it is required that if $\mathcal{C}(\alpha) = a$ and $\mathcal{C}(\gamma) = c$, then $c = a * b$.

The quandle element $\mathcal{C}(r)$ assigned to an arc r by a coloring $\mathcal{C}$ is called a *color* of the arc. Note that locally the colors do not depend on the orientation of the under-arc. This definition of colorings on knot diagrams has been known, see [15, 19] for

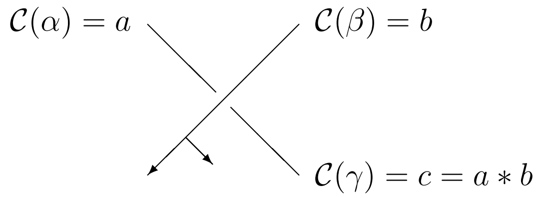

Figure 9. Quandle relation at a crossing

example. The set of colorings of a knot diagram K by a quandle X is denoted by $\mathrm{Col}_X(K)$. The cardinality of all such colorings is denoted by $|\mathrm{Col}_X(K)|$. Henceforth, all the quandles that are used to color diagrams will be finite.

Example 2.4.2 Let X be R_3 (the dihedral quandle of order 3) with quandle operation $a * b = 2b - a$, where $a, b, c \in X$. Then, the trefoil K is colored by R_3 as depicted in Figure 10. It is seen that $|\mathrm{Col}_X(K)| = 9$.

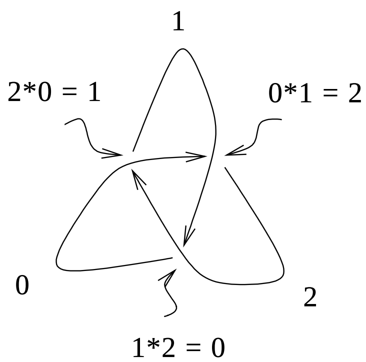

Figure 10. A trefoil colored by the quandle R_3

Proposition 2.4.3 [8] The number of colorings $|\mathrm{Col}_X(K)|$ is a knot invariant.

Consider the dihedral quandle of n elements, R_n. At a crossing we have the colors $\mathcal{C}(\lambda_r) = a$, $\mathcal{C}(\lambda_q) = b$ and $\mathcal{C}(\lambda_s) = a * b$. If $a = b$, then $a * b = a * a = a$. Otherwise, $a * b \equiv 2b - a \pmod{n}$. We see that the quandle operation is equivalent to the requirement of the n-colorability condition defined in Definition 1.3.4. Therefore, an n-coloring is a quandle coloring of a link by R_n. The n-colorability is equivalent to $|\mathrm{Col}_X(K)| > |X|$. The classical result that a knot is non-trivially (Fox) n-colorable (for n prime) if $n|\Delta(-1)$, where $\Delta(T)$ denotes the Alexander polynomial, has been generalized by Inoue [24] to the following:

Theorem 2.4.4 [24] Let $\Delta_K^{(i)}(T)$ denote the greatest common divisor of all $(n-i-1)$ minor determinants of the presentation matrix for the knot module obtained via the Fox calculus [29]. Let p be a prime number and J an ideal of the ring $\Lambda_p = \mathbb{Z}_p[T, T^{-1}]$. For each $i \geq 0$, put $e_i(T) = \Delta_K^{(i)}(T)/\Delta_K^{(i+1)}(T)$. Then, the number of colorings by the Alexander quandle Λ_p/J is equal to the cardinality of the module $\Lambda_p/J \oplus \oplus_{i=0}^{n-2}\{\Lambda_p/(e_i(T), J)\}$.

Classical knots have fundamental quandles that are defined via generators and relations. The definitions of the fundamental quandle are found in [15, 27, 28, 33], for example. Here we give a brief description.

A *presentation* $\langle\ S \mid R\ \rangle$ of a rack or a quandle is defined in a similar way as for groups as follows [15, 27, 28]. The free rack $FR(S)$ is as a set $S \times F(S)$, where $F(S)$ is the free group on S. The rack operation is defined by $(a, w) * (b, z) = (a, wz^{-1}bz)$. The set of relations R is given and it consists of identities of the form $x = y$. Define a congruence relation $\sim$ on a rack Y to be an equivalence relation such that if $a \sim b \in Y$, then $a * c \sim b * c$ and $c * a \sim c * b$, for any $c \in Y$. Let $\sim$ on $FR(S)$ be the smallest congruence containing R, that is, $\sim$ is the smallest congruence such that if $x = y$ is in R, then $x \sim y$. Then, the rack with given presentation is defined by

$$X = \langle\ S \mid R\ \rangle = FR(S)/\sim .$$

For a presentation of a quandle we require $a * a \sim a$, for any $a \in FR(S)$.

The fundamental quandle is defined in a similar way to the fundamental group, as follows. The generators, $x_1, \ldots, x_m$, are assigned to arcs of a given knot or link diagram. A relation is assigned to each crossing as depicted in Figure 9. Specifically, if x_i is the generator assigned to the under-arc away from which the normal of the over-arc points, x_k is assigned to the other under-arc, and x_j is assigned to the over-arc, then the relation $r_h : x_i * x_j = x_k$ is assigned to obtain the set of relations $r_1, \ldots, r_n$ from all the crossings (see Figure 11).

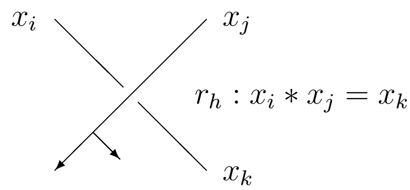

Figure 11. Wirtinger relation of the fundamental quandle

The quandle $Q(K)$ defined by the thus obtained presentation

$$\langle\ x_1, \ldots, x_m \mid r_1, \ldots, r_n\ \rangle$$

is called the *fundamental quandle* of a classical knot K, or simply the *quandle of* K.

Note the similarity of the Wirtinger presentation to the fundamental group [30], where the relation $x_i * x_j = x_k$ corresponds to $x_j^{-1} x_i x_j = x_k$ in the fundamental group.

Example 2.4.5 From a standard diagram of a trefoil K as depicted in Figure 6, we obtain a presentation for the fundamental quandle of the trefoil

$$Q(K) = \langle\ x_1,\ x_2,\ x_3 \mid x_3 * x_2 = x_1,\ x_2 * x_1 = x_3,\ x_1 * x_3 = x_2\ \rangle.$$

A coloring of a classical knot diagram by a quandle X gives rise to a quandle homomorphism from the fundamental quandle to the quandle X [24]. Knot diagrams colored by quandles can be used to study quandle homology groups. This viewpoint was developed in [16, 18, 23] for rack homology and homotopy, and generalized to quandle homology in [12]. Quandle homomorphisms and virtual knots are applied to this homology theory [10]. State-sum invariants using quandle cocycles as Boltzmann weights are defined [8] and computed for important families of classical knots and knotted surfaces [9]. The invariants were applied to studying knots, for example, in detecting non-invertible knotted surfaces [8].

The next chapter discusses homology and cohomology theories of quandles, and cocycle knot invariants.

CHAPTER 3

COHOMOLOGY THEORY OF QUANDLES AND COCYCLE KNOT INVARIANTS

In this chapter, following [10] and [25], we first define the homology and cohomology theories for quandles needed to understand this work. Then, motivated by the main problem of knot theory of distinguishing different knots, we consider new link invariants. The new invariants are defined using colorings of link diagrams by quandles and quandle cocycles.

3.1 Homology and cohomology of quandles

Originally, rack homology and homotopy theories were defined and studied in [16], and a modification to a quandle homology theory was given in [8] to define a knot invariant in a state-sum form. The most general form of the quandle homology known to date is given in [2]. The cohomology theory has found applications to the classification of Nichols algebras [2]. Computations are found in [9, 10, 14, 31, 34], for example.

For $n > 0$, let $C^{\mathrm{R}}_n(X)$ be the free abelian group generated by n-tuples $(x_1, \ldots, x_n)$ of elements of a quandle X. If $n \leq 0$, let $C^{\mathrm{R}}_n(X) = 0$. Define a homomorphism $\partial_n : C^{\mathrm{R}}_n(X) \to C^{\mathrm{R}}_{n-1}(X)$ by

$$\begin{aligned} \partial_n(x_1, x_2, \ldots, x_n) &= \sum_{i=2}^{n} (-1)^i \left[(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \right. \\ &\quad \left. - (x_1 * x_i, x_2 * x_i, \ldots, x_{i-1} * x_i, x_{i+1}, \ldots, x_n) \right], \end{aligned}$$

for $n \geq 2$ and $\partial_n = 0$ for $n \leq 1$. Direct calculations show that $\partial \circ \partial = 0$. Then, $C^{\mathrm{R}}_*(X) = \{C^{\mathrm{R}}_n(X), \partial_n\}$ is a chain complex.

Let $C_n^{\mathrm{D}}(X)$ be the subset of $C_n^{\mathrm{R}}(X)$ generated by n-tuples $(x_1, \ldots, x_n)$ with $x_i = x_{i+1}$ for some $i \in \{1, \ldots, n-1\}$ if $n \geq 2$; otherwise let $C_n^{\mathrm{D}}(X) = 0$. If X is a quandle, then $\partial_n(C_n^{\mathrm{D}}(X)) \subset C_{n-1}^{\mathrm{D}}(X)$ and $C_*^{\mathrm{D}}(X) = \{C_n^{\mathrm{D}}(X), \partial_n\}$ is a sub-complex of $C_*^{\mathrm{R}}(X)$. Put $C_n^{\mathrm{Q}}(X) = C_n^{\mathrm{R}}(X)/C_n^{\mathrm{D}}(X)$ and $C_*^{\mathrm{Q}}(X) = \{C_n^{\mathrm{Q}}(X), \partial'_n\}$, where ∂'_n is the induced homomorphism. Henceforth, all boundary maps will be denoted by ∂_n.

For an abelian group G, define the chain and cochain complexes

$$
\begin{aligned}
C_*^{\mathrm{W}}(X;G) &= C_*^{\mathrm{W}}(X) \otimes G, \qquad && \partial = \partial \otimes \mathrm{id}; \\
C^*_{\mathrm{W}}(X;G) &= \mathrm{Hom}(C_*^{\mathrm{W}}(X), G), \qquad && \delta = \mathrm{Hom}(\partial, \mathrm{id})
\end{aligned}
$$

in the usual way, where W = D, R, Q.

Definition 3.1.1 [8] The nth *quandle homology group* and the nth *quandle cohomology group* of a quandle X with coefficient group G are

$$H_n^{\mathrm{Q}}(X;G) = H_n(C_*^{\mathrm{Q}}(X;G)), \quad H^n_{\mathrm{Q}}(X;G) = H^n(C^*_{\mathrm{Q}}(X;G)).$$

The cycle and boundary groups are denoted by $Z_n^{\mathrm{W}}(X;G)$ and $B_n^{\mathrm{W}}(X;G)$, so that $H_n^{\mathrm{W}}(X;G) = Z_n^{\mathrm{W}}(X;G)/B_n^{\mathrm{W}}(X;G)$, where W is one of D, R, or Q. The cocycle and coboundary groups are denoted by $Z^n_{\mathrm{W}}(X;G)$ and $B^n_{\mathrm{W}}(X;G)$, respectively, so that

$$H^n_{\mathrm{W}}(X;G) = Z^n_{\mathrm{W}}(X;G)/B^n_{\mathrm{W}}(X;G).$$

The coefficient group G is omitted if $G = \mathbb{Z}$.

In the following sections we discuss cocycle knot invariants from the viewpoint of coloring knot diagrams by quandles.

3.2 Cocycle knot invariants

The notion of state-sum originated from statistical mechanics. From the mathematical point of view, the state-sum invariants have been studied in relation to the Jones polynomial and generalizations (see, for example, [35]). State-sum invariants using quandle cocycles as Boltzmann weights were defined in [8] and computed for important families of classical knots and knotted surfaces in [9]. The invariants were applied to studying knots, for example, in detecting non-invertible knotted surfaces [8]. Here we describe such invariants.

In Figure 12, the two possible oriented and co-oriented crossings are depicted. On the left the crossing is positive and on the right is negative. Let τ denote a crossing and $\mathcal{C}$ denote a coloring. Let r be the over-arc at τ, and r_1, r_2 be under-arcs such that the normal to r points from r_1 to r_2. Let $x = \mathcal{C}(r_1)$ and $y = \mathcal{C}(r)$. Pick a quandle 2-cocycle $\phi \in Z^2(X;G)$. Then, define $B(\tau, \mathcal{C}) = \phi(x,y)^{\varepsilon(\tau)}$ to be the *Boltzmann weight*, where $\varepsilon(\tau) = 1$ or -1, if τ is positive or negative crossing, respectively.

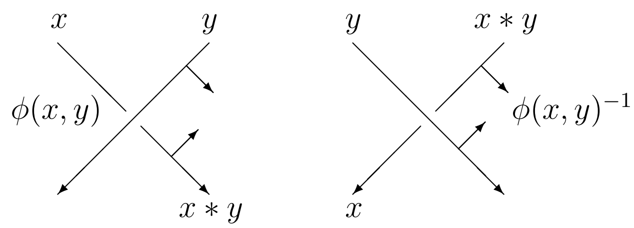

Figure 12. Weights for positive and negative crossings

Definition 3.2.1 [8] The *state-sum*, or *partition function*, is defined by

$$\Phi_\phi(K) = \sum_{\mathcal{C}} \prod_{\tau} B(\tau, \mathcal{C}).$$

The product is taken over all crossings of the given diagram, and the sum is taken over all possible colorings.

Note that the state-sum depends on the choice of the 2-cocycle ϕ. The values of the state-sum are taken to be in the group ring $\mathbb{Z}[G]$ where G is the coefficient group (written multiplicatively). This is proved [8] to be a knot invariant, called the *(quandle) cocycle invariant.*

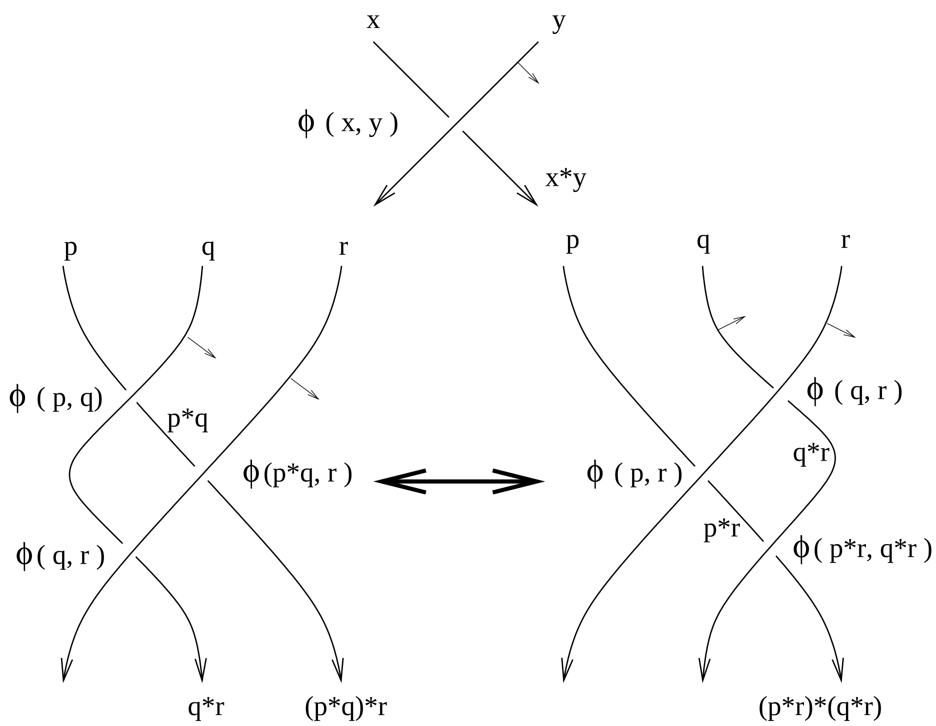

Figure 13. Type III move and the quandle identity

Figure 13 shows the invariance of the state-sum under the Reidemeister type III move. The products of cocycles, equated before and after the move, is the 2-cocycle condition

$$\phi(p,q)\phi(p*q,r) = \phi(p,r)\phi(p*r,q*r).$$

Example 3.2.2 Consider the diagram of the (4,2)-torus link K as depicted in Figure 14. Let $X = R_4$ (the dihedral quandle of order 4), and let $G = \mathbb{Z}_2$ be the coefficient group with generator t. Define the characteristic function by

$$\chi_x(y) = \begin{cases} t & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases},$$

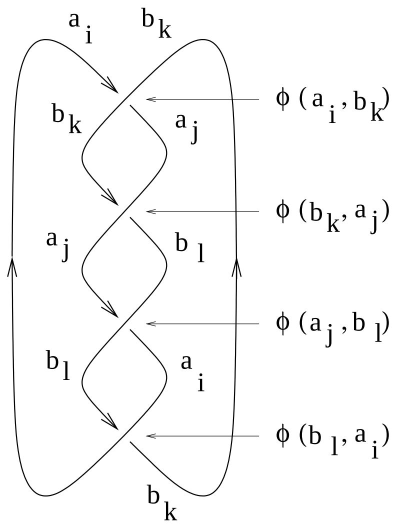

Figure 14. The (4,2)-torus link

where x, y denote pairs (a, b) of quandle elements. Let ϕ be a function $\phi : \{0, 1, 2, 3\} \times \{0, 1, 2, 3\} \to \{1, t^{\pm 1}, t^{\pm 2}, \ldots\}$ given by $\phi = \chi_{0,1} + \chi_{0,3}$. It is known that ϕ satisfies the 2-cocycle condition [8]. With this cocycle, the color $a_i = 0, b_k = 1$ in Figure 14 contributes t to the state-sum. Note that the choice of colors for a_i and b_k uniquely determines a coloring of L. There are 16 colorings in total. The state-sum invariant, as defined in Definition 3.2.1, is computed by considering all possible colorings as follows:

$$\begin{aligned} \sum_{\mathcal{C}} \prod_{\tau} B(\tau, \mathcal{C}) &= \sum_{\mathcal{C}} \phi(a_i, b_k)\phi(b_k, a_j)\phi(a_j, b_\ell)\phi(b_\ell, a_i) \\ &= \sum_{\mathcal{C}} t^q, \ q \in \mathbb{Z}_2 \\ &= 8 + 8t. \end{aligned}$$

Therefore, we find that the state-sum is $\Phi_\phi(T(4, 2)) = 8 + 8t$.

3.3 Variations of cocycle knot invariants

Motivated by state-sum invariants of knot diagrams, we generalize cocycle knot invariants to invariants of classical links component-wise.

Definition 3.3.1 Let $L = K_1 \cup \cdots \cup K_r$ be a link diagram and let $\mathcal{T}_i$, $i = 1, \ldots, r$, denote the set of crossings at which the under-arcs belong to the component K_i. Define the state-sum $\Phi_i(L) = \sum_{\mathcal{C}} \prod_{\tau \in \mathcal{T}_i} B(\tau, \mathcal{C})$ for each $i = 1, \ldots, r$. The vector $\vec{\Phi}(L) = (\Phi_i(L))_{i=1}^r$ of the state-sum invariants is called the *component-wise* (quandle) cocycle invariant of L.

It was observed [5] that $\vec{\Phi}(L) = (\sum_{\mathcal{C}} \prod_{\tau \in \mathcal{T}_i} B(\tau, \mathcal{C}))_{i=1}^r$ is a link invariant, strictly stronger than the single state-sum.

Example 3.3.2 We caclulate the component-wise cocycle invariant, as given in Definition 3.3.1, for the (4,2)-torus link shown in Figure 14. The link has two components; let K_1 be the component on the left of the figure and K_2 the component on the right. Let $\tau_1, \tau_2, \tau_3, \tau_4$ be the four crossings from top to bottom, and $\phi \in Z^2_{\rm Q}(R_4; \mathbb{Z}_2)$ be given by $\chi_{0,1} + \chi_{0,3}$. Let t be the generator of the group $\mathbb{Z}$. Then, $\mathcal{T}_1 = \{\tau_1, \tau_3\}$ and $\mathcal{T}_2 = \{\tau_2, \tau_4\}$.

For the first component, K_1, we have

$$\begin{aligned} \Phi_1(L) &= \sum_{\mathcal{C}} \prod_{\tau \in \mathcal{T}_1} B(\tau, \mathcal{C}) \\ &= \sum_{\mathcal{C}} \phi(a_i, b_k)\phi(a_j, b_\ell) \\ &= 12 + 4t. \end{aligned}$$

Similarly for the second component K_2 we get $\Phi_2(L) = \sum_{\mathcal{C}} \phi(b_k, a_j)\phi(b_\ell, a_i) = 12+4t$. Therefore, the cocycle invariant of the (4,2)-torus link is given by

$$\vec{\Phi}(T(4,2)) = (\Phi_1(T(4,2)), \Phi_2(T(4,2))) = (12 + 4t, 12 + 4t).$$

Lopes [32] observed that the family $\{\prod_\tau B(\tau, \mathcal{C})\}_{\mathcal{C} \in \mathrm{Col}_X(K)}$, where $\mathrm{Col}_X(K)$ denotes the set of colorings of the knot K, is a knot invariant without taking summation. In particular, infinite quandles can be used for coloring in this case. Moreover, he defined for r-component links the vector version $(\{\prod_{\tau \in \mathcal{T}_i} B(\tau, \mathcal{C})\}_{\mathcal{C}})_{i=1}^{r}$.

We combine the above variations together with the one given by Definition 3.3.1 to define the following generalized cocycle invariant.

Definition 3.3.3 [7] Let X be a quandle, $\phi \in Z^2_{\mathrm{Q}}(X; A)$, where A is an abelian group, let $\mathcal{C}$ be a coloring of L by X, and $B(\tau, \mathcal{C})$ the Boltzmann weight at a crossing τ. Let $L = K_1 \cup \cdots \cup K_r$ be a link and $\mathcal{T}_i$, $i = 1, \ldots, r$, be the set of crossings of L such that the under-arcs belong to K_i. Define

$$\vec{\Psi}(L) = \left\{ \left(\prod_{\tau \in \mathcal{T}_1} B(\tau, \mathcal{C}), \ldots, \prod_{\tau \in \mathcal{T}_r} B(\tau, \mathcal{C}) \right) \right\}_{\mathcal{C} \in \mathrm{Col}_X(L)},$$

where $\mathrm{Col}_X(L)$ denotes the set of colorings, i.e. $\mathrm{Col}_X(L) = \{\mathcal{C}\}$.

This version of a family of vectors is potentially stronger than Lopes's version of a vector of families. For example, the two distinct families of vectors $\{(1, t), (t, 1)\}$ and $\{(1, 1), (t, t)\}$ give rise to the same vector of families $(\{1, t\}, \{1, t\})$. As an example, we evaluate the invariant for the $(4, 2)$-torus link.

Example 3.3.4 We apply Definition 3.3.3 to Figure 14 of the $L = (4, 2)$-torus link. The link $L = K_1 \cup K_2$, where K_1 is the component on the left of the figure, and K_2 is the component on the right. Let X be the quandle $X = R_4$ and the cocycle $\phi = \chi_{0,1} + \chi_{0,3}$, where $\phi \in Z^2_{\mathrm{Q}}(R_4; \mathbb{Z}_2)$, with $\mathbb{Z}_2 = \{1, t\}$. The generalized cocycle invariant is calculated to be

$$\begin{aligned} \vec{\Psi}(L) &= \left\{ \left(\prod_{\tau \in \mathcal{T}_1} B(\tau, \mathcal{C}), \prod_{\tau \in \mathcal{T}_2} B(\tau, \mathcal{C}) \right) \right\}_{\mathcal{C} \in \mathrm{Col}_X(L)} \\ &= \{(\phi(a_i, b_k)\phi(a_j, b_\ell), \phi(b_k, a_j)\phi(b_\ell, a_i))\}_{\mathrm{Col}_X(L)} \\ &= \{\underbrace{(1,1), \ldots, (1,1)}_{8 \text{ copies}}, \underbrace{(1,t), \ldots, (1,t)}_{4 \text{ copies}}, \underbrace{(t,1), \ldots, (t,1)}_{4 \text{ copies}}\}. \end{aligned}$$

CHAPTER 4
EXTENSIONS OF QUANDLES BY 2-COCYCLES

In this chapter we discuss abelian extensions of quandles. Constructions of extensions of quandles using cocycles were given in [11], which are similar to extensions of groups using group cocycles [4]. We develop methods of constructing cocycles from extensions. This is the opposite direction of [11] where an extension was constructed from a 2-cocycle.

4.1 Abelian extensions

For a quandle X, an abelian group A, and a 2-cocycle $\phi \in Z^2_{\rm Q}(X;A)$, the *abelian extension* $E = E(X, A, \phi)$ was defined in [11] as the set $A \times X$, with the quandle operation defined by $(a_1, x_1) * (a_2, x_2) = (a_1\phi(x_1, x_2), x_1 * x_2)$. Here, the abelian group operation of A in the first factor is denoted by multiplicative notation. The following lemma is the converse of the fact proved in [11] that $E(X, A, \phi)$ is a quandle.

Lemma 4.1.1 *Let X, E be finite quandles, and A be a finite abelian group written multiplicatively. Suppose there exists a bijection $f : E \to A \times X$ with the following property. There exists a function $\phi : X \times X \to A$ such that for any $e_i \in E$ $(i = 1, 2)$, if $f(e_i) = (a_i, x_i)$, then $f(e_1 * e_2) = (a_1\phi(x_1, x_2), x_1 * x_2)$. Then, $\phi \in Z^2_{\rm Q}(X;A)$.*

Proof. For any $x \in X$ and $a \in A$, there is $e \in E$ such that $f(e) = (a, x)$, and

$$(a, x) = f(e) = f(e * e) = (a\phi(x, x), x),$$

so that we have $\phi(x, x) = 1$ for any $x \in X$.

By identifying $A \times X$ with E by f, the quandle operation $*$ on $A \times X$ is defined for any (a_i, x_i), $i = 1, 2$, by

$$(a_1, x_1) * (a_2, x_2) = (a_1\phi(x_1, x_2), x_1 * x_2).$$

Since $A \times X$ is quandle isomorphic to E under this $*$, we have

$$\begin{aligned}
&[(a_1, x_1) * (a_2, x_2)] * (a_3, x_3) \\
&= (a_1\phi(x_1, x_2), x_1 * x_2) * (a_3, x_3) \\
&= (a_1\phi(x_1, x_2)\phi(x_1 * x_2, x_3), (x_1 * x_2) * x_3),
\end{aligned}$$

and

$$\begin{aligned}
&[(a_1, x_1) * (a_3, x_3)] * [(a_2, x_2) * (a_3, x_3)] \\
&= (a_1\phi(x_1, x_3), x_1 * x_3) * (a_2\phi(x_2, x_3), x_2 * x_3) \\
&= (a_1\phi(x_1, x_3)\phi(x_1 * x_3, x_2 * x_3), (x_1 * x_3) * (x_2 * x_3))
\end{aligned}$$

are equal for any (a_i, x_i), $i = 1, 2, 3$. Hence, ϕ satisfies the 2-cocycle condition. □

Then Lemma 4.1.1 implies that, under the same assumptions, we have $E = E(X, A, \phi)$, where $\phi \in Z^2_{\mathrm{Q}}(X; A)$. Next we identify such examples.

Theorem 4.1.2 *For any positive integers q and m, $U_{m+1} = \mathbb{Z}_{q^{m+1}}[T, T^{-1}]/(T - 1 + q)$ is an abelian extension $E = E(\mathbb{Z}_{q^m}[T, T^{-1}]/(T - 1 + q), \mathbb{Z}_q, \phi)$ of $X = U_m = \mathbb{Z}_{q^m}[T, T^{-1}]/(T - 1 + q)$ for some cocycle $\phi \in Z^2_{\mathrm{Q}}(X; \mathbb{Z}_q)$.*

Proof. Represent the elements of $\mathbb{Z}_{q^{m+1}}$ by $\{0, 1, \ldots, q^{m+1} - 1\}$ and express them in their q-ary expansion:

$$A = A_m q^m + \cdots + A_1 q + A_0 \in \mathbb{Z}_{q^{m+1}},$$

where $0 \leq A_j < q$, $j = 0, \ldots, m$. With this convention, the A_j's are uniquely determined integers. Define $f : E \to \mathbb{Z}_q \times X$ by

$$f(A) = (A_m \pmod{q},\ \overline{A} \pmod{q^m}),$$

where $\overline{A} = \sum_{j=0}^{m-1} A_j q^j$. Then, for $A, B \in \mathbb{Z}_{q^{m+1}}$, the quandle operation is computed in $\mathbb{Z}_{q^{m+1}}$ by

$$\begin{aligned}
A * B &= TA + (1-T)B \\
&= (1-q)(A_m q^m + \cdots + A_1 q + A_0) \\
&\quad + q(B_m q^m + \cdots + B_1 q + B_0) \\
&= (A_m - A_{m-1} + B_{m-1})q^m + (A_{m-1} - A_{m-2} + B_{m-2})q^{m-1} \\
&\quad + \ldots + (A_1 - A_0 + B_0)q + A_0 \\
&= (A_m - A_{m-1} + B_{m-1})q^m + \sum_{j=0}^{m-1}(A_j - A_{j-1} + B_{j-1})q^j,
\end{aligned}$$

where A_{-1}, B_{-1} are understood to be zeros in the last summation. Define a set-theoretic section $s : \mathbb{Z}_{q^m} \to \mathbb{Z}_{q^{m+1}}$ by

$$s\left(\sum_{j=0}^{m-1} X_j q^j\right) = 0 \cdot q^m + \sum_{j=0}^{m-1} X_j q^j.$$

For $X, Y \in \mathbb{Z}_{q^m}$ define

$$\phi(X, Y) = [s(X) * s(Y) - s(X * Y)]/q^m \in \mathbb{Z}_q.$$

Division by q^m means to consider these elements as integers that are divisible by q^m, divide by q^m, and compute the residue class modulo q. Note that $\overline{s(X)} * \overline{s(Y)} =$

$\overline{s(X*Y)}$. Hence, $s(X)*s(Y)-s(X*Y)$ is divisible by q^m. Then, we have

$$f(A*B) = (A_m + \phi(\overline{A}, \overline{B}), \overline{A} * \overline{B}).$$

Therefore, f yields an isomorphism

$$\mathbb{Z}_{q^{m+1}}[T, T^{-1}]/(T-1+q) \to E(\mathbb{Z}_{q^m}[T, T^{-1}]/(T-1+q), \mathbb{Z}_q, \phi).$$

We can also make the following observation. In $\mathbb{Z}_{q^m}$, we have that

$$\overline{A} * \overline{B} = \sum_{j=0}^{m-1} (A_j - A_{j-1} + B_{j-1}) q^j.$$

Moreover, the right-hand side of this equality is a uniquely determined integer for a given $A, B \in \mathbb{Z}_{q^{m+1}}$. If this integer is positive, then $A * B$ can be rewritten as a q-ary expansion with $A_m - A_{m-1} + B_{m-1}$ as the leading coefficient, and we have

$$f(A*B) = (A_m - A_{m-1} + B_{m-1}, \sum_{j=0}^{m-1} (A_j - A_{j-1} + B_{j-1}) q^j) \in \mathbb{Z}_q \times X.$$

If this integer is negative, then rewrite $A * B$ in terms of q-ary expansion with positive coefficients, and apply f to get that

$$f(A*B) = (A_m - A_{m-1} + B_{m-1} - 1, \sum_{j=0}^{m-1} (A_j - A_{j-1} + B_{j-1}) q^j) \in \mathbb{Z}_q \times X.$$

Thus, define $\delta : \mathbb{Z}_{q^m} \times \mathbb{Z}_{q^m} \to \{0, -1\}$ by

$$\delta(\overline{A}, \overline{B}) = \begin{cases} 0 & \text{if } \sum_{j=0}^{m-1} (A_j - A_{j-1} + B_{j-1}) q^j \geq 0, \\ -1 & \text{if } \sum_{j=0}^{m-1} (A_j - A_{j-1} + B_{j-1}) q^j < 0. \end{cases}$$

Then, we rewrite $f(A * B)$ as

$$f(A * B) = (A_m + \phi(\overline{A}, \overline{B}), \overline{A} * \overline{B}),$$

where $\phi(\overline{A}, \overline{B}) = B_{m-1} + \delta(\overline{A}, \overline{B})$. Hence, f yields an isomorphism

$$f : \mathbb{Z}_{q^{m+1}}[T, T^{-1}]/(T - 1 + q) \to E(\mathbb{Z}_{q^m}[T, T^{-1}]/(T - 1 + q), \mathbb{Z}_q, \phi). \quad \square$$

Theorem 4.1.3 *For any positive integer q and m, the quandle $W_{m+1} = \mathbb{Z}_q[T, T^{-1}]/(1 - T)^{m+1}$ is an abelian extension of $X = W_m = \mathbb{Z}_q[T, T^{-1}]/(1 - T)^m$ over $\mathbb{Z}_q$: $E = E(X, \mathbb{Z}_q, \phi)$, for some $\phi \in Z^2_{\mathrm{Q}}(X; \mathbb{Z}_q)$.*

Proof. Represent elements of E by $A = A_m(1-T)^m + \cdots + A_1(1-T) + A_0$, where $A_j \in \mathbb{Z}_q$, $j = 0, \ldots, m$. Define $f : E \to \mathbb{Z}_q \times X$ by

$$f(A) = (A_m \pmod{q},\ \overline{A} \ (\mathrm{mod}\ (1 - T)^m)),$$

where $\overline{A} = \sum_{j=0}^{m-1} A_j(1 - T)^j$. Then, for $A, B \in W_{m+1}$, the quandle operation is computed to be

$$\begin{aligned}
A * B &= TA + (1 - T)B \\
&= [1 - (1 - T)](A_m(1 - T)^m + \cdots + A_1(1 - T) + A_0) \\
&\quad +(1 - T)(B_m(1 - T)^m + \cdots + B_1(1 - T) + B_0) \\
&= (A_m - A_{m-1} + B_{m-1})(1 - T)^m + (A_{m-1} - A_{m-2} + B_{m-2})(1 - T)^{m-1} \\
&\quad + \cdots + (A_1 - A_0 + B_0)(1 - T) + A_0 \\
&= (A_m - A_{m-1} + B_{m-1})(1 - T)^m + \sum_{j=0}^{m-1}(A_j - A_{j-1} + B_{j-1})(1 - T)^j,
\end{aligned}$$

where A_{-1}, B_{-1} are understood to be zeros in the last summation, and the coefficients are in $\mathbb{Z}_q$. Note that in $\mathbb{Z}_q[T, T^{-1}]/(1-T)^m$ the quandle operation gives

$$\overline{A} * \overline{B} = \sum_{j=0}^{m-1} (A_j - A_{j-1} + B_{j-1})(1-T)^j.$$

Therefore, we have

$$f(A * B) = (A_m - A_{m-1} + B_{m-1}, \sum_{j=0}^{m-1} (A_j - A_{j-1} + B_{j-1})(1-T)^j) \in \mathbb{Z}_q \times X.$$

Then, we can write

$$f(A * B) = (A_m + \phi(\overline{A}, \overline{B}), \overline{A} * \overline{B}),$$

where $\phi(\overline{A}, \overline{B}) = B_{m-1} - A_{m-1}$. Hence f yields an isomorphism

$$\mathbb{Z}_q[T, T^{-1}]/(1-T)^{m+1} \to E(\mathbb{Z}_q[T, T^{-1}]/(1-T)^m, \mathbb{Z}_q, \phi).$$

The cocycle ϕ has a similar description to the one in Theorem 4.1.2. Let

$$s : \mathbb{Z}_q[T, T^{-1}]/(1-T)^m \to \mathbb{Z}_q[T, T^{-1}]/(1-T)^{m+1}$$

be a set-theoretic section defined by

$$s\left(\sum_{j=0}^{m-1} A_j (1-T)^j \mod (1-T)^m\right) = \sum_{j=0}^{m-1} A_j (1-T)^j \mod (1-T)^{m+1}.$$

Then, we have $\overline{s(X)} * \overline{s(Y)} = \overline{s(X * Y)}$ for any $X, Y \in \mathbb{Z}_q[T, T^{-1}]/(1-T)^m$. Hence, $[s(X) * s(Y) - s(X * Y)]$ is divisible by $(1-T)^m$, and we get

$$\phi(\overline{A}, \overline{B}) = [s(A) * s(B) - s(A * B)]/(1-T)^m \in \mathbb{Z}_q. \quad \square$$

Example 4.1.4 1. Consider the case $q = 2$, $m = 2$ in Theorem 4.1.2. In this case

$$\mathbb{Z}_4[T, T^{-1}]/(T + 1) = R_4, \quad \text{and}$$

$$\mathbb{Z}_8[T, T^{-1}]/(T + 1) = R_8 = E(R_4, \mathbb{Z}_2, \phi)$$

for some $\phi \in Z^2_{\mathrm{Q}}(R_4; \mathbb{Z}_2)$. In order to construct the cocycle, we use the bijection $f : R_8 \to \mathbb{Z}_2 \times R_4$ defined by

$$f(0) = (0, 0), \quad f(1) = (0, 1), \quad f(2) = (0, 2), \quad f(3) = (0, 3),$$

$$f(4) = (1, 0), \quad f(5) = (1, 1), \quad f(6) = (1, 2), \quad f(7) = (1, 3).$$

By Lemma 4.1.1, if $f(e_i) = (a_i, x_i)$, then $f(e_1 * e_2) = (a_1 + \phi(x_1, x_2), x_1 * x_2)$, using additive notation, for any $e_1, e_2 \in R_8$.

Let $e_1 = 0, e_2 = 1$. Then $f(e_1 * e_2) = (0 + \phi(0, 1), 0 * 1) = (0 + \phi(0, 1), 2)$. On the other hand, $f(e_1 * e_2) = f(0 * 1) = f(2) = (0, 2)$. By equating the corresponding parts in the last two relations, we observe that $\phi(0, 1) = 0$.

Now choose $e_1 = 0$ and $e_2 = 2$. By similar calculations we get that $f(e_1 * e_2) = (0 + \phi(0, 2), 0 * 2) = (0 + \phi(0, 2), 0)$, and $f(e_1 * e_2) = f(0 * 2) = f(4) = (1, 0)$. Note that the first factors in the two relations differ by 1. This must be the contribution of $\phi(0, 2)$. Therefore, the characteristic function $\chi_{0,2}$ appears in the cocycle ϕ, where

$$\chi_{a,b}(x, y) = \begin{cases} 1 & \text{if } (x, y) = (a, b), \\ 0 & \text{if } (x, y) \neq (a, b) \end{cases}$$

denotes the characteristic function.

By carrying out similar computations for all pairs, we obtain an explicit formula for this cocycle ϕ:

$$\phi = \chi_{0,2} + \chi_{0,3} + \chi_{1,0} + \chi_{1,3} + \chi_{2,0} + \chi_{2,3} + \chi_{3,0} + \chi_{3,1}.$$

Similar computations yield the following formulas.

2. In case $m = 1$ and $q = 3$, the cocycle constructed is of the form

$$\phi = \chi_{0,1} + \chi_{1,2} + \chi_{2,0} + 2(\chi_{0,2} + \chi_{1,0} + \chi_{2,1}).$$

3. In case $m = 2$ and $q = 3$, the cocycle is

$$\begin{aligned}
\phi &= \chi_{0,3} + \chi_{0,4} + \chi_{0,5} + 2\chi_{0,6} + 2\chi_{0,7} + 2\chi_{0,8} \\
&+ 2\chi_{1,0} + \chi_{1,4} + \chi_{1,5} + \chi_{1,6} + 2\chi_{1,7} + 2\chi_{1,8} \\
&+ 2\chi_{2,0} + 2\chi_{2,1} + \chi_{2,5} + \chi_{2,6} + \chi_{2,7} + 2\chi_{2,8} \\
&+ 2\chi_{3,0} + 2\chi_{3,1} + \chi_{3,5} + \chi_{3,6} + \chi_{3,7} + 2\chi_{3,8} \\
&+ 2\chi_{4,0} + 2\chi_{4,1} + 2\chi_{4,2} + \chi_{4,6} + \chi_{4,7} + \chi_{4,8} \\
&+ \chi_{5,0} + 2\chi_{5,1} + 2\chi_{5,2} + 2\chi_{5,3} + \chi_{5,7} + \chi_{5,8} \\
&+ \chi_{6,0} + 2\chi_{6,1} + 2\chi_{6,2} + 2\chi_{6,3} + \chi_{6,7} + \chi_{6,8} \\
&+ \chi_{7,0} + \chi_{7,1} + 2\chi_{7,2} + 2\chi_{7,3} + 2\chi_{7,4} + \chi_{7,8} \\
&+ \chi_{8,0} + \chi_{8,1} + \chi_{8,2} + 2\chi_{8,3} + 2\chi_{8,4} + 2\chi_{8,5}.
\end{aligned}$$

4. Consider the case $q = 2$ and $m = 2$ in Theorem 4.1.3. The quandle $\mathbb{Z}_2[T, T^{-1}]/(1-T)^2$ is isomorphic to R_4 by the correspondence $0 \leftrightarrow 0(1-T)+0$, $1 \leftrightarrow 0(1-T)+1$, $2 \leftrightarrow 1(1-T)+0$, and $3 \leftrightarrow 1(1-T)+1$. This is a special case of the isomorphism

$$\mathbb{Z}_n[T, T^{-1}]/(1-T)^2 \cong \mathbb{Z}_{n^2}[T, T^{-1}]/(T-(kn+1)) \quad \text{if} \quad \gcd(n, k) = 1$$

given in [31]. Then, the quandle $\mathbb{Z}_2[T,T^{-1}]/(1-T)^3$ is an abelian extension $E(R_4;\mathbb{Z}_2,\phi')$ for some $\phi' \in Z^2_{\mathrm{Q}}(R_4;\mathbb{Z}_2)$. Moreover, the cocycle $\phi'(\overline{A},\overline{B}) = B_1 - A_1$ is 1 if and only if the pair $(\overline{A},\overline{B})$ has distinct coefficients for $(1-T)$, and we obtain

$$\phi' = \chi_{0,2} + \chi_{2,0} + \chi_{1,2} + \chi_{2,1} + \chi_{0,3} + \chi_{3,0} + \chi_{1,3} + \chi_{3,1}.$$

The cocycles $\phi_0 = \chi_{2,1} + \chi_{2,3}$, $\phi_1 = \chi_{1,0} + \chi_{1,2}$, and ϕ as constructed in Example 4.1.4(1) are linearly independent (evaluate on the cycles defined in Remark 4.1.9), and $\phi' = \phi + \phi_0 + \phi_1$.

Abelian extensions define surjective homomorphisms $E(X,A,\phi) = A \times X \to X$ defined by the projection onto the second factor. It was proved in [11] that two abelian extensions $E(X,A,\phi)$ and $E(X,A,\phi')$ are isomorphic if and only if ϕ is cohomologous to ϕ'.

Proposition 4.1.5 *The cocycles ϕ, ϕ' obtained from the abelian extensions*

$$\mathbb{Z}_{q^{m+1}}[T,T^{-1}]/(T-1+q) = E(\mathbb{Z}_{q^m}[T,T^{-1}]/(T-1+q),\mathbb{Z}_q,\phi),$$

$$\mathbb{Z}_q[T,T^{-1}]/(1-T)^{m+1} = E(\mathbb{Z}_q[T,T^{-1}]/(1-T)^m,\mathbb{Z}_q,\phi'),$$

respectively, are not coboundaries.

Proof. Direct computations show that the chains

$$c = (0,1) + (q, q^{m-1} + q - 1) \quad \in Z_2^{\mathrm{Q}}(X;\mathbb{Z}_q) \quad \text{and}$$

$$c' = (0,1) + (1-T, (1-T)^{m-1} + (1-T) - 1) \ \in Z_2^{\mathrm{Q}}(X;\mathbb{Z}_q)$$

are cycles for $X = \mathbb{Z}_{q^m}[T,T^{-1}]/(T-1+q)$ and $X = \mathbb{Z}_q[T,T^{-1}]/(1-T)^m$, respectively.

Then, it is computed that $\phi(c) = 1$ and $\phi'(c') = 1$, and hence ϕ and ϕ' are not coboundaries, and the result follows. □

An extension theory of quandles for twisted cohomology cocycles was developed in [6], and it provided more general extension theories. In the twisted case, the coefficient group is taken to be a Λ-module, thus has an Alexander quandle structure and the extension $AE(X, A, \phi) = (A \times X, *)$ is defined by $(a_1, x_1) * (a_2, x_2) = (a_1 * a_2 + \phi(x_1, x_2), x_1 * x_2)$ for $\phi \in Z^2_{\rm TQ}(X; A)$, and is called an *Alexander extension* of X by (A, ϕ). For example, R_{p^e} is an Alexander extension of $R_{p^{e-1}}$ by R_p such that $R_{p^e} = AE(R_{p^{e-1}}, R_p, \phi)$, for some $\phi \in Z^2_{\rm TQ}(R_{p^{e-1}}; R_p)$.

Remark 4.1.6 The quandle structure of a dihedral quandle R_n is defined using the ring structure of $\mathbb{Z}_n$. The product quandle $R_m \times R_n$ is defined by component-wise operation, so that it is defined from the ring structure of $\mathbb{Z}_m \times \mathbb{Z}_n$ as well. Consequently, two quandles $R_m \times R_n$ and R_{mn} are isomorphic if $\mathbb{Z}_m \times \mathbb{Z}_n$ and $\mathbb{Z}_{mn}$ are isomorphic as rings. Hence, if $n = p_1^{e_1} \cdots p_k^{e_k}$ is the prime decomposition, then R_n is isomorphic to $R_{p_1^{e_1}} \times \cdots \times R_{p_k^{e_k}}$. For $p = 2$, the result of this section shows that R_{p^e} is described succesively as an extension of $R_{p^{e-1}}$.

Ohtsuki [37] defined an extension theory and a new cohomology theory for quandles, together with a list of problems in the subject.

The following lemma follows from definitions.

Lemma 4.1.7 [5] *Let X, Y be quandles and A be an abelian group. If E is an abelian extension of X for $\phi \in Z^2_{\rm Q}(X; A)$: $E = E(X, A, \phi)$, then $E \times Y$ is an abelian extension of $X \times Y$ for $p^{\#}\phi \in Z^2_{\rm Q}(X \times Y; A)$: $E \times Y = E(X \times Y, A, p^{\#}\phi)$, where $p : X \times Y \to X$ is the projection to the first factor.*

Corollary 4.1.8 *For any positive integer n, $E = R_{4n}$ is an abelian extension $E = E(R_{2n}, \mathbb{Z}_2, \phi)$ of $X = R_{2n}$ for some cocycle $\phi \in Z^2_{\rm Q}(R_{2n}; \mathbb{Z}_2)$.*

Proof. Let $2n = 2^m k$ for an odd integer k. Then $R_{2n} \cong R_{2^m} \times R_k$ by Remark 4.1.6, and by Lemma 4.1.7, $R_{4n} \cong R_{2^{m+1}} \times R_k$ is an abelian extension of R_{2n} if $R_{2^{m+1}}$ is an abelian

extension of R_{2^m}. This follows from Theorem 4.1.2 since $R_{2^m} \cong \mathbb{Z}_{2^m}[T, T^{-1}]/(T+1)$.
□

Remark 4.1.9 By Lemma 4.1.1 and Corollary 4.1.8, there is a cocycle $\phi \in Z^2_{\rm Q}(R_{4n}; \mathbb{Z}_2)$ such that R_{8n} is isomorphic to $E(R_{4n}, \mathbb{Z}_2, \phi)$.

Let $\phi_{0,1}, \phi_{1,0} \in Z^2_{\rm Q}(R_{4n}; \mathbb{Z}_2)$ be cocycles defined by

$$\phi_{0,1} = p^{\#}(\chi_{0,1} + \chi_{0,3}), \quad \text{and} \quad \phi_{1,0} = p^{\#}(\chi_{1,0} + \chi_{1,2}),$$

respectively, where $p : R_{4n} \to R_4$ is a natural map $p(x \bmod (4n)) = x \bmod (4)$. Here, it is known [8] that

$$\chi_{0,1} + \chi_{0,3}, \quad \text{and} \quad \chi_{1,0} + \chi_{1,2}$$

are cocycles in $Z^2_{\rm Q}(R_4; \mathbb{Z}_2)$. It is directly computed that

$$c_{0,1} = (0,1)+(2,1), \quad c_{1,0} = (1,0)+(4n-1,0), \quad c'_{0,1} = (0,1)+(2,2n+1) \in Z_2^{\rm Q}(R_{4n}; \mathbb{Z}_2)$$

are cycles. Then, we have

$$\begin{array}{rclrclrcl} \phi_{0,1}(c_{0,1}) &=& 1, & \phi_{0,1}(c_{1,0}) &=& 0, & \phi_{0,1}(c'_{0,1}) &=& 1, \\ \phi_{1,0}(c_{0,1}) &=& 0, & \phi_{1,0}(c_{1,0}) &=& 1, & \phi_{1,0}(c'_{0,1}) &=& 0, \\ \phi\ \ (c_{0,1}) &=& 0, & \phi\ \ (c_{1,0}) &=& 0, & \phi\ \ (c'_{0,1}) &=& 1, \end{array}$$

where ϕ is computed in Example 4.1.4. Hence, we see that the cocycles $\phi_{0,1}, \phi_{1,0}$, and ϕ are linearly independent.

Lemma 4.1.10 [5] *Let X be a quandle, $\phi_i \in Z^2_{\rm Q}(X; A)$, $i = 0, 1$, where A is an abelian group, and $E_1 = E(X, \phi_1)$ be an extension, and let $p : E_1 \to X$ be the natural homomorphism (the projection onto the second factor). Then, $E_0 = E(E_1, A, p^{\#}\phi_0)$ is isomorphic to $E_2 = E(X, (A, \phi_0), (A, \phi_1))$.*

Proof. As sets, $E_0 = A \times E_1 = A \times (A \times X)$ and $E_2 = A \times A \times X$. Let $f : E_0 \to E_2$ be the "identity" map between these sets. We show that f is an isomorphism. It only needs to be checked that f is a homomorphism. One computes

$$\begin{aligned}
& f((a_1,(b_1,x_1)) * (a_2,(b_2,x_2))) \\
&= f(a_1 p^* \phi_0((b_1,x_1),(b_2,x_2)),(b_1,x_1) * (b_2,x_2)) \\
&= f(a_1 \phi_0(p(b_1,x_1),p(b_2,x_2)),(b_1 \phi_1(x_1,x_2),x_1 * x_2)) \\
&= f(a_1 \phi_0(x_1,x_2),((b_1 \phi_1(x_1,x_2),x_1 * x_2)) \\
&= (a_1 \phi_0(x_1,x_2),b_1 \phi_1(x_1,x_2),x_1 * x_2) \\
&= f((a_1,(b_1,x_1))) * f((a_2,(b_2,x_2)))
\end{aligned}$$

as desired. □

Lemma 4.1.11 *Let* $p : R_{4n} \to T_2$ *be the quotient homomorphism defined by* $p(x) = x \pmod 2$*, and* $\xi \in Z^2_{\mathrm{Q}}(T_2;\mathbb{Z}_2)$*. Then,*

(a) $E(R_{4n},\mathbb{Z}_2,p^*\xi)$ *has a subquandle isomorphic to* T_4*, and*

(b) $E(R_{4n},\mathbb{Z}_2,p^*\xi)$ *is not isomorphic to* R_{8n}*.*

Proof. By Lemmas 4.1.8 and 4.1.10, $E = E(R_{4n},\mathbb{Z}_2,p^*\xi)$ is isomorphic to $E_0 = E(R_{2n},(\mathbb{Z}_2,\xi),(\mathbb{Z}_2,\phi))$, since $R_{4n} = E(R_{2n},\mathbb{Z}_2,\phi)$ for some $\phi \in Z^2_{\mathrm{Q}}(R_{2n};\mathbb{Z}_2)$. Then, the subset

$$\{(x,y,0) \in \mathbb{Z}_2 \times \mathbb{Z}_2 \times R_{2n} = E_0\}$$

forms a subquandle isomorphic to T_4. This proves (a).

To prove (b), it is sufficient to prove that R_{8n} does not contain a subquandle isomorphic to T_4. If $a * b = a \in R_{8n}$, then $2b - a \equiv a \pmod{4n}$, hence $a \equiv b \pmod{2n}$. However, there are only two such integers mod $4n$. Hence, the largest subquandle isomorphic to a trivial quandle has two elements. □

4.2 Evaluations of cocycle invariants using extension cocycles

As examples, we evaluate the previously defined invariants for Whitehead link and Borromean rings, using extension cocycles constructed in Section 4.1. We use the coefficient group $A = \mathbb{Z}_q = \{t^n | n = 0, 1, \ldots, q-1\}$ for a positive integer q.

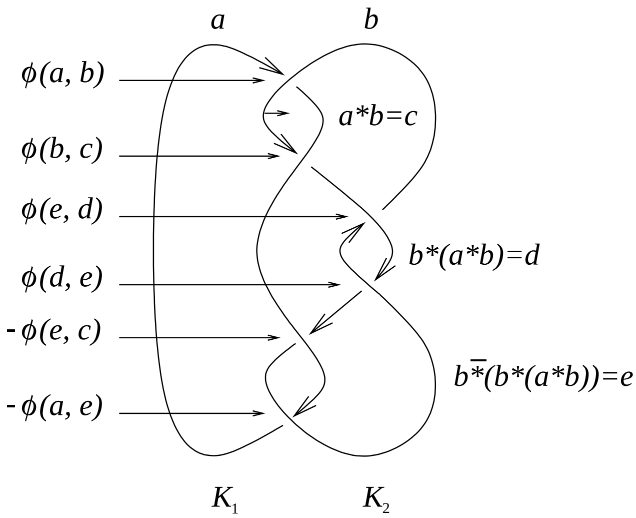

Figure 15. A colored Whitehead link

Example 4.2.1 In Figure 15, a Whitehead link $L = K_1 \cup K_2$ is depicted. Let $\phi \in Z^2_{\rm Q}(R_8; \mathbb{Z}_2)$ be the cocycle defined by Corollary 4.1.8. We evaluate the component-wise cocycle invariant $\vec{\Phi}(L) = (\Phi_1(L), \Phi_2(L))$, which was given in Definition 3.3.1. Denote the multiplicative generator of the coefficient group $\mathbb{Z}_2$ by t, so that $\mathbb{Z}_2 = \{1, t\}$ and the invariant takes the form $\vec{\Phi}(L) = (A_1 + B_1 t, A_2 + B_2 t)$, where A_i, B_i, $i = 1, 2$, are non-negative integers.

The colors assigned to arcs are represented by the letters a through e. From the figure, it is seen that all the colors are determined by the colors a and b assigned to the top two arcs. We observe from the calculations that for any choice of two elements a and b of R_8, there is a unique coloring of L by R_8 that restricts to the chosen elements a and b. Therefore, there are $8^2 = 64$ colorings of L by R_8.

We show that the state-sum term $\prod_{\tau \in \mathcal{T}_1} B(\tau, \mathcal{C})$ is trivial if and only if a and b have the same parity (both even or both odd).

Suppose that a and b are both even, so that $a = 2\alpha$, $b = 2\beta$. Then, one computes that $c = 4\beta - 2\alpha$, $d = 6\beta - 4\alpha$, and we obtain $e = 2\beta = b$. Similar computations show that $e = b$, if a and b are both odd. From the figure, the state-sum term for $\mathcal{T}_1$ is $\phi(a, b) - \phi(a, e)$, which is equal to $\phi(a, b) - \phi(a, b) = 0$, in this case. Suppose now that a and b have opposite parities. By setting $a = 2\alpha + 1$ and $b = 2\beta$ (and vice versa), we compute that $e = b + 4$, so that we obtain the state-sum term $\phi(a, b) - \phi(a, e) = \phi(a, b) - \phi(a, b + 4)$. We claim that this is t.

Using the formula at the end of the proof of Theorem 4.1.2, we have $\phi(a, b) = [s(a) * s(b) - s(a * b)]/8$. Here, $s(a) * s(b)$ is $2b - a$ computed modulo 16, and $s(a * b)$ is $2b - a$ computed modulo 8, then regarded as an element modulo 16. Since $a * (b + 4) = 2(b + 4) - a = (2b - a) + 8$ modulo 16, we have

$$\phi(a, b) - \phi(a, b+4) = [s(a) * s(b) - s(a * b)]/8 - [s(a) * s(b) + 8 - s(a * b)]/8 = 1 \pmod{2},$$

written additively. This proves the above claim. There are 32 colorings with the same parity, and 32 with distinct parities. Hence, we obtain $\vec{\Phi}(L) = (32 + 32t, 32 + 32t)$.

The following examples deal with the generalized definitions of cocycle invariants as those were described in Definitions 3.3.1 and 3.3.3.

Example 4.2.2 Let $X = W_m = \mathbb{Z}_q[T, T^{-1}]/(1 - T)^m$ or $X = U_m = \mathbb{Z}_{q^m}[T, T^{-1}]/(T - 1 + q)$, and L the Whitehead link. Then the generalized cocycle invariant is

$$\vec{\Psi}(L) = \begin{cases} \{\underbrace{(1,1), \ldots, (1,1)}_{q^{2m} \text{ copies}}\} & \text{for } m = 1, 2, \\ \{\underbrace{(t^n, t^{-n}), \ldots, (t^n, t^{-n})}_{q^{m+2} \text{ copies}}\}_{n \in \{0,1,\ldots,q-1\}} & \text{for } m \geq 3. \end{cases}$$

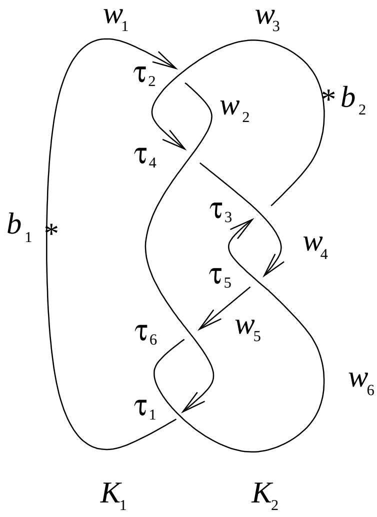

Figure 16. Whitehead link

Consequently,

$$\vec{\Phi}(L) = \begin{cases} (q^{2m}, q^{2m}) & \text{for } m = 1, 2, \\ (q^{m+2}(t^{q-1} + \cdots + t + 1), q^{m+2}(t^{q-1} + \cdots + t + 1)) & \text{for } m \geq 3. \end{cases}$$

Proof. Let $X = \mathbb{Z}_q[T, T^{-1}]/(1 - T)^m$. The case for $X = \mathbb{Z}_{q^m}[T, T^{-1}]/(T - 1 + q)$ is similar. Pick base points b_1 and b_2 on the components K_1 and K_2, respectively, of the Whitehead link $L = K_1 \cup K_2$ as depicted in Figure 16, and trace each component in the given orientation of the link. The colors assigned to the arcs are elements of X and appear in this order w_1, w_2 for K_1, and $w_3, \ldots, w_6$ for K_2 as shown in the figure. The crossing at the tail of the arc colored by w_i is defined to be τ_i. First, we determine the set of colorings: For $m \geq 3$ and for two elements w_1, $w_3 \in W_m$ assigned to the top two arcs of the Whitehead link L, there is a coloring of L by X which restricts to the given w_1, w_3 if and only if

$$w_3 - w_1 \equiv 0 \pmod{(1 - T)^{m-3}}.$$

For $m = 1, 2$, there is such a coloring for any $w_1, w_3 \in X$. This can be computed as follows.

Represent the elements of $X = \mathbb{Z}_q[T, T^{-1}]/(1-T)^m$ by $a = a_{m-1}(1-T)^{m-1} + \cdots + a_1(1-T) + a_0$, where $a_j \in \mathbb{Z}_q$. Note that $(1-(1-T))(1+(1-T)+\cdots+(1-T)^{m-1}) = 1$ in X, so $T^{-1} = 1+(1-T)+\cdots+(1-T)^{m-1}$. Note also that $a \mathbin{\bar{*}} b = T^{-1}a + (1-T^{-1})b$. We have the following calculations for each arc:

$$
\begin{aligned}
w_2 &= w_1 * w_3 = w_1 + (1-T)(w_3 - w_1), \\
w_4 &= w_3 * w_2 = w_3 + (1-T)(w_2 - w_3) \\
&= (w_3 - w_1)(1-T)^2 - (w_3 - w_1)(1-T) + w_3, \\
w_6 &= w_3 \mathbin{\bar{*}} w_4 \\
&= T^{-1}w_3 + (1-T^{-1})w_4 = (w_3 - w_1)(1-T)^2 + w_3, \\
w_5 &= w_4 * w_6 = w_4 + (1-T)(w_6 - w_4) \\
&= 2(w_3 - w_1)(1-T)^2 - (w_3 - w_1)(1-T) + w_3.
\end{aligned}
$$

These relations are obtained using the top four crossings (τ_2, τ_4, τ_3, and τ_5). The bottom two crossings (τ_6 and τ_1) of the link give rise to relations. The first relation is $w_6 * w_2 = w_5$ for the second bottom crossing, giving $(w_1 - w_3)(1-T)^3 \equiv 0 \pmod{(1-T)^m}$. The second relation that corresponds to the bottom crossing is $w_1 * w_6 = w_2$ giving $(w_3 - w_1)(1-T)^3 \equiv 0 \pmod{(1-T)^m}$, as claimed above.

Now we determine the contribution to the invariant for each coloring. Recall that $\phi(w_1, w_3) = [s(w_1) * s(w_3) - s(w_1 * w_3)]/(1-T)^m$. Since $(w_3 - w_1)(1-T)^3 \equiv 0 \pmod{(1-T)^m}$, we see that the contribution is

$$
\begin{aligned}
&\phi(w_1, w_3) - \phi(w_1, w_6) \\
&\quad = [s(w_1) * s(w_3) - s(w_1 * w_3)]/(1-T)^m \\
&\qquad -[s(w_1) * s(w_3) + (w_3 - w_1)(1-T)^3 - s(w_1 * w_3)]/(1-T)^m
\end{aligned}
$$

$$= \ -(w_3 - w_1)(1-T)^3/(1-T)^m \pmod q,$$

for the first component, and for the second component, computations show that

$$\phi(w_3, w_2) - \phi(w_6, w_2) + \phi(w_6, w_4) + \phi(w_4, w_6) = (w_3 - w_1)(1-T)^3/(1-T)^m \pmod q.$$

For $m = 1, 2$ the contributions for the first and the second component are both 0, and we have q^m choices for both w_1 and w_3, therefore $\vec{\Psi}(L) = (\underbrace{(1,1), \ldots, (1,1)}_{q^{2m} \text{ copies}})$ and $\vec{\Phi}(L) = (q^{2m}, q^{2m})$.

For $m \geq 3$, if w_1 and w_3 color L, then $(w_3 - w_1)(1-T)^3$ is 0 as an element of X, so that $w_3 - w_1$ is uniquely written as $w_3 - w_1 = k(1-T)^{m-3}$, where $k = k_0 + k_1(1-T) + k_2(1-T)^2$, and $k_0, k_1, k_2 \in \{0, 1, \ldots, q-1\}$. Then

$$\begin{aligned} &(w_3 - w_1)(1-T)^3 = k(1-T)^m \\ &= \ (k_0 + k_1(1-T) + k_2(1-T)^2)(1-T)^m = k_0(1-T)^m \ \in W_{m+1}. \end{aligned}$$

Thus, the contribution to the invariant for the first and second components are t^{-k_0} and t^{k_0}, respectively.

To find the number of colorings conrtibuting to t^{-k_0} and t^{k_0}, fix k_0. We have q^m choices for w_1 and q^2 choices for k. Then, w_3 is uniquely determined by $w_3 = w_1 + k(1-T)^{m-3}$. In total, the contribution is $q^m q^2 = q^{m+2}$ for each t^{-k_0} and t^{k_0}. Setting $n = -k_0$ we obtain the result. □

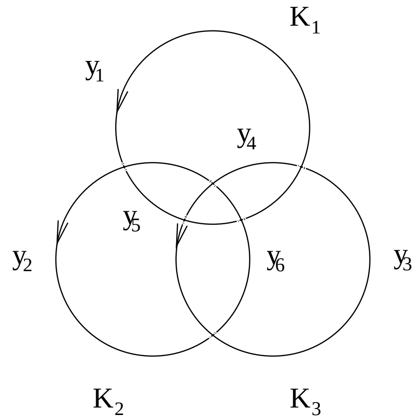

Figure 17. Borromean rings

Example 4.2.3 Let $X = W_m = \mathbb{Z}_q[T, T^{-1}]/(1-T)^m$ or $X = U_m = \mathbb{Z}_{q^m}[T, T^{-1}]/(T - 1 + q)$, and L the Borromean rings. Then, the generalized cocycle invariant is

$$\vec{\Psi}(L) = \begin{cases} \{\underbrace{(1,1,1), \ldots, (1,1,1)}_{q^{3m} \text{ copies}}\} & \text{for } m = 1, \\ \{\underbrace{(t^{-k_0}, t^{-\ell_0}, t^{k_0+\ell_0}), \ldots, (t^{-k_0}, t^{-\ell_0}, t^{k_0+\ell_0})}_{q^{m+2} \text{ copies}}\}_{k_0, \ell_0 \in \{0,1,\ldots,q-1\}} & \text{for } m \geq 2. \end{cases}$$

Consequently,

$$\vec{\Phi}(L) = \begin{cases} (q^{3m}, q^{3m}, q^{3m}) & \text{for } m = 1, \\ (q^{m+2}(t^{q-1} + \cdots + 1), q^{m+2}(t^{q-1} + \cdots + 1), q^{m+2}(t^{q-1} + \cdots + 1)) & \\ & \text{for } m \geq 2. \end{cases}$$

Proof. Let $X = W_m$ and let L be the Borromean rings as depicted in Figure 17. The case for $X = U_m$ is similar. Calculations are similar to the preceding example and we give a sketch. First, we determine the set of colorings: For three elements $y_1, y_2, y_3 \in X$ assigned to each outer arc in the diagram of L, there is a coloring of L by X which restricts to the given y_1, y_2, y_3 if and only if

$$(y_2 - y_3)(1-T)^2 \equiv 0, \quad (y_1 - y_2)(1-T)^2 \equiv 0 \quad \text{and} \quad (y_3 - y_1)(1-T)^2 \equiv 0.$$

The outer three crossings are used to describe y_4, y_5, y_6 in terms of y_1, y_2, y_3, and the inner three crossings give the above relations.

Contributions to the invariant are computed as follows. The contribution for the first component of L colored by y_1 is $\phi(y_1, y_2) - \phi(y_1, y_2 * y_3) = -(y_3 - y_2)(1 - T)^2/(1-T)^m \pmod q$. For $m = 1$, the contribution is trivial, and the total number of colorings is q^{3m}. For $m \geq 2$, $(y_3 - y_2)(1 - T)^2$ is divisible by $(1 - T)^m$, so $y_3 - y_2$ is uniquely written as $y_3 - y_2 = k(1 - T)^{m-2}$, where $k = k_0 + k_1(1 - T)$ and $k_0, k_1 \in \{0, 1, \ldots, q - 1\}$. Therefore,

$$(y_3 - y_2)(1 - T)^2 = k(1 - T)^m = (k_0 + k_1 q)(1 - T)^m = k_0(1 - T)^m,$$

and the first component contributes t^{-k_0} to the invariant. For the second component of L colored by y_2, similar calculations as above give the contribution $\phi(y_2, y_3) - \phi(y_2, y_3 * y_1) = -(y_1 - y_3)(1-T)^2$, which is divisible by $(1-T)^m$ so $y_1 - y_3 = (\ell_0 + \ell_1(1-T))(1-T)^{m-2}$ and therefore $-(y_1 - y_3)(1 - T)^2 = -\ell_0(1 - T)^m$. The we obtain $y_2 - y_1 = -[(k_0 + \ell_0) + (k_1 + \ell_1)(1 - T)](1 - T)^{m-2}$, so that the third component contributes $t^{k_0+\ell_0}$. Finally, the contribution to the invariant is the vector $(t^{-k_0}, t^{-\ell_0}, t^{k_0+\ell_0})$, where the entries correspond to the components K_1, K_2, K_3, respectively. The result follows. $\square$

CHAPTER 5

EXTENDING COLORINGS OF KNOTS

In Chapter 2, we demonstrated how a knot diagram can be colored by a quandle. Furthermore, in Chapter 4 we introduced the notion of extensions of quandles. Since extensions of quandles are also quandles, we are led to ask: when can a coloring of a knot by a quandle be extended to a coloring by an extension of the quandle? We investigate this problem in the following chapter.

5.1 Extensions of colorings

Definition 5.1.1 Let K be a classical knot or link. Let $\mathcal{C}$ be a coloring of K by X. Let E be a quandle with a surjective homomorphism $p : E \to X$. If there is a coloring $\mathcal{C}'$ of K by E, such that for every arc a of K it holds that $p(\mathcal{C}'(a)) = \mathcal{C}(a)$, then $\mathcal{C}'$ is called an *extension* of $\mathcal{C}$.

Example 5.1.2 Let K be the $(4, 2)$-torus link shown in Figure 14. Let $X = R_4$ and let $E(R_4, \mathbb{Z}_2, \phi) = R_8$, where ϕ is constructed in Example 4.1.4(1). Note that any pair of elements of X assigned to a_i and b_k, uniquely determine the colors assigned to all the arcs in the diagram. Let $\mathcal{C}$ be a coloring by R_4 determined by $a_i = 0$ and $b_k = 2$. Thus, $a_j = 0$ and $b_\ell = 2$. Observe that $a_i = 0$ and $b_k = 2$ also uniquely determine a coloring by R_8, with $a_j = 4$ and $b_\ell = 6$. Call this coloring $\mathcal{C}'$. Then, $\mathcal{C}'$ is an extension of $\mathcal{C}$.

Now let $\mathcal{C}$ be the coloring of K by R_4 determined by $a_i = 0$ and $b_k = 1$. Let $a_i = 0$ and $b_k = 1$ in R_8. Then, the top two crossings require that $a_j = 2$ and $b_\ell = 3$ in R_8. This assignment though does not satisfy the requirement of a coloring for the bottom two crossings, and therefore $\mathcal{C}$ does not extend to a coloring by R_8 for this choice. By checking all the possibilities $(a_i, b_k) = \{(0,1),(0,5),(4,1),(4,5)\}$ we conclude that $\mathcal{C}$ does not extend.

By checking all the cases, we find that a coloring $\mathcal{C}$ by R_4 extends to a coloring $\mathcal{C}'$ by R_8 if and only if the pair (a_i, b_k) is from the set

$$\{(0,0),(0,2),(1,1),(1,3),(2,0),(2,2),(3,1),(3,3)\}.$$

5.2 Cocycle knot invariants as obstructions to extending colorings

Let K be a knot and denote by $\Phi_\phi(K)$ the state-sum invariant of K, as was given in Definition 3.2.1, with respect to a quandle X, an abelian group A, and a cocycle $\phi \in Z^2_{\rm Q}(X; A)$. Let $E = E(X, A, \phi)$ be the abelian extension of X by ϕ. We characterize when the state-sum invariant defined from this cocycle is non-trivial, if the cocycles used are those defined from abelian extensions. For characterizations on the triviality of colorings, see [24].

Theorem 5.2.1 [5] *Let $C_0(K, X)$ be the constant term (a positive integer) of $\Phi_\phi(K)$, and $C(K, X)$ be the number of all colorings of K by X. Then, the number of colorings of K by X that extend to colorings of K by $E(X, A, \phi)$ is equal to $C_0(K, X)$, and the number of colorings that do not extend is $C(K, X) - C_0(K, X)$.*

Proof. Let $\mathcal{C}$ be a coloring whose contribution to $\Phi_\phi(K)$ is 1. Fix this coloring in what follows. Pick a base point b_0 on a knot diagram of K. Let $x \in X$ be the color on the arc α_0 containing b_0. Let α_i, $i = 1, \ldots, n$, be the set of arcs that appear in this order when the diagram of K is traced in the given orientation of K, starting from b_0. Pick an element $a \in A$ and give a color (a, x) on α_0, so that we define a coloring $\mathcal{C}'$ by E on α_0 by $\mathcal{C}'(\alpha_0) = (a, x) \in E$. We try to extend it to the entire diagram by traveling the diagram from b_0 along the arcs α_i, $i = 1, \ldots, n$, in this order, by induction.

Assume $\mathcal{C}'(\alpha_i)$ is defined for $0 \le i < k$. Define $\mathcal{C}'(\alpha_{k+1})$ as follows. Suppose that the crossing τ_k separating α_k and α_{k+1} is positive, and the over-arc at τ_k is α_j. Let $\mathcal{C}'(\alpha_k) = (a, x)$ and $\mathcal{C}(\alpha_j) = y \in X$. Then, we have $\mathcal{C}(\alpha_{k+1}) = x * y \in X$. Define $\mathcal{C}'(\alpha_{k+1}) = (a\phi(x, y), x * y)$ in this case.

Suppose that the crossing τ_k is negative. Let $\mathcal{C}'(\alpha_k) = (a, x)$ and $\mathcal{C}(\alpha_j) = y \in X$. Hence, if $\mathcal{C}(\alpha_{k+1}) = z$, then we have $z * y = x$. Define $\mathcal{C}'(\alpha_{k+1}) = (a\phi(z, y)^{-1}, z)$ in this case.

Define $\mathcal{C}'(\alpha_i)$ inductively for all $i = 0, \dots, n$. Regard α_0 as α_{n+1}, and repeat the above construction at the last crossing τ_n to come back to α_0. By the construction we have $\mathcal{C}'(\alpha_{n+1}) = (a \prod_\tau B(\tau, \mathcal{C}), \mathcal{C}(\alpha_0))$, where $\prod_\tau B(\tau, \mathcal{C})$ is the state-sum contribution (the product of Boltzmann weights over all crossings) of $\mathcal{C}$. This contribution is equal to 1 by the assumption that $\prod_\tau B(\tau, \mathcal{C}) = 1$, and we have a well-defined coloring $\mathcal{C}'$. Hence, this color extends to $E(X, A, \phi)$.

Conversely, if a coloring $\mathcal{C}$ by X extends to a coloring by $E(X, A, \phi)$, then from the above argument, we have that $(a, x) = (a \prod_\tau B(\tau, \mathcal{C}), x)$, if (a, x) is the color on the base point b_0. Hence, $\prod_\tau B(\tau, \mathcal{C}) = 1$. □

Thus, the non-trivial value of $\Phi(K)$ is the obstruction to extending colorings of K by X to $E(X, A, \phi)$, in the following sense: there is a coloring $\mathcal{C}$ of K by X which does not extend to a coloring by $E(X, A, \phi)$, if and only if $\Phi_\phi(K)$ is not a positive integer.

Example 5.2.2 For $X = \mathbb{Z}_2[T, T^{-1}]/(T^2+T+1)$ we have a cocycle $\phi = \prod_{a,b \neq T} \chi_{(a,b)} \in Z^2_{\mathrm{Q}}(X; \mathbb{Z}_2)$ [8], let $E = E(X, \mathbb{Z}_2, \phi)$. Then, $\Phi_\phi(K) = a + bt$, where $X = S_4$, has the above characterization.

Specifically, in [9], it was computed that among knots in the table up to 9 crossings, the state-sum invariant with the above quandle and the cocycle takes the value $4 + 12t$

for the knots

$$3_1, 4_1, 7_2, 7_3, 8_1, 8_4, 8_{11}, 8_{13}, 9_1, 9_{12}, 9_{13}, 9_{14}, 9_{21}, 9_{23}, 9_{35}, 9_{37},$$

and the value $16+48t$ for $8_{18}, 9_{40}$. Hence, for the knots in the former list, the number of colorings by X which extend to those by E is 4 (trivial colorings, by a single color), and those that do not extend is 12 (all non-trivial colorings do not extend). For the knots in the latter list, there are 16 colorings that extend, and 48 colorings that do not.

Corollary 5.2.3 *For $A = \mathbb{Z}_2$ and some cocycle $\phi \in Z^2_{\mathrm{Q}}(X; A)$, $\Phi_\phi(K) = a+bt$, where t is the variable, i.e. the generator of $\mathbb{Z}_2$, is determined by the number of colorings with X and E: a is the number of colorings of X that extend to colorings by E, and b is the number of those that do not.*

Let $L = K_1 \cup \cdots \cup K_r$ be a link diagram. Recall from Definition 3.3.1 the generalization of the state-sum invariant to links component-wise. We observe here that Theorem 5.2.1 in this section applies to component-wise cocycle invariants.

Theorem 5.2.4 *Let $\vec{\Phi}(L) = (\Phi_i(L))_{i=1}^r$ be the component-wise cocycle invariant of a link $L = K_1 \cup \cdots \cup K_r$ with a quandle X and a cocycle $\phi \in Z^2_{\mathrm{Q}}(X; A)$ for an abelian group A. Then, $\Phi_i(L)$ is not a positive integer for some i if and only if there is a coloring of L by X that does not extend to a coloring of L by $E(X, A, \phi)$.*

Example 5.2.5 Let L be a colored Whitehead link $L = K_1 \cup K_2$, as depicted in Figure 15. We have seen in Example 4.2.1 that the component-wise cocycle invariant is $\vec{\Phi}(L) = (32 + 32t, 32 + 32t)$, where the cocycle ϕ defines the extension $E = R_{16} = E(R_8, \mathbb{Z}_2, \phi)$.

Theorem 5.2.4 implies that there are colorings by R_8 that do not extend to colorings by R_{16}. In fact, from the proof of Theorem 5.2.1, we see that 32 colorings having the same parity for a and b extend to R_{16}, and those 32 colorings with the opposite parities do not. This fact can be computed directly, and gives an alternate method of computing the above invariant using Corollary 5.2.3.

Example 5.2.6 Let L be the $(4,2)$-torus link (see Figure 14) colored by the quandle $X = R_4$. The extension $E = R_8 = E(R_4, \mathbb{Z}_2, \phi)$ is defined by the cocycle ϕ (see Example 4.1.4(1)), where $\phi = \chi_{0,2} + \chi_{0,3} + \chi_{1,0} + \chi_{1,3} + \chi_{2,0} + \chi_{2,3} + \chi_{3,0} + \chi_{3,1}$. The component-wise invariant is computed to be $\vec{\Phi}(L) = (8+8t, 8+8t)$. Then, by Theorem 5.2.4 we can assert that there are 8 colorings by R_4 that extend to colorings by R_8, and 8 that do not extend. The colorings that do extend are listed in Example 5.1.2.

From the examples 4.2.2 and 4.2.3, we see that the cocycle invariant is non-trivial when the given link is colored by $X = \mathbb{Z}_q[T, T^{-1}]/(1-T)^m$, but not by $E = \mathbb{Z}_q[T, T^{-1}]/(1-T)^{m+1}$, and the descrepancy in extending the coloring contributes to the invariant. This is the case in general, as proved in [5] for the knot case. We rephrase the theorem in our situation and include a similar proof for reader's convenience.

Theorem 5.2.7 [7] Let $\vec{\Psi}(L) = \{(\prod_{\tau \in \mathcal{T}_1} B(\tau, \mathcal{C}), \ldots, \prod_{\tau \in \mathcal{T}_r} B(\tau, \mathcal{C}))\}_{\mathcal{C} \in \mathrm{Col}_X(L)}$ be the generalized cocycle invariant of a link $L = K_1 \cup \cdots \cup K_r$ with a quandle X and a cocycle $\phi \in Z^2_{\mathrm{Q}}(X; A)$, for an abelian group A. Then, $(\prod_{\tau \in \mathcal{T}_1} B(\tau, \mathcal{C}), \ldots, \prod_{\tau \in \mathcal{T}_r} B(\tau, \mathcal{C}))$ is a vector with every entry 1 for a coloring $\mathcal{C}$ if and only if the coloring $\mathcal{C}$ extends to a coloring of L by $E(X, A, \phi)$.

Proof. Let $\mathcal{C}$ be a coloring whose contribution to $\vec{\Psi}(L)$ is $(1, \ldots, 1)$. Fix this coloring in what follows. Pick a base point b_0 on a component K_i of L. Let $x \in X$ be the color on the arc α_0 containing b_0. Let α_i, $i = 1, \ldots, n$, be the set of arcs that appear in this order when the diagram K is traced in the given orientation of K_i, starting from b_0. Pick an element $a \in A$ and give a color (a, x) on α_0, so that we define a coloring $\mathcal{C}'$ by E on α_0 by $\mathcal{C}'(\alpha_0) = (a, x) \in E$. We try to extend it to the entire diagram by traveling the diagram from b_0 along the arcs α_i, $i = 1, \ldots, n$, in this order, by induction.

Assume $\mathcal{C}'(\alpha_i)$ is defined for $0 \leq i < k$. Define $\mathcal{C}'(\alpha_{k+1})$ as follows. Suppose that the crossing τ_k separating α_k and α_{k+1} is positive, and the over-arc at τ_k is α_j. Let $\mathcal{C}'(\alpha_k) = (a, x)$ and $\mathcal{C}(\alpha_j) = y \in X$. Then, we have $\mathcal{C}(\alpha_{k+1}) = x * y \in X$. Define $\mathcal{C}'(\alpha_{k+1}) = (a\phi(x, y), x * y)$ in this case.

Suppose that the crossing τ_k is negative. Let $\mathcal{C}'(\alpha_k) = (a, x)$ and $\mathcal{C}(\alpha_j) = y \in X$. Therefore, if $\mathcal{C}(\alpha_{k+1}) = z$, then we have $z * y = x$. Define $\mathcal{C}'(\alpha_{k+1}) = (a\phi(z, y)^{-1}, z)$ in this case.

Define $\mathcal{C}'(\alpha_i)$ inductively for all $i = 0, \ldots, n$. Regard α_0 as α_{n+1}, and repeat the above construction at the last crossing τ_n to come back to α_0. By the construction we have $\mathcal{C}'(\alpha_{n+1}) = (a \prod_\tau B(\tau, \mathcal{C}), \mathcal{C}(\alpha_0))$, where $\prod_\tau B(\tau, \mathcal{C})$ is the state-sum contribution of $\mathcal{C}$. This contribution is equal to 1 by the assumption that $\prod_\tau B(\tau, \mathcal{C}) = 1$, and we have a well-defined coloring $\mathcal{C}'$. Hence, this color extends to $E(X, A, \phi)$.

Conversely, if a coloring $\mathcal{C}$ by X extends to a coloring by $E(X, A, \phi)$, then from the above argument, we have that $(a, x) = (a \prod_\tau B(\tau, \mathcal{C}), x)$, if (a, x) is the color on the base point b_0. Hence, $\prod_\tau B(\tau, \mathcal{C}) = 1$. □

Example 5.2.8 Let L be a colored Whitehead link shown in Figure 15. The generalized cocycle invariant $\vec{\Psi}(L)$ is given by $\vec{\Psi}(L) = \{\underbrace{(1,1), \ldots, (1,1)}_{8 \text{ copies}}, \underbrace{(t,t), \ldots, (t,t)}_{8 \text{ copies}}\}$, where the cocycle ϕ defines the extension $E = R_8 = E(R_4, \mathbb{Z}_2, \phi)$. By Theorem 5.2.7, a vector with every entry 1 is a coloring that extends to a coloring of the torus link L by its extension $E = R_8$. Hence, we have 8 such coloring extensions.

CHAPTER 6

RELATIONS TO ALEXANDER MATRICES

The discovery of the Alexander matrix and the Alexander polynomial was one of the early achievements in knot theory. In this chapter, we point out relations of the cocycle invariants to Alexander matrices. We examine closely the two examples given in Section 3.3 from this new point of view.

6.1 Cocycle invariants and Alexander matrices

For a link diagram D_L let $B_{D_L} = \sum_{i=1}^{n} B_i$ be an $(n \times n)$-matrix, where B_i is the $(n \times n)$-matrix corresponding to each crossing point τ_i (see Figure 15) such that the (k_i, i) entry is T^{ε_i}, the (ℓ_i, i) entry is $1 - T^{\varepsilon_i}$ and otherwise is 0. Here, ε_i denotes the sign of the crossing point τ_i. Set $A_{D_L} = B_{D_L} - E_n$, where E_n denotes the n-dimensional identity matrix. It follows from the definitions [24] that A_{D_L} is an Alexander matrix. Recall that a coloring is a function $\mathcal{C} : R \to X$, where R is the set of over-arcs in the diagram and X is a fixed Alexander quandle Λ/J for an ideal J. A coloring which assigns w_i to an arc a_i ($\mathcal{C}(a_i) = w_i$) is represented by the vector $\vec{w} = (w_1, \ldots, w_n)$ satisfying $\vec{w}A_{D_L}^{(X)} = \vec{0}$. These descriptions are given in [24] to prove Theorem 2.4.4.

Proposition 6.1.1 *Let $L = K_1 \cup \cdots \cup K_r$ be a link and $X = \Lambda_q/J$ be an Alexander quandle. Suppose $E = \Lambda_{q'}/J'$ is an abelian extension of X, where q, q' are positive integers. Let $A_{D_L}^{(X)}$ (respectively $A_{D_L}^{(E)}$) be the matrix A_{D_L} regarded as a matrix over X (respectively over E). Then, a coloring $\vec{w}$ of L by X contributes a non-trivial value to the invariant $\vec{\Psi}(L)$ if and only if $\vec{w}A_{D_L}^{(X)} = \vec{0}$ and $s(\vec{w})A_{D_L}^{(E)} = \vec{x} \neq \vec{0}$, where $s : X \to E$ is a set-theoretic section.*

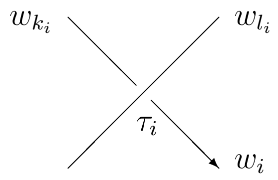

Figure 18. Labeling a crossing

Proof. Let $\psi : (\Lambda_q/J)^n \to (\Lambda_q/J)^n$ be the map which takes a row vector $\vec{w}$ to $\vec{w}A_{D_L}$. By Inoue's description given above, the set of all quandle colorings is equal to $\ker A_{D_L}^{(X)}$. If $\vec{w}A_{D_L}^{(X)} = \vec{0}$ and $s(\vec{w})A_{D_L}^{(E)} = \vec{x} \neq 0$, then by Theorem 5.2.7 we obtain that $\vec{\Psi}(L)$ is non-trivial. □

Next, we compute the non-trivial contributions using Alexander matrices, for the extensions discussed in Section 4.1. Let $X = W_m = \Lambda_q/(1-T)^m$ or $X = U_m = \Lambda_{q^m}/(T-1+q)$, and let $E = W_{m+1}$ or $E = U_{m+1}$ be their abelian extensions, respectively. For this purpose, we fix the following convention in numbering crossings and arcs of a given diagram.

Let $L = K_1 \cup \cdots \cup K_r$ be a link with n crossings. Pick a base point b_i on K_i, for $i = 1, \ldots, r$. Let $a_1, \ldots, a_{i_1}$ be the arcs of K_1 such that a_1 contains b_1 and that they appear in this order when one traces K_1 in the given orientation of K_1 starting from b_1. Then, let a_{i_1+1} be the arc of K_2 containing b_2 and $a_{i_1+2}, \ldots, a_{i_2}$ be the arcs of K_2 similarly defined from the given orientation. Repeat this process for the remaining components to obtain the arcs $a_1, \ldots, a_{i_1}, a_{i_1+1}, \ldots, a_{i_2}, a_{i_2+1}, \ldots, a_{i_{r-1}+1}, \ldots, a_{i_r} = a_n$. Let $\mathcal{C} : R \to X$ be a coloring of L by X. Let $w_i = \mathcal{C}(a_i)$ and τ_i be the crossing such that the outcoming under-arc is a_i for $i = 1, \ldots, n$ (see Figure 18). This convention is used in Figure 15.

Let $s : X \to E$ be the section defined in Section 4.1 by

$$s\left(\sum_{j=0}^{m-1} A_j(1-T)^j \mod (1-T)^m\right) = \sum_{j=0}^{m-1} A_j(1-T)^j \mod (1-T)^{m+1} \quad \text{for } W_m,$$

and

$$s\left(\sum_{j=0}^{m-1} X_j q^j\right) = 0 \cdot q^m + \sum_{j=0}^{m-1} X_j q^j \quad \text{for } U_m.$$

For the following proposition, let $\vec{\Psi}(L)$ be the generalized cocycle invariant defined with the cocycle $\phi \in Z^2_{\mathrm{Q}}(X;\mathbb{Z}_q)$ corresponding to the extension $p: E \to X$ specified by Definition 3.3.3.

Proposition 6.1.2 *Let A_{D_L} be the Alexander matrix obtained from a diagram D_L with the above choice of order of w_i and τ_i.*

A given coloring represented by a vector $\vec{w}$ contributes a non-trivial vector to the invariant $\vec{\Psi}(L)$ if and only if $\vec{w}A^{(X)}_{D_L} = \vec{0}$ and $s(\vec{w})A^{(E)}_{D_L} = \vec{z} \neq \vec{0}$. This contribution is

$$(t^{\sum_{j=1}^{i_1} \eta(\tau_j)z_j/(1-T)^m}, \ldots, t^{\sum_{j=i_{r-1}+1}^{i_r} \eta(\tau_j)z_j/(1-T)^m}) \qquad \text{for } X = W_m,$$

and

$$(t^{\sum_{j=1}^{i_1} \eta(\tau_j)z_j/q^m}, \ldots, t^{\sum_{j=i_{r-1}+1}^{i_r} \eta(\tau_j)z_j/q^m}) \qquad \text{for } X = U_m,$$

respectively, where $\eta(\tau) = 1$ for a positive crossing τ and $\eta(\tau) = T$ for a negative crossing τ.

Proof. We consider the case $X = W_m$, as the other case is similar. Let $\psi : (\Lambda_q/(1-T)^m)^n \to (\Lambda_q/(1-T)^m)^n$ be the map which takes a row vector $\vec{w}$ to $\vec{w}A^{(X)}_{D_L}$. Assume that $\vec{w}A^{(X)}_{D_L} = \vec{0}$ and $s(\vec{w})A^{(E)}_{D_L} = \vec{z} \neq \vec{0}$. The contribution to the invariant at a positive crossing τ_i is given by

$$\begin{aligned}
\phi(w_{k_i}, w_{\ell_i}) &= [s(w_{k_i}) * s(w_{\ell_i}) - s(w_{k_i} * w_{\ell_i})]/(1-T)^m \\
&= [s(w_{k_i}) * s(w_{\ell_i}) - s(w_i)]/(1-T)^m,
\end{aligned}$$

where w_{ℓ_i} is the color on the over-arc at the crossing τ_i, and w_{k_i} is the color on the incoming under-arc at τ_i if τ_i is positive (see Figure 18). Since $\vec{w}$ is in the kernel, $w_{k_i} * w_{\ell_i} - w_i = Tw_{k_i} + (1-T)w_{\ell_i} - w_i = 0 \mod(1-T)^m$ and we have $[s(w_{k_i}) * s(w_{\ell_i}) - s(w_i)]/(1-T)^m = z_i/(1-T)^m$.

Suppose τ_i is negative. Then, the contribution is

$$\begin{aligned} -\phi(w_i, w_{\ell_i}) &= -[s(w_i) * s(w_{\ell_i}) - s(w_i * w_{\ell_i})]/(1-T)^m \\ &= -[s(w_i) * s(w_{\ell_i}) - s(w_{k_i})]/(1-T)^m \\ &= -[Tw_i + (1-T)w_{\ell_i} - w_{k_i}]/(1-T)^m. \end{aligned}$$

On the other hand,

$$z_i = T^{-1}w_{k_i} + (1-T^{-1})w_{\ell_i} - w_i = -T^{-1}[Tw_i + (1-T)w_{\ell_i} - w_{k_i}]$$

so that the contribution is Tz_i, in this case. Hence, the total contribution of the invariant for the component K_r is

$$t^{\sum_{j=i_{r-1}+1}^{i_r} \eta(\tau_j) z_j/(1-T)^m},$$

where $\{z_1, \ldots, z_{i_r}\} \in K_r$. $\square$

Example 6.1.3 We consider the Whitehead link $L = K_1 \cup K_2$ depicted in Figure 15. Let $X = W_m$ and $E = W_{m+1}$. Use the letters w_i $(i = 1, \ldots, 6)$ as shown in the figure as colors assigned to the arcs, as well as generators for the Alexander matrix. Then, the Alexander matrix $A_{D_L} = B_{D_L} - E_n$ with respect to the columns corresponding to $(\tau_1, \ldots, \tau_6)$ and rows corresponding to $(w_1, \ldots, w_6)$ is given by

$$A_{D_L} = \begin{pmatrix} -1 & T & 0 & 0 & 0 & 0 \\ T^{-1} & -1 & 0 & 1-T & 0 & 1-T^{-1} \\ 0 & 1-T & -1 & T & 0 & 0 \\ 0 & 0 & 1-T & -1 & T & 0 \\ 0 & 0 & 0 & 0 & -1 & T^{-1} \\ 1-T^{-1} & 0 & T & 0 & 1-T & -1 \end{pmatrix}.$$

After some row and column permutations we obtain

$$A_0 = \begin{pmatrix} -1 & 1-T & 0 & 0 & 1-T^{-1} & T^{-1} \\ 0 & -1 & 1-T & T & 0 & 0 \\ 0 & 0 & T & 1-T & -1 & 1-T^{-1} \\ 0 & 0 & 0 & -1 & T^{-1} & 0 \\ T & 0 & 0 & 0 & 0 & -1 \\ 1-T & T & -1 & 0 & 0 & 0 \end{pmatrix},$$

with respect to the columns corresponding to $(\tau_2, \tau_4, \tau_3, \tau_5, \tau_6, \tau_1)$ and rows corresponding to $(w_2, w_4, w_6, w_5, w_1, w_3)$. This permutation is performed so that we can diagonalize the first four rows and columns by column reductions to obtain

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -T & -T+T^2 & (1-T)^2 & 1-3T+2T^2 & -T^{-1}(1-T)^3 & T^{-1}(1-T)^3 \\ -1+T & -1+T-T^2 & -1-(1-T)^2 & -2+3T-2T^2 & T^{-1}(1-T)^3 & -T^{-1}(1-T)^3 \end{pmatrix}.$$

The solution set

$$(w_2, w_4, w_6, w_5, w_1, w_3)A_1^{(X)} = (0, 0, 0, 0, 0, 0),$$

is written by

$$\begin{aligned}
w_2 &= Tw_1 + (1-T)w_3 \\
w_4 &= T(1-T)w_1 + (T + (1-T)^2)w_3 \\
w_6 &= -(1-T)^2 w_1 + (1 + (1-T)^2)w_3 \\
w_5 &= (T(1-T) - (1-T)^2)w_1 + (T + 2(1-T)^2)w_3 \\
0 &= (w_3 - w_1)T^{-1}(1-T)^3
\end{aligned}$$

where $A_1^{(X)}$ denotes the matrix A_1 regarded as a matrix over X. The set of colorings is represented by vectors in the kernel of $A_1^{(X)}$. Specifically, the kernel is the set of vectors $\vec{w}$ with w_1 and w_3 satisfying $(1-T)^3(w_3 - w_1) = 0$ in X and w_2, w_4, w_6, w_5 determined accordingly as above. This matches the computations in Example 4.2.2. The contribution to the invariant is obtained by computing

$$\begin{aligned}
\vec{z} &= s(\vec{w})A_{D_L}^{(E)} \\
&= (-T^{-1}(1-T)^3(w_3 - w_1),\ 0,\ 0,\ 0,\ 0,\ T^{-1}(1-T)^3(w_3 - w_1)).
\end{aligned}$$

By Proposition 6.1.2, the non-trivial contribution to $\vec{\Psi}(L)$ is

$$(t^{\sum_{j=1}^{2} \eta(\tau_j) z_j/(1-T)^3}, t^{\sum_{j=3}^{6} \eta(\tau_j) z_j/(1-T)^3}) = (t^{-s}, t^{s})$$

for some s, for $0 \leq s \leq q-1$, depending on the value of $w_3 - w_1$. This result matches the one in Example 4.2.2.

Example 6.1.4 Let $L = K_1 \cup K_2 \cup K_3$ be the Borromean rings as depicted in Figure 17. The Alexander matrix A_{D_L}, where the rows correspond to the crossings $\tau_1, \ldots, \tau_6$, and the columns correspond to $y_1, \ldots, y_6$ from left to right, respectively, is given by the matrix

$$A_{D_L} = \begin{pmatrix} -1 & 0 & 0 & 1-T & T & 0 \\ 0 & -1 & 0 & 0 & 1-T & T \\ 0 & 0 & -1 & T & 0 & 1-T \\ 0 & 1-T^{-1} & T^{-1} & -1 & 0 & 0 \\ T^{-1} & 0 & 1-T^{-1} & 0 & -1 & 0 \\ 1-T^{-1} & T^{-1} & 0 & 0 & 0 & -1 \end{pmatrix}.$$

The vector $\vec{y}$ has 3 independent entries y_1, y_2, y_3, and the other three entries y_4, y_5, y_6 are linear combinations of these. The solution vector is given by

$$\left(y_1,\ y_2,\ y_3,\ y_4 = (1-T)y_1 + Ty_3,\ y_5 = Ty_1 + (1-T)y_2,\ y_6 = Ty_2 + (1-T)y_3 \right).$$

The solution set $\vec{y}$ can also be obtained by column reductions as was done in the previous example. By rearranging rows and columns, we obtain a new matrix A_1 in such a way that the rows of A_1 correspond to $(y_4, y_5, y_6, y_1, y_2, y_3)$ and the columns of A_1 correspond to $(\tau_4, \tau_5, \tau_6, \tau_1, \tau_2, \tau_3)$ so that

$$A_1 = \begin{pmatrix} -1 & 0 & 0 & 0 & 1-T^{-1} & T^{-1} \\ 0 & -1 & 0 & T^{-1} & 0 & 1-T^{-1} \\ 0 & 0 & -1 & 1-T^{-1} & T^{-1} & 0 \\ 1-T & T & 0 & -1 & 0 & 0 \\ 0 & 1-T & T & 0 & -1 & 0 \\ T & 0 & 1-T & 0 & 0 & -1 \end{pmatrix}.$$

By this rearrangement, we can diagonalize the first three rows and columns by column reductions to obtain

$$A_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ T-1 & -T & 0 & 0 & -T^{-1}(1-T)^2 & T^{-1}(1-T)^2 \\ 0 & T-1 & -T & T^{-1}(1-T)^2 & 0 & -T^{-1}(1-T)^2 \\ -T & 0 & T-1 & -T^{-1}(1-T)^2 & T^{-1}(1-T)^2 & 0 \end{pmatrix}.$$

Hence, the solution set is written by

$$(y_4, y_5, y_6, y_1, y_2, y_3)A_2^{(X)} = (0, 0, 0, 0, 0, 0)$$

and we obtain

$$\begin{aligned} & y_4 = (1-T)y_1 + Ty_3, \\ & y_5 = Ty_1 + (1-T)y_2, \\ & y_6 = Ty_2 + (1-T)y_3, \\ & \left.\begin{aligned} & y_1, \\ & y_2, \\ & y_3 \end{aligned}\right\} \text{free variables.} \end{aligned}$$

Then, $s(\vec{y})A_{D_L} = \vec{x}$, where the vector $\vec{x}$ is given by

$$\vec{x} = \left(\; T^{-1}(1-T)^2(y_2 - y_3), \; T^{-1}(1-T)^2(y_3 - y_1), \; T^{-1}(1-T)^2(y_1 - y_2), \; 0, \; 0, \; 0 \; \right).$$

The first statement of Theorem 6.1.1 (the coloring condition) implies that L is colored by $X = \Lambda_q/J$ if and only if $\vec{x} = \vec{0}$ in X, i.e., $(1-T)^2(y_2-y_3) = 0,\ (1-T)^2(y_3-y_1) = 0,\ (1-T)^2(y_1-y_2) = 0$ in X. This condition matches the one found in Example 4.2.2

for the quandles X considered therein. The second statement of Theorem 6.1.1 implies, in particular, that the cocycle invariant is non-trivial for $X = \Lambda_q/(1-T)^2$ and $E = \Lambda_q/(1-T)^3$, as we have seen in Example 4.2.3. Note that if we use the rearrangement of the arcs $(w_1, \ldots, w_6)$ we defined before Theorem 6.1.2, we get $w_1 = y_1,\ w_2 = y_5,\ w_3 = y_2,\ w_4 = y_6,\ w_5 = y_3,\ w_6 = y_4$ with appropriate base points. As we can see from Figure 17, y_1, y_5 are colors assigned to the component K_1, y_2, y_6 are for K_2, and y_3, y_4 are for K_3.

$$
\begin{aligned}
\vec{z} &= (z_1, \cdots, z_6) \\
&= (T^{-1}(1-T)^2(z_3 - z_5),\ 0,\ T^{-1}(1-T)^2(z_5 - z_1),\ 0,\ T^{-1}(1-T)^2(z_1 - z_3),\ 0),
\end{aligned}
$$

where z_1, z_2 belong to K_1, z_3, z_4 belong to K_2, and z_5, z_6 belong to component K_3. By Theorem 6.1.2, the non-trivial contribution to $\vec{\Psi}(L)$ is

$$
(t^{\sum_{j=1}^{2}\eta(\tau_j)z_j/(1-T)^2}, t^{\sum_{j=3}^{4}\eta(\tau_j)z_j/(1-T)^2}, t^{\sum_{j=5}^{6}\eta(\tau_j)z_j/(1-T)^2}) = (t^k, t^\ell, t^{-(k+\ell)})
$$

for some k, ℓ, where $0 \le k, \ell \le q-1$, depending on the values of $w_3 - w_5$ and $w_5 - w_1$. Note that this matches the conclusion of Example 4.2.3.

6.2 A relation to Alexander-Conway polynomial

In this section we describe a relation of cocycle invariants originating from extensions to the Conway polynomial.

Let $\Delta_L(T) \in \mathbb{Z}[T^{-\frac{1}{2}}, T^{\frac{1}{2}}]$ be the *Conway-normalized Alexander polynomial* [30]. In our case, let A'_{D_L} be the matrix obtained from A_{D_L} by deleting the ith column and ith row for some i, $i = 1, \ldots, n$, let $f(T) = \det(A'_{D_L}) \in \mathbb{Z}[T^1, T^{-1}]$ and μ and ν be the maximal and minimal degree of f respectively. Then, $\Delta_L(T) = T^{-\frac{\mu+\nu}{2}} f(T)$.

The *Conway polynomial* $\nabla_L(z) \in \mathbb{Z}[z]$ is defined by $\nabla_L(T^{-\frac{1}{2}} - T^{\frac{1}{2}}) = \Delta_L(T)$, where $z = T^{-\frac{1}{2}} - T^{\frac{1}{2}}$.

Proposition 6.2.1 *Let the minimal degree of* $\nabla_L(z)$ *be denoted by* min-deg$\nabla_L(z)$*, then it satisfies* min-deg$\nabla_L(z) \geq m$*, where* m *is the smallest integer such that the cocycle invariant defined from the extension of* $\mathbb{Z}_q[T, T^{-1}]/(1-T)^m$ *to* $\mathbb{Z}_q[T, T^{-1}]/(1-T)^{m+1}$ *is non-trivial.*

Proof. Assume that $\vec{y}A^{(X)}_{D_L} = \vec{0}$ and $s(\vec{y})A^{(E)}_{D_L} = \vec{x} \neq \vec{0}$. Then $\vec{y}$ contributes a non-trivial value to the invariant $\vec{\Psi}(L)$ as in Proposition 6.1.1. Since $\vec{x} \neq \vec{0}$ there exists i, $1 \leq i \leq n$, such that $x_i \neq 0$. Let j be an integer, $1 \leq j \leq n$, with $j \neq i$. Let $\vec{x'}$ be the vector $\vec{x}$ with the x_j entry deleted. Then, there exists $\vec{y'} \neq \vec{0}$, where $\vec{y'}$ is the vector $\vec{y}$ with the jth entry deleted, such that $\vec{y'}A'^{(X)}_{D_L} = \vec{0}$. This implies that $\det A'^{(X)}_{D_L} = 0$. Hence, $\det A'_{D_L} \equiv 0 \pmod{(1-T)^m}$, and we have min-deg$\nabla_L(z) \geq m$. □

REFERENCES

[1] J. W. Alexander. A lemma on a system of knotted curves. *Proc. Natl. Acad. Sci. USA*, 9:93–95, 1928.

[2] N. Andruskiewitsch and M. Graña. From racks to pointed hopf algebras. *Adv. Math.*, to appear, preprint at http://arxiv.org/abs/math.QA/0202084.

[3] E. Brieskorn. Automorphic sets and singularities. *Contemp. Math.*, 78:45–115, 1988.

[4] K. S. Brown. *Cohomology of Groups*, volume 87 of *Grad. Texts Math.* Springer-Verlag, New York-Berlin, 1982.

[5] J. S. Carter, M. Elhamdadi, M. Appiou Nikiforou, and M. Saito. Extensions of quandles and cocycle knot invariants. *J. Knot Theory Ramifications*, to appear, preprint at http://arxiv.org/abs/math.GT/0107021.

[6] J. S. Carter, M. Elhamdadi, and M. Saito. Twisted quandle homology theory and cocycle knot invariants. *Algebr. Geom. Topol.*, pages 95–135, 2002.

[7] J. S. Carter, A. Harris, M. Appiou Nikiforou, and M. Saito. Cocycle knot invariants, quandle extensions, and alexander matrices. *Kyoto University, (RIMS Kokyuroku 1272)*, pages 12–35, June 2002.

[8] J. S. Carter, D. Jelsovsky, S. Kamada, L. Langford, and M. Saito. Quandle cohomology and state-sum invariants of knotted curves and surfaces, preprint at http://arxiv.org/abs/math.GT/9903135.

[9] J. S. Carter, D. Jelsovsky, S. Kamada, and M. Saito. Computations of quandle cocycle invariants of knotted curves and surfaces. *Adv. Math.*, 157(1):36–94, 2001.

[10] J. S. Carter, D. Jelsovsky, S. Kamada, and M. Saito. Quandle homology groups, their betti numbers, and virtual knots. *J. Pure Appl. Algebra*, 157:135–155, 2001.

[11] J. S. Carter, S. Kamada, and M. Saito. Diagrammatic computations for quandles and cocycle knot invariants, preprint at http://arxiv.org/abs/math.GT/0102092.

[12] J. S. Carter, S. Kamada, and M. Saito. Geometric interpretations of quandle homology. *J. Knot Theory Ramifications*, 10(3):345–358, 2001.

[13] J. H. Conway. An enumeration of knots and links, and some of their algebraic properties. In *Computational Problems in Abstract Algebra*, Proc. Conf., Oxford, 1967, pages 329–358, Pergamon, Oxford, 1970.

[14] P. Etingof and M. Graña. On rack cohomology. *J. Pure Appl. Algebra*, to appear, preprint at http://arxiv.org/abs/math.QA/0201290.

[15] R. Fenn and C. Rourke. Racks and links in codimension two. *J. Knot Theory Ramifications*, 1(4):343–406, 1992.

[16] R. Fenn, C. Rourke, and B. Sanderson. James bundles and applications, preprint at http://www.maths.warwick.ac.uk/~bjs.

[17] R. Fenn, C. Rourke, and B. Sanderson. Trunks and classifying spaces. *Appl. Categ. Struct.*, 3(4):321–356, 1995.

[18] J. Flower. *Cyclic Bordism and Rack Spaces.* PhD thesis, Warwick, 1995.

[19] R. H. Fox. A quick trip through knot theory. In *Topology of 3-manifolds and related topics*, pages 120–167. Prentice-Hall, 1962.

[20] N. D. Gilbert and T. Porter. *Knots and Surfaces.* Oxford University Press, United Kingdom, 1994.

[21] M. Graña. Indecomposable racks of order p^2, preprint at http://arxiv.org/abs/math.QA/0203157.

[22] M. Graña. On Nichols algebras of low dimension. *New trends in Hopf algebra theory, Contemp. Math.*, 267:111–136, 2000.

[23] M. T. Green. *Some Results in Geometric Topology and Geometry.* PhD thesis, Warwick, 1997.

[24] A. Inoue. Quandle homomorphisms of knot quandles to alexander quandles. *J. Knot Theory Ramifications*, 10(6):813–821, 2001.

[25] D. Jelsovsky. *Quandle Cohomology and State-Sum Invariants of Knots.* PhD thesis, University of South Florida, August 2000.

[26] V. F. R. Jones. Hecke algebra representations of braid groups and link polynomials. *Ann. Math. 2*, 126:335–388, 1987.

[27] D. Joyce. A classifying invariant of knots, the knot quandle. *J. Pure Appl. Algebra*, 23:37–65, 1982.

[28] L. H. Kauffman. *Knots and Physics*, volume 1 of *Series on knots and everything.* World Scientific, 1991.

[29] A. Kawauchi. *A Survey of Knot Theory.* Birkhäuser, 1996.

[30] W. B. R. Lickorish. *An introduction to knot theory*, volume 175 of *Grad. Texts Math.* Springer-Verlag New York Inc., 1997.

[31] R. A. Litherland and S. Nelson. The betti numbers of some finite racks, preprint at http://arxiv.org/abs/math.GT/0106165.

[32] P. Lopes. Quandles at finite temperatures I. *J. Knot Theory Ramifications*, to appear, preprint at http://arxiv.org/abs/math.QA/0105099.

[33] S. V. Matveev. Distributive groupoids in knot theory. *Math. USSR-Sbornik*, 47:73–83, 1982. In Russian.

[34] T. Mochizuki. Some calculations of cohomology groups of finite alexander quandles, preprint.

[35] K. Murasugi. *Knot Theory and Its Applications.* Birkhäuser Boston, Inc., 1996. Translated from the 1993 Japanese original by B. Kurpita.

[36] S. Nelson. Classification of finite alexander quandles, preprint at http://arxiv.org/abs/math.GT/0202281.

[37] T. Ohtsuki. Problems on invariants of knots and 3-manifolds, preprint.

[38] P. G. Tait. On knots I, II, III. *Scientific Papers*, 1:273–437, 1898. London, Cambridge University Press.

[39] M. Takasaki. Abstraction of symmetric transformations. *Tohoku Math. J.*, 49:145–207, 1942/43. In Japanese.

ABOUT THE AUTHOR

Marina Appiou Nikiforou was born on October 9, 1974 in Platres, Cyprus. She received a Bachelor's Degree in Mathematics from University of Cyprus in 1996. She earned the following degrees from University of South Florida: Master's Degree in Mathematics in 1998, and her Doctor of Philosophy in Mathematics with emphasis in Knot Theory. Mrs. Nikiforou received three scholarships and worked as a teaching assistant.