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A Study of the Critical Condition of a Battened Column and a Frame by Classical Methods

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of a Battened Column and a Frame

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by

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ABSTRACT

Knowledge of structural stability theory is of paramount importance to the practicing structural engineer. In many instances, buckling is the primary consideration in the design of various structural configurations. The first chapter introduces a simplified treatment of the elastic stability of a battened column using classical methods without getting involved with lengthy and complicated mathematical operations. In chapter two, a treatment of the elastic stability of a frame is presented, including effects of elastic restraints. In this study, a theoretical treatment is given which although approximate, is believed to constitute a satisfactory solution of the structure.

Chapter One: Classical Solution of the Battened Column

Introduction

One of the commonest open-panel structures is the battened column that consists of two flanges connected by evenly spaced flat batten plates. Fig. 1 shows diagrammatically the column to be studied. Two equal flange members are connected by flat batten plates in such a way that the joints are rigid, i.e. there will be no relative movements between plates and flanges at their junctions. It will be assumed that the general design is such as to eliminate the possibility of any secondary failure, the batten plates might fail in bending if they were made thin or insufficiently wide. It will also be assumed that the distance between batten plates is much smaller than the length of the column and that the flanges, instead of being joined by a finite number of batten plates, are connected by a web medium which can apply flexural restraint to the flanges.

The flexural rigidity of this web will be taken as $E'I'$ per unit length of column.

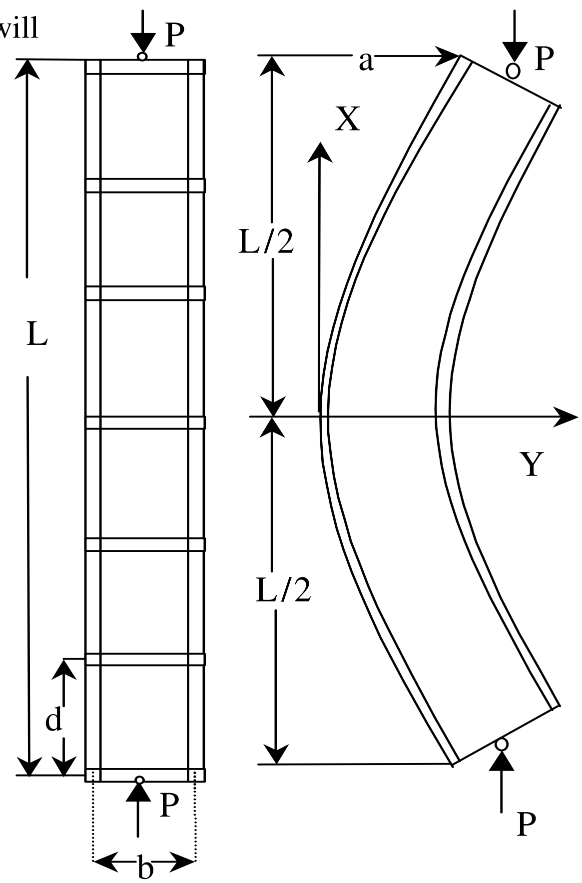


Figure 1. Battened column

Analysis

Let P be the critical axial compression on the pin-ended column, i.e. the compression which would produce instability of the column as a whole,

A, the area of each flange

I, the second moment of area of each flange

Ic, the second moment of area of batten plates.

I′, the second moment of area of the web medium about the axis of bending per unit length

L, the length of the column between pins

b, the length of the batten plates

d, the distance between batten plates

n, the number of batten plates

n′, the stiffness ratio $\frac{E_C I_C}{EI}\frac{L}{d}$

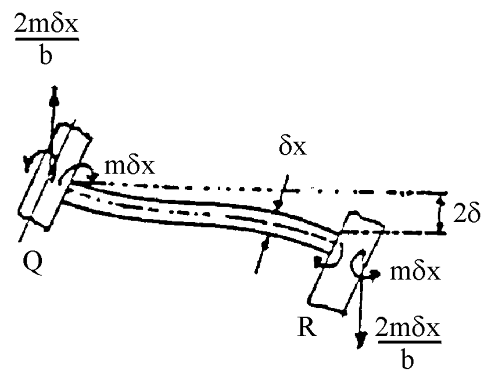


Figure 2. Unit element of web

Q, the critical load for one flange $= \frac{\pi^2 EI}{L^2}$

When flexure occurs the conditions at one of the batten plates are shown in Fig.2. The plate is considered to be an element of the web of length δx. When the column is deflected from its initial position the element is distorted as shown and applies couples of magnitude $m\delta x$ to each flange and also reactive forces, $\frac{2m\delta x}{b}$. The reactions increase the length of the left flange and decrease the length of the right flange. The slope of the flanges at any point is therefore the sum of two components; ϕ, caused by these longitudinal strains and θ, due to the flexure of the batten plates. It will be assumed that the contribution of ϕ is very small compared with that due to the cross beams and it will therefore be neglected. To determine θ, let QR in Fig. 2. represent a unit element of the web carrying couples m at each end and equilibrating forces $\frac{2m}{b}$ as shown.

At a distance x from Q

$$E_C I_C \frac{d^2y}{dx^2} = -m + \frac{2mx}{b}$$

$$E_C I_C \frac{dy}{dx} = -mx + \frac{mx^2}{b} + A$$

$$E_C I_C y = -\frac{mx^2}{2} + \frac{mx^3}{3b} + Ax + B$$

The second constant of integration B being 0 since y is 0 at Q. From the symmetry of the loading there can be no deflection at the midpoint of the element, i.e. y = 0 when x = b/2 and so $A = \frac{mb}{6}$.

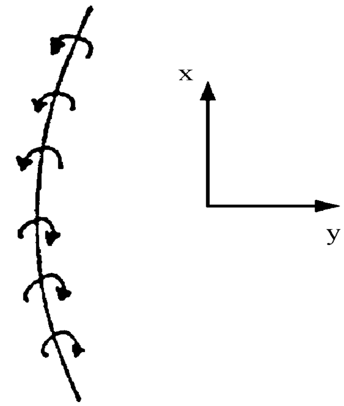


Figure 3. Bending moments on column

Therefore when x = 0, the slope at the end of the element is $\frac{dy}{dx} = \frac{mb}{6E_C I_C}$. This slope is the same as that of the flanges at Q and R.

If bending moments required to produce curvature, as shown in Fig. 3 are taken to be positive, the slope at any point, $\frac{dy}{dx}$, is positive.

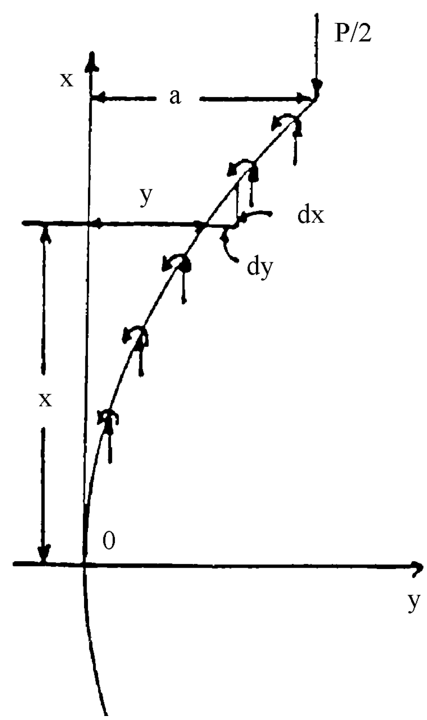


Figure 4. Displaced column

Assuming that $d \leq L$ and if there are n equal batten plates in a length L of the structure the total flexural rigidity is $nE_C I_C$ and the flexural rigidity of the medium per unit length is $n\,E_C I_C / L$, which will be denoted by $E'I'$.

And so, if the equation for the slope derived for a batten plate be rewritten in terms of the continuous medium, we have for either flange

$$EI\frac{dy}{dx} = \frac{mb\,EI}{6E'I'} \quad \text{..............................} \ (1)$$

If the column is displaced from its normal position when carrying a load less than the critical value, it will recover its straightness when the displacing force is removed. When it carries the critical load it will remain in its displaced position. Fig. 4 shows the column in this state having been displaced an arbitrary amount a at the center.

Since the flanges are assumed to be similar the load is evenly divided between them and each carries P/2. If an origin be taken at the center of the displaced column and x and y measured in the direction shown then the moment in each flange at x from the origin, due to the web reactions and to the applied load P/2, is

$$EI\frac{d^2y}{dx^2} = \frac{P}{2}(a-y) - \int_x^{L/2} m\,dx - \int_y^a \int_x^{L/2} \frac{2m}{b}dx \cdot \frac{dy}{dx} \cdot dx$$

From (1), $m = \frac{dy}{dx}\frac{6E'I'}{b}$, therefore the third term on right hand side of the above equation can be neglected as being a small quantity of the second order.

so $$EI\frac{d^2y}{dx^2} = \frac{P}{2}(a-y) - \int_x^{L/2} m\,d\,x$$

and $$EI\frac{d^3y}{dx^3} = -\frac{P}{2}\frac{dy}{dx} + m$$

hence $$m = EI\frac{d^3y}{dx^3} + \frac{P}{2}\frac{dy}{dx} \quad \text{.................................. (2)}$$

From (1)

$$m = \frac{6E'I'}{b}\frac{dy}{dx}$$

This, on substitution in (2), gives

$$\frac{d^3y}{dx^3} + c_1\frac{dy}{dx} = 0$$

Where $$c_1 = \left(\frac{P}{2EI} - \frac{6E'I'}{6EI}\right)^{1/2}$$

The appropriate solution of the equation, taking account of symmetry is

$y = A\cos c_1 x + B$ .... (3)

When x = 0, y = 0; and when x = L/2, y = a.

Using these conditions the constants are

$$A = \left\{ \frac{a}{\cos (c_1 L/2) - 1} \right\}$$

$$B = -\left\{ \frac{a}{\cos (c_1 L/2) - 1} \right\}$$

Also $\frac{d^2 y}{dx^2} = 0$ when x = L/2, and if this condition is used in equation (3),

$$-Ac_1^2 \cos (c_1 L/2) = 0$$

or $$\left\{ \frac{-a\, c_1^2}{\cos (c_1 L/2) - 1} \right\} \cos(c_1 L/2) = 0$$

This must be satisfied for any value of a, which is an arbitrary displacement, and so

$$\cos (c_1 L/2) = 0$$

Hence, the condition that determines the value of the critical load P is

$$c_1 L = \pi \quad \text{..................................} \ (4)$$

substitution of this in the constants of integration gives

$$A = -a \qquad \text{and} \qquad B = a$$

and the deflected form of the column is, from (3)

$$y = a(1 - \cos \frac{\pi x}{L}) \quad \text{..................................} \ (5)$$

Then, from (2)

$$m = \frac{\pi a}{L} (\frac{P}{2} - Q) \sin \frac{\pi x}{L} \quad \text{..................................} \ (6)$$

and from (4)

$$c_1^2 = \frac{\pi^2}{L^2}$$

Substitution of the value of c_1 in the above expression yields

$$\frac{P}{2EI} - \frac{6E'I'}{bEI} = \frac{\pi^2}{L^2}$$

or $$P = \frac{\pi^2 2EI}{L^2} + \frac{12E'I'}{b} \quad \text{....................................} \ (7)$$

and $$\frac{P}{Q} = 2 + \frac{12E'I'L^2}{\pi^2 EIb} \quad \text{.....................................} \quad (8)$$

by substitution, above expression can also be re-written as:

$$\frac{P}{Q} = 2 + \frac{12nE_C I_C L}{\pi^2 EIb}$$

or $$\frac{P}{Q} = 2 + \frac{12n'}{\pi^2}\frac{L}{b} \quad \text{where } n' = n\frac{E_C I_C}{EI}$$

Fig. 5 is a plot of $\frac{P}{Q}$ V/S n' with $\frac{L}{b}$ taken equal to 4.

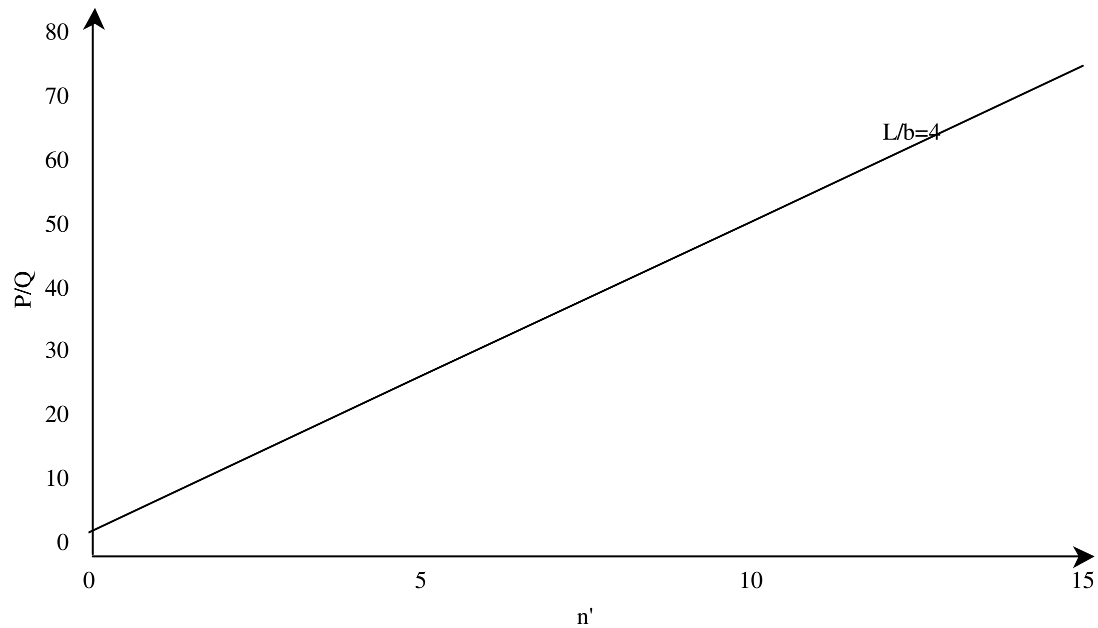


Figure 5. P/Q v/s stiffness ration; L/b=4

Figure 6 is the plot of the couple m at different locations along the column and for various values of the stiffness ratio n'.

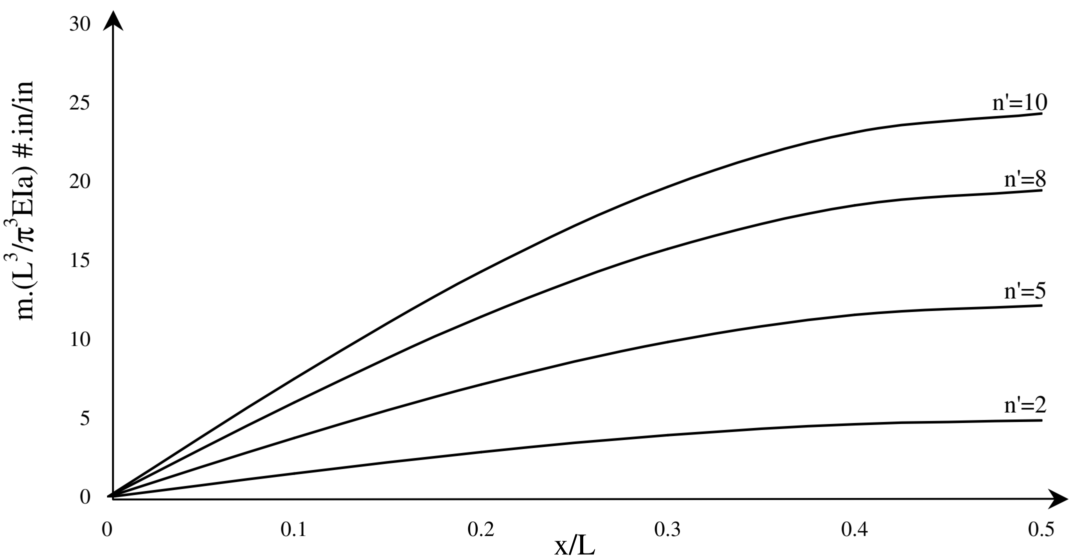


Figure 6. Bending moment for various stiffness ratios

Chapter Two: Buckling of Clamped Frame

Purpose of Study

Frames of various types are used in structural configurations such as buildings and bridges. These frames are subjected to concentrated and distributed loads, which in many cases, may cause buckling of an element or group of elements of the frame. Because the members are rigidly connected to other members, flexural deformations in one element cause deformations in the neighboring elements. Knowledge of the critical condition is essential in the design of both simple and complex frames. Neglecting longitudinal strains, the foregoing analysis presents a simplified method that can be successfully used to arrive at the critical condition and the equation of the corresponding deflected shape.

Case Study

Let us consider the frame shown in Fig. 7. We are interested in finding the smallest possible loads, P_A and P_C that will cause both columns AB and CD of the frame to buckle.

To accomplish this, we must consider all possible modes of buckling and establish through a comparison the corresponding buckling mode.

The frame under study is symmetric and the different buckling modes are shown in fig. 8. Noting that there is no possibility of a sway buckling mode at the mid-span when the horizontal bar buckles symmetrically, only antisymmetric sway buckling modes will be further discussed.

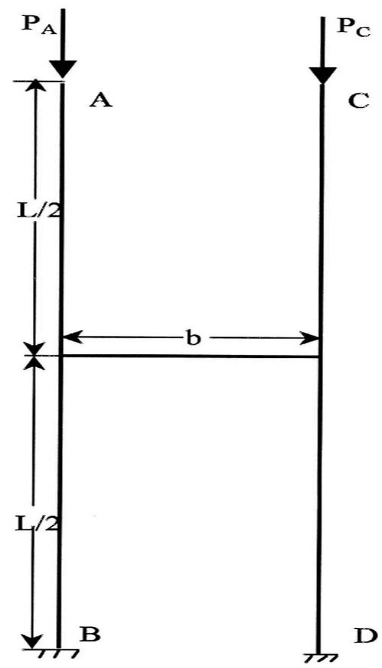
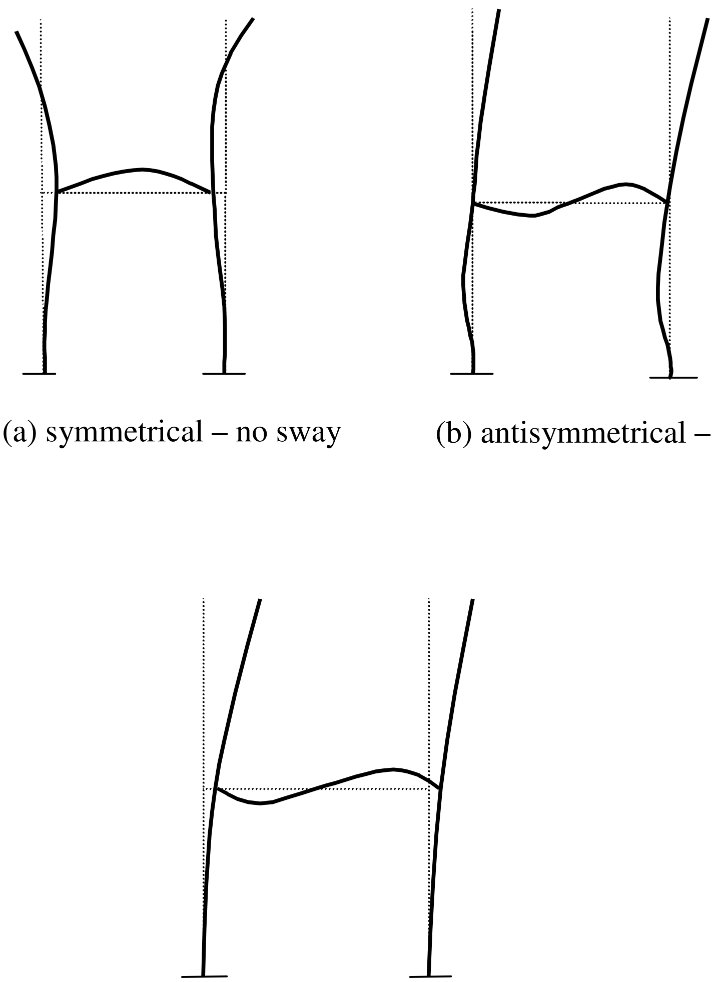


Figure 7. Frame under loads P_A and P_C



(a) symmetrical – no sway (b) antisymmetrical – no sway

(c) antisymmetrical – sway allowed

Figure 8. Buckling of frame

Analysis

When a bar such as AB in fig. 9 is initially straight and of perfect geometry and it is subjected to the action of a compressive force without eccentricity, the column is compressed but remains straight. We then need to know if the column will remain straight no matter what the level of the applied force is. To determine this, we seek nontrivial solutions ($w \neq 0$) for the equations governing the bending of this column under an axial compressive load P and subject to the singularities at midspan and to the particular set of boundary conditions where:

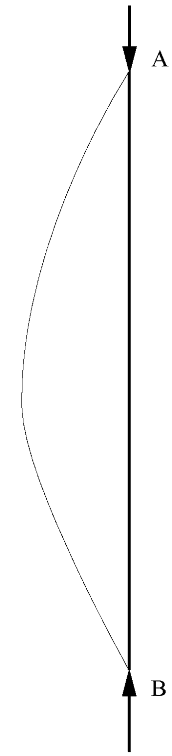


Figure 9. Bar under compressive forces

EI, the bending stiffness of each column.

$E'I'$, the bending stiffness of the horizontal beam.

L, the length of each column.

b, the length of the horizontal beam.

R, the stiffness ratio $= \dfrac{E'I'}{EI}\dfrac{L^2}{b^2}$

P_{Acr}, the critical load for column AB.

P_{Ccr}, the critical load for column CD.

P_{cr}, the total critical load for the frame.

Note that in deriving the governing differential equations, it was assumed that the applied compressive loads remained parallel to their original direction and there was no eccentricity in either the geometry or the applied loads. The mathematical formutation of this problem is given below.

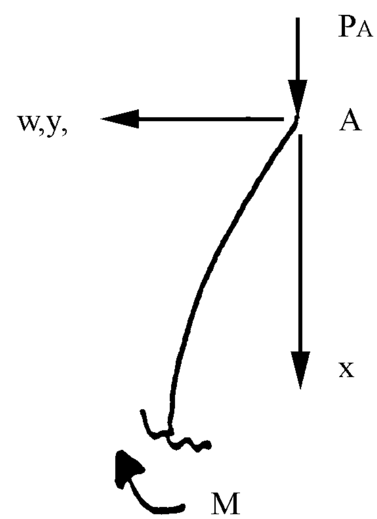


Figure 10. Bending of top half of column AB

The equation governing the bending of the top half of column AB. (See Fig. 10.) is given by

D.E. $EI\, y_1'' = -P_A y_1 \qquad (0 \le x \le L/2)$

B.C. $y_1''(0) = 0, \quad y_1(0) = 0$

Assuming that the bending stiffness EI of the column is constant and introducing the parameter $K_1^2 = P_A / EI$ allows us to write the governing differential equation in the following form: $y_1'' + K_1^2 y_1 = 0$

The general solution of this equation is given by

$$y_1 = A_1 \sin K_1 x + B_1 \cos K_1 x$$

This solution must satisfy the prescribed boundary conditions. This requirement leads to two linear homogeneous algebraic equations in the two constants A_1 and B_1

$$-A_1 K_1^2 \sin(0) - B_1 K_1^2 \cos(0) = 0$$

$$A_1 \sin(0) + B_1 \cos(0) = 0$$

or $\quad B_1 = 0$

and $\quad y_1(x) = A_1 \sin K_1 x$

The mathematical formulation of the bottom half of the column is given below (see Fig. 11)

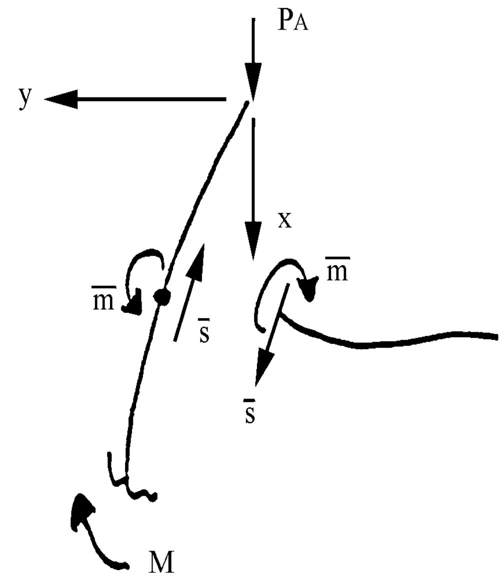


Figure 11. Bending of bottom half of column AB

D.E. $EI\, y_2'' = -P_A y_2 + \overline{m} + \overline{s}\,[y_2 - y_2(L/2)] \qquad L/2 \le x \le L$

B.C. $y_2'(L) = 0$

introducing the parameter $K_2^2 = \dfrac{P_A - \overline{s}}{EI}$

allows us to write the governing differential equation in the following form

$$y_2'' + K_2^2 y_2 = \frac{\overline{m} - \overline{s} y_2(L/2)}{EI}$$

where $\quad \overline{m} = \dfrac{6E'I'}{b} y_2'(L/2) \quad$ and $\quad \overline{s} = \dfrac{2m}{b}$

or $\quad \overline{s} = \dfrac{12E'I'}{b^2} y_2'(L/2)$

The general solution of this equation is given by

$$y_2(x) = A_2 \sin K_2 x + B_2 \cos K_2 x + \frac{\overline{m} - \overline{s} y_2(L/2)}{EIK_2^2}$$

This solution must satisfy compatibility at midspan i.e.

$$y_1''(L/2) = y_2''(L/2) - \frac{\overline{m}}{EI}$$

$$y_1'(L/2) = y_2'(L/2)$$

and $$y_1(L/2) = y_2(L/2)$$

The first two conditions above lead to two linear homogenous algebraic equations in the two constants A_2 and B_2

$$-A_1K_1^2 \sin K_1L/2 = -A_2K_2^2 \sin K_2L/2 - B_2K_2^2 \cos K_2L/2 - \frac{\overline{m}}{EI}$$

and

$$A_1K_1 \cos K_1L/2 = A_2K_2 \cos K_2L/2 - B_2K_2 \sin K_2L/2$$

Solution of the above equations yields

$$A_2 = A_1 \frac{K_1}{K_2}\left(\frac{K_1}{K_2} \sin K_1L/2 \sin K_2L/2 + \cos K_1L/2 \cos K_2L/2\right) - \frac{\overline{m} \sin K_2L/2}{EIK_2^2}$$

$$B_2 = A_1 \frac{K_1}{K_2}\left(\frac{K_1}{K_2} \sin K_1L/2 \cos K_2L/2 - \cos K_1L/2 \sin K_2L/2\right) - \frac{\overline{m} \cos K_2L/2}{EIK_2^2}$$

hence,

$$y_2(x) = \left\{A_1 \frac{K_1}{K_2}\left(\frac{K_1}{K_2} \sin K_1L/2 \sin K_2L/2 + \cos K_1L/2 \cos K_2L/2\right) - \frac{\overline{m} \sin K_2L/2}{EIK_2^2}\right\} \sin K_2x$$

$$+\left\{A_1 \frac{K_1}{K_2}\left(\frac{K_1}{K_2} \sin K_1L/2 \cos K_2L/2 - \cos K_1L/2 \sin K_2L/2\right) - \frac{\overline{m} \cos K_2L/2}{EIK_2^2}\right\} \cos K_2x$$

$$+\frac{\overline{m} - \overline{s}\ y_2(L/2)}{EIK_2^2}$$

Where

$$\overline{m} = \frac{6E'I'}{b} y_2'(L/2) = \frac{6E'I'}{b} y_1'(L/2)$$

$$\overline{s} = \frac{12E'I'}{b^2} y_2'(L/2) = \frac{12E'I'}{b^2} y_1'(L/2)$$

$$y_2(L/2) = y_1(L/2)$$

Substitution yields:

$$y_2(x) = A_1 \left\{\left[\frac{K_1}{K_2}\left(\frac{K_1}{K_2} \sin K_1L/2 \sin K_2L/2 + \cos K_1L/2 \cos K_2L/2\right)\right.\right.$$

$$-\frac{6E'I'K_1}{b\,EIK_2^2}\cos K_1L/2\sin K_2L/2\Bigg]\sin K_2x$$

$$+\left[\frac{K_1}{K_2}\left(\frac{K_1}{K_2}\sin K_1L/2\cos K_2L/2-\cos K_1L/2\sin K_2L/2\right)\right.$$

$$-\frac{6E'I'K_1}{b\,EIK_2^2}\cos K_1L/2\cos K_2L/2\Bigg]\cos K_2x$$

$$+\frac{6E'I'K_1}{b\,EIK_2^2}\cos K_1L/2(1-\frac{2}{b}A_1\sin K_1L/2)\Bigg\}$$

The necessary boundary condition at the fixed-end leads to the characteristic equation:

$$0=-\frac{K_1^2}{K_2^2}\sin K_1L/2\sin K_2L/2+\frac{K_1}{K^2}\cos K_1L/2\cos K_2L/2$$

$$+\frac{6E'I'K_1}{b\,EIK_2^2}\cos K_1L/2\sin K_2L/2\Bigg\}$$

but $K_2^2=\dfrac{P_A-s}{EI}$ and $K_1^2=\dfrac{P_A}{EI}$

$\therefore$ $K_2=(K_1^2-\dfrac{12E'I'K_1}{EIb^2}A_1\cos K_1L/2)^{1/2}$

This upon substitution in the above characteristic equation yields the critical parameter K_1 and hence the critical load P_A for column AB. Identical work was done for member CD of the structure noting however that the reactive shear $\bar{s}$ assumes an opposite direction in this case (See Fig. 12).

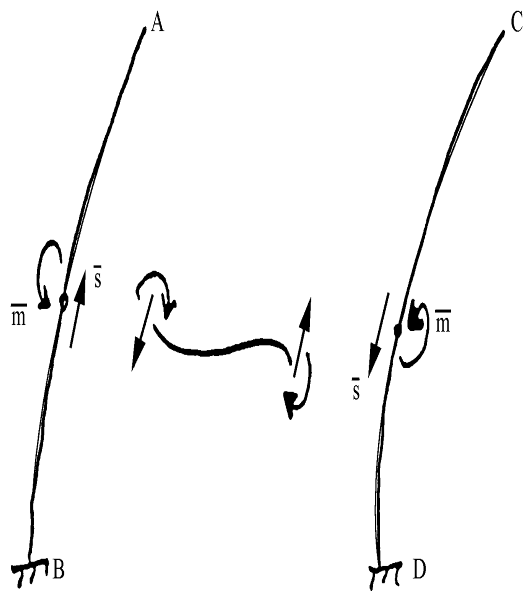


Figure 12. Reactive shear at midspan

Chapter Three: Findings

Description of Findings

Solution of the characteristic equation for column AB is plotted in Fig. 13. An arbitrary deflection at midspan relative to the free end at point A was taken equal to 0.1L. The stiffness ratio $R = \frac{E'I'}{EI}\frac{L^2}{b^2}$ was taken equal to 10.

Fig. 14. is the solution of the characteristic equation for column CD, considering that at mid span and due to the buckling symmetry of the horizontal bar, both members AB and CD have equal slopes.

Upon substitution, where $P_{cr} = (K_1L)^2 EI/L^2$, values of both critical loads P_{Acr} and P_{Ccr} for columns AB and CD respectively can be derived. Fig. 15 is a plot of the characteristic equation solution v/s the total critical load $P_{cr} = P_{Acr} + P_{Ccr}$.

The critical values for the first and second mode shapes are summarized in Table. 1.

Fig. 16 plots the values of the critical parameter K_1 for column AB against the stiffness ratio R. We can see that when R=0, then K_1 assumes the value $\pi/2$, the same as for a column with a fixed end at support. In addition, when R approaches infinity the horizontal bar acts like a fixed support at mid span and $K_1 = \pi$.

Fig. 17 and 18 are first mode plots of the critical parameter K_1 and critical load P_{Acr} for column AB v/s stiffness ratio R. A logarithmic scale was used to enable us cover a wide range of the stiffness ratio. Same is plotted in Fig. 19 and 20 for column CD.

Values of the total critical load for the Frame in the first mode shape, $P_{cr} = P_{Acr} + P_{Ccr}$, are plotted against the stiffness ratio R in Fig. 21 also using a logarithmic scale.

Fig. 22 and 23 show the deflected shape of the structure for the first and second modes shapes with R taken equal to 10.

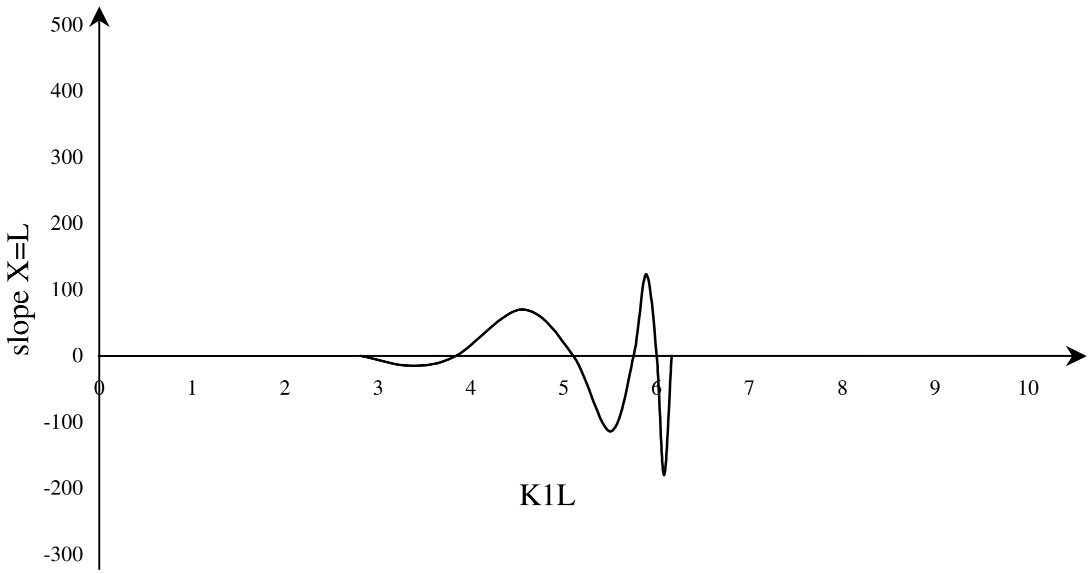


Figure 13. Critical coditions for column AB

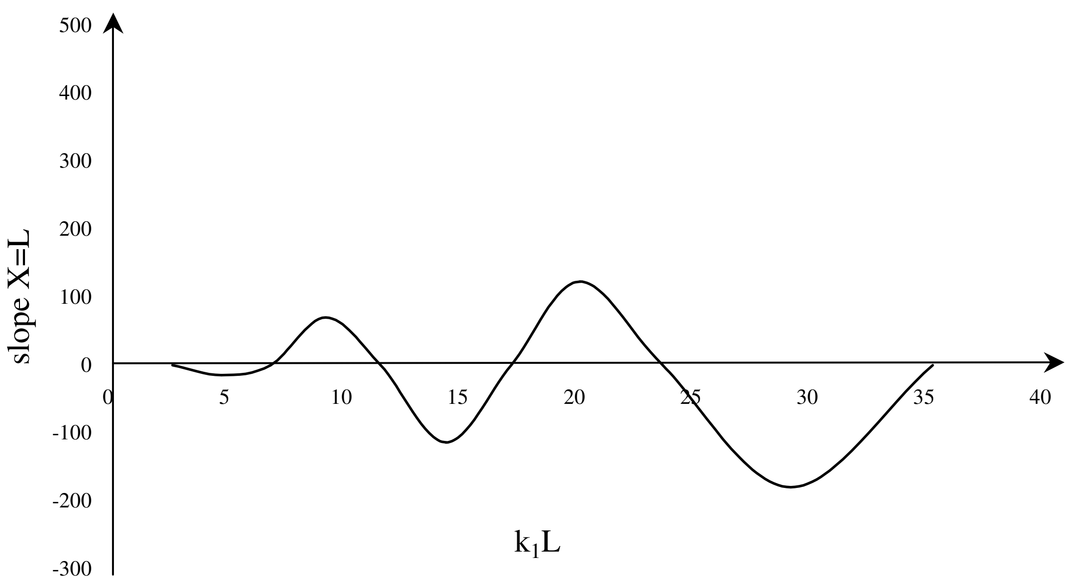


Figure 14. Critical conditions for column CD

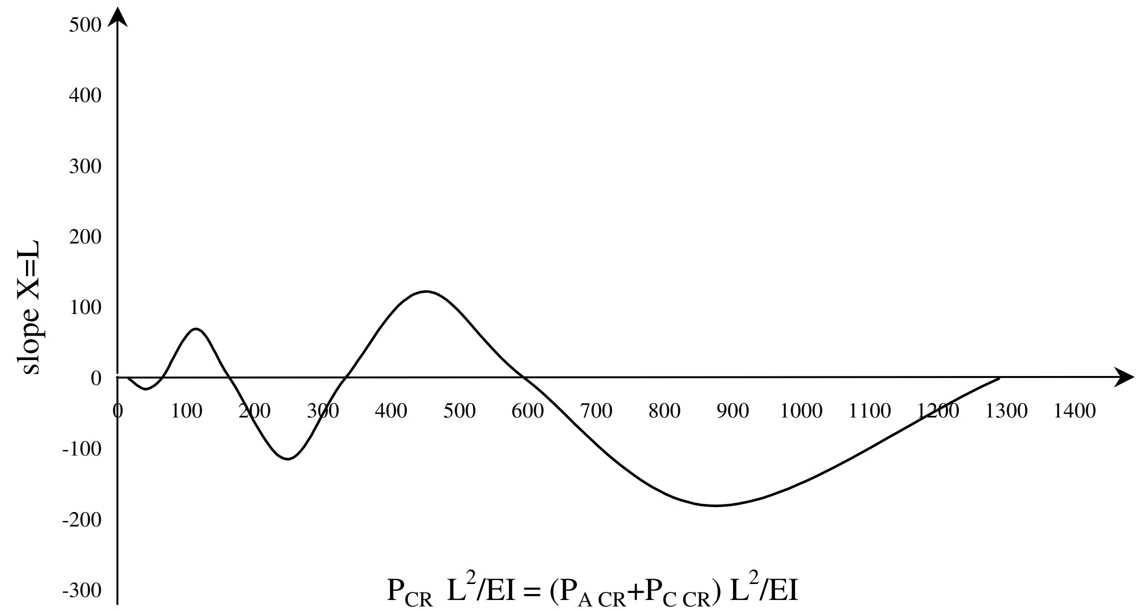


Figure 15. Total critical load

Table 1. Critical parameters, column AB

R	MODE SHAPE	K_1L	K_2L	CRITICAL LOAD $P_{A\,CR}\,L^2/EI$	REACTION AT SUPPORT	ERROR
0	1	1.571383	1.570783	2.469243166	2.467358611	5.44E-09
0	2	4.712185	4.712784	22.20468296	22.21033642	-3.81E-09
10	1	2.806983	1.480042	7.879155976	2.190524884	1.06E-07
10	2	3.900811	5.821642	15.21632685	33.89152118	7.68E-08
1000	1	3.137414	1.406191	9.843367003	1.977373286	8.54E-08
1000	2	3.157128	6.276601	9.967455295	39.39571664	1.71E-07

Table 2. Critical parameters, column CD

R	MODE SHAPE	K_1L	K_2L	CRITICAL LOAD $P_{C\,CR}\,L^2/EI$	REACTION AT SUPPORT	ERROR
0	1	1.571165	1.571764	2.468558546	2.4704431	-9.94E-09
0	2	1.5716	1.569801	2.469927619	2.464274153	1.17E-09
10	1	2.754776	3.643819	7.588788777	13.27741987	1.12E-08
10	2	6.964238	5.461266	48.50061534	29.82542101	2.71E-07
1000	1	3.137403	4.208241	9.843294547	17.70928826	1.48E-07
1000	2	8.294618	6.274745	68.80069147	39.37243012	-5.93E-08

Table 3. Critical load for frame

R	MODE SHAPE	TOTAL CRITICAL LOAD $P_{CR}\,L^2/EI=(P_{A\,CR}+P_{C\,CR})\,L^2/EI$
0	1	4.937802
0	2	24.67461
10	1	15.46794
10	2	63.71694
1000	1	19.68666
1000	2	78.76815

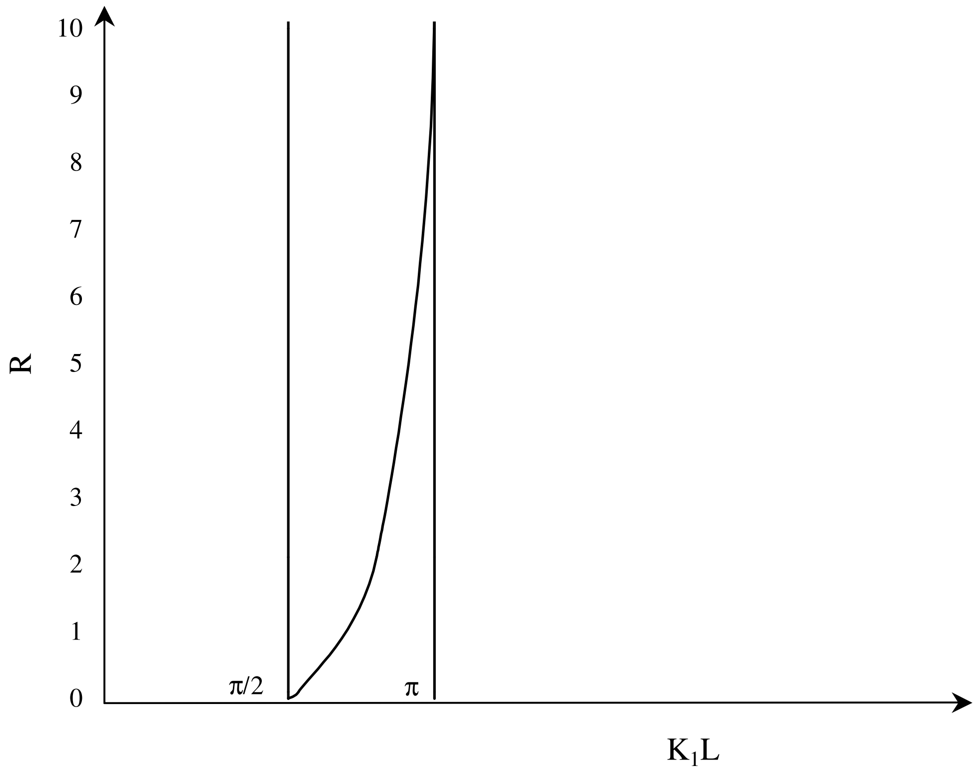


Figure 16. First mode critical parameter v/s stiffness ratio for column AB

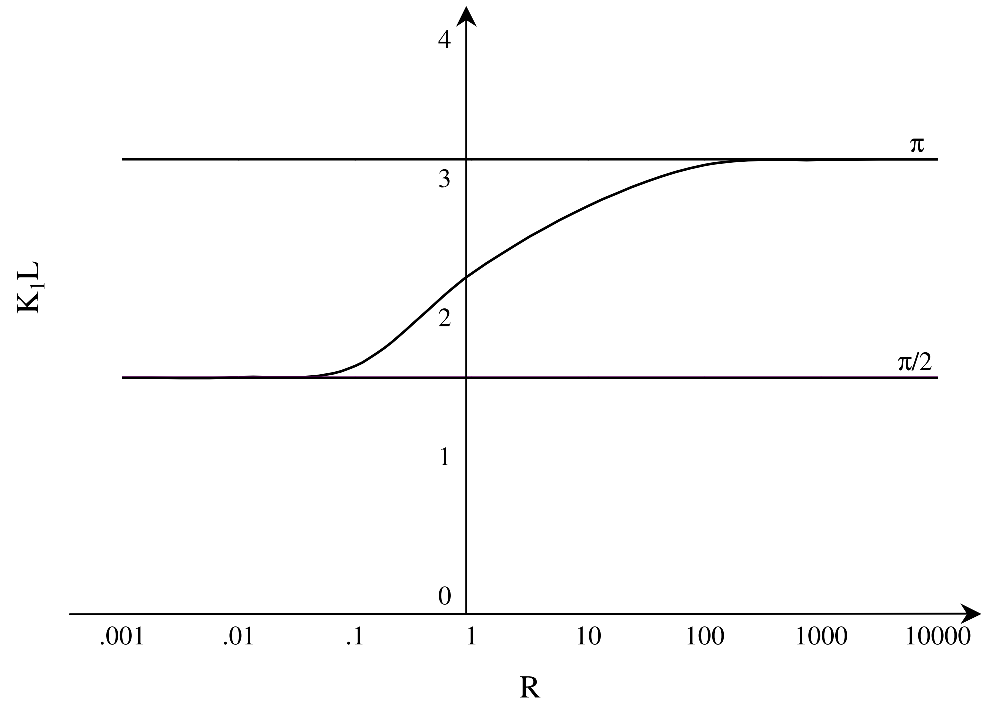


Figure 17. Critical parameter for column AB, first mode shape

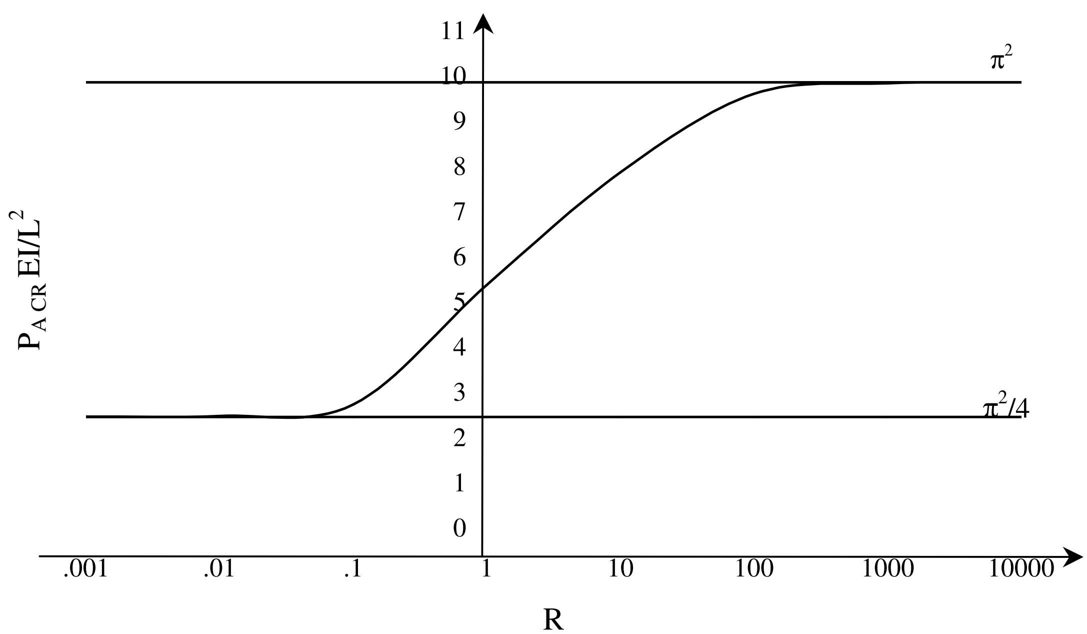


Figure 18. Critical load for column AB, first mode shape

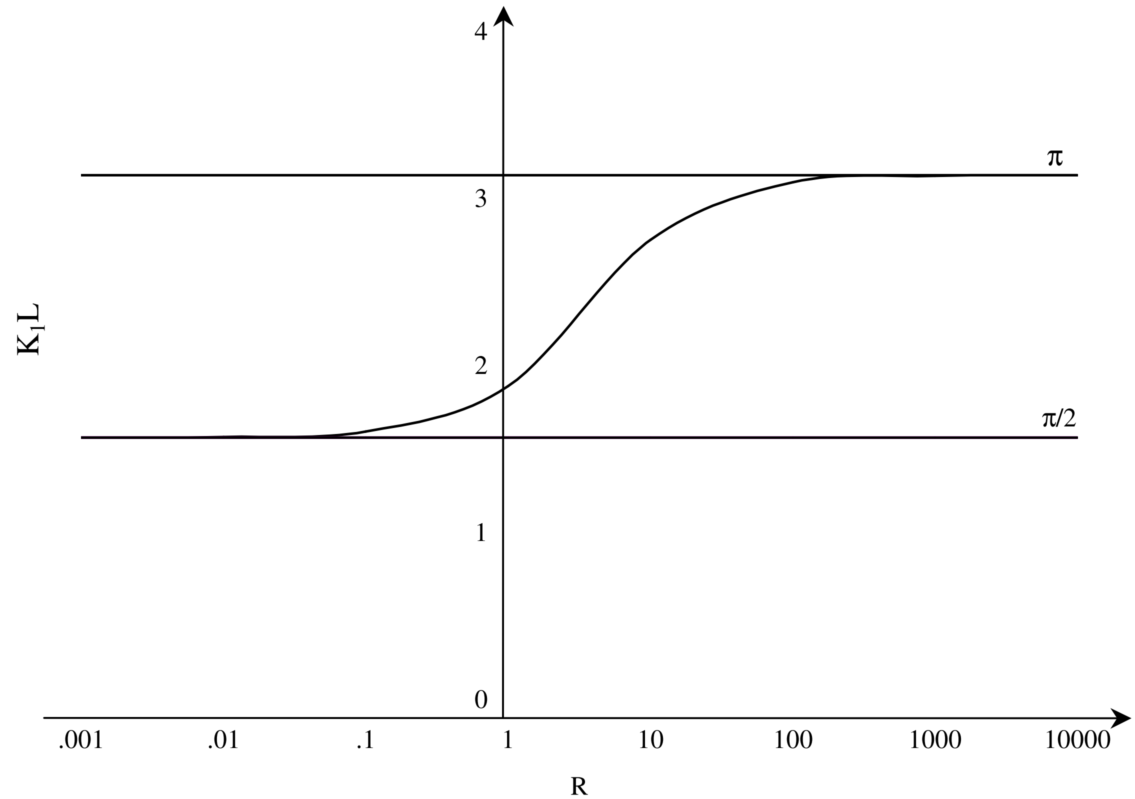


Figure 19. Critical parameter for column CD, first mode shape

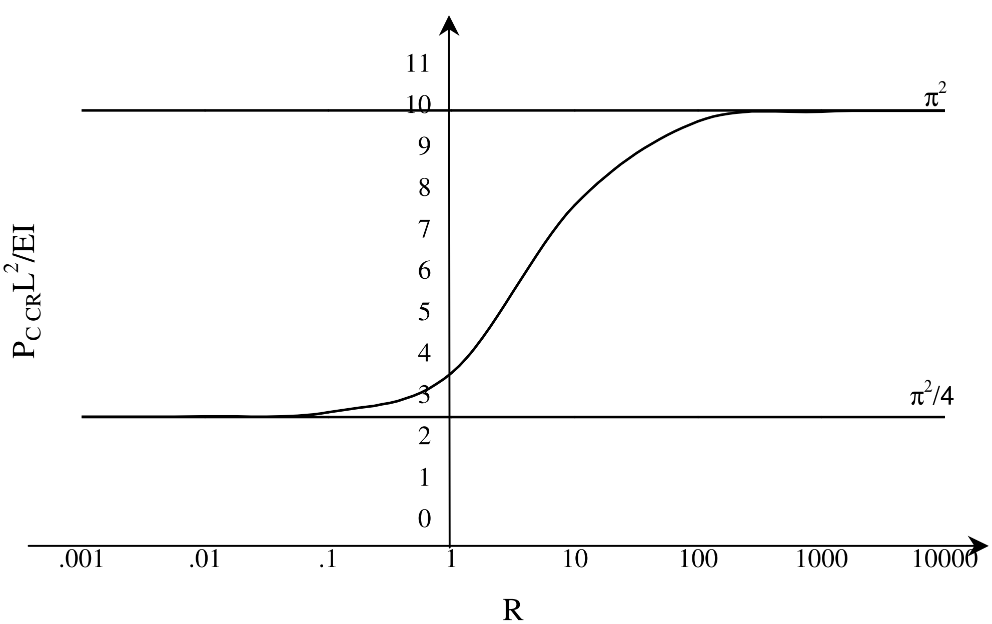


Figure 20. Critical load for column CD, first mode shape

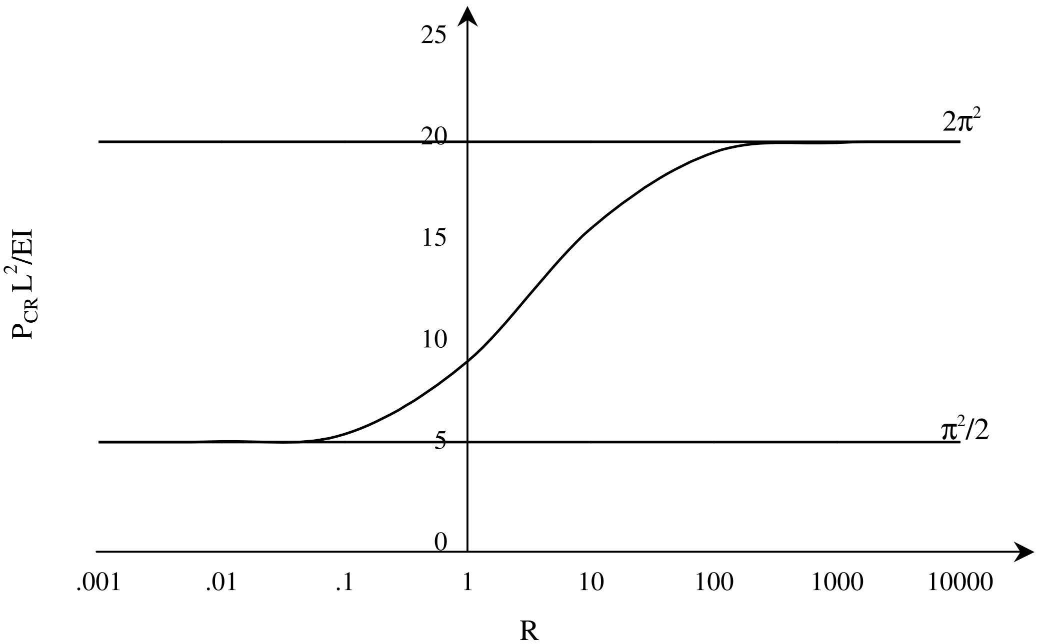


Figure 21. Total critical load, first mode shape

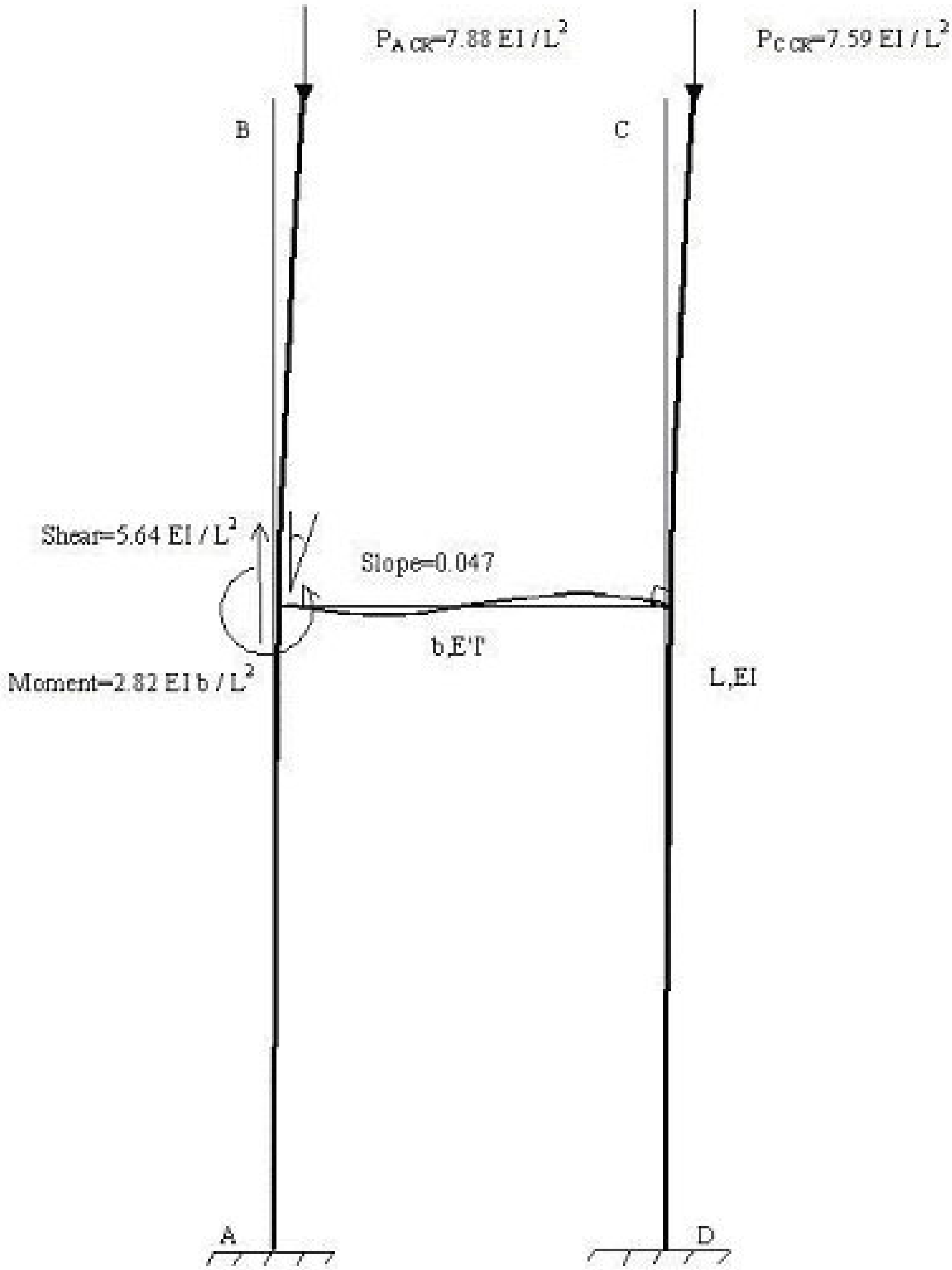


Figure 22. Deflected frame under critical loads, first mode shape

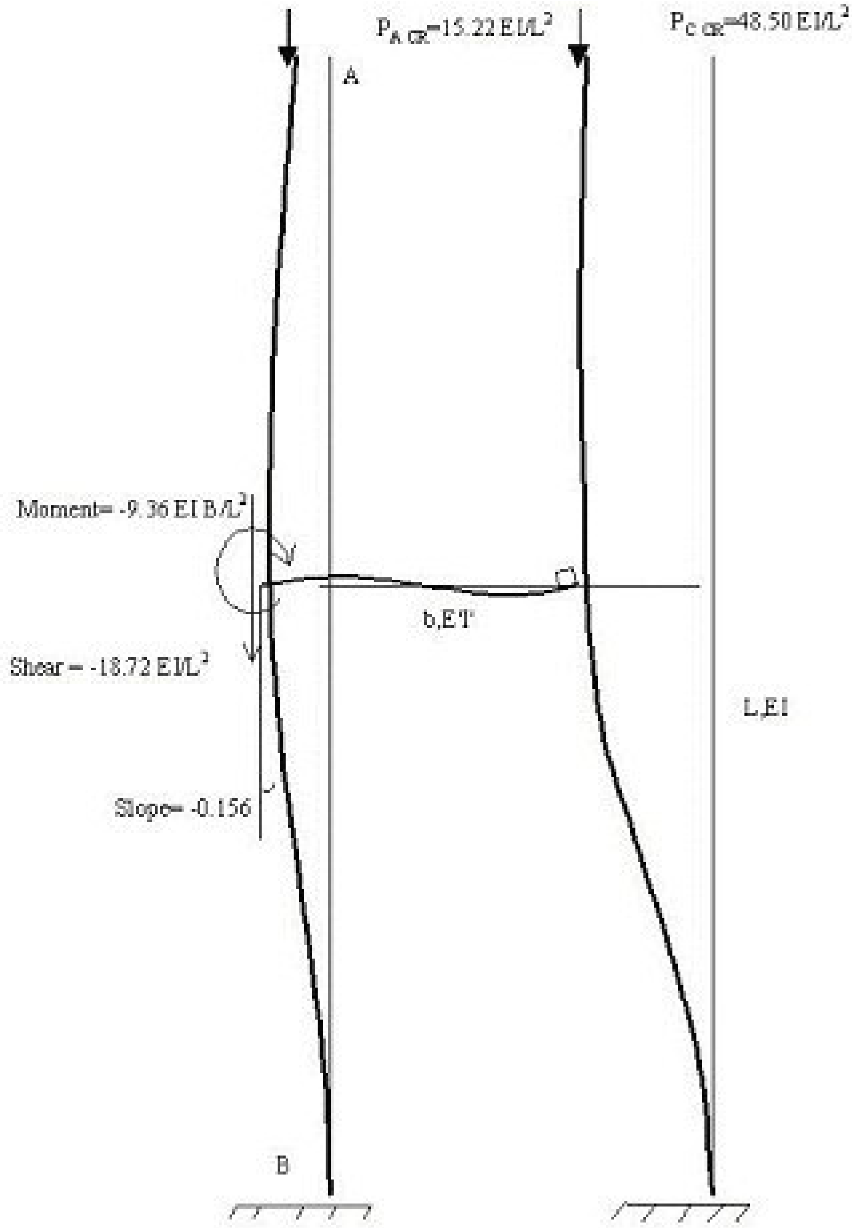


Figure 23. Deflected frame under critical loads, second mode shape

References

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