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Hölder Continuity of Green's Functions

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by

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Dedication

To my parents

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ABSTRACT

We investigate local properties of the Green function of the complement of a compact set E.

First we consider the case $E \subset [0, 1]$ in the extended complex plane. We extend a result of V. Andrievskii which claims that if the Green function satisfies the Hölder-1/2 condition locally at the origin, then the density of E at 0, in terms of logarithmic capacity, is the same as that of the whole interval $[0, 1]$. We give an integral estimate on the density in terms of the Green function, which also provides a necessary condition for the optimal smoothness. Then we extend the results to the case $E \subset [-1, 1]$. In this case the maximal smoothness of the Green function is Hölder-1 and a similar integral estimate and necessary condition hold as well.

In the second part of the paper we consider the case when E is a compact set in $\mathbf{R}^d$, $d > 2$. We give a Wiener type characterization for the Hölder continuity of the Green function, thus extending a result of L. Carleson and V. Totik. The obtained density condition is necessary, and it is sufficient as well, provided E satisfies the cone condition. It is also shown that the Hölder condition for the Green function at a boundary point can be equivalently stated in terms of the equilibrium measure and the solution to the corresponding Dirichlet problem. The results solve a long standing open problem - raised by Maz'ja in the 1960's - under the simple cone condition.

1 Introduction

The continuity of Green's functions at boundary points has been extensively studied for a long time. The aim of this research is to give conditions for the stronger Hölder continuity in terms of the geometry of the set. We consider both the planar and the higher dimensional case. For the concepts and notions in this Chapter see Section 2.1.

Suppose that $E \subset \mathbf{C}$ is a compact set with positive logarithmic capacity $\mathrm{cap}(E) > 0$. Let $\Omega := \overline{\mathbf{C}} \setminus E$, where $\overline{\mathbf{C}} := \{\infty\} \cup \mathbf{C}$ is the extended complex plane. Denote by $g_\Omega(z) = g_\Omega(z, \infty)$, $z \in \Omega$, the Green function of Ω with pole at ∞. We extend g_Ω to $\partial\Omega$ in the usual way by

$$g_\Omega(z, \infty) = \limsup_{w \to z,\ w \in \Omega} g_\Omega(w, \infty),$$

and to $\overline{\mathbf{C}} \setminus \overline{\Omega}$ by setting $g_\Omega(z, \infty) = 0$ there. This way g_Ω becomes a subharmonic function on $\mathbf{C}$. We are interested in the behavior of g_Ω at 0.

Suppose that 0 is a regular point of E, i.e., $g_\Omega(z)$ is continuous at 0 and $g_\Omega(0) = 0$. First consider the case $E \subset [0, 1]$. The monotonicity of the Green function yields

$$g_\Omega(z) \geq g_{\overline{\mathbf{C}} \setminus [0,1]}(z), \quad z \in \mathbf{C} \setminus [0, 1],$$

that is, if E has the "highest density" at 0, then g_Ω has the "highest smoothness" at the origin. In particular

$$g_\Omega(-r) \geq g_{\overline{\mathbf{C}} \setminus [0,1]}(-r) > \sqrt{r}, \quad 0 < r < 1.$$

In this regard, we would like to explore properties of E whose Green function has the "highest smoothness" at 0, that is, E conforming to the following condition

$$g_\Omega(z) \le C|z|^{1/2}, \qquad z \in \mathbf{C},$$

which is known to be the same as

$$g_\Omega(-r) \le Cr^{1/2}, \qquad 0 < r < 1 \tag{1.0.1}$$

(c.f. [1, Theorem 3.6]). Various sufficient conditions for (1.0.1) in terms of metric properties of E are stated in [4], where the reader can also find further references.

There are compact sets $E \subset [0,1]$ of linear Lebesgue measure 0 with property (1.0.1) (see e.g. [4, Corollary 5.2]), hence (1.0.1) may hold, though the set E is not dense at 0 in terms of linear measure. On the contrary, V. Andrievskii [2] proved that if E satisfies (1.0.1) then its density in a small neighborhood of 0, measured in terms of logarithmic capacity, is arbitrary close to the density of $[0,1]$ in that neighborhood, i.e. (1.0.1) implies

$$\lim_{r\to 0} \frac{\operatorname{cap}(E \cap [0,r])}{r} = \frac{1}{4}. \tag{1.0.2}$$

In Chapter 2 we will prove an integral estimate for the density via the Green function, from which (1.0.2) easily follows.

Andrievskii also constructed a regular compact set $E \subset [0,1]$ such that

$$\lim_{r\to 0} \frac{g_\Omega(-r)}{r^{1/2-\varepsilon}} = 0, \quad 0 < \varepsilon < \frac{1}{2}$$

holds but

$$\liminf_{r\to 0} \frac{\operatorname{cap}(E \cap [0,r])}{r} = 0. \tag{1.0.3}$$

Furthermore he proved that conversely, (1.0.2) does not imply (1.0.1).

Now let's turn to the case $E \subset [-1,1]$. In this case

$$g_\Omega(ir) \ge g_{\overline{\mathbf{C}}\setminus[-1,1]}(ir) > \frac{r}{2}, \quad 0 < r < 1,$$

therefore in this case the optimal smoothness for Green functions is Hölder 1 and we are interested in sets E satisfying

$$g_\Omega(z) \le C|z|, \qquad 0 < |z| < 1.$$

This is equivalent to

$$g_\Omega(ir) \le Cr, \qquad 0 < r < 1 \tag{1.0.4}$$

because $g_\Omega(x+iy)$ is monotone in y. As we will see, the necessary condition for the optimal smoothness can be generalized to this case, as well.

Let us consider now the more general setting when E is an arbitrary compact subset of $\mathbf{C}$. Assume that 0 is a boundary point of Ω. Several equivalent conditions are known for the regularity of 0 (see e.g. ([13, Appendix A2.]). One of them is due to Wiener. It characterizes the regularity with the capacity of the sets

$$E^n = E \cap (\overline{D}_{2^{-n+1}} \setminus D_{2^{-n}}) = \Big\{ z \in E \ : \ 2^{-n} \le |z| \le 2^{-n+1} \Big\}.$$

Theorem 1.0.1 $g_\Omega(0) = 0$ *if and only if*

$$\sum_{n=1}^{\infty} \frac{n}{\log(1/\mathrm{cap}(E^n))} = \infty, \tag{1.0.5}$$

where $\mathrm{cap}(E^n)$ *denotes the logarithmic capacity of* E^n.

L. Carleson and V. Totik (see [4]) characterized in a similar manner the stronger Hölder continuity:

$$g_\Omega(z, \infty) \le C|z|^\kappa \tag{1.0.6}$$

with some positive numbers C, κ.

For $\varepsilon > 0$ set

$$\mathcal{N}_E(\varepsilon) = \{ n \in \mathbf{N} \ : \ \ \mathrm{cap}(E^n) \ge \varepsilon 2^{-n} \}, \tag{1.0.7}$$

and we say that a subsequence $\mathcal{N} = \{n_1 < n_2 < \ldots\}$ of the natural numbers is of positive lower density if

$$\liminf_{N\to\infty} \frac{|\mathcal{N} \cap \{0, 1, \ldots, N\}|}{N+1} > 0,$$

which is clearly the same condition as $n_k = O(k)$.

Theorem 1.0.2 (Carleson, Totik) *Suppose that the compact set E satisfies the cone condition. Then Green's function g_Ω is Hölder continuous at 0 if and only if $\mathcal{N}_E(\varepsilon)$ is of positive lower density for some $\varepsilon > 0$.*

The Hölder continuity of the Green function can be stated as an equivalent condition in terms of the harmonic and equilibrium measure and the solution to the corresponding Dirichlet problem as well (see [4, Proposition 1.4]).

Totik suggested that these results could be extended to the higher dimensional case, i.e. when $E \subset \mathbf{R}^d$. For this case a Wiener type condition like in Theorem 1.0.2 was already defined by Maz'ja (see [7]- [10]). Maz'ja proved its sufficiency for the Hölder continuity of the solution to the Dirichlet problem and showed that in general it is not necessary. In Chapter 3 we will prove the sufficiency of this condition for the Hölder continuity of the Green function and show that it is also necessary provided E satisfies the cone condition. We also give an equivalent characterization in terms of the equilibrium measure. In other words, under the cone condition we completely characterize Hölder continuity, which has been a long standing open problem.

2 Optimal Smoothness for $E \subset [0,1]$

2.1 Notations, Definitions

We shall use $c, c_0, c_1, c_2, \ldots,$ $C, C_0, C_1, C_2, \ldots$ and $d_1, d_2, \ldots$ to denote positive constants. These constants may be either absolute or they may depend on E depending on the context. We may use the same symbol for different constants if this does not lead to confusion.

$|F|$ denotes the linear Lebesgue measure of a measurable subset $F \subset \mathbf{R}$ of the real line $\mathbf{R}$.

$\mathbf{D} := \{z \ : \ |z| < 1\}$ is the unit disk, $\mathbf{T} = \partial\mathbf{D}$ is the unit circle and for $z_1, z_2 \in \mathbf{C}$, $z_1 \neq z_2$ let

$$[z_1, z_2] := \{tz_2 + (1-t)z_1 : \ 0 \le t \le 1\}$$

be the interval between these points.

For the notions of logarithmic potential theory see e.g. [12] or [13]. In what follows μ_E denotes the equilibrium measure of E,

$$U^{\nu}(z) := \int \log \frac{1}{|z-t|} d\nu(t)$$

the logarithmic potential of the measure ν, $g_G(z,a)$ the Green function of the domain G with pole at a, $\omega(x, H, G)$ the harmonic measure in G corresponding to the set $H \subseteq \partial G$. We shall frequently use the relation

$$g_{\mathbf{C}\setminus E}(z) = \log \frac{1}{\operatorname{cap}(E)} - U^{\mu_E}(z), \quad z \in \mathbf{C} \setminus E \tag{2.1.1}$$

valid for any compact set E of positive capacity.

Let G be a domain with compact boundary and with $\mathrm{cap}(\partial G) > 0$, and let ν be a measure supported on $\overline{G}$. We shall need the concept of balayage (or sweeping) of ν out of G (sometimes we say balayage onto ∂G), see e.g. [13, Sec. II.4]. It is the unique measure $\overline{\nu}$ supported on ∂G with the property that

$$U^{\overline{\nu}}(z) = U^{\nu}(z) + \mathrm{const} \tag{2.1.2}$$

for $z \in \partial G$ with the exception of a set of capacity 0. For regular G the exceptional set is empty. If G is bounded, then the constant is 0 ([13, Ch. II, Theorem 4.1]), and if G is unbounded, then it is ([13, Ch. II, Theorem 4.4])

$$\mathrm{const} = \int_G g_G(a, \infty)\, d\nu(a). \tag{2.1.3}$$

We shall use the notation $\mathrm{Bal}\Big(\nu, G\Big)$ for the balayage measure $\overline{\nu}$.

There is a connection between harmonic and balayage measures: if $K \subseteq \partial G$ are compact sets, then for $x \in G$ the equality

$$\mathrm{Bal}\Big(\delta_x, G\Big)(K) = \omega(x, K, G) \tag{2.1.4}$$

holds, where δ_x denotes the point mass (Dirac measure) placed at the point x (see e.g. [13, Appendix A3, (3.3)]). Therefore, in what follows we shall interchangeably use the harmonic measure and balayage notations.

2.2 Results

Let $E \subset [0, 1]$ be a compact set with positive (logarithmic) capacity and let $\Omega := \overline{\mathbf{C}} \setminus E$.

Recall that $\mathrm{cap}(I) = |I|/4$ for any interval I, where $|I|$ denotes the length (Lebesgue measure) of I.

For $0 < \varepsilon < 1/2$ we set

$$E_\varepsilon(t) = (E \cap [0,t]) \cup [0, \varepsilon t] \cup [(1-\varepsilon)t, t]. \tag{2.2.5}$$

Our first result is

Theorem 2.2.1 *For any $\varepsilon > 0$*

$$\int_r^1 \left(\frac{1}{4} - \frac{\operatorname{cap}(E_\varepsilon(t))}{t} \right)^2 \frac{1}{t} dt < C_0 \frac{g_\Omega(-r)}{\sqrt{r}} \tag{2.2.6}$$

where C_0 is independent of r.

Corollary 2.2.2 *If E satisfies* (1.0.1), *then for any $\varepsilon > 0$*

$$\int_0^1 \left(\frac{1}{4} - \frac{\operatorname{cap}(E_\varepsilon(t))}{t} \right)^2 \frac{1}{t} dt < \infty. \tag{2.2.7}$$

Since $\operatorname{cap}(E_\varepsilon) \geq |E_\varepsilon|/4$, condition (2.2.7) is somewhat weaker than

$$\int_0^1 \left(\frac{1}{4} - \frac{|E_\varepsilon(t)|}{4t} \right)^2 \frac{1}{t} dt < \infty, \tag{2.2.8}$$

which is known to be sufficient for (1.0.1) (see [4, Theorem 2.1]).

Andrievskii's theorem is a consequence of Corollary 2.2.2 (see Lemma 2.5.1).

Condition (2.2.7) is not sufficient for Hölder continuity, it does not imply (1.0.1). Indeed, let $\sum_k \theta_k^2 = \infty$ but $\sum_k \theta_k^3 < \infty$, and consider a set E of the form

$$E = [0,1] \setminus \bigcup_{k=1}^{\infty} ((1-\theta_k)2^{-n_k}, 2^{-n_k})$$

with some very fast increasing sequence $\{n_k\}$ (say $n_{k+1} > k^2 n_k$). One can verify that for this set (2.2.8) is true because of $\sum_k \theta_k^3 < \infty$, but it was shown in ([4, Corollary 3.3]) that (1.0.1) does not hold, due to $\sum \theta_k^2 = \infty$.

The method used in the proof of Theorem 2.2.1 can be applied to the case $E \subset [-1,1]$ as well. The highest smoothness of the Green function at the origin (Lipschitz condition) implies the highest density at 0. Namely, let $E \subset [-1,1]$ and set $E_\varepsilon(t)$ as in (2.2.5) and

$$E_\varepsilon(-t) = (E \cap [-t,0]) \cup [-t,(1-\varepsilon)(-t)] \cup [-\varepsilon t, 0].$$

Theorem 2.2.3 *If $E \subseteq [-1,1]$ and $\varepsilon > 0$ then*

$$\int_r^1 \left(\frac{1}{4} - \frac{\operatorname{cap}(E_\varepsilon(t))}{t}\right)^2 \frac{1}{t} dt < C_0 \frac{g_\Omega(ir)}{r} \tag{2.2.9}$$

The same is true for $E_\varepsilon(-t)$.

Corollary 2.2.4 *If $E \subseteq [-1,1]$ satisfies*

$$g_\Omega(ir) \le Cr, \qquad 0 < r < 1, \tag{2.2.10}$$

then for any $\varepsilon > 0$ (2.2.7) holds for $E_\varepsilon(t)$ and $E_\varepsilon(-t)$.

Corollary 2.2.5 *If E satisfies (2.2.10) then*

$$\lim_{r \to 0} \frac{\operatorname{cap}(E \cap [-r,r])}{r} = \frac{1}{2}. \tag{2.2.11}$$

2.3 Proof of Theorem 2.2.1

We divide the proof into three steps.

Step I. First we are going to verify the following: *let $I_j = [a_j, b_j]$, $j \in \mathbf{N}$ be disjoint closed subintervals of $(0,1]$ such that $b_j \le C_1|I_j|$, $j \in \mathbf{N}$ for some C_1, and for $\varepsilon > 0$ set*

$$F_j = (I_j \cap E) \cup [a_j, a_j + (\varepsilon/2)|I_j|] \cup [b_j - (\varepsilon/2)|I_j|, b_j]. \tag{2.3.12}$$

Then

$$\sum_{j:I_j\subseteq[r,1]}\left(\frac{1}{4}-\frac{\mathrm{cap}(F_j)}{|I_j|}\right)^2 < c_0\frac{g_\Omega(-r)}{\sqrt{r}}. \tag{2.3.13}$$

For the proof first of all notice that (1.0.1) implies

$$\mu_{E\cup[0,r]}([0,r]) \le C_2 g_\Omega(-r), \qquad 0<r<1, \tag{2.3.14}$$

for some $C_2>0$ (recall that $\mu_{E\cup[0,r]}$ denotes the equilibrium measure of $E\cup[0,r]$). This is immediate, since (see (2.1.1))

$$\begin{aligned}
g_\Omega(-r) &\ge g_{\overline{\mathbf{C}}\setminus(E\cup[0,r])}(-r)\\
&= \log\frac{1}{\mathrm{cap}(E\cup[0,r])} - U^{\mu_{E\cup[0,r]}}(-r)\\
&= U^{\mu_{E\cup[0,r]}}(0) - U^{\mu_{E\cup[0,r]}}(-r) = \int \log\frac{t+r}{t}\,d\mu_{E\cup[0,r]}(t)\\
&\ge \log 2\int_0^r d\mu_{E\cup[0,r]}(t) = (\log 2)\mu_{E\cup[0,r]}([0,r]).
\end{aligned}$$

Next we use that $\mu_{E\cup[0,r]}$ is the balayage of $\mu_{[0,1]}$ onto $E\cup[0,r]$ ([13, Theorem IV.1.6, (e)]), and so

$$\begin{aligned}
\mu_{E\cup[0,r]}([0,r]) &= \mathrm{Bal}\Big(\mu_{[0,1]}, \mathbf{C}\setminus(E\cup[0,r])\Big)([0,r])\\
&\ge \mathrm{Bal}\Big(\mu_{[0,1]}\big|_{[r,1]\setminus E}, \mathbf{C}\setminus(E\cup[0,r])\Big)([0,r])\\
&\ge \sum_{I_j\subseteq[r,1]}\mathrm{Bal}\Big(\mu_{[0,1]}\big|_{I_j\setminus F_j}, \mathbf{C}\setminus(E\cup[0,r])\Big)([0,r]).
\end{aligned}$$

Thus, if we can prove that with

$$\theta_j = \frac{1}{4}-\frac{\mathrm{cap}(F_j)}{|I_j|}$$

we have for $I_j\subseteq[r,1]$

$$\mathrm{Bal}\Big(\mu_{[0,1]}\big|_{I_j\setminus F_j}, \mathbf{C}\setminus(E\cup[0,r])\Big)([0,r]) \ge c\theta_j^2\sqrt{r} \tag{2.3.15}$$

with some positive constant c, then we get from the preceding inequality and from (2.3.14)

$$\sum_{I_j \subseteq [r,1]} c\theta_j^2 \sqrt{r} \le C_2 g_\Omega(-r)$$

and the relation (2.3.13) follows.

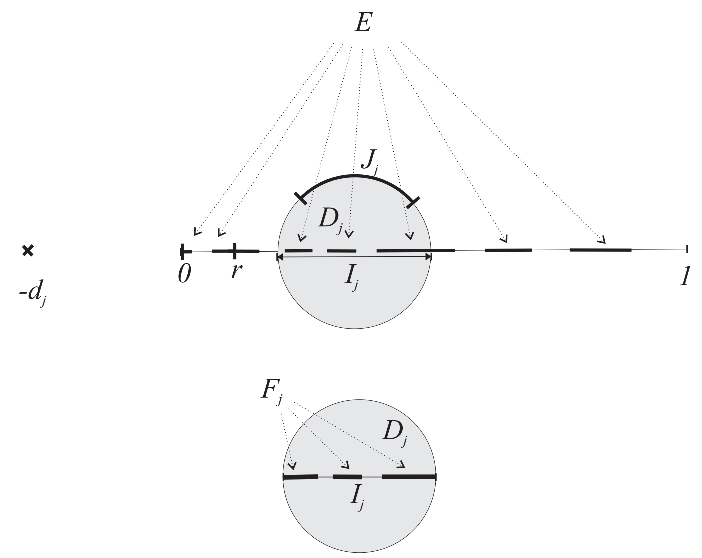

Figure 2.1: The disk D_j and the set F_j

Let D_j be the open disk with diameter I_j, and let J_j be the middle third part of the arc $\partial D_j \cap \{\Im z > 0\}$ (see Figure 2.1). Taking balayage of some measure supported in D_j onto $E \cup [0,r]$ can be done in two steps: first take balayage onto $\partial(D_j \setminus (E \cup [0,r]))$, and then onto $E \cup [0,r]$. By Lemma 2.5.2 below (transform the result in Lemma 2.5.2 into a result on the disk D_j) if

$$H_j^* = \{x \in I_j \ : \ \omega(x, \partial D_j, D_j \setminus F_j) > \theta_j/3\},$$

then $|H_j^*| \ge \theta_j |I_j|$. On the other hand, taking into account the equality (2.1.4), Lemma 2.5.3 below gives that for $x \in H_j^*$ we have

$$\begin{aligned}\mathrm{Bal}\Big(\delta_x, D_j \setminus F_j\Big)(J_j) &\ge c_\varepsilon \mathrm{Bal}\Big(\delta_x, D_j \setminus F_j\Big)(\partial D_j) \\ &= c_\varepsilon \omega(x, \partial D_j, D_j \setminus F_j) \ge c_\varepsilon \theta_j/3.\end{aligned}$$

Thus,

$$
\begin{aligned}
&\mathrm{Bal}\Big(\mu_{[0,1]}\big|_{I_j \setminus F_j}, D_j \setminus F_j\Big)(J_j) = \int_{I_j \setminus F_j} \mathrm{Bal}\Big(\delta_x, D_j \setminus F_j\Big)(J_j) d\mu_{[0,1]}(x) \\
&\geq \int_{H_j^*} (c_\varepsilon \theta_j/3) d\mu_{[0,1]}(x) \geq (c_\varepsilon \theta_j/3)|H_j^*| \frac{1}{4C_1\sqrt{|I_j|}} \\
&\geq (c_\varepsilon \theta_j/3)(\theta_j|I_j|)\frac{1}{4C_1\sqrt{|I_j|}} = \frac{c_\varepsilon}{12C_1}\theta_j^2\sqrt{|I_j|}, \qquad (2.3.16)
\end{aligned}
$$

where in the last but one inequality we used that

$$
d\mu_{[0,1]}(t) = \frac{1}{\pi\sqrt{t(1-t)}}dt,
$$

and hence for $t \in I_j = [a_j, b_j]$ we have

$$
\frac{1}{\pi\sqrt{t(1-t)}} \geq \frac{1}{\pi\sqrt{b_j}} \geq \frac{1}{4C_1\sqrt{|I_j|}}
$$

by the assumption $b_j \leq C_1|I_j|$.

Now since $[0, r]$ is outside D_j,

$$
\begin{aligned}
&\mathrm{Bal}\Big(\mu_{[0,1]}\big|_{I_j \setminus F_j}, \mathbf{C} \setminus (E \cup [0, r])\Big)([0, r]) \qquad (2.3.17) \\
= &\int_{\partial D_j} \mathrm{Bal}\Big(\delta_y, \mathbf{C} \setminus (E \cup [0, r])\Big)([0, r]) d\mathrm{Bal}\Big(\mu_{[0,1]}\big|_{I_j \setminus F_j}, D_j \setminus (E \cup [0, r])\Big)(y),
\end{aligned}
$$

and here for $y \in \partial D_j$

$$
\mathrm{Bal}\Big(\mu_{[0,1]}\big|_{I_j \setminus F_j}, D_j \setminus (E \cup [0, r])\Big)(y) \geq \mathrm{Bal}\Big(\mu_{[0,1]}\big|_{I_j \setminus F_j}, D_j \setminus F_j\Big)(y).
$$

Therefore, we can continue (2.3.17) as

$$
\begin{aligned}
&\mathrm{Bal}\Big(\mu_{[0,1]}\big|_{I_j \setminus F_j}, \mathbf{C} \setminus (E \cup [0, r])\Big)([0, r]) \\
&\geq \int_{J_j} \mathrm{Bal}\Big(\delta_y, \mathbf{C} \setminus (E \cup [0, r])\Big)([0, r]) d\mathrm{Bal}\Big(\mu_{[0,1]}\big|_{I_j \setminus F_j}, D_j \setminus F_j\Big)(y)
\end{aligned}
$$

$$\geq \left(\inf_{y\in J_j} \mathrm{Bal}\Big(\delta_y, \mathbf{C}\setminus (E\cup[0,r])\Big)([0,r])\right) \mathrm{Bal}\Big(\mu_{[0,1]}\Big|_{I_j\setminus F_j}, D_j\setminus F_j\Big)(J_j)$$

$$\geq \left(\inf_{y\in J_j} \mathrm{Bal}\Big(\delta_y, \mathbf{C}\setminus (E\cup[0,r])\Big)([0,r])\right) \frac{c_\varepsilon}{12C_1}\theta_j^2\sqrt{|I_j|}. \qquad (2.3.18)$$

For $y\notin E\cup[0,r]$ the quantity

$$\mathrm{Bal}\Big(\delta_y, \mathbf{C}\setminus (E\cup[0,r])\Big)([0,r]) = \omega\Big(y,[0,r],\mathbf{C}\setminus(E\cup[0,r])\Big)$$

is a nonnegative harmonic function of y, hence by Harnack's inequality we have for $y\in J_j$ and $d_j=|I_j|$ the inequality

$$\mathrm{Bal}\Big(\delta_y, \mathbf{C}\setminus (E\cup[0,r])\Big)([0,r]) \geq c_1\mathrm{Bal}\Big(\delta_{-d_j}, \mathbf{C}\setminus (E\cup[0,r])\Big)([0,r])$$

with some absolute constant $c_1>0$ because $\mathrm{dist}(J_j,0)\sim \mathrm{dist}(J_j,[0,1])\sim |I_j|=d_j$. By ([13, Ch. II, (4.47)]) we have

$$\begin{aligned}\mathrm{Bal}\Big(\delta_{-d_j}, \mathbf{C}\setminus (E\cup[0,r])\Big)([0,r]) &\geq \mathrm{Bal}\Big(\delta_{-d_j}, \mathbf{C}\setminus [0,1]\Big)([0,r]) \qquad (2.3.19)\\ &= \frac{1}{\pi}\int_0^r \frac{\sqrt{d_j}\sqrt{1+d_j}}{\sqrt{t(1-t)}(t+d_j)}dt\\ &\geq \frac{1}{\pi}\frac{\sqrt{r}}{\sqrt{d_j}} = \frac{1}{\pi}\frac{\sqrt{r}}{\sqrt{|I_j|}}.\end{aligned}$$

This, the previous inequality and (2.3.18) give

$$\mathrm{Bal}\Big(\mu_{[0,1]}\Big|_{I_j\setminus F_j}, \mathbf{C}\setminus (E\cup[0,r])\Big)([0,r]) \geq \frac{c_1c_\varepsilon}{12\pi C_1}\theta_j^2\sqrt{r},$$

which proves (2.3.15), and the proof of (2.3.13) is complete.

Step II. Let $E\subseteq[0,1]$ be compact and for $\varepsilon>0$, $0<t<1$ set

$$E_\varepsilon^*(t) = (E\cap[\varepsilon t/2,t])\cup[\varepsilon t/2,\varepsilon t]\cup[(1-\varepsilon/2)t,t]. \qquad (2.3.20)$$

Then for $0 < q < 1$

$$\sum_{m:q^m>\frac{2r}{\varepsilon}} \left(\frac{1}{4} - \frac{\operatorname{cap}(E_\varepsilon^*(q^m))}{q^m(1-\varepsilon/2)}\right)^2 < c_0 \frac{g_\Omega(-r)}{\sqrt{r}}, \tag{2.3.21}$$

where c_0 *depends only on* ε *and* q.

To prove this let the integer M be so large that $q^M < \varepsilon/2$. Clearly, it is sufficient to show that for each $l = 1, \ldots, M$ the sum for the subsequence $m = jM + l$, $j \in \mathbf{N}$ satisfies

$$\sum_{j:q^{jM+l}>\frac{2r}{\varepsilon}} \left(\frac{1}{4} - \frac{\operatorname{cap}(E_\varepsilon^*(q^{jM+l}))}{q^{jM+l}(1-\varepsilon/2)}\right)^2 < c_l \frac{g_\Omega(-r)}{\sqrt{r}}.$$

But this immediately follows from the result proved in Part 1, since the intervals $I_j = [\varepsilon q^{jM+l}/2, q^{jM+l}]$, $j \in \mathbf{N}$ are pairwise disjoint and the set F_j defined in (2.3.12) for these intervals is contained in $E_\varepsilon^*(q^{jM+l})$.

Step III. Finally, we complete the proof of Theorem 2.2.1. Let $\varepsilon > 0$ and $0 < u < 1$. If $u \le t \le u(1-\varepsilon/2)/(1-\varepsilon)$, then for the sets (2.2.5) and (2.3.20) the relation $E_\varepsilon(t) \cap [\varepsilon u/2, u] \supseteq E_\varepsilon^*(u)$ holds, and so

$$\frac{\operatorname{cap}(E_\varepsilon^*(u))}{u(1-\varepsilon/2)} \le \frac{\operatorname{cap}(E_\varepsilon(t) \cap ([\varepsilon u/2, u]))}{u(1-\varepsilon/2)}. \tag{2.3.22}$$

But $E_\varepsilon(t) = [0, \varepsilon u/2] \cup (E_\varepsilon(u) \cap [\varepsilon u/2, u]) \cup [u, t]$, i.e. $E_\varepsilon(t)$ is obtained from $E_\varepsilon(t) \cap [\varepsilon u/2, u]$ by attaching one-one intervals to the right and to the left. Therefore, we can apply Lemma 2.5.4 below $\Big($(2.5.42),twice$\Big)$ to conclude

$$\frac{\operatorname{cap}(E_\varepsilon(u) \cap ([\varepsilon u/2, u]))}{u(1-\varepsilon/2)} \le \frac{\operatorname{cap}(E_\varepsilon(t))}{t},$$

which, together with (2.3.22), gives

$$\frac{1}{4} - \frac{\operatorname{cap}(E_\varepsilon(t))}{t} \le \frac{1}{4} - \frac{\operatorname{cap}(E_\varepsilon^*(u))}{u(1-\varepsilon/2)}. \tag{2.3.23}$$

This is true for all $u \le t \le u(1-\varepsilon/2)/(1-\varepsilon)$, therefore if we square both sides, divide by t and integrate with respect to t over the interval $[u, u(1-\varepsilon/2)/(1-\varepsilon)]$ then we obtain with $q = (1-\varepsilon)/(1-\varepsilon/2)$

$$\int_u^{u/q} \left(\frac{1}{4} - \frac{\mathrm{cap}(E_\varepsilon(t))}{t}\right)^2 \frac{1}{t} dt \le \left(\log \frac{1-\varepsilon/2}{1-\varepsilon}\right) \left(\frac{1}{4} - \frac{\mathrm{cap}(E^*_\varepsilon(u))}{u(1-\varepsilon/2)}\right)^2 . \qquad (2.3.24)$$

Let k be the largest integer for which $q^k > \frac{2r}{\varepsilon}$. Summing up (2.3.24) for the values $u = q, q^2, q^3, \ldots, q^k$ and making use of (2.3.21) we obtain

$$\int_{q^k}^1 \left(\frac{1}{4} - \frac{\mathrm{cap}(E_\varepsilon(t))}{t}\right)^2 \frac{1}{t} dt < C_3 \frac{g_\Omega(-r)}{\sqrt{r}}.$$

Since

$$q^k \le \frac{2r}{\varepsilon}\frac{1}{q} = \frac{2r}{\varepsilon}\frac{1-\frac{\varepsilon}{2}}{1-\varepsilon} \le \frac{4r}{\varepsilon},$$

we can change the limit of the integral to $\frac{4r}{\varepsilon}$. Then, changing $\frac{4r}{\varepsilon}$ for r we can use Harnack's inequality to obtain

$$g_\Omega(-\frac{\varepsilon r}{4}) \le C_4 g_\Omega(-r),$$

where C_4 depends only on ε. This completes the proof of Theorem 2.2.1.

■

2.4 Proof of Theorem 2.2.3

First of all notice that in the proof of Theorem 2.2.1 we used the fact that $E \subset [0,1]$ only in Step I. Actually, we used it at two steps: proving (2.3.16) (using the equilibrium measure of $[0,1]$) and establishing (2.3.19). Therefore, we will only mention the steps where the proof differs from that of Theorem 2.2.2.

We are going to use the notations of Step I. Instead of (2.3.14) now we have

$$\mu_{E\cup[0,r]}([0,r]) \le C_2 g_\Omega(ir), \qquad 0 < r < 1. \qquad (2.4.25)$$

Indeed,

$$\begin{aligned}
g_\Omega(ir) &\geq g_{\overline{\mathbf{C}}\setminus(E\cup[0,r])}(ir)\\
&= U^{\mu_{E\cup[0,r]}}(0) - U^{\mu_{E\cup[0,r]}}(ir) = \int \log\left|\frac{ir-t}{t}\right| d\mu_{E\cup[0,r]}(t)\\
&\geq \log\sqrt{2}\int_0^r d\mu_{E\cup[0,r]}(t) = \Big(\log\sqrt{2}\Big)\,\mu_{E\cup[0,r]}([0,r]).
\end{aligned}$$

Replacing $\mu_{[0,1]}$ by $\mu_{[-1,1]}$ in the argument before (2.3.15) one can see that it suffices to prove that for $I_j \subseteq [r,1]$, $|I_j| \geq r$

$$\mathrm{Bal}\Big(\mu_{[-1,1]}\big|_{I_j\setminus F_j}, \mathbf{C}\setminus(E\cup[0,r])\Big)([0,r]) \geq c\theta_j^2 r \tag{2.4.26}$$

holds with some positive constant c.

Now (c.f. (2.3.16)) we have

$$\mathrm{Bal}\Big(\mu_{[-1,1]}\big|_{I_j\setminus F_j}, D_j\setminus F_j\Big)(J_j) \geq \int_{H_j^*}(c_\varepsilon\theta_j/3)\,d\mu_{[-1,1]}(x) \geq \frac{c_\varepsilon\theta_j^2}{3\pi}|I_j|, \tag{2.4.27}$$

since

$$d\mu_{[-1,1]}(t) = \frac{1}{\pi\sqrt{(1+t)(1-t)}}dt,$$

$$\frac{1}{\pi\sqrt{(1+t)(1-t)}} \geq \frac{1}{\pi}$$

and $|H_j^*| \geq \theta_j|I_j|$.

In (2.3.19) we used δ_{-d_j}. Now, since $-d_j$ may be in E, let us change it for id_j. By Harnack's inequality we have for $y \in J_j$

$$\begin{aligned}
\mathrm{Bal}\Big(\delta_y, \mathbf{C}\setminus(E\cup[0,r])\Big)([0,r]) &\geq c_1\mathrm{Bal}\Big(\delta_{id_j}, \mathbf{C}\setminus(E\cup[0,r])\Big)([0,r])\\
&\geq c_1\mathrm{Bal}\Big(\delta_{id_j}, \mathbf{C}\setminus[-1,1]\Big)([0,r])\\
&= c_1\omega\Big(id_j, [0,r], \mathbf{C}\setminus[-1,1]\Big).
\end{aligned} \tag{2.4.28}$$

Applying the transformation $\varphi(z) = z - \sqrt{z^2-1}$ and using ([13, Ch. II,(4.8)]) we have

$$\begin{aligned}
\omega\Big(id_j,[0,r],\mathbf{C}\setminus[-1,1]\Big) &= \omega\Big(i\Big(d_j-\sqrt{1+d_j^2}\Big),A,D\Big)\\
&= \frac{1}{2\pi}\Big(\int\limits_{\arccos r}^{\frac{\pi}{2}} P\Big(\zeta,i\Big(d_j-\sqrt{1+d_j^2}\Big)\Big)dt\\
&+ \int\limits_{\frac{3\pi}{2}}^{2\pi-\arccos r} P\Big(\zeta,i\Big(d_j-\sqrt{1+d_j^2}\Big)\Big)dt\Big),
\end{aligned}$$

where $\zeta=e^{it}$, P is the Poisson kernel and A is the intersection $\mathbf{T}\cap\{z\ :\ 0\le\Re(z)\le r\}$ consisting of two arcs on the unit circle. Thus,

$$\begin{aligned}
&\omega\Big(id_j,[0,r],\mathbf{C}\setminus[-1,1]\Big)\\
&\ge \frac{1}{2\pi}\int\limits_{\frac{3\pi}{2}}^{2\pi-\arccos r}\frac{1-\Big(1+2d_j^2-2d_j\sqrt{1+d_j^2}\Big)}{1-2(\sqrt{1+d_j^2}-d_j)\cos\Big(t+\frac{\pi}{2}\Big)+(1+2d_j^2-2d_j\sqrt{1+d_j^2})}dt\\
&\ge \frac{1}{\pi}r\frac{d_j(\sqrt{1+d_j^2}-d_j)}{1-2(\sqrt{1+d_j^2}-d_j)\sqrt{1-r^2}+(1+2d_j^2-2d_j\sqrt{1+d_j^2})}.
\end{aligned}$$

Assuming $d_j\ge r$ we get

$$\begin{aligned}
&\omega\Big(id_j,[0,r],\mathbf{C}\setminus[-1,1]\Big)\\
&\ge \frac{r}{\pi}\frac{d_j(\sqrt{1+d_j^2}-d_j)}{1-2(\sqrt{1+d_j^2}-d_j)\sqrt{1-d_j^2}+(1+2d_j^2-2d_j\sqrt{1+d_j^2})}\\
&= \frac{r}{2\pi}\frac{d_j(\sqrt{1+d_j^2}-d_j)}{1-(\sqrt{1+d_j^2}-d_j)(\sqrt{1-d_j^2}+d_j)}\\
&= \frac{r}{2\pi}f(d_j),
\end{aligned}$$

where

$$f(x)=\frac{x(\sqrt{1+x^2}-x)}{1-(\sqrt{1+x^2}-x)(\sqrt{1-x^2}+x)}=\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{2x}.$$

$xf(x)$ is monotone decreasing on $[0,1]$, hence

$$f(x) \geq \frac{f(1)}{x} = \frac{\sqrt{2}}{2}\left(\frac{1}{x}\right),$$

which gives

$$\omega\Big(id_j, [0,r], \mathbf{C} \setminus [-1,1]\Big) \geq \frac{\sqrt{2}}{4\pi}\left(\frac{r}{d_j}\right).$$

This, (2.4.28) and (2.4.27) give

$$\mathrm{Bal}\Big(\mu_{[-1,1]}\Big|_{I_j \setminus F_j}, \mathbf{C} \setminus (E \cup [0,r])\Big)([0,r]) \geq \frac{c_1 c_\varepsilon \sqrt{2}}{12\pi^2}\theta_j^2 r,$$

which proves (2.4.26), and the proof of Theorem 2.2.3 is complete.

■

The proof of Corollary 2.2.5 is immediate from Lemmas 2.5.1 and 2.5.4. First of all, Lemma 2.5.1 implies (1.0.2) and

$$\lim_{r\to 0} \frac{\mathrm{cap}(E \cap [-r,0])}{r} = \frac{1}{4}.$$

Then, taking $I = [-r,0]$, $J = [0,r]$ and $F = E \cap [-r,0]$ in (2.5.42) we get

$$\lim_{r\to 0} \frac{\mathrm{cap}\Big((E \cap [-r,0]) \cup [0,r]\Big)}{r} = \frac{1}{2}. \tag{2.4.29}$$

Next, taking $I = [0,r]$, $J = [-r,0]$, $F = E \cap [0,r]$ and $G = E \cap [-r,0]$ in (2.5.41) we can infer

$$\frac{\mathrm{cap}(E \cap [-r,r])}{\mathrm{cap}\Big((E \cap [-r,0]) \cup [0,r]\Big)} \geq \frac{4\mathrm{cap}(E \cap [0,r])}{r} \to 1. \tag{2.4.30}$$

Finally, (2.2.11) is a direct consequence of (2.4.29) and (2.4.30).

■

2.5 Lemmas

Lemma 2.5.1 *If* (2.2.7) *is true for every* $\varepsilon > 0$ *then* (1.0.2) *holds.*

Proof. Let $\eta > 0$ be arbitrary such that $1+\eta \le (1-\varepsilon/2)/(1-\varepsilon)$. For $t/(1+\eta) \le u \le t$ we have $E_{\varepsilon/2}(u) \subseteq E_\varepsilon(t)$, therefore

$$\begin{aligned}\frac{1}{4} - \frac{\operatorname{cap}(E_{\varepsilon/2}(u))}{u} &\ge \frac{1}{4} - \frac{\operatorname{cap}(E_\varepsilon(t))}{u} \ge \frac{1}{4} - \frac{\operatorname{cap}(E_\varepsilon(t))}{t}(1+\eta) \\ &\ge \frac{1}{4} - \frac{\operatorname{cap}(E_\varepsilon(t))}{t} - \eta.\end{aligned}$$

On adding η to the first and last term in this inequality, squaring, dividing by u both sides and integrating with respect to u over the interval $t/(1+\eta) \le u \le t$ we obtain

$$\left(\frac{1}{4} - \frac{\operatorname{cap}(E_\varepsilon(t))}{t}\right)^2 \le 2\frac{1}{\log(1+\eta)} \int_{t/(1+\eta)}^{t} \left(\frac{1}{4} - \frac{\operatorname{cap}(E_{\varepsilon/2}(u))}{u}\right)^2 \frac{1}{u} dt + 2\eta^2.$$

Therefore, the finiteness of the integral in (2.2.7) (for $\varepsilon/2$ rather than for ε) gives

$$\limsup_{t\to 0} \left(\frac{1}{4} - \frac{\operatorname{cap}(E_\varepsilon(t))}{t}\right) \le \sqrt{2}\eta,$$

and since here $\eta > 0$ can be arbitrary small, it follows that

$$\lim_{t\to 0} \frac{\operatorname{cap}(E_\varepsilon(t))}{t} = \frac{1}{4}. \tag{2.5.31}$$

Now let $\{t_n\}$ be an arbitrary positive sequence tending to 0 and set

$$F_n = E_\varepsilon(t_n)/t_n, \qquad \nu_n = \frac{1}{\mu_{F_n}([\varepsilon, 1-\varepsilon])} \mu_{F_n}\big|_{[\varepsilon, 1-\varepsilon]}.$$

We have just proved that $\operatorname{cap}(F_n) \to 1/4$ as $n \to \infty$, and below we verify that this implies the convergence $\mu_{F_n} \to \mu_{[0,1]}$ in the weak* topology. Since

$$\mu_{[0,1]}((\varepsilon, 1-\varepsilon)) > 1 - 2\sqrt{\varepsilon},$$

there is an n_0 such that for $n \ge n_0$ we have $\mu_{F_n}((\varepsilon, 1-\varepsilon)) \ge 1 - 2\sqrt{\varepsilon}$. There is also an n_1 such that for $n \geq n_1$ the inequality

$$U^{\mu_{F_n}}(x) = \log \frac{1}{\mathrm{cap}(F_n)} \leq (1+\varepsilon) \log 4, \qquad x \in F_n,$$

holds, which implies for $n \geq \max(n_0, n_1)$

$$U^{\nu_n}(x) \leq \frac{1}{1 - 2\sqrt{\varepsilon}} (1+\varepsilon) \log 4, \qquad x \in F_n.$$

But the measure ν_n is supported on $F_n \cap [\varepsilon, 1-\varepsilon]$ and has mass 1, hence the preceding inequality gives

$$\log \frac{1}{\mathrm{cap}(F_n \cap ([\varepsilon, 1-\varepsilon])} \leq \int U^{\nu_n} d\nu_n \leq \frac{1}{1-2\sqrt{\varepsilon}} (1+\varepsilon) \log 4, \qquad x \in F_n,$$

i.e.

$$\mathrm{cap}((E_n \cap [0, t_n])/t_n) \geq \mathrm{cap}(F_n \cap [\varepsilon, 1-\varepsilon]) \geq \left(\frac{1}{4}\right)^{(1+\varepsilon)/(1-2\sqrt{\varepsilon})}.$$

Since here $\varepsilon > 0$ is arbitrary, it follows that $\mathrm{cap}(E_n \cap [0, t_n])/t_n) \to 1/4$ as $n \to \infty$, and this is (1.0.2).

In the preceding argument we used that as $n \to \infty$, we have $\mu_{F_n} \to \mu_{[0,1]}$ in the weak* topology on measures. In fact, let σ be a weak* limit of some subsequence, say $\mu_{F_{n_l}} \to \sigma$ as $l \to \infty$. Then σ is supported in $[0, 1]$, has total mass 1, and all we have to show is that $\sigma = \mu_{[0,1]}$. We know that

$$U^{\mu_{F_n}}(x) = \log \frac{1}{\mathrm{cap}(F_n)} \tag{2.5.32}$$

for $x \in F_n$ with the exception of a set of capacity 0, and the same is true for $[0, 1]$. Since $F_n \subset [0, 1]$, it follows that

$$U^{\mu_{F_n}}(x) \leq U^{\mu_{[0,1]}}(x) + \log \frac{\mathrm{cap}([0,1])}{\mathrm{cap}(F_n)}$$

for $x \in F_n$ with the exception of a set of capacity 0, and since every set of zero capacity has zero μ_{F_n}-measure (see [13, Remark I.1.7, p. 28]), it follows that this inequality is true μ_{F_n}-almost everywhere. But then by the principle of domination [13, Theorem II.3.2] the same inequality is true for all $x \in \mathbf{C}$. Fixing such an $x \notin [0,1]$ and letting n tend to infinity through the subsequence $\{n_l\}$ it follows from $\mathrm{cap}(F_n) \to 1/4 = \mathrm{cap}(E)$ that

$$U^{\sigma}(x) \leq U^{\mu_{[0,1]}}(x).$$

Thus, this inequality is true for all $x \in \mathbf{C} \setminus [0,1]$.

However, the function

$$U^{\mu_{[0,1]}}(x) - U^{\sigma}(x)$$

vanishes at infinity, so it is harmonic there, and an appeal to the minimum principle on the domain $\overline{\mathbf{C}} \setminus [0,1]$ yields that we must have

$$U^{\sigma}(x) \equiv U^{\mu_{[0,1]}}(x), \qquad x \in \mathbf{C} \setminus [0,1].$$

Now we can conclude $\sigma = \mu_{[0,1]}$ from the unicity theorem [13, Theorem II.4.13].

■

Lemma 2.5.2 *Let $F \subseteq [-1,1]$ be a compact set,*

$$\theta = \frac{1}{4} - \frac{\mathrm{cap}(F)}{2}$$

and

$$H^* = \{x \in [-1,1] \ : \ \omega(x, \mathbf{T}, \mathbf{D} \setminus F) > \theta/3\}.$$

Then $|H^*| \geq 2\theta$.

Proof. It is enough to show that if $\theta \leq 1/8$ and

$$H = \{x \in [-1,1] \ : \ \omega(x, \mathbf{T}, \mathbf{D} \setminus F) > 2\theta/3\},$$

then

$$|H| \geq 4\theta. \tag{2.5.33}$$

In fact, for $\theta \le 1/8$ this is better than the claim in the lemma, while for $1/4 \ge \theta > 1/8$ the claim follows from the $\theta = 1/8$ special case of (2.5.33).

Let thus be $\theta \le 1/8$. Suppose to the contrary that (2.5.33) is not true, i.e. $|H| < 4\theta$. Then

$$\mathrm{cap}([-1,1] \setminus H) \ge \frac{|[-1,1] \setminus H|}{4} > \frac{1-2\theta}{2}. \tag{2.5.34}$$

Let $\mathbf{D}_2$ denote the open disk about the origin and of radius 2. If $x \in [-1,1] \setminus H$, we have

$$\omega(x, \partial\mathbf{D}_2, \mathbf{D}_2 \setminus F) \le \omega(x, \mathbf{T}, \mathbf{D} \setminus F) < \frac{2\theta}{3}.$$

But for $|z| = 2$

$$g_{\overline{\mathbf{C}} \setminus F}(z) \le \log \frac{1}{\mathrm{cap}(F)} + \log 3,$$

and by $\theta \le 1/8$ here $\mathrm{cap}(F) \ge 1/4$. Thus,

$$g_{\overline{\mathbf{C}} \setminus F}(z) \le \log 12 \le 3 \qquad z \in \partial\mathbf{D}_2,$$

which means that on the boundary of the set $\mathbf{D}_2 \setminus F$ we have

$$g_{\overline{\mathbf{C}} \setminus F}(z) \le 3\omega(z, \partial\mathbf{D}_2, \mathbf{D}_2 \setminus F),$$

so this inequality pertains for all $z \in \mathbf{D}_2 \setminus F$. In particular, for $x \in [-1,1] \setminus H$ we have

$$g_{\overline{\mathbf{C}} \setminus F}(x) \le 2\theta,$$

which can be rewritten (see (2.1.1)) as

$$\log \frac{1}{\mathrm{cap}(F)} - U^{\mu_F}(x) \le 2\theta = \log \frac{1}{\mathrm{cap}([-1,1] \setminus H)} - U^{\mu_{[-1,1]\setminus H}}(x) + 2\theta.$$

But then by the principle of domination ([13, Ch. II, Theorem 3.2]) this inequality holds true for all $x \in \mathbf{C}$, and for $x \to \infty$ we obtain

$$\log \frac{1}{\mathrm{cap}(F)} \le \log \frac{1}{\mathrm{cap}([-1,1] \setminus H)} + 2\theta,$$

that is

$$\mathrm{cap}(F) \geq \mathrm{cap}([-1,1] \setminus H)e^{-2\theta}.$$

Now this gives via (2.5.34)

$$\begin{aligned} 2\theta = \frac{1}{2} - \mathrm{cap}(F) &\leq \frac{1}{2} - \mathrm{cap}([-1,1] \setminus H)e^{-2\theta} < \frac{1}{2} - \frac{1-2\theta}{2}e^{-2\theta} \\ &< \frac{1}{2}(1-e^{-2\theta}) + \theta \leq \frac{2\theta}{2} + \theta = 2\theta, \end{aligned}$$

which is a contradiction. This contradiction proves (2.5.33).

■

Lemma 2.5.3 *Let $J = \{e^{i\varphi} \; : \; \pi/3 \leq \varphi \leq 2\pi/3\}$ be the middle third of the upper part of the unit circle. For every $\varepsilon > 0$ there is a constant $c_\varepsilon > 0$ with the following property: if $F \subset [-1,1]$ is any compact set with $[-1,-1+\varepsilon] \cup [1-\varepsilon, 1] \subseteq F$, then for $x \in [-1,1] \setminus F$ the inequality*

$$\omega(x, J, \mathbf{D} \setminus F) \geq c_\varepsilon \omega(x, \mathbf{T}, \mathbf{D} \setminus F) \tag{2.5.35}$$

holds.

Proof. First we verify the lemma in the special case when $[-1,1] \setminus F = I = (u,v)$ is an interval. Let $\alpha \in I$ be the point for which

$$-\frac{u-\alpha}{1-\alpha u} = \frac{v-\alpha}{1-\alpha v},$$

and apply the conformal map $\psi_1(z) = (z-\alpha)/(1-\alpha z)$. This maps the unit circle into itself, F into a set F' of type $[-1,-a] \cup [a,1]$, and J into some arc J' of the upper half circle $\mathbf{T}_+ = \{e^{i\varphi} \; : \; 0 \leq \varphi \leq \pi\}$ (see Figure 2.2). It is easy to see that there is constant $b_\varepsilon > 0$ depending only ε such that F' contains the intervals $[-1,-1+b_\varepsilon]$ and $[1-b_\varepsilon, 1]$ and the both the arc length of J' and the distance of J' from the points ± 1 is $\geq b_\varepsilon$. Map now $\mathbf{D} \setminus F'$ conformally onto $\mathbf{D}$ via the mapping ψ_2 normalized by $\psi_2(0) = 0$, $\psi_2'(0) > 0$. The image of $[-1,0] \cap F' = [-1,-a]$ is an arc on $\mathbf{T}$ symmetric about the point -1, and similarly the image of $[0,1] \cap F' = [a,1]$ is an arc on $\mathbf{T}$

symmetric about the point 1, furthermore the length of these arcs are bounded from below by some constant $d_\varepsilon > 0$. $\mathbf{T}$ is mapped into the complementary arcs of $\mathbf{T}$, and let us denote the complementary arc lying on the upper half plane by A'' (which is the image of the upper half circle $\mathbf{T}_+$ under ψ_2, i.e. $A'' = \psi_2(\mathbf{T}_+)$). The image J'' of J' is a subarc of A'', and its length is comparable to the length of the latter, i.e. with some $\delta_\varepsilon > 0$ we have

$$\text{(arc length of } J'') \geq \delta_\varepsilon \text{(arc length of } A'').$$

If $y = \psi_2(\psi_1(x)) \in (-1, 1)$ is the image of x, then using the conformal invariance of harmonic measures, (2.5.35) takes the form

$$\omega(y, J'', \mathbf{D}) \geq c_\varepsilon \omega(y, \psi_2 \circ \psi_1(\mathbf{T}), \mathbf{D}),$$

which, using the symmetry of the image $\psi_2 \circ \psi_1(\mathbf{T}) = A'' \cup (-A'')$, is equivalent to

$$\omega(y, J'', \mathbf{D}) \geq 2c_\varepsilon \omega(y, A'', \mathbf{D}).$$

But in $\mathbf{D}$ harmonic measures are given by the Poisson kernel, so the preceding inequality is the same as

$$\frac{1}{2\pi} \int_{J''} \frac{1 - y^2}{|\xi - y|^2} |d\xi| \geq 2c_\varepsilon \frac{1}{2\pi} \int_{A''} \frac{1 - y^2}{|\xi - y|^2} |d\xi|, \tag{2.5.36}$$

which is clear with some $c_\varepsilon > 0$, since $y \in [-1, 1]$ and on the two sides during integration ξ runs trough two arcs of comparable length both of which lie of distance $\geq d_\varepsilon/4$ from $[-1, 1]$. Thus, (2.5.36) is true with some $c_\varepsilon > 0$, and this gives (2.5.35).

Next we turn to the general case, i.e. when $[-1, 1] \setminus F = [-1 + \varepsilon, 1 - \varepsilon] \setminus F$ is an arbitrary open set. Since the constant c_ε should be independent of the set F (depending only on ε with $[-1, -1 + \varepsilon] \cup [1 - \varepsilon, 1] \subseteq F$), without loss of generality we may assume F to consist of finitely many intervals, in which case $[-1, 1] \setminus F$ consists of finitely many open intervals, say $I_1, \ldots, I_m$.

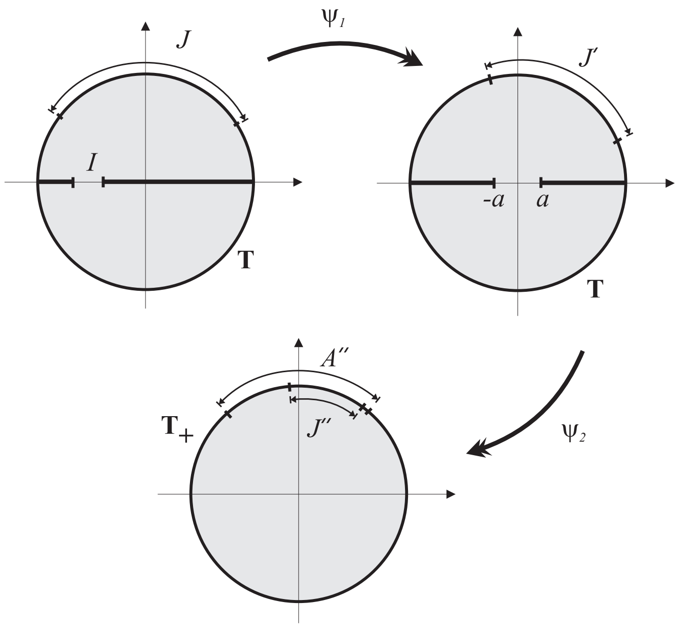

Figure 2.2: The mappings ψ_1 and ψ_2

According to (3.1.7), what we have to prove is that there is a constant $c_\varepsilon > 0$ such that for $x \in [-1, 1] \setminus F$ we have

$$\mathrm{Bal}\Big(\delta_x, \mathbf{D} \setminus F\Big)(J) \geq c_\varepsilon \mathrm{Bal}\Big(\delta_x, \mathbf{D} \setminus F\Big)(\mathbf{T}). \tag{2.5.37}$$

We show that the constant c_ε verified above for the special case when $[-1, 1] \setminus F$ was an interval, is appropriate. To this end, starting from $\nu_0 = \delta_x$, we successively define the measures ν_n by

$$\nu_{n+1} = \mathrm{Bal}\Big(\nu_n, \mathbf{D} \setminus ([-1, 1] \setminus I_{j_n})\Big),$$

where $j_n \in \{1, 2, \ldots, m\}$ is the index j for which $\nu_n(I_j)$ is maximal for $j = 1, \ldots, m$. Each ν_n is supported on $\mathbf{T} \cup [-1, 1]$, and on $\mathbf{T} \cup F$ the measures ν_n form a monotone increasing sequence of measures. Note also that on $\mathbf{T} \cup F$ we have

$$\nu_{n+1} - \nu_n = \int_{I_{j_n}} \mathrm{Bal}\Big(\delta_y, \mathbf{D} \setminus ([-1, 1] \setminus I_{j_n})\Big) d\nu_n(y),$$

and by the special case proved in the first part of this proof, here we have for all

$x \in I_{j_n}$ the inequality

$$\mathrm{Bal}\Big(\delta_y, \mathbf{D} \setminus ([-1,1] \setminus I_{j_n})\Big)(J) \geq c_\varepsilon \mathrm{Bal}\Big(\delta_y, \mathbf{D} \setminus ([-1,1] \setminus I_{j_n})\Big)(\mathbf{T}).$$

Therefore the same is true of $\nu_{n+1} - \nu_n$, i.e. we have

$$\nu_{n+1}(J) - \nu_n(J) \geq c_\varepsilon(\nu_{n+1}(\mathbf{T}) - \nu_n(\mathbf{T})), \qquad n = 0, 1, \ldots.$$

Since $\nu_0(J) = \nu_0(\mathbf{T}) = 0$, induction gives

$$\nu_{n+1}(J) \geq c_\varepsilon \nu_{n+1}(\mathbf{T}), \qquad n = 0, 1, \ldots,$$

therefore (2.5.37) will follow from here if we show that $\nu_n \to \mathrm{Bal}\Big(\delta_x, \mathbf{D} \setminus F\Big)$ as $n \to \infty$. As $\{\nu_n|_{\mathbf{T} \cup F}\}_{n=0}^{\infty}$ is an increasing sequence of measures on $\mathbf{T} \cup F$, it converges to some measure ν supported on $\mathbf{T} \cup F$, and to complete the proof we show that $\nu = \mathrm{Bal}\Big(\delta_x, \mathbf{D} \setminus F\Big)$ and $\nu_n([-1,1] \setminus F) \to 0$ as $n \to 0$. Since the total mass of each ν_n is 1, it is clear that the total mass of ν is at most 1. Also, by the properties of balayage measures, for $z \in C \cup F$ and for all n we have the equality

$$U^{\nu_{n+1}}(z) = U^{\nu_n}(z) = \cdots = U^{\nu_0}(z) = \log \frac{1}{|z - x|},$$

and it is easy to see that then the same is true of ν, i.e.

$$U^{\nu}(z) = \log \frac{1}{|z - x|}, \qquad z \in \mathbf{T} \cup F. \tag{2.5.38}$$

Now $\mathrm{Bal}\Big(\delta_x, \mathbf{D} \setminus F\Big)$ is the unique measure supported on $\mathbf{T} \cup F$ which has mass 1 and its logarithmic potential is $\log 1/|z - x|$, thus the proof will be complete if we show that ν has mass 1, i.e. $\nu(\mathbf{T} \cup F) = 1$, which is the same as

$$\lim_{n\to\infty} \nu_n([-1,1] \setminus F) = 0$$

which we wanted to prove anyway. This will be done by showing that in each step when going from ν_n to ν_{n+1} a fixed portion of the mass $\nu_n\Big|_{[-1,1]\setminus F}$ is moved to F, i.e. with some $\gamma < 1$ we have

$$\nu_{n+1}([-1,1]\setminus F) \le \gamma\nu_n([-1,1]\setminus F). \tag{2.5.39}$$

Let $I_j = [a_j, b_j]$, and let $\tau > 0$ be so small that all the intervals $[a_j - \tau, a_j]$ and $[b_j, b_j + \tau]$ are part of $(-1,1)$ and they are disjoint. For $I = I_{j_n}$ and $y \in I$ the value

$$\mathrm{Bal}\Big(\delta_y, \mathbf{D}\setminus([-1,1]\setminus I)\Big)([a_{j_n} - \tau, a_{j_n}] \cup [b_{j_n}, b_{j_n} + \tau]),$$

which is the same as

$$\omega\Big(y, [a_{j_n} - \tau, a_{j_n}] \cup [b_{j_n}, b_{j_n} + \tau], \mathbf{D}\setminus([-1,1]\setminus I)\Big)$$

is bounded from below by a constant ρ independent of n and $y \in I = I_{j_n}$. In fact, consider the conformal maps ψ_1, ψ_2 from the first part of the proof. Under $\psi_2 \circ \psi_1$ the set $[a_{j_n} - \tau, a_{j_n}] \cup [b_{j_n}, b_{j_n} + \tau]$ is mapped into the union of two arcs $A_\pm$, one-one around ± 1, of length bounded from below by a positive constant depending only on ε and τ. Now the inequality

$$\omega\Big(y, [a_{j_n} - \tau, a_{j_n}] \cup [b_{j_n}, b_{j_n} + \tau], \mathbf{D}\setminus([-1,1]\cup I)\Big) \ge \rho \tag{2.5.40}$$

with some positive constant ρ follows from the fact that here the left hand side is

$$\omega(z, A_- \cup A_+, \mathbf{D}) = \frac{1}{2\pi}\int_{A_-\cup A_+} \frac{1 - z^2}{|\xi - z|^2}|d\xi|, \qquad z = \psi_2(\psi_1(y)),$$

and the integral is bounded from below by a positive constant ρ for any point $z \in [-1,1]$ (and hence in particular also for the point $z = \psi_2(\psi_1(y))$).

We obtain from (2.5.40)

$$\mathrm{Bal}\Big(\delta_y, \mathbf{D}\setminus([-1,1]\setminus I)\Big)(F) \ge \\ \mathrm{Bal}\Big(\delta_y, \mathbf{D}\setminus([-1,1]\setminus I)\Big)([a_{j_n} - \tau, a_{j_n}] \cup [b_{j_n}, b_{j_n} + \tau]) \ge \rho,$$

which gives

$$\begin{aligned}\mathrm{Bal}\Big(\nu_n\Big|_{I_{j_n}}, \mathbf{D}\setminus([-1,1]\setminus I)\Big)(F) &= \int_{I_{j_n}} \mathrm{Bal}\Big(\delta_y, \mathbf{D}\setminus([-1,1]\setminus I)\Big)(F)d\nu_n(y)\\ &\geq \int_{I_{j_n}} \rho\, d\nu_n(y) = \rho\nu_n(I_{j_n}),\end{aligned}$$

and here the right hand side is at least $\rho\nu_n([-1,1]\setminus F)/m$ by the choice of the interval I_{j_n}. Thus,

$$\begin{aligned}\nu_{n+1}([-1,1]\setminus F) &\leq \nu_n([-1,1]\setminus F) - \mathrm{Bal}\Big(\nu_n\Big|_{I_{j_n}}, \mathbf{D}\setminus([-1,1]\setminus I)\Big)(F)\\ &\leq \nu_n([-1,1]\setminus F) - \rho\nu_n([-1,1]\setminus F)/m\\ &= \nu_n([-1,1]\setminus F)(1-\rho/m).\end{aligned}$$

This proves (2.5.39) and the proof is complete.

∎

Lemma 2.5.4 *Let I be a closed interval in $\mathbf{R}$ and let J be a closed interval that is attached either from the left or from the right to I. Let F and G be closed subsets of I and J, respectively. Then*

$$\frac{\mathrm{cap}(F)}{|I|} \leq \frac{\mathrm{cap}(F\cup G)}{4\mathrm{cap}(I\cup G)}. \tag{2.5.41}$$

In particular, if $G = J$ then

$$\frac{\mathrm{cap}(F)}{|I|} \leq \frac{\mathrm{cap}(F\cup J)}{|I|+|J|}. \tag{2.5.42}$$

Proof. Without loss of generality we may assume F and G to be regular compact sets (or to consist of finitely many closed intervals if we wish). The equilibrium measure μ_F is obtained from μ_I by adding to $\mu_I\Big|_F$ the balayage of $\nu := \mu_I\Big|_{I\setminus F}$ out of $\mathbf{C}\setminus F$ (see [13, Theorem IV.1.6, (e)]), and in this balayage process the potential on F increases by a constant value. More precisely (see (2.1.2), (2.1.3)) for $x \in F$ and

$\overline{\nu} := \mathrm{Bal}\Big(\nu, \mathbf{C} \setminus F\Big)$ we have

$$U^{\overline{\nu}}(x) = U^{\nu}(x) + \int_{I \setminus F} g_{\overline{\mathbf{C}} \setminus F}(a) d\nu(a),$$

and this gives

$$U^{\mu_F}(x) = U^{\mu_I}(x) + \int_{I \setminus F} g_{\overline{\mathbf{C}} \setminus F}(a) d\mu_I(a).$$

Taking into account that for $x \in F$ the equilibrium potentials on the left and right hand sides are the constants $\log 1/\mathrm{cap}(F)$ and $\log 1/\mathrm{cap}(I) = \log 4/|I|$, respectively, we obtain the identity

$$\log \frac{1}{\mathrm{cap}(F)} - \log \frac{4}{|I|} = \int_{I \setminus F} g_{\overline{\mathbf{C}} \setminus F}(a) d\mu_I(a). \tag{2.5.43}$$

The analogous formula for $F \cup G$ and $I \cup G$ reads as

$$\begin{aligned} \log \frac{1}{\mathrm{cap}(F \cup G)} - \log \frac{1}{\mathrm{cap}(I \cup G)} &= \int_{(I \cup G) \setminus (F \cup G)} g_{\overline{\mathbf{C}} \setminus (F \cup G)}(a) d\mu_{I \cup G}(a) \\ &= \int_{I \setminus F} g_{\overline{\mathbf{C}} \setminus (F \cup G)}(a) d\mu_{I \cup G}(a), \end{aligned} \tag{2.5.44}$$

where we used that $(I \cup G) \setminus (F \cup G) = I \setminus F$, so the integration is over the same set on the right hand sides of (2.5.43) and (2.5.44). Since the measure μ_I is the balayage of $\mu_{I \cup G}$ onto I (see [13, Theorem IV.1.6, (e)]), we have on $I \setminus F$ the inequality $d\mu_{I \cup G}(a) \le d\mu_I(a)$. At the same time $g_{\overline{\mathbf{C}} \setminus (F \cup G)}(a) \le g_{\overline{\mathbf{C}} \setminus F}(a)$, and these show that the integral on the right hand side of (2.5.44) is not larger than the integral on the right hand side of (2.5.43). This gives

$$\log \frac{1}{\mathrm{cap}(F \cup G)} - \log \frac{1}{\mathrm{cap}(I \cup G)} \le \log \frac{1}{\mathrm{cap}(F)} - \log \frac{4}{|I|},$$

which is the same as (2.5.41).

■

3 A Wiener-type Condition in $\mathbf{R}^d$

3.1 Preliminaries

We shall use $c, c_0, c_1, c_2, \ldots$ to denote positive constants. $B_r(x)$ resp. $\overline{B}_r(x)$ denote the open resp. closed ball about the point x of radius r, and $S_r(x)$ is the bounding surface of these balls. $||\mu||$ denotes the total mass of the measure μ.

For the notions of classical potential theory in $\mathbf{R}^d$ see e.g. [6]. The Newtonian potential of the measure ν is defined as

$$U^\nu(x) := \int \frac{1}{|x-t|^{d-2}} d\nu(t),$$

and the energy integral is

$$I(\nu) := \int\int \frac{1}{|x-t|^{d-2}} d\nu(t) d\nu(x).$$

The capacity of a compact set E is the number

$$\operatorname{cap}(E) := \frac{1}{\inf I(\nu)},$$

where the infimum is taken over all probability measures on E. There is a unique measure λ for which the infimum (minimum) is attained. $\mu_E = \operatorname{cap}(E)\lambda$ is called the equilibrium measure of E. E.g. the equilibrium measure of $\overline{B}_r$ (and S_r) is

$$\mu_{\overline{B}_r} = r^{d-2}\sigma_{S_r}, \tag{3.1.1}$$

where σ_{S_r} is the $(d-1)$-dimensional normalized surface area measure on S_r.

If the compact set E has positive capacity then for the Newtonian potential of the equilibrium measure we have

$$U^{\mu_E}(z) = 1, \qquad \text{for q.e. } x \in E, \tag{3.1.2}$$

where q.e. means "quasi-everywhere", i.e. with the exception of a set of zero capacity.

If E is of positive capacity, then μ_E has finite energy. Hence a set of zero capacity has zero μ_E-measure, and so if a property holds quasi-everywhere, i.e. with the exception of a set of zero capacity, then it also holds μ_E-almost everywhere.

If σ is a measure supported on the compact set F and $U^\sigma(x) \leq 1$ for all $x \in \mathbf{R}^d$, then the set

$$K := \{x \ : \ U^\sigma(x) \geq \gamma\} \tag{3.1.3}$$

has capacity at most $(1/\gamma)\mathrm{cap}(F)$. In fact, if K is of positive capacity, then the inequality

$$\frac{U^\sigma(x)}{\mathrm{cap}(F)} \geq \frac{U^{\mu_K}(x)}{\mathrm{cap}(K)} + \frac{\gamma}{\mathrm{cap}(F)} - \frac{1}{\mathrm{cap}(K)}$$

holds true for quasi-every $x \in K$. Hence this is true for μ_K-almost all x, and then the principle of domination ([6, Theorem 1.27]) gives the same inequality for all $x \in \mathbf{R}^d$. Now

$$\mathrm{cap}(K) \leq \frac{1}{\gamma}\mathrm{cap}(F) \tag{3.1.4}$$

follows if we let x tend to infinity.

We shall also need the following result. There is a positive constant c such that if $A \subseteq S_1$ and $\beta(A)$ denotes the $(d-1)$-dimensional surface area measure of A then

$$\beta(A) \leq c\sqrt{\mathrm{cap}(A)}. \tag{3.1.5}$$

Indeed, if λ denotes the normalized surface area measure on S_1 then based on the definition of capacity:

$$\frac{1}{\mathrm{cap}(A)} \quad \leq \quad \frac{1}{\beta(A)^2}\int_A\int_A \frac{1}{|x-t|^{d-2}} d\lambda(x)d\lambda(t)$$

$$\leq \frac{1}{\beta(A)^2} \int_{S_1} \int_{S_1} \frac{1}{|x-t|^{d-2}} d\lambda(x) d\lambda(t).$$

Hence, (3.1.5) follows with

$$c = \sqrt{\int_{S_1} \int_{S_1} \frac{1}{|x-t|^{d-2}} d\lambda(x) d\lambda(t)}.$$

Let G be a domain with compact boundary and with $\mathrm{cap}(\partial G) > 0$, and let ν be a Borel measure supported on G (by which we mean that $\nu(\mathbf{R}^d \setminus G) = 0$). We shall again need the concept of balayage of ν out of G, see e.g. [13, Sec. II.4] or [6, Chapter IV]. The definition is slightly different from the two dimensional case. It is the unique Borel measure $\overline{\nu}$ supported on ∂G with the properties:

- $\|\overline{\nu}\| \leq \|\nu\|$, where $\|\nu\|$ denotes the total mass of ν,
- for all $x \in \partial G$ with the exception of a set of capacity 0

$$U^{\overline{\nu}}(x) = U^{\nu}(x), \tag{3.1.6}$$

- $\overline{\nu}$ is so called C-continuous, i.e. the $\overline{\nu}$-measure of any set of zero capacity is zero.

For regular G the exceptional set is empty. If G is bounded, then $\overline{\nu}$ has the same total mass as ν. If ν is not supported on G, then taking its balayage out of G is understood in the sense that we take the balayage of $\nu\big|_G$ and leave the rest of ν unchanged. In this sense if $G_1 \subseteq G_2$, then taking balayage out of G_2 can be done in two steps: first take balayage out of G_1, and then take the balayage of the resulting measure out of G_2.

Perhaps the most important connection between equilibrium and balayage measures is the fact that if $E \subseteq F$ are compact sets of positive capacity, then μ_E is the balayage of μ_F onto E (i.e. out of the unbounded component of $\overline{\mathbf{R}}^d \setminus E$).

If $K \subseteq \partial G$ are compact sets of positive capacity, then the harmonic measure $\omega(x, K, G)$ is the unique solution of the generalized Dirichlet-problem in G corresponding to the characteristic function of K in ∂G. There is a connection between harmonic and balayage measures: for $a \in G$ the equality

$$\overline{\delta_a}(K) = \omega(a, K, G) \tag{3.1.7}$$

holds, where δ_a denotes the point mass (Dirac measure) placed at the point a and $\overline{\delta_a}$ denotes its balayage out of G (see e.g. [13, Appendix A3, (3.3)] or [6, IV.3]).

Green's function of G with pole at $y \in G$ is defined as

$$g_G(x, y) = U^{\delta_y}(x) - U^{\overline{\delta_y}}(x).$$

Let $0 < r < R$, $x \in S_R$ and let $\overline{\delta_x}$ be the balayage of δ_x out of $\mathbf{R}^d \setminus \overline{B}_r$. This measure is given by the formula

$$\frac{d\overline{\delta_x}(y)}{d\sigma_{S_r}} = r^{d-2} \frac{R^2 - r^2}{|x-y|^d}, \tag{3.1.8}$$

where $y \in S_r$ and σ_{S_r} is the normalized surface area measure on S_r. Indeed, Poisson's formula (see e.g. [3, Section 1.3, (1.3.1)]) gives

$$\frac{d\overline{\delta_x}(y)}{d\sigma} = \frac{1}{\omega_n r} \frac{R^2 - r^2}{|x-y|^d},$$

where σ is the surface area measure (not normalized) on S_r and $\omega_n = \sigma(S_1)$. Multiplying by $d\sigma / d\sigma_{S_r} = \omega_n r^{d-1}$ we obtain (3.1.8). Thus, for the density of $\overline{\delta_x}$ with respect to σ_{S_r} we have the inequalities

$$r^{d-2} \frac{R-r}{(R+r)^{d-1}} \leq \frac{d\overline{\delta_x}(y)}{d\sigma_{S_r}} \leq r^{d-2} \frac{R+r}{(R-r)^{d-1}}. \tag{3.1.9}$$

Multiplying by R^{d-2} and letting $R \to \infty$ we get that $\overline{\delta_\infty}$ can be understood as r^{d-2} -times the normalized surface area measure on S_r, which is the equilibrium measure of S_r ($\overline{B}_r$). On applying this for a large r containing the set E of positive capacity we can see that if $\widehat{\cdot}$ denotes balayage onto E, then $\mu_E = \widehat{\delta_\infty}$. It also follows that $r^{d-2} \widehat{\sigma_{S_r}} = \mu_E$. But $\widehat{\sigma_{S_r}} = \int \widehat{\delta_a} d\sigma_{S_r}(a)$, so it follows from Harnack's inequality that there are constants c_r, C_r such that for $a \in S_r$ we have $c_r\mu_E \le \widehat{\delta_a} \le C_r\mu_E$. Another application of Harnack's inequality gives

$$c_a\mu_E \le \widehat{\delta_a} \le C_a\mu_E \tag{3.1.10}$$

for any a lying in the unbounded component of $\mathbf{R}^d \setminus E$ with some constants c_a, C_a.

Let μ be a measure on S_r. The lower Radon-Nikodym derivative (density) of μ with respect to normalized surface area measure on S_r is defined as follows (see e.g. [5, Chapter 3] or [11, Chapter VII]). Let $x_0 \in S_r$ and $0 < \tau < 1$. Then the cone

$$C(x_0,\tau) := \{x \in \mathbf{R}^d \ : \ \frac{\langle x, x_0\rangle}{r\|x\|} \ge 1-\tau\}$$

determines a closed polar cap $K(x_0,\tau) = C(x_0,\tau) \cap S_r$ centered at x_0. The lower derivative of μ with respect to σ_{S_r} at x_0 is

$$v(x_0) := \liminf_{\sigma(K)\to 0} \mu(K)/\sigma(K),$$

where K is an arbitrary closed polar cap containing $x_0 \in S_r$. Wherever the ordinary Radon-Nikodym derivative exists, it agrees with v. Therefore, $v(y)d\sigma_{S_r}(y) \le d\mu(y)$.

Finally, let us recall that the Newtonian capacity is subadditive: if $F = \cup_{i=1}^k F_i$, then

$$\mathrm{cap}(F) \le \sum_{i=1}^k \mathrm{cap}(F_i). \tag{3.1.11}$$

In particular, one of the sets F_i must have capacity $\ge \mathrm{cap}(F)/k$. On the other hand, if the distance between the sets F_1 and F_2 is at least l, then

$$\mathrm{cap}(F_1 \cup F_2) \ge \frac{\mathrm{cap}(F_1) + \mathrm{cap}(F_2)}{1 + 2\frac{\mathrm{cap}(F_1)\mathrm{cap}(F_2)}{l^{d-2}(\mathrm{cap}(F_1)+\mathrm{cap}(F_2))}}. \tag{3.1.12}$$

Indeed, set

$$\nu = \frac{1-t}{\mathrm{cap}(F_1)}\mu_{F_1} + \frac{t}{\mathrm{cap}(F_2)}\mu_{F_2},$$

where t is between 0 and 1. Then ν is a probability measure and

$$I(\nu) \leq \frac{(1-t)^2}{\text{cap}(F_1)} + \frac{t^2}{\text{cap}(F_2)} + \frac{2t(1-t)}{l^{d-2}\text{cap}(F_1)\text{cap}(F_2)}||\mu_{F_1}||||\mu_{F_2}||.$$

This yields with $||\mu_{F_1}|| = \text{cap}(F_1)$ and $||\mu_{F_2}|| = \text{cap}(F_2)$

$$\text{cap}(F_1 \cup F_2) \geq \frac{1}{\frac{(1-t)^2}{\text{cap}(F_1)} + \frac{t^2}{\text{cap}(F_2)} + \frac{2t(1-t)}{l^{d-2}}}.$$

Now $t = \text{cap}(F_2)/(\text{cap}(F_1) + \text{cap}(F_2))$ gives (3.1.12).

3.2 Results

Let $E \subset \mathbf{R}^d$ be a compact set of positive Newtonian capacity, Ω the unbounded component of $\mathbf{R}^d \setminus E$ and $g_\Omega(x,a)$ the Green's function of Ω with pole at $a \in \Omega$. We extend g_Ω to $\partial\Omega$ by

$$g_\Omega(x,a) = \limsup_{w\to x,\ w\in\Omega} g_\Omega(w,a),$$

and to $\mathbf{R}^d \setminus \overline{\Omega}$ by setting $g_\Omega(x,a) = 0$ there. We are interested in the behavior of g_Ω at a boundary point of Ω, which we assume to be 0, i.e. let $0 \in \partial\Omega$.

Let $B_r = B_r(0)$ be the ball of radius r about the origin, and we shall denote its closure by $\overline{B}_r$ and its boundary (the sphere of center 0 and radius r) by S_r. With

$$E^n = E \cap (\overline{B}_{2^{-n+1}} \setminus B_{2^{-n}}) = \left\{x \in E\ :\ 2^{-n} \leq |x| \leq 2^{-n+1}\right\}$$

the regularity of the boundary point 0 was characterized by Wiener (see e.g. [6, Theorem 5.2]): Green's function $g_G(x,a)$ $(a \in \Omega)$ is continuous at $0 \in \partial\Omega$ (i.e. 0 is a regular boundary point of E) if and only if

$$\sum_{n=1}^{\infty} \text{cap}(E^n) 2^{n(d-2)} = \infty, \tag{3.2.13}$$

where $\text{cap}(E^n)$ denotes the (d-dimensional) Newtonian capacity of E^n. We would like to characterize in a similar manner the stronger Hölder continuity:

$$g_\Omega(x,a) \le C|x|^\kappa \tag{3.2.14}$$

with some positive numbers C, κ.

Following the definitions in [4], for $\varepsilon > 0$ set

$$\mathcal{N}_E(\varepsilon) = \{n \in \mathbf{N} \ : \ \ \text{cap}(E^n) \ge \varepsilon 2^{-n(d-2)}\}, \tag{3.2.15}$$

and we say that a subsequence $\mathcal{N} = \{n_1 < n_2 < \ldots\}$ of the natural numbers is of positive lower density if

$$\liminf_{N\to\infty} \frac{|\mathcal{N} \cap \{0,1,\ldots,N\}|}{N+1} > 0.$$

Let $x_0 \in S_1$, $0 < \tau < 1$, $\ell > 0$ and set

$$C(x_0,\tau,\ell) := \{x \in B_\ell \ : \ \ \frac{\langle x, x_0\rangle}{\|x\|} \ge 1-\tau\}. \tag{3.2.16}$$

This is a cone with vertex at 0 and x_0 as the direction of its axis. We say that E satisfies the cone condition if

$$C(x_0,\tau,\ell) \subset \Omega \tag{3.2.17}$$

with some $x_0 \in S_1$, τ and $\ell > 0$, which means that Ω contains a cone with vertex at 0.

Theorem 3.2.1 **a)** *If $\mathcal{N}_E(\varepsilon)$ is of positive lower density for some $\varepsilon > 0$ then Green's function g_Ω is Hölder continuous at* 0.

b) *If Green's function g_Ω is Hölder continuous at* 0 *and E satisfies the cone condition then $\mathcal{N}_E(\varepsilon)$ is of positive lower density for some $\varepsilon > 0$.*

The importance of the Hölder property is explained by the following result. Let G be a domain in $\mathbf{R}^d$ with compact boundary such that 0 is on the boundary of G. We may assume that $G \not\subseteq B_1$, and set $E = \overline{B}_1 \setminus G$. Then $\Omega := \mathbf{R}^d \setminus E = G \cup (\mathbf{R}^d \setminus \overline{B}_1)$ is a domain larger than G and 0 is on the boundary of Ω. If f is a bounded Borel function on the boundary of G, then let u_f denote the Perron-Wiener-Brelot solution of the Dirichlet problem in G with boundary function f. We think u_f to be extended to ∂G as $u_f = f$ there.

Lemma 3.2.2 *Suppose that* 0 *is a regular boundary point of* G. *Then the following are equivalent.*

1) $g_G(\cdot, a)$ *is Hölder continuous at* 0 *for* $a \in G$.

2) $\mu_E(\overline{B}_r) \le Cr^{d-2+\kappa}$ *for some* C, $\kappa > 0$ *and all* $r < 1$.

If, in addition, G *satisfies the cone condition at* 0*, then* **1)** - **2)** *are also equivalent to*

3) *If* f *is Hölder continuous at* 0*, then so is* u_f.

Note also that it is indifferent if "for $a \in G$" in **1)** is understood as "for some $a \in G$" or as "for all $a \in G$".

3.3 Proof of Theorem 3.2.1

Proof of a) in Theorem 3.2.1

Let us suppose that $\mathcal{N}_E(\varepsilon)$ is of positive lower density for some $\varepsilon > 0$. Clearly then 0 is a regular boundary point of Ω, hence by the equivalence of **1)** and **2)** in Lemma 3.2.2 it is sufficient to verify $\mu_E(\overline{B}_r) \le r^{d-2+\kappa}$ for some $\kappa > 0$ and sufficiently small r.

Let F be a compact set such that 0 is on the boundary of the unbounded component of $\mathbf{R}^d \setminus F$, and let $\widehat{\nu}$ denote the balayage of some measure ν out of $\mathbf{R}^d \setminus (F \cup \overline{B}_1)$. First we verify that if $\mathrm{cap}(F \cap (\overline{B}_8 \setminus B_4)) \ge 4\varepsilon$, with some $\varepsilon \le 1/8$, then

$$\widehat{\sigma_{S_8}}\Big|_{\overline{B}_1} \le \frac{1}{8^{d-2}} \left(1 - \frac{\varepsilon}{9^d}\right) \sigma_{S_1}. \tag{3.3.18}$$

In fact, let $F_1 = F \cap (\overline{B}_8 \setminus B_4)$, and $F_2 = F_1 \cup \overline{B}_1$. We enlarge the balayage measure on the left in (3.3.18) if we replace the domain $\mathbf{R}^d \setminus (F \cup \overline{B}_1)$ with $\mathbf{R}^d \setminus (F_1 \cup \overline{B}_1)$, hence we may suppose $F = F_1$, $F_2 = F \cup \overline{B}_1$. Let $\overline{\nu}$ be the balayage of some measure ν out of $\mathbf{R}^d \setminus \overline{B}_1$. Then $\overline{\nu} = \overline{\widehat{\nu}}$, i.e.

$$\overline{\nu} = \widehat{\nu}\Big|_{\overline{B}_1} + \overline{\widehat{\nu}\Big|_F},$$

and we apply this with $\nu = \sigma_{S_8}$. Thus,

$$\widehat{\sigma_{S_8}}\Big|_{\overline{B}_1} = \overline{\sigma_{S_8}} - \overline{\widehat{\sigma_{S_8}}\Big|_F}. \tag{3.3.19}$$

The left hand side is what is on the left of (3.3.18), and since $\sigma_{S_8} = \mu_{S_8}/8^{d-2}$, and $\overline{\mu_{S_8}} = \mu_{S_1} = \sigma_{S_1}$, the first term on the right hand side is $\sigma_{S_1}/8^{d-2}$. Therefore, it has left to estimate from below the second measure on the right of (3.3.19).

For every $a \in S_8$ (3.1.9) with $r = 1$ and $R = 8$ shows that

$$\overline{\delta_a} \geq \frac{7}{9^{d-1}}\sigma_{S_1} > \frac{1}{9^d}\sigma_{S_1},$$

and so

$$\overline{\widehat{\sigma_{S_8}}\Big|_F} \geq \frac{\widehat{\sigma_{S_8}}(F)}{9^d}\sigma_{S_1}, \tag{3.3.20}$$

and we have to estimate how much of $\widehat{\sigma_{S_8}}$ goes on to F. Since we assumed $F_2 = F \cup \overline{B}_1 \subseteq \overline{B}_8$, and, as we have just remarked, $\sigma_{S_8} = \mu_{S_8}/8^{d-2}$, it follows that

$$\widehat{\sigma_{S_8}}(F) = \frac{1}{8^{d-2}}\mu_{F_2}(F). \tag{3.3.21}$$

The distance of the sets $F \subseteq \overline{B}_8 \setminus B_4$ and $\overline{B}_1$ is at least 3, so (3.1.12) yields for the capacities of F, $\overline{B}_1$ and $F_2 = F \cup \overline{B}_1$ the inequality

$$\operatorname{cap}(F_2) \geq \frac{\operatorname{cap}(F) + \operatorname{cap}(\overline{B}_1)}{1 + \frac{2\operatorname{cap}(F)\operatorname{cap}(\overline{B}_1)}{3^{d-2}(\operatorname{cap}(F)+\operatorname{cap}(\overline{B}_1))}} \geq \frac{1 + \operatorname{cap}(F)}{1 + \frac{2}{3^{d-2}}\operatorname{cap}(F)},$$

because $\operatorname{cap}(\overline{B}_1) = 1$. The latter expression is monotone increasing in $\operatorname{cap}(F)$, and the assumption gives $\mathrm{cap}(F) \geq 4\varepsilon$ and $\varepsilon \leq 1/8$, thus

$$\mathrm{cap}(F_2) \geq \frac{1+4\varepsilon}{1+\frac{2}{3^{d-2}}4\varepsilon} \geq 1+\varepsilon.$$

Therefore

$$\mu_{F_2}(F) + \mu_{F_2}(\overline{B}_1) = ||\mu_{F_2}|| = \mathrm{cap}(F_2) \geq 1+\varepsilon.$$

Here $\mu_{F_2}(\overline{B}_1) \leq 1$ because $\mu_{\overline{B}_1}(B_1) = 1$ and $\mu_{\overline{B}_1}$ is obtained by taking the balayage of μ_{F_2} onto $\overline{B}_1$. Hence $\mu_{F_2}(F) \geq \varepsilon$ follows. This and (3.3.20)–(3.3.21) give

$$\overline{\widehat{\sigma_{S_8}}\Big|_F} \geq \frac{1}{8^{d-2}}\frac{\varepsilon}{9^d}\sigma_{S_1}.$$

Now all from (3.3.19) imply

$$\widehat{\sigma_{S_8}}\Big|_{\overline{B}_1} \leq \frac{1}{8^{d-2}}\sigma_{S_1} - \frac{1}{8^{d-2}}\frac{\varepsilon}{9^d}\sigma_{S_1} = \frac{1}{8^{d-2}}\left(1-\frac{\varepsilon}{9^d}\right)\sigma_{S_1},$$

and (3.3.18) has been verified.

We shall use (3.3.18) in a scaled form, namely if E is compact, 0 is on the boundary of the unbounded component of $\mathbf{R}^d \setminus E$ and

$$\mathrm{cap}(E \cap (\overline{B}_{2^{-n}} \setminus B_{2^{-n-1}})) \geq 4\varepsilon 2^{(-n-3)(d-2)}, \tag{3.3.22}$$

then we have

$$\widehat{\sigma_{S_{2^{-n}}}}\Big|_{\overline{B}_{2^{-n-3}}} \leq \frac{1}{8^{d-2}}\left(1-\frac{\varepsilon}{9^d}\right)\sigma_{S_{2^{-n-3}}}, \tag{3.3.23}$$

where now $\widehat{\cdot}$ denotes balayage out of $\mathbf{R}^d \setminus (E \cup \overline{B}_{2^{-n-3}})$.

After this preparation let us return to the set $\mathcal{N}_E(\varepsilon)$ which was assumed to be of positive lower density. Then there is an $\eta > 0$ such that for large N the set $\mathcal{N}_E(\varepsilon)$ has at least ηN elements smaller than N. For large N then we can select a subset

$$K \subseteq \mathcal{N}_E(\varepsilon) \cap \{2, \ldots, N-2\}$$

such that it has $k \geq \eta(N+1)/5$ elements, and if $n_1, n_2, \ldots, n_k$ is the increasing enumeration of K, then $n_{i+1} > n_i + 3$ for all $i < k$.

We set

$$E_n = E \cup \overline{B}_{2^{-n}}, \qquad \mu_n = \mu_{E_n}.$$

Our aim is to estimate the quantity $\mu_E(\overline{B}_{2^{-N}})$, which is at most as large as $\mu_N(\overline{B}_{2^{-N}})$ (recall that μ_E is the balayage of μ_N onto E, and during this we sweep out of $\mathbf{R}^d \setminus E$ the portion of μ_N sitting on $\overline{B}_{2^{-N}} \setminus E$, so the measure of $\overline{B}_{2^{-N}}$ is not increasing during this sweeping process). We shall recursively estimate $\mu_n\big|_{S_{2^{-n}}}$ by $\sigma_{S_{2^{-n}}}$ and the $n = N$ case will give the desired result.

First note that $\mu_0\big|_{S_1} \leq \sigma_{S_1}$ ($\sigma_{S_1} = \mu_{S_1}$ is the balayage of μ_0 onto $\overline{B}_1$). Suppose $\mu_n\big|_{S_{2^{-n}}} \leq c\sigma_{S_{2^{-n}}}$ holds true with some constant c. The measure μ_{n+1} is the balayage of μ_n onto E_{n+1} and during this balayage we sweep out only $\mu_n\big|_{S_{2^{-n}}}$ onto $S_{2^{-n-1}} \cup \big(E \cap (\overline{B}_{2^{-n}} \setminus B_{2^{-n-1}})\big)$. This balayage measure is not less than the balayage of $\mu_n\big|_{S_{2^{-n}}}$ onto $S_{2^{-n-1}}$. Therefore if $\overline{\cdot}$ denotes the balayage out of $\mathbf{R}^d \setminus \overline{B}_{2^{-n-1}}$ then we have (see (3.1.1))

$$\mu_{n+1}\big|_{S_{2^{-n-1}}} \leq \overline{\mu_n\big|_{S_{2^{-n}}}} \leq c\overline{\sigma_{S_{2^{-n}}}} = c\frac{1}{2^{d-2}}\sigma_{S_{2^{-n-1}}}. \tag{3.3.24}$$

On the other hand, if $n = n_i - 1$, with $n_i \in K$, then (3.3.22) is true, hence for such n we have (3.3.23). Again, the measure μ_{n+3} is the balayage of μ_n onto E_{n+3}, and in taking this balayage we sweep out only the part of μ_n that is sitting on $S_{2^{-n}} \setminus E$. Thus, if $\widehat{\cdot}$ denotes the balayage onto E_{n+3}, then

$$\begin{aligned} \mu_{n+3}\big|_{S_{2^{-n-3}}} &= \widehat{\mu_n\big|_{S_{2^{-n}}}}\Big|_{S_{2^{-n-3}}} \leq c\widehat{\sigma_{S_{2^{-n}}}}\Big|_{S_{2^{-n-3}}} \\ &\leq c\frac{1}{8^{d-2}}\left(1 - \frac{\varepsilon}{9^d}\right)\sigma_{S_{2^{-n-3}}}. \end{aligned}$$

This estimate holds for all n with $n+1 \in K$, and consecutive numbers in K differ by at least 3, hence this estimate for going from n to $n+3$ can be applied at least $k \geq (N+1)\eta/5$ times. For other n we just use (3.3.24) ($N - 3k$ times altogether).

Thus, we eventually obtain

$$\begin{aligned}\mu_N(\overline{B}_N) &= \mu_N(S_{2^{-N}}) \le \left(\left(\frac{1}{8}\right)^{d-2}\left(1-\frac{\varepsilon}{9^d}\right)\right)^k \left(\frac{1}{2^{d-2}}\right)^{N-3k} \sigma_{S_{2^{-N}}}(S_{2^{-N}}) \\ &\le \left(\frac{1}{2}\right)^{N(d-2)}\left(1-\frac{\varepsilon}{9^d}\right)^k \le \left(\frac{1}{2}\right)^{N(d-2)}\left(1-\frac{\varepsilon}{9^d}\right)^{\eta(N+1)/5}.\end{aligned}$$

This is the desired inequality, for it immediately implies for $2^{-N-1} < r \le 2^{-N}$ that

$$\begin{aligned}\mu_E(\overline{B}_r) &\le \mu_E(\overline{B}_{2^{-N}}) \le \mu_N(\overline{B}_{2^{-N}}) \\ &\le \left(\frac{1}{2}\right)^{N(d-2)}\left(1-\frac{\varepsilon}{9^d}\right)^{\eta(N+1)/5} \le r^{d-2+\kappa},\end{aligned}$$

provided

$$N > \frac{-10(d-2)\log 2}{\eta \log(1-\varepsilon/9^d)} - 1$$

and κ is defined by the equation

$$2^{-\kappa} = \left(1-\frac{\varepsilon}{9^d}\right)^{\eta/10}.$$

■

Remark 3.3.1 Note that the previous proof was effective in the sense that *if* $\varepsilon > 0$ *and* $\eta > 0$ *are given, then there are an* N_0 *and a* $\kappa > 0$ *such that if for a particular* $M \ge N_0$ *we have* $|\mathcal{N}_E(\varepsilon) \cap \{0, 1, \ldots, M\}| \ge \eta M$, *then*

$$\mu_E(\overline{B}_{2^{-M}}) \le \mu_M(\overline{B}_{2^{-M}}) \le (2^{-M})^{d-2+\kappa}. \tag{3.3.25}$$

Proof of b) in Theorem 3.2.1

The proof is rather long, therefore we break it into several steps.

Step I

Let $L \geq 2$ be a fixed natural number, $F \subseteq \overline{B}_{2^{-1}} \setminus B_{2^{-L-1}}$ a compact set such that

$$\operatorname{cap}(F \cap (\overline{B}_{2^{-j+1}} \setminus B_{2^{-j}})) \leq \varepsilon 2^{-j(d-2)}, \qquad j = 2, \dots, L+1, \tag{3.3.26}$$

and let $\widehat{\delta_a}$ be the balayage of δ_a out of the domain $(B_{2^{-1}} \setminus \overline{B}_{2^{-L-1}}) \setminus F$. We shall estimate from below this balayage measure on $S_{2^{-L-1}}$ for $a \in S_{2^{-L}}$; namely we shall show that for large L and small $\varepsilon > 0$, disregarding a small subset of $S_{2^{-L}}$, for $a \in S_{2^{-L}}$ the measure $\widehat{\delta_a}\Big|_{S_{2^{-L-1}}}$ has almost full density (i.e. as in the case $F = \emptyset$) on a large (almost full) subset of $S_{2^{-L-1}}$.

For notational convenience let $\Delta_1 = B_{2^{-1}}$, $\Delta_L = B_{2^{-L}}$, $\Delta_{L+1} = B_{2^{-L-1}}$, $\Delta_{3/2} = B_{3 \cdot 2^{-L-2}}$, and let the bounding surfaces of these balls be denoted by T_1, T_L, T_{L+1} and $T_{3/2}$, respectively. Set also $F_{3/2} = F \cap \overline{\Delta}_{3/2}$. We shall take the balayage out of different sets, and for the convenience of the reader we list them here:

- $\widehat{\cdot}$ is the balayage out of $(\Delta_1 \setminus \overline{\Delta}_{L+1}) \setminus F$,
- $\widetilde{\cdot}$ is the balayage out of $(\mathbf{R}^d \setminus \overline{\Delta}_{L+1}) \setminus F_{3/2}$,
- $\overline{\cdot}$ is the balayage out of $\mathbf{R}^d \setminus \overline{\Delta}_{L+1}$.

We start from the representation

$$F = \bigcup_{j=2}^{L+1} F \cap (\overline{B}_{2^{-j+1}} \setminus B_{2^{-j}}),$$

hence (3.1.11) gives

$$\operatorname{cap}(F) \leq \sum_{j=2}^{L+1} \varepsilon 2^{-j(d-2)} \leq \varepsilon. \tag{3.3.27}$$

Now let $a \in \Delta_1 \setminus \overline{\Delta}_{L+1}$, and let $\nu = \nu_a = \widehat{\delta_a}$ be the balayage measure of δ_a out of $(\Delta_1 \setminus \overline{\Delta}_{L+1}) \setminus F$. This measure has total mass 1 and it is supported on $T_{L+1} \cup F \cup T_1$. First we verify that it has small mass on F.

Without loss of generality we may assume that F is of positive capacity (otherwise enlarge it), and then we can write

$$\nu(F) = \int_F U^{\mu_F} d\nu = \int U^{\mu_F} d\nu_{\big|F} = \int U^{\nu_{\big|F}} d\mu_F \le \int U^{\nu} d\mu_F. \tag{3.3.28}$$

The potential $U^{\nu}(x)$ agrees with

$$U^{\delta_a}(x) = \frac{1}{|z-a|^{d-2}}$$

for quasi-every $x \in F$ (see (3.1.6)) and hence for μ_F-almost all x, therefore the last integral on the right of (3.3.28) is $U^{\mu_F}(a)$. This gives that if

$$U^{\mu_F}(a) \le \frac{1}{L},$$

then

$$\nu_a(F) \le \frac{1}{L}. \tag{3.3.29}$$

We shall need a similar reasoning for the balayage $\nu^* = \nu^*_a := \widetilde{\delta_a}$ of δ_a out of $(\mathbf{R}^d \setminus \overline{\Delta}_{L+1}) \setminus F_{3/2}$. In fact, replace in (3.3.28) F by $F_{3/2}$ and ν by ν^*. This gives that if

$$U^{\mu_{F_{3/2}}}(a) \le \frac{1}{L},$$

then

$$\nu^*_a(F_{3/2}) \le \frac{1}{L}. \tag{3.3.30}$$

Thus, if

$$K := \left\{ a \ : \ U^{\mu_F}(a) \ge \frac{1}{L} \right\}, \tag{3.3.31}$$

then for $a \notin K$ we have (3.3.29), and if

$$K_{3/2} := \left\{ a \ : \ U^{\mu_{F_{3/2}}}(a) \ge \frac{1}{L} \right\}, \tag{3.3.32}$$

then for $a \in T_L \setminus K_{3/2}$ we have (3.3.30). For the capacity of K we get from (3.1.3)–

(3.1.4) and (3.3.27) the inequality

$$\mathrm{cap}(K) \le L\mathrm{cap}(F) \le \varepsilon L, \tag{3.3.33}$$

and similarly we get

$$\mathrm{cap}(K_{3/2}) \le L\mathrm{cap}(F_{3/2}) \le \varepsilon L. \tag{3.3.34}$$

If σ_L denotes the $(d-1)$-dimensional normalized surface area measure on T_L, then by (3.1.5) we have

$$\begin{aligned} \sigma_L(K \cap T_L) \le c\sqrt{2^L \mathrm{cap}(K \cap T_L)} &\le c2^{L/2}\sqrt{\mathrm{cap}(K)} \\ &\le c2^{L/2}\sqrt{\varepsilon L}. \end{aligned} \tag{3.3.35}$$

An identical inequality is true for $K_{3/2}$ (c.f. (3.3.34)):

$$\sigma_L(K_{3/2} \cap T_L) \le c2^{L/2}\sqrt{\varepsilon L}. \tag{3.3.36}$$

Let $a, b \in T_L$, and let $\widetilde{\delta_a}$, $\widetilde{\delta_b}$ be the balayage of δ_a, δ_b out of the domain $\mathbf{R}^d \setminus (F_{3/2} \cup \overline{\Delta}_{L+1})$. This balayage is obtained by first taking balayage of δ_a, δ_b out of $\mathbf{R}^d \setminus \Delta_{3/2}$, and if these balayage measures are denoted by α_a and α_b, then take the balayage of α_a and α_b (which are supported on $T_{3/2}$) out of $\mathbf{R}^d \setminus (F_{3/2} \cup \overline{\Delta}_{L+1})$. The measures α_a and α_b are given by the formula (3.1.8) with $r = 3 \cdot 2^{-L-2}$ and $R = 2^{-L}$, hence (3.1.9) gives the inequality

$$\alpha_a \le \left(\frac{1 + 3/4}{1 - 3/4}\right)^d \alpha_b = 7^d \alpha_b,$$

therefore we also have $\widetilde{\delta_a} \le 7^d \widetilde{\delta_b}$. Now $\widehat{\delta_a}$ is the balayage out of $(\Delta_1 \setminus \overline{\Delta}_{L+1}) \setminus F$, while $\widetilde{\delta_a}$ is the balayage out of the larger domain $(\mathbf{R}^d \setminus \overline{\Delta}_{L+1}) \setminus F_{3/2}$, hence

$$\widehat{\delta_a}\Big|_{\overline{\Delta}_{L+1} \cup F_{3/2}} \le \widetilde{\delta_a}.$$

These give for all $a, b \in T_L$ the inequality

$$\widehat{\delta_a}\Big|_{\overline{\Delta}_{L+1} \cup F_{3/2}} \leq 7^d \widetilde{\delta_b}. \tag{3.3.37}$$

Choose and fix a $b \in T_L \setminus K_{3/2}$ (see (3.3.32)). By (3.3.36) if ε is sufficiently small compared to L, then there is such a b. In this case (3.3.30) gives $\widetilde{\delta_b}(F_{3/2}) \leq 1/L$, hence the balayage $\tau := \overline{\widetilde{\delta_b}\Big|_{F_{3/2}}}$ of $\widetilde{\delta_b}\Big|_{F_{3/2}}$ out of $\mathbf{R}^d \setminus \overline{\Delta}_{L+1}$ also has total mass at most $1/L$. Therefore, if we define

$$H^* = \left\{ y \in T_{L+1} \ : \ \frac{d\tau(y)}{d\sigma_{L+1}} \geq \frac{1}{\sqrt{L}} \right\}, \tag{3.3.38}$$

then

$$\sigma_{L+1}(H^*) \leq \frac{1}{\sqrt{L}}.$$

Taking into account (3.3.37) we obtain for the measures $\rho_a := \overline{\widehat{\delta_a}\Big|_{F_{3/2}}}$ the inequality

$$\frac{d\rho_a(x)}{d\sigma_{L+1}} \leq \frac{7^d}{\sqrt{L}} \tag{3.3.39}$$

for all $a \in T_L$ and all $x \in T_{L+1} \setminus H^*$.

Next consider the balayage $\rho_a^* := \overline{\widehat{\delta_a}\Big|_{F \setminus F_{3/2}}}$ of the restriction $\widehat{\delta_a}\Big|_{F \setminus F_{3/2}}$ out of $\mathbf{R}^d \setminus \overline{\Delta}_{L+1}$. The set $F \setminus F_{3/2}$ lies outside $\Delta_{3/2}$, and for each c outside $\Delta_{3/2}$ the inequality (3.1.9) gives for the density of the balayage $\overline{\delta_c}$ of δ_c out of $\mathbf{R}^d \setminus \overline{\Delta}_{L+1}$ the estimate

$$\frac{d\overline{\delta_c}}{d\sigma_{L+1}} \leq c_0 \left(\frac{1}{2^{L+1}} \right)^{d-2} \frac{3/2^{L+2} + 1/2^{L+1}}{(3/2^{L+2} - 1/2^{L+1})^{d-1}} = 5c_0 2^{d-2}.$$

Hence for $a \in T_L \setminus K$ we get

$$\begin{aligned} \frac{d\rho_a^*}{d\sigma_{L+1}} = \int_{F \setminus F_{3/2}} \frac{d\overline{\delta_c}}{d\sigma_{L+1}} d\widehat{\delta_a}(c) \ &\leq \ 5c_0 2^{d-2} \cdot \widehat{\delta_a}(F \setminus F_{3/2}) \\ &\leq \ 5c_0 2^{d-2} \cdot \widehat{\delta_a}(F) \leq \frac{5c_0 2^{d-2}}{L}, \end{aligned} \tag{3.3.40}$$

where we used (3.3.29) which is valid for $a \notin K$.

In a similar fashion we obtain for $\rho_a^{**} := \overline{\widehat{\delta}_a\Big|_{T_1}}$ the estimate

$$\frac{d\rho_a^{**}}{d\sigma_{L+1}} \leq \frac{c_0(2^L+1)}{(2^L-1)^{d-1}} \cdot \widehat{\delta}_a(T_1) \leq \frac{c_0(2^L+1)}{(2^L-1)^{d-1}}. \tag{3.3.41}$$

Now note that

$$\rho_a + \rho_a^* + \rho_a^{**} + \overline{\widehat{\delta}_a\Big|_{T_{L+1}}} = \overline{\widehat{\delta}_a} = \overline{\delta_a},$$

and the last term on the left hand side is actually $\widehat{\delta}_a\Big|_{T_{L+1}}$. Thus, (3.3.39), (3.3.40) and (3.3.41) give that for all $a \in T_L \setminus K$ and $y \in T_{L+1} \setminus H^*$ we have

$$\frac{d\widehat{\delta}_a(y)}{d\sigma_{L+1}} \geq \frac{d\overline{\delta_a}(y)}{d\sigma_{L+1}} - \frac{7^d}{\sqrt{L}} - \frac{5c_0 2^{d-2}}{L} - \frac{c_0(2^L+1)}{(2^L-1)^{d-1}},$$

which, in view of (3.1.8) gives for $a \in T_L \setminus K$ and for $y \in T_{L+1} \setminus H^*$

$$\frac{d\widehat{\delta}_a(y)}{d\sigma_{L+1}} \geq \frac{3}{2^{d(L+1)}|a-y|^d} - \frac{c_1}{\sqrt{L}}. \tag{3.3.42}$$

This derivation used the existence of $b \in T_L \setminus K_{3/2}$, and it is valid if ε is sufficiently small compared to L (see (3.3.36)).

Step II

We follow the notations from Step I, in particular suppose that F is a compact set with (3.3.26).

Let $\delta > 0$. Suppose that μ is a measure on T_L such that

$$\frac{d\mu(y)}{d\sigma_L} \geq 1 \qquad \text{for } y \in T_L \setminus H, \tag{3.3.43}$$

where $H \subseteq T_L$ is of (normalized surface area) measure at most δ. Let $\widehat{\mu}$ be the balayage of μ out of $(\Delta_1 \setminus \overline{\Delta}_{L+1}) \setminus F$. We are going to show that for large L and small $\varepsilon > 0$ the measure $\widehat{\mu}$ satisfies a similar condition as (3.3.43) but on T_{L+1}, namely we verify

$$\frac{d\widehat{\mu}(y)}{d\sigma_{L+1}} \geq \frac{1}{2^{d-2}}(1 - c_2\delta) \qquad \text{for } y \in T_{L+1} \setminus H^*, \tag{3.3.44}$$

where H^* is a set of (normalized surface area) measure at most δ and $c_2 > 0$ is a constant depending only on d.

First of all note that we have (3.3.42) for $a \in T_L \setminus K$ and $y \in T_{L+1} \setminus H^*$, where H^* is the fixed set defined in (3.3.38), and also note that the integral over T_{L+1} of the first term on the right of (3.3.42) with respect to $d\sigma_L$ is

$$\begin{aligned}
\int_{T_L} \frac{3}{2^{d(L+1)}|a-y|^d} d\sigma_L(a) &= \int_{T_L} \frac{d\overline{\delta_a}(y)}{d\sigma_{L+1}} d\sigma_L(a) \\
&= \frac{d\overline{\sigma_L}(y)}{d\sigma_{L+1}} = \frac{(2^L)^{d-2} d\overline{\mu_{T_L}}(y)}{d\sigma_{L+1}} \\
&= \frac{2^{L(d-2)} d\mu_{T_{L+1}}(y)}{2^{(L+1)(d-2)} d\mu_{T_{L+1}}} = \left(\frac{1}{2}\right)^{d-2},
\end{aligned}$$

where μ_{T_L} denotes the equilibrium measure of T_L. Therefore we have

$$\int_{T_L} \left(\frac{3}{2^{d(L+1)}|a-y|^d} - \frac{c_1}{\sqrt{L}} \right) d\sigma_L(a) \geq \frac{1}{2^{d-2}} - \frac{c_1}{\sqrt{L}}. \tag{3.3.45}$$

We write with $a \in T_L$ and $y \in T_{L+1}$

$$\frac{d\widehat{\mu}(y)}{d\sigma_{L+1}} = \int_{T_L} \frac{d\widehat{\delta_a}(y)}{d\sigma_{L+1}} d\mu(a).$$

The integral element $d\mu(a)$ is at least as large as $(d\mu(y)/d\sigma_L)d\sigma_L$, and here $d\mu(y)/d\sigma_L \geq 1$ if $y \in T_L \setminus H$. Furthermore, as we have just mentioned, for $a \notin K$ the integrand is at least as large as the integrand in (3.3.45), and these give for $y \in T_{L+1} \setminus H^*$

$$\frac{d\widehat{\mu}(y)}{d\sigma_{L+1}} \geq \frac{1}{2^{d-2}} - \frac{c_1}{\sqrt{L}} - A$$

where

$$A = \left(\int_H + \int_K \right) \frac{3}{2^{d(L+1)}|a-y|^d} d\sigma_L(a).$$

The integrand on the right hand side is at most 3, hence the integral is bounded by 3 times the normalized surface area measure of H and those $a \in T_L$ for which $a \in K$, which is at most $\sigma_L(H) + \sigma_L(K \cap T_L)$. Thus, the assumption $\sigma_L(H) \leq \delta$ and the inequality (3.3.35) give

$$A \leq 3\Big(\sigma_L(H) + \sigma_L(K \cap T_L)\big) \leq 3\Big(\delta + c2^{L/2}\sqrt{\varepsilon L}\Big).$$

Thus, if $y \in T_{L+1} \setminus H^*$ then

$$\begin{aligned} \frac{d\widehat{\mu}(y)}{d\sigma_{L+1}} &\geq \frac{1}{2^{d-2}} - \frac{c_1}{\sqrt{L}} - 3\Big(\delta + c2^{L/2}\sqrt{\varepsilon L}\big) \\ &\geq \frac{1}{2^{d-2}}(1 - 2^d\delta), \end{aligned} \tag{3.3.46}$$

provided

$$\frac{c_1}{\sqrt{L}} + 3c2^{L/2}\sqrt{\varepsilon L} \leq \delta. \tag{3.3.47}$$

This condition should be understood in the sense that first we choose L large enough, then for fixed L choose $\varepsilon > 0$ small to satisfy (3.3.47). Furthermore, assuming this condition, H^* has measure at most

$$\sigma_{L+1}(H^*) \leq \frac{1}{\sqrt{L}} \leq \delta. \tag{3.3.48}$$

Thus, with such a choice for L and ε the estimate (3.3.44) holds with $c_2 = 2^d$.

Step III

We follow the notations from steps I and II, and assume that F is a compact set with (3.3.26).

Let $c_3/4 > \delta > 0$, where c_3 is a constant to be chosen later, and suppose that μ is a measure on T_L such that

$$\frac{d\mu(y)}{d\sigma_L} \geq 1 \qquad \text{for } y \in H', \tag{3.3.49}$$

where H' is of (normalized surface area) measure at least $c_3 - \delta$. Thus, we consider the same situation as in step II, but there the assumption on the density of μ with respect to σ_L was on a large set (namely on $T_L \setminus H$ of measure $\geq 1-\delta$), while here the assumption is on a set H' of measure at least $c_3 - \delta$.

Let, as before, $\widehat{\mu}$ be the balayage of μ out of $(\Delta_1 \setminus \overline{\Delta}_{L+1}) \setminus F$. We are going to show that for large L and small $\varepsilon > 0$ the measure $\widehat{\mu}$ satisfies

$$\frac{d\widehat{\mu}(y)}{d\sigma_{L+1}} \geq c_4 \qquad \text{for } y \in T_{L+1} \setminus H^*, \tag{3.3.50}$$

where H^* is a set of (normalized surface area) measure at most δ and c_4 depends only on d.

For a proof just follow the proof in step II. We have (3.3.42) for $a \in T_L \setminus K$ with H^* given in (3.3.38), and note that

$$\frac{3}{2^{d(L+1)}|a-y|^d} \geq \frac{3}{2^{2d}}.$$

Therefore, if $a \in T_L \setminus K$ and $y \in T_{L+1} \setminus H^*$ then (3.3.42) yields

$$\frac{d\widehat{\delta_a}(y)}{d\sigma_{L+1}} \geq \frac{3}{2^{2d}} - \frac{c_1}{\sqrt{L}} \geq c_5$$

provided L is large enough. Integrating this inequality with respect to $\mu(a)$ for $a \in H' \setminus K$, we obtain for $y \in T_{L+1}$ as in (3.3.46)

$$\frac{d\widehat{\mu}(y)}{d\sigma_{L+1}} \geq c_5 \left(c_3 - \delta - c2^{L/2}\sqrt{\varepsilon L} \right),$$

where we used (3.3.35) and the fact that the measure of H' is at least $c_3 - \delta$. Since $\delta < c_3/4$, we get that if $c2^{L/2}\sqrt{\varepsilon L} < \delta < c_3/4$ then (3.3.50) follows for all $y \in T_{L+1} \setminus H^*$ where H^* is the set defined in (3.3.38) of measure at most $1/\sqrt{L}$. If, in addition, $1/\sqrt{L} < \delta$, then $\sigma_{L+1}(H^*) \leq \delta$, as was claimed in (3.3.50).

Note that both of these conditions are satisfied if L is sufficiently large and ε is sufficiently small.

■

Step IV

The estimate below will be used when our compact set omits the cone $C_{2\tau}$, where

$$C_\tau := \{x \in \mathbf{R}^d \ : \ \frac{\langle x, x_0\rangle}{\|x\|} \geq 1-\tau\}. \tag{3.3.51}$$

Consider the domain $(B_2\setminus\overline{B}_{1/2})\cap C_{2\tau}$, and let $a \in S_1\cap C_\tau$. It is clear (by Harnack's inequality) that there is a positive constant c_τ depending only on τ such that if $\breve{\delta}_a$ is the balayage of δ_a out of $(B_2\setminus\overline{B}_{1/2})\cap C_{2\tau}$, then on $S_{1/2}\cap C_\tau$ this balayage has density at least c_τ, i.e.

$$\frac{d\breve{\delta}_a(y)}{d\sigma_{1/2}} \geq c_\tau, \qquad y \in S_{1/2}\cap C_\tau, \quad a \in C_\tau \cap S_1.$$

Thus, if $c_3 = \sigma_1(C_\tau\cap S_1)$, $\delta < c_3/2$ and μ is a measure on S_1 such that $d\mu(y)/d\sigma_1 \geq 1$ on a set $H'' \subseteq C_\tau\cap S_1$ of measure at least $c_3-\delta$, then

$$\frac{d\breve{\mu}(y)}{d\sigma_{1/2}} \geq \frac{c_3 c_\tau}{2}, \qquad y \in S_{1/2}\cap C_\tau. \tag{3.3.52}$$

Step V

Now we can complete the proof of the necessity direction part (**b**) in Theorem 3.2.1. Let us suppose that $\mathcal{N}_E(\varepsilon)$ is of zero lower density for every $\varepsilon > 0$ and that E satisfies the cone condition. We may assume that the cone that E omits is $C_{2\tau}\cap B_1$ with C_τ defined in (3.3.51), and first let us suppose that E is contained in the unit ball. Then $E\cap C_{2\tau} = \emptyset$.

Let $\delta < c_3/2 < 1/2$, and select the integer L and $\varepsilon > 0$ in such a way that all the estimates in steps II–IV hold.

Let $E_n = E\cup\overline{B}_{2^{-n}}$, $\mu_n = \mu_{E_n}$, and let

$$v_n(y) = \frac{d\mu_n(y)}{d\sigma_n}$$

be the lower Radon-Nikodym derivative (density) of μ_n on $S_{2^{-n}}$ with respect to normalized surface area measure on $S_{2^{-n}}$. Thus, $v_n(y)d\sigma_n(y) \leq d\mu_n(y)$. Note that μ_0 is the normalized surface area measure on S_1, hence $v_0(y)\equiv 1$.

Let

- $\Sigma_0 = \{n \geq L \ : \ \ n+1, n, \ldots, n-L+1 \not\in \mathcal{N}_E(\varepsilon)\}$,
- $\Sigma_1 = \{n \geq L \ : \ \ n+1, n, \ldots, n-L+2 \not\in \mathcal{N}_E(\varepsilon), \ n-L+1 \in \mathcal{N}_E(\varepsilon)\}$,
- $\Sigma_2 = \{n \in \mathbf{N} \ : \ \ n < L$ or one of $n+1, n, \ldots, n-L+2$ belongs to $\mathcal{N}_E(\varepsilon)\}$.

These give a partition of the integers. For every natural number n we define a number A_n as follows. If $n > L$ and $n, n-1, \ldots, n-L+1 \not\in \mathcal{N}_E(\varepsilon)$, then let A_n be the largest number with the property that $v_n(y) \geq A_n$ for all $y \in S_{2^{-n}}$ with the exception of a set of normalized surface measure $\leq \delta$. Let us call this case for A_n of the first type. If, however, $n \leq L$ or one of $n, n-1, \ldots, n-L+1$ belongs to $\mathcal{N}_E(\varepsilon)$, then let A_n be the largest number with the property that $v_n(y) \geq A_n$ for all $y \in S_{2^{-n}} \cap C_\tau$ with the exception of a set of normalized surface measure $\leq \delta$. Let us call this case for A_n of the second type.

We want to compare A_{n+1} with A_n for $n \geq L$. Let $\overline{\cdot}$ denote the balayage out of $\overline{\mathbf{R}}^d \setminus (E \cup \overline{B}_{2^{-n-1}})$. Then $\mu_{n+1} = \overline{\mu_n}$ and $\mu_{n+1}\Big|_{S_{2^{-n-1}}} = \overline{\mu_n\Big|_{S_{2^{-n}}}}\Big|_{S_{2^{-n-1}}}$. Thus, v_{n+1} is at least as large as the density (on $S_{2^{-n-1}}$) of the balayage $\overline{v_n(y)d\sigma_n}$ of $v_n(y)d\sigma_n$. If $\widehat{\cdot}$ denotes balayage out of the narrower domain $B_{2^{n-L+1}} \setminus (E \cup \overline{B}_{2^{-n-1}})$, then the density of $\widehat{v_n(y)d\sigma_n}$ is not larger on $S_{2^{-n-1}}$ than the density of the balayage $\overline{v_n(y)d\sigma_n}$, which (as we have just seen) is not larger than v_{n+1}. Now if $n \in \Sigma_0$, then both A_n and A_{n+1} are of first type (i.e. $v_n(y) \geq A_n$ and $v_{n+1}(y) \geq A_{n+1}$ for all $y \in S_{2^{-n}}$ and $y \in S_{2^{-n-1}}$, respectively, with the exception of a set of measure $\leq \delta$), hence (3.3.44) can be applied (in a scaled form) for the measure $d\mu(y) = \widehat{v_n(y)d\sigma_n}$ to conclude that $A_{n+1} \geq (1/2^{d-2})(1-c_2\delta)A_n$.

In a completely similar manner, if $n \in \Sigma_1$, then A_n is of the second type while A_{n+1} is of the first type, i.e. $v_n(y) \geq A_n$ for all $y \in S_{2^{-n}} \cap C_\tau$ with the exception of a set of measure $\leq \delta$ and $v_{n+1}(y) \geq A_{n+1}$ for all $y \in S_{2^{-n-1}}$ with the exception of a set of measure $\leq \delta$. Now instead of (3.3.44) we apply (3.3.50) to conclude that $A_{n+1} \geq c_4 A_n$.

Finally, if $n \in \Sigma_2$, $n \geq L$, then A_{n+1} is definitely of the second type, but A_n

may be of the first or second type (of the first type only if $n+1 \in \mathcal{N}_E(\varepsilon)$, but $n, n-1, \dots n-L+2 \notin \mathcal{N}_E(\varepsilon)$). In either case $v_n(y) \geq A_n$ for all $y \in S_{2^{-n}} \cap C_\tau$ with the exception of a set of measure $\leq \delta$, and hence we can apply (3.3.52) to conclude $A_{n+1} \geq (c_3 c_\tau/2)A_n$. This is also the estimate we use for all $n < L$.

In summary, we have $A_{n+1} \geq (1/2^{d-2})(1-c_2\delta)A_n$ for $n \in \Sigma_0$, $A_{n+1} \geq c_4 A_n$ for $n \in \Sigma_1$ and $A_{n+1} \geq (c_3 c_\tau/2)A_n$ for $n \in \Sigma_2$. If $s = s_N$ denotes the number of elements of $\mathcal{N}_E(\varepsilon) \cup \{0\}$ not larger than N, then there are at most s elements of Σ_1 and at most sL elements of Σ_2 not larger than N. Thus, we can conclude

$$A_{N+1} \geq \left(\frac{1}{2^{d-s}}\right)^N (1-c_2\delta)^N (c_4)^{s_N} \left(\frac{c_3 c_\tau}{2}\right)^{s_N L} A_0.$$

Since $\mathcal{N}_E(\varepsilon)$ is of zero lower density, the limit inferior of s_N/N is zero, hence there are infinitely many N's for which

$$(c_4)^{s_N} \left(\frac{c_3 c_\tau}{2}\right)^{s_N L} A_0 \geq \frac{2}{c_3}(1-c_2\delta)^N. \tag{3.3.53}$$

For all such N we can conclude that

$$A_{N+1} \geq (2/c_3)(1/2^{d-2})^N (1-c_2\delta)^{2N},$$

which implies

$$\mu_{N+1}(S_{N+1}) \geq \left(\frac{1}{2^{d-2}}\right)^N (1-c_2\delta)^{2N} \tag{3.3.54}$$

because, independently if A_N is of the first or second type, we have $v_N(y) \geq A_N$ on a set of measure at least $c_3 - \delta \geq c_3/2$.

Now we can easily complete the proof. Let Ω_{N+1} be the unbounded component of $\mathbf{R}^d \setminus E_{N+1}$. Consider Green's function with pole at $y_0 \in \Omega_{N+1}$ and integrate it over the sphere S_r with $r = r_N = 2^{-N}$.

$$\begin{aligned}\int_{S_r} g_{\Omega_{N+1}}(x,y_0)\,d\sigma_{S_r}(x) &= \int_{S_r} \big(g_{\Omega_{N+1}}(x,y_0) - g_{\Omega_{N+1}}(0,y_0)\big)\,d\sigma_{S_r}(x) \\ &= \int_{S_r} \big(U^{\delta_{y_0}}(x) - U^{\delta_{y_0}}(0)\big)\,d\sigma_{S_r}(x)\end{aligned}$$

$$+\quad \int_{S_r}\left(U^{\widetilde{\delta_{y_0}}}(0)-U^{\widetilde{\delta_{y_0}}}(x)\right)d\sigma_{S_r}(x), \tag{3.3.55}$$

where $\widetilde{\cdot}$ denotes the balayage out of Ω_{N+1}.

Here the first integrand is

$$\frac{1}{|x-y_0|^{d-2}}-\frac{1}{|y_0|^{d-2}}\le c_6|x|, \tag{3.3.56}$$

where c_6 depends only on y_0 and d. For the second integral we have

$$\begin{aligned}&\int_{S_r}\left(U^{\widetilde{\delta_{y_0}}}(0)-U^{\widetilde{\delta_{y_0}}}(x)\right)d\sigma_{S_r}(x)\\ &\quad=\int_{E_{N+1}}\frac{1}{|t|^{d-2}}d\widetilde{\delta_{y_0}}(t)-\int_{S_r}\int_{E_{N+1}}\frac{1}{|x-t|^{d-2}}d\widetilde{\delta_{y_0}}(t)d\sigma_{S_r}(x).\end{aligned} \tag{3.3.57}$$

Since

$$\int_{S_r}\frac{1}{|x-t|^{d-2}}d\sigma_{S_r}(x)=\min\left(\frac{1}{|t|^{d-2}},\frac{1}{r^{d-2}}\right)$$

(see e.g. [3, Example 4.2.9]) and there exists c_7 depending only on y_0 and d such that $d\widetilde{\delta_{y_0}}\ge c_7 d\mu_{N+1}$ (see (3.1.10)), we get from (3.3.55), (3.3.56) and (3.3.57)

$$\begin{aligned}\int_{S_r} g_{\Omega_{N+1}}(x,y_0)d\sigma_{S_r}(x) &\ge \int_{E_{N+1}}\left(\frac{1}{|t|^{d-2}}-\min\left(\frac{1}{|t|^{d-2}},\frac{1}{r^{d-2}}\right)\right)d\widetilde{\delta_{y_0}}(t)-c_6 r\\ &\ge c_7\int_{\overline{B}_{2^{-N-1}}}\left(\frac{1}{|t|^{d-2}}-\frac{1}{r^{d-2}}\right)d\mu_{N+1}(t)-c_6 r\\ &= (2^{d-2}-1)c_7\frac{1}{r^{d-2}}\mu_{N+1}(S_{N+1})-c_6 r\\ &\ge (2^{d-2}-1)c_7(1-c_2\delta)^{2N}-c_6 r\ge r^{\kappa},\end{aligned} \tag{3.3.58}$$

provided $\delta<(1-\sqrt{2}/2)/c_2$,

$$\begin{aligned}N\ge\max\Bigg(&\frac{\log(2c_6)-\log\left((2^{d-2}-1)c_7\right)}{\log\left(2(1-c_2\delta)^2\right)},\\ &\frac{1}{\delta\log 2}\left(\log 2-\log\left((2^{d-2}-1)c_7\right)\right)\Bigg)\end{aligned} \tag{3.3.59}$$

and

$$\kappa = \delta + \frac{2}{\log 2} \log \frac{1}{1 - c_2\delta}.$$

Hence, there is an x_N such that $g_{\Omega_{N+1}}(x_N, y_0) \geq r^\kappa$, and this implies

$$g_\Omega(x, y_0) \geq g_{\Omega_{N+1}}(x, y_0) \geq r_N^\kappa.$$

Here $\kappa > 0$ can be arbitrarily small since $\delta > 0$ is as close to 0 as we wish, and this inequality is true for a sequence $r_N = 2^{-N} \to 0$ (for which s_N satisfies (3.3.53) and (3.3.59)). Therefore Green's function g_Ω is not Hölder continuous at 0.

The proof above used that E is contained in the unit ball and omits the cone $C_{2\tau}$. The general case can be similarly handled. In fact, let Ω contain the cone $C = C(x_0, 2\tau, \ell)$. Select a sphere S_{r_0}, $r_0 < l/2$, that intersects C. Without loss of generality (use a dilation) we may assume that $S_{r_0} = S_1$, and let $J = S_1 \cap C_\tau$ be the middle part of $S_1 \cap C$. Then $\mu_{E \cup \overline{B}_1}$ has strictly positive density on J, say (with the notations of the preceding proof) $v_0(y) \geq c_9 > 0$ for $y \in J$. Now the preceding proof can be repeated, the only difference is that in this case the starting value of A_0 is c_9 (note that for $n = 0$ the number A_n is of the first type).

■

Remark 3.3.2 The preceding proof was effective in the following sense. *Let y_0, τ, ℓ, c, r_0, $\kappa > 0$, $r_0 < \ell/2$ be given. Then there are $\varepsilon > 0$, $\eta > 0$ and M that depend only on $d, y_0, \tau, \ell, c, r_0, \kappa$, with the following property. Let E be a compact set of positive capacity, Ω the unbounded component of $\mathbf{R}^d \setminus E$, $0 \in \partial\Omega$, and assume that Ω contains a cone $C(x_0, 2\tau, \ell)$. If for the measure $\mu_0 = \mu_{E \cup \overline{B}_{r_0}}$ the condition*

$$\frac{d\mu_0(y)}{d\sigma_{S_{r_0}}} \geq c, \qquad y \in S_{r_0} \cap C(x_0, \tau, \ell)$$

holds, and if for a particular $N \geq M$ we have $|\mathcal{N}_E(\varepsilon) \cap \{0, 1, \ldots, N\}| \leq \eta N$, then there is an $x \in S_{2^{-N}}$ such that

$$g_\Omega(x, y_0) \geq (2^{-N})^\kappa.$$

3.4 Proof of Lemma 3.2.2

First we show that **1)** is equivalent to **2)**. As at the end of the proof of Theorem 3.2.1 in Section 3.3 (see (3.3.58)), we can write

$$\int_{S_r} g_G(x,a)d\sigma_{S_r}(x) \geq \frac{c_1}{r^{d-2}}\mu_E(\overline{B}_r) - c_2 r$$

with some constants c_1 and c_2. If $g_G(\cdot,a)$ is Hölder continuous at 0, then the left-hand side is less than $c_3 r^\kappa$ for some positive constants c_3 and $\kappa < 1$. Therefore it follows for $r < 1$ that

$$\mu_E(\overline{B}_r) \leq c_4 r^{d-2+\kappa}$$

for some constant c_4 and this shows that **1)** implies **2)**.

Conversely, suppose **2)**. Let $|x| = r$ be small, and $E^* = E \setminus B_{2r}$. Let $\overline{\cdot}$ and $\widehat{\cdot}$ denote the balayage out of $\mathbf{R}^d \setminus E^*$ and $\mathbf{R}^d \setminus E$, respectively. Since 0 is a regular point, $g_\Omega(0,a) = 0$. Therefore

$$\begin{aligned} g_{\mathbf{R}^d\setminus E^*}(0,a) &= g_{\mathbf{R}^d\setminus E^*}(0,a) - g_\Omega(0,a) \\ &= (U^{\delta_a}(0) - U^{\overline{\delta_a}}(0)) - (U^{\delta_a}(0) - U^{\widehat{\delta_a}}(0)) \\ &= U^{\widehat{\delta_a}}(0) - U^{\overline{\delta_a}}(0) \\ &= \int \frac{1}{|y|^{d-2}}(d\widehat{\delta_a}(y) - d\overline{\delta_a}(y)). \end{aligned}$$

Now, $\overline{\delta_a}$ is the balayage of $\widehat{\delta_a}$ onto E^*, and so $\widehat{\delta_a}\Big|_{E^*} \leq \overline{\delta_a}$. Thus, we do not decrease the integral by integrating only over B_{2r} with respect to $\widehat{\delta_a}$, i.e.

$$g_{\mathbf{R}^d\setminus E^*}(0,a) \leq \int_{B_{2r}} \frac{1}{|y|^{d-2}} d\widehat{\delta_a}(y). \tag{3.4.60}$$

Furthermore, using **2)** we obtain

$$\begin{aligned}\int_{B_{2r}} \frac{1}{|y|^{d-2}} d\mu_E(y) &= \sum_{i=0}^{\infty} \int_{B_{2r/2^i} \setminus \overline{B}_{r/2^i}} \frac{1}{|y|^{d-2}} d\mu_E(y) \\ &\leq \sum_{i=0}^{\infty} \left(\frac{2^i}{r}\right)^{d-2} \mu_E(B_{2r/2^i}) \leq \sum_{i=0}^{\infty} \frac{2^{(d-2)i}}{r^{d-2}} C \left(\frac{2r}{2^i}\right)^{d-2+\kappa} \\ &= \frac{C2^{d-2+2\kappa}}{2^\kappa - 1} r^\kappa.\end{aligned}$$

This, (3.4.60) and (3.1.10) give $g_{\mathbf{R}^d \setminus E^*}(0,a) \leq c_5 r^\kappa$ for all small $r > 0$ with some constant c_5. Now the ball B_{2r} is contained in $\mathbf{R}^d \setminus E^*$, hence Harnack's inequality gives $g_{\mathbf{R}^d \setminus E^*}(x,a) \leq c_6 g_{\mathbf{R}^d \setminus E^*}(0,a) \leq c_6 c_5 r^\kappa$ for all $|x| = r$. Since here $g_{\mathbf{R}^d \setminus E^*}(x,a) \geq g_\Omega(x,a) \geq g_G(x,a)$, the Hölder continuity of $g_G(x,a)$ follows, and this proves **2)** $\Rightarrow$ **1)**.

Remark 3.4.1 The proof just given is effective in the following sense. *If for some $r > 0$ we have $\mu_E(\overline{B}_t) \leq Ct^{d-2+\kappa}$ for $t \leq 2r$, then for $|x| = r$*

$$g_G(x,a) \leq C_1 r^\kappa, \qquad C_1 = \frac{C_a C c_6 2^{d-2+2\kappa}}{2^\kappa - 1}. \tag{3.4.61}$$

Next we show that **3)** implies **1)**. Let R be so large that $\partial G \subset B_R$ and construct a domain T in the following way. If $a \notin \overline{B}_R$ then set $T = B_R \cap G$. Otherwise take a small ball $B_a \subset G$ centered at a and set $T = (B_R \setminus \overline{B}_a) \cap G$. Let r be small and set $f = 0$ on $\overline{B}_r$ and $f = 1$ on $\partial G \setminus \overline{B}_r$. Compare u_f and g_G in the region T. Both are harmonic in T and positive on S_R and ∂B_a. Hence an application of the maximum principle shows that $g_G(x,a) \leq c_7 u_f(x)$ in T, and this proves **3)** $\Rightarrow$ **1)**.

It is left to show **1)** $\Rightarrow$ **3** under the cone condition. Under the cone condition **1)** implies the positive lower density of $\mathcal{N}_E(\varepsilon)$ for some $\varepsilon > 0$, i.e. there is an η and an N_1 such that $|\mathcal{N}_E(\varepsilon) \cap \{0, 1, \ldots, N\}| \geq 4\eta N$ for $N \geq N_1$. Then for large N, say for $N \geq N_2$, we also have

$$|\mathcal{N}_E(\varepsilon) \cap \{[(2\eta)N] + 1, [(2\eta)N] + 2, \ldots, M\}| \geq \eta M$$

for any $M \geq N$. Set $r_N = 2^{-[2\eta N]}$ and $F = \overline{B}_{r_N} \cap (\mathbf{R}^d \setminus G)$. For this F the preceding inequality gives

$$|\mathcal{N}_F(\varepsilon) \cap \{0, 1, \ldots, M\}| \geq \eta M, \qquad \text{for } M \geq N \geq N_2,$$

hence, by the proof of **a**) in Theorem 3.2.1, see in particular Remark 3.3.1, there are a $\kappa > 0$ and an $N_0 \geq N_2$ (depending only on ε and η) such that for all $M \geq N \geq N_0$ the inequality $\mu_F(\overline{B}_{2^{-M+1}}) \leq (2^{-M+1})^{d-2+\kappa}$ is true. This implies $\mu_F(\overline{B}_t) \leq 2^{d-2+\kappa} t^{d-2+\kappa}$ for $t \leq 2{\cdot}2^{-N}$. Hence, by the effective form of the implication **2**) $\Rightarrow$ **1**) given in Remark 3.4.1, we can conclude for $|x| = r = 2^{-N}$ the inequality $g_{\mathbf{R}^d \setminus F}(x, a) \leq C_1 (2^{-N})^\kappa$ with $C_1 = C_a c_6 2^{2(d-2+2\kappa)}/(2^\kappa - 1)$.

■

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