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Probability Theory on Semihypergroups

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by

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I praise you, Father, Lord of heaven and earth, because you have hidden these things from the wise and learned, and revealed them to little children.

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ABSTRACT

Motivated by the work of Hognas and Mukherjea on semigroups [HM95], we study semihypergroups, which are structures closer to semigroups than hypergroups in the sense that they do not require an identity or an involution. Dunkl[Du73] calls them hypergroups (without involution), and Jewett[Je75] calls them semiconvos. A semihypergroup does not assume any algebraic operation on itself. To generalize results from semigroups to semihypergroups, we first put together the fundamental algebraic concept a semihypergroup inherits from its measure algebra. Among other things, we define the Rees convolution product, and prove that if X, Y are non-empty sets and H is a hypergroup, then with the Rees convolution product, $X \times H \times Y$ is a completely simple semihypergroup which has all its idempotent elements in its center. We also point out striking differences between semigroups and semihypergroups. For instance, we construct an example of a commutative simple semihypergroup, which is not completely simple. In a commutative semihypergroup S, we solve the Choquet equation $\mu * \nu = \nu$, under certain mild conditions.We also give the most general result for the non-commutative case.We give an example of an idempotent measure on a commutative semihypergroup whose support does not contain an idempotent element and so could not be completely simple. This is in contrast with the context of semigroups, where idempotent measures have completely simple supports.

The results of Hognas and Mukherjea [HM95] on the weak convergence of the sequence of averages of convolution powers of probability measures is generalized to semihypergroups. We use these results to give an alternative method of solving the Choquet equation on hypergroups (which was initially solved in [BH95] with many steps). We show that If S is a compact semihypergroup and μ is a probability measure with $S = \overline{[\bigcup_{n=1}^{\infty} Supp(\mu)^n]}$, then for any open set $G \supset K$ where K is the kernel of S

$$lim_{n \longrightarrow \infty} \mu^n(G) = 1.$$

Finally, we extend to hypergroups basic techniques on multipliers set forth for groups in [HR70], namely propositions 5.2.1 and 5.2.2 , we give a proof of an extended version of Wendel's theorem for locally compact commutative hypergroups and show that this version also holds for compact non-commutative hypergroups. For a compact commutative hypergroup H, we establish relationships between semigroup $\mathcal{S} = \{T(\xi) : \xi > 0\}$ of operators on $L_p(H)$, $1 \leq p < \infty$, which commutes with translations, and semigroup $\mathcal{M} = \{E_\xi : \xi > 0\}$ of $L_p(H)$ multipliers. These results generalize those of [HP57] for the circle groups and [B074] for compact abelian groups.

1 PRELIMINARIES

1.1 Introduction

The origin of hypergroup can be traced back to the time of the rise of group theory in 1900 with the work of Frobenious. In the mid thirties, F Marty [MA] and M. S. Wall [WA] introduced the concept of an algebraic hypergroup, mainly within the theory of non-abelian groups and related structures of spaces of conjugacy classes and double cosets. The term hypergroup in a topological context was first used by Delsarte [DE] when he introduced the theory of generalized translation operators. His theory was used by Levitan [LE] and later on Berzanski [BK63] in their theory of hypercomplex system. In 1956, I.I. Hirshman, Jr.[Hi56a], pointed out that the structure for harmonic analysis exists in a setting where certain orthogonal polynomials could play the role of the exponentials in classical Fourier analysis. A product formula on the system of orthogonal functions is used to defined a convolution on the vector space of Radon measures. I.M. Gelfand [Ge50], S. Bochner [B074], [Bo54] and I.I. Hirshman Jr [Hi56a], obtained product formulas and a number of very interesting consequences for ultraspherical polynomials and, in particular, for Legendre polynomials and for Bessel functions. For Jacobi polynomials product formulas were found by G. Gasper [GA70],[GA71], [GA72].

About 3 decades ago, Harmonic Analysts and Probability Theorists were faced with the question of which topological spaces have enough structures so that a convolution for all finite regular Borel measures could be defined on these spaces. Charles Dunkl [Du73], Robert Jewett[Je75] and René Spector[Sp78] independently addressed this question and set down basic axioms, defining such a structure, which has come to be called DJS-Hypergroups. Hypergroups generalize in many ways locally compact semigroups. A hypergroup requires an identity element and an involution (which acts as the group inverse). In this chapter we introduced the preliminary notations and basic concepts. We will also present the axioms of hypergroups as set down by each of the founders. We will end with what is today generally accepted as the DJS-hypegroup which will be our definition of reference throughout this dissertation.

1.2 Basic Concepts

1.2.1 Notations

We first recall some standard notations. Let S be a locally compact Hausdorff space:

i. $C(S)$: the space of complex continuous functions on S,

ii. $C_b(S)$: the space of bounded elements of $C(S)$

iii. $C_0(S)$: the space of elements of $C_b(S)$ which tends to 0 at ∞

iv. $C_c(S)$: the space of elements of $C_0(S)$ with compact support

v. $C_c^+(S)$: the space of nonnegative elements of $C_c(S)$.

vi. $M(S)$ denote the set of finite regular Borel measures.

vii. $M_+(S)$: the space of all nonnegative elements of $M(S)$

viii. $M_1(S)$ denote the set of probability measures.

ix. If $\mu \in M(S)$, then $Supp(\mu) = \{x \in S :$ if V is any open set containing x, then $\mu(V) > 0\}$

x. An unspecified topology on $M_+(S)$ is the cone topology.

xi. If $x \in S$ then δ_x denotes the point mass at x

xii. $B_\infty^+(S)$ denotes the extended nonnegative real-valued Borel functions.

xiii. If A is any subset of S, $\overline{A}$ is the closure of A, and A^c is the complement of A.

1.2.2 The Michael topology

Let S be a locally compact space and let $\mathcal{C}(S)$ be the space of all compact subsets of S. For $A, B \subset S$, $\mathcal{C}_A(B) = \{C \in \mathcal{C}(S) : C \cap A \neq \emptyset \text{ and } C \subset B\}$ Then $\mathcal{C}(S)$ can be given the topology generated by the sub-basis of all $\mathcal{C}_U(V)$ for which U and V are open subsets of S. This topology which was developed by Michael[Mi55] has the following properties [Je75]

Properties 1.2.1 **i.** *If S is compact, then $\mathcal{C}(S)$is compact.*

ii. *$\mathcal{C}(S)$ is a locally compact space.*

iii. *The mapping $x \mapsto \{x\}$ is a homeomorphism of S onto a closed subset of $\mathcal{C}(S)$.*

iv. *The collection of nonempty finite subsets of S is a dense subset of $\mathcal{C}(S)$.*

v. *If Ω is a compact subset of $\mathcal{C}(S)$, then $B = \bigcup\{A : A \in \Omega\}$ is a compact subset of S.*

vi. *If S is metrizable with metric d, then the Michael topology on $\mathcal{C}(S)$ is stronger than the Hausdorff topology given by the Hausdorff metric ρ which for $A, B \in \mathcal{C}(S)$ is defined by $\rho(A, B) = max\{h(A, B), h(B, A)\}$ where $h(A, B) = sup\{d(x, B) : x \in A\}$*

1.3 Definitions of a Hypergroups

1.3.1 Dunkl's Definition of a Hypergroup

A locally compact space H is called a hypergroup if there is a map $\lambda : H \times H \to M_1(H)$ with the following properties:

D_1. For each $f \in C_c(H)$ the map

$$(x, y) \longmapsto \int_H f d\lambda(x, y)$$

is in $C_b(H \times H)$ and

$$x \longmapsto \int_H f d\lambda(x, y)$$

is in $C_b(H)$ for each $y \in H$

D_2. The convolution on $M(H)$ defined implicitly by

$$\int_H f d\mu * \nu = \int_H d\mu(x) \int_H d\nu(y) \int_H f d\lambda(x, y)$$

$\mu, \nu \in M(H)$, $f \in C_0(H)$ is associative.

D_3. There is a point (the identity) $e \in H$ such that

$$\lambda(x, e) = \delta_x \qquad (x \in H)$$

D_4. The hypergroup H is said to be commutative if

$$\lambda(x, y) = \lambda(y, x) \qquad \forall x, y \in H$$

Remark 1.3.1 *Dunkl does not require the existence of an involution in his definition, rather he called any hypergroup ,with an involution, which possesses an invariant measure, a $*$-hypergroup. He does not require that the support of the convolution of two point masses be compact and consequently does not use the Michael topology in his definition.*

1.3.2 Jewett's Definition of a Convos

A pair $(K, *)$ will be called a **semiconvo** if the following five conditions are satisfied:

J_1. K is a nonvoid locally compact Hausdorff space.

J_2. The symbol $*$ denotes a binary operation on $M(K)$ and with this operation $M(K)$ is a complex (associative) algebra.

J_3. The bilinear mapping $(\mu, \nu) \longmapsto \mu * \nu$ is positive-continuous. (That is, $\mu * \nu \geq 0$ whenever $\mu \geq 0$ and $\nu \geq 0$ and the convolution restricted to $M_+(S) \times M_+(S) \longrightarrow M_+(S)$ is continuous).

J_4. If $x, y \in K$ then $\delta_x * \delta_y$ is a probability measure with compact support.

J_5. The mapping $(x, y) \longmapsto Supp(\delta_x * \delta_y)$ from $K \times K$ to $\mathcal{C}(K)$ (with the Michael topology) is continuous.

If in addition we also have,

J_6. There exists a (necessarily unique) element e of K such that $\delta_x * \delta_e = \delta_e * \delta_x = \delta_x$ for all $x \in K$

J_7. There exists a (necessarily unique) involution $x \longmapsto x^-$ of K such that (for $x, y \in K$) the element e is in the support of $\delta_x * \delta_y$ if and only if $x = y^-$, the semiconvo will be called a **convo**.

Remark 1.3.2 *Dunkl's definition of hypergroup is a commutative semiconvo with identity. Jewett's commutative convo is according to Dunkl's definition, a $*$-hypergroup.*

1.3.3 Spector's Definition of a Hypergroup

A hypergroup is a locally compact space X, together with a convolution $*$ that makes $M(X)$ a Banach algebra and satisfy the following properties:

S_1. $M_1(X) * M_1(X) \subset M_1(X)$

S_2. $*$ is separately continuous from $M_1(X) \times M_1(X)$ to $M_1(X)$ with the weak topology defined by the duality between $M(X)$ and $C_0(X)$ $(\sigma(M(X), C_0))$.

S_3. The map $(x,y) \longmapsto \delta_x * \delta_y$ is continuous from $X \times X$ onto $M_1(X)$ with the weak topology induced by $\sigma(M(X), C_0)$

S_4. There is a necessarily unique point e called the "identity element of the hypergroup X", such that δ_e is the identity element of the convolution $*$.

S_5. There is an involutive homeomorphism of X onto X, denoted by $x \longmapsto x^-$ with natural extension to $M(X)$ satisfying $(\mu * \nu)^- = \nu^- * \mu^-$; in particular $e^- = e$ this homeomorphism will be called the "symmetry of the hypergroup".

S_6. For every $x, y \in X$, $e \in Supp(\delta_x * \delta_y)$ if and only if $x = y^-$

S_7. For any compact subset K of X and any neighborhood V of K there exists a neighborhood U of e such that

(1) $Supp(\mu) \subset K$ and $Supp(\nu) \subset U$ imply $Supp(\mu * \nu) \subset V$ and $Supp(\nu * \mu) \subset V$

(2)$Supp(\mu) \subset K$ and $Supp(\nu) \subset U^c$ imply that the support of $\mu * \nu^-$,$\mu^- * \nu$,$\nu * \mu^-$,and $\nu^- * \mu$ are disjoint with U.

Remark 1.3.3 *Spector does not require the support of the convolution of two point masses to be compact, which leads sometimes to some technical complications as he acknowledges himself. Actually he also acknowledges not having any substantial example where this condition fails. Consequently there is no use of the Michael topology in his proofs.*

We now give a general definition of a hypergroup which is now called the DJS-hypergroup. To this end, we start with the definition of a semihypergroup and give simple examples of semihypergroups and hypergroups.

1.4 The DJS-Hypergroup

A nonempty locally compact Hausdorff space S will be called a **semihypergroup** if the following conditions are satisfied:

(SH_1) $(M_b(S),+,*)$ is a Banach algebra.

(SH_2) For all $x,y\in S$, $\delta_x*\delta_y$ is a probability measure with compact support contained in S.

(SH_3) The mapping $(x,y)\mapsto\delta_x*\delta_y$ of $S\times S$ into $M_1(S)$, where $S\times S$ has the product topology and $M_1(S)$ has the weak topology, is continuous.

(SH_4) The mapping$(x,y)\mapsto Supp(\delta_x*\delta_y)$ of $S\times S$ into $\mathcal{C}(S)$ is continuous, where $\mathcal{C}(S)$ is the space of compact subsets of S endowed with the Michael topology, that is the topology generated by the subbasis of all $\mathcal{C}_U(V)=\{C\in\mathcal{C}(S):C\cap U\neq\emptyset$ and $C\subset V\}$ where U and V are open subsets of S.

Remark If $\delta_x*\delta_y=\delta_y*\delta_x$ for all $x,y\in S$, then we say that $(S,*)$ is a commutative semihypergroup. If, in addition, we also have:

SH_5 there exists $e\in S$ such that $\delta_x*\delta_e=\delta_e*\delta_x=\delta_x$ $\forall x\in S$, and

SH_6 There exists a topological involution (a homeomorphism) from S onto S such that $(x^-)^-=x$ $\forall x\in S$, with $(\delta_x*\delta_y)^-=\delta_{y^-}*\delta_{x^-}$ and $e\in Supp(\delta_x*\delta_y)$ if and only if $x=y^-$ where for any Borel set B, $\mu^-(B)=\mu(\{x^-:x\in B\})$,

then $(S,*)$ is called a **hypergroup**

Example 1.4.1

1. If $(S,.)$ is a topological semigroup, where S is a locally compact Hausdorff space then with convolution defined by $\delta_x*\delta_y=\delta_{xy}$, $(S,*)$ is a semihypergroup. Also if a semihypergroup is such that the convolution of two point masses is a point mass, then it is a semigroup.

2. Let $S = \{x, y\}$ with the discrete topology. Then S is a locally compact space we can define the convolution of point masses by

$$\delta_x * \delta_x = a\delta_x + b\delta_y$$

$$\delta_y * \delta_y = b'\delta_x + a'\delta_y$$

$$\delta_x * \delta_y = p\delta_x + p'\delta_y$$

$$\delta_y * \delta_x = q\delta_x + q'\delta_y$$

where $a, b, a', b', p, p', q, q'$ are non-negative real numbers such that $a + b = a' + b' = p + p' = q + q' = 1$ (for the convolution product of two point masses to be a probability measure) and $bb' = pp' = qq'$ (for the convolution product to be associative). Then $(S, *)$ is a semihypergroup.

3. Let $S = \{e, a, b\}$. Let e be the identity element and let us define

$$\delta_a * \delta_a = \frac{1}{2}\delta_a + \frac{1}{2}\delta_b$$

$$\delta_b * \delta_b = \delta_a$$

$$\delta_a * \delta_b = \delta_b * \delta_a = \frac{1}{2}\delta_e + \frac{1}{2}\delta_b$$

Then $(S, *)$ is a semihypergroup and if we defined an involution by $a' = b$ and $b' = a$ we have

$$(\delta_a * \delta_a)' = \frac{1}{2}\delta_{a'} + \frac{1}{2}\delta_{b'} = \frac{1}{2}\delta_b + \frac{1}{2}\delta_a$$

But

$$\delta_{a'} * \delta_{a'} = \delta_b * \delta_b = \delta_a \neq \frac{1}{2}\delta_a + \frac{1}{2}\delta_b$$

So although $e \in Supp(\delta_a * \delta_b)$ this involution does not satisfy the condition $(\delta_a * \delta_b)' = \delta_{b'} * \delta_{a'}$ this semihypergroup is almost (though not) a hypergroup and it is called a regular semihypergroup [On93].

4. Let $H = \{e, x, y\}$ and let e be the identity element , the identity function is considered as the involution, and a commutative convolution is defined on H by

$$\delta_x * \delta_x = a\delta_e + b\delta_x + c\delta_y$$

$$\delta_y * \delta_y = a'\delta_e + c'\delta_x + b'\delta_y$$

$$\delta_x * \delta_y = \delta_y * \delta_x = q\delta_x + q'\delta_y$$

Then $(H, *)$ is a hypergroup provided $a + b + c = a' + b' + c' = q + q' = 1$ (for the convolution of two point masses to be a probability measure, and $a'c = aq$ (for associativity of convolution).

Other examples of hypergroups and semihypergroups could be found in [Du73], [Je75], [Sp78]. In the next section, we will give some interesting examples of semihypergroups and hypergroups.

2.1 Some finite-element semihypergroups

In this section we consider two-element semihypergroups and two, three and four-element hypergroups. We show that there are non-commutative two-element semihypergroups but for $1 \leq n < 5$ every n-element hypergroup is commutative. (The proof of this latter result, though well-known and part of the folklore, is not available in print, to the best of our knowledge.)

2.1.1 Two-element semihypergroups

Let $X = \{x, y\}$. The most general convolution product on elements of X is given by

$$\delta_x * \delta_x = a\delta_x + b\delta_y$$

$$\delta_y * \delta_y = b'\delta_x + a'\delta_y$$

$$\delta_x * \delta_y = p\delta_x + p'\delta_y$$

$$\delta_y * \delta_x = q'\delta_x + q\delta_y$$

where $a, b, a', b', p, p', q, q'$ are non-negative real numbers.

Now we observe that for the convolution product of two point masses to be a probability measure we must have $a + b = a' + b' = p + p' = q + q' = 1$.

Remark 2.1.1 *Since the convolution is associative we have*

$$(\delta_x * \delta_x) * \delta_x = \delta_x * (\delta_x * \delta_x)$$

But

$$(\delta_x * \delta_x) * \delta_x = a\delta_x * \delta_x + b\delta_x * \delta_y$$

and

$$\delta_x * (\delta_x * \delta_x) = a\delta_x * \delta_x + b\delta_y * \delta_x$$

so that we have

$$b\delta_x * \delta_y = b\delta_y * \delta_x$$

and if $b \neq 0$ *the convolution is commutative. In a similar way, we can show that if* $b' \neq 0$ *the convolution is commutative. So we can only expect commutativity when either* b *or* b' *is non-zero.*

To have associativity the following relations also hold

$$(\delta_x * \delta_y) * \delta_x = \delta_x * (\delta_y * \delta_x)$$

$$(\delta_x * \delta_x) * \delta_y = \delta_x * (\delta_x * \delta_y)$$

$$(\delta_y * \delta_x) * \delta_y = \delta_y * (\delta_x * \delta_y)$$

$$(\delta_y * \delta_y) * \delta_x = \delta_y * (\delta_y * \delta_x)$$

which are respectively equivalents to the systems

$$\begin{cases} bp = bq' \\ aq' + pq = ap + p'q' \end{cases} \tag{2.1}$$

$$\begin{cases} pp' = bb' \\ bp + p'^2 = ap' + ba' \end{cases} \tag{2.2}$$

$$\begin{cases} b'p' = b'q \\ a'q + p'q' = a'p' + pq \end{cases} \tag{2.3}$$

$$\begin{cases} bb' = qq' \\ ab' + a'q' = q'^2 + b'q \end{cases} \tag{2.4}$$

which are all equivalent to the relation $bb' = pp' = qq'$

Proposition 2.1.1 *If a two-element semihypergroup is not commutative, then it is a semigroup.*

Proof

From the remark above if either b or b' is non-zero the semihypergroup is commutative. Now let us assume that b and b' are both zero. Then, associativity of convolution implies that $pp' = qq' = 0$ So that the semihypergroup is actually a semigroup since one of p, p' is zero and one of q, q' is zero. Thus we most have $\delta_x * \delta_x = \delta_x, \delta_y * \delta_y = \delta_y, \delta_x * \delta_y = \delta_y, \delta_y * \delta_x = \delta_x$ which is a non commutative semigroup.

Remark

We have observed above that unless a two-element semihypergroup is a semigroup, it is commutative. We easily observe also that if $X = \{e, x\}$ is a two element hypergroup, then it is commutative. In fact the convolution product will be defined by

$$\delta_x * \delta_x = t\delta_e + (1 - t)\delta_x$$

Where t is any positive real number, e is the identity element and the involution is the identity function. X is a hermitian hypergroup. We now prove the same result for three and four-element hypergroups.

2.1.2 Three-element hypergroups

Let $H = \{e, x, y\}$ be a three-element hypergroup with identity element e. If H is Hermitian then it is commutative. Now lets assume that the involution on H is defined by $x^- = y$ and $y^- = x$. The involution property $(\delta_x * \delta_y)^- = \delta_{y^-} * \delta_{x^-}$ implies that the convolution of point masses are defined by

$$\delta_x * \delta_x = a\delta_x + b\delta_y$$

$$\delta_y * \delta_y = b\delta_x + a\delta_y$$

$$\delta_x * \delta_y = m\delta_e + p(\delta_x + \delta_y)$$

$$\delta_y * \delta_x = m'\delta_e + p'(\delta_x + \delta_y)$$

Where m, m', a, b, p, p' are nonnegative real numbers. Since H so define is a hypergroup, the convolution of two point masses is a probability measure, so that $a + b = m + 2p = m' + 2p' = 1$ with $mm' \neq 0$ The associativity axiom leads to the following system of equations.

$$\begin{cases} bp = bp' \\ m'p = mp' \\ m + ap + pp' = m' + ap' + pp' \end{cases} \tag{2.5}$$

$$\begin{cases} b^2 = m + p^2 \\ am = mp \\ ap + ab = pb + p^2 \end{cases} \tag{2.6}$$

And the dual systems

$$\begin{cases} bp' = pb \\ mp' = m'p \\ m' + pp' + ap' = m + pp' + ap \end{cases} \tag{2.7}$$

$$\begin{cases} b^2 = m' + p'^2 \\ am' = m'p' \\ ap' + ab = p'b + p'^2 \end{cases} \tag{2.8}$$

Since $mm' \neq 0$ and p, q, p', q' are nonnegative, $b \neq 0$ and it follows that $p = p'$ so that $m = m'$. Therefore H is commutative.

Remark

This example also shows that there are non Hermitian finite hypergroups.

2.1.3 Four-element hypergroups

Let $H = \{e, x, y, z\}$ be a four-element hypergroup with identity element e. If H is Hermitian then it is commutative. Now lets assume that the involution on H is defined by $x^- = x$,$y^- = z$ and $z^- = y$. Then involution property $(\delta_x * \delta_y)^- = \delta_{y^-} * \delta_{x^-}$ implies that the convolution of point masses are defined by

$$\delta_x * \delta_x = a\delta_e + b\delta_x + c(\delta_y + \delta_z)$$

$$\delta_x * \delta_y = p\delta_x + q\delta_y + r\delta_z$$

$$\delta_x * \delta_z = s\delta_x + t\delta_y + u\delta_z$$

$$\delta_y * \delta_y = p'\delta_x + q'\delta_y + r'\delta_z$$

$$\delta_y * \delta_x = s\delta_x + u\delta_y + t\delta_z$$

$$\delta_y * \delta_z = a'\delta_e + b'\delta_x + c'(\delta_y + \delta_z)$$

$$\delta_z * \delta_z = p'\delta_x + r'\delta_y + q'\delta_z$$

$$\delta_z * \delta_x = p\delta_x + r\delta_y + q\delta_z$$

$$\delta_z * \delta_y = a''\delta_e + b''\delta_x + c''(\delta_y + \delta_z)$$

where , $a, b, c, a', b', c', a'', b'', c'', p, q, r, p', q', r', s, t, u$ are nonnegative real numbers. Since H so define is a hypergroup, the convolution of two point masses is a probability measure, so that

$$a + b + 2c = a' + b' + 2c' = a'' + b'' + 2c'' = p + q + r = p' + q' + r' = s + t + u = 1$$

with

$$aa'a'' \neq 0$$

From associativity property,

$$(\delta_x * \delta_y) * \delta_x = \delta_x * (\delta_y * \delta_x)$$

which implies

$$\begin{cases} pa = as \\ pb + qs + rp = bs + up + ts \\ pc + qu + r^2 = cs + uq + t^2 \\ pc + qt + rq = cs + ur + tu \end{cases} \tag{2.9}$$

Since $aa'a'' \neq 0$,$p = s$ so that (2.9) becomes

$$\begin{cases} p = s \\ pb + qp + rp = bp + up + tp \\ pc + qu + r^2 = cp + uq + t^2 \\ pc + qt + rq = cp + ur + tu \end{cases} \tag{2.10}$$

which is equivalent to

$$\begin{cases} p = s \\ qp + rp = up + tp \\ r^2 = t^2 \\ qt + rq = ur + tu \end{cases} \tag{2.11}$$

so that $r^2 = t^2 \iff r = t$ since r, t are nonnegative. but

$$p + q + r = s + t + u$$

and since $p = s$ and $r = t$ we have $q = u$ so that

$$\delta_x * \delta_y = \delta_y * \delta_x$$

and also

$$\delta_x * \delta_z = \delta_z * \delta_x$$

From associativity we also have

$$(\delta_y * \delta_z) * \delta_y = \delta_y * (\delta_z * \delta_y)$$

which leads to

$$\begin{cases} c'a'' = c''a' \\ pb' + c'p' + c'b'' = b''s + c''p' + c''b' \\ a' + b'q + c'q' + c'c'' = a'' + b''u + c''q' + c''c' \\ b'r + c'r' + c'c'' = b''t + c''r' + c'c'' \end{cases} \tag{2.12}$$

We also have

$$(\delta_y * \delta_y) * \delta_z = \delta_y * (\delta_y * \delta_z)$$

which leads to the equation

$$\begin{cases} q'a' = c'a' \\ p'c + q'b' + r'p' = b's + c'p' + c'b' \\ p't + c'q' + r'^2 = a' + b'u + c'q' + c'^2 \\ p'u + c'q' + r'q' = b't + c'r' + c'^2 \end{cases} \tag{2.13}$$

and

$$(\delta_z * \delta_z) * \delta_y = \delta_z * (\delta_z * \delta_y)$$

implies

$$\begin{cases} q'a'' = c'a'' \\ p'p + r'p' + q'b'' = b''p + c''b'' + c''p' \\ p'q + r'q' + q'c'' = b''r + c''^2 + c''r' \\ p'r + r'^2 + q'c'' = a'' + b''q + c''^2 + c''q' \end{cases} \tag{2.14}$$

and since $a'a'' \neq 0$, $q' \neq 0$ and we have $q' = c' = c''$ also from (2.9)but we know that $a' = a''$ $a' + b' + 2c' = a'' + b'' + 2c''$ so $b' = b''$ therefore

$$\delta_y * \delta_z = \delta_z * \delta_y$$

Hence, H is commutative.

2.2 Product Formula

In this section we present examples of hypergroups generated by simple functions, via some product formula . We also give an example of a hypergroup generated by a Sturm Liouville problem, namely, the Chebli-Trimèche hypergroup. We end with an example of a semihypergroup generated by certain partial differential operators on the space $X_n = [0, +\infty] \times [-n\pi, n\pi]$. We show that unless $n = 1$ the semihypergroup is not a hypergroup.

Definition 2.2.1 *Let $\{P_\lambda\}_{\lambda \in \mathbb{R}}$, be a family of orthogonal functions on the real interval I. We say that $\{P_\lambda\}_{\lambda \in \mathbb{R}}$ has a product formula if for each $s, t \in I$, there is a Borel measure $\sigma_{s,t}$ with $Supp(\sigma_{s,t}) \subset I$ such that*

$$\int P_\lambda d\sigma_{s,t} = P_\lambda(s) P_\lambda(t)$$

for every $\lambda \in \mathbb{R}$

Example 2.2.1

Let $X = [0, +\infty)$ and $\varphi_\lambda(x) = \cos \lambda x$, $\lambda \in [0, +\infty)$. Then we have the relation

$$\varphi_\lambda(x)\varphi_\lambda(y) = \frac{1}{2}[\varphi_\lambda(x+y) + \varphi_\lambda(x-y)]$$

for all $\lambda \in [0, +\infty)$. Since φ_λ is an even function, this relation is equivalent to

$$\varphi_\lambda(x)\varphi_\lambda(y) = \frac{1}{2}[\varphi_\lambda(x+y) + \varphi_\lambda(|x-y|)]$$

Let $\sigma_{x,y} = \frac{1}{2}[\delta_{x+y} + \delta_{|x-y|}]$. Then $\{\varphi_\lambda\}$ satisfies the product formula

$$\varphi_\lambda(x)\varphi_\lambda(y) = \int \varphi_\lambda(z)\sigma_{x,y}(dz)$$

Now given two Radon measures μ and ν on X we can define a convolution

$$\mu * \nu(f) = \int\int\int f(z)\sigma_{x,y}(z)\mu(dx)\nu(dy)$$

for all $f \in C_c(X)$. With this convolution, $M(X)$ is a Banach algebra [Tr97]. Notice that taking $\mu = \delta_x$ and $\nu = \delta_y$, this gives us:

$$\delta_x * \delta_y = \frac{1}{2}[\delta_{x+y} + \delta_{|x-y|}]$$

So that $(X, *)$ is a hypergroup with identity element 0, the involution here being the identity function.

2.2.1 Legendre Polynomial

The Legendre polynomials $\{P_n\}_{n\in\mathbb{N}_0}$ are orthogonal with respect to the Lebesgue measure on $I = [-1, 1]$ and are normalized by requiring that $P_n(1) = 1$. They also satisfy a product formula

$$P_n(x)P_n(y) = \int_I K(x,y,z)P_n(z)dz \qquad\qquad (-1 \le x, y \le 1)$$

with

$$K(x,y,z) = \begin{cases} \pi^{-1}(1-x^2-y^2-z^2+2xyz)^{-\frac{1}{2}} & \text{if } 1-x^2-y^2-z^2+2xyz > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Obviously $K(x,y,z) \ge 0$ and since $P_0(x) = 1$, it follows from the product formula that

$$\int_I K(x,y,z)dz = 1.$$

For $f, g \in L^1(I, dx)$ define

$$f * g(z) = \int_I \int_I K(x,y,z)f(x)g(y)dxdy$$

so that

$$\int_I (f*g)(x)P_n(x)dx = [\int_I f(x)P_n(x)dx].[\int_I g(x)P_n(x)dx],$$

and it follows that $(L^1, *)$ is a Banach algebra (with respect to the measure $P_n(x)dx$). The operation is easily extended to the point masses by defining

$$d(\delta_x * \delta_y)(z) = K(x,y,z)dz, \qquad\qquad (-1 < x, y < 1),$$

$$\delta_x * \delta_1 = \delta_x \quad \textit{and} \quad \delta_x * \delta_{-1} = \delta_{-x}, \quad (x \in I).$$

Now given two Radon measures μ and ν on I we can define a convolution

$$\mu * \nu(f) = \int_I \int_I \int_I f d(\delta_x * \delta_y)\mu(dx)\nu(dy)$$

for all $f \in C_c(I)$. Hirschman also discusses these constructions for ultraspherical polynomials $\{P_n^{(\alpha)}(x)\}$ that are orthogonal on I with respect to the measure $(1-x^2)^{\alpha-\frac{1}{2}}dx$; the Legendre polynomials are ultraspherical polynomials with $\alpha = \frac{1}{2}$. The polynomials $R_n^{(\alpha)}(x) = \frac{P_n^{(\alpha)}(x)}{P_n^{(\alpha)}(1)}$ are used in place of the Legendre polynomial $P_n(x)$. In that case

$$K(x,y,z) = \begin{cases} \frac{2^{1-2\alpha}(1-x^2-y^2-z^2+2xyz)^{\alpha-1}}{\Gamma^2(\alpha)[(1-x^2)(1-y^2)(1-z^2)]^{\alpha-\frac{1}{2}}} & \text{if } 1-x^2-y^2-z^2+2xyz > 0, \\ 0 & \text{otherwise.} \end{cases}$$

So for each $\alpha \geq -\frac{1}{2}$ Hirschman [Hi56a] obtains a measure algebra that we denote by $(I, *_\alpha)$. It is important to note that $*_\alpha$ is a distinct convolution for each $\alpha \geq -\frac{1}{2}$, hence a continuum of Banach algebras is built on the single Banach space $M(I)$. The algebraic structure does not depend on any arithmetic in the underlying space I.[See [Hi56a][Hi56b]]

2.2.2 Polynomial Hypergroups

Let p_n, q_n and r_n be three sequences of real numbers such that $p_n > 0, r_n \geq 0, q_{n+1} > 0, q_0 = 0$ and $p_n + q_n + r_n = 1$ for all $n \in \mathbb{N}$. The polynomials defined by $P_0 \equiv 1, P_1(x) = x$ and

$$xP_n(x) = q_nP_{n-1}(x) + r_nP_n(x) + p_nP_{n+1}(x)(n \geq 1) \tag{2.15}$$

are orthogonal polynomials on $[-1,1]$ with respect to some measure $d\Pi(x)$. If their linearization coefficients are nonnegative (i.e. for m,n we have

$$P_m(x)P_n(x) = \sum_{r=|m-n|}^{m+n} c(m,n,r)P_r(x)$$

with $c(m,n,r) \geq 0$ for all r),we can define an Hermitian hypergroup structure $(\mathbb{N}, *)$ on $\mathbb{N}$ with $e = 0$ and

$$\delta_m * \delta_n = \sum_{r=|m-n|}^{m+n} c(m,n,r)\delta_r. \tag{2.16}$$

It is called a polynomial hypergroup with parameters $(p_n), (q_n)$ and (r_n). The Haar measure is given by $\omega = \sum_{n=0}^{\infty} \omega(n)\delta_n$ with

$$\omega(n) = \frac{p_0 p_1 \ldots p_{n-1}}{q_1 q_2 \ldots q_n} \tag{2.17}$$

$(n \geq 1)(\omega(0) = 1)$.

The characters are the functions on $\mathbb{N} : n \longmapsto P_n(x)$ with $x \in [-1, 1]$ (see [BH95])

2.2.3 Kingman's Hypergroup

Consider a pair of independent random variables $\mathbf{X}, \mathbf{Y}$ in $\mathbb{R}^2$, with lengths X and Y, but with directions uniformly distributed. The sum $\mathbf{Z} = \mathbf{X} + \mathbf{Y}$ also has uniformly distributed direction, but its length $Z = |\mathbf{Z}|$ is a random number with the range $|X - Y| \leq Z \leq X + Y$. In general, if $\mathbf{X}$ and $\mathbf{Y}$ are independent random variables in $\mathbb{R}^2$ with uniformly distributed direction, but with lengths X and Y having probability distributions $\mu, \nu \in M_1(\mathbb{R}_+)$, then Z is a random variable in $\mathbb{R}_+$ with a probability distribution depending on μ and ν, denoted by $\mu \circ \nu$, and we write $Z = X \oplus Y$. The operation $\circ$ can be extended to all of $M(\mathbb{R}_+)$ so that $(M(\mathbb{R}_+), \circ)$ becomes a hypergroup measure algebra that is isometrically isomorphic to the subalgebra of the group convolution algebra $M(\mathbb{R}^2), *)$, consisting of the measures invariant with respect to rotations of the plane. The characters are indexed by $\mathbb{R}_+$ and given by $\phi_y(x) = J_0(xy)$ where J_α is the Bessel function of the first kind of order α. These satisfy a product formula that yields

$$\int_{\mathbb{R}_+} \phi_y d(\mu * \nu) = [\int_{\mathbb{R}_+} \phi_y d\mu][\int_{\mathbb{R}_+} \phi_y d\nu]$$

so that the useful substitute for the characteristic function of the random variable X is $\Phi_X(y) = \int_{\mathbb{R}_+} \phi_y d\mu$. The product formula for the Bessel functions also ensures the fundamental property of characteristic equations $\Phi_{X \oplus Y} = \Phi_X \Phi_Y$ when X and Y are independent random variables in $\mathbb{R}_+$. Kingman actually describes a continuum of Hermitian hypergroups $(\mathbb{R}_+, \circ_\alpha)$(of course , he never uses the word "hypergroup"). The identity element is 0, and the characters are given by

$$\phi_y(x) = \mathcal{J}_\alpha(yx) = 2^\alpha \Gamma(\alpha+1)(yx)^{-\alpha} J_\alpha(yx)$$

for $y \in \mathbb{R}_+$.

When $n = 2\alpha + 2$ is an integer, $(\mathbb{R}_+, \circ_\alpha)$ is isometrically isomorphic to the algebras of rotation invariant measures on $\mathbb{R}^n$. There is again no useful algebraic structure in the underlying spaces. Nevertheless Kingman is able to define random walk and Brownian motion, and obtain a law of large numbers, a central limit theorem, a recurrence theorem, and characterizations of infinitely divisible and stable distributions. When $n = 2\alpha + 2$ is an integer, all of this is an inheritance from the group structure on $\mathbb{R}^n$, but Kingman obtains his results for all real $\alpha \geq -\frac{1}{2}$ with no reference to the group case except for inspiration [see [Ki63]].

2.2.4 Chébli-Trimèche hypergroups

Let A be an increasing unbounded real valued function on $\mathbb{R}_+$ such that $A(0) = 0$. We suppose A differentiable, A'/A non-increasing on $\mathbb{R}_+^*$,

$$\lim_{x \to +\infty} A'(x)/A(x) = 2\rho \geq 0$$

and $A'(x)/A(x) = \alpha/x + B(x)$ in a neighborhood of zero, with $\alpha > 0, B$ an infinitely differentiable odd function on $\mathbb{R}$. Let us consider the operator

$$\Delta = \frac{d^2}{dx^2} + \frac{A'(x)}{A(x)} \frac{d}{dx}.$$

For every infinitely differentiable even function f on $\mathbb{R}$, the solution u on $\mathbb{R}_+ \times \mathbb{R}_+$ of the Cauchy hyperbolic problem $\Delta_x u = \Delta_y u$, with initial condition $u(x,0) = f(x)$ and $\frac{\partial}{\partial y}u(x,0) = 0$, can be written in the form

$$u(x,y) = \int_0^{+\infty} f(t)\sigma_{xy}(dt),$$

where $\sigma_{xy} \in M_1(\mathbb{R}_+)$ is a unique probability measure with support the interval $[|x - y|, x+y]$. Now let $\delta_x * \delta_y = \sigma_{xy}$, the involution be defined by $x^- = x$, and the identity element $e = 0$. Then $(\mathbb{R}_+, *)$ with the usual topology is called the Chébli-Trimèche hypergroup with function A. The Haar measure is $\omega(dx) = A(x)dx$ and characters are the functions $\varphi_\lambda(\lambda \in \mathbb{C})$ that are solutions of the eigenvalue problem

$$\Delta\varphi_\lambda = -(\lambda^2 + \rho^2)\varphi_\lambda, \qquad \varphi_\lambda(0) = 1, \quad \varphi'_\lambda(0) = 1$$

Moreover the dual $\hat{\mathbb{R}}_+$ consists of characters φ_λ with $\lambda \in \mathbb{R}_+ \cup i[0,\rho]$[see [BH95]].

2.2.5 Semihypergroups and Hypergroups associated with Partial differential operators

For fixed $n \in \mathbb{N}$ we denote

$$U_n = \begin{cases} (-n\pi, 0) \cup (0, n\pi), & \text{if } n \in \mathbb{N}, \\ \mathbb{R} & \text{if } n = 0. \end{cases}$$

$$V_n^{\mathbb{C}} = \begin{cases} \mathbb{Z}/n, & \text{if } n \in \mathbb{N}, \\ \mathbb{C} & \text{if } n = 0. \end{cases}$$

Let $X_n = [0, +\infty] \times \overline{U}_n$. Consider the following partial differential operators

$$D_1 = \frac{\partial}{\partial\theta}$$

$$D_2 = \frac{\partial^2}{\partial y^2} + [(2\alpha+1)\coth y + \tanh y]\frac{\partial}{\partial y} - \frac{1}{\cosh^2 y}\frac{\partial^2}{\partial \theta^2} + (\alpha+1)^2$$

where $(y,\theta) \in (0,+\infty) \times U_n$ and $\alpha \geq 0$. The unique solution of the system

$$\begin{cases} D_1 u = i\lambda u, & \lambda \in V_n^{\mathbb{C}} \\ D_2 u = -\mu^2 u, & \mu \in \mathbb{C} \\ u(0,0) = 1, \quad \frac{\partial u}{\partial y}(0,\theta) = 0 & \text{for all } \theta \in U_n \end{cases} \tag{2.18}$$

denoted by $\varphi_{\lambda,\mu}(y,\theta)$ is given by

$$\varphi_{\lambda,\mu}(y,\theta) = e^{i\lambda\theta}(\cosh y)^{\lambda}\varphi_\mu^{(\alpha,\lambda)}(y) \tag{2.19}$$

where $\varphi_\mu^{(\alpha,\lambda)}$ is a Jacobi function, that is, the unique solution of the equation

$$\begin{cases} \Delta_{\alpha,\lambda}\varphi_\mu^{(\alpha,\lambda)}(x) = -(\mu^2+\rho^2)\varphi_\mu^{(\alpha,\lambda)}(x) \\ \varphi_\mu^{(\alpha,\lambda)}(0) = 1, \quad \frac{d}{dx}\varphi_\mu^{(\alpha,\lambda)}(0) = 0, \end{cases} \tag{2.20}$$

where $\Delta_{\alpha,\lambda}$ is the Jacobi differential operator

$$\Delta_{\alpha,\lambda} = \frac{1}{A_{\alpha,\lambda}(x)}\frac{d}{dx}[A_{\alpha,\lambda}(x)\frac{d}{dx}]$$

with

$$A_{\alpha,\lambda}(x) = 2^{2\rho}(\sinh x)^{2\alpha+1}(\cosh x)^{2\lambda+1}$$

and $\rho = \alpha+\lambda+1$. The function $\varphi_{\lambda,\mu}$, $(\lambda,\mu) \in V_n^{\mathbb{C}} \times \mathbb{C}$, satisfies the following product formulas

i. if $\alpha > 0$, then for all $(y,\theta),(t,\phi) \in X_n$

$$\varphi_{\lambda,\mu}(y,\theta)\varphi_{\lambda,\mu}(t,\phi) = \frac{\alpha}{\pi}\int_D \varphi_{\lambda,\mu}[\cosh y\cosh t e^{i(\theta+\phi)} + \sinh y \sinh t\xi](1-|\xi|^2)^{\alpha-1}dm(\xi)$$

where D is the open unit disc of $\mathbb{C}$ center at 0 and $dm(\xi_1+i\xi_2) = d\xi_1 d\xi_2$.

ii. If $\alpha = 0$, then for all $(y,\theta),(t,\phi) \in X_n$

$$\varphi_{\lambda,\mu}(y,\theta)\varphi_{\lambda,\mu}(t,\phi) = \frac{1}{2\pi}\int_0^{2\pi} \varphi_{\lambda,\mu}[\cosh y \cosh t e^{i(\theta+\phi)} + \sinh y \sinh t e^{i\psi}]d\psi.$$

Now let $C_*(X_n)$ denote the space of continuous functions on $\mathbb{R}\times\overline{U}_n$ even with respect to the first variable and such that for $n \neq 0$, the function $\theta \longmapsto f(y,\theta)$ is $2n\pi$-periodic on $\mathbb{R}$. Let $f \in C_*(X_n)$, we defined the convolution of two point masses of X_n by

i. if $\alpha > 0$, for all $(y,\theta),(t,\phi) \in X_n$

$$\delta_{(y,\theta)} * \delta_{(t,\phi)}(f) = \frac{\alpha}{\pi}\int_D f[\cosh y \cosh t e^{i(\theta+\phi)} + \sinh y \sinh t\xi](1-|\xi|^2)^{\alpha-1}dm(\xi)$$

where D is the open unit disc of $\mathbb{C}$ center at 0 and $dm(\xi_1 + i\xi_2) = d\xi_1 d\xi_2$.

ii. If $\alpha = 0$, then for all $(y,\theta),(t,\phi) \in X_n$

$$\delta_{(y,\theta)} * \delta_{(t,\phi)}(f) = \frac{1}{2\pi}\int_0^{2\pi} f[\cosh y \cosh t e^{i(\theta+\phi)} + \sinh y \sinh t e^{i\psi}]d\psi$$

The convolution of point masses has the following properties:

i. For all $\theta \in U_n, (t,\phi) \in X_n$

$$\delta_{(0,\theta)} * \delta_{(t,\phi)} = \delta_{(t,\theta+\phi)}$$

ii. for all $(y,\theta),(t,\phi) \in X_n$

$$\delta_{(y,\theta)} * \delta_{(t,\phi)} = \delta_{(t,\phi)} * \delta_{(y,\theta)}$$

iii. For all $(y,\theta) \in X_n$

$$\delta_{(y,\theta)} * \delta_{(0,0)} = \delta_{(y,\theta)}$$

So $\delta_{(0,0)}$ is the identity element.

Remark 2.2.1

Let W_α, $\alpha > 0$ be the function defined on $X_n \times X_n \times \mathbb{C}$ by

$$W_\alpha((y,\theta),(t,\phi),\bar{z}) =$$

$$\begin{cases} \frac{\alpha}{2^{2(\alpha+1)}\pi}[1-\cosh^2 y - \cosh^2 t - |z|^2 + 2Re(\cosh y \cosh t e^{i(\theta+\phi)}\bar{z}]^{\alpha-1}, & \text{if } z \in D_{(y,\theta),(t,\phi)}, \\ \\ 0 & \text{if } z \notin D_{(y,\theta),(t,\phi)}. \end{cases}$$

where $D_{(y,\theta),(t,\phi)}$ is the disc of $\mathbb{C}$ centered at $\cosh y \cosh t e^{i(\theta+\phi)}$ with radius $\sinh y \sinh t$. Then for all $(y,\theta),(t,\phi) \in X_n$

i. if $\alpha > 0$

$$\delta_{(y,\theta)} * \delta_{(t,\phi)}(f) = \int_{D_{(y,\theta),(t,\phi)}} f(z) W_\alpha((y,\theta),(t,\phi),\bar{z}) dm_n(z)$$

where $dm_n(z) = 2^{2(\alpha+1)}(x^2+y^2-1)^\alpha dxdy$ if $z = x+iy$

ii. if $\alpha = 0$

$$\delta_{(y,\theta)} * \delta_{(t,\phi)}(f) = \int_{C_{(y,\theta),(t,\phi)}} f(z) W_0((y,\theta),(t,\phi),dz))$$

where $C_{(y,\theta),(t,\phi)}$ is the disc of $\mathbb{C}$ centered at $\cosh y \cosh t e^{i(\theta+\phi)}$ with radius $\sinh y \sinh t$ and $W_0((y,\theta),(t,\phi),dz))$ the measure given by

$$W_0((y,\theta),(t,\phi),dz)) = \frac{dz}{z - \cosh y \cosh t e^{i(\theta+\phi)}}$$

Remark 2.2.2

For all $(y,\theta),(t,\phi) \in X_n$ satisfying $y,t \neq 0$ we have

i. The function $W_\alpha((y,\theta),(t,\phi),\bar{z})$ is positive and we have

$$\int_{D_{(y,\theta),(t,\phi)}} W_\alpha((y,\theta),(t,\phi),\bar{z}) dm_n(z) = 1$$

ii. we also have

$$\int_{C_{(y,\theta),(t,\phi)}} W_0((y,\theta),(t,\phi),dz)) = 1$$

iii. The measure dm_n can be written for $z = \cosh re^{i\omega}$ with $r \geq 0$, $\omega \in U_n$, in the form

$$dm_n(r,\omega) = 2^{2(\alpha+1)}(\sinh r)^{2\alpha+1}\cosh r dr d\sigma_n(\omega)$$

where

$$d\sigma_n(\omega) = \begin{cases} \frac{1}{2n\pi}d\omega, & \text{if } n \in \mathbb{N}, \\ \\ d\omega & \text{if } n = 0. \end{cases}$$

We now state the following theorem [Tr97],[Si95]

Theorem 2.2.1 *With convolution defined by*

$$\mu * \nu(f) = \int_{X_n}\int_{X_n} \delta_{(y,\theta)} * \delta_{(t,\phi)}(f)\mu(d(y,\theta)\nu(d(t,\phi)$$

for all $\mu, \nu \in M(X_n)$ *and for all* $f \in C_b(X_n)$, $(M(X_n, *)$ *is a commutative Banach algebra with identity element* $\delta_{(0,0)}$ *and with an involution define on* X_n *by* $(y,\theta)^- = (y,-\theta)$ *and the Haar measure is* m_n.

The next proposition is from [Tr97][Si95]

Proposition 2.2.1 *For all* $(y,\theta),(t,\phi) \in X_n$, *we have*

i.

$$Supp(\delta_{(y,\theta)} * \delta_{(t,\phi)}) = \begin{cases} \{(r,s) \in X_n : \cosh re^{is} \in D_{(y,\theta),(t,\phi)}\}, & \text{for } \alpha > 0, \\ \\ \{(r,s) \in X_n : \cosh re^{is} \in C_{(y,\theta),(t,\phi)}\} & \text{otherwise.} \end{cases}$$

ii. $Supp(\delta_{(y,\theta)} * \delta_{(t,\phi)})$ *is compact if and only if* $n \in \mathbb{N}$.

iii. $(0,0) \in Supp(\delta_{(y,\theta)} * \delta_{(t,\phi)})$ *if and only if* $y = t$ *and* $\cos(\theta + \phi) = 1$

Remark 2.2.3 **i.** *If* $n = 0$, $(X_0, *)$ *is not a semihypergroup because the support of the convolution of two point masses is not compact.*

ii. *If* $n = 1$ *then* $(X_1, *)$ *is a hypergroup (called the hypergroup of the exterior of the unit disc)*

iii. *If* $n \geq 2$, *then* $(X_n, *)$ *is a semihypergroup which is not a hypergroup. In fact all the axioms of a hypergroup are satisfied except the property*

$(0, 0) \in Supp(\delta_{(y,\theta)} * \delta_{(t,\phi)})$ *if and only if* $(y, \theta) = (t, \phi)^-$. *In fact from proposition 2.2.1(iii)*,$(0, 0) \in Supp(\delta_{(y,\theta)} * \delta_{(t,\phi)})$ *if and only if* $y = t$ *and* $\cos(\theta + \phi) = 1$, *that is* $\theta + \phi = 2n\pi$ *for all* $n \in \mathbb{N}$ *so for* $n = 1$ $\theta + \phi = 0$ *and for* $n \geq 2$, *we can have* $\theta + \phi \neq 0$.

3 Topological Semihypergroup

3.1 Introduction

As observed in the previous chapter, a semihypergroup does not assume any algebraic operation on itself. So to be able to work on semihypergroups, we induced from the convolution defined on its measure algebra an algebraic operation, which enables us to generalize many results from semigroups to semihypergroups. We set down in this section basic results necessary to do harmonic analysis or probability theory on semihypergroups.

3.2 Preliminaries

Definition 3.2.1 *1. An element $e \in S$ is called a left (right) identity element of S if $\delta_e * \delta_x = \delta_x$ ($\delta_x * \delta_e = \delta_x$) for every $x \in S$. An element e is called a two sided identity of S or simply an identity of S, if it is both a left and right identity. The identity, when it exists, is unique.*

2. An element $z \in S$ is called a left(right) zero element of S if $\delta_z * \delta_x = \delta_z$ ($\delta_x * \delta_z = \delta_z$) for all $x \in S$. If z is both left and right zero, we simply call it the zero of S. A semihypergroup has at most one zero.

3. An element $a \in S$ is called an idempotent element of S if $\delta_a * \delta_a = \delta_a$

Remark 3.2.1 *The only idempotent element in a hypergroup is the identity element. For if there is an idempotent element, its point mass would be an idempotent measure* *and its support a singleton subhypergroup ([Je75] 10.2E).*

Definition 3.2.2 *Let $(S, *)$ be a semihypergroup*

1. *If $x \in S$ and A, B are subsets of S we define*

$$Ax = \bigcup_{y \in A} Supp(\delta_y * \delta_x)$$

$$xA = \bigcup_{y \in A} Supp(\delta_x * \delta_y)$$

$$A * B = \bigcup_{x \in A, y \in B} Supp(\delta_x * \delta_y)$$

Remark 3.2.2 *A closed nonempty subset F of S can be verified to be a subsemihypergroup of S if and only if $F * F \subset F$*

The next lemma is from [Je75]

Lemma 3.2.1 *Let S be a semihypergroup and $A, B, C \subset S$. Then*

i. $\bar{A} * \bar{B} \subset \overline{A * B}$.

ii. *If A and B are compact then $A * B$ is compact*

iii. *Convolution is a continuous operation on $\mathcal{C}(S)$*

iv. *If A and B are compact and U is an open set containing $A * B$, then there exist open sets V and W such that $A \subset V$, $B \subset W$ and $V * W \subset U$*

v. $(A * B) * C = A * (B * C)$

*$\mathcal{C}(S)$ with * so defined is a topological semigroup.*

Remark 3.2.3 *The following remark is from [Je75]*

1. *If $\{x_\beta\}$ is a net in a hypergroup S, then the expression $x_\beta \to \infty$ means that $x_\beta \in S - A$ eventually for each compact subset A of S.*

2. *If $\{A_\beta\}$ is a net in $\mathcal{C}(S)$, then the expression $A_\beta \to \{\infty\}$ means that $A_\beta \subset S - A$ eventually for each compact subset A of S.*

Note that $A_\beta \to \infty$ and $A_\beta \to \{\infty\}$ have different meanings.

The next lemma is stated without proof in [Je75]. We give here a detailed proof.

Lemma 3.2.2 *[Jewett]*

If H is a hypergroup and A, B, C are subsets of H, then

i. *$e \in A^- * B$ if and only if $A \cap B \neq \emptyset$; also $e \in A * B^-$ if and only if $A \cap B \neq \emptyset$*

ii. *$(A * B) \cap C \neq \emptyset$ if and only if $B \cap (A^- * C) \neq \emptyset$ if and only if $A \cap (C * B^-) \neq \emptyset$*

iii. *If B is open, then $A * B$ is open and $\bar{A} * B = A * B$*

iv. *If A is compact and B is closed, then $A * B$ is closed.*

Proof

i. Suppose $e \in A^- * B$. Then there exists $x \in A$ and $y \in B$ such that $e \in Supp(\delta_{x^-} * \delta_y)$ which implies $x = y$ (from SH_6), so $A \cap B \neq \emptyset$. Now if $A \cap B \neq \emptyset$ then there exists $x \in A \cap B$, and so $e \in Supp(\delta_{x^-} * \delta_x)$. Therefore, $e \in A^- * B$

ii. $(A * B) \cap C \neq \emptyset$ if and only if $e \in (A * B)^- * C$ if and only if $e \in B^- * (A^- * C)$ if and only if $B \cap (A^- * C) \neq \emptyset$ if and only if $e \in B * (C^- * A) = (B * C^-) * A$ if and only if $A \cap (C * B^-) \neq \emptyset$

iii. Suppose B is open. Let $a \in A$, then $x \in \{a\} * B$ if and only if $B \cap \{a^-\} * \{x\} \neq \emptyset$ (from ii above). Since the map $x \longmapsto \{a^-\} * \{x\}$ is continuous (from SH_4), the set $\mathcal{C}_B(H)$ is an open set in the Michael topology which contains $\{a^-\} * \{x\}$ (because $B \cap \{a^-\} * \{x\} \neq \emptyset$ and $\{a^-\} * \{x\} \subset H$) so its inverse image by the continuous function $x \longmapsto \{a^-\} * \{x\}$ is open, which is, $\{y \in H : \{a^-\} * \{y\} \cap B \neq \emptyset\} = \{a\} * B$. Thus $\{a\} * B$ is an open subset of H.

iv. Let (x_n) be a sequence of elements of $A*B$ converging to an element $x \in S$. Then there are sequences $(a_n) \subset A$ and $(b_n) \subset B$ such that $x_n \in \{a_n\} * \{b_n\}$ for each n. This is equivalent to $b_n \in \{a_n^-\} * \{x_n\}$ for each n (from (ii) above see also the remark (i) below). Since A is compact, the sequence (a_n) has a convergent subsequence say, (a_k) such that $b_k \in \{a_k^-\} * \{x_n\}$ for each k. Furthermore (a_k^-) and (x_k) are relatively compact (As convergent sequences). So (b_k) has a convergent subsequence converging to a point $b \in B$ (since B is closed). Now from SH_4 if $a_k \longrightarrow a \in A$ then $\{a_k^-\} * \{x_k\} \longrightarrow \{a^-\} * \{x\}$. So $b \in \{a^-\} * \{x\}$ (since $b_k \in \{a_k^-\} * \{x\}$ for all k). And again from (ii) above $b \in \{a^-\} * \{x\}$ if and only if $x \in \{a\} * \{b\} \subset A * B$. Thus $A * B$ is closed.

Remark 3.2.4 **i.** *From (ii) above we also have $z \in Supp(\delta_x * \delta_y)$ if and only if $y \in Supp(\delta_{x^-} * \delta_z)$ if and only if $x \in Supp(\delta_z * \delta_{y^-})$*

ii. *If H is a compact hypergroup, then $H * A = A * H = H$ for all $A \in \mathcal{C}(H)$ so H is the zero of $(\mathcal{C}(H), *)$. But if H is not compact, then H is not an element of $\mathcal{C}(H)$.This result is not always true for compact semihypergroups as we will see below with the definition of ideals in semihypergroups.*

Definition 3.2.3 *1. A **homomorphism of semihypergroups** is defined via measure algebra as follows: Let S and T be two semihypergroups. A mapping ϕ from S into T is called a semihypergroup homomorphism if and only if $\phi : (M_1(S), *) \longrightarrow (M_1(T), \bullet)$ is a semigroup homomorphism. That is, $\phi(\mu * \nu) = \phi(\mu) \bullet \phi(\nu)$, $\forall \mu, \nu \in M_1(S)$, such that $\phi(\delta_x)$ is a point mass in $M_1(T)$, $\forall x \in S$. If in addition ϕ is one to one and onto, it is referred to as an isomorphism.*

*2. **Product of semihypergroups**. Let $(S, *), (T, \bullet)$ be two semihypergroups. The set $S \times T$ with the product topology is a locally compact space, and this can be made into a semihypergroup by defining*

$$\delta_{(x,y)} \circ \delta_{(s,t)} = \delta_x * \delta_s \otimes \delta_y \bullet \delta_t$$

where $(x, y), (s, t) \in S \times T$ and $\delta_{(x,y)} \circ \delta_{(s,t)}$ is a product measure on $S \times T$.

Definition 3.2.4 *Let S be a locally compact semihypergroup. The center of S is defined by $Z(S) = \{x \in S : Supp(\delta_x * \delta_y)$ is a singleton, for all $y \in S\}$*

Remark 3.2.5 *For a hypergroup H the center is the maximum subgroup defined by Jewett as $Z(H) = \{x \in H : \delta_x * \delta_{x^-} = \delta_{x^-} * \delta_x = \delta_e\}$ To see this, suppose that $\delta_x * \delta_{x^-} = \delta_{x^-} * \delta_x = \delta_e$and let $y \in H$ be arbitrarily chosen, and assume that $a, b \in Supp(\delta_x * \delta_y)$ then since $a \in \{x\} * \{y\}$ from lemma (3.2.2) 1.1.7(ii) $\{x\} * \{y\} \cap \{a\} \neq \emptyset$ which is equivalent to $y \in \{x^-\} * \{a\}$; similarly, $y \in \{x^-\} * \{b\}$ which means that $\{x^-\} * \{a\} \cap \{x^-\} * \{b\} \neq \emptyset$ and this is equivalent to $\{a\} \cap \{x\} * \{x^-\} * \{b\} \neq \emptyset$ and since $\delta_x * \delta_{x^-} = \delta_{x^-} * \delta_x = \delta_e$ it follows that $\{a\} \cap \{b\} \neq \emptyset$ that is $a = b$ so that $Supp(\delta_x * \delta_y)$ is a singleton, for all $y \in H$*

Conversely suppose an element x is such that $Supp(\delta_x * \delta_y)$ is a singleton, for all $y \in H$ then $Supp(\delta_x * \delta_{x^-})$is a singleton and since by definition it contains e we have $\delta_x * \delta_{x^-} = \delta_e$

Example 3.2.1

i. Every semigroup is a semihypergroup and its center is the entire semigroup. Also every group is a hypergroup which is the maximum subgroup(equivalently the center) of itself.

ii. If H is a hypergroup, then $e \in H$ so the center of a hypergroup is nonempty. When $Z(H) = \{e\}$, the center is said to be trivial.

iii. Let $S = \{x, y\}$ with convolution defined by

$$\delta_x * \delta_x = \delta_y$$

$$\delta_y * \delta_y = \frac{1}{4}\delta_x + \frac{3}{4}\delta_y$$

$$\delta_x * \delta_y = \delta_y * \delta_x = \frac{1}{2}\delta_x + \frac{1}{2}\delta_y$$

from example 1.4.1(ii) S is a semihypergroup with a void center

iii. Consider the segment $[0,1]$ with convolution defined by

$$\delta_r * \delta_s = \frac{1}{2}\delta_{|r-s|} + \frac{1}{2}\delta_{1-|1-r-s|}$$

for all $r, s \in [0,1]$ Zeuner [Ze89] proved that $([0,1], *)$ is a hypergroup with a nontrivial center $\{0,1\}$

3.3 Ideals of semihypergroups

Definition 3.3.1 *(Ideals)*

1. *A subsemihypergroup L (R) of a semihypergroup S is called a left (right) ideal of S if $S * L \subset L$ $(R * S \subset R)$; I is called an ideal of S if and only if it is both a right and left ideal.*

2. *S is called, left (right) simple if it contains no proper left (right) ideal. S is said to be simple if it contains no proper ideal. A left (right) ideal is said to be a principal left (right) ideal if it is of the form $\{a\} \cup Sa$ ($\{a\} \cup aS$)for some $a \in S$ (Recall that we write Sa to mean $S * \{a\}$.*

3. *$\forall a, b \in S$ we say that the equation $xa = b$ is **solvable** if and only if there exists $x_0 \in S$ such that $b \in Supp(\delta_{x_0} * \delta_a)$*

Proposition 3.3.1 *S is left simple if and only if $\forall a, b \in S$ the equation $xa = b$ is solvable.*

Proof:

First, assume S is left simple. Then $\forall a \in S$, Sa is a left ideal of S and since S is left simple $S = Sa$ and it follows that $\forall b \in S$, $\exists x_0 \in S$ such that $b \in Supp(\delta_{x_0} * \delta_a)$ so $xa = b$ is soluble. Now assume that $xa = b$ is soluble for all $a, b \in S$, and L is a left ideal of S. Then given $a \in L, Sa \subset L$. Also given $b \in S$ the equation $xa = b$ is soluble so $\exists x_0 \in S$ such that $b \in Supp(\delta_{x_0} * \delta_a)$ which is a subset of Sa, so $S \subset Sa \subset L$ therefore $S = L$ and so S is left simple. We can also make a similar argument for right ideals.

Remark 3.3.1 **i.** *Every left (right) ideal contains a left (right) ideal of the form Sa (aS) for some $a \in S$. For if L is a left ideal then for any $a \in L$, Sa is a left ideal contain in L. A similar statement holds for right ideals.*

ii. *A semihypergroup can be left and right simple without being a hypergroup. An example is the following semihypergroup. Let $S = \{x, y\}$ with convolution defined by*

$$\delta_x * \delta_x = \delta_y$$

$$\delta_y * \delta_y = \frac{1}{4}\delta_x + \frac{3}{4}\delta_y$$

$$\delta_x * \delta_y = \delta_y * \delta_x = \frac{1}{2}\delta_x + \frac{1}{2}\delta_y$$

From example 1.1.4(ii) S so defined is a semihypergroup with no proper ideal but is not a hypergroup since it has no identity element.

Definition 3.3.2 *1. An idempotent element in a semihypergroup S is said to be **a primitive idempotent element** if it is in the center of the semihypergroup and is minimal with respect to the partial order $\leq$ on $E(S)$ (the set of idempotent elements of S), defined by*

$$e \leq f \Longleftrightarrow \delta_e * \delta_f = \delta_f * \delta_e = \delta_e$$

*2. **A completely simple semihypergroup** is a simple semihypergroup which contains a primitive idempotent element.*

Remark 3.3.2 *The order defined on $E(S)$ uses convolution of point masses to compare idempotent elements of S. Note that if a is a primitive idempotent of S, δ_a is not* *necessarily a primitive idempotent in $M_1(S)$, according to the definition of primitive idempotents in the semigroup (with respect to convolution)$M_1(S)$.*

Lemma 3.3.1 *Let S be a compact semihypergroup. Then each left (right) ideal of S contains at least one minimal left (right) ideal, and each minimal left (right) ideal is principal that is, of the form Sa (aS) for some $a \in S$. Every compact semihypergroup has a minimal two sided ideal.*

Proof:

Given a left ideal I, define the family $\mathcal{F} = \{J : J \text{ is a left ideal of } S,\ J \subset I\}$. For any $a \in I$, Sa is a left ideal of S and hence an element of $\mathcal{F}$. The usual inclusion relation is a partial order on $\mathcal{F}$. Furthermore, any linearly ordered subfamily of $\mathcal{F}$ has a minimal element since sets in $\mathcal{F}$ are compact. By Zorn's lemma, there exists at least one minimal element (with respect to inclusion) in $\mathcal{F}$. Call this minimal left ideal I_0. Clearly, for any x in I_0, $Sx = I_0$.

Proposition 3.3.2 *Let S be a compact semihypergroup. Then S has a kernel K, that is, a minimal two sided ideal.*

Proof:

By the compactness of S, there is a minimal 2-sided ideal K_0. Let $K = \bigcup\{xS : xS$ is a minimal right ideal of $S\}$. Notice that $K_0 * S$ is also an ideal contains in K_0. and therefore, $K_0 = K_0 * S$. Therefore $K_0 \subseteq \bigcup\{yS : y \in K_0\}$. Since $xS * K_0 \subset K_0 \cap xS$, $K_0 \cap xS$ is a right ideal contains in xS therefore $xS \subseteq K_0$, and it follows that $K_0 = K$.

We will now define the Rees convolution product which will be used to construct a class of completely simple semihypergroups with non empty and infinite center.

3.4 Rees Convolution Product

Let $(H, *)$ be a hypergroup with center Z and X,Y be two nonempty sets. Let $\phi : Y \times X \longrightarrow Z$ be a mapping . Let us define a convolution on point masses of $X \times H \times Y$

by

$$\delta_{(x,h,y)} \bullet \delta_{(x',h',y')} = \delta_x \otimes (\delta_h * \delta_{\phi(y,x')} * \delta_{h'}) \otimes \delta_{y'}$$

This product will be referred to as the Rees convolution product.

Proposition 3.4.1 *If H is a hypergroup, and X and Y are two nonempty locally compact Hausdorff spaces, then the space $X \times H \times Y$ is a semihypergroup with the Rees convolution product, as defined above.*

Proof:

Let $K = X \times H \times Y$ and $(x, h, y), (x', h', y')$ be two points in K. Then

$$[\delta_{(x,h,y)} \bullet \delta_{(x',h',y')}](K) = [\delta_x \otimes (\delta_h * \delta_{\phi(y,x')} * \delta_{h'}) \otimes \delta_{y'}](K) =$$

$$\delta_x(X)[\delta_h * \delta_{\phi(y,x')} * \delta_{h'}(H)]\delta_{y'}(Y) = 1$$

Since $\delta_h * \delta_{\phi(y,x')}$ is a probability measure with compact support in H, $\delta_h * \delta_{\phi(y,x')} * \delta_{h'}$ is a probability measure with compact support in H and it follows that $\delta_x \otimes (\delta_h * \delta_{\phi(y,x')} * \delta_h') \otimes \delta_{y'}$ is a probability measure with compact support in K.

Next we have to show that $\bullet$ is associative. Let $(x, h, y), (x', h', y')$ and (x'', h'', y'') be three arbitrary elements of K then

$$[\delta_{(x,h,y)} \bullet \delta_{(x',h',y')}] \bullet \delta_{(x'',h'',y'')} = [\delta_x \otimes (\delta_h * \delta_{\phi(y,x')} * \delta_{h'}) \otimes \delta_{y'}] \bullet \delta_{(x'',h'',y'')} =$$

$$\delta_x \otimes ((\delta_h * \delta_{\phi(y,x')} * \delta_{h'}) * \delta_{\phi(y',x'')} * \delta_{h''}) \otimes \delta_{y''}$$

And

$$\delta_{(x,h,y)} \bullet [\delta_{(x',h',y')} \bullet \delta_{(x'',h'',y'')}] = \delta_{(x,h,y)} \bullet [\delta_{x'} \otimes \delta_{h'} * \delta_{\phi(y',x'')} * \delta_{h''} \otimes \delta_{y''} =$$

$$\delta_x \otimes (\delta_h * \delta_{\phi(y,x')} * (\delta_{h'} * \delta_{\phi(y',x'')} * \delta_{h''})) \otimes \delta_{y''}$$

And we can easily see that

$$[\delta_{(x,h,y)} \bullet \delta_{(x',h',y')}] \bullet \delta_{(x'',h'',y'')} = \delta_{(x,h,y)} \bullet [\delta_{(x',h',y')} \bullet \delta_{(x'',h'',y'')}]$$

This shows that $(K, \bullet)$ is a semihypergroup.

Up to this point we have considered $\phi : Y \times X \longrightarrow H$ and have not used the fact that ϕ maps $Y \times X$ into Z, the center of H. We will require this in what follows.

Lemma 3.4.1 *An element $(x, h, y) \in K$ is an idempotent element if and only if $h = \phi(y, x)^-$. Furthermore, idempotent elements of K are in its center.*

Proof:

Let (x, h, y) be an idempotent element of K. Then, we have :

$$\delta_{(x,h,y)} \bullet \delta_{(x,h,y)} = \delta_x \otimes \delta_h * \delta_{\phi(y,x)} * \delta_h \otimes \delta_y = \delta_x \otimes \delta_h \otimes \delta_y$$

That is,

$$\delta_h * \delta_{\phi(y,x)} * \delta_h = \delta_h$$

Multiplying both sides of the equality above by $\delta_{\phi(y,x)}$ on the left, we have

$$(\delta_{\phi(y,x)} * \delta_h) * (\delta_{\phi(y,x)} * \delta_h) = \delta_{\phi(y,x)} * \delta_h$$

This shows that $(\delta_{\phi(y,x)} * \delta_h)$ is an idempotent element of the hyperpgroup H and so is the identity of H, therefore, $h = \phi(y, x)^-$

We note here that if we did not assume that $\phi(y, x)$ was in the center of H this result will still hold as $(\delta_{\phi(y,x)} * \delta_h)$ will be considered an idempotent probability measure and so its support is a subhypergroup ([Je75] 10.2E) of H containing the identity so that $h = \phi(y,x)^-$, by axiom SH_6 in the definition of a hypergroup.

Next we need to show that $\forall x \in X$ and $y \in Y$ $(x, \phi(y,x)^-, y)$ is an idempotent element of K for

$$\delta_{(x,\phi(y,x)^-,y)} \bullet \delta_{(x,\phi(y,x)^-,y)} = \delta_x \otimes \delta_{\phi(y,x)^-} * \delta_{\phi(y,x)} * \delta_{\phi(y,x)^-} \otimes \delta_y =$$

$$\delta_x \otimes \delta_{\phi(y,x)^-} \otimes \delta_y = \delta_{(x,\phi(y,x)^-,y)}$$

since $\delta_{\phi(y,x)^-}$ is in the center of H(this is the first time we have used the center property of Z), let $(x, \phi(y,x)^-, y)$ and (x', h', y') be two arbitrary elements of K. Then,

$$\delta_{(x,\phi(y,x)^-,y)} \bullet \delta_{(x',h',y')} =$$

$$\delta_x \otimes \delta_{\phi(y,x)^-} * \delta_{\phi(y,x')} * \delta_{h'} \otimes \delta_{y'}$$

Notice that by the center property of Z, $\delta_{\phi(y,x)^-} * \delta_{\phi(y,x')} * \delta_{h'}$ is a point mass. Thus, $(x, \phi(y,x)^-, y)$ is in the center of K.

Proposition 3.4.2 *If H is a hypergroup, and X and Y are two nonempty locally compact Hausdorff spaces, then the semihypergroup $K = X \times H \times Y$ with the Rees convolution product, as defined above, is completely simple.*

Proof:

First, we need to show that K is simple. Let I be an ideal of K and let $(x, h, y) \in K$ be a point in K. We will show that $(x, h, y) \in I$ which shows that $K = I$. To do this, let (x_1, h_1, y_1) be any point of I. Then the support of the probability measure $\delta_{(x,h,y)} \bullet \delta_{(x_1,h_1,y_1)} \bullet \delta_{(x,h,y)}$ is a subset of I. We will prove that the point $(x, h, y) \in I$.

By definition of the convolution product on K

$$\delta_{(x,h,y)} \bullet \delta_{(x_1,h_1,y_1)} \bullet \delta_{(x,h,y)} =$$

$$\delta_x \otimes \delta_h * \delta_{\phi(y,x_1)} * \left(\delta_{h_1} * \delta_{\phi(y_1,x)} * \delta_h\right) \otimes \delta_y$$

And observe that

$$Supp(\delta_{(x,h,y)} \bullet \delta_{(x_1,h_1,y_1)} \bullet \delta_{(x,h,y)}) = \{x\} \times Supp(\delta_h * \delta_{\phi(y,x_1)} * (\delta_{h_1} * \delta_{\phi(y_1,x)} * \delta_h) \times \{y\}$$

Thus whenever $(x,h,y) \in K$,$\{x\} \times Supp(\delta_h * \delta_{\phi(y,x_1)} * (\delta_{h_1} * \delta_{\phi(y_1,x)} * \delta_h) \times \{y\} \subset I$ Since $\delta_{\phi(y,x_1)} * (\delta_{h_1} * \delta_{\phi(y_1,x)}) = \delta_k$ for some $k \in H$,we have $(x, k^-, y) \in \{x\} \times Supp(\delta_{k^-} * \delta_{\phi(y,x_1)} * (\delta_{h_1} * \delta_{\phi(y_1,x)}) * \delta_{k^-}) \times \{y\} \subset I$ for some $k \in H$. Now if $\delta_{k^-} * \delta_{\phi(y,x)} = \delta_u$, then $(x, u^-, y) \in K$ and $(x, e, y) \in \{(x, k^-, y)\} \bullet \{(x, u^-, y)\} \subset I$. Now for any $h \in H$, $(x, \{\phi(y,x)^-\} * \{h\}, y) \in K$ and we have

$$(x, e, y) \bullet (x, \{\phi(y,x)^-\} * \{h\}, y) = (x, h, y) \in I.$$

This shows that $I = K$, and thus K is simple.

Next we need to show that K contains a primitive idempotent element.

Now suppose $(x, \phi(y,x)^-, y)$ and $(x', \phi(y',x')^-, y')$ are two idempotent elements of K such that $(x, \phi(y,x)^-, y) \leq (x', \phi(y',x')^-, y')$ then

$$\delta_{(x,\phi(y,x)^-,y)} \bullet \delta_{(x',\phi(y',x')^-,y')} = \delta_{(x,\phi(y,x)^-,y)}$$

which is equivalent to

$$\delta_x \otimes \delta_{\phi(y,x)^-} * \delta_{\phi(y,x')} * \delta_{\phi(y',x')^-} \otimes \delta'_y = \delta_x \otimes \delta_{\phi(y,x)^-} \otimes \delta_y$$

so that $y' = y$

And

$$\delta_{(x',\phi(y',x')^-,y')} \bullet \delta_{(x,\phi(y,x)^-,y)} = \delta_{(x,\phi(y,x)^-,y)}$$

which is equivalent to

$$\delta_{x'} \otimes \delta_{\phi(y',x')^-} * \delta_{\phi(y',x)} * \delta_{\phi(y,x)^-} \otimes \delta_y = \delta_x \otimes \delta_{\phi(y,x)^-} \otimes \delta_y$$

so that $x' = x$. Combining these two results we see that $(x, \phi(y,x)^-, y) = (x', \phi(y',x')^-, y')$ That is $(x', \phi(y',x')^-, y')$ is a minimal idempotent element. And similarly we can show that $(x, \phi(y,x)^-, y)$ is a minimal idempotent element. So all idempotent elements of K are primitives, so K is a completely simple semihypergroup.

Remark 3.4.1 *If the operation $\bullet$ defined above is commutative then, X and Y are each a singleton set and in this case K is a hypergroup. This is proved in the corollary below.*

Corallory 3.4.1 *Let $(H, *)$ be a hypergroup and s, t two elements. Then $\{s\} \times H \times \{t\}$ with the Rees convolution product is a cell hypergroup with identity element $(s, \phi(t,s)^-, t)$ and the involution defined by $(s,h,t)^\vee = (s,h',t)$ if and only if*

$$\delta_{h'} = \delta_{\phi(t,s)^-} * \delta_{h^-} * \delta_{\phi(t,s)^-}$$

Proof:

First we need to show that $(s, \phi(t,s)^-, t)$ is the identity of $\{s\} \times H \times \{t\}$ Let $h \in H$ then

$$\delta_{(s,\phi(t,s)^-,t)} \bullet \delta_{(s,h,t)} =$$

$$\delta_s \otimes \delta_{\phi(t,s)^-} * \delta_{\phi(t,s)} * \delta_h \otimes \delta_t =$$

$$\delta_s \otimes \delta_h \otimes \delta_t$$

And since $\delta_{\phi(t,s)^-} * \delta_{\phi(t,s)}$ is the identity in H this equality hold.

Next we need to show that for all $h \in H$ $(s,h,t)^{\vee\vee} = (s,h,t)$, $(s, \phi(t,s)^-, t) \in Supp(\delta_{(s,h,t)} \bullet \delta_{(s,h',t)}$ if and only if $(s,h,t)^\vee = (s,h',t)$.

Suppose $(s,h,t)^\vee = (s,h',t)$ where

$$\delta_{h'} = \delta_{\phi(t,s)^-} * \delta_{h^-} * \delta_{\phi(t,s)^-}$$

Suppose also that $(s,h',t)^\vee = (s,h'',t)$ where

$$\delta_{h''} = \delta_{\phi(t,s)^-} * \delta_{h'^-} * \delta_{\phi(t,s)^-}$$

then

$$\delta_{h'^-} = \delta_{\phi(t,s)} * \delta_h * \delta_{\phi(t,s)}$$

So that

$$\delta_{h''} = \delta_{\phi(t,s)^-} * \delta_{\phi(t,s)} * \delta_h * \delta_{\phi(t,s)} * \delta_{\phi(t,s)^-} = \delta_h$$

So $h = h''$ and therefore $(s,h,t)^{\vee\vee} = (s,h,t)$

Next suppose $(s,\phi(t,s)^-,t) \in Supp(\delta_{(s,h,t)} \bullet \delta_{(s,h',t)}$, that is $\phi(t,s)^- \in \{h\} * \{\phi(t,s)\} * \{h'\}$ which is equivalent to $h' \in \{\phi(t,s)^-\} * \{h^-\} * \{\phi(t,s)^-\}$ but $\{\phi(t,s)^-\} * \{h^-\} * \{\phi(t,s)^-\}$ is a singleton as $\phi(t,s)^-$ is in the center of H so $\delta_{h'} = \delta_{\phi(t,s)^-} * \delta_{h^-} * \delta_{\phi(t,s)^-}$ which shows that $(s,h,t)^\vee = (s,h',t)$.

Now suppose $(s,h,t)^\vee = (s,h',t)$ then $\delta_{h'} = \delta_{\phi(t,s)^-} * \delta_{h^-} * \delta_{\phi(t,s)^-}$ which implies that $h' \in \phi(t,s)^- * \{h^-\} * \{\phi(t,s)^-\}$ which is equivalent to $\phi(t,s)^- \in \{h\} * \{\phi(t,s)\} * \{h'\}$ which shows that $(s,\phi(t,s)^-,t) \in \{s\} \times \{h\} * \{\phi(t,s)\} * \{h'\} \times \{t\}$ That is $(s,\phi(t,s)^-,t) \in Supp(\delta_{(s,h,t)} \bullet \delta_{(s,h',t)})$.

Next we need to show that

$$(\delta_{(s,h,t)} \bullet \delta_{(s,g,t)})^\vee = \delta_{(s,g,t)^\vee} \bullet \delta_{(s,h,t)^\vee}$$

Note that by the definition of involution on the $\{s\} \times H \times \{t\}$, if $\mu \in M(H)$ then

$$(\delta_s \otimes \mu \otimes \delta_t)^\vee = \delta_s \otimes \delta_{\phi(t,s)^-} * \mu^- * \delta_{\phi(t,s)^-} \otimes \delta_t$$

Now

$$(\delta_{(s,h,t)} \bullet \delta_{(s,g,t)})^{\vee} = (\delta_s \otimes \delta_h * \delta_{\phi(t,s)} * \delta_g \otimes \delta_t)^{\vee} =$$

$$\delta_s \otimes \delta_{\phi(t,s)^-} * (\delta_h * \delta_{\phi(t,s)} * \delta_g)^- \delta_{\phi(t,s)^-} \otimes \delta_t =$$

$$\delta_s \otimes \delta_{\phi(t,s)^-} * \delta_g^- * \delta_{\phi(t,s)^-} * \delta_h^- * \delta_{\phi(t,s)^-} \otimes \delta_t =$$

$$\delta_s \otimes \delta_{\phi(t,s)^-} * \delta_g^- * \delta_{\phi(t,s)^-} * \delta_{\phi(t,s)} * \delta_{\phi(t,s)^-} * \delta_h^- * \delta_{\phi(t,s)^-} \otimes \delta_t =$$

$$\delta_s \otimes \delta_{g'} * \delta_{\phi(t,s)} * \delta_{h'} \otimes \delta_t = \delta_{(s,g,t)^{\vee}} \bullet \delta_{(s,h,t)^{\vee}}$$

Which completes the proof.

4.1 Preliminary Results

In this section S will denote a locally compact Hausdorff second-countable semihypergroup. (Often assertions in various results of this section are valid in more general topological structures; however this is not pointed out explicitly). We recall that (from Banach-Alaoglu's theorem in functional analysis that the unit ball in the dual of $C_c(S)$ is weak* compact) the set

$B(S) \equiv \{\mu : \mu \in M(S)^{+}\text{with } \mu(S) \leq 1\}$

is compact in the weak* topology. Recall: A net (μ_α) in $B(S)$, w^* converges to μ in $B(S)$ if and only if for every f in $C_c(S)$, $\int f d\mu_\alpha \to \int f d\mu$. However, $P(S) \equiv \{\mu \in B(S) : \mu(S) = 1\}$ need not be weak* compact, unless S is compact. Note that in $P(S)$, weak* compactness is equivalent to weak compactness, and thus $P(S)$ is weak* compact if and only if S is compact. For a subset $\Gamma \subset P(S)$, the weak* closure of Γ in $P(S)$ is weak* compact, if Γ is tight; that is , given $\epsilon > 0$, there is a compact subset $K_\epsilon \subset S$ such that

$$\mu \in \Gamma \Rightarrow \mu(K_\epsilon) > 1 - \epsilon$$

The reason for this is obvious since $\mu \in w^*$-closure of Γ and Γ is tight only if $\mu \in P(S)$ and since $B(S)$ is w^*-compact.

Definition 4.1.1 *If f is a Borel function on S and $x, y \in S$, then we define*

$$f(x * y) \equiv f_x(y) \equiv f^y(x) = \int_S f d(\delta_x * \delta_y)$$

If this integral exists, even when it is not finite, f_x is called the left translation of f and f^x is called the right translation of f.

The next two lemmas are proved in [Je75]

Lemma 4.1.1 *Let f be a continuous function on S and let $x \in S$*

i. *The mapping $(x, y) \mapsto f(x * y)$ is a continuous function on $S \times S$*

ii. *f_x and f^x are continuous functions on S.*

Lemma 4.1.2 *Let $f \in B_\infty(S)$, $\mu, \nu \in M_+(S)$ and $x, y, z \in S$*

i. *The mapping $(x, y) \mapsto f(x * y)$ is a Borel function on $S \times S$*

ii. *f_x and f^x are Borel functions in S*

iii. $\int_S f d(\mu * \nu) = \int_S \int_S f(x * y)\mu(dx)\nu(dy)$

iv. $\int_S f_x d\mu = \int_S f d(\delta_x * \mu)$

v. $f_x(y * z) = f^z(x * y)$

Notation 4.1.1 *Let S be a locally compact semihypergroup.*

Then $\forall x \in S, \mu \in M_1(S)$, and $f \in C(S)$, we write:

$$\delta_x * \mu(f) = \int_S f_x d\mu$$

($\equiv \mu(f_x)$, say)

and also,

$$\mu * \delta_x(f) = \mu(f^x)$$

Definition 4.1.2 *Let S be a locally compact semihypergroup and B be a Borel subset of S. Then*

$$Bx^- = \{y \in S : Supp(\delta_y * \delta_x) \cap B \neq \emptyset\}$$

Similarly,

$$x^- B = \{y \in S : Supp(\delta_x * \delta_y) \cap B \neq \emptyset\}$$

Proposition 4.1.1 *Let B be a Borel subset of a semihypergroup S. Then for any $x \in S$, the sets Bx^- and $x^- B$ are also Borel subsets of S.*

Proof:

We only prove that Bx^- is Borel whenever B is Borel. To this end, first notice that if B is open, then we have:

$$Supp(\delta_y * \delta_x) \cap B \neq \emptyset$$

implies that $\delta_y * \delta_x(B) > 0$. Since the map $(x, y) \longmapsto \delta_y * \delta_x$ is a continuous map (with respect to weak topology in $M_1(S)$) by axiom SH_3, there is an open subset $N(y)$ containing y such that for each $y' \in N(y)$, $\delta_{y'} * \delta_x(B) > 0$. This means that

$$Supp(\delta_{y'} * \delta_x) \cap B \neq \emptyset$$

for each $y' \in N(y)$ so that $N(y) \subset Bx^-$; consequently, Bx^- is open whenever B is open. Let us now suppose that B is a closed subset of S. Let $x \in S$ and $y \in (Bx^-)^c$. Then we have:

$$Supp(\delta_y * \delta_x) \cap B = \emptyset$$

so that $Supp(\delta_y * \delta_x)$, which is compact, is contained in the open set B^c. Since by SH_4, the map $(x, y) \longmapsto \delta_y * \delta_x$ is continuous with respect to the product topology in the domain and the Michael topology for the compact subsets in the range, the set

$$\{y' : Supp(\delta_{y'} * \delta_x) \subset B^c\}$$

is an open set containing y; in other words, $(Bx^-)^c$ is open, and this means that Bx^- is closed whenever B is closed.

Now let us define the class $\mathcal{F}$ by

$\mathcal{F} = \{B : Bx^- \text{ is Borel whenever } B \text{ is Borel and } x \in S\}$. Then $\mathcal{F}$ contains all open and all closed subsets of S. Furthermore, if V is an open set and W is a closed set, then since S is locally compact Hausdorff second countable, there is a sequence $\{F_n\}$ of closed sets such that $V = \bigcup_{n=1}^{\infty} F_n$.

$$[(V \cap W)x^-]^c = \{y : Supp(\delta_y * \delta_x) \cap (V \cap W) = \emptyset\} =$$

$$\{y : Supp(\delta_y * \delta_x) \subset W^c\} \bigcup \{y : Supp(\delta_y * \delta_x) \cap W \subset V^c\} =$$

$$\{y : Supp(\delta_y * \delta_x) \subset W^c\} \bigcup [\bigcap_{n=1}^{\infty} \{y : Supp(\delta_y * \delta_x) \cap W \subset F_n^c\}] =$$

$$\{y : Supp(\delta_y * \delta_x) \subset W^c\} \bigcup [\bigcap_{n=1}^{\infty} \{\{y : Supp(\delta_y * \delta_x) \subset F_n^c \cup W^c\} - \{y : Supp(\delta_y * \delta_x) \subset W^c\}\}]$$

Now the mapping $\gamma : S \times S \longrightarrow \mathcal{C}(S) : (y, x) \longmapsto Supp(\delta_y * \delta_x)$ is continuous, and since the sets $\mathcal{C}_S(W^c)$, $\mathcal{C}_S(F_n^c \cup W^c)$ are open sets (in the Michael topology, $\gamma^{-1}(\mathcal{C}_S(W^c)) = \{y : Supp(\delta_y * \delta_x) \subset W^c\}$ and $\gamma^{-1}(\mathcal{C}_S(F_n^c \cup W^c)) = \{y : Supp(\delta_y * \delta_x) \subset F_n^c \cup W^c\}$ are open, so are Borel sets. It follows that $[(V \cap W)x^-]^c$ is a Borel set. Therefore, $(V \cap W)x^-$ is a Borel set.

This means that the algebra $\mathcal{A}$ (finite intersections and complements) generated by all open subsets of S is contained in $\mathcal{F}$. It is also clear that

$$(\bigcup_{n=1}^{\infty} B_n)x^- = \bigcup_{n=1}^{\infty} (B_n x^-)$$

whenever $B_n \in \mathcal{F}, n \geq 1$, and $x \in S$. This means that the monotone class generated by $\mathcal{A}$, which is a σ-algebra and which contains all Borel subsets of S, is contained in $\mathcal{F}$.

Lemma 4.1.3 *Let S be a locally compact semihypergroup, $B \subset S$ and $x \in S$. Then*

$$B \subset (Bx)x^-$$

Proof:

$(Bx)x^- = \{y \in S : Supp(\delta_y * \delta_x) \cap Bx \neq \emptyset\}$ Since $Bx = \bigcup_{b \in B} Supp(\delta_b * \delta_x)$, If $y \in B$, then $Supp(\delta_y * \delta_x) \subset Bx$; therefore, $Supp(\delta_y * \delta_x) \cap Bx \neq \emptyset$ so $y \in (Bx)x^-$, which implies $B \subset (Bx)x^-$

Lemma 4.1.4 *Let S be a locally compact semihypergroup and C be a compact subset of S. If $B \subset S$,*

$$(B - Cx)x^- \subset Bx^- - C$$

Proof:

If $y \in (B - Cx)x^-$ then $Supp(\delta_y * \delta_x) \cap (B - Cx) \neq \emptyset \Longrightarrow Supp(\delta_y * \delta_x) \cap B \neq \emptyset$ and $Supp(\delta_y * \delta_x) \cap (Cx)^c \neq \emptyset \Longrightarrow y \in Bx^-$ and $Supp(\delta_y * \delta_x)$ is not entirely in Cx that is $y \notin C$ (for if $y \in C$ then $Supp(\delta_y * \delta_x) \subset Cx) \Longrightarrow y \in Bx^- - C \Longrightarrow (B - Cx)x^- \subset (Bx^- - C)$.

The next lemma was proved for hypergroups in [BH95]. The same result holds for semihypergroups with the same proof which we reproduce here.

Lemma 4.1.5 *Let S be a locally compact space and $\mu \in M_1(S)$. Then $\forall x \in S$ and compact $C \subset S$*

$$\delta_x * \mu(C) \leq \mu(x^- C)$$

Proof:

By definition,

$$x^- C = \{y \in S : Supp(\delta_x * \delta_y) \cap C \neq \emptyset\}$$

So $y \in x^- C$ if and only if $Supp(\delta_x * \delta_y) \cap C \neq \emptyset$. Thus,

$$\delta_x * \mu(C) = \int_S \delta_x * \delta_y(C)\mu(dy) = \int_{x^- C} \delta_x * \delta_y(C)\mu(dy) \leq \mu(x^- C)$$

since $\delta_x * \delta_y(C) \leq 1$

Remark

As pointed out in [BH95] we cannot expect equality here even when S is compact.

4.2 Convolution Equation

Proposition 4.2.1 *Let S be a locally compact Hausdorff semihypergroup. Let $f \in C_c(S)$ and $\mu \in M_1(S)$. Consider the function g defined by $g(x) = \delta_x * \mu(f)$. Then for every such f and each $\mu \in M_1(S)$, the function $g \in C_0(S)$ if and only if the convolution on $M_1(S)$ is separately continuous on the right in the weak-star topology (that is, if a sequence of probability measures μ_n on the Borel subsets of S weak-star converges to a nonnegative (not necessarily a probability) measure μ_0, then for any probability measure ν, the sequence $\mu_n * \nu$ weak-star converges to $\mu_0 * \nu$).*

Proof:

(The only if part) We assume that all functions of the type $g(x)$, as described above in the proposition, vanish at infinity on S. Let μ_n be a sequence of probability measures $w*$-converging to the measure μ_0. Let ν be any given probability measure on S. Let $f \in C_c^+(S)$. Notice that the function defined by

$$\int f(x * y)\nu(dy) = \delta_x * \nu(f),$$

as a function of x on S, vanishes at infinity, by our hypothesis (in the "only if" part). Thus,

$$\int \delta_x * \nu(f)\mu_n(dx) \longrightarrow \int \delta_x * \nu(f)\mu_0(dx)$$

as $n \longrightarrow \infty$. But notice that

$$\int f(u)\mu_n * \nu(du) = \int [\int f(x * y)\nu d(y)]\mu_n(dx) = \int \delta_x * \nu(f)\mu_n(dx)$$

and similarly

$$\int f(u)\mu_0 * \nu(dx) = \int \delta_x * \nu(f)\mu_0(dx).$$

The desired separate continuity of convolution follows and the proof of this part is now complete.

(The "if part")

Suppose that convolution is separately continuous in the sens described above. Let f be a continuous function on S with compact support. Let μ be any probability measure on S and $g(x) = \delta_x * \mu(f)$. We claim that g vanishes at infinity on S. If not, then there must exist $b > 0$ such that the set$\{x : g(x) > b\}$ is not relatively compact. This means that there exists a sequence of elements x_n in S such that for each n, $g(x_n) > b$ and the sequence δ_{x_n} of unit masses at x_n $w*$-convergences to the $0-$measure. But then, by the assumption of separate continuity of convolution, it follows that for each $y \in S$

$$\int f(u)\delta_{x_n} * \delta_y(du) \longrightarrow 0$$

as $n \longrightarrow \infty$. This would then mean that

$$g(x_n) = \int [\int f(u)\delta_{x_n} * \delta_y(du)]\mu(dy)$$

must also converge to zero as n goes to infinity (by the dominated convergence theorem). But this is a contradiction since each $g(x_n) > b > 0$. The proof of the proposition is now complete.

The proof of the next proposition is adapted from the proof for semigroups [HM95].

Proposition 4.2.2 *Let $\mu \in M_1(S)$, B a Borel subset of S and V an open (closed or compact) subset of S. Then*

i. *$g(x) = \mu * \delta_x(V)$ is a lower (upper) semi-continuous function of x and*

ii. *$g(x) = \mu * \delta_x(B)$ is Borel measurable.*

Proof:

From proposition 4.2.1 if $f \in C(S)$ and $\mu \in M_1(S)$ then $g(x) = \mu * \delta_x(f)$ is continuous. Moreover $\mu * \delta_x(V) = sup\{\int f(y)\mu * \delta_x(dy) : f \in C(S), 0 \leq f \leq 1, f = 0$ on $S - V\}$. This implies that $g(x) = \mu * \delta_x(V)$ is lower semi-continuous (see [[HR70] theorem 11.10]. This also implies that $\mathcal{F} = \{B \in \mathcal{B} : \mu * \delta_x(B)$ is a Borel measurable function of x} contains all open subsets and all closed subsets of S, furthermore $\mathcal{F}$ is a monotone class containing the class $\mathcal{F}_\circ = \{V \cap W : V$ is an open set in $\mathcal{B}$, and W is a closed set in $\mathcal{B}\}$. Here $\mathcal{B}$ is the family of Borel subsets of S. We observe that $\mathcal{F}_\circ$ is closed under finite intersections and the complement of any set in $\mathcal{F}_\circ$ is a finite disjoint union of sets in $\mathcal{F}_\circ$, (in fact $(V \cap W)^c = V^c \cup (W^c - V^c)$ is a finite disjoint union of sets in $\mathcal{F}_\circ$). Thus it belongs to $\mathcal{F}$. Since $\mathcal{F}$ contains the algebra generated by $\mathcal{F}_\circ$, and $\mathcal{B}$ is the smallest σ-algebra generated by open sets, $\mathcal{B} \subset \mathcal{F}$.

Remark 4.2.1 *Let S be a locally compact semihypergroup and $\mu, \nu \in M_1(S)$ and $f \in C(S)$, define g by $g(x) = \delta_x * \nu(f)$. Then $\delta_x * \mu(g) = \delta_x * \mu * \nu(f)$. Since*

$$\delta_x * \mu(g) = \int g(y)\delta_x * \mu(dy) = \int \delta_y * \nu(f)\delta_x * \mu(dy) =$$

$$\int\int f(y * u)\nu(du)\delta_x * \mu(dy) = \delta_x * \mu * \nu(f)$$

Theorem 4.2.1 *Let S be a commutative semihypergroup where convolution is separately continuous in the weak*-topology, and $\mu, \nu \in M_1(S)$ then, we have:*
$\mu * \nu = \nu * \mu = \nu$ if and only if $\nu = \nu * \delta_y = \delta_y * \nu$ $\forall y \in [Supp(\mu)]$

Proof:

Suppose $\delta_y * \nu = \nu$ $\forall y \in Supp(\mu)$. Let $f \in C_c(S)$. Then

$$\mu * \nu(f) = \int\int f(x * y)\mu(dx)\nu(dy) = \int\int f(x * y)\nu(dy)\mu(dx) =$$

$$\int \delta_x * \nu(f)\mu(dx) = \int \nu(f)\mu(dx) = \nu(f)$$

Since this is true for any $f \in C_c(S)$,$\mu * \nu = \nu$.

Conversely, suppose $\mu * \nu = \nu$ and let $f \in C_c^+(S)$. Let g be defined by $g(x) = \delta_x * \nu(f)$ for all $x \in S$. Then, from remark 4.2.1, $\delta_x * \mu(g) = \delta_x * \mu * \nu(f) = \delta_x * \nu(f) = g(x)$. Now since $g \in C_0(S)$ by proposition 4.2.1 , and $[Supp(\mu)]$ is closed, there exists

$b \in [Supp(\mu)]$ such that $\|g\|_{[Supp(\mu)]} = g(b) = max\{x; x \in [Supp(\mu)]\}$. However,

$$g(b) = \delta_b * \mu(g) = \int\int g(b * y)\mu(dy) \tag{4.1}$$

Also for all $y \in Supp(\mu)$,

$$g(b * y) = \int g(u)\delta_b * \delta_y(du) \leq \|g\|_{[Supp(\mu)]} = max\{x; x \in [Supp(\mu)]\} = g(b)$$

so that for any $y \in Supp(\mu)$,

$$g(b * y) \leq g(b). \tag{4.2}$$

It follows from (4.1) and (4.2)that

$$g(b) = g(b * y) \tag{4.3}$$

for all $y \in Supp(\mu)$. Since $g(b*y) = \int g(u)\delta_b * \delta_y(du)$, by the same argument as above we have $g(u) = g(b)$ for all $u \in Supp(\delta_b * \delta_y)$ and hence for all $u \in \{b\} * Supp(\mu)$.We also know that $\nu * \mu = \nu \Longrightarrow \nu * \mu^n = \nu$ for all n so that the argument above remains valid when μ is replaced by μ^n. This means that

$$g(y) = g(b) \tag{4.4}$$

for all $y \in \bigcup_{n=1}^{\infty} \{b\} * Supp(\mu^n)$. Let $H = \overline{\bigcup_{n=1}^{\infty} Supp(\mu^n)}$. Then H is nonempty and $H * H \subset H$ so that H is a semihypergroup containing $Supp(\mu)$. Furthermore, H is closed and it follows from the continuity of g that $g(b) = g(y)$ for all $y \in \{b\}*[Supp(\mu)]$. Moreover, $b \in [Supp(\mu)]$, so $\{b\} * [Supp(\mu)] \subset [Supp(\mu)]$. Notice that so far we have not used commutativity. Now $\mu * \nu = \nu$ implies that for each $n \geq 1$, $\mu^n * \nu = \nu$. If the sequence (μ^n) is not tight, then since the set $\{\beta : \beta$ is a nonnegative Borel measure on S and $\beta(S) \leq 1\}$ is a compact in the weak*-topology, there is a subsequence (n_k) of positive integers such that μ^{n_k} weak*-converges to some measure β. By the assumption of continuity of convolution in the weak*topology, $\beta * \nu = \nu$. But this implies that β must be a probability measure on S. In other words, the sequence (μ_n) must be tight. Thus, the weak limit μ_0 of the sequence $\frac{1}{n}\sum_{k=1}^{n}\mu^k$ exists, $\mu_0 \in M_1(S)$, and furthermore, $\mu_0 * \mu_0 = \mu_0$ and $Supp\mu_0 = [Supp\mu]$. Since $\mu_0 * \nu = \nu$, following the same argument as before, it follows that for any y and z in $[Supp(\mu)]$, $g(y) = g(z)$ since $[Supp(\mu)]$ is simple (by [Du73] theorem 1.13).

Now from (4.3)

$$g(b) = \int g(u)\delta_b * \delta_y(du) = \int \delta_u * \nu(f)\delta_b * \delta_y(du) =$$

$$\int\int f(u * x)\nu(dx)\delta_b * \delta_y(du) = \delta_b * \delta_y * \nu(f)$$

for all $y \in Supp(\mu)$, that is $g(b) = \delta_b * \delta_y * \nu$ for all $y \in [Supp(\mu)]$ and from (4.4)we have that

$$g(x) = \delta_x * \delta_y * \nu(f)$$

for almost all $x, y \in [Supp(\mu)]$, that is $\delta_x * \nu(f) = \delta_x * \delta_y * \nu(f)$ so that

$$\nu(f) = \mu * \nu(f) = \int\int f(x * u)\mu(dx)\nu(du) = \int \delta_x * \nu(f)\mu(dx) =$$

$$\int \delta_x * \delta_y * \nu(f)\mu(dx) = \int f(x * u)\delta_y * \nu(du)\mu(dx) =$$

$$\delta_y * \nu * \mu(f) = \delta_y * \nu(f)$$

for all $y \in [Supp(\mu)]$.

The next result consider the Choquet equation for non commutative semihypergroup.

Proposition 4.2.3 *Let S be a semihypergroup, $\mu, \nu \in M_1(S)$ such that $\nu = \mu * \nu = \nu * \mu$. Then for $x \in Supp(\nu), y \in Supp(\mu)$ we have*

$$\nu * \delta_x = \nu * \delta_y * \delta_x$$

and

$$\delta_x * \nu = \delta_x * \delta_y * \nu$$

Proof:

Since $\mu * \nu = \nu * \mu = \nu$ for all $f \in C(S)$,$\nu(f) = \nu * \mu(f) = \mu * \nu(f)$. Let $x \in Supp(\nu)$ and K be any compact subset of S. Let $\delta_x * \nu(K) = a$ and $\epsilon > 0$ be given. Since $\delta_x * \nu$ is regular, there exist open sets G and W and a closed set C such that $K \subset W \subset C \subset G$ such that $\delta_x * \nu(G) \leq a + \epsilon$. Note that the set $A = \{s \in S : \delta_s * \nu(C) < a + \epsilon\}$ is open and contains x so that $\nu(A) > 0$ (since $x \in Supp(\nu)$).

Define the function g on S by $g(s) = max\{\delta_s * \nu(C) - a - \epsilon, 0\}$ then

$$\int g(t)\nu(dt) = \int \int g(s * t)\mu(ds)\nu(dt)$$

so that

$$\int [\int g(s * t)\mu(ds) - g(t)]\nu(dt) = 0 \tag{4.5}$$

Using the fact that $\mu * \nu = \nu * \mu = \nu$ we have for $t \in S, \nu * \delta_t(C) = \nu * \mu * \delta_t(C)$ (Recall $\nu * \mu(B) = \int \nu * \delta_s(B)\mu(ds)$, for all Borel set B).

Now if $h(t) = \nu * \delta_t(C)$ then

$$h(t) = \nu * \delta_t(C) = \nu * \mu * \delta_t(C) =$$

$$\int \nu * \delta_s(C)\mu * \delta_t(ds) = \int h(s)\mu * \delta_t(ds) =$$

$$\int h(s * t)\mu(ds) = \mu * \delta_t(h)$$

And again since $g(s) = max\{h(t) - a - \epsilon, 0\}$ and

$$0 \leq h(t) - a - \epsilon = \int [h(s * t) - a - \epsilon]\mu(ds) \leq$$

$$\int g(s*t)\mu(ds) \geq 0$$

so that for $t \in S$ $g(t) \leq \int g(s*t)\mu(ds)$ that is

$$\int g(s*t)\mu(ds) - g(t) \geq 0 \tag{4.6}$$

Combining (4.5) and (4.6) we have that $g(t) = \int g(s*t)\mu(ds)$ ν- almost surely. For $t \in A, h(t) - a - \epsilon < 0$ so $g(t) = 0$ that is $\int g(s*t)\mu(ds) = 0$ so $g(s*t) = 0$ for almost all t (with respect to ν) in A. Also

$$h(s*t) = \int h(x)\delta_s * \delta_t(dx) = \int \nu * \delta_x(C)\delta_s * \delta_t(dx) =$$

$$\int \delta_u * \delta_x(C)\nu(du)\delta_s * \delta_t(dx) = \nu * \delta_s * \delta_t(C)$$

So

$$g(s*t) = 0 \Longrightarrow h(s*t) \leq a + \epsilon \Longrightarrow \nu * \delta_s * \delta_t(C) \leq a + \epsilon$$

which implies

$$\nu * \delta_s * \delta_t(W) \leq a + \epsilon$$

as $W \subset C$ and since W is open the functions

$$s \mapsto \nu * \delta_s * \delta_t(W)$$

and

$$t \mapsto \nu * \delta_s * \delta_t(W)$$

are both lower semi-continuous, then it follows that for all $t \in A$ and $s \in Supp(\mu)$ we must have $\nu * \delta_s * \delta_t(W) \leq a + \epsilon$. Since again $x \in A$ and $K \subset W$, $\nu * \delta_s * \delta_x(K) \leq \nu * \delta_s * \delta_x(W) \leq a + \epsilon$ But $a = \nu * \delta_x(K)$ so $\nu * \delta_s * \delta_x(K) \leq \nu * \delta_x(K) + \epsilon$ So $\nu * \delta_s * \delta_x(K) \leq \nu * \delta_x(K)$ for all $s \in Supp(\mu)$.

Now lets define $f(x) = \nu * \delta_x(K)$ since $\nu * \mu = \nu$,

$$f(x) = \nu * \mu * \delta_x(K) = \int \nu * \delta_s(K)\mu * \delta_x(ds) =$$

$$\int f(s)\mu * \delta_x(ds) = \int f(s * x)\mu(ds).$$

And since

$$f(s * x) = \nu * \delta_s * \delta_x(K)$$

we have

$$f(x) = \int f(s * x)\mu(ds) = \int \nu * \delta_s * \delta_x(K)\mu(ds)$$

Which implies

$$\int [f(x) - \nu * \delta_s * \delta_x(K)]\mu(ds) = 0$$

It follows that for all s (with respect to μ) $\nu * \delta_x(K) = \nu * \delta_s * \delta_x(K)$

From the upper semi-continuity of the function $s \mapsto \nu * \delta_s * \delta_x(K)$ we have $\nu * \delta_x(K) = \nu * \delta_s * \delta_x(K)$ whenever $x \in Supp(\nu)$ and $s \in Supp(\mu)$. Since ν is regular, $\nu * \delta_x, \nu * \delta_s * \delta_x$ are also regular and we have $\nu * \delta_x(B) = \nu * \delta_s * \delta_x(B)$ for $x \in Supp(\nu), s \in Supp(\mu)$ and B any Borel set.

Therefore $\nu * \delta_x = \nu * \delta_s * \delta_x$ for $x \in Supp(\nu), s \in Supp(\mu)$.

The second inequality follows in the same manner.

4.3 Invariant and Idempotent Measures

Definition 4.3.1 *Let S be a locally compact semihypergroup. A measure m on S (not necessarily bounded) will be called left subinvariant if $\delta_x * m$ is defined and $\delta_x * m \leq m$ for all $x \in S$. If we have $\delta_x * m = m$, m will be called a left invariant measure on S. (Right invariant measures are defined the same way).*

Example 4.3.1 i. *The space $(\mathbb{R}, +)$ is a locally compact group so is a hypergroup with the appropriate convolution and has a left invariant measure which is the*

Lebesgue measure.

ii. *The Sturm Liouville hypergroup is defined from a Sturm Liouville problem whose eigenfunctions are orthogonal with respect to a weight function $dm(x) = \rho(x)dx$, this measure is the invariant measure of the Sturm Liouville Hypergroup.*

iii. *The semihypergroup $S = \{e, x, y\}$ where e is considered the neutral element and convolution is defined by*

$$\delta_x * \delta_x = \delta_y$$

$$\delta_y * \delta_y = \frac{1}{4}\delta_x + \frac{3}{4}\delta_y$$

$$\delta_x * \delta_y = \delta_y * \delta_x = \frac{1}{2}\delta_x + \frac{1}{2}\delta_y$$

Has invariant measure $m = \frac{1}{3}\delta_x + \frac{2}{3}\delta_y$ we can also observe that $Supp(m) \neq S$ although $Supp(m)$ is a simple ideal which is not completely simple (it contains no idempotent element) this is in contrast with semigroups where the support of the invariant measure of an abelian semigroup is a group. For compact commutative semihypergroup we have the following theorem proved by Dunkl.

The next theorem is from [Du73]

Theorem 4.3.1 *If S is a compact commutative semihypergroup then S has a unique invariant measure m, the support of m is a simple subsemihypergroup.*

Proof:

Since S is compact, the set $M_1(S)$ of probability measures on S is weak* compact and convex. Further $M_1(S)$ acts as a commutative semigroup of (weak*) continuous linear operators on itself by convolution [[Du73] proposition 1.8] so by the Markov-Kakutani Theorem, there exists $m \in M_1(S)$ such that $m * \mu = m$ for all $\mu \in M_1(S)$ in particular $m * \delta_x = \delta_x * m = m$. Now suppose that μ is such that $\mu * \delta_x = \mu$ $(x \in S)$, then $\int f d(\mu * m) = \int dm(S) \int f^x d\mu = \int f d\mu$ so $\mu * m = \mu$ but $\mu * m = m$ so $m = \mu$.

Next we observe that if I_1 and I_2 are ideals of S then $I_1 * I_2 \subset I_1 \cap I_2$ and since S is compact, S has a minimal ideal say I. For $x \in I$, $m * \delta_x = m$ which implies that

$Supp(m) = \{x\} * S \subset I$. Conversely for any $x \in S$ $\{x\} * Supp(m) \subset Supp(m)$ so $Supp(m)$ is a closed ideal contained in I and since I is minimal, $Supp(m) = I$.

Remark 4.3.1 *i. Jewett [Je75] and Spector [Sp78] proved that every locally compact hypergroup H has a left subinvariant measure m and $Supp(m) = H$. For a locally compact group the existence of a subinvariant measure implies that of an invariant measure in fact if G is a group and m is such that $\delta_x * m \leq m$ then given any Borel set A, $\delta_x * m(A) \leq m(A)$ but $m(A) = \delta_e * m(A) = \delta_x * \delta_{x^-} * m(A) \leq \delta_x * m(A)$ (In groups $\delta_x * \delta_{x^-} = \delta_e$ and we have $\delta_{x^-} * m(A) \leq m(A)$) and it follows that $\delta_x * m = m$. This is not the case for hypergroup (see an example of Naimark in [Je75] 9.5) though it is easy to prove that when a compact hypergroup has a subinvariant measure it is also invariant. Both authors also proved the existence of invariant measures for discrete hypergroups. Spector [Sp78] proved that if a hypergroup is commutative it has an invariant measure.*

ii. Jewett's conjecture [Je75] that there exist a left invariant measure on all locally compact hypergroup is yet to be proved.

iii. Onipchuk [On93] announced the proof of this conjecture but in reading through it we realized that he is using commutativity implicitly in his assumptions.Precisely the enveloping algebra $A \otimes A'$ is not involutive unless the semihypergroup is commutative.

The next two results, generalize to semihypergroups results given by [BH95] for hypergroup.

Proposition 4.3.1 *Let C be a compact subset of the semihypergroup S and $z \in C$. If C is a subsemihypergroup of S, then there exists $\mu \in M_1(S)$ with $Supp(\mu) \subset C$, $\delta_z * \mu = \mu$ and $\mu * \mu = \mu$.*

Proof:

For all $n \geq 1$, define

$$\mu_n = \frac{1}{n}(\delta_z + \delta_z * \delta_z + \ldots + \delta_z^n) \tag{4.7}$$

Then

$$\|\delta_z * \mu_n - \mu_n\| = \frac{1}{n}\|\delta_z - \delta_z^n\| \leq \frac{2}{n}$$

Now by assumption, we have $Supp(\mu) \subset C$ for all $n \in \mathbb{N}$ which just says that (μ_n) is uniformly tight. By the Prohorov's Theorem (μ_n) is relatively compact (in the weak topology) and hence there exists $\mu \in M_1(S)$ with $\mu_n \longrightarrow_w \mu$. Clearly $Supp(\mu) \subset C$ and $\delta_z * \mu = \mu$. By equation (4.7) we have that $\mu_n * \mu = \mu$ for all $n \in \mathbb{N}$ and hence $\mu * \mu = \mu$

Definition 4.3.2 *Let S be a locally compact semihypergroup. A probability measure μ is said to be idempotent if and only if $\mu * \mu = \mu$*

Example 4.3.2 *Consider the semihypergroup $S = \{e, x, y\}$ where e is considered the neutral element and convolution is defined by*

$$\delta_x * \delta_x = \delta_y$$

$$\delta_y * \delta_y = \frac{1}{4}\delta_x + \frac{3}{4}\delta_y$$

$$\delta_x * \delta_y = \delta_y * \delta_x = \frac{1}{2}\delta_x + \frac{1}{2}\delta_y$$

The measure $\mu = \frac{1}{3}\delta_x + \frac{2}{3}\delta_y$ is an idempotent measure (as an invariant measure) with $Supp(\mu) = \{x, y\}$.

Theorem 4.3.2 *Let S be a commutative semihypergroup such that the convolution is separately continuous in the weak*-topology. For $\mu \in M_b(S)$ let $L(\mu) = \{x \in S : \delta_x * \mu = \mu\}$ Then $L(\mu)$ is a compact subsemihypergroup of S.*

Proof:

Let $f \in C_c^+(S)$ be fixed and let g be such that $g(x) = \delta_x * \mu(f)$ for all $x \in S$. Then $\forall x \in L(\mu)$ $g(x) = \mu(f)$. Thus, g is constant on $L(\mu)$ and since $g \in C_0(S)$, $L(\mu)$ is compact.

Next $\forall x, y \in L(\mu)$ $\delta_x * \delta_y * \mu = \delta_x * \mu = \mu$. This implies that $\nu = \delta_x * \delta_y$ is a probability measure satisfying $\nu * \mu = \mu$ with $\nu = \delta_x * \delta_y$. Since S is commutative, for all $z \in Supp(\delta_x * \delta_y)$, $\delta_z * \mu = \mu$ (see the Choquet equation theorem 4.2.1) which implies $z \in L(\mu)$ that is $Supp(\delta_x * \delta_y) \subset L(\mu)$ for all $x, y \in L(\mu)$, so $L(\mu)$ is a subsemihypergroup of S.

Remark

Every probability measure on a semihypergroup invariant on its support is an idempotent measure. The converse is not always true. Jewett [Je75] proved in the case of a hypergroup(having an invariant measure) the following theorem which we give without proof.

Theorem 4.3.3 *Let H be a hypergroup with an invariant measure. If $\mu \in M^+(H)$ $\mu \neq 0$ and $\mu * \mu = \mu$ (μ is an idempotent measure on H), then $\mu^- = \mu$, the set $G = Supp(\mu)$ is a compact subhypergroup of H, and μ is the normalized invariant measure on G.*

Remark 4.3.2 *When S is a commutative semihypergroup. Dunkl [Du73] proved that an idempotent measure is invariant on its support which is a compact simple semihypergroup. Onipchuk [On89] proved Dunkl's result for compact non-commutative semihypergroup, with the additional condition that for any idempotent measure μ, $\mu * \delta_x = \delta_x * \mu$ for all $x \in Supp(\mu)$. We prove below that Dunkl's result extends to locally compact semihypergroup (not necessarily commutative) with a more relaxed condition.*

Theorem 4.3.4 *Let S be a locally compact semihypergroup with the condition that for any idempotent measure μ, $Supp(\mu) * \{x\} = \{x\} * Supp(\mu)$ for all $x \in Supp(\mu)$. Suppose also that convolution is separately continuous in the weak*-topology. If $\mu \in$*

$M_1(S)$ is an idempotent measure, then $Supp(\mu)$ is a compact subsemihypergroup of S with no two sided proper ideals. Furthermore, μ is invariant on its support.

Proof:

Let $K = Supp(\mu)$ then since $\mu * \mu = \mu$ we obviously have $K * K \subset K$ so $(K, *)$ is a subsemihypergroup of S.

Claim if $f \in C_c^+(S)$ and $g(x) = \delta_x * \mu(f)$ then if f is not identically zero on K, $g(z) > 0$ for some $z \in K$.To see this suppose $g(z) = 0$ for all $z \in K$ then $\int f(z * y)\mu(dy) = 0$ for all $z \in K$ which implies that $f(z * y) = 0$ for all $z, y \in K$ so that $f \equiv 0$ on $K * K = K$ a contradiction so $\int f(z * y)\mu(dy) > 0$ for some $z \in K$.

Next let $f \in C_c^+(S)$, f not identically zero on S. Let g be defined by $g(x) = \delta_x * \mu(f)$ then from proposition (4.2.1) $g \in C_0(S)$. Further $g \geq 0$, and is not identically zero on S. We have from Remark 4.2.1 that

$$\delta_x * \mu(g) = \delta_x * \mu(f) = g(x)$$

Since $g \in C_0(S)$ and K is a closed subset of S there exists $x_0 \in K$ such that $g(x_0) = \delta_{x_0} * \mu(g) = \int g(x_0 * y)\mu(dy)$ for all $y \in K$ which in turns implies that $g(z) = g(x_0)$ for all $z \in x_0 * K$. Since $x_0 * K$ is an ideal in S and g is constant on $x_0 * K$, it is a compact two sided ideal of K(since $g \in C_0(S)$ and g is not identically zero). Further since $Supp(\mu) * \{x\} = \{x\} * Supp(\mu)$ for all $x \in K$, K has a minimum compact nonempty ideal $I \subset x_0 * K$ and for each $f \in C_c^+(S), g$ defined as above is constant on I with value $\|g\|_K$.

Now suppose $I \neq K$ then there exists $z \in K$, $z \notin I$, hence there exists $f \in C_0(S)$ such that $f(z) \neq 0$ and $f(I) = 0$. If we define a function g as above, $g \in C_0(S)$ and there exists $x_0 \in K$ such that $g(x_0) = Sup_{x \in K} |g(x)|$ and since $g(x_0) = \delta_{x_0} * \mu(g)$ we have $g(x_0) = g(x_0 * y)$ for all $y \in K$. If $y \in I$ $Supp(\delta_{x_0} * \delta_y) \subset I$. So

$$g(x_0 * y) = \int g d(\delta_{x_0} * \delta_y) =$$

$$\int g(u)\delta_{x_0} * \delta_y(du) =$$

$$\int \delta_u * \mu(f)\delta_{x_0} * \delta_y(du) =$$

$$\int\int f(u*v)\mu(du)\delta_{x_0} * \delta_y(du)$$

But $u \in I$ so $Supp(\delta_u * \delta_v) \subset I$ so $g(x_0 * y) = 0$ which implies $g(x_0) = 0$ but f is not identically zero on K this is a contradiction so $K = I$ therefore K is a compact simple semihypergroup.

Finally let $f \in C_c^+(S)$ be given and still define the function g as $g(x) = \delta_x * \mu(f)$ for all $x \in S$. Then as we have seen above there exists $x_0 \in K$ such that $g(x_0) = Sup_{x\in K}|g(x)|$ and g is constant on K. Moreover if $\nu \in M_1(K)$

$$\nu * \mu(f) = \int_K \delta_x * \mu(f)\nu(dx) = \int_K g(x)\nu(dx) = g(x) = \delta_x * \mu(f)$$

Since g is constant on K and ν is a probability measure on K. This shows that $\nu * \mu(f) = \delta_x * \mu(f)$ for all $x \in$ $Supp(\nu)$. Since ν was arbitrarily chosen, in particular for $\nu = \mu$ we will have $\mu = \mu * \mu = \delta_x * \mu$ so μ is invariant on $K = Supp(\mu)$.

Remark 4.3.3 *It is well known that if S is a commutative semigroup and μ is idempotent in S, then $Supp(\mu)$ is a group. This result fail in semihypergroups in general. In example 4.3.2 we have a 3 points commutative semihypergroup with and idempotent measure whose support does not contain an idempotent element and so cannot be a hypergroup.*

4.4 Weak Convergence of Convolution Products of Probability Measures on Semihypergroups

4.4.1 Concretization for Semihypergroups

Definition 4.4.1 *A triplet (X, μ, Φ) consisting of a compact space X, a probability measure $\mu \in M_1(X)$, and a Borel-measurable mapping $\Phi : S \times S \times X \to S$ is called a concretization of the semihypergroup $(S, *)$ if*

$$\mu(\{z \in X : \phi(x, y, z) \in A\}) = \delta_x * \delta_y(A)$$

For all $x, y \in S$ and $A \in \mathcal{B}(S)$.

Example 4.4.1 1. *Let G be a locally compact group with multiplication, a convolution $*$ and a neutral element e. The triplet (X, μ, Φ) defined by $X = \{e\}$, $\mu = \delta_e$ and $\Phi(x, y, e) := xy$ for all $x, y \in G$ is a concretization of G.*

2. *Consider the hypergroup $K = \mathbb{R}_+$ with convolution defined by*

$$\delta_x * \delta_y = \frac{1}{2}\delta_{|x-y|} + \frac{1}{2}\delta_{x+y}$$

for all $x, y \in K$ we obtain the concretization (X, μ, Φ) where

$$X = \{-1, 1\}, \mu = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$$

and

$$\Phi(x, y, -1) = |x - y|$$

$$\Phi(x, y, 1) = x + y$$

Since Φ is Borel measurable we just need to check that

$$\mu(\{z \in X : \Phi(x, y, z) \in A\}) = \delta_x * \delta_y(A)$$

Actually

$$\mu(\{z \in X : \phi(x,y,z) \in A\}) =$$

$$\delta_x * \delta_y(A) = \frac{1}{2}\delta_{|x-y|}(A) + \frac{1}{2}\delta_{x+y}(A)$$

And since $X = \{-1,1\}$, *then if* $\Phi(x,y,z) \notin A \ \forall z \in \{-1,1\}$, *then* $|x-y| \notin A$ *and* $x+y \notin A$ *so that* $\mu(\{z \in X : \Phi(x,y,z) \in A\}) = 0$ *and* $\delta_x * \delta_y(A) = 0$. *If* $\Phi(x,y,-1) \in A$ *and* $\Phi(x,y,1) \notin A$, *then* $\mu(\{-1\}) = \frac{1}{2}$ *and* $\delta_x * \delta_y(A) = \frac{1}{2}\delta_{|x-y|}(A) = \frac{1}{2}$ *and if* $\Phi(x,y,-1) \notin A$ *and* $\Phi(x,y,1) \in A$ *then* $\mu(\{1\}) = \frac{1}{2}$ *and* $\delta_x * \delta_y(A) = \frac{1}{2}\delta_{x+y}(A) = \frac{1}{2}$. *Finally if* $\Phi(x,y,-1) \in A$ *and* $\Phi(x,y,1) \in A$ *then* $\mu(\{-1,1\}) = 1$ *and* $\delta_x * \delta_y(A) = 1$. *So we have* (X,μ,Φ) *as defined above is a concretization of* $(\mathbb{R}_+, *)$.

The next theorem is from [BH95] it is also valid for semihypergroups with the same proof.

Theorem 4.4.1 *Let S be a second countable semihypergroup. There exists a measurable mapping Φ from $S \times S \times [0,1]$ into S such that $([0,1], \lambda_{[0,1]}, \Phi)$ is a concretization of S.*

Remark 4.4.1 *In the special case of one dimensional semihypergroup $S = \mathbb{R}_+$ we may assume without loss of generality that*

$$minsupp(\delta_x * \delta_y) = |x-y|$$

$$maxsupp(\delta_x * \delta_y) = x+y$$

whenever $x,y \in K$ The measurable mapping $\Phi : S \times S \times [0,1] \longrightarrow S$ established in Theorem 4.4.1 also satisfies the following five properties:

1. $\Phi(x,y,0) = |x-y|$
2. $\Phi(x,y,1) = x+y$

3. $\Phi(x,y,t) = \Phi(y,x,t)\ \forall t \in [0,1]$

4. $\Phi(0,x,t) = \Phi(x,0,t) = x\ \forall t \in]0,1]$

5. The mapping $\Phi(.,.,t) : S \times S \longrightarrow S$ *is lower semicontinuous.*

Now let S be a semihypergroup with a fixed concretization (X, μ, Φ) and $(\Omega, \mathcal{A}, P)$ denote an arbitrary probability space.

Definition 4.4.2 *For any S-valued random variables X and Y on $(X_n)_{n\geq 1}$ and an (auxiliary) X-valued random variable ξ on (Ω, A, P) such that ξ is (stochastically) independent of $X \otimes Y$ and has distribution $P_\xi = \mu$ we define the randomized sum of X with Y by $X \hat{+} Y = \Phi(X, Y, \xi)$.*

Remark 4.4.2 *This definition can be extended to sequences $(X_n)_{n\geq 1}$ of X-valued random variables on (Ω, A, P) provided all random variables occurring in the sequence $(X_n)_{n\geq 1}$ and $(\xi_n)_{n\geq 1}$ are independent and $P_{\xi_n} := \mu$ for all $n \geq 1$ in fact by the recurrence*

$$\sum_{j=1}^{0} \hat{X}_j := e$$

$$\sum_{j=1}^{n} \hat{X}_j := X_n \hat{+} \sum_{j=1}^{n-1} \hat{X}_j, n \geq 1$$

the randomized sums $S_n = \sum_{j=1}^{n} \hat{X}_j$, $n \geq 1$ are introduced again as S-valued random variables on (Ω, A, P), which form a (non homogeneous) Markov chain $(S_n)_{n\geq 0}$ with corresponding sequence $(N_n)_{n\geq 1}$ of transition kernels on $(S, \mathcal{B}(S))$ satisfying

$$N_n(x, A) = (P_{X_n} * \delta_x)(A) =$$

$$P(S_n \in A : S_{n-1} = x)$$

For $P_{S_{n-1}}$-almost all $x \in S, A \in \mathcal{B}(S)$ and $n \geq 1$

Proposition 4.4.1 *Let X and Y be S-valued random variables and let ξ be an X-valued random variable on (Ω, A, P) with $P_{\xi_n} := \mu$ such that X, Y, ξ are independent then $P_{X \hat{+} Y} = P_X * P_Y$*

Proof:

$\forall A \in (S, \mathcal{B}(S))$

$$P_{X\hat{+}Y}(A) = P(\Phi(X, Y, \xi) \in A) =$$

$$\int\int P(\Phi(X, Y, \xi) \in A) P_X(dx) P_Y(dy)$$

$$\int\int \mu[\Phi(X, Y, \xi) \in A)] P_X(dx) P_Y(dy)$$

$$\int\int \delta_x * \delta_y(A) P_X(dx) P_Y(dy)$$

$$P_X * P_Y(A)$$

So $P_{X\hat{+}Y} = P_X * P_Y$

Remark 4.4.3 *Forming randomized sums is generally not an associative operation although convolution obviously is. While randomized sum $X\hat{+}Y$ clearly depends on the particular choice of the underlying concretization of S the joint distribution of the random variables X, Y and $X\hat{+}Y$ does not.*

4.4.2 Sequence of Convolution of Measures

Theorem 4.4.2 *Let S be a locally compact semihypergroup. Assume $\mu \in M_1(S)$ and suppose that the sequence (μ^n) is tight. Suppose also that*

$$S = \overline{[\bigcup_{n=1}^{\infty} Supp(\mu)^n]}$$

let $\mathcal{K} = \{\mu \in M_1(S) : \mu$ is a weak limit point of the sequence $(\mu)^n\}$ also let us define

$$S_0 = \bigcup\{Supp(\lambda) : \lambda \in \mathcal{K}\}$$

and $S_1 = \bar{S}_0$ *then the sequence*

$$\frac{1}{n}\sum_{k=1}^{n}\mu^k$$

converges weakly to a probability measure ν *such that* $\nu = \nu * \nu = \mu * \nu = \nu * \mu$ *and* $Supp(\nu)$ *is the closed minimal ideal of* S.

Proof:

Write $\mu_n = \frac{1}{n}\sum_{k=1}^{n}\mu^k$ then for $k \geq 1$, $\mu^k * \mu_n = \mu_n * \mu^k$ and

$$lim_{n\to\infty}\|\mu_n - \mu^k * \mu_n\| = 0 \tag{4.8}$$

its follows , since the sequence μ^n is tight, that the sequence (μ_n) is also tight so that $\{(\mu_n) : n \geq 1\}$ is weakly relatively compact. Let ν_1 and ν_2 be two limit points of (μ_n) then by (4.8)

$$\mu^k * \nu_1 = \nu_1 * \mu^k = \nu_1$$

$$\mu^k * \nu_2 = \nu_2 * \mu^k = \nu_2$$

It follows that

$$\frac{1}{n}\sum_{k=1}^{n}\mu_k * \nu_1 = \frac{1}{n}\sum_{k=1}^{n}\nu_1 * \mu_k = \nu_1$$

and

$$\frac{1}{n}\sum_{k=1}^{n}\mu_k * \nu_2 = \frac{1}{n}\sum_{k=1}^{n}\nu_2 * \mu_k = \nu_2$$

That is

$$\mu_n * \nu_1 = \nu_1 * \mu_n = \nu_1$$

$$\mu_n * \nu_2 = \nu_2 * \mu_n = \nu_2$$

which then implies that $\nu_1 = \nu_2(\equiv \nu)$ and $\mu * \nu = \nu * \mu = \nu = \nu * \nu$ and since ν is an idempotent measure it is a simple semihypergroup and since $\overline{Supp(\mu) * Supp(\nu)} = \overline{Supp(\nu) * Supp(\mu)} = Supp(\nu)$,$Supp(\nu)$ is the minimal ideal of $S = \overline{[\bigcup_{n=1}^{\infty} Supp(\mu)^n]}$

Remark

If μ^n converges weakly then $liminf(Supp(\mu)^n)$ is nonempty. To see this suppose $\mu_n \longrightarrow_w \nu$ then claim $Supp(\nu) \subset liminf(Supp(\mu)^n)$ for let $x \in Supp(\nu)$ then for every neighborhood U of x, $\nu(U) > 0$ but $\nu(U) \leq liminf \mu^n(U)$ so $liminf \mu^n(U) > 0$ which implies that $x \in liminf(Supp(\mu)^n)$ which implies that $Supp(\nu) \subset liminf Supp(\mu^n)$ therefore $liminf(Supp(\mu)^n) \neq \emptyset$.

We now solve the Choquet equation for not necessarily commutative hypergroups (an alternative proof can also be found in [BH95] but required lots of steps).

Corallory 4.4.1 *Suppose H is a hypergroup with an invariant measure and $\mu, \nu \in M_1(H)$. Then $\mu = \mu * \nu$ if and only if $\mu = \mu * \delta_x$ for all $x \in [Supp(\nu)]$(the smallest subhypergroup of H containing $Supp(\nu)$*

Proof:

The if part is trivial. Now suppose that $\mu = \mu * \nu$ then $\mu = \mu * \nu^n$. Given $\epsilon > 0$, let K be a compact subset of H such that $\mu(K) > 1 - \epsilon$ Then

$$1 - \epsilon < \mu(K) = \mu = \mu * \nu^n(K) = \int \delta_x * \nu^n(K)\mu(dx) \leq$$

$$\int \nu^n(x^- K)\mu(dx) \leq \int \nu^n(x^- K)\mu(dx) + \epsilon \leq$$

$$\nu^n(K^- * K) + \epsilon$$

Where $K^- * K = \cup_{x \in K} x^- K$. Since $K^- * K \subset H$, $K^- * K$ is compact, and consequently the sequence ν^n is tight. We can now use theorem (4.4.2)since $\frac{1}{n} \sum_{k=1}^{n} \nu^k$ converges weakly to some idempotent probability measure β. Also since $\mu = \mu * \nu^n$ we have $\mu = \mu * (\frac{1}{n} \sum_{k=1}^{n} \nu^k)$and since convolution is separately continuous with respect to weak topology we have $\mu = \mu * \beta$, where $\beta = \beta * \beta$, and consequently, $Supp(\beta)$ is a compact subhypergroup of H [[Je75] theorem 10.2E] containing $Supp(\nu)$. And since $\nu * \beta = \beta * \nu = \beta$ (β is the Haar measure of $[Supp(\beta)]$). Now suppose $\mu = \mu * \beta$ Let

$f \in C_c(H)$ and define g by $g(x) = \mu * \delta_x(f)$ for all $x \in H$ then

$$\beta * \delta_x(g) = \int g(y)\beta * \delta_x(dy) = \int \mu * \delta_y(f)\beta * \delta_x(dy) =$$

$$\int\int f(z * y)\mu(dz)\beta * \delta_x(dy) =$$

$$\mu * \beta * \delta_x(f) = \mu * \delta_x(f) = g(x)$$

Since $g \in C_0(H)$ and $Supp(\beta)$is compact, there exists $x_0 \in Supp(\beta)$ such that

$$g(x_0) = \|g\|_{Supp(\beta)} = Sup_{x\in Supp(\beta)} |g(x)|$$

Now $\beta*\delta_{x_0}(g) = g(x_0)$ so that $g(x_0) = \int g(y*x_0)\beta(dy)$ which implies $g(x_0) = g(y*x_0)$ for all $y \in Supp(\beta)$ which implies

$$g(x_0) = g(y * x_0) = \int g(u)\delta_y * \delta_{x_0}(du) =$$

$$\int \mu * \delta_u(f)\delta_y * \delta_{x_0}(du) =$$

$$\int\int f(z * u)\mu(dz)\delta_y * \delta_{x_0}(du) =$$

$$\mu * \delta_y * \delta_{x_0}(f)$$

Since $g(x_0) = g(y * x_0)$ for all $y \in Supp(\beta)$ g is constant on $Supp(\beta) * x_0 \subset Supp(\beta)$ which is a right ideal of $Supp(\beta)$ so contains the neutral element e. So we have

$$g(e) = \mu * \delta_y * \delta_e(f) = \mu * \delta_y(f)$$

and since $g(e) = \mu * \delta_e(f)$ we have

$$\mu * \delta_e(f) = \mu * \delta_y * (f)$$

so that $\mu(f) = \mu * \delta_y(f)$ for all $f \in C_c(H)$. Therefore $\mu = \mu * \delta_y$ for all $y \in Supp(\beta)$ and since $Supp(\nu) \subset Supp(\beta)$ we have that $\mu = \mu * \delta_x$ for all $x \in [Supp(\nu)]$

Corallory 4.4.2 *Let S be a semihypergroup and $\nu \in M_1(S)$ be such that the sequence (ν^n) is tight and $S = \overline{[\bigcup_{n=1}^{\infty} Supp(\nu)^n]}$. Let $\mu \in M_1(S)$ such that $\mu * \nu = \mu$ Then the following assertions are valid.*

i. *S has a simple ideal $K = Supp(\nu_0)$, where ν_0 is the weak limit of $\frac{1}{n}\sum_{k=1}^{n} \nu^k$ and*

$\nu * \nu_0 = \nu_0 * \nu = \nu_0$

ii. *$Supp(\mu) \subset K$ and $\mu = \mu * \mu$*

Proof:

Assertion (i) follows from theorem (4.4.2). Suppose now that $\mu * \nu = \mu$ for some $\mu \in M_1(S)$. Then

$$\mu * (\frac{1}{n}\sum_{k=1}^{n} \nu^k) = \mu, n \geq 1$$

and it follows that $\mu * \nu_0 = \mu$ and $Supp(\mu) = \overline{Supp(\mu)Supp(\nu_0)} \subset Supp(\nu_0) = K$

Now let $x \in Supp(\mu)$ and $f \in C_b(H)$ then $\mu * \nu_0 = \mu$ implies

$$\delta_x * \mu(f) = \delta_x * \mu * \nu_0(f) =$$

$$\int \delta_x * \delta_y * \nu_0(f)\mu(dy) = \int \delta_x * \nu_0(f)\mu(dy) = \delta_x * \nu_0(f)$$

We have $\delta_x * \delta_y * \nu_0 = \delta_x * \nu_0$ since $Supp(\mu) \subset Supp(\nu_0)$ and $\nu_0 = \nu_0 * \nu_0$ by proposition (4.2.3).And it follows that

$$\mu * \mu(f) = \int \delta_x * \mu(f)\mu(dx) =$$

$$\int \delta_x * \nu_0(f)\mu(dx) = \mu * \nu_0(f) = \mu(f)$$

So that $\mu = \mu * \mu$. is an idempotent measure so $Supp(\mu)$ is a simple subsemihypergroup of $K = Supp(\nu_0)$

Corallory 4.4.3 *Let S be a semihypergroup and $\nu \in M_1(S)$,$S = \overline{[\bigcup_{n=1}^{\infty} Supp(\nu)^n]}$. Suppose that S satisfies the following compactness condition*

K is compact, $x \in S \Longrightarrow x^{-}K$ is compact

Let $\mu \in M_1(S)$ such that $\mu * \nu = \mu$ then $\frac{1}{n}\sum_{k=1}^{n} \nu^k$ converges weakly to $\nu_0 \in M_1$, and consequently all the results in corollary 4.4.3 remain valid.

Proof:

Let λ be a weak* limit points of the sequence

$$\nu_n = \frac{1}{n}\sum_{k=1}^{n} \nu^k$$

If all such weak* limit points are probability measures, then it follows from theorem 4.4.2, that the sequence $\frac{1}{n}\sum_{k=1}^{n} \nu^k$ converges weakly to some ν_0 in $M_1(S)$, and the rest of corollary 4.4.3 then follows exactly as in corollary 4.4.2. Thus it suffices to show that $\lambda \in M_1(S)$.

Let $f \in C_c(S)$ and $x \in S$. Then $f_x \in C_c(S)$. Let (n_k) be the subsequence such that (ν_{n_k}) weak* converges to λ. Then let us define the function g_k and g by

$$g_k(x) = \delta_x * \nu_{n_k}(f)$$

and

$$g(x) = \delta_x * \lambda(f)$$

Since convolution is separately continuous $\delta_x * \nu_{n_k} \rightarrow_{w*} \delta_x * \lambda$, so $g_k(x) \longrightarrow g(x)$ as $k \longrightarrow \infty$ therefore by the bounded convergence theorem, for $f \in C_c(S)$ we have

$$\mu(f) = \int f(x)\mu(dx) = \mu * \nu_{n_k}(f) =$$

$$\int \delta_x * \nu_{n_k}(f)\mu(dx) =$$

$$\int g_k(x)\mu(dx) \longrightarrow \int g(x)\mu(dx) =$$

$$\int \delta_x * \lambda(f)\mu(dx) = (\mu * \lambda)(f)$$

So that $\mu = \mu * \lambda$. That is $\mu(S) = \mu(S)\lambda(S)$ which implies that $\lambda(S) = 1$ so $\lambda \in M_1(S)$.

Proposition 4.4.2 *Suppose S is a compact semihypergroup, with a continuous concretization, $\mu \in M_1(S)$ and*
$S = \overline{[\bigcup_{n=1}^{\infty} Supp(\mu)^n]}$ then for any open set G containing the kernel K of S,

$$lim_{n\to\infty}\mu^n(G) = 1$$

Proof:

Let $K \subset G$, G open , since $K * S \subset G$, S, K are compact, there exists an open set V containing K such that $V * S \subset G$. Notice that if

$$lim_{k\to\infty}\mu^{n_k}(V) = 1, \tag{4.9}$$

then $\forall \epsilon > 0$ there exists k_0 such that $m > n_{k_0}$ implies

$$\mu^m(G) \geq \mu^{n_{k_0}}(V)\mu^{m-n_{k_0}}(S) > 1 - \epsilon$$

which means that

$$lim_{n\to\infty}\mu^n(G) = 1$$

Therefore it is enough to established (4.9) for some subsequence (n_k). To this end let $x \in K$ then since $SxS \subset K \subset V$ there exists an open set W such that $x \in W$ and $S * W * S \subset V$ since $x \in W \subset S = \overline{[\bigcup_{n=1}^{\infty} Supp(\mu)^n]}$ there exists $m > 0$ such that $\mu^m(W) > 0$.

Let (X_n) be a sequence of independent S-valued random variable each with distribution μ^m. Then we have

$$\sum P(X_n \in W) = \infty$$

and by Borel Cantelli lemma we have

$$P(X_n \in W, i.o) = 1$$

Since $\{X_n \in W\}$ are independent, $\forall \epsilon > 0\ \exists m_0$ such that

$$P(\bigcup_{n=0}^{m_0}\{X_n \in W\} > 1 - \epsilon.$$

Now if

$$x \in \bigcup_{n=0}^{m_0}\{X_n \in W\}.$$

$\exists n_0$ such that $X_{n_0}(x) \in W$, let $S_n = \sum_{k=1}^{n} \hat{X}_k\ n \geq m_0$.

Note that if X and Y are two random variables such that X is A-valued and Y is B-valued then $X\hat{+}Y$ is AB-valued. For $X\hat{+}Y = \Phi(X, Y, \xi)$ when ξ is $[0,1]$-valued so that $(X\hat{+}Y)(x) = \Phi(X(x), Y(x), \xi(t))$. Set

$$X(x) = z, Y(x) = y, \xi(t) = s$$

Claim: $\Phi(z, y, s) \in Supp(\delta_z * \delta_y) \subset A * B$ for all $s \in [0,1]$.To see this suppose $x \in A, y \in B$ let V be an open set containing $\Phi(z, y, s)$, $s \in [0,1]$ then $\delta_x * \delta_y(V) = \lambda\{s : \Phi(z, y, s) \in V\}$ and since Φ is continuous, $\lambda\{s : \Phi(z, y, s) \in V\} > 0$ so that $\delta_x * \delta_y(V) > 0$, that is $\Phi(z, y, s) \in Supp(\delta_z * \delta_y) \subset A * B$. So $X\hat{+}Y$ is AB-valued and by the definition of the randomized sum

$$S_n = X_1\hat{+}X_2\hat{+}X_3\hat{+}\ldots\hat{+}X_n$$

Since X_{n_0} is K-valued S_n will be V-valued so that

$$\bigcup_{n=0}^{m_0}\{X_n \in W\} \subset \{X_1\hat{+}X_2\hat{+}X_3\hat{+}\ldots\hat{+}X_n \in V\}, n \geq m_0$$

Since $S * W * S \subset V$.

Now as $X_1 \hat{+} X_2 \hat{+} X_3 \hat{+} \ldots \hat{+} X_n$ has distribution μ^{mn}, it is clear that for $n \geq m_0 (mn > m_0)$ so that $\mu^{mn}(V) > 1 - \epsilon$ for all $\epsilon > 0$. So $\mu^{mn}(V) = 1$.

Proposition 4.4.3 *Let I be a Borel set that is an ideal of S. Suppose that for some positive integer m, $\mu^m(I) > 0$ for some $\mu \in M_1(S)$. Then the sequence $(\mu^n(I))$ monotonically increases to* 1.

Proof:

$I * S \subset I$ so $\mu^{n+1}(I) \geq \mu^n(I)\mu(S) = \mu^n(I)$

For all positive integer n. Now the prove follows as above since $S * I * S \subset I$.

Proposition 4.4.4 *Let μ_n be a sequence in $M_1(S)$ such that the subsequence μ_{0,n_t} where*

$$\mu_{k,n} = \mu_{k+1} * \ldots * \mu_n$$

has at least one weak limit point in $M_1(S)$. Suppose that S has the property such that convolution as a map from $M_1(S) \times B(S) \longrightarrow B(S)$ is continuous in the weak* sense. Then there is a sequence $(p_t) \subset (n_t)$ such that for each positive integer k*

$$\mu_{k,p_t} \longrightarrow_w \lambda_k \in M_1(S)$$

$$\lambda_{p_t} \longrightarrow_w \lambda * \lambda \in M_1(S)$$

$$\lambda_k * \lambda = \lambda_k$$

Where $B(S) = \{\mu : \mu$ is a nonnegative regular Borel measure with $\mu(S) \leq 1\}$

Proof:

Suppose $\mu_{0,n_t} \longrightarrow_w \lambda_0 \in M_1(S)$. Note that w*-convergence is weak convergence when the limit is in $M_1(S)$. Now for each positive integer t

$$y_{n_t} \equiv (\mu_{0,n_t}, \mu_{1,n_t}, \ldots, \mu_{n_t - 1, n_t}, 0, 0, 0, \ldots)$$

are elements in the product space

$$Y = \prod_{i=1}^{\infty} X_i, \qquad X_i = B(S)$$

with weak*topology, where Y has the product topology and is therefore compact, since $B(S)$ is w*compact. Since Y is compact (and first countable), there is a subsequence $(m_t) \subset (n_t)$ such that $y_{m_t} \longrightarrow y \in Y$, in the topology of Y. This means that for each $k \geq 0$, there exists $\lambda_k \in B(S)$ such that

$$\mu_{k,m_t} \longrightarrow \lambda_k$$

Since convolution is continuous as a map from $M_1(S) \times B(S) \longrightarrow B(S)$ it follows that for each $k \geq 1$

$$\mu_{0,m_t} = \mu_1 * \mu_2 * \ldots mu_k * \mu_{k+1} * \ldots * \mu_{m_t} =$$

$$\mu_{0,k} * \mu_{k,m_t} \longrightarrow \mu_{0,k} * \lambda_k$$

in the weak* sense and this means that

$$\mu_{0,k} * \lambda_k = \lambda_0, k \geq 1$$

(since $\mu_{0,m_t} \longrightarrow \lambda_0$)

However since $\lambda_0 \in M_1(S)$ this implies that $\lambda_k \in M_1(S)$ for each $k \geq 1$. Let $(p_t) \subset (m_t)$ be a subsequence such that $\lambda_{p_t} \longrightarrow \lambda \in B(S)$ in the weak*sens. Now for fixed integer s and $t > s$ such that $p_s > k$, we have

$$\mu_{k,p_s} * \mu_{p_s,p_t} = \mu_{k,p_t}$$

Again by the continuity of convolution, it follows that given $k \geq 0$ for each s such that $p_s > k$

$$\mu_{k,p_s} * \lambda_{p_s} = \lambda_k$$

which in turn implies that $\lambda_k * \lambda = \lambda_k$, $k \geq 1$, since $\lambda_k \in M_1(S)$. The last equation implies that $\lambda * \lambda = \lambda$.

Proposition 4.4.5 *Suppose S satisfies the following compactness condition.*

K compact and $x \in S$ implies $x^- K$ is compact.

If $\mu_n \longrightarrow \mu$ weakly in $M_1(S)$ and $\nu_n \longrightarrow \nu \in B(S)$ in the weak sense with $\nu_n \in M_1(S)$ then $\mu_n * \nu_n \longrightarrow \mu * \nu$ in the weak* topology.*

Proof:

Let $f \in C_c(S)$. Then for each $s \in S$, $t \mapsto f_s(t)$ is in $C_c(S)$. Hence if

$$g_n(s) \equiv \int f(s * t)\nu_n(dt)$$

$$g(s) \equiv \int f(s * t)\nu(dt)$$

Then

$$lim_{n\to\infty} g_n(s) = g(s)$$

Since ν is a regular measure, it is easily seen that g is a bounded continuous function on S. Also by Egoroff's theorem in analysis, given $\epsilon > 0$ there exists a compact set K such that $\mu(K) < \epsilon$ and on $S - K$, $g_n \longrightarrow g$ uniformly. Since $\mu_n \longrightarrow \mu$ weakly,

$$limsup_{n\to\infty} \mu_n(K) \leq \mu(K) < \epsilon$$

Then we have

$$\left|\int g_n d\mu_n - \int g d\mu\right| \leq$$

$$\left|\int_K g_n d\mu_n - \int_K g d\mu_n\right| + \int_{K^c} |g_n - g|\, d\mu_n + \left|\int g d\mu_n - \int g d\mu\right|$$

which shows that

$$lim_{n\to\infty} \int g_n d\mu_n = \int g d\mu$$

because

$$\int g_n d\mu_n - \int g d\mu = \int g_n d\mu_n - \int g d\mu_n + \int g d\mu_n - \int g d\mu =$$

$$\int_K g_n d\mu_n + \int_{K^c} g_n d\mu_n - \int_K g d\mu_n - \int_{K^c} g d\mu_n + \int g\mu_n - \int g d\mu. =$$

$$\int_K g_n d\mu_n - \int_K g d\mu_n + \int_{K^c} (g_n - g) d\mu_n + \int g d\mu_n - \int g d\mu$$

So

$$lim_{n\to\infty} \int g_n d\mu_n = \int g d\mu$$

This means that

$$\int f d\mu_n * \nu_n = \int \int f(s*t)\mu_n(ds)\nu_n(dt) =$$

$$\int [\int f(s*t)\nu_n(dt)]\mu_n(ds) =$$

$$\int g_n(s)\mu_n(ds) \longrightarrow \int g d\mu = \int f d\mu * \nu$$

So $\mu_n * \nu_n \longrightarrow \mu * \nu$

5 Semigroups of Multipliers associated with semigroups of Operators in $L_p(H)$

5.1 Introduction

In this chapter we extend to hypergroups basic techniques on multipliers set forth for groups in [HR70], namely propositions 5.2.1 and 5.2.2, as well as an extended version of Wendel's theorem. Let H be a compact commutative hypergroup and $\hat{H}$ its dual space. We establish relationships between semigroups $\mathcal{S} = \{T(\xi) : \xi > 0\}$ of operators on $L_p(H)$, $1 \leq p < \infty$, which commutes with translations, and semigroups $\mathcal{M} = \{E_\xi : \xi > 0\}$ of $L_p(H)$ multipliers. These results generalize to hypergroups those of [HP57] for the circle groups and [B074] for compact abelian groups.

Remark 5.1.1 *Let H be a locally compact hypergroup.*

Then $\forall x \in H, \mu \in M(H)$, and $f \in C(H)$

$$\delta_x * \mu(f) = \int_H f_x d\mu$$

($\equiv \mu(f_x)$, say)

and similarly

$$\mu * \delta_x(f) = \mu(f^x)$$

Definition 5.1.1 *A character χ on a hypergroup H is a continuous complex-valued* *function on H which is not identically zero and satisfies*

$$\int_H \chi d\delta_x * \delta_y = \chi(x)\chi(y)$$

for all $x, y \in H$. A character χ is said to be ***Hermitian*** *if and only if $\chi(x^-) = \overline{\chi(x)}$.*

Remark 5.1.2 *For commutative hypergroups, the dual $\hat{H}$ of H is the set of all Hermitian characters on H. $\hat{H}$ is a hypergroup provided it is a hypergroup with respect to pointwise multiplication. That is, with a convolution defined such that $\delta_{\chi_1} * \delta_{\chi_2}(x) = \chi_1(x)\chi_2(x)$ for all $x \in H$. Unlike in the group case, $\hat{H}$ is not always a hypergroup even in the commutative case [Je75]. Also if $\hat{H}$ is a hypergroup, $\hat{\hat{H}}$ may not necessarily be a hypergroup.* ***If H admits a dual hypergroup structure, it is called a hypergroup of type D. If $\hat{H}$ is a hypergroup , $H \subset \hat{\hat{H}}$ in a natural manner, and if in addition, $H = \hat{\hat{H}}$ holds, we call H a strong hypergroup.*** *We will assume throughout this chapter that $\hat{H}$ is supported by its invariant measure.*

Definition 5.1.2 *Let $(H, *)$ be a commutative hypergroup with invariant measure m. The Fourier Stieljes transform $\hat{\mu}$ of μ is defined on $\hat{H}$ by*

$$\hat{\mu}(\chi) = \int_H \bar{\chi} d\mu$$

And for all $f \in L_1(H)$, the Fourier transform $\hat{f}$ of f with respect to m is defined by

$$\hat{f}(\chi) = \int_H f\bar{\chi} dm$$

Also the convolution of a function f with a measure μ is given by

$$f * \mu(x) = \int_H \int_H f d\delta_x * \delta_{y^-} \mu(dy)$$

5.2 Multipliers on Hypergroups

Definition 5.2.1 *Let H be a locally compact hypergroup with invariant measure m, and $L_p(H)$ has its usual meaning $1 \leq p \leq \infty$. Let $\mathcal{U} = C(H)$ or $L_p(H)$. Given $A \subset \mathcal{U}$, we define, by $\hat{A}$, the set of all Fourier transforms $\hat{f}$ of $f \in A$. A complex valued function φ on the dual space $\hat{H}$ of H is called an (A, B)-multiplier if and only if $\varphi\hat{f} \in \hat{B}$ ($\varphi\hat{f}$ is a pointwise multiplication) for each $f \in A$ where A, B are subsets of $\mathcal{U}$. The set of all (A, B)-multiplier is denoted by $\mathcal{M}(A, B)$.By multiplier we here really mean a left multiplier. Right multipliers are defined in a similar way.*

Remark 5.2.1 *First we point out that in the case of a compact non commutative hypergroup, $\hat{H}$ denote the dual object of H, that is, the set of continuous unitary representations U of H. Suppose $U \in \hat{H}$ and $\{\tau_j\}_{j=1}^{d_U}$ is an orthonormal basis for H_U (the Hilbert space associated with U with dimension d_U). We define coordinate functions for U as in [Vr79] by*

$$u_{jk}(x) =< u_x\tau_k, \tau_j >$$

where $1 \leq j, k \leq d_U$. For details about representations on compact hypergroups, see [Vr79]. $Trig_U(H)$ is the linear span of coordinate functions of U and $Trig(H) = \bigcup\{Trig_U(H) : U \in \hat{H}\}$.

Further the $*-$algebra $\prod_{U\in\hat{H}} B(H_U)$ will be denoted by $\mathfrak{E}(\hat{H})$; scalar multiplication, addition, multiplication and adjoint of an element are defined coordinate wise. Let $E = (E_U)$ be an element of $\mathfrak{E}(\hat{H})$. For $1 \leq p < \infty$ we define

$$\|E\|_p = (\sum_{U\in\hat{H}} k_U\|E_U\|^p_{\varphi_p})^{\frac{1}{p}}$$

and

$$\|E\|_\infty = sup\{\|E_U\|_{\varphi_\infty}\}$$

The norms $\|.\|_{\varphi_P}$ are the operator norms of [[HR70] D.37, D.36(e)] and the notations

$\mathfrak{E}_p(\hat{H})$, $\mathfrak{E}_{00}(\hat{H})$, and $\mathfrak{E}_0(\hat{H})$ is as in [[HR70] 28.24].

As in the group case [HR70], many of the argument used in the theory of multipliers are based on the closed graph theorem. The next result put into one place all those closed graph theorem arguments and also list all the spaces that we deal with in this section.

Proposition 5.2.1 *Let H be a compact hypergroup. Let $\mathcal{U}$ and $\mathcal{B}$ be any of the spaces*

i. *$\mathfrak{E}_p(\hat{H})$, $(1 \leq p \leq \infty)$, $\mathfrak{E}_0(\hat{H})$,*

ii. *$L_p(H)$, $(1 \leq p \leq \infty)$,$C_b(H)$, $M(H)$*

where $\hat{H}$ denote the dual object of H.

Let E be a $(\mathcal{U}, \mathcal{B})$-multiplier. Define the mapping

$$T : \mathcal{U} \longrightarrow \mathcal{B}$$

by the following rules

iii. *$T(g) = Eg$ for $g \in \mathcal{U}$ if $\mathcal{U}$ and $\mathcal{B}$ are chosen from (i)*

iv. *$\widehat{T(g)} = Eg$ for $g \in \mathcal{U}$, $\mathcal{U}$ chosen from (i) and $\mathcal{B}$ from (ii)*

v. *$T(f) = E\hat{f}$ for $f \in \mathcal{U}$[or $T(\mu) = E\hat{\mu}$ for $\mu \in M(H)$] if $\mathcal{U}$ is chosen from(ii) and $\mathcal{B}$ from (i)*

vi. *$\widehat{T(f)} = E\hat{f}$ for $f \in \mathcal{U}$ [or $\widehat{T(\mu)} = E\hat{\mu}$ for $\mu \in M(H)$] if $\mathcal{U}$ and $\mathcal{B}$ are chosen from (ii).*

If $\mathcal{U}$ and $\mathcal{B}$ are given their usual norm, then T is a bounded linear transformation from $\mathcal{U}$ to $\mathcal{B}$

Proof:

The proof is adapted from [[HR70]35.2] for the group case.

First, we need to show that T is well-defined in (iii),(iv),(v),(vi).

For (iii), since $\mathcal{U}$ and $\mathcal{B}$ are chosen from (i), by the definition of a $(\mathcal{U},\mathcal{B})$-multiplier, for all $g \in \mathcal{U}$, $Eg \in \mathcal{B}$ uniquely so that T is well defined.

In (v) $\mathcal{U}$ is from (ii) so that $\hat{\mathcal{U}}$ is a subset of a set in (i) and E is a $(\hat{\mathcal{U}},\mathcal{B})$-multiplier so for all $f \in \mathcal{U}$, $\hat{f} \in \mathcal{U}$ and $E\hat{f} \in \mathcal{B}$ uniquely, therefore T is well defined as $\hat{f}$ is uniquely defined.

For (iv) $\mathcal{U}$ is chosen from (i) and $\mathcal{B}$ from (ii)so E is a $(\mathcal{U},\hat{\mathcal{B}})$-multiplier so $\forall g \in \mathcal{U}$, $Eg \in \mathcal{B}$. Now if $\widehat{T(g)} = Eg$, by the uniqueness of the Fourier transform [[Je75] 7.3E], $T(g)$ is well defined. Similarly in (vi) for $(\mathcal{U},\mathcal{B})$ from (ii) E is a $(\hat{\mathcal{U}},\hat{\mathcal{B}})$-multiplier that is $\forall f \in \mathcal{U}$,$\hat{f} \in \hat{\mathcal{U}}$ and $E\hat{f} \in \mathcal{B}$ so if $\widehat{T(f)} = E\hat{f}$ then by the uniqueness of the Fourier Stieltjes transform $T(f)$ is unique, T is then well defined.

Now if $\mathcal{U}$ is chosen from (i) then we have $\mathcal{U} \subset \mathfrak{C}_\infty(\hat{H})$ and [[HR70] 28.32(iv)] shows that for all $g \in \mathcal{U}$

$$||g||_\infty \leq ||g||_{\mathcal{U}} \tag{5.1}$$

If $\mathcal{U}$ is chosen from (ii) $\mathcal{U} \subset M(H)$ and as $M(H)$ is isomorphic with $\mathfrak{C}_\infty(\hat{H})$ [[Vr79] 3.2] and since the isomorphism is norm-decreasing we have

$$||\hat{\mu}|_\infty \leq ||\mu|| \leq ||\mu||_{\mathcal{U}}$$

For all $g \in \hat{\mathcal{U}}$ (This is obtained by writing $||\hat{\mu}||_{\hat{\mathcal{U}}}$ for $||\mu||_{\mathcal{U}}$ in the previous inequality), we have

$$||g||_\infty \leq ||g||_{\hat{\mathcal{U}}} \tag{5.2}$$

Relations (5.1) and (5.2) shows that in all cases $\mathcal{U}$ can be regarded as a subspace of $\mathfrak{C}_\infty(\hat{H})$ for which (5.1) holds. The same remark evidently holds for $\mathcal{B}$. Thus we may consider $\mathcal{U}$ and $\mathcal{B}$ as linear subspaces of $\mathfrak{C}_\infty(\hat{H})$ with complete norm $||.||_{\mathcal{U}}$ and $||.||_{\mathcal{B}}$ satisfying the inequality (5.1). So we will just prove that the mapping T defined from $\mathcal{U}$ to $\mathcal{B}$ by $T(g) = Eg$ is a bounded linear transformation carrying $\mathcal{U}$ to $\mathcal{B}$ for all subspaces of $\mathfrak{C}_\infty(\hat{H})$ having complete norms that satisfy (5.1). Since $E, g \in \mathfrak{C}_\infty(\hat{H})$ we have for $g_1, g_2 \in \mathcal{U}$, $g_1, g_2 \in \mathfrak{C}_\infty(\hat{H})$ and $E(g_1 + g_2) = Eg_1 + Eg_2$ so $T(g) = Eg$ is a linear transformation. Now let $g \in \mathcal{U}$ and $\{g^{(n)}\}_{n=1}^{\infty}$ be a sequence in $\mathcal{U}$ such that

$$lim_{n\to\infty}\|g^{(n)} - g\|_{\mathcal{U}} = 0 \tag{5.3}$$

Suppose that g' is a limit point of $\mathcal{B}$ such that

$$lim_{n\to\infty}\|T(g^{(n)}) - g'\|_{\mathcal{B}} = 0 \tag{5.4}$$

Then from (5.1) applied to $\mathcal{B}$ and (5.4)

$$lim_{n\to\infty}\|Eg^{(n)} - g'\|_{\infty} =$$

$$lim_{n\to\infty}\|T(g^{(n)}) - g'\|_{\infty} = 0 \tag{5.5}$$

For each $U \in \hat{H}$, (5.5) shows that

$$lim_{n\to\infty}\|E_U g_U^{(n)} - g'_U\|_{\varphi\infty} = 0 \tag{5.6}$$

and from [HR70] D.52i and (5.1)

$$\|E_U g_U^{(n)} - E_U g_U\|_{\varphi\infty} \leq$$

$$\|E_U\|_{\varphi\infty}\|g_U^{(n)} - g_U\|_{\varphi\infty}$$

From (5.3) we have

$$lim_{n\to\infty}\|E_U g_U^{(n)} - E_U g_U\|_{\varphi\infty} = 0 \tag{5.7}$$

For each $U \in \hat{H}$. The inequalities (5.6) and (5.7) imply that $Eg = g'$, that is, T has a closed graph in $\mathcal{U} \times \mathcal{B}$. and from the closed graph theorem T is continuous.

Our next result describes the duality properties of multipliers. It provides useful shortcuts in computations involving multipliers.

Proposition 5.2.2 *Suppose that $\mathcal{U}$ and $\mathcal{B}$ and also their conjugate spaces $\mathcal{U}^*$, $\mathcal{B}^*$ are* *among the spaces listed in Proposition (5.2.1) [(i),(ii)][Thus neither $\mathcal{U}$ nor $\mathcal{B}$ can be $\mathfrak{C}_\infty(\hat{H})$, $L_\infty(H)$ or $M(H)$] Then we have*

$$\mathcal{M}(\mathcal{U}, \mathcal{B}^*) = \mathcal{M}(\mathcal{B}, \mathcal{U}^*)^{\tilde{}}$$

Proof:

It is enough to show that $E \in \mathcal{M}(\mathcal{U}, \mathcal{B}^*)$ implies $E^{\tilde{}} \in \mathcal{M}(\mathcal{B}, \mathcal{U}^*)$
We will examine four cases according to whether $\mathcal{U}$ and $\mathcal{B}$ are chosen from Proposition (5.2.1)(i) or Proposition (5.2.1)(ii). Throughout this prove, let T be as defined in Proposition (5.2.1) for the element $E \in \mathcal{M}(\mathcal{U}, \mathcal{B}^*)$.
First suppose that both $\mathcal{U}$ and $\mathcal{B}$ are chosen from Proposition (5.2.1)(i), consider a fixed $B \in \mathcal{B}$. For each $A \in \mathcal{U}$ define

$$\phi(A) = < T(A), B >$$

where $<, >$ is defined as in [[HR70] 28.28i]. Holder's inequality [[HR70] 28.28ii] and the boundedness of T show that

$$|\phi(A)| = |< T(A), B >| = \|T(A)\|_{\mathcal{B}^*} \|B\|_{\mathcal{B}} \leq \|T\| \|A\|_{\mathcal{U}} \|B\|_{\mathcal{B}}$$

For all $A \in \mathcal{U}$. The space $\mathfrak{C}_{00}(\hat{H})$ is contained in $\mathcal{U}$ and so for $A \in \mathfrak{C}_\infty(\hat{H})$ we have

$$< A, C > = < T(A), B > = < EA, B > = < A, E^{\tilde{}} B >$$

It follows that $E^{\tilde{}} B = C$ and consequently that $E^{\tilde{}} \in \mathcal{M}(\mathcal{B}, \mathcal{U}^*)^{\tilde{}}$
Next suppose that $\mathcal{U}$ is chosen from Proposition (5.2.1)(i) and $\mathcal{B}$ from Proposition (5.2.1)(ii). Consider a fixed but arbitrary $f \in \mathcal{B}$. For each $A \in \mathcal{U}$, $T(A)$ belongs to $M(H)$ [In case $T(A)$ is a function $C \in L_{p'}(H)$ we mean by this that $T(A)$ is a

measure such that $dT(A) = gdm$], we define ϕ on $\mathcal{U}$ by

$$\phi(A) = \int \bar{f} dT(A) \tag{5.8}$$

for all $A \in \mathcal{U}$ then

$$\|\phi(A)\| \leq \|\bar{f}\|_{\mathcal{B}} \|T(A)\| \leq \|\bar{f}\|_{\mathcal{B}} \|T\| \|A\|$$

so ϕ is bounded because T is bounded. ϕ being a bounded linear functional on $\mathcal{U}$, there is a $C \in \mathcal{U}^*$ for which

$$\phi(A) =< A, C > \tag{5.9}$$

for all $A \in \mathcal{U}$.

Since $\mathfrak{E}_{00}(\hat{H}) \subset \mathcal{U}$, $T(A_0)$ is defined for $A_0 \subset \mathfrak{E}_{00}(\hat{H})$. The element EA_0 is in $\mathfrak{E}_{00}(\hat{H})$ and the definition Proposition (5.2.1)iv of T shows that $dT(A_0) = gdm$ where $g \in Trig(H)$ and $\hat{g} = EA_0$. Applying (5.8), (5.9) and [[HR70] 34.33] we obtain

$$< A_0, C >= \int \bar{f} dT(A_0) = \int \bar{f} gdm =< \hat{g}, \hat{f} >=< EA_0, \hat{f} >=< A_0, E^{\tilde{}} \hat{f} > . \tag{5.10}$$

Since f is arbitrary in $\mathcal{B}$ and A_0 is arbitrary in $\mathfrak{E}_{00}(\hat{H})$, (5.10) shows that $E^{\tilde{}}$ carries $\hat{\mathcal{B}}$ into $\mathcal{U}^*$; that is $E^{\tilde{}}$ is in $\mathcal{M}(\mathcal{B}, \mathcal{U}^*)$.
Third suppose that $\mathcal{U}$ is chosen from Proposition (5.2.1)(ii) and $\mathcal{B}$ from Proposition (5.2.1)(i), for a fixed but arbitrary $B \in \mathcal{B}$ define ϕ on $\mathcal{U}$ by

$$\phi(f) =< T(f), B >$$

for all $\mathcal{U}$. As defined before ϕ is a bounded linear functional , applying [[HR70] 14.10], if $\mathcal{U} = C(H)$] and [[HR70] 12.18] if $\mathcal{U} = L_p(H)$], we define a measure μ (which has the form gdm if $\mathcal{U} = L_p(H)$) such that

$$\overline{\phi(\bar{f})} = \int f d\mu$$

for all $f \in \mathcal{U}$

for each $f \in Trig(H) \subset \mathcal{U}$ we have

$$< \hat{f}, \hat{\mu} >= \overline{< \hat{\mu}, \hat{f} >} = \int_H \bar{f} d\mu =$$

$$\phi(f) =< T(f), B >=< E\hat{f}, B >=< \hat{f}, E^{\tilde{}} B >$$

and hence $E^{\tilde{}} B = \hat{\mu}$. Thus again $E^{\tilde{}}$ belongs to $\mathcal{M}(\mathcal{B},\mathcal{U}^*)$. Suppose finally that $\mathcal{U}$ and $\mathcal{B}$ are chosen from Proposition (5.2.1)(ii), and consider $g \in \mathcal{B}$. For $f \in \mathcal{U}$ define $\phi(f) = \int \bar{g} dT(f)$ as in the previous case, there is a $\mu \in \mathcal{U}^*$ such that $\overline{\phi(\bar{f})} = \int f d\mu$ for $f \in \mathcal{U}$. For $f \in Trig(H)$ we have $dT(f) = hdm$ where $h \in Trig(H)$ and $\hat{h} = E\hat{f}$. Hence we can write

$$< \hat{f}, \hat{\mu} >= \phi(f) = \int \hat{g} dT(f) = \int \bar{g} h dm =$$

$$< \hat{h}, \hat{g} >=< E\hat{f}, \hat{g} >=< \hat{f}, E^{\tilde{}} \hat{g} >$$

once again $E^{\tilde{}} \hat{g} = \hat{\mu}$ and $E^{\tilde{}}$ is in $\mathcal{M}(\mathcal{B},\mathcal{U}^*)$

We now consider a version of Wendel's theorem. This theorem tells us when bounded linear operators on $L_1(H)$ commute with translation operators. It was stated and proved for locally compact abelian groups in [Lr71]. A statement of this theorem for locally compact commutative hypergroups is in [Ls82]. We give here a complete proof.

Theorem 5.2.1 *Let H be a locally compact commutative hypergroup. Suppose $T : L_1(H) \to L_1(H)$ is a bounded linear transformation. Then the following statements are equivalent:*

i. *T commutes with right translation operator that is $T(f^s) = T(f)^s$ for all $s \in H$*

ii. *$T(f * g) = T(f) * g$ for each $f, g \in L_1(H)$*

iii. *There exists a unique transformation φ on $\hat{H}$ such that $\widehat{T(f)} = \varphi\hat{f}$ for each $f \in L_1(H)$.*

iv. *There exists a unique measure $\mu \in M(H)$ such that $\widehat{T(f)} = \hat{\mu}\hat{f}$ for each $f \in L_1(H)$*

v. *There exists a unique measure $\mu \in M(H)$ such that $T(f) = f * \mu$ for each $f \in L_1(H)$*

Proof:

(i)implies (ii)

Suppose T commute with right translation, let $k \in L_\infty(H)$ then the mapping defined on $L_1(H)$ by

$$f \mapsto \int T(f)(t)k(t^-)dm(t)$$

is a linear functional on $L_1(H)$ moreover

$$\| \int T(f)(t)k(t^-)dm(t)\| \leq \|k\|_\infty \|T(f)\|_1 \leq \|k\|_\infty \|T\| \|f\|_1$$

where $\|T\|$ denotes the usual operator norm of T. Consequently there exists a function $h \in L_\infty(H)$ such that

$$\int T(f)(t)k(t^-)dm(t) = \int f(t)h(t^-)dm(t) \tag{5.11}$$

by virtue of ([HK75] 20.20). If f, $g \in L_1(H)$ we have

$$\int [T(f) * g](t)k(t^-)dm(t) = \int [\int T(f)(t * s)g(s^-)dm(s)]k(t^-)dm(t) =$$

$$\int [T(f)^s(t)g(s^-)dm(s)]k(t^-)dm(t) =$$

$$\int [\int T(f^s)(t) g(s^-) dm(s)] k(t^-) dm(t)$$

and from Fubini's theorem

$$= \int g(s^-) \int T(f^s)(t) k(t^-) dm(t) dm(s)$$

and from (5.11)

$$\int g(s^-) \int f^s(t) h(t^-) dm(t) dm(s) =$$

$$\int h(t^-) \int f^s(t) g(s^-) dm(s) dm(t) =$$

$$\int (f * g)(t) h(t^-) dm(t) = \int T(f * g)(t) k(t^-) dm(t).$$

And since k was arbitrarily chosen in $L_\infty(H)$ it follows that $T(f) * g = T(f * g)$ for all $f, g \in L_1(H)$. At this point commutativity is not assume and will be assume now

(ii)implies (iii)

Suppose $T(f) * g = T(f * g)$ for all $f, g \in L_1(H)$. Then since H is commutative, $L_1(H)$ is commutative that is $f*g = g*f$ for all $f, g \in L_1(H)$ so that $T(f*g) = T(g*f)$ and we have

$$T(f * g) = T(f) * g = T(g * f) = T(g) * f$$

In particular for all $f, g \in L_1(H)$ we have

$$(\widehat{T(f) * g}) = (\widehat{T(g) * f}) \Rightarrow \widehat{T(f)}\hat{g} = \widehat{T(g)}\hat{f}$$

Now for all $\chi \in \hat{H}$ choose $g \in L_1(H)$ such that $\hat{g}(\chi) \neq 0$ (see Hille and Philips [HP57] 4.15 for the existence of such g) define $\varphi(\chi) = \frac{\widehat{(T(g)}(\chi)}{\hat{g}(\chi)}$ then the equation

$$\widehat{T(f)}\hat{g} = \hat{f}\widehat{T(g)} \Rightarrow \frac{\widehat{T(f)}}{\hat{f}}(\chi) = \frac{\widehat{T(g)}}{\hat{g}}(\chi)$$

Therefore φ is independent of g and we have

$$\widehat{T(f)}\hat{g}(\chi) = \hat{f}\widehat{T(g)}(\chi)$$

which implies

$$\widehat{T(f)}(\chi) = \hat{f}(\chi)(\frac{\widehat{T(g)}}{\hat{g}})(\chi) =$$

$$\varphi(\chi)\hat{f}(\chi) = (\varphi\hat{f})(\chi)$$

Therefore $\widehat{T(f)} = \varphi\hat{f}$

(iii) implies (iv)

Suppose that $\widehat{T(f)} = \varphi\hat{f}$ for all $f \in L_1(H)$. That is $\varphi\hat{f} \in \widehat{L_1(H)}$. It follows that $\varphi\hat{f}$ is a Fourier transform (of $T(f)$)and since $\varphi \in C(\hat{H})$ [$\varphi\hat{f}$ is continuous] φ is a Fourier Stieltjes transform [[Ls82] Theorem 2.1.3], that is, there exists $\mu \in M(H)$ such that $\varphi = \hat{\mu}$ so $\widehat{T(f)} = \hat{\mu}\hat{f}$

(iv) implies (v)

$\widehat{T(f)} = \hat{\mu}\hat{f} = \widehat{\mu * f}$ Now $(T(f) - \mu * f)\hat{} = 0$ implies $T(f) = \mu * f$

Finally (v) implies (i)

Since $f^s \in L_1(H)$ there exists $\mu \in M(H)$ such that

$$T(f^s) = \mu * f^s = \mu * (f * \delta_{s^-}) = \mu * f * \delta_{s^-}$$

but $(\mu * \nu) * f = \mu * (\nu * f)$ so that

$$T(f^s) = (\mu * f) * \delta_{s^-} = (\mu * f)^s = T(f)^s$$

So $T(f^s) = T(f)^s$.

The next theorem is a reduced form of Wendel's theorem for locally compact non-commutative hypergroups It is stated without proof in [[BH95] Theorem 1.6.24.]

Theorem 5.2.2 *Suppose H is a locally compact (not necessarily commutative) and T*

is a bounded linear transformation of $L_1(H)$ into itself. Then the following statements are equivalent

i. *T commutes with right translation operator that is $T(f^s) = T(f)^s$ for all $s \in H$*

ii. *$T(f * g) = T(f) * g$ for each $f, g \in L_1(H)$*

iii. *There exists a unique measure $\mu \in M(H)$ such that $T(f) = f * \mu$ for each $f \in L_1(H)$*

Proof:

(i) implies (ii) and (iii) implies (i) follow exactly as above and (ii) implies (iii) follows as in [[HR70] Theorem 35.5].

We now give a proof of Theorem (5.2.1) for compact (not necessarily commutative) hypergroups.

Theorem 5.2.3 *Let H be a compact (not necessarily commutative) hypergroup. Suppose $T : L_1(H) \to L_1(H)$ is a bounded linear transformation. Then the following statements are equivalent:*

i. *T commutes with right translation operators that is $T(f^s) = T(f)^s$ for all $s \in H$*

ii. *$T(f * g) = T(f) * g$ for each $f, g \in L_1(H)$*

iii. *There exists a unique transformation φ on $\hat{H}$ such that $\widehat{T(f)} = \varphi\hat{f}$ for each $f \in L_1(H)$.*

iv. *There exists a unique measure $\mu \in M(H)$ such that $\widehat{T(f)} = \hat{\mu}\hat{f}$ for each $f \in L_1(H)$*

v. *There exists a unique measure $\mu \in M(H)$ such that $T(f) = f * \mu$ for each $f \in L_1(H)$*

Proof:

From Theorem (5.2.2) (i),(ii), (v) are equivalent. We now show that (ii) and (iii) are equivalent.

(ii)implies (iii) Let $\hat{H}$ be the dual object of H, suppose $U \in \hat{H}$ and $\{\tau_j\}_{j=1}^{d_U}$ is an orthonormal basis for H_U (The Hilbert space associated with U with dimension d_U). With coordinate functions defined for U by

$$u_{jk}(x) =< u_x \tau_k, \tau_j >$$

where $1 \leq j, k \leq d_U$ then if $U, V \in \hat{H}$, there exists a constant k_U with $k_U \geq d_U$ such that

$$\int u_{jk}(\bar{v_{rs}})dm = \begin{cases} k_u^{-1} & \text{when } U = V, \text{ j=r,k=s,} \\ 0 & \text{otherwise.} \end{cases}$$

moreover if H is a compact hypergroup then $k_U = d_U$[Vr79] theorem 2.6.

Now let $\chi_U = k_U^{-1} I_U$ where $k_U \geq d_U$. Then χ_U is in the center $Z(L_1(H))$ of $L_1(H)$ that is $f * \chi_U = \chi_U * f$ because

$$(\widehat{f * \chi_U}) = \hat{f}\hat{\chi_U} = \hat{f} k_U^{-1} \hat{I_U} = k_U^{-1} \hat{I_U} \hat{f}$$

$$(\widehat{k_U^{-1} I_U})\hat{f} = [(k_U^{-1} I_U) * f\widehat{]} = (\widehat{\chi_U * f})$$

So $f * \chi_U = \chi_U * f$

We can now define $\varphi(U) = T(\widehat{k_U \chi_U})(U)$

$$\widehat{Tf}(U) = \widehat{Tf}(U)\hat{I_U}(U) = \widehat{Tf}(U)(\widehat{k_U \chi_U})(U) =$$

$$[(Tf) * k_U \chi_U\widehat{]}(U) = [T(f * k_U \chi_U)\widehat{]}(U) =$$

$$[T(k_U \chi_U * f\widehat{]}(U) = [T(k_U \chi_U) * f\widehat{]}(U) =$$

$$T(\widehat{k_U \chi_U})(U)\hat{f}(U) = \varphi(U)\hat{f}(U)$$

that is $(Tf\hat{)}(U) = \varphi(U)\hat{f}(U) = (\varphi\hat{f})(U)$ which implies $Tf = \varphi\hat{f}$.

Now (iii) implies (ii), assume $T(f*g)=T(f)*g$ for each $f,g\in L_1(H)$ then There exists a unique transformation φ on $\hat{H}$ such that $\widehat{T(f)}=\varphi\hat{f}$

$$\widehat{T(f*g)}=\varphi\widehat{f*g}=\varphi\hat{f}\hat{g}=\widehat{T(f)}\hat{g}=\widehat{T(f)*g}$$

so that $T(f*g)=T(f)*g$

The equivalence of (iv) and (v) is obtained from the isomorphism of the Fourier transform.

Remark

When H is a compact commutative hypergroup and $1<p\leq\infty$, then $L_p(H)\subset L_1(H)$. So the above characterizations of multipliers apply to elements of $L_p(H)$ as well.

Remark

The next section deals with multipliers on $L_p(H)$ spaces for $1\leq p<\infty$, where H is a compact commutative hypergroup.

5.3 Semigroups of Operators, Semigroups of Multipliers on $L_p(H)$

Definition 5.3.1 *Let X be a Banach space. Denote by $B(X)$ the Banach algebra of all bounded linear operators on X, with the operator norm. A family $\mathcal{S}=\{T(\xi):\xi>0\}$ of operators in $B(X)$ is called a semigroup of operators on X if and only if*

$$T(\xi_1+\xi_2)=T(\xi_1)T(\xi_2)$$

for all $\xi_1,\xi_2>0$

*The **infinitesimal operator** A_0 of $\mathcal{S}$ is defined as the limit in norm as $\eta\to 0+$ of*

$$A_\eta x=\frac{1}{\eta}[T(\eta)-I]x$$

whenever it exists. In general A_0 is an unbounded linear operator; however the domain of A_0 is dense in the union of the range spaces of $\{T(\alpha); \alpha > 0\}$. The operator A_0 is in general not closed; its closure A, when it exists, will be called the ***infinitesimal generator*** *of $\mathcal{S}$..*

A comprehensive account of semigroups of operators on Banach spaces can be found in Hille and Phillips [HP57] ,where all undefined terms used in this work in connection with such semigroups are explained.

Theorem 5.3.1 *Let $\mathcal{S} = \{T(\xi) : \xi > 0\}$ be a semigroup of bounded linear operators on $\mathcal{U} = L_p(H)$. Suppose that for each $\xi > 0$, the operator $T(\xi)$ commute with translations. Then $\mathcal{S}$ defines a semigroup $\mathcal{M} = \{E_\xi : \xi > 0\}$ of $(\mathcal{U},\mathcal{U})$-multipliers such that*

i. *For each $\xi > 0$, $E_\xi \hat{f} = (\widehat{T(\xi)f})$ for each $f \in \mathcal{U}$; and*

ii. $E_{\xi_1+\xi_2}(\chi) = E_{\xi_1}(\chi)E_{\xi_2}(\chi)$*, $\xi_1, \xi_2 > 0$ and $\chi \in \hat{H}$*

If moreover, $T(\xi)$ is weakly measurable, then there exists a subset $\hat{H}_0$ of $\hat{H}$ and a mapping $\varphi : \chi \mapsto \varphi_\chi$ of $\hat{H}_0$ into $\mathbb{C}$ such that

$$E_\xi(\chi) = \begin{cases} e^{\varphi(\chi)\xi} & \text{if } \chi \in \hat{H}_0, \\ 0 & \text{if } \chi \notin \hat{H}_0. \end{cases}$$

for each $\xi > 0$

Proof:

i. For all $\xi > 0$, $T(\xi)$ is a continuous linear operator on $\mathcal{U}$ which commutes with translation and from Theorem (5.2.1), there is a unique E_ξ on $\hat{H}$ such that $\widehat{T(\xi)f} = E_\xi \hat{f}$ for all $f \in \mathcal{U}$

ii. Now for all $\xi_1, \xi_2 > 0$ $T\widehat{(\xi_1+\xi_2)f} = E_{\xi_1+\xi_2}\hat{f}$ But

$$T\widehat{(\xi_1+\xi_2)f} = [T\widehat{(\xi_1)T(\xi_2)}]f =$$

$$E_{\xi_1}(\widehat{T(\xi_2)f}) = E_{\xi_1}(E_{\xi_2}\hat{f})$$

Since f is arbitrary, $E_{\xi_1+\xi_2} = E_{\xi_1}E_{\xi_2}$ That is $E_{\xi_1}(\chi)E_{\xi_2}(\chi) = E_{\xi_1+\xi_2}(\chi)$ which proves (ii).

Suppose now that $T(\xi)$ is weakly measurable. Then for each continuous linear functional ψ on $\mathcal{U}$ and for each $f \in \mathcal{U}$,

$$\xi \mapsto \psi(T(\xi)f)$$

from $\mathbb{R} \to \mathbb{C}$ is Lebesgue measurable. In particular if for each $\chi \in \hat{H}$ we define ψ_χ by $\psi_\chi(f) = \hat{f}(\chi)$, $f \in \mathcal{U}$, then ψ_χ is a continuous linear operator on $\mathcal{U}$ such that the mapping

$$\xi \mapsto \psi_\chi(T(\xi)\chi) = E_\xi(\chi)\hat{\chi}(\chi)$$

is measurable. It follows that for each χ, $E_\xi(\chi)$ is measurable. Since

$$E_{\xi_1+\xi_2}(\chi) = E_{\xi_1}(\chi)E_{\xi_2}(\chi)$$

$E_\xi(\chi)$ is a measurable character and from [Hille and Phillips [HP57] corollary to theorem 4.17.3] it follows that for nontrivial characters

$$E_\xi(\chi) = e^{\varphi(\chi)\xi}$$

for some complex numbers $\varphi(\chi)$. Now we can set

$$\hat{H}_0 = \{\chi \in \hat{H} : E_\xi(\chi) \neq 0\}$$

and

$$\varphi : \chi \mapsto \varphi(\chi) : \hat{H}_0 \to \mathbb{C}$$

is a well defined mapping such that

$$E_\xi(\chi) = \begin{cases} e^{\varphi(\chi)\xi} & \text{if } \chi \in \hat{H}_0, \\ 0 & \text{if } \chi \notin \hat{H}_0. \end{cases}$$

for each $\xi > 0$.Which ends the proof.

Now let $\hat{H}_0$ be a fixed subset of $\hat{H}$ and let

$$E_\xi(\chi) = \begin{cases} e^{\varphi(\chi)\xi} & \text{if } \chi \in \hat{H}_0, \\ 0 & \text{if } \chi \notin \hat{H}_0. \end{cases}$$

for each $\xi > 0$

Assume that E_ξ as defined here is a $(\mathcal{U},\mathcal{U})$-multiplier, then we have

Theorem 5.3.2 *For each $\xi > 0$, define a mapping $T(\xi)$ of $\mathcal{U}$ into itself by $\widehat{T(\xi)f} = E_\xi \hat{f}$, $f \in \mathcal{U}$ then*

i. *$\mathcal{S} = \{T(\xi) : \xi > 0\}$ defines a semigroup of bounded linear operators on $\mathcal{U}$, the elements of which commute with translations and are continuous in the strong operator topology for $\xi > 0$.*

ii *For each $f \in D(A_0)$ and $\chi \notin \hat{H}_0$ we have $\hat{f}(\chi) = 0$ where A_0 denotes the infinitesimal operator of $\mathcal{S}$ and $D(A_0)$ is the domain of A_0. Moreover, φ is a $(D(A_0),\mathcal{U})$-multiplier since $\widehat{A_0 f} = \varphi\hat{f}$ for all $f \in D(A_0)$*

iii *If $\mathcal{S}$ is of class (A), $\overline{\hat{H}_0} = \hat{H}$ and*

$$D(A) = \{f \in \mathcal{U} : \varphi\hat{f} \in \hat{\mathcal{U}}\}$$

That is φ is a $(D(A),\mathcal{U})$-multiplier and moreover $\widehat{Af} = \varphi\hat{f}$ for all $f \in D(A)$, where A is the infinitesimal generator of $\mathcal{S}$

Proof:

i. From proposition (5.2.1)(vi), $T(\xi)$ is a bounded linear operator for each $\xi > 0$, moreover we have

$$\widehat{T(\xi_1 + \xi_2)}f = E_{\xi_1+\xi_2}\hat{f} = E_{\xi_1}(E_{\xi_2}\hat{f}) =$$

$$E_{\xi_1}(\widehat{T(\xi_2)})f = \widehat{T(\xi_1)T(\xi_2)}f$$

So $T(\xi_1 + \xi_2) = T(\xi_1)T(\xi_2)$ and $\mathcal{S} = \{T(\xi) : \xi > 0\}$is a semigroup of bounded linear operators on $\mathcal{U}$. And from the definition of $T(\xi)$ each of the operators $T(\xi)$ commutes with translation.

Now to prove that $T(\xi)$ is continuous in the strong operator topology for $\xi > 0$, first suppose that $t \in I(H)$,the set of all finite complex linear combination of continuous characters on H. Thus t is of the form $t = \sum_{i=1}^{n} \alpha_i\chi_i$, the orthogonality of $I(H)$ implies $T(\xi)$ is defined by

$$\widehat{T(\xi)}t(x) = \sum_{i=1}^{n} \alpha_i e^{\varphi(\chi_i)\xi}(x)$$

$x \in H$ then we have

$$\|T(\xi)t - T(\xi_0)t\| = \| \sum_{i=1}^{n}[\alpha_i e^{\varphi(\chi_i)\xi}\chi_i - \alpha_i e^{\varphi(\chi_i)\xi_0}\chi_i]\| \leq$$

$$\sum_{i=1}^{n} |\alpha_i||e^{\varphi(\chi_i)\xi} - e^{\varphi(\chi_i)\xi_0}| \to 0$$

as

$$\xi \to \xi_0$$

Suppose now that f is arbitrarily chosen in $\mathcal{U}$ and let $\epsilon > 0$ be given. Then, there exists $t \in I(H)$ such that $\|f - t\| < \varepsilon$ since

$$\|T(\xi)f - T(\xi_0)t\| \leq \|T(\xi)\|\|f - t\|$$

for each $\xi > 0$, $T(\xi)$ is strongly measurable [HP57] 3.5.4. Hence by [HP57] 10.2.3, $T(\xi)$ is continuous in the strong operator topology for $\xi > 0$. This complete the proof for (i)

ii. Let $f \in D(A_0)$, then

$$\lim_{\eta \to 0^+} \frac{1}{\eta}(T(\eta)f - f)$$

exists. Let this limit be g, then $g = A_0 f$, the limit being taken in the norm topology. For each ξ

$$\frac{1}{\eta}[\widehat{T(\eta)f}(\chi) - \hat{f}(\chi)] \to \hat{g}(\chi)$$

and since $\widehat{T(\eta)f} = E_\eta \hat{f}$ we have

$$\frac{1}{\eta}[E_\eta(\chi) - 1]\hat{f}(\chi) \to \hat{g}(\chi)$$

as $\eta \to 0^+$ but

$$E_\eta(\chi) = \begin{cases} e^{\varphi(\chi)\eta} & \text{if } \chi \in \hat{H}_0, \\ 0 & \text{if } \chi \notin \hat{H}_0. \end{cases}$$

so that

$$\hat{f}(\chi) = lim_{\eta \to 0^+} \eta \hat{g}(\chi) = 0$$

if $\chi \notin \hat{H}_0$, that is $\hat{f}(\chi) = 0$ and $E_\eta(\chi) = e^{\varphi(\chi)\eta}$ if $\chi \in \hat{H}_0$, so

$$\lim_{\eta \to 0^+} \frac{1}{\eta}[E_\eta(\chi) - 1] = \lim_{\eta \to 0^+} \frac{1}{\eta}[e^{\varphi(\chi)\eta} - 1] = \varphi(\chi)$$

And we have

$$\lim_{\eta\to 0^+}\frac{1}{\eta}[E_\eta(\chi)-1]\hat{f}(\chi)=\varphi(\chi)\hat{f}(\chi)$$

That is $\widehat{A_0 f}=\varphi\hat{f}$.

iii. Suppose that $\{T(\xi):\xi>0\}$ is of class (A) with infinitesimal generator $A=\bar{A}_0$, the smallest closed extension of its infinitesimal operator A_0. Then $\mathcal{U}_0=\{T(\xi)f: f\in\mathcal{U},\xi>0\}$ and $D(A_0)$ are dense in $\mathcal{U}$. Suppose there exists $\chi_0\in\hat{H}$ such that $\chi_0\notin\hat{H}_0$ choose $f\in\mathcal{U}$ such that $\hat{f}(\chi_0)\neq 0$, then given $\varepsilon>0$ there exists an $f'\in D(A_0)$ such that $\|f'-f\|<\varepsilon$ then

$$|\hat{f}'(\chi_0)-\hat{f}(\chi_0)|\le\|f'-f\|<\varepsilon$$

and since this is true for all $\epsilon>0$, $\hat{f}'(\chi_0)=\hat{f}(\chi_0)=0$ a contradiction. Hence $\hat{H}_0=\hat{H}$. Finally let

$$\omega_0=inf\frac{1}{\xi}\log\|T(\xi)\|=\lim_{\xi\to\infty}\frac{1}{\xi}\log\|T(\xi)\|$$

That is $\mathcal{S}$ is of type ω_0. For λ with $Re(\lambda)>\omega_0$, let $R(\lambda:A)$ denote the resolvent of the infinitesimal generator A of $\mathcal{S}$ then there exists a $\omega_1>\omega_0$ such that

$$R(\lambda:A)f=\int_0^\infty e^{-\lambda\xi}T(\xi)f\,d\xi$$

$f\in\mathcal{U}_0$, $Re(\lambda)>\omega_1$

since $\forall\chi\in\hat{H}$, the mapping $f\longmapsto\hat{f}(\chi)$ is a bounded linear functional on $\mathcal{U}$, we have for all $f\in\mathcal{U}_0$

$$(R(\lambda:A)f)(\chi)=\int_0^\infty e^{-\lambda\xi}\widehat{T(\xi)f}(\chi)d\xi=$$

$$\int_0^\infty e^{-\lambda\xi}e^{\varphi(\chi)}\xi\hat{f}(\chi)d\xi=$$

$$\frac{1}{\varphi(\chi)-\lambda}e^{\varphi(\chi)-\lambda}\xi\hat{f}(\chi)|_{\xi=0}^{\xi=\infty} = (\lambda-\varphi(\chi))^{-1}\hat{f}(\chi)$$

for each $\chi \in \hat{H}$. Since $\mathcal{U}_0$ is dense in $\mathcal{U}$, we have

$$(\widehat{R(\lambda;A)}f)(\chi) = (\lambda-\varphi(\chi))^{-1}\hat{f}(\chi) \tag{5.12}$$

for all $f \in \mathcal{U}$ with $Re(\lambda) > \omega_1$.

Let $\lambda > \omega_1$ be fixed and suppose that $f \in D(A)$. Then there exists a $g \in \mathcal{U}$ such that $f = R(\lambda;A)g$ and we have for each $\chi \in \hat{H}$

$$\widehat{Af}(\chi) = [\lambda R(\lambda;A)g - \widehat{g]}(\chi) = [\lambda R(\lambda;A)\widehat{g]}(\chi) - \hat{g}(\chi) =$$

$$\lambda(\lambda-\varphi(\chi))^{-1}\hat{g}(\chi) - \hat{g}(\chi) = (\frac{\lambda}{(\lambda-\varphi(\chi))} - 1)\hat{g}(\chi)$$

$$(\frac{\lambda-\lambda+\varphi(\chi)}{\lambda-\varphi(\chi)})\hat{g}(\chi) =$$

$$\varphi(\chi)\widehat{R(\lambda;A)}g(\chi) = \varphi(\chi)\hat{f}(\chi)$$

Thus whenever $f \in D(A)$, $\varphi\hat{f} \in \hat{\mathcal{U}}$.

Conversely, suppose that f is an element of $\mathcal{U}$ such that $\varphi\hat{f} \in \hat{\mathcal{U}}$. This means that there exists an $h \in \mathcal{U}$ such that

$$\varphi(\chi)\hat{f}(\chi) = \hat{h}(\chi)$$

for all $\chi \in \hat{H}$. Then $g = \lambda f - h \in \mathcal{U}$ and for all $\chi \in \hat{H}$

$$[R(\lambda;A)\widehat{g]}(\chi) = (\lambda-\varphi(\chi))^{-1}\hat{g}(\chi) = [\lambda-\varphi(\chi)]^{-1}[\lambda\hat{f}(\chi) - \varphi(\chi)\hat{f}(\chi)] =$$

$$[\lambda-\varphi(\chi)]^{-1}[\lambda-\varphi(\chi)]\hat{f}(\chi) = \hat{f}(\chi)$$

Which implies that $R(\lambda;A)g = f$ which also implies that $f \in D(A)$

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