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Fibonacci Vectors

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by

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Dedication

For the three major role models in my life. My mom whose courage and strength make me so proud. My dad whose unconditional love is my guiding light. And for the best teacher in the world, Dr. Stone.

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ABSTRACT

By the n-th Fibonacci (respectively Lucas) vector of length m, we mean the vector whose components are the n-th through $(n+m-1)$-st Fibonacci (respectively Lucas) numbers. For arbitrary m, we express the dot product of any two Fibonacci vectors, any two Lucas vectors, and any Fibonacci vector and any Lucas vector in terms of the Fibonacci and Lucas numbers. We use these formulas to deduce a number of identities involving the Fibonacci and Lucas numbers.

1 Introduction

Seldom, in the study of mathematics, does one come across a topic so fascinating that it captivates the minds of mathematicians and non-mathematicians alike. The Fibonacci sequence, though over 700 years old, still contains many secrets yet to be discovered. The famous sequence has intrigued so many people that *The Fibonacci Quarterly*, a journal devoted solely to the study of anything Fibonacci was created in 1963, shortly after the formation of the Fibonacci Society in 1962.

Very famous during his life, Fibonacci is considered the greatest European mathematician of the middle ages. Fibonacci, short for Filius Bonacci–son of Bonacci–was born around 1170, to a Pisan Merchant who freely traveled the expanse of the Byzantine Empire. Due to the extensive traveling, Leonardo of Pisa was frequently exposed to Islamic scholars and the mathematics of the Islamic world. After his return to Pisa, Fibonacci spent the next 25 years writing books that included much of what he had learned in his travels. Though many works were lost, three main works were preserved. They are: *Liber abaci* (1202, 1228), the *Practica geometriae* (1220), and the *Liber quadratorum* (1225). Fibonacci is credited with being one of the first people to introduce the Hindu-Arabic number system into Europe–the system we now use today–based of ten digits with its decimal point and a symbol for zero: 1, 2, 3, 4, 5, 6, 7, 8, 9 and 0.

It is in *Liber abaci* that Fibonacci poses his famous rabbit question,

> A certain man put a pair of rabbits in a place surrounded on all sides by a wall. How many pairs of rabbits can be produced from that pair in a year if it is supposed that every month each pair begets a new pair which from

the second month on becomes productive?

The answer to this question involves the famous Fibonacci sequence: 1, 1, 2, 3, 5, 8, 13,... where the n-th Fibonacci number, denoted F_n, is defined for all integers n by the recurrence relation $F_{n+2} = F_{n+1} + F_n$, with starting values $F_1 = F_2 = 1$. What seems like such a simple concept appears in almost every scientific field of study from botany to architecture, and biology to painting. We are but beings swimming in the sea of Fibonacci where all we must perform is a single stroke to meet a Fibonacci number. It is this sequence of numbers that inspires all of the ideas in this paper and to no suprise we find that whenever one works with Fibonacci numbers we obtain beautiful results. We will look at vectors of the form $\vec{f_n} = \langle F_n, F_{n+1}, F_{n+2}, ..., F_{n+m-1}\rangle$, and call them Fibonacci vectors. We will study in depth the Fibonacci vectors in arbitrary dimensions. Simultaneously, we shall consider analagous results for Lucas vectors. Edouard Lucas was born in France in 1842. Following service as an artillery officer during the Franco-Prussian War (1870-1871), Lucas became professor of mathematics at the Lycée Saint Louis in Paris. He later became professor of mathematics at the Lycée Charlemagne, also in Paris. Lucas is best known for his studies in number theory. We will study the Lucas sequence: 1, 3, 4, 7, 11, 18, 29,... where the n-th Lucas number, denoted L_n, is defined for all integers n by the recurrence relation $L_{n+2} = L_{n+1} + L_n$, with starting values $L_1 = 1$, $L_2 = 3$. We look at analogous Lucas vectors of the form $\vec{\ell_n} = \langle L_n, L_{n+1}, L_{n+2}, ..., L_{n+m-1}\rangle$.

We will begin our study by recalling some well-known facts concerning the Fibonacci and Lucas vectors in two dimensions. We then generalize these facts to arbitrary dimension. From very elementary linear algebraic considerations we shall derive a number of nontrivial relations involving the Fibonacci and Lucas numbers. It turns out that vectors in a fixed dimension lie in a single plane. We consider angles between the vectors of interest.

2 Fibonacci and Lucas numbers

2.1 Fundamental Fibonacci and Lucas facts

The Fibonacci numbers F_n are defined for all integers n by the second order recurrence relation

$$F_{n+2} = F_{n+1} + F_n \tag{2.1.1}$$

and initial conditions

$$F_1 = F_2 = 1. \tag{2.1.2}$$

The Lucas numbers L_n are defined for all integers n by the same second order recurrence relation as the Fibonacci numbers

$$L_{n+2} = L_{n+1} + L_n \tag{2.1.3}$$

but initial conditions

$$L_1 = 1, \qquad L_2 = 3. \tag{2.1.4}$$

We recall some relations involving the Fibonacci and Lucas numbers. These facts are well-known and can be found in most basic references, e.g. [3, 4].

Theorem 2.1.1 ([3]) *For all integers $n \geq 1$,*

$$F_{2n} = F_{n+1}^2 - F_{n-1}^2, \tag{2.1.5}$$

$$
\begin{aligned}
F_{2n-1}F_{2n+1} &= F_{2n}^2+1, && (2.1.6)\\
F_n &= F_kF_{n-k+1}+F_{k-1}F_{n-k}, && (2.1.7)\\
F_{2n+1} &= F_n^2+F_{n+1}^2, && (2.1.8)\\
F_n &= L_{n-1}+L_{n+1}, && (2.1.9)\\
F_{-n} &= (-1)^{n+1}F_n, && (2.1.10)\\
L_{-n} &= (-1)^nL_n. && (2.1.11)
\end{aligned}
$$

2.2 A linear algebraic perspective

We recall a well-known linear algebraic approach to the Fibonacci numbers. This can be found in [4] and many elementary linear algebra books, e.g. [7, 9]. This approach appears to be due to Binet. Set

$$T = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}. \tag{2.2.12}$$

Theorem 2.2.1 ([4]) *Let T be as in Eqn.* (2.2.12)*. For all integers n, let $\vec{f}_n = [F_n, F_{n+1}]^{\mathrm{t}}$. Then*

$$\vec{f}_{n+1} = T\vec{f}_n.$$

In particular, for all integers k

$$\vec{f}_{n+1} = T^{n-k+1}\vec{f}_k. \tag{2.2.13}$$

Theorem 2.2.1 can be used to prove a number of Fibonacci identities. Observe that the rows and columns of T are simply $\vec{f}_0$ and $\vec{f}_1$, so that T^{n+1} has rows and columns $\vec{f}_{n-1}$ and $\vec{f}_n$. Thus, the simple observation that $T^n = T^{n-k}T^k$ gives the identity (2.1.7). See [4]. Recently, Askey [1, 2] and Huang [5] have given matrix theoretic proofs in this spirit of other Fibonacci identities as an interesting application of matrix multiplication.

Our work applies this sort of argument in arbitrary dimension. Rather than use matrix multiplication, we consider the dot product of vectors with components consisting of consecutive Fibonacci numbers. This is a natural generalization since the entries in a matrix product can be viewed as a dot product of a row with a column. We shall treat the Lucas vectors in the same manner.

2.3 The Binet formulas

We recall closed-form formulas for the Fibonacci and Lucas numbers known as the Binet Formulas. These formulas are often found as an application of Theorem 2.2.1 in elementary linear algebra textbooks. We recall these formulas and the common linear algebraic derivation now.

Lemma 2.3.1 *The matrix T of Eqn.* (2.2.12) *has characteristic polynomial x^2-x-1. Thus it has eigenvalues $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. The associated eigenvectors are respectively*

$$\begin{bmatrix} 1 \\ \alpha \end{bmatrix}, \qquad \begin{bmatrix} 1 \\ \beta \end{bmatrix}.$$

Lemma 2.3.2 *For all integers n,*

$$\begin{aligned} \vec{f_n} &= \frac{1}{\alpha-\beta}\left(\alpha^n \begin{bmatrix} 1 \\ \alpha \end{bmatrix} - \beta^n \begin{bmatrix} 1 \\ \beta \end{bmatrix}\right), \\ \vec{\ell_n} &= \alpha^n \begin{bmatrix} 1 \\ \alpha \end{bmatrix} + \beta^n \begin{bmatrix} 1 \\ \beta \end{bmatrix}. \end{aligned}$$

Proof. Observe that

$$\begin{aligned} \vec{f_0} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} &= \frac{1}{\alpha-\beta}\left(\alpha^0 \begin{bmatrix} 1 \\ \alpha \end{bmatrix} - \beta^0 \begin{bmatrix} 1 \\ \beta \end{bmatrix}\right), \\ \vec{\ell_0} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} &= \alpha^0 \begin{bmatrix} 1 \\ \alpha \end{bmatrix} + \beta^0 \begin{bmatrix} 1 \\ \beta \end{bmatrix}. \end{aligned}$$

Now,

$$\begin{aligned}
\vec{f}_n = T^n \vec{f}_0 &= \frac{1}{\alpha-\beta}\left(T^n \begin{bmatrix} 1 \\ \alpha \end{bmatrix} - T^n \begin{bmatrix} 1 \\ \beta \end{bmatrix}\right) && \text{by Eqn. (2.2.13)} \\
&= \frac{1}{\alpha-\beta}\left(\alpha^n \begin{bmatrix} 1 \\ \alpha \end{bmatrix} - \beta^n \begin{bmatrix} 1 \\ \beta \end{bmatrix}\right) && \text{by Lem. 2.3.1.}
\end{aligned}$$

■

Theorem 2.3.3 *For all integers n,*

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \tag{2.3.14}$$

$$L_n = \alpha^n + \beta^n. \tag{2.3.15}$$

Proof. Equate the first components on each side of these equations of Lemma 2.3.2. ■

Equations (2.3.14) and (2.3.15) are generally known as the *Binet formulas for F_n and L_n* (although previously known to Euler and Daniel Bernoulli [4]). We shall make extensive use of the Binet formulas, so we present some formulas involving α and β and some alternate versions of the Binet formulas.

Lemma 2.3.4

$$\alpha\beta = -1, \tag{2.3.16}$$

$$\alpha + \beta = 1, \tag{2.3.17}$$

$$\alpha - \beta = \sqrt{5}, \tag{2.3.18}$$

$$\alpha^2 + 1 = \sqrt{5}\alpha, \tag{2.3.19}$$

$$\beta^2 + 1 = -\sqrt{5}\beta, \tag{2.3.20}$$

$$\alpha = 2 - \beta^2, \tag{2.3.21}$$

$$\beta = 2 - \alpha^2. \tag{2.3.22}$$

Proof. Immediate since α and β are the roots of $x^2 - x - 1$. ∎

Lemma 2.3.5 *[4] For all integers n,*

$$\alpha^n = F_n\alpha + F_{n-1}, \tag{2.3.23}$$

$$\beta^n = F_n\beta + F_{n-1}. \tag{2.3.24}$$

Lemma 2.3.6 *For all integers n_1 and n_2,*

$$\alpha^{n_1}\beta^{n_2} + \alpha^{n_2}\beta^{n_1} = (-1)^{n_1}L_{n_2-n_1}, \tag{2.3.25}$$

$$\alpha^{n_1}\beta^{n_2} - \alpha^{n_2}\beta^{n_1} = (-1)^{n_1+1}(\alpha-\beta)F_{n_2-n_1}. \tag{2.3.26}$$

Proof. Observe that

$$\begin{aligned}
\alpha^{n_1}\beta^{n_2} &= \frac{1}{2}\left(\alpha^{n_1}\beta^{n_2} + \alpha^{n_1}\beta^{n_2}\right) \\
&= \frac{1}{2}\big((-1)^{n_1}\beta^{n_2-n_1} + (-1)^{n_2}\alpha^{n_1-n_2}\big) \qquad \text{by Eqn. (2.3.16)}, \\
\alpha^{n_2}\beta^{n_1} &= \frac{1}{2}\left(\alpha^{n_2}\beta^{n_1} + \alpha^{n_2}\beta^{n_1}\right) \\
&= \frac{1}{2}\big((-1)^{n_2}\beta^{n_1-n_2} + (-1)^{n_1}\alpha^{n_2-n_1}\big) \qquad \text{by Eqn. (2.3.16)}.
\end{aligned}$$

Thus

$$\begin{aligned}
\alpha^{n_1}\beta^{n_2} + \alpha^{n_2}\beta^{n_1} &= \frac{1}{2}\big((-1)^{n_1}(\alpha^{n_2-n_1} + \beta^{n_2-n_1}) + (-1)^{n_2}(\alpha^{n_1-n_2} + \beta^{n_1-n_2})\big) \\
&= \frac{(-1)^{n_1}L_{n_2-n_1} + (-1)^{n_2}L_{n_1-n_2}}{2} \qquad \text{by Eqn. (2.3.15)} \\
&= (-1)^{n_1}L_{n_2-n_1} \qquad \text{by Eqn. (2.1.11)} \\
&= (-1)^{n_2}L_{n_1-n_2} \qquad \text{by Eqn. (2.1.11)}.
\end{aligned}$$

Similarly,

$$\alpha^{n_1}\beta^{n_2} - \alpha^{n_2}\beta^{n_1} = \frac{1}{2}\big((-1)^{n_1}(-\alpha^{n_2-n_1} + \beta^{n_2-n_1}) + (-1)^{n_2}(\alpha^{n_1-n_2} - \beta^{n_1-n_2}\big)$$

$$
\begin{aligned}
&= \frac{(\alpha-\beta)}{2}\left((-1)^{n_1+1}F_{n_2-n_1}+(-1)^{n_2}F_{n_1-n_2}\right) \qquad \text{by Eqn. (2.3.14)}\\
&= (-1)^{n_1+1}(\alpha-\beta)F_{n_2-n_1} \qquad \text{by Eqn. (2.1.10)}\\
&= (-1)^{n_2}(\alpha-\beta)F_{n_1-n_2} \qquad \text{by Eqn. (2.1.10).}
\end{aligned}
$$

■

Corollary 2.3.7 *For all integers n,*

$$
\begin{aligned}
\left(\frac{\alpha}{\beta}\right)^n + \left(\frac{\beta}{\alpha}\right)^n &= (-1)^n L_{2n},\\
\left(\frac{\alpha}{\beta}\right)^n - \left(\frac{\beta}{\alpha}\right)^n &= (-1)^n(\alpha-\beta)F_{2n}.
\end{aligned}
$$

Proof. Take $n_1 = n$ and $n_2 = -n$ in (2.3.25) and (2.3.26).

■

Lemma 2.3.8 *For all integers n,*

$$
\begin{aligned}
F_n &= \frac{1}{\alpha^{n-1}}\left(\frac{\alpha^{2n}-(-1)^n}{\alpha^2-1}\right) &(2.3.27)\\
&= \frac{1}{\beta^{n-1}}\left(\frac{\beta^{2n}-(-1)^n}{\beta^2-1}\right), &(2.3.28)\\
L_n &= \frac{\alpha^{2n}+(-1)^n}{\alpha^n} &(2.3.29)\\
&= \frac{\beta^{2n}+(-1)^n}{\beta^n}. &(2.3.30)
\end{aligned}
$$

Proof. Compute

$$
\begin{aligned}
F_n &= \frac{\alpha^n-\beta^n}{\alpha-\beta} \qquad \text{by Eqn. (2.3.14)}\\
&= \frac{\alpha^{2n}-\alpha^n\beta^n}{\alpha^2-\alpha\beta}\frac{\alpha}{\alpha^n}\\
&= \frac{1}{\alpha^{n-1}}\left(\frac{\alpha^{2n}-(-1)^n}{\alpha^2-1}\right) \qquad \text{by Eqn. (2.3.16),}\\
L_n &= \alpha^n+\beta^n
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\alpha^{2n} + \alpha^n \beta^n}{\alpha^n} \\
&= \frac{\alpha^{2n} + (-1)^n}{\alpha^n}
\end{aligned}
$$

Equations (2.3.29) and (2.3.30) are derived similarly.

$\blacksquare$

3 Fibonacci and Lucas vectors

In this chapter we study Fibonacci and Lucas vectors in arbitrary dimension from a linear algebraic perspective.

3.1 Linear algebraic set up

Throughout this section m shall be a fixed positive integer.

Define an $m \times m$ matrix T by

$$T = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 1 \end{bmatrix}. \tag{3.1.1}$$

Lemma 3.1.1 *The matrix T of Eqn.* (3.1.1) *has characteristic polynomial $x^{(m-1)}(x^2 - x - 1)$. Thus the non-zero eigenvalues of T are $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$, each with geometric and algebraic multiplicity 1. The eigenspaces associated with α*

and β are spanned respectively by

$$\vec{a} = \begin{bmatrix} 1 \\ \alpha \\ \alpha^2 \\ \vdots \\ \alpha^{m-1} \end{bmatrix} \text{ and } \vec{b} = \begin{bmatrix} 1 \\ \beta \\ \beta^2 \\ \vdots \\ \beta^{m-1} \end{bmatrix}.$$

Proof. Elementary linear algebra. ■

Observe that the matrix T and vectors $\vec{a}$ and $\vec{b}$ depend upon the dimension m. Since m will always be clear from context, we shall suppress this dependence in notation. We shall frequently use the formula for the sum of a finite geometric series.

Lemma 3.1.2 *For all real numbers $c \neq 1$ and all positive integers m,*

$$\sum_{j=0}^{m-1} c^j = \frac{c^m - 1}{c - 1}. \tag{3.1.2}$$

Lemma 3.1.3

$$\vec{a} \cdot \vec{a} = \begin{cases} F_m(\alpha - \beta)\alpha^{m-1} & \text{if } m \text{ is even,} \\ L_m \alpha^{m-1} & \text{if } m \text{ is odd,} \end{cases} \tag{3.1.3}$$

$$\vec{b} \cdot \vec{b} = \begin{cases} -F_m(\alpha - \beta)\beta^{m-1} & \text{if } m \text{ is even,} \\ L_m \beta^{m-1} & \text{if } m \text{ is odd,} \end{cases} \tag{3.1.4}$$

$$\vec{a} \cdot \vec{b} = \begin{cases} 0 & \text{if } m \text{ is even,} \\ 1 & \text{if } m \text{ is odd.} \end{cases} \tag{3.1.5}$$

Proof. By definition of dot product

$$\begin{aligned} \vec{a} \cdot \vec{a} &= \sum_{j=0}^{m-1} \alpha^{2j} \\ &= \sum_{j=0}^{m-1} \left(\frac{-\alpha}{\beta} \right)^j \qquad \text{by Eqn. (2.3.16)} \end{aligned}$$

$$
\begin{aligned}
&= \frac{(-\alpha/\beta)^m - 1}{(-\alpha/\beta) - 1} \qquad \text{by Eqn. (3.1.2)} \\
&= \frac{1}{\beta^{m-1}} \frac{(-1)^m \alpha^m - \beta^m}{-\alpha - \beta} \\
&= \alpha^{m-1} \left(\alpha^m - (-1)^m \beta^m\right) \qquad \text{by Lem. 2.3.4.}
\end{aligned}
$$

Suppose m is even. Then

$$
\begin{aligned}
\vec{a} \cdot \vec{a} &= \alpha^{m-1}(\alpha - \beta) \left(\frac{\alpha^m - \beta^m}{\alpha - \beta} \right) \qquad \text{by Eqn. (2.3.16)} \\
&= \alpha^{m-1}(\alpha - \beta) F_m \qquad \text{by Eqn. (2.3.14).}
\end{aligned}
$$

Now suppose m is odd. Then

$$
\begin{aligned}
\vec{a} \cdot \vec{a} &= \alpha^{m-1} \left(\alpha^m + \beta^m\right) \qquad \text{by Eqn. (2.3.16)} \\
&= \alpha^{m-1} L_m \qquad \text{by Eqn. (2.3.15).}
\end{aligned}
$$

The computation of $\vec{b} \cdot \vec{b}$ is similar, so we omit it.

By definition of dot product

$$
\begin{aligned}
\vec{a} \cdot \vec{b} &= \sum_{j=0}^{m-1} \alpha^j \beta^j \\
&= \sum_{j=0}^{m-1} (-1)^j \qquad \text{by Eqn. (2.3.16)} \\
&= \begin{cases} 0 & \text{if } m \text{ is even,} \\ 1 & \text{if } m \text{ is odd.} \end{cases}
\end{aligned}
$$

■

Recall that for any vector $\vec{v}$, the square of the length of $\vec{v}$ is

$$
\|\vec{v}\|^2 = \vec{v} \cdot \vec{v}.
$$

Corollary 3.1.4

$$\|\vec{a}\| = \begin{cases} \sqrt{F_m(\alpha-\beta)\alpha^{m-1}} & \textit{if } m \textit{ is even,} \\ \sqrt{L_m\alpha^{m-1}} & \textit{if } m \textit{ is odd,} \end{cases} \tag{3.1.6}$$

$$\left\|\vec{b}\right\| = \begin{cases} \sqrt{-F_m(\alpha-\beta)\beta^{m-1}} & \textit{if } m \textit{ is even,} \\ \sqrt{L_m\beta^{m-1}} & \textit{if } m \textit{ is odd,} \end{cases} \tag{3.1.7}$$

$$\|\vec{a}\|\left\|\vec{b}\right\| = \begin{cases} F_m(\alpha-\beta) & \textit{if } m \textit{ is even,} \\ L_m & \textit{if } m \textit{ is odd,} \end{cases} \tag{3.1.8}$$

3.2 Fibonacci and Lucas vectors

Definition 3.2.1 For all positive integers m and for all integers n, define

$$\vec{f}_n^m = \begin{bmatrix} F_n \\ F_{n+1} \\ \vdots \\ F_{n+m-1} \end{bmatrix}, \qquad \vec{\ell}_n^m = \begin{bmatrix} L_n \\ L_{n+1} \\ \vdots \\ L_{n+m-1} \end{bmatrix}.$$

We refer to $\vec{f}_n^m$ and $\vec{\ell}_n^m$ as the *n-th Fibonacci and Lucas vectors of length m*, respectively.

The vectors $\vec{f}_n$ and $\vec{\ell}_n$ of Section 2.2 are just $\vec{f}_n^2$ and $\vec{\ell}_n^2$ in the present notation. We shall carry the dimension m in the notation as we shall have occasion below to use more than one value of m in the same equation.

A number of other relations among the Fibonacci and Lucas numbers generalize to the corresponding vectors. Observe that the $\vec{f}_n^m$ and $\vec{\ell}_n^m$ satisfy the vector recurrence relation

$$\vec{x}_{n+2} = \vec{x}_{n+1} + \vec{x}_n.$$

The analog of Theorem 2.2.1 is the following.

Lemma 3.2.2 *Let T be as in Eqn.* (3.1.1)*. Then for all integers n,*

$$\vec{f}_{n+1}^m = T\vec{f}_n^m,$$

$$\vec{\ell}^{m}_{n+1} = T\vec{\ell}^{m}_{n}.$$

Thus for all integers $k \geq n+1$,

$$\begin{aligned}\vec{f}^{m}_{n+1} &= T^{n-k+1}\vec{f}^{m}_{k},\\ \vec{\ell}^{m}_{n+1} &= T^{n-k+1}\vec{\ell}^{m}_{k}.\end{aligned}$$

Vector versions of the Binet formulas hold here.

Theorem 3.2.3 *For all integers n,*

$$\vec{f}^{m}_{n} = \frac{1}{\alpha-\beta}\left(\alpha^n\vec{a}-\beta^n\vec{b}\right), \tag{3.2.9}$$

$$\vec{\ell}^{m}_{n} = \alpha^n\vec{a}+\beta^n\vec{b}. \tag{3.2.10}$$

In particular, $\vec{f}^{m}_{n}$ and $\vec{\ell}^{m}_{n}$ lie in the plane spanned by $\vec{a}$ and $\vec{b}$.

Proof. The j-th entry of $\vec{f}^{m}_{n}$ is $F_{n+j-1} = (\alpha^{n+j-1}-\beta^{n+j-1})/(\alpha-\beta)$. The j-th entry of $\left(\alpha^n\vec{a}-\beta^n\vec{b}\right)/(\alpha-\beta)$ is $(\alpha^n\alpha^{j-1}-\beta^n\beta^{j-1})/(\alpha-\beta)$. The first equation follows since both sides have the same entries. A similar argument gives the second equation. ∎

The analog of Lemma 2.3.5 is the following.

Lemma 3.2.4 *For all integers n,*

$$\begin{aligned}\alpha^n\vec{a} &= \alpha\vec{f}^{m}_{n}+\vec{f}^{m}_{n-1},\\ \beta^n\vec{b} &= \beta\vec{f}^{m}_{n}+\vec{f}^{m}_{n-1}.\end{aligned}$$

Proof. Compute corresponding entries on each side of the equations and note that they are equal by Lemma 2.3.5. ∎

3.3 Dot products

We compute the dot products of Fibonacci and Lucas vectors.

Theorem 3.3.1 *For all positive integers m and for all integers n_1 and n_2,*

$$\vec{f}^m_{n_1} \cdot \vec{f}^m_{n_2} = \begin{cases} F_m F_{n_1+n_2+m-1} & \text{if } m \text{ is even,} \\ \frac{1}{5}\left(L_m L_{n_1+n_2+m-1} - (-1)^{n_1} L_{n_2-n_1}\right) & \text{if } m \text{ is odd.} \end{cases} \tag{3.3.11}$$

Proof.

$$\begin{aligned} \vec{f}^m_{n_1} \cdot \vec{f}^m_{n_2} &= \frac{1}{\alpha-\beta}\left(\alpha^{n_1}\vec{a} - \beta^{n_1}\vec{b}\right) \cdot \frac{1}{\alpha-\beta}\left(\alpha^{n_2}\vec{a} - \beta^{n_2}\vec{b}\right) \qquad \text{by Eqn. (3.2.9)} \\ &= \frac{1}{5}\left(\alpha^{n_1+n_2}\vec{a}\cdot\vec{a} + \beta^{n_1+n_2}\vec{b}\cdot\vec{b} - (\alpha^{n_1}\beta^{n_2} + \alpha^{n_2}\beta^{n_1})\vec{a}\cdot\vec{b}\right). \end{aligned}$$

Suppose m is even. Then

$$\begin{aligned} \vec{f}^m_{n_1} \cdot \vec{f}^m_{n_2} &= \frac{F_m}{\alpha-\beta}\left(\alpha^{n_1+n_2+m-1} - \beta^{n_1+n_2+m-1}\right) \qquad \text{by Lem. 3.1.3} \\ &= F_m F_{n_1+n_2+m-1} \qquad \text{by Eqn. (2.3.14).} \end{aligned}$$

Now suppose m is odd. Then

$$\begin{aligned} \vec{f}^m_{n_1} \cdot \vec{f}^m_{n_2} &= \frac{1}{5}\left(L_m(\alpha^{n_1+n_2+m-1} + \beta^{n_1+n_2+m-1}) - (\alpha^{n_1}\beta^{n_2} + \alpha^{n_2}\beta^{n_1})\right) \\ &= \frac{1}{5}\left(L_m L_{n_1+n_2+m-1} - (-1)^{n_1} L_{n_2-n_1}\right) \qquad \text{by Eqns. (2.3.14), (2.3.25).} \end{aligned}$$

■

Corollary 3.3.2 *For all positive integers m and for all integers n_1 and n_2,*

$$\left\|\vec{f}^m_n\right\|^2 = \begin{cases} F_m F_{2n+m-1} & \text{if } m \text{ is even,} \\ \frac{1}{5}\left(L_m L_{2n+m-1} - 2(-1)^n\right) & \text{if } m \text{ is odd.} \end{cases} \tag{3.3.12}$$

Proof. Observe that $\left\|\vec{f}^m_n\right\|^2 = \vec{f}^m_n \cdot \vec{f}^m_n$.

■

Theorem 3.3.3 *For all positive integers m and for all integers n_1 and n_2,*

$$\vec{\ell}^{m}_{n_1} \cdot \vec{\ell}^{m}_{n_2} = \begin{cases} 5F_m F_{n_1+n_2+m-1} & \text{if } m \text{ is even,} \\ L_m L_{n_1+n_2+m-1} + (-1)^{n_1} L_{n_2-n_1} & \text{if } m \text{ is odd.} \end{cases} \tag{3.3.13}$$

Proof.

$$\begin{aligned} \vec{\ell}^{m}_{n_1} \cdot \vec{\ell}^{m}_{n_2} &= \left(\alpha^{n_1}\vec{a} + \beta^{n_1}\vec{b}\right) \cdot \left(\alpha^{n_2}\vec{a} + \beta^{n_2}\vec{b}\right) \qquad \text{by Eqn. (3.2.9)} \\ &= \left(\alpha^{n_1+n_2}\vec{a}\cdot\vec{a} + \beta^{n_1+n_2}\vec{b}\cdot\vec{b} + (\alpha^{n_1}\beta^{n_2} + \alpha^{n_2}\beta^{n_1})\vec{a}\cdot\vec{b}\right). \end{aligned}$$

Suppose m is even. Then

$$\begin{aligned} \vec{\ell}^{m}_{n_1} \cdot \vec{\ell}^{m}_{n_2} &= F_m(\alpha-\beta)\left(\alpha^{n_1+n_2+m-1} - \beta^{n_1+n_2+m-1}\right) \qquad \text{by Lem. 3.1.3} \\ &= 5F_m\left(\alpha^{n_1+n_2+m-1} - \beta^{n_1+n_2+m-1}\right)/(\alpha-\beta) \qquad \text{by Eqn. (2.3.18)} \\ &= 5F_m F_{n_1+n_2+m-1} \qquad \text{by Eqn. (2.3.14).} \end{aligned}$$

Now suppose m is odd. Then

$$\begin{aligned} \vec{\ell}^{m}_{n_1} \cdot \vec{\ell}^{m}_{n_2} &= L_m(\alpha^{n_1+n_2+m-1} + \beta^{n_1+n_2+m-1}) + (\alpha^{n_1}\beta^{n_2} + \alpha^{n_2}\beta^{n_1}) \qquad \text{by Lem. 3.1.3} \\ &= L_m L_{n_1+n_2+m-1} + (-1)^{n_1} L_{n_2-n_1} \qquad \text{by Eqn. (2.3.15).} \end{aligned}$$

■

Corollary 3.3.4 *For all positive integers m and for all integers n,*

$$\left\|\vec{\ell}^{m}_{n}\right\|^2 = \begin{cases} 5F_m F_{2n+m-1} & \text{if } m \text{ is even,} \\ L_m L_{2n+m-1} + 2(-1)^n & \text{if } m \text{ is odd.} \end{cases} \tag{3.3.14}$$

Corollary 3.3.5 *For all positive integers m and for all integers n_1 and n_2,*

$$\vec{\ell}^{m}_{n_1} \cdot \vec{\ell}^{m}_{n_2} = 5\vec{f}^{m}_{n_1} \cdot \vec{f}^{m}_{n_2}. \tag{3.3.15}$$

Proof. Clear from (3.3.11) and (3.3.13). ∎

Theorem 3.3.6 *For all positive integers m and for all integers n_1 and n_2,*

$$\vec{f}^m_{n_1} \cdot \vec{\ell}^m_{n_2} = \begin{cases} 5F_m L_{n_1+n_2+m-1} & \text{if } m \text{ is even,} \\ L_m F_{n_1+n_2+m-1} + (-1)^{n_1+1} F_{n_2-n_1} & \text{if } m \text{ is odd.} \end{cases} \tag{3.3.16}$$

Proof.

$$\begin{aligned} \vec{f}^m_{n_1} \cdot \vec{\ell}^m_{n_2} &= \frac{1}{\alpha-\beta}\left(\alpha^{n_1}\vec{a} - \beta^{n_1}\vec{b}\right) \cdot \left(\alpha^{n_2}\vec{a} + \beta^{n_2}\vec{b}\right) \qquad \text{by Thm. 3.2.3} \\ &= \frac{1}{\alpha-\beta}\left(\alpha^{n_1+n_2}\vec{a}\cdot\vec{a} - \beta^{n_1+n_2}\vec{b}\cdot\vec{b} + (\alpha^{n_1}\beta^{n_2} - \alpha^{n_2}\beta^{n_1})\vec{a}\cdot\vec{b}\right). \end{aligned}$$

Suppose m is even. Then

$$\begin{aligned} \vec{f}^m_{n_1} \cdot \vec{\ell}^m_{n_2} &= F_m(\alpha-\beta)\left(\alpha^{n_1+n_2+m-1} + \beta^{n_1+n_2+m-1}\right) \qquad \text{by Lem. 3.1.3} \\ &= 5F_m L_{n_1+n_2+m-1} \qquad \text{by Eqns. (2.3.15) and (2.3.18).} \end{aligned}$$

Now suppose m is odd. Then

$$\begin{aligned} \vec{f}^m_{n_1} \cdot \vec{\ell}^m_{n_2} &= \frac{1}{\alpha-\beta} L_m\left(\alpha^{n_1+n_2+m-1} - \beta^{n_1+n_2+m-1}\right) + (\alpha^{n_1}\beta^{n_2} - \alpha^{n_2}\beta^{n_1}) \qquad \text{by Thm. 3.2.3} \\ &= L_m F_{n_1+n_2+m-1} + (-1)^{n_1+1} F_{n_2-n_1} \qquad \text{by Eqn. (2.3.14).} \end{aligned}$$

∎

3.4 Fibonacci and Lucas identities

In this section we state a number of identities involving the Fibonacci and Lucas numbers which follow from the dot product formulas of the previous section.

Theorem 3.4.1 *For all positive integers m and for all integers n_1 and n_2,*

$$\sum_{j=0}^{m-1} F_{n_1+j} F_{n_2+j} = \begin{cases} F_m F_{n_1+n_2+m-1} & \text{if } m \text{ is even,} \\ \frac{1}{5}\left(L_m L_{n_1+n_2+m-1} - (-1)^{n_1} L_{n_2-n_1}\right) & \text{if } m \text{ is odd.} \end{cases} \tag{3.4.17}$$

Proof. Observe that both sides are equal to $\vec{f}^m_{n_1} \cdot \vec{f}^m_{n_2}$ by Theorem 3.3.1 and the definition of dot product.

■

From this basic identity a number of other identities can be derived. We state two of the simplest consequences now.

Corollary 3.4.2 *For all positive integers m and for all integers n,*

$$\sum_{j=0}^{m} F_{n+j}^2 = \begin{cases} F_m F_{2n+m-1} & \text{if } m \text{ is even,} \\ \frac{1}{5}\left(L_m L_{2n+m-1} - 2(-1)^n\right) & \text{if } m \text{ is odd.} \end{cases}$$

Proof. Take $n_1 = n_2$ in Eqn. (3.4.17).

■

Corollary 3.4.3 *For all positive integers m and for all integers k and n,*

$$F_{n+m-2}F_{n-k+m-1} + F_{n-1}F_{n-k} = \begin{cases} F_{m-1}F_{2n-k+m-2} & \text{if } m \text{ is even,} \\ \frac{1}{5}\left(L_{m-1}L_{2n-k+m} - 2(-1)^{n-k}L_{k-1}\right) & \text{if } m \text{ is odd.} \end{cases}$$

Proof. By the definition of dot product, the left-hand side is $\vec{f}^m_{n-1} \cdot \vec{f}^m_{n-k} - \vec{f}^{m-2}_n \cdot \vec{f}^{m-2}_{n-k+1}$. For m even, Theorem 3.3.1 gives

$$\begin{aligned} \vec{f}^m_{n-1} \cdot \vec{f}^m_{n-k} - \vec{f}^{m-2}_n \cdot \vec{f}^{m-2}_{n-k+1} &= F_m F_{2n-k+m-2} - F_{m-2}F_{2n-k+m-2} \\ &= (F_m - F_{m-2})F_{2n-k+m-2} \\ &= F_{m-1}F_{2n-k+m-2}. \end{aligned}$$

For m odd, Theorem 3.3.1 gives that $\vec{f}^m_{n-1} \cdot \vec{f}^m_{n-k} - \vec{f}^{m-2}_n \cdot \vec{f}^{m-2}_{n-k+1}$ equals

$$\begin{aligned} &\frac{1}{5}\left(L_m L_{n-1+n-k+m-1} - (-1)^{n-1}L_{n-k-n+1}\right) \\ &\quad - \frac{1}{5}\left(L_{m-2}L_{n+n-k+1+m-2-1} - (-1)^n L_{n-k+1-n}\right) \\ = &\ \frac{1}{5}\left(L_m L_{2n-k+m-2} - (-1)^{n-1}L_{-k+1}\right) - \frac{1}{5}\left(L_{m-2}L_{2n-k+m-2} - (-1)^n L_{-k+1}\right) \\ = &\ \frac{1}{5}L_m L_{2n-k+m-2} - \frac{1}{5}L_{m-2}L_{2n-k+m-2} - \frac{1}{5}(-1)^{n-1}L_{-k+1} + \frac{1}{5}(-1)^n L_{-k+1} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{5}L_{2n-k+m-2}\left(L_m - L_{m-2}\right) - \frac{1}{5}L_{-k+1}2(-1)^{n-1} \\
&= \frac{1}{5}L_{m-1}L_{2n-k+m-2} - 2(-1)^{n-1}L_{-k+1}.
\end{aligned}$$

■

The idea of the previous corollary can be used to derive a number of other more complicated identities involving more summands of the same sort.

Theorem 3.4.4 *For all integers n_1 and n_2 and for all positive integers m,*

$$\sum_{j=0}^{m} L_{n_1+j}L_{n_2+j} = \begin{cases} 5F_mF_{n_1+n_2+m-1} & \text{if } m \text{ is even,} \\ L_mL_{n_1+n_2+m-1} + (-1)^{n_1}L_{n_2-n_1} & \text{if } m \text{ is odd.} \end{cases} \tag{3.4.18}$$

Proof. Observe that both sides are equal to $\vec{f}^{\,m}_{n_1} \cdot \vec{f}^{\,m}_{n_2}$ by Theorem 3.3.3 and the definition of dot product.

■

Corollary 3.4.5 *For all positive integers m and for all integers n,*

$$\sum_{j=0}^{m} L_{n+j}^2 = \begin{cases} 5F_mF_{2n+m-1} & \text{if } m \text{ is even,} \\ L_mL_{2n+m-1} + 2(-1)^n & \text{if } m \text{ is odd.} \end{cases}$$

Proof. Take $n_1 = n_2$ in Eqn. (3.4.18).

■

Corollary 3.4.6 *For all positive integers m and for all integers k and n,*

$$L_{n+m-2}L_{n-k+m-1} + L_{n-1}L_{n-k} = \begin{cases} 5L_{m-1}L_{2n-k+m-2} & \text{if } m \text{ is even,} \\ L_{m-1}L_{2n-k+m-2} + 2(-1)^{n-1}L_{-k+1} & \text{if } m \text{ is odd.} \end{cases}$$

Proof. By the definition of dot product, the left-hand side is $\vec{\ell}^{\,m}_{n-1} \cdot \vec{\ell}^{\,m}_{n-k} - \vec{\ell}^{\,m-2}_{n} \cdot \vec{\ell}^{\,m-2}_{n-k+1}$. For m is even, Theorem 3.3.3 gives

$$\vec{\ell}^{\,m}_{n-1} \cdot \vec{\ell}^{\,m}_{n-k} - \vec{\ell}^{\,m-2}_{n} \cdot \vec{\ell}^{\,m-2}_{n-k+1} \quad = \quad 5F_mF_{2n-k+m-2} - 5F_{m-2}F_{2n-k+m-2}$$

$$
\begin{aligned}
&= 5(F_m - F_{m-2})F_{2n-k+m-2} \\
&= 5F_{m-1}F_{2n-k+m-2}.
\end{aligned}
$$

For m odd, Theorem 3.3.3 gives that $\vec{\ell}^m_{n-1} \cdot \vec{\ell}^m_{n-k} - \vec{\ell}^{m-2}_n \cdot \vec{\ell}^{m-2}_{n-k+1}$ equals

$$
\begin{aligned}
& \big(L_m L_{n-1+n-k+m-1} + (-1)^{n-1} L_{n-k-n+1}\big) \\
& \quad - (L_{m-2} L_{n+n-k+1+m-2-1} + (-1)^n L_{n-k+1-n}) \\
= & \big(L_m L_{2n-k+m-2} + (-1)^{n-1} L_{-k+1}\big) - (L_{m-2} L_{2n-k+m-2} + (-1)^n L_{-k+1}) \\
= & L_m L_{2n-k+m-2} - L_{m-2} L_{2n-k+m-2} + (-1)^{n-1} L_{-k+1} - (-1)^n L_{-k+1} \\
= & L_{2n-k+m-2}\,(L_m - L_{m-2}) + 2(-1)^{n-1} L_{-k+1} \\
= & L_{m-1} L_{2n-k+m-2} + 2(-1)^{n-1} L_{-k+1}.
\end{aligned}
$$

■

Theorem 3.4.7 *For all integers n_1 and n_2 and for all positive integers m,*

$$
\sum_{j=0}^{m} F_{n_1+j} L_{n_2+j} = \begin{cases} 5F_m L_{n_1+n_2+m-1} & \text{if } m \text{ is even,} \\ L_m F_{n_1+n_2+m-1} + (-1)^{n_1+1} F_{n_2-n_1} & \text{if } m \text{ is odd.} \end{cases} \tag{3.4.19}
$$

Proof. Observe that both sides are equal to $\vec{f}^m_{n_1} \cdot \vec{\ell}^m_{n_2}$ by Theorem 3.3.6 and the definition of dot product.

■

Corollary 3.4.8 *For all positive integers m and for all integers n,*

$$
\sum_{j=0}^{m} F_{n+j} L_{n+j} = \begin{cases} 5F_m L_{2n+m-1} & \text{if } m \text{ is even,} \\ L_m F_{2n+m-1} + (-1)^{n+1} & \text{if } m \text{ is odd.} \end{cases}
$$

Proof. Take $n_1 = n_2$ in Eqn. (3.4.19).

■

Corollary 3.4.9 *For all integers k, n and for all postive integers m,*

$$F_{n+m-2}L_{n-k+m-1} + F_{n-1}L_{n-k} = \begin{cases} 5F_{m-1}L_{2n-k+m-2} & \text{if } m \text{ is even} \\ L_{m-1}F_{2n-k+m-2} + 2(-1)^n F_{-k+1} & \text{if } m \text{ is odd.} \end{cases}$$

Proof. By the definition of dot product, the left-hand side is $\vec{f}^{\,m}_{n-1} \cdot \vec{\ell}^{\,m}_{n-k} - \vec{f}^{\,m-2}_{n} \cdot \vec{\ell}^{\,m-2}_{n-k+1}$. For m even, Theorem 3.3.6 gives

$$\begin{aligned} \vec{f}^{\,m}_{n-1} \cdot \vec{\ell}^{\,m}_{n-k} - \vec{f}^{\,m-2}_{n} \cdot \vec{\ell}^{\,m-2}_{n-k+1} &= 5F_mL_{2n-k+m-2} - 5F_{m-2}L_{2n-k+m-2} \\ &= 5(F_m - F_{m-2})L_{2n-k+m-2} \\ &= 5F_{m-1}L_{2n-k+m-2}. \end{aligned}$$

For m odd, Theorem 3.3.6 gives that $\vec{f}^{\,m}_{n-1} \cdot \vec{\ell}^{\,m}_{n-k} - \vec{f}^{\,m-2}_{n} \cdot \vec{\ell}^{\,m-2}_{n-k+1}$ equals

$$\begin{aligned} & \big(L_mF_{n-1+n-k+m-1} + (-1)^{n-1+1}F_{n-k-n+1}\big) \\ & \quad - \big(L_{m-2}F_{n+n-k+1+m-2-1} + (-1)^{n+1}F_{n-k+1-n}\big) \\ = \; & (L_mF_{2n-k+m-2} + (-1)^nF_{-k+1}) - \big(L_{m-2}F_{2n-k+m-2} + (-1)^{n+1}F_{-k+1}\big) \\ = \; & L_mF_{2n-k+m-2} - L_{m-2}F_{2n-k+m-2} + (-1)^nF_{-k+1} - (-1)^{n+1}F_{-k+1} \\ = \; & F_{2n-k+m-2}\,(L_m - L_{m-2}) + 2(-1)^nF_{-k+1} \\ = \; & L_{m-1}F_{2n-k+m-2} + 2(-1)^nF_{-k+1}. \end{aligned}$$

∎

4 Angles in even dimension

In this chapter we study the angles between the various vectors which we studied so far. We shall restrict our attention to the case of even dimension as the formulas become much more involved in odd dimension. We shall comment on the odd dimensional case in Chapter 5. Recall that the angle θ between arbitrary vectors v_1 and v_2 satisfies

$$\cos\theta = \frac{v_1 \cdot v_2}{\|v_1\| \, \|v_2\|}.$$

4.1 Dot products with $\vec{a}$ and $\vec{b}$

In this section, we compute the dot products of the vectors $\vec{a}$ and $\vec{b}$ with the Fibonacci and Lucas vectors.

Lemma 4.1.1 *For all integers n and for all positive even integers m,*

$$\vec{f}_n^m \cdot \vec{a} = F_m \alpha^{n+m-1}, \tag{4.1.1}$$

$$\vec{f}_n^m \cdot \vec{b} = F_m \beta^{n+m-1}. \tag{4.1.2}$$

Proof.

$$\begin{aligned}
\vec{f}_n^m \cdot \vec{a} &= \frac{1}{\alpha-\beta}\left(\alpha^n \vec{a} - \beta^n \vec{b}\right) \cdot \vec{a} && \text{by Thm. 3.2.3} \\
&= \frac{1}{\alpha-\beta}\left(\alpha^n \vec{a} \cdot \vec{a} - \beta^n \vec{b} \cdot \vec{a}\right) \\
&= \frac{1}{\alpha-\beta}\alpha^n (F_m(\alpha-\beta)\alpha^{m-1}) && \text{by Lem. 3.1.3} \\
&= F_m \alpha^{n+m-1}.
\end{aligned}$$

■

Lemma 4.1.2 *For all integers n and for all positive even integers m,*

$$\vec{\ell}_n^m \cdot \vec{a} = F_m(\alpha - \beta)\alpha^{n+m-1}, \tag{4.1.3}$$

$$\vec{\ell}_n^m \cdot \vec{b} = F_m(\alpha - \beta)\beta^{n+m-1}. \tag{4.1.4}$$

Proof. Compute

$$\begin{aligned} \vec{\ell}_n^m \cdot \vec{a} &= (\alpha^n \vec{a} + \beta^n \vec{b}) \cdot \vec{a} \qquad \text{by Thm. 3.2.3} \\ &= \alpha^n \vec{a} \cdot \vec{a} + \beta^n \vec{b} \cdot \vec{a} \\ &= \alpha^n F_m(\alpha - \beta)\alpha^{m-1} \qquad \text{by Lem. 3.1.3} \\ &= F_m(\alpha - \beta)\alpha^{n+m-1}. \end{aligned}$$

Equation (4.1.4) is proved similarly.

■

4.2 Cosines of angles with $\vec{a}$ and $\vec{b}$

In light of Theorem 3.2.3, it is interesting to consider the angles of the Fibonacci and Lucas vectors with $\vec{a}$ and $\vec{b}$. Recall that all the vectors of interest lie in the same plane.

Lemma 4.2.1 *For all integers k and all positive integers m, $\vec{f}_{2k}^m$ and $\vec{\ell}_{2k+1}^m$ are on the opposite side of $\vec{a}$ as $\vec{b}$, and $\vec{f}_{2k+1}^m$ and $\vec{\ell}_{2k}^m$ are on the same side of $\vec{a}$ as $\vec{b}$.*

Proof. Observe that $\beta < 0$, so β^{2k} is positive and β^{2k-1} is negative. The result follows from Theorem 3.2.3.

■

Lemma 4.2.2 *Let $\phi_{n,m}$ and $\chi_{n,m}$ denote the respective angles between $\vec{f}_n^m$ and $\vec{a}$ and* *between $\vec{f}_n^m$ and $\vec{b}$. Then*

$$\begin{aligned} \cos\phi_{n,m} &= (\alpha-\beta)^{-1/2}\sqrt{\frac{\alpha^{2n+m-1}}{F_{2n+m-1}}}, && (4.2.5) \\ \cos\chi_{n,m} &= (\alpha-\beta)^{-1/2}\sqrt{\frac{-\beta^{2n+m-1}}{F_{2n+m-1}}}. && (4.2.6) \end{aligned}$$

Proof. Compute

$$\begin{aligned} \cos\phi_{n,m} &= \frac{\vec{f}_m^n \cdot \vec{a}}{\left\|\vec{f}_m^n\right\| \|\vec{a}\|} \\ &= \frac{F_m\alpha^{n+m-1}}{\sqrt{F_mF_{2n+m-1}}\sqrt{F_m(\alpha-\beta)\alpha^{m-1}}} && \text{by Lem. 4.1.1, Cors. 3.1.4, 3.3.2} \\ &= (\alpha-\beta)^{-1/2}\sqrt{\frac{\alpha^{2n+m-1}}{F_{2n+m-1}}}. \end{aligned}$$

Also

$$\begin{aligned} \cos\chi_{n,m} &= \frac{\vec{f}_m^n \cdot \vec{b}}{\left\|\vec{f}_m^n\right\| \left\|\vec{b}\right\|} \\ &= \frac{F_m\alpha^{n+m-1}}{\sqrt{F_mF_{2n+m-1}}\sqrt{-F_m(\alpha-\beta)\beta^{m-1}}} && \text{by Lem. 4.1.1, Cors. 3.1.4, 3.3.2} \\ &= (\alpha-\beta)^{-1/2}\sqrt{\frac{-\beta^{2n+m-1}}{F_{2n+m-1}}}. \end{aligned}$$

∎

Lemma 4.2.3 *Let $\lambda_{n,m}$ and $\mu_{n,m}$ denote the respective angles between $\vec{\ell}_n^m$ and $\vec{a}$ and between $\vec{\ell}_n^m$ and $\vec{b}$. Then*

$$\begin{aligned} \cos\lambda_{n,m} &= (\alpha-\beta)^{-1/2}\sqrt{\frac{\alpha^{2n+m-1}}{F_{2n+m-1}}}, && (4.2.7) \\ \cos\mu_{n,m} &= (\alpha-\beta)^{-1/2}\sqrt{\frac{\beta^{2n+m-1}}{F_{2n+m-1}}}. && (4.2.8) \end{aligned}$$

Proof. Compute

$$\begin{aligned}
\cos\lambda_{n,m} &= \frac{\vec{\ell}^{\,n}_m \cdot \vec{a}}{\left\|\vec{\ell}^{\,n}_m\right\| \|\vec{a}\|} \\
&= \frac{F_m(\alpha-\beta)\alpha^{n+m-1}}{\sqrt{5F_mF_{2n+m-1}}\sqrt{F_m(\alpha-\beta)\alpha^{m-1}}} \qquad \text{by Lem. 4.1.1, Cors. 3.1.4, 3.3.2} \\
&= (\alpha-\beta)^{-1/2}\sqrt{\frac{\alpha^{2n+m-1}}{F_{2n+m-1}}}.
\end{aligned}$$

Equation (4.2.8) is proved similarly.

∎

Corollary 4.2.4 *For all integers n and for all even integers m, $|\phi_{n,m}| > |\phi_{n+1,m}|$.*

Proof. Observe that $\phi_{n,m}$ is accute since the coefficient of $\vec{a}$ in the expression of f^m_n in (3.2.9) is positive. To show that $|\phi_{n,m}| > |\phi_{n+1,m}|$, it suffices to show that $\cos\phi_{n,2k} < \cos\phi_{n+1,m}$. Note that $\alpha^j/F_j > \alpha^{j+1}/F_{j+1}$, so the result follows from Equation (4.2.5).

∎

Corollary 4.2.5 *For all integers n and for all positive integers t, $\phi_{n,2t} = \phi_{n+t,2} = -\lambda_{n+t} = -\lambda_{n,2t}$. In particular, the sequences of angles $\{\phi_{j,2t}\}_{j=1}^{\infty}$, $\{-\lambda_{j,2t}\}_{j=1}^{\infty}$, and $\{\phi_{t+j,2}\}_{j=1}^{\infty}$ are equal.*

Proof. Immediate from (4.2.5) and (4.2.7).

∎

Since the sequences of angles are simply tails of those in dimension two, we shall take another look at the Fibonacci vectors of length 2 in the Section 4.5.

4.3 Angles between vectors

In this section we study the angles between Fibonacci and Lucas vectors. We fix m to be a positive even integer.

Lemma 4.3.1 *For all integers n_1 and n_2, let $\zeta_{n_1,n_2,m}$ denote the angle between $\vec{f}^m_{n_1}$ and $\vec{f}^m_{n_2}$. Then*

$$\cos\zeta_{n_1,n_2,m} = \frac{F_{n_1+n_2+m-1}}{\sqrt{F_{2n_1+m-1}F_{2n_2+m-1}}}.$$

Proof. Follows directly from Theorem 3.3.1 and Corollary 3.3.2. ∎

Lemma 4.3.2 *For all integers n_1 and n_2, let $\eta_{n_1,n_2,m}$ denote the angle between $\vec{\ell}^m_{n_1}$ and $\vec{\ell}^m_{n_2}$. Then*

$$\cos\eta_{n_1,n_2,m} = \frac{F_{n_1+n_2+m-1}}{\sqrt{F_{2n_1+m-1}F_{2n_2+m-1}}}.$$

Proof. Follows directly from Theorem 3.3.3 and Corollary 3.3.4. ∎

Lemma 4.3.3 *For all integers n_1 and n_2, let $\theta_{n_1,n_2,m}$ denote the angle between $\vec{f}^m_{n_1}$ and $\vec{\ell}^m_{n_2}$. Then*

$$\cos\theta_{n_1,n_2,m} = \frac{5L_{n_1+n-2+m-1}}{\sqrt{F_{2n_1+m-1}F_{2n_2+m-1}}}.$$

Proof. Follows directly from Theorem 3.3.6 and Corollaries 3.3.2 and 3.3.4. ∎

The various angles which we have studied are not independent. Observe that the angle between $f^m_{n_1}$ and $f^m_{n_2}$ can be computed by either adding or subtracting their respective angles with $\vec{a}$. If n_1 and n_2 differ by an even number they are on the same side of $\vec{a}$ so we subtract, and if n_1 and n_2 differ by an odd number they are on opposite sides of $\vec{a}$ so we add. Thus for $n_1 > n_2$,

$$\zeta_{n_1,n_2,m} = \phi_{n_1,m} + (-1)^{n_1-n_2}\phi_{n_2,m}.$$

Similarly,

$$\eta_{n_1,n_2,m} = \lambda_{n_1,m} + (-1)^{n_1-n_2+1}\lambda_{n_2,m},$$

$$\theta_{n_1,n_2,m} = \phi_{n_1,m} + (-1)^{n_1-n_2}\lambda_{n_2,m}.$$

4.4 Angles with the axes

Let $\vec{x}_i$ denote the unit vector in the direction of the i-th coordinate axis and let m be a positive even integer. Observe that $\vec{f}_n^m \cdot \vec{x}_i = F_{n+i-1}$ and $\vec{\ell}_n^m \cdot \vec{x}_i = L_{n+i-1}$.

Lemma 4.4.1 *For all integers n, let $\omega_{m,n,i}$ be the angle between $\vec{f}_n^m$ and the i-th standard unit vector then*

$$\cos \omega_{m,n,i} = \frac{F_{n+i-1}}{\sqrt{F_m F_{2n+m-1}}}.$$

Lemma 4.4.2 *For all integers n, let $\iota_{m,n,i}$ be the angle between $\vec{l}_n^m$ and the i-th standard unit vector then*

$$\cos \iota_{m,n,i} = \frac{L_{n+i-1}}{\sqrt{5F_m F_{2n+m-1}}}.$$

4.5 Dimension two

In this section we discuss the Fibonacci and Lucas vectors of length two. This is directly applicable to all even dimensional cases as discussed in Section 4.2. We begin with a result of Lucas which gives a nice geometric condition on the Fibonacci vectors of length 2.

Theorem 4.5.1 *(Lucas) The endpoints of $\vec{f}_n^2$ lie on two hyperbolas given by the equation $y^2 - yx - x^2 = \pm 1$.*

In Chapter 6 we will consider the limit of the sequence of angles of the Fibonacci and Lucas vectors with $\vec{a}$. In dimension 2 we can also easily consider the angles with the axes and then deduce the angles with $\vec{a}$.

Lemma 4.5.2 *[4] For all integers k,*

$$\lim_{n\to\infty} \frac{F_{n+k}}{F_n} = \alpha^k. \tag{4.5.9}$$

Lemma 4.5.3 *For all integers n,*

$$\begin{aligned}
\lim_{n\to\infty} \cos\omega_{2,n,x} &= \frac{1}{1+\alpha^2}, \\
\lim_{n\to\infty} \cos\omega_{2,n,y} &= \frac{\alpha}{1+\alpha^2}, \\
\lim_{n\to\infty} \cos\iota_{2,n,x} &= \frac{1}{1+\alpha^2}, \\
\lim_{n\to\infty} \cos\iota_{2,n,y} &= \frac{\alpha}{1+\alpha^2}.
\end{aligned}$$

Proof. By Theorem 4.4.1

$$\begin{aligned}
\cos\omega_{2,n,x} &= \frac{F_n}{\sqrt{F_n^2+F_{n+1}^2}} \\
&= \frac{1}{\sqrt{1+(F_{n+1}/F_n)^2}}.
\end{aligned}$$

Thus by Lemma 4.5.2

$$\lim_{n\to\infty} \cos\omega_{2,n,x} = 1/(1+\alpha^2).$$

The remaining limits are proven similarly.

∎

It follows from the previous lemma that the limiting unit vectors for the directions of f_n^2 and ℓ_n^2 are both

$$\frac{1}{1+\alpha^2}\begin{bmatrix} 1 \\ \alpha \end{bmatrix} = \frac{1}{1+\alpha^2}\vec{a}.$$

In otherwords the unit directions approach that of $\vec{a}$. We shall see that this is the case in all dimensions.

We note that in dimension two $\vec{a}$ and $\vec{b}$ form an orthogonal basis for the whole vector space. From Theorem 3.2.3, we see that the transformation matrix from $\vec{a}$, $\vec{b}$ to $f_{n_1}^2$ and $f_{n_2}^2$ is

$$\begin{bmatrix} \alpha^{n_1} & -\beta^{n_1} \\ \alpha^{n_2} & -\beta^{n_2} \end{bmatrix}.$$

Since this matrix is essentially Vandermonde it is invertible. In other words, any two disinct Fibonacci vectors form a basis for $\mathbb{R}^2$. Similarly for Lucas vectors.

5 Comments on angles in odd dimension

Although we do not carry out the computations of the cosines of angles in odd dimension because the results are not so nice, some computations do turn out nicely. We present them in this Chapter.

5.1 The angle between $\vec{a}$ and $\vec{b}$

Recall that $\vec{a}$ and $\vec{b}$ form a basis for the plane containing all of the Fibonacci and Lucas vectors. When m is even, $\vec{a}$ and $\vec{b}$ are orthogonal. When m is odd we have the following.

Theorem 5.1.1 *Assume that m is odd. Then the cosine of the angle between $\vec{a}$ and $\vec{b}$ is $1/L_m$.*

Proof. Let τ denote the angle between $\vec{a}$ and $\vec{b}$. Then

$$\begin{aligned}\cos\tau &= \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\left\|\vec{b}\right\|} \\ &= \frac{1}{L_m} \qquad \text{by Cor. 3.1.4.}\end{aligned}$$

■

Lemma 5.1.2 *Suppose m is odd. Then the vector $-\vec{a}+(L_m\alpha^{m-1})\vec{b}$ is orthogonal to $\vec{a}$ and on the same side of $\vec{a}$ as $\vec{b}$.*

Proof. Compute

$$\begin{aligned}
\vec{a} \cdot (-\vec{a} + (L_m \alpha^{m-1})\vec{b}) &= -\vec{a} \cdot \vec{a} + (L_m \alpha^{m-1})\vec{a} \cdot \vec{b} \\
&= -L_m \alpha^{m-1} + L_m \alpha^{m-1} \qquad \text{by Lem. 3.1.3} \\
&= 0.
\end{aligned}$$

∎

5.2 Three-dimensional Fibonacci Vectors

In this section we discuss the geometric interpretations of these results in 3 dimensions. This case is nicer than the general odd dimensional case.

Example 5.2.1 With the notation of Lemma 4.4.1,

$$\begin{aligned}
\lim_{n\to\infty} \cos \omega_{3,n,x} &= 1/(2\alpha), \\
\lim_{n\to\infty} \cos \omega_{3,n,y} &= 1/2, \\
\lim_{n\to\infty} \cos \omega_{3,n,z} &= \alpha/2.
\end{aligned}$$

So, the limiting unit vector is $\left\langle \frac{1}{2\alpha}, \frac{1}{2}, \frac{\alpha}{2} \right\rangle = \frac{1}{2\alpha} \left\langle 1, \alpha, \alpha^2 \right\rangle$. Thus, geometrically the limiting line is given by $\vec{r}(t) = t(1, \alpha, \alpha^2)$ or $x = t, y = \alpha t$, and $z = \alpha^2 t$. Now consider the determinant of the matrix

$$\begin{vmatrix} \vec{f}_n \\ \vec{f}_m \\ \vec{f}_r \end{vmatrix} = \begin{vmatrix} F_n & F_{n+1} & F_{n+2} \\ F_m & F_{m+1} & F_{m+2} \\ F_r & F_{r+1} & F_{r+2} \end{vmatrix}.$$

The latter determinant equals zero since the last column is the sum of the first two columns. Therefore, the vectors are dependent (coplanar). We assume without loss of generality $n < m < r$, then $\vec{f}_r = (-1)^{m+n}(\frac{F_{r-m}}{F_{m-n}})\vec{f}_n + (\frac{F_{r-n}}{F_{m-n}})\vec{f}_m$. Taking $\vec{f}_n = \vec{f}_1$ and $\vec{f}_m = \vec{f}_2$ we write $\vec{f}_r = F_{r-2}\vec{f}_1 + F_{r-1}\vec{f}_2$. Thus, all vectors $\vec{f}_n$ fall in the plane determined by $\vec{f}_1$ and $\vec{f}_2$. This plane has normal vector $\vec{n} = \langle -1, -1, 1 \rangle$ and equation

$z = x + y$. Each vector $\vec{f}_n / \left\| \vec{f}_n \right\|$ lies in the intersection of this plane with the unit sphere $x^2 + y^2 + z^2 = 1$–this intersection is a great circle of this sphere.

6 Asymptotics of angles

We consider the ultimate behavior of the angles of the Fibonacci and Lucas vectors with $\vec{a}$ and $\vec{b}$.

6.1 Angles with $\vec{a}$

Lemma 6.1.1 *With reference to Lemma 4.2.2, for all even integers m,*

$$\lim_{n\to\infty} \cos\phi_{n,m} = 1.$$

That is to say the direction of the Fibonacci vectors approaches that of $\vec{a}$.

Proof. By Lemma 4.2.5, $\cos\phi_{n,m} = (\alpha-\beta)^{-1/2}\sqrt{\alpha^{2n+m-1}/F_{2n+m-1}}$ Thus we consider F_k/α^k. By Equation (2.3.14), $F_k/\alpha^k = (\alpha-\beta)^{-1}(\alpha^k-\beta^k)/\alpha^k = (\alpha-\beta)^{-1}(1-(\beta/\alpha)^k)$. Observe that $|\beta/\alpha| < 1$ and $\alpha - \beta$. Thus

$$\lim_{k\to\infty} F_k/\alpha^k = \frac{1}{\alpha-\beta}.$$

It follows that

$$\lim_{n\to\infty} \cos\phi_{n,m} = 1.$$

■

Lemma 6.1.2 *With reference to Lemma 4.2.3, for all even integers m,*

$$\lim_{n\to\infty} \cos\lambda_{n,m} = 1.$$

That is to say the direction of the Lucas vectors approaches that of $\vec{a}$.

Proof. Similar to that of Lemma 6.1.1. ∎

6.2 Dominant eigenvalues

In the last section we saw that in even dimension, the sequences of the Fibonacci and Lucas vectors approach in direction that of the vector $\vec{a}$. We now give another proof of this fact which does not depend upon the parity of the dimension. We use a variation of the well-known power method to do so.

Definition 6.2.1 Suppose λ is an eigenvalue of a square matrix A that is larger in absolute value than any other eigenvalue of A. Then λ is called the *dominant eigenvalue* of A. An eigenvector corresponding to λ is called a dominant eigenvector of A.

Dominant eigenvalues play an important role in the study of matrices. The power method provides a means of finding the dominant eigenvalue and eigenvector. We state one of the more general forms of the power method.

Theorem 6.2.2 *Assume that the $m \times m$ complex matrix A has a dominant eigenvalue λ and a unique dominant eigenvector $\vec{v}$ up to scalar multiples. Then the sequence $A\vec{x}/\|A\vec{x}\|$, $A^2\vec{x}/\|A^2\vec{x}\|$, $A^3\vec{x}/\|A^3\vec{x}\|$, $\ldots$, converges to $\vec{v}/\|\vec{v}\|$ for any intitial vector $\vec{x}$ except for a set of measure zero.*

In the literature one finds various criteria on the initial vector for the convergence of the above sequence to a dominant eigenvector. Clearly, any vector in the sum of eigenspaces other than that of the dominant eigenvalue cannot converge a dominant eigenvalue. For a diagonalizable matrix, this is almost a sufficient condition.

Theorem 6.2.3 *Assume that the $m \times m$ complex matrix A is diagonalizable and has a dominant eigenvalue λ and a unique dominant eigenvector $\vec{v}$ up to scalar multiples. Let $\vec{v}_1 = \vec{v}$, $\vec{v}_2$, $\ldots$, $\vec{v}_m$ denote a basis of eigenvectors for the complex vector space of* *columns vectors with complex entries and length m. Let $\vec{x} = c_1 v_1 + c_2 v_2 + \cdots + c_m v_m$ with $c_1 \neq 0$. Then the sequence $A\vec{x}/\left\|A\vec{x}\right\|$, $A^2\vec{x}/\left\|A^2\vec{x}\right\|$, $A^3\vec{x}/\left\|A^3\vec{x}\right\|$, $\ldots$ converges to $\vec{v}/\left\|\vec{v}\right\|$.*

Proof. Compute

$$\begin{aligned} A\vec{x} &= A(c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m) \\ &= c_1 A\vec{v}_1 + c_2 A\vec{v}_2 + \cdots + c_m A\vec{v}_m \\ &= c_1\lambda_1\vec{v}_1 + c_2\lambda_2\vec{v}_2 + \cdots + c_m\lambda_m\vec{v}_m, \end{aligned}$$

where λ_i is the eigenvalue associated with $\vec{v}_i$ for all i. By repeated multiplication by A,

$$A^k\vec{x} = c_1\lambda_1^k\vec{v}_1 + c_2\lambda_2^k\vec{v}_2 + \cdots + c_m\lambda_m^k\vec{v}_m.$$

Thus

$$A^k\vec{x} = \lambda^k\left[c_1\vec{v}_1 + c_2\left(\frac{\lambda_2}{\lambda}\right)^k\vec{v}_2 + \cdots + c_r\left(\frac{\lambda_r}{\lambda}\right)^k\vec{v}_r\right].$$

But λ is a dominant eigenvalue, so it is larger in absolute value than all other eigenvalues, so that $|\lambda_i/\lambda| < 1$ for $i = 2, 3, \ldots, r$. Thus each fraction $(\lambda_i/\lambda)^k$ approaches 0 as k goes to infinity.

■

Let us modify the previous theorem slightly to make it more applicable to our situation.

Theorem 6.2.4 *Assume that the $m \times m$ complex matrix A as a dominant eigenvalue λ and dominant eigenvector $\vec{v}$. Let $\vec{v}_1 = \vec{v}$, $\vec{v}_2$, $\ldots$, $\vec{v}_r$ denote a maximal set of linearly independent nonzero eigenvectors and let $\vec{v}_{r+1}$, $\vec{v}_{r+2}$, $\ldots$, $\vec{v}_s$ denote a basis of generalized eigenvectors for the generalized eigenspace associated with zero. Let $\vec{x} = c_1 v_1 + c_2 v_2 + \cdots + c_r v_r$ with $c_1 \neq 0$. Then the sequence $A\vec{x}/\left\|A\vec{x}\right\|$, $A^2\vec{x}/\left\|A^2\vec{x}\right\|$, $A^3\vec{x}/\left\|A^3\vec{x}\right\|$, $\ldots$ converges to $\vec{v}/\left\|\vec{v}\right\|$.*

Proof. Observe that for some j, $A^j\vec{v}_i$ is an eigenvector of A for all i $(1 \leq i \leq s)$. Now apply Theorem 6.2.3 with A restricted to the eigenspaces and $\vec{y} = A^j\vec{x}$ in place of $\vec{x}$. Now the result follows since $A^k\vec{y} = A^{k+j}\vec{x}$.

■

Observe that the matrix T of Eqn. (3.1.1) has eigenvalues 0, α, β with respective algebraic multiplicites $m-2$, 1, 1 and respective geometric multiplicites 1, 1, 1. Thus the complex vector space of complex vectors of length m has a basis consisting of nonzero eigenvectors and generalized eigenvectors associated with 0. A basis of the generalized eigenspace associated with 0 has a basis of generalized eigenvectors consisting of

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix}, \cdots, e_{m-2} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Thus the column vector space has a basis $\vec{a}$, $\vec{b}$, e_1, e_2, $\ldots$, e_{m-2}. This gives us the following.

Corollary 6.2.5 *Let T be as in Eqn.* (3.1.1)*. Let $\vec{x} = c_a\vec{a} + c_v\vec{b} + c_1e_1 + \cdots c_{m-2}e_{m-2}$ with $c_\alpha \neq 0$. Then the sequence $T\vec{x}/\left\|T\vec{x}\right\|$, $T^2\vec{x}/\left\|T^2\vec{x}\right\|$, $T^3\vec{x}/\left\|T^3\vec{x}\right\|$, $\ldots$ converges to $\vec{a}/\left\|\vec{a}\right\|$.*

Corollary 6.2.6 *The sequences of directions of the Fibonacci and Lucas vectors approach that of the vector $\vec{a}$*

Proof. Immediate from Lemma 3.2.2 and Corollary 6.2.5.

■

6.3 Further directions

We have begun to consider a number of variations on the work presented in this thesis.

- Consider the reverse Fibonacci and Lucas vectors

$$\vec{r}_n^m = \begin{bmatrix} F_{n+m-1} \\ F_{n+m-2} \\ \vdots \\ F_n \end{bmatrix}, \qquad \vec{t}_n^m = \begin{bmatrix} L_{n+m-1} \\ L_{n+m-2} \\ \vdots \\ L_n \end{bmatrix}.$$

Identical results hold for these vectors as did for $\vec{f}_n^m$, $\vec{\ell}_n^m$. However, a number of very interesting identities arise when we consider $\vec{f}_n^m \cdot \vec{r}_n^m$, $\vec{f}_n^m \cdot \vec{t}_n^m$, $\vec{\ell}_n^m \cdot \vec{r}_n^m$, and $\vec{\ell}_n^m \cdot \vec{t}_n^m$.

- Consider the Fibonacci and Lucas vectors with a step of p:

$$\vec{f}_n^{m,p} = \begin{bmatrix} F_n \\ F_{n+1p} \\ \vdots \\ F_{n+(m-1)p} \end{bmatrix}, \qquad \vec{\ell}_n^{m,p} = \begin{bmatrix} L_n \\ L_{n+1p} \\ \vdots \\ L_{n+(m-1)p} \end{bmatrix}.$$

Our computations still work out since this constant step gives rise to geometric series in α and β as before. Again, interesting identities arise from the computations of dot products. Different steps can be mixed, as well as reverse vectors considered.

- To some extent our computations can be carried out for any second order recurrence $x_{n+2} = cx_n + dx_{n+1}$. However, when $c \neq 1$ (so $\alpha\beta \neq -1$) many nice properties vanish.

A number of other problems suggest themselves

- Consider the Kronecker products of the Fibonacci vectors. We expect that such considerations will give rise to identities involving sums of longer products.

- Reduce Fibonacci and Lucas vectors mod k for any modulus k. These vectors cycle through a finite collection rather than approach a given line. There are many deep open problems concerning the Fibonacci numbers modulo k. Perhaps we may gain some insight from a consideration of these vectors.

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About the Author

Growing up I would have never dreamed that I would write a thesis in mathematics. After all, the one subject I feared the most was mathematics. Then one day it all changed while I was in college at Georgia Southern University. It was there that I met people who were willing to take the time out to explain the many things that confused me and it was there that I decided I wanted to become a math professor. For many years my fear of mathematics made me shy away from learning about several other subjects. It is my goal to make sure that no student of mine will ever let the subject of mathematics get in the way of the dreams they wish to pursue. I hope that in my lifetime I can be half the teacher that my professors were to me.