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# Analysis on a Class of Carnot Groups of Heisenberg Type

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Analysis on a Class of Carnot Groups of Heisenberg Type

by

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A dissertation submitted in partial fulfillment  
of the requirements for the degree of  
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ANALYSIS ON A CLASS OF CARNOT GROUPS OF HEISENBERG TYPE

MEAGAN MCNAMEE

ABSTRACT

In this thesis, we examine key geometric properties of a class of Carnot groups of Heisenberg type. After first computing the geodesics, we consider some partial differential equations in such groups and discuss viscosity solutions to these equations.

## 1 BACKGROUND AND MOTIVATION

We consider  $\mathbb{R}^n$  endowed with the Euclidean norm  $\|\cdot\|$  defined for  $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$  by

$$\|x\| = (x_1^2 + x_2^2 + \dots + x_n^2)^{\frac{1}{2}}$$

and with the vector fields

$$\left\{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n} \right\}.$$

Note that these vector fields are also the standard directional derivatives.

If  $f$  is a sufficiently smooth function on  $\mathbb{R}^n$ , then the gradient of  $f$  is given by

$$\nabla f = \left( \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right)$$

and the second order derivative matrix, denoted  $D^2 f$ , has  $ij$ -th entry

$$(D^2 f)_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}.$$

Note that this matrix is symmetric, since mixed partials are equal.

Since  $f$  is sufficiently smooth, its Taylor Polynomial at a point  $p_0 = (x_1^0, x_2^0, \dots, x_n^0)$  is given by

$$f(p) = f(p_0) + \langle \nabla f(p_0), p - p_0 \rangle + \frac{1}{2} \langle D^2 f(p_0)(p - p_0), p - p_0 \rangle + o(\|p - p_0\|^2)$$

where  $p$  is near  $p_0$  and  $\langle \cdot, \cdot \rangle$  is the Euclidean inner product related to the norm  $\|\cdot\|$ .

If a function is not sufficiently smooth, then the Taylor polynomial does not exist.

This possible lack of existence motivates us to define second order superjets, denoted  $J^{2,+}$ , and second order subjets, denoted  $J^{2,-}$  by considering the following inequality:

$$f(p) \leq f(p_0) + \langle \eta, p - p_0 \rangle + \frac{1}{2} \langle X(p - p_0), p - p_0 \rangle + o(\|p - p_0\|^2) \quad (1.0.1)$$

for a vector  $\eta \in \mathbb{R}^n$  and  $X \in S^n$ , where  $S^n$  is the symmetric  $n \times n$  matrices. Note that if a function is sufficiently smooth, the vector  $\eta$  can be replaced by  $\nabla f(p_0)$  and the symmetric matrix  $X$  can be replaced by  $D^2 f(p_0)$ . If  $f$  is not sufficiently smooth, the vector  $\eta$  and the symmetric matrix  $X$  play the role of a generalized derivative, acting as a substitute for the derivatives that do not exist. It should be noted that there is no guarantee that a pair  $(\eta, X)$  does indeed exist.

The second order superjet and subjet of  $f$  at a point  $p_0$  are defined by

$$\begin{aligned} (\eta, X) \in J^{2,+} f(p_0) &\Leftrightarrow (1.0.1) \text{ holds.} \\ J^{2,-} f(p_0) &= -J^{2,+}(-f)(p_0). \end{aligned}$$

As mentioned above, these jets may be empty. We denote the set-theoretic closure of the superjet by  $\bar{J}^{2,+}$ . That is,  $(\eta, X) \in \bar{J}^{2,+} u(p)$  if there is a sequence  $\{p_n, \eta_n, X_n\}$  so that  $(\eta_n, X_n) \in J^{2,+} u(p_n)$  and

$$\{p_n, \eta_n, X_n\} \xrightarrow{n \rightarrow \infty} (p, \eta, X).$$

Using these jets, we can define viscosity solutions for a class of non-linear partial differential equations. We let  $S^n$  be the set of  $n \times n$  symmetric matrices. Given a continuous function

$$F : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times S^n \mapsto \mathbb{R},$$

we require that

$$F(p, r, \eta, X) \leq F(p, s, \eta, Y)$$

when  $r \leq s$  and  $Y \leq X$ . That is,  $F$  is proper in the sense of [3]. We then form the

class of non-linear partial differential equations

$$F(p, f(p), \nabla f(p), D^2 f(p)) = 0.$$

Examples of such equations include the infinite Laplacian, denoted  $\Delta_\infty$ , which is defined by

$$\Delta_\infty f(p) = -\langle D^2 f(p) \nabla f(p), \nabla f(p) \rangle.$$

We may also consider the  $P$ -Laplacian, which is defined for  $2 \leq P < \infty$  and given by

$$\Delta_P f = -\left( \|\nabla f(p)\|^2 \operatorname{tr}(D^2 f(p)) - (P - 2)\Delta_\infty f(p) \right).$$

We then define viscosity solutions as follows:

**Definition 1.0.1** A continuous function  $f$  is a viscosity subsolution of

$$F(p, f(p), \nabla f(p), D^2 f(p)) = 0$$

if for all  $(\eta, X) \in \overline{J}^{2,+} f(p)$  we have

$$F(p, f(p), \eta, X) \leq 0$$

A continuous function  $f$  is a viscosity supersolution of

$$F(p, f(p), \nabla f(p), D^2 f(p)) = 0$$

if for all  $(\mu, Y) \in \overline{J}^{2,-} f(p)$  we have

$$F(p, f(p), \mu, Y) \geq 0$$

The continuous function  $f$  is a viscosity solution if it is both a viscosity subsolution and a viscosity supersolution.

We shall focus on a special subclass of such functions  $F$ . We call the subclass  $\mathcal{F}$

those functions  $F$  for which there is a universal  $\delta > 0$  so that for each real  $r$  and  $s$  with  $r < s$ , we have

$$0 < \delta(s - r) \leq F(p, s, \eta, X) - F(p, r, \eta, X)$$

and we further assume that there exists  $w_1, w_2, w_3 : [0, \infty] \rightarrow [0, \infty]$ ,  $w_i(0^+) = 0$  for  $i = 1, 2, 3$  so that

$$\begin{aligned} F(p, r, \eta, Y) - F(p, r, \eta, X) &\leq w_1(\|X - Y\|_M) \\ F(p, r, \eta, X) - F(p, r, \gamma, X) &\leq w_2(|\|\eta\| - \|\gamma\||) \\ F(p, r, \eta, X) - F(q, r, \eta, X) &\leq w_3(\|p - q\|). \end{aligned}$$

Here  $\|\cdot\|_M$  is the standard matrix norm.

An example of such an  $F \in \mathcal{F}$  is

$$F(p, r, \eta, X) = -\|AX\|_M + \|\eta\| + cr$$

where  $A$  is a symmetric matrix such that  $A \geq 0$  and  $c > 0$  is a real constant. In this case,  $\delta = c$ .

Concerning our subclass  $\mathcal{F}$ , the following theorem is known in  $\mathbb{R}^n$ .

**Theorem 1.0.2 (CIL)** *Let  $F \in \mathcal{F}$  and  $\Omega$  be a bounded open set in  $\mathbb{R}^n$ . Then if  $u$  is a viscosity subsolution to*

$$F(p, u(p), \nabla u(p), D^2u(p)) = 0$$

*and  $v$  is a viscosity supersolution to*

$$F(p, v(p), \nabla v(p), D^2v(p)) = 0$$

*with  $u \leq v$  on  $\partial\Omega$ . Then  $u \leq v$  in  $\Omega$ .*

In this thesis, we look to extend this theorem to a class of Carnot groups, which



are natural generalizations of  $\mathbb{R}^n$ . As discussed below,  $\mathbb{R}^n$  is a specific Carnot group, but the class we consider possesses a rich geometry that is much different from  $\mathbb{R}^n$ .

## 2 CARNOT GROUPS

### 2.1 Geometry

A Lie Group  $G$  is a real, connected, manifold with a non-abelian group structure such that the map  $x \cdot y^{-1}$  is smooth. The Lie Algebra  $g$  of a Lie Group  $G$  is the tangent space at the origin  $e$ . It can be endowed with a Lie bracket, which is an antisymmetric bilinear form  $[\cdot, \cdot] : g \times g \mapsto g$  obeying the Jacobi identity

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0.$$

The Lie Algebra is identified with the left invariant (under the group structure) vector fields on  $G$  and  $[X, Y]$  is then the Lie bracket of such vector fields  $X$  and  $Y$ . Under an appropriate change of variables, we may identify the points in the Lie Algebra with the points in the Lie Group. That is, if  $\{X_i\}_{i=1}^n$  is a basis of the Lie Algebra, we have

$$\sum_{i=1}^n x_i X_i \Leftrightarrow (x_1, x_2, x_3, \dots, x_n).$$

This identification has two main purposes. First, the practical effect of identifying  $G$  with  $g$  is that it mimics the important property that  $\mathbb{R}^n$  is both the manifold and tangent space. Secondly, it allows us to write the group multiplication law in a workable form. Namely,

$$p \cdot q = p + q + \frac{1}{2}[p, q] + H(p, q) \tag{2.1.1}$$

where  $H(p, q)$  consists of Lie brackets of order 2 and higher. Note that  $H(p, q)$  need

not be non-zero. In fact, a Lie Algebra  $g$  is *nilpotent* if  $H(\cdot, \cdot)$  consists of only a finite number of non-zero brackets and a Lie Group  $G$  is *nilpotent* if its corresponding Lie Algebra is nilpotent.

We are now ready to define a Carnot group.

**Definition 2.1.1** A nilpotent Lie Group  $G$  with a Lie Algebra  $g$  that satisfies

$$g = V_1 \oplus V_2 \oplus \cdots \oplus V_l$$

with  $[V_1, V_j] = V_{1+j}$  is called a Carnot group, or homogeneous group. Note that this requires  $V_1$  to generate  $g$  as a Lie Algebra, since  $V_2 = [V_1, V_1]$ ,  $V_3 = [V_1, V_2] = [[V_1, V_1], V_1]$  and so on.

It should be noted that from linear algebra, we can find a basis of  $V_i$ , denoted  $\{X_{ij}\}_{j=1}^{d_i}$ , where  $d_i = \dim V_i$ , so that the identification of  $G$  with  $g$  holds.

The Carnot group  $G$  has two natural dimensions to measure its size. The topological dimension is simply the number of elements in the basis of  $g$ , namely,

$$D = \sum_{i=1}^l \dim V_i = \sum_{i=1}^l \sum_{j=1}^{d_i} X_{ij}.$$

This dimension does not reflect the stratification in the definition of Carnot groups. A more natural choice would be one that takes into account the fact that  $V_1$  generates  $g$  as a Lie Algebra. This motivates the concept of the homogenous dimension, denoted  $Q$ , and given by

$$Q = \sum_{i=1}^l i d_i.$$

Note that this is no smaller than the topological dimension of  $G$  and, in general, is much larger.

Every Lie Algebra of a Carnot group has at least one non-isotropic linear dilation, denoted  $\delta_s$  for  $s > 0$ , that has the property

$$\delta_s(V_i) = s^i V_i.$$

Using the above identification of  $G$  with  $g$ , that is,

$$\sum_{i=1}^l \sum_{j=1}^{d_i} x_{ij} X_{ij} \in g \leftrightarrow (x_{11}, x_{12}, \dots, x_{ld_l}) \in G,$$

the dilation on  $g$  induces a dilation on  $G$ , also denoted  $\delta_s$ , defined by

$$\delta_s p = (sx_{11}, sx_{12}, \dots, sx_{1d_1}, s^2x_{21}, s^2x_{22}, \dots, s^l x_{ld_l}) \quad (2.1.2)$$

Additionally, we may choose an invariant Riemannian inner product denoted by  $\langle \cdot, \cdot \rangle$ , and its associated norm  $\| \cdot \|$  on the vector space  $g$ . A curve  $\gamma : \mathbb{R} \mapsto G$  is called *horizontal* if the tangent vector  $\gamma'(t)$  is in  $V_1$ . Define the Carnot-Carathéodory distance by the following:

$$d_C(p, q) = \inf_{\Gamma} \int_0^1 \|\gamma'(t)\| dt$$

where  $p$  and  $q$  are in  $G$  and  $\Gamma$  is the set of all horizontal curves  $\gamma$  such that  $\gamma(0) = p$  and  $\gamma(1) = q$ . It is a non-trivial fact (Chow's theorem, [1]) that any two points in a Carnot group can be connected by a horizontal curve, which makes  $d_C(p, q)$  a left-invariant metric on  $G$ . In addition, there exists at least one shortest curve (minimizing geodesic) connecting points  $p$  and  $q$ . It should be noted that  $d_C(p, q)$  is independent of the choice of  $\| \cdot \|$ . Define a Carnot-Carathéodory ball of radius  $r$  centered at a point  $p_0$  by

$$B = B(p_0, r) = \{p \in G : d_C(p, p_0) < r\}.$$

## 2.2 Calculus

Let  $G$  be a Carnot group with Lie Algebra  $g$ . Choose an orthonormal basis with respect to  $\langle \cdot, \cdot \rangle$  for  $V_i$  denoted  $\{X_{i1}, X_{i2}, \dots, X_{id_i}\}$ . We have two important gradients. The first is the horizontal gradient of  $f$ , defined by

$$\nabla_0 f = (X_{11}f, X_{12}f, \dots, X_{1d_1}f).$$

This is the projection onto  $V_1$  of the usual gradient of  $f$ , which is given by

$$\nabla f = (X_{11}f, X_{12}f, \dots, X_{1d_1}f, X_{21}f, X_{22}f, \dots, X_{2d_2}f, \dots, X_{l1}f, X_{l2}f, \dots, X_{ld_l}f).$$

The second is the semi-horizontal gradient, defined by

$$\nabla_1 f = (X_{11}f, X_{12}f, \dots, X_{1d_1}f, X_{21}f, X_{22}f, \dots, X_{2d_2}f).$$

This is the projection of the gradient onto  $V_1 \oplus V_2$  and it has first and second order derivatives. Another important second order derivative is the symmetrized second order matrix, denoted  $(D^2 f)^*$ , which is a  $d_1 \times d_1$  matrix with entries

$$((D^2 f)^*)_{ij} = \frac{1}{2}(X_i X_j + X_j X_i)f.$$

We note that for technical reasons, we need a symmetric second order derivative matrix. Because of the non-abelian nature of the Carnot group,  $D^2 f$ , which is the matrix given by

$$(D^2 f)_{ij} = X_i X_j f$$

need not be symmetric.

A function  $f$  is  $C^1$  if  $X_i f$  is continuous for all  $i$ . A function  $f$  is  $C^2$  if it is  $C^1$  and  $X_i X_j f$  is continuous for all  $i, j$  (which implies  $\nabla_1 f$  has continuous components).

Lastly, we note all integration is done with respect to Lebesgue measure.

### 3 GROUPS OF HEISENBERG TYPE

#### 3.1 Definition

A Carnot group is of Heisenberg type [4] if its Lie Algebra  $g$  satisfies  $g = V_1 \oplus V_2$  and  $g$  has an inner product  $\langle \cdot, \cdot \rangle$  with associated norm  $\| \cdot \|$  so that for each  $z \in V_2$ , the linear map  $J_z : V_1 \rightarrow V_1$ , defined by

$$\langle J_z(v), w \rangle = \langle z, [v, w] \rangle$$

satisfies the property

$$J_z^2 = -\|z\|^2 Id.$$

The function  $J_z$  also has the following useful property[4].

**Proposition 3.1.1**  $\|J_z w\|^2 = \|z\|^2 \|w\|^2$ .

$$\begin{aligned} \langle w, J_z v \rangle &= \langle J_z v, w \rangle \\ &= \langle z, [v, w] \rangle \\ &= \langle -z, [w, v] \rangle \\ &= -\langle J_z w, v \rangle. \end{aligned}$$

This then leads to  $\|J_z v\|^2 = \langle J_z v, J_z v \rangle = -\langle J_z^2 v, v \rangle$  and the proposition follows from the property above.

We note that when  $\dim V_2 = 1$ , this guarantees the existence of the  $J_z$ , and so in this case, we have a group of Heisenberg type. [4]

In groups of Heisenberg type, the semi-horizontal gradient and full gradient are equal, and both are denoted by  $\nabla f$ .

### 3.2 A Class of Groups of Heisenberg Type

Let  $n \geq 3$  and consider  $\mathbb{R}^{n+1}$  spanned by the linearly independent vector fields  $\{X_i, T\}$ , where the index  $i$  ranges from 1 to  $n$ , defined by

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x_1} - \frac{x_2}{2} \frac{\partial}{\partial t} \\ X_2 &= \frac{\partial}{\partial x_2} + \frac{x_1}{2} \frac{\partial}{\partial t} \\ X_3 &= \frac{\partial}{\partial x_3} \\ &\vdots \\ X_n &= \frac{\partial}{\partial x_n} \\ T &= \frac{\partial}{\partial t} \end{aligned}$$

This Lie algebra, denoted  $h_n$ , has the property that for  $i < j$

$$[X_i, X_j] = \begin{cases} T & \text{if } i = 1, j = 2 \\ 0 & \text{otherwise} \end{cases}$$

and for all  $i$ ,

$$[X_i, T] = 0.$$

Thus,  $h_n$  decomposes as a direct sum

$$h_n = V_1 \oplus V_2$$

where  $V_1$  is spanned by the  $X_i$ 's and  $V_2$  is spanned by  $T$ .

From the previous section, the corresponding Lie Group, denoted  $H_n$ , is a Carnot group of Heisenberg type. We note that although  $\mathbb{R}^n$  is a Carnot group with  $l = 1$ , it is not a group of Heisenberg type, because  $V_2 = \{0\}$ .

Using the group law formula (2.1.1), we see that the group law for our class of groups of Heisenberg type is

$$p \cdot q = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n, t + s + \frac{1}{2}(x_1 y_2 - x_2 y_1)) \quad (3.2.1)$$

where  $p = (x_1, x_2, \dots, x_n, t)$  and  $q = (y_1, y_2, \dots, y_n, s)$ .

### 3.2.1 Geodesics

We first wish to explore the geodesics for the class of groups  $H_n$ . Because  $\mathbb{R}^n$  is not a group of Heisenberg type, we expect the geodesics to be different from those in  $\mathbb{R}^n$ . In order to compute the geodesics of a Carnot group, we invoke the Pontrjagin maximum principle, as used in [5] and [1]. The following lemma produces a formula for these geodesics.

**Lemma 3.2.1** *The geodesics starting at the origin have the following equations:*

$$\begin{aligned} x_1(\tau) &= -\frac{A_2}{B}(1 - \cos B\tau) + \frac{A_1}{B} \sin B\tau \\ x_2(\tau) &= \frac{A_1}{B}(1 - \cos B\tau) + \frac{A_2}{B} \sin B\tau \\ x_j(\tau) &= A_j \tau, \quad j = 3, 4, \dots, n \\ t(\tau) &= \frac{A_1^2 + A_2^2}{2B^2}(B\tau - \sin B\tau) \end{aligned}$$

where the  $A_i$  and  $B$  are constants. We note that by periodicity,  $B \in (-\pi, \pi]$ . Also, when  $B = 0$ , by continuity, we may take the limit as  $B \rightarrow 0$  and obtain  $T(\tau) = 0$  and for all  $i$ ,  $x_i(\tau) = A_i \tau$ .

**Proof:**

By left multiplication under the group law, we may assume all geodesics are based at the origin, which is denoted  $0 = (0, 0, 0, \dots, 0)$ . Because a point  $p \in H_n$  has



coordinates  $p = (x_1, x_2, x_3, \dots, x_n, t)$ , we parameterize a geodesic curve as  $(x_1(\tau), x_2(\tau), \dots, x_n(\tau), t(\tau))$  for  $0 \leq \tau \leq 1$ . The geodesics start at the origin ( $\tau = 0$ ) and will terminate at the point corresponding to  $\tau = 1$ .

We wish to set up a system of equations involving the vectors  $X_i$  and  $T$  as above and the corresponding covectors  $\epsilon_i$ . Note that by the definition of horizontal curve, we do not need the vector  $T$  in our system of equations. We may also treat the covectors as components and write

$$\epsilon(\tau) = (\epsilon_1(\tau), \epsilon_2(\tau), \epsilon_3(\tau), \dots, \epsilon_n(\tau), \epsilon_t(\tau)).$$

Before computing, we note that by the above discussion, our initial conditions are  $x_i(0) = 0$  and  $\epsilon_i(0) = A_i$  where the  $A_i$  are constants. In order to begin computing, we need to define the operator that is the basis of our calculations. This operator uses both vectors and covectors and is defined by

$$a(p(\tau), \epsilon(\tau)) = \sum_{i=1}^n \langle X_i(p(\tau)), \epsilon(\tau) \rangle^2.$$

Writing this out in coordinate form, we obtain

$$\begin{aligned} a(p(\tau), \epsilon(\tau)) &= (\epsilon_1(\tau) - \frac{x_2(\tau)}{2}\epsilon_t(\tau))^2 + (\epsilon_2(\tau) + \frac{x_1(\tau)}{2}\epsilon_t(\tau))^2 \\ &\quad + (\epsilon_3(\tau))^2 + (\epsilon_4(\tau))^2 + \dots + (\epsilon_n(\tau))^2. \end{aligned}$$

We normalize this operator by multiplying  $a(p(\tau), \epsilon(\tau))$  by  $\frac{1}{2}$ , to obtain

$$\begin{aligned} \frac{1}{2}a(p(\tau), \epsilon(\tau)) &= \frac{1}{2}(\epsilon_1(\tau) - \frac{x_2(\tau)}{2}\epsilon_t(\tau))^2 + \frac{1}{2}(\epsilon_2(\tau) + \frac{x_1(\tau)}{2}\epsilon_t(\tau))^2 \\ &\quad + \frac{1}{2}(\epsilon_3(\tau))^2 + \frac{1}{2}(\epsilon_4(\tau))^2 + \dots + \frac{1}{2}(\epsilon_n(\tau))^2. \end{aligned}$$

We differentiate the function  $a(p(\tau), \epsilon(\tau))$  to obtain

$$\begin{aligned}\frac{\partial a}{\partial \epsilon} &\equiv p' = (x_1'(\tau), x_2'(\tau), x_3'(\tau), \dots, x_n'(\tau)) \\ -\frac{\partial a}{\partial p} &\equiv \epsilon' = (\epsilon_1'(\tau), \epsilon_2'(\tau), \epsilon_3'(\tau), \dots, \epsilon_n'(\tau), \epsilon_t'(\tau)).\end{aligned}$$

Using the Pontrjagin maximum principle ([5] and [1]), we have the relations

$$\begin{aligned}x_1'(\tau) &= \epsilon_1(\tau) - \frac{x_2(\tau)}{2}\epsilon_t(\tau) \\ x_2'(\tau) &= \epsilon_2(\tau) + \frac{x_1(\tau)}{2}\epsilon_t(\tau) \\ x_3'(\tau) &= \epsilon_3(\tau) \\ x_4'(\tau) &= \epsilon_4(\tau) \\ &\vdots \\ x_n'(\tau) &= \epsilon_n(\tau) \\ t'(\tau) &= -\frac{x_2(\tau)}{2}(\epsilon_1(\tau) - \frac{x_2(\tau)}{2}\epsilon_t(\tau)) + \frac{x_1(\tau)}{2}(\epsilon_2(\tau) + \frac{x_1(\tau)}{2}\epsilon_t(\tau)) \\ \epsilon_1'(\tau) &= -\frac{1}{2}\epsilon_t(\tau)(\epsilon_2(\tau) + \frac{x_1(\tau)}{2}\epsilon_t(\tau)) \\ \epsilon_2'(\tau) &= \frac{1}{2}\epsilon_t(\tau)(\epsilon_1(\tau) - \frac{x_2(\tau)}{2}\epsilon_t(\tau)) \\ \epsilon_3'(\tau) &= 0 \\ \epsilon_4'(\tau) &= 0 \\ &\vdots \\ \epsilon_n'(\tau) &= 0 \\ \epsilon_t'(\tau) &= 0.\end{aligned}$$

We now need to solve this system of differential equations, using the above initial conditions. We easily conclude that  $\epsilon_t(\tau) = B$  and  $\epsilon_i(\tau) = A_i$  for  $i = 3$  to  $n$ . Thus,

Thus  $x_j(\tau) = A_j\tau$  for  $j = 3$  to  $n$ . We are then left to consider

$$\begin{aligned}\epsilon_1'(\tau) &= -\frac{1}{2}\epsilon_t(\tau)(\epsilon_2(\tau) + \frac{x_1(\tau)}{2}\epsilon_t(\tau)) \\ &= -\frac{\epsilon_t(\tau)}{2}x_2'(\tau) \\ \epsilon_2'(\tau) &= \frac{1}{2}\epsilon_t(\tau)(\epsilon_1(\tau) - \frac{x_2(\tau)}{2}\epsilon_t(\tau)) \\ &= \frac{\epsilon_t(\tau)}{2}x_1'(\tau).\end{aligned}$$

Simplifying, we have

$$\begin{aligned}\epsilon_1(\tau) &= -\frac{\epsilon_t(\tau)}{2}x_2(\tau) + A_1 \\ \epsilon_2(\tau) &= \frac{\epsilon_t(\tau)}{2}x_1(\tau) + A_2.\end{aligned}$$

We need to solve for  $x_1(\tau)$  and  $x_2(\tau)$ ,

$$x_1'(\tau) = \epsilon_1(\tau) - \frac{x_2(\tau)}{2}\epsilon_t(\tau).$$

Plugging in  $\epsilon_1(\tau) = -\frac{\epsilon_t(\tau)}{2}x_2(\tau) + A_1$  and  $\epsilon_t(\tau) = B$ , we obtain

$$\begin{aligned}x_1'(\tau) &= -Bx_2(\tau) + A_1 \\ x_2'(\tau) &= \epsilon_2(\tau) + \frac{x_1(\tau)}{2}\epsilon_t(\tau).\end{aligned}$$

Similarly, plugging in  $\epsilon_2(\tau) = \frac{\epsilon_t(\tau)}{2}x_1(\tau) + A_2$  and  $\epsilon_t(\tau) = B$ , we have

$$x_2'(\tau) = Bx_1(\tau) + A_2.$$

We conclude that since

$$x_1'(\tau) = -Bx_2(\tau) + A_1,$$

$$x_1''(\tau) = -Bx_2'(\tau)$$

and substituting  $x_2'(\tau) = Bx_1(\tau) + A_2$ , we conclude

$$x_1''(\tau) = -B^2x_1(\tau) - BA_2.$$

By the method of undetermined coefficients

$$x_1(\tau) = \alpha \cos B\tau + \gamma \sin B\tau + \mu.$$

But  $x_1(0) = 0$  so  $\alpha = -\mu$  and so

$$x_1(\tau) = \mu(1 - \cos B\tau) + \gamma \sin B\tau.$$

To solve for  $\mu$  we take the second derivative of  $x_1(\tau)$  and compare it to  $x_1''(\tau)$  obtained earlier and see that  $x_1''(\tau) = -B^2x_1(\tau) - BA_2$  and  $x_1(\tau) = \mu(1 - \cos B\tau) + \gamma \sin B\tau$ .

Thus

$$x_1''(\tau) = -B^2(\mu(1 - \cos B\tau) + \gamma \sin B\tau) - BA_2$$

and so

$$x_1''(\tau) = -B^2\mu + B^2\mu \cos B\tau - B^2\gamma \sin B\tau - BA_2.$$

Taking the second derivative of the  $x_1(\tau)$  we obtained, we conclude

$$x_1''(\tau) = B^2\mu \cos B\tau - \gamma B^2 \sin B\tau.$$

Thus we have

$$-B^2\mu + B^2\mu \cos B\tau - B^2\gamma \sin B\tau - BA_2 = B^2\mu \cos B\tau - \gamma B^2 \sin B\tau$$

which implies

$$-B^2\mu - BA_2 = 0.$$

That is,

$$\mu = -\frac{A_2}{B}.$$

Thus

$$x_1(\tau) = -\frac{A_2}{B}(1 - \cos B\tau) + \gamma \sin B\tau.$$

We now solve for  $x_2(\tau)$ . From above, we have

$$x_2'(\tau) = Bx_1(\tau) + A_2.$$

This implies

$$x_2(\tau) = \int (Bx_1(\tau) + A_2) d\tau.$$

This implies

$$\begin{aligned} x_2(\tau) &= \int (B(-\frac{A_2}{B}(1 - \cos B\tau) + \gamma \sin B\tau) + A_2) d\tau \\ &= \gamma B \int \sin B\tau d\tau + A_2 \int \cos B\tau d\tau \end{aligned}$$

and so

$$x_2(\tau) = -\gamma \cos B\tau + \frac{A_2}{B} \sin B\tau + \delta.$$

But  $x_2(0) = 0$ , so  $\delta = \gamma$ , giving us

$$x_2(\tau) = \gamma(1 - \cos B\tau) + \frac{A_2}{B} \sin B\tau.$$

To solve for  $\gamma$  we compare the derivatives of  $x_1(\tau)$ . We know  $x_1'(\tau) = -Bx_2(\tau) + A_1$  and by substituting  $x_2(\tau) = \gamma(1 - \cos B\tau) + \frac{A_2}{B} \sin B\tau$  we get

$$x_1'(\tau) = -\gamma B + \gamma B \cos B\tau - A_2 \sin B\tau + A_1.$$

We found that

$$x_1(\tau) = -\frac{A_2}{B}(1 - \cos B\tau) + \gamma \sin B\tau.$$

Thus

$$x_1'(\tau) = \gamma B \sin B\tau - A_2 \sin B\tau.$$

We conclude

$$-B\gamma + A_1 = 0$$

and so

$$\gamma = \frac{A_1}{B}.$$

Plugging  $\gamma$  into  $x_1(\tau)$  and  $x_2(\tau)$ , we have

$$\begin{aligned} x_1(\tau) &= -\frac{A_2}{B}(1 - \cos B\tau) + \frac{A_1}{B} \sin B\tau \\ x_2(\tau) &= \frac{A_1}{B}(1 - \cos B\tau) + \frac{A_2}{B} \sin B\tau. \end{aligned}$$

We now need to solve for  $t(\tau)$ .

$$t'(\tau) = -\frac{x_2(\tau)}{2}(\epsilon_1(\tau) - \frac{x_2(\tau)}{2}\epsilon_t(\tau)) + \frac{x_1(\tau)}{2}(\epsilon_2(\tau) + \frac{x_1(\tau)}{2}\epsilon_t(\tau))$$

which then implies

$$\begin{aligned}
t'(\tau) &= -\frac{x_2(\tau)x_1'(\tau)}{2} + \frac{x_1(\tau)x_2'(\tau)}{2} \\
&= \frac{1}{2}(x_1(\tau)x_2'(\tau) - x_2(\tau)x_1'(\tau)) \\
&= \frac{1}{2}\left(\left(-\frac{A_2}{B}(1 - \cos B\tau) + \frac{A_1}{B}\sin B\tau\right)(A_1\sin B\tau + A_2\cos B\tau) \right. \\
&\quad \left. - \left(\frac{A_1}{B}(1 - \cos B\tau) + \frac{A_2}{B}\sin B\tau\right)(A_1\cos B\tau - A_2\sin B\tau)\right) \\
&= \frac{1}{2}\left(\left(\frac{A_1}{B}\sin B\tau - \frac{A_2}{B} + \frac{A_2}{B}\cos B\tau\right)(A_1\sin B\tau + A_2\cos B\tau) \right. \\
&\quad \left. - \left(\frac{A_1}{B} - \frac{A_1}{B}\cos B\tau + \frac{A_2}{B}\sin B\tau\right)(A_1\cos B\tau - A_2\sin B\tau)\right) \\
&= \frac{1}{2}\left(\frac{A_1^2}{B}\sin^2 B\tau - \frac{A_1A_2}{B}\sin B\tau + \frac{A_1A_2}{B}\sin B\tau\cos B\tau \right. \\
&\quad \left. + \frac{A_1A_2}{B}\sin B\tau\cos B\tau - \frac{A_2^2}{B}\cos B\tau + \frac{A_2^2}{B}\cos^2 B\tau \right. \\
&\quad \left. - \frac{A_1^2}{B}\cos B\tau + \frac{A_1^2}{B}\cos^2 B\tau - \frac{A_1A_2}{B}\sin B\tau\cos B\tau \right. \\
&\quad \left. + \frac{A_1A_2}{B}\sin B\tau - \frac{A_1A_2}{B}\sin B\tau\cos B\tau + \frac{A_2^2}{B}\sin^2 B\tau\right) \\
&= \frac{1}{2}\left(\frac{A_1^2}{B} + \frac{A_2^2}{B} - \frac{A_2^2}{B}\cos B\tau - \frac{A_1^2}{B}\cos B\tau\right) \\
&= \frac{1}{2}\left(\frac{A_1^2 + A_2^2}{B}\right) - \frac{1}{2}\left(\frac{A_1^2 + A_2^2}{B}\right)\cos B\tau
\end{aligned}$$

so

$$t'(\tau) = \frac{1}{2}\left(\frac{A_1^2 + A_2^2}{B}\right) - \frac{1}{2}\left(\frac{A_1^2 + A_2^2}{B}\right)\cos B\tau.$$

Integrating, we arrive at

$$\begin{aligned}
t(\tau) &= \frac{1}{2}\left(\frac{A_1^2 + A_2^2}{B}\right)\int 1 \, d\tau - \frac{1}{2}\left(\frac{A_1^2 + A_2^2}{B}\right)\int \cos B\tau \, d\tau \\
&= \frac{1}{2}\left(\frac{A_1^2 + A_2^2}{B}\right)\tau - \frac{1}{2}\left(\frac{A_1^2 + A_2^2}{B^2}\right)\sin B\tau + \delta.
\end{aligned}$$

But  $t(0) = 0$ , so  $\delta = 0$  and so

$$\begin{aligned} t(\tau) &= \frac{1}{2} \left( \frac{A_1^2 + A_2^2}{B} \right) \tau - \frac{1}{2} \left( \frac{A_1^2 + A_2^2}{B^2} \right) \sin B\tau \\ &= \frac{A_1^2 + A_2^2}{2B^2} (B\tau - \sin B\tau). \end{aligned}$$

■

### 3.2.2 Taylor Polynomials

Taylor polynomials in  $\mathbb{R}^n$  are useful to approximate functions. This role will be extended to groups of Heisenberg type. However, in this setting, the Taylor polynomials must be modified to compensate for the group operation (2.1.1) and the structure of the Heisenberg type groups. The following proposition gives the formula for the Taylor polynomial based at an arbitrary point.

**Proposition 3.2.2** *Let  $f : H_n \mapsto \mathbb{R}$  be a  $C^2$  function. Let the base point be denoted by  $p_0$ . Then,*

$$\begin{aligned} f(p) &= f(p_0) + \sum_{i=1}^n X_i f(p_0) (x_i - x_i^0) + T f(p_0) (t - t^0 + \frac{1}{2} (x_1 x_2^0 - x_1^0 x_2)) \\ &\quad + \frac{1}{2} \sum_{i,j=1}^n \frac{1}{2} (X_i X_j + X_j X_i) f(p_0) (x_i - x_i^0) (x_j - x_j^0) + o(d_C(p_0, p))^2 \\ &= P(p) + o(d_C(p_0, p))^2. \end{aligned}$$

By the definition of the Taylor Polynomial,  $f(p)$  should satisfy each of the following:

$$\begin{aligned} X_j P(p_0) &= X_j f(p_0) \\ X_i X_j P(p_0) &= X_i X_j f(p_0) \\ T P(p_0) &= T f(p_0). \end{aligned}$$



So, we will check each of these. Clearly,

$$TP(p_0) = Tf(p_0)$$

and we have agreement. We now turn to  $X_1$ .

$$\begin{aligned} X_1P(p) &= X_1f(p_0) + \frac{x_2^0}{2}Tf(p_0) \\ &\quad + \frac{1}{2} \sum_{j=1}^n \frac{1}{2}(X_1X_j + X_jX_1)f(p_0)(x_j - x_j^0) \\ &\quad + \frac{1}{2} \sum_{i=1}^n \frac{1}{2}(X_iX_1 + X_1X_i)f(p_0)(x_i - x_i^0) - \frac{x_2}{2}Tf(p_0). \end{aligned}$$

This gives us

$$X_1P(p_0) = X_1f(p_0) + \frac{x_2^0}{2}Tf(p_0) - \frac{x_2}{2}Tf(p_0)$$

and so we have agreement. Looking at  $X_2$ , we have

$$\begin{aligned} X_2P(p) &= X_2f(p_0) + \frac{-x_1^0}{2}Tf(p_0) \\ &\quad + \frac{1}{2} \sum_{j=1}^n \frac{1}{2}(X_2X_j + X_jX_2)f(p_0)(x_j - x_j^0) \\ &\quad + \frac{1}{2} \sum_{i=1}^n \frac{1}{2}(X_iX_2 + X_2X_i)f(p_0)(x_i - x_i^0) - \frac{x_1}{2}Tf(p_0). \end{aligned}$$

This gives us

$$X_2P(p_0) = X_2f(p_0) - \frac{x_1^0}{2}Tf(p_0) + \frac{x_1}{2}Tf(p_0)$$

and we have agreement.

We now let  $k \geq 3$ .

$$\begin{aligned} X_k P(p) &= X_k f(p_0) + \frac{1}{2} \sum_{j=1}^n \frac{1}{2} (X_k X_j + X_j X_k) f(p_0) (x_j - x_j^0) \\ &\quad + \frac{1}{2} \sum_{i=1}^n \frac{1}{2} (X_i X_k + X_k X_i) f(p_0) (x_i - x_i^0) \end{aligned}$$

and so we have agreement. Thus, all the  $X_i$  and  $T$  derivatives coincide. We now focus on the derivatives of the form  $X_i X_j$ . Letting  $i = j = 1$ , we have, using the above first order calculations,

$$\begin{aligned} X_1 X_1 P(p) &= \frac{1}{2} \left( \frac{1}{2} (X_1 X_1 + X_1 X_1) f(p_0) + \frac{1}{2} (X_1 X_1 + X_1 X_1) f(p_0) \right) \\ &= \frac{1}{2} (2 X_1 X_1 f(p_0)) \\ &= X_1 X_1 f(p_0) \end{aligned}$$

and so

$$X_1 X_1 P(p_0) = X_1 X_1 f(p_0).$$

Now, letting  $i = j = 2$ , we have, using the above first order calculations,

$$\begin{aligned} X_2 X_2 P(p) &= \frac{1}{2} \left( \frac{1}{2} (X_2 X_2 + X_2 X_2) f(p_0) + \frac{1}{2} (X_2 X_2 + X_2 X_2) f(p_0) \right) \\ &= \frac{1}{2} (2 X_2 X_2 f(p_0)) \\ &= X_2 X_2 f(p_0) \end{aligned}$$

and so

$$X_2 X_2 P(p_0) = X_2 X_2 f(p_0).$$

Now, for  $3 \leq m \leq n$  and  $3 \leq k \leq n$ , we have

$$\begin{aligned} X_m X_k P(p) &= \frac{1}{2} \left( \frac{1}{2} (X_k X_m + X_m X_k) f(p_0) + \frac{1}{2} (X_m X_k + X_k X_m) f(p_0) \right) \\ &= \frac{1}{2} ((X_k X_m + X_m X_k) f(p_0)). \end{aligned}$$

Since  $[X_k, X_m] = 0$ , we have  $X_k X_m = X_m X_k$ , so

$$\begin{aligned} X_m X_k P(p) &= \frac{1}{2}(2X_m X_k f(p_0)) \\ &= X_m X_k f(p_0) \end{aligned}$$

thus

$$X_m X_k P(p_0) = X_m X_k f(p_0).$$

Looking at  $X_1 X_2$ , we have

$$X_1 X_2 P(p) = \frac{1}{2}\left(\frac{1}{2}(X_2 X_1 + X_1 X_2)f(p_0) + \frac{1}{2}(X_1 X_2 + X_2 X_1)f(p_0)\right) + \frac{1}{2}Tf(p_0).$$

Now,  $T = X_1 X_2 - X_2 X_1$ , so we have

$$\begin{aligned} X_1 X_2 P(p) &= \frac{1}{2}\left(\frac{1}{2}(X_2 X_1 + X_1 X_2)f(p_0)\right) + \frac{1}{2}\left(\frac{1}{2}(X_1 X_2 - X_2 X_1)f(p_0)\right) \\ &= \frac{1}{2}\left(\frac{1}{2}(X_2 X_1 + X_1 X_2 + X_1 X_2 - X_2 X_1)f(p_0)\right) \\ &= \frac{1}{2}(2X_1 X_2 f(p_0)) \\ &= X_1 X_2 f(p_0) \end{aligned}$$

therefore

$$X_1 X_2 P(p_0) = X_1 X_2 f(p_0).$$

Looking at  $X_2 X_1$ , we have

$$X_2 X_1 P(p) = \frac{1}{2}\left(\frac{1}{2}(X_1 X_2 + X_2 X_1)f(p_0) + \frac{1}{2}(X_2 X_1 + X_1 X_2)f(p_0)\right) - \frac{1}{2}Tf(p_0).$$

Now  $T = X_1X_2 - X_2X_1$ , so we have

$$\begin{aligned}
X_2X_1P(p) &= \frac{1}{2}((X_1X_2 + X_2X_1)f(p_0)) - \frac{1}{2}((X_1X_2 - X_2X_1)f(p_0)) \\
&= \frac{1}{2}((X_1X_2 + X_2X_1 - X_1X_2 + X_2X_1)f(p_0)) \\
&= \frac{1}{2}(2X_2X_1f(p_0)) \\
&= X_2X_1f(p_0)
\end{aligned}$$

and so

$$X_2X_1P(p_0) = X_2X_1f(p_0).$$

Looking at  $X_1X_k$  for  $3 \leq k \leq n$ , we have

$$\begin{aligned}
X_1X_kP(p) &= \frac{1}{2}\left(\frac{1}{2}(X_kX_1 + X_1X_k)f(p_0) + \frac{1}{2}(X_1X_k + X_kX_1)f(p_0)\right) \\
&= \frac{1}{2}((X_kX_1 + X_1X_k)f(p_0)).
\end{aligned}$$

Since  $[X_1, X_k] = 0$ , we have  $X_1X_k = X_kX_1$ , thus

$$\begin{aligned}
X_1X_kP(p) &= \frac{1}{2}(2X_1X_kf(p_0)) \\
&= X_1X_kf(p_0)
\end{aligned}$$

hence

$$X_1X_kP(p_0) = X_1X_kf(p_0).$$

Looking at  $X_kX_1$  for  $3 \leq k \leq n$ , we have

$$\begin{aligned}
X_kX_1P(p) &= \frac{1}{2}\left(\frac{1}{2}(X_1X_k + X_kX_1)f(p_0) + \frac{1}{2}(X_kX_1 + X_1X_k)f(p_0)\right) \\
&= \frac{1}{2}((X_1X_k + X_kX_1)f(p_0)).
\end{aligned}$$

Since  $[X_1, X_k] = 0$ , we have  $X_1X_k = X_kX_1$ , so

$$\begin{aligned} X_kX_1P(p) &= \frac{1}{2}(2X_kX_1f(p_0)) \\ &= X_kX_1f(p_0) \end{aligned}$$

therefore

$$X_kX_1P(p_0) = X_kX_1f(p_0).$$

Looking at  $X_2X_k$  for  $3 \leq k \leq n$ , we have

$$\begin{aligned} X_2X_kP(p) &= \frac{1}{2}\left(\frac{1}{2}(X_kX_2 + X_2X_k)f(p_0) + \frac{1}{2}(X_2X_k + X_kX_2)f(p_0)\right) \\ &= \frac{1}{2}\left((X_kX_2 + X_2X_k)f(p_0)\right). \end{aligned}$$

Since  $[X_2, X_k] = 0$ , we have  $X_2X_k = X_kX_2$ , so we have

$$\begin{aligned} X_2X_kP(p) &= \frac{1}{2}(2X_2X_kf(p_0)) \\ &= X_2X_kf(p_0) \end{aligned}$$

and so

$$X_2X_kP(p_0) = X_2X_kf(p_0).$$

Looking at  $X_kX_2$  for  $3 \leq k \leq n$ , we have

$$\begin{aligned} X_kX_2P(p) &= \frac{1}{2}\left(\frac{1}{2}(X_2X_k + X_kX_2)f(p_0) + \frac{1}{2}(X_kX_2 + X_2X_k)f(p_0)\right) \\ &= \frac{1}{2}\left((X_2X_k + X_kX_2)f(p_0)\right). \end{aligned}$$

Since  $[X_2, X_k] = 0$ , we have  $X_2X_k = X_kX_2$ , so we have

$$\begin{aligned} X_kX_2P(p) &= \frac{1}{2}(2X_kX_2f(p_0)) \\ &= X_kX_2f(p_0) \end{aligned}$$

thus

$$X_k X_2 P(p_0) = X_k X_2 f(p_0).$$



## 4 VISCOSITY SOLUTIONS

### 4.1 A Class of Partial Differential Equations

We look to extend the results in  $\mathbb{R}^n$  from Chapter 1 to our class of groups of Heisenberg type. We recall the calculus on Carnot groups from Chapter 2. The horizontal gradient is given by

$$\nabla_0 f(p) = (X_1 f(p), X_2 f(p), X_3 f(p), \dots, X_n f(p))$$

and the gradient is given by

$$\nabla f(p) = (X_1 f(p), X_2 f(p), X_3 f(p), \dots, X_n f(p), T f(p)).$$

We also recall that this is also the semi-horizontal gradient. In addition, we consider the symmetrized second order derivative matrix  $(D^2 f)^*(p)$  with entries

$$(D^2 f)_{ij}^* = \frac{1}{2}(X_i X_j + X_j X_i) f.$$

Note that neither the horizontal gradient nor the second order derivative matrix use the vector  $T$  directly.

Following Chapter 1 and recalling that  $S^n$  is the set of  $n \times n$  symmetric matrices, we consider a class of partial differential equations of the form

$$F(p, u(p), \nabla u(p), (D^2 u(p))^*) = 0$$

where continuous

$$F : H_n \times \mathbb{R} \times h_n \times S^n \mapsto \mathbb{R}$$

has the property that  $F(p, r, \eta, X) \leq F(p, s, \eta, Y)$  when  $r \leq s$  and  $Y \leq X$ . Again, this class of equations is proper in the sense of [3].

We may consider the same examples in Chapter 1, substituting the Carnot group derivatives in place of the  $\mathbb{R}^n$  derivatives.

## 4.2 Jets

Let  $\eta = (\eta_1, \eta_2, \dots, \eta_n, \eta_t) \in h_n$  and  $X \in S^n$ . Given a function  $f : H_n \mapsto \mathbb{R}$ , consider the following inequality based on the Taylor Polynomial from the previous section:

$$\begin{aligned} f(p) \leq & f(p_0) + \sum_{i=1}^n \eta_i(x_i - x_i^0) + \eta_t(t - t^0 + \frac{1}{2}(x_1x_2^0 - x_1^0x_2)) \\ & + \frac{1}{2} \sum_{i,j=1}^n X_{ij}(x_i - x_i^0)(x_j - x_j^0) + o(d_C(p_0, p))^2 \end{aligned} \quad (4.2.1)$$

Following Chapter 1, we define the second order superjet of  $f$  at  $p_0$ , denoted  $J^{2,+}f(p_0)$ , and the second order subjet of  $f$  at  $p_0$ , denoted  $J^{2,-}f(p_0)$ , by the following:

$$(\eta, X) \in J^{2,+}f(p_0) \iff (4.2.1) \text{ holds.}$$

$$J^{2,-}f(p_0) = -J^{2,+}(-f)(p_0).$$

We note as in Chapter 1, that jets may be empty at a point, and so there is no guarantee that such a pair  $(\eta, X)$  exists. Additionally, we again define  $\bar{J}^{2,+}u(p)$  as the set-theoretic closure of  $J^{2,+}u(p)$ .

We then define viscosity solutions as follows:

**Definition 4.2.1** A continuous function  $f$  is a viscosity subsolution of

$$F(p, f(p), \nabla f(p), (D^2 f)^*(p)) = 0$$



if for all  $(\eta, X) \in \overline{J}^{2,+} f(p)$  we have

$$F(p, f(p), \eta, X) \leq 0$$

A continuous function  $f$  is a viscosity supersolution of

$$F(p, f(p), \nabla f(p), (D^2 f)^*(p)) = 0$$

if for all  $(\mu, Y) \in \overline{J}^{2,-} f(p)$  we have

$$F(p, f(p), \mu, Y) \geq 0$$

The continuous function  $f$  is a viscosity solution if it is both a viscosity subsolution and a viscosity supersolution.

As in Chapter 1, we focus on a special subclass of such functions  $F$ . We call the subclass  $\mathcal{F}$  those functions  $F$  for which there is a universal  $\delta > 0$  so that for each real  $r$  and  $s$  with  $r < s$ , we have

$$0 < \delta(s - r) \leq F(p, s, \eta, X) - F(p, r, \eta, X)$$

and we further assume that there exists  $w_1, w_2, w_3 : [0, \infty] \rightarrow [0, \infty]$ ,  $w_i(0^+) = 0$  for  $i = 1, 2, 3$  so that

$$\begin{aligned} F(p, r, \eta, \mathcal{Y}) - F(p, r, \eta, \mathcal{X}) &\leq w_1(\|\mathcal{X} - \mathcal{Y}\|_M) \\ F(p, r, \eta, \mathcal{X}) - F(p, r, \gamma, \mathcal{X}) &\leq w_2(|\|\eta\| - \|\gamma\||) \\ F(p, r, \eta, \mathcal{X}) - F(q, r, \eta, \mathcal{X}) &\leq w_3(d_C(p, q)). \end{aligned}$$

Again, we let  $\|\cdot\|_M$  be the matrix norm.

We wish to prove the following theorem.

**Theorem 4.2.2** *Let  $F \in \mathcal{F}$  and  $\Omega$  be a bounded open set in  $H_n$ . Then if  $u$  is a viscosity subsolution to*

$$F(p, u(p), \nabla u(p), (D^2u)^*(p)) = 0$$

*and  $v$  is a viscosity supersolution to*

$$F(p, v(p), \nabla v(p), (D^2v)^*(p)) = 0$$

*with  $u \leq v$  on  $\partial\Omega$ . Then  $u \leq v$  in  $\Omega$ .*

Our proof will not be as easy as the  $\mathbb{R}^n$  case, because the technical lemmas are unproven in groups of Heisenberg type. Instead, we need the Bieske-Manfredi[2] twisting lemma that connects jets on our groups of Heisenberg type to jets in  $\mathbb{R}^n$ . We state the full version of the twisting lemma.

**Lemma 4.2.3** *Let  $DL_{p_0}$  be the differential of the left multiplication map at the point  $p_0$ , let  $J_{eucl}^{2,+}u(p_0)$  be the traditional Euclidean superjet of  $u$  at the point  $p_0$  (as defined in Chapter 1) and let  $(\eta, X) \in \mathbb{R}^{n+1} \times S^{n+1}$ . Then,*

$$(\eta, X) \in \bar{J}_{eucl}^{2,+}u(p_0)$$

*gives the element*

$$\left( DL_{p_0}\eta, (DL_{p_0} X (DL_{p_0})^T)_n \right) \in \bar{J}^{2,+}u(p_0)$$

*with the convention that for any matrix  $M$ ,  $M_n$  is the  $n \times n$  principal minor.*

For any point  $p = (x_1, x_2, x_3, \dots, t)$ , we can compute  $DL_p$  directly for our class of groups of Heisenberg type because the multiplication law for our class is given explic-

itly by Equation (3.2.1). If we let  $v$  be the  $n \times 1$  vector given by

$$v = \begin{pmatrix} -\frac{x_2}{2} \\ \frac{x_1}{2} \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

and  $\mathcal{O}$  be the  $1 \times n$  vector given by  $\mathcal{O} = (0, 0, 0, \dots, 0)$ , then we have

$$DL_p = \begin{bmatrix} I_n & v \\ \mathcal{O} & 1 \end{bmatrix}.$$

If  $\eta$  is the vector  $(\eta_1, \eta_2, \eta_3, \dots, \eta_{n+1})$ , we can then compute  $DL_p\eta$  explicitly to obtain

$$DL_p\eta = \begin{pmatrix} \eta_1 - \frac{x_2}{2}\eta_{n+1} \\ \eta_2 + \frac{x_1}{2}\eta_{n+1} \\ \eta_3 \\ \eta_4 \\ \vdots \\ \eta_n \\ \eta_{n+1} \end{pmatrix}.$$

In addition, if the matrix  $X$  has  $ij$ -th entry  $X_{ij}$ , then we can compute the  $n \times n$  principal minor of  $DL_pXDL_p^T$ , denoted  $(DL_pXDL_p^T)_n$ , directly also. The matrix is

given by

$$\begin{aligned}
((DL_p X DL_p^T)_n)_{1,1} &= x_{1,1} - \frac{x_2}{2} x_{n+1,1} + (x_{1,n+1} - \frac{x_2}{2} x_{n+1,n+1}) (-\frac{x_2}{2}) \\
((DL_p X DL_p^T)_n)_{1,2} &= x_{1,2} - \frac{x_2}{2} x_{n+1,2} + (x_{1,n+1} - \frac{x_2}{2} x_{n+1,n+1}) (\frac{x_1}{2}) \\
((DL_p X DL_p^T)_n)_{1,j} &= x_{1,j} - \frac{x_2}{2} x_{n+1,j}, \quad j = 3, 4, \dots, n \\
((DL_p X DL_p^T)_n)_{2,1} &= x_{2,1} + \frac{x_1}{2} x_{n+1,1} + (x_{2,n+1} + \frac{x_1}{2} x_{n+1,n+1}) (-\frac{x_2}{2}) \\
((DL_p X DL_p^T)_n)_{2,2} &= x_{2,2} + \frac{x_1}{2} x_{n+1,2} + (x_{2,n+1} + \frac{x_1}{2} x_{n+1,n+1}) (\frac{x_1}{2}) \\
((DL_p X DL_p^T)_n)_{2,j} &= x_{2,j} + \frac{x_1}{2} x_{n+1,j} \quad j = 3, 4, \dots, n \\
((DL_p X DL_p^T)_n)_{i,1} &= x_{i,1} + x_{i,n+1} (-\frac{x_2}{2}) \quad i = 3, 4, \dots, n \\
((DL_p X DL_p^T)_n)_{i,2} &= x_{i,2} + x_{i,n+1} (\frac{x_1}{2}) \quad i = 3, 4, \dots, n \\
((DL_p X DL_p^T)_n)_{i,j} &= x_{i,j} \quad i, j = 3, 4, \dots, n
\end{aligned}$$

This twisting will allow us to use the maximum principle from the  $\mathbb{R}^n$  environment, but it will not produce estimates that are as easy to control. The maximum principle works by using a “penalty function” that guarantees the existence of jet elements at certain points close to the desired point. We present a weak version of the maximum principle in  $H_n$ . [2]

**Theorem 4.2.4** *Let  $u$  and  $v$  be continuous functions in a bounded domain  $\Omega \subset H_n$ . If  $u - v$  has a positive interior local maximum*

$$\sup_{\Omega} (u - v) > 0$$

then we have:

For  $\tau > 0$  we can find points  $p_\tau, q_\tau \in H_n$  such that

i)

$$\lim_{\tau \rightarrow \infty} \tau \psi(p_\tau, q_\tau) = 0,$$

where

$$\psi(p, q) = \|p - q\|_{eucl}^2.$$

ii) There exists a point  $\hat{p} \in \Omega$  such that  $p_\tau \rightarrow \hat{p}$  (and so does  $q_\tau$  by (i)) and

$$\sup_{\Omega}(u - v) = u(\hat{p}) - v(\hat{p}) > 0,$$

iii) there exist symmetric matrices  $\mathcal{X}_\tau, \mathcal{Y}_\tau$  and vectors  $\eta_\tau^+, \eta_\tau^-$  so that

iv)

$$(\eta_\tau^+, \mathcal{X}_\tau) \in \bar{J}^{2,+}(u, p_\tau),$$

v)

$$(\eta_\tau^-, \mathcal{Y}_\tau) \in \bar{J}^{2,-}(v, q_\tau),$$

vi)

$$\eta_\tau^+ - \eta_\tau^- = \mathcal{R}_\tau$$

where  $\mathcal{R}_\tau \rightarrow 0$  as  $\tau \rightarrow \infty$ . and

vii)

$$\mathcal{X}_\tau \leq \mathcal{Y}_\tau + \mathcal{Z}_\tau$$

where  $\mathcal{Z}_\tau \rightarrow 0$  as  $\tau \rightarrow \infty$ .

**Proof:** By the  $\mathbb{R}^n$  version [3],

$$(\tau D_p \psi(p_\tau, q_\tau), X_\tau) \in \bar{J}_{\text{eucl}}^{2,+} u(p_\tau)$$

and

$$(-\tau D_q \psi(p_\tau, q_\tau), Y_\tau) \in \bar{J}_{\text{eucl}}^{2,-} v(q_\tau)$$

where  $D_p$  and  $D_q$  are euclidean derivatives and as symmetric matrices,  $X_\tau \leq Y_\tau$ . By

the above twisting lemma, we have

$$\begin{aligned}
\eta_\tau^+ &= \tau DL_{p_\tau} D_p \psi(p_\tau, q_\tau) \\
\eta_\tau^- &= -\tau DL_{q_\tau} D_q \psi(p_\tau, q_\tau) \\
\mathcal{X}_\tau &= (DL_{p_\tau} X_\tau DL_{p_\tau}^T)_n \\
\mathcal{Y}_\tau &= (DL_{q_\tau} Y_\tau DL_{q_\tau}^T)_n
\end{aligned}$$

Let  $\xi$  be a fixed  $n \times 1$  vector and let  $\bar{\xi}$  be the  $(n+1) \times 1$  vector given by  $\bar{\xi} = (\xi, 0)$ .

Then

$$\begin{aligned}
\langle \mathcal{X}_\tau \xi, \xi \rangle - \langle \mathcal{Y}_\tau \xi, \xi \rangle &= \langle (DL_{p_\tau} X_\tau DL_{p_\tau}^T)_n \xi, \xi \rangle - \langle (DL_{q_\tau} Y_\tau DL_{q_\tau}^T)_n \xi, \xi \rangle \\
&= \langle DL_{p_\tau} X_\tau DL_{p_\tau}^T \bar{\xi}, \bar{\xi} \rangle - \langle DL_{q_\tau} Y_\tau DL_{q_\tau}^T \bar{\xi}, \bar{\xi} \rangle \\
&= \langle X_\tau DL_{p_\tau}^T \bar{\xi}, DL_{p_\tau}^T \bar{\xi} \rangle - \langle Y_\tau DL_{q_\tau}^T \bar{\xi}, DL_{q_\tau}^T \bar{\xi} \rangle.
\end{aligned}$$

By the results from  $\mathbb{R}^n$  [3], we have

$$\langle X_\tau DL_{p_\tau}^T \bar{\xi}, DL_{p_\tau}^T \bar{\xi} \rangle - \langle Y_\tau DL_{q_\tau}^T \bar{\xi}, DL_{q_\tau}^T \bar{\xi} \rangle \leq 3\tau \|DL_{p_\tau}^T \bar{\xi} - DL_{q_\tau}^T \bar{\xi}\|^2.$$

We therefore obtain

$$\begin{aligned}
\langle \mathcal{X}_\tau \xi, \xi \rangle - \langle \mathcal{Y}_\tau \xi, \xi \rangle &\leq 3\tau \|(DL_{p_\tau}^T - DL_{q_\tau}^T) \bar{\xi}\|^2 \\
&= 3\|\bar{\xi}\|^2 \tau \|(DL_{p_\tau}^T - DL_{q_\tau}^T)\|^2 \\
&\sim \tau \psi(p_\tau, q_\tau)
\end{aligned}$$

and we arrive at the result by letting  $\tau \rightarrow \infty$ .

Examining the vector difference, the fact that  $D_p\psi = -D_q\psi$  gives us

$$\begin{aligned}
\left| \|\eta_\tau^+\| - \|\eta_\tau^-\| \right| &= \tau \left| \|DL_{p_\tau}D_p\psi(p_\tau, q_\tau)\| - \|-DL_{q_\tau}D_q\psi(p_\tau, q_\tau)\| \right| \\
&= \tau \left| \|DL_{p_\tau}D_p\psi(p_\tau, q_\tau)\| - \|DL_{q_\tau}D_p\psi(p_\tau, q_\tau)\| \right| \\
&\leq \tau \|(DL_{p_\tau} - DL_{q_\tau})D_p\psi(p_\tau, q_\tau)\| \\
&\leq \tau \|(DL_{p_\tau} - DL_{q_\tau})\| \|D_p\psi(p_\tau, q_\tau)\| \\
&\sim \tau \psi(p_\tau, q_\tau)^{\frac{1}{2}} \psi(p_\tau, q_\tau)^{\frac{1}{2}}.
\end{aligned}$$

The result follows by taking  $\tau \rightarrow \infty$ .

■

We are now ready to restate and prove our theorem.

**Theorem 4.2.5** *Let  $\Omega \subset H_n$  be a bounded open set. Let  $F \in \mathcal{F}$ . If  $u$  is a viscosity subsolution to*

$$F(p, u(p), \nabla u(p), (D^2u(p))^*) = 0$$

*and  $v$  is a viscosity supersolution to*

$$F(p, v(p), \nabla v(p), (D^2v(p))^*) = 0$$

*such that  $u \leq v$  on  $\partial\Omega$  then  $u \leq v$  in  $\Omega$ .*

**Proof:** Suppose there is a  $\hat{p}$  inside  $\Omega$  so that  $u(\hat{p}) > v(\hat{p})$ . By the maximum principle and continuity of  $u$  and  $v$ , we have  $u(p_\tau) > v(q_\tau)$  for large  $\tau$ . In addition,  $p_\tau$  and  $q_\tau$  are interior points.

Using the fact that  $F \in \mathcal{F}$ , we obtain

$$\begin{aligned}
0 < \sigma(u(p_\tau) - v(q_\tau)) &\leq F(p_\tau, u(p_\tau), \tau\eta_{p_\tau}, \mathcal{X}_\tau) - F(p_\tau, v(q_\tau), \tau\eta_{p_\tau}, \mathcal{X}_\tau) \\
&= F(p_\tau, u(p_\tau), \tau\eta_{p_\tau}, \mathcal{X}_\tau) - F(q_\tau, v(q_\tau), \tau\eta_{q_\tau}, \mathcal{Y}_\tau) \\
&+ F(q_\tau, v(q_\tau), \tau\eta_{q_\tau}, \mathcal{Y}_\tau) - F(p_\tau, v(q_\tau), \tau\eta_{q_\tau}, \mathcal{Y}_\tau) \\
&+ F(p_\tau, v(q_\tau), \tau\eta_{q_\tau}, \mathcal{Y}_\tau) - F(p_\tau, v(q_\tau), \tau\eta_{p_\tau}, \mathcal{Y}_\tau) \\
&+ F(p_\tau, v(q_\tau), \tau\eta_{p_\tau}, \mathcal{Y}_\tau) - F(p_\tau, v(q_\tau), \tau\eta_{p_\tau}, \mathcal{X}_\tau) \\
&\leq 0 + w_3(d_C(p_\tau, q_\tau)) + w_2 \left| \|\tau\eta_{p_\tau}\| - \|\tau\eta_{q_\tau}\| \right| \\
&\quad + w_1(\|\mathcal{X}_\tau - \mathcal{Y}_\tau\|_M).
\end{aligned}$$

We note the estimate on the first difference comes from the fact that  $u$  is a subsolution and  $v$  is a supersolution.

Letting  $\tau \rightarrow \infty$  we arrive at a contradiction via the maximum principle and the properties of the functions  $w_i$ . Thus, the maximum can not occur at the interior point  $\hat{p}$ , and so  $u \leq v$  in  $\Omega$ .

■

Using the Theorem, we have the following corollary.

**Corollary 4.2.6** *Let  $F(p, r, \eta, X)$  and  $\Omega$  be as in the theorem. Let  $\nu : \partial\Omega \rightarrow \mathbb{R}$  be continuous. Then, there is at most one solution to*

$$\begin{cases} F(p, f(p), \nabla f(p), (D^2 f)^*(p)) = 0 & \text{in } \Omega \\ f = \nu & \text{on } \partial\Omega \end{cases}$$

**Proof:** Suppose  $u_1, u_2$  are two solutions.  $u_1$  is both a subsolution and supersolution and  $u_2$  is both a subsolution and supersolution. Then  $u_1 \leq u_2$  and  $u_2 \leq u_1$  on  $\partial\Omega$ . By the theorem,  $u_1 \leq u_2$  and  $u_2 \leq u_1$  in  $\Omega$  and so  $u_1 = u_2$ .

■



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