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Identification of the Parameters When the Density of the Minimum is Given

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by

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Dedication

To the loving memory of my parents, John C. Davis Jr. (1921-1999) and Mattie E. Davis (1922-2006), who sacrificed for, nurtured, and were the largest contributors to the learning experiences of their children.

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ABSTRACT

Let (X_1, X_2, X_3) be a tri-variate normal vector with a non-singular co-variance matrix Σ, where for $i \neq j$, $\Sigma_{ij} < 0$. It is shown here that it is then possible to determine the three means, the three variances and the three correlation coefficients based only on the knowledge of the probability density function for the minimum variate $Y = \min\{X_1, X_2, X_3\}$. We will present a method for identifying the nine parameters which consists of careful determination of the asymptotic orders of various bivariate tail probabilities.

Chapter 1
Introduction

The identification of parameters problem is one in which the underlying type of distribution is known but the values of the parameters is unknown. To be identifiable the parameters must have unique values under the specified conditions. A. P. Basu [5] defined *Identifiable* as follows: "Let U be an observable random variable whose distribution function belongs to a family $\mathcal{F}_\Omega = \{F_\theta : \theta \in \Omega\}$ of distribution functions indexed by a parameter θ. Here θ could be scalar or vector valued. We shall say θ is nonidentifiable by U if there are distinct parameter values, θ and θ', such that $F_\theta(u) = F_{\theta'}(u)$ for all u. In the contrary case we shall say θ is identifiable."

The identification of parameters by the distribution of the extremum problem involves finding the unique values of parameters for a set of random variables when we know only the underlying type of distribution of the random variables but we know exactly the distribution of the minimum variate (or maximum variate).

Suppose we are given the distribution function F and we know there exists a random vector $(X_1, X_2, \ldots, X_n)$ such that the minimum variate of this vector, $X = \min\{X_1, X_2, \ldots, X_n\}$, has F as its distribution function. Furthermore, suppose we know there is a family, $\mathcal{G}$, to which the distribution functions of $X_1, X_2, \ldots, X_n$ all belong, but the specific parameters for each of the distribution functions are not known. The questions addressed by the identification of parameters problem are:

1. Is the vector $(X_1, X_2, \ldots, X_n)$ unique in its relationship to F or can there be another vector $(Y_1, Y_2, \ldots, Y_n)$ whose components' distribution functions are also in $\mathcal{G}$ and whose minimum variate $Y = \min\{Y_1, Y_2, \ldots, Y_n\}$ has distribution function $F_Y = F$?
2. Can the values of the parameters of $(X_1, X_2, \ldots, X_n)$ be determined from F ?

These types of problems can be found in several areas of application. An econometric model for supply and demand in a state of disequilibrium can be stated as an identification of parameters by the distribution of the minimum problem. Let X_1 be the amount of a commodity that consumers will purchase at price p and let X_2 represent the amount of this commodity that producers are willing to supply at price p. Assuming EX_1 is a downward sloping function of p and EX_2 is an upward sloping function of p we write:

$$X_1 = \alpha_1 p + \beta_1 + e_1$$
$$X_2 = \alpha_2 p + \beta_2 + e_2$$

Here α_1 and β_1 represent the slope and intercept for EX_1 while e_1 is $N(0, \sigma_1^2)$. Similarly, α_2 and β_2 represent the slope and intercept for EX_2 while e_2 is $N(0, \sigma_2^2)$. $\alpha_1, \beta_1, \sigma_1, \alpha_2, \beta_2, \sigma_2$ are the parameters for this model. In a state of disequilibrium when the amount demanded is less than the amount produced ($X_1 < X_2$) then the amount actually purchased is X_1 and there is an excess amount produced that goes unsold. On the other hand, when the amount produced is less than the amount demanded ($X_2 < X_1$) then the amount actually purchased is X_2 and there is a shortage. If we denote by Y the amount purchased at price p we can write $Y = \min\{X_1, X_2\}$. Here Y is the observable random variable and the question is, does the distribution of Y uniquely determine the six parameters and therefore give the distribution for the demand schedule (X_1) and the distribution for the supply schedule (X_2)? Anderson and Ghurye [1] prove that the answer to this question is yes. It is shown that the mean and variance of X_1 and X_2 are identified by Y. Observe that

$$X_1 \sim N(\alpha_1 p + \beta_1, \sigma_1^2)$$
$$X_2 \sim N(\alpha_2 p + \beta_2, \sigma_2^2)$$

From an observation of Y at price $p = p^*$ the mean and variance of X_1 are identified as, say, μ^* and σ_1^2 while the mean and variance of X_2 are identified as ν^* and σ_2^2. From another observation of Y at price $p = p^{**}$ the means of X_1 and X_2 are identified as μ^{**} and ν^{**} respectively. The parameters $\alpha_1, \beta_1, \alpha_2, \beta_2$ can be identified using the following equations:

$$\alpha_1 p^* + \beta_1 = \mu^*$$
$$\alpha_1 p^{**} + \beta_1 = \mu^{**}$$
$$\alpha_2 p^* + \beta_2 = \nu^*$$
$$\alpha_2 p^{**} + \beta_2 = \nu^{**}$$

Rather than prove directly that the parameters of (X_2, X_1) can be identified by the *minimum* variate, the following statement is made: "Since the normal distribution is symmetric, the question is mathematically equivalent to the question poised in terms of the maximum of X_1 and X_2." This can be seen by observing:

$$Y = \min\{X_1, X_2\} = -\max\{-X_1, -X_2\}$$

When Y is observable, $Z = -Y$ is also observable. Now $Z = \max\{V, W\}$ where $V = -X_1, W = -X_2$. Therefore if

$$X_1 \sim N(\mu_1, \sigma_1^2)$$
$$X_2 \sim N(\mu_2, \sigma_2^2)$$

then

$$V \sim N(-\mu_1, \sigma_1^2)$$
$$W \sim N(-\mu_2, \sigma_2^2)$$

Hence if observable maximum variate Z identifies the parameters of (V, W) then $Y = \min\{X_1, X_2\}$ identifies the parameters of (X_2, X_1). So Anderson and Ghurye prove that the distribution of the observable maximum variate, $Z = \max\{X_1, X_2\}$, identifies the parameters of (X_2, X_1) when X_1 and X_2 are independent normal random variables. In fact, they prove the following, more general theorem: Suppose $\mathcal{G}$ is a family of probability density functions on the real number line with the two properties (1) each element of $\mathcal{G}$ is continuous and positive to the right of some point A, (2) for any two distinct elements of $\mathcal{G}$, say f and g, as $x \to \infty$, $\frac{f(x)}{g(x)}$ either converges to 0 or diverges to ∞. Let $X_1, X_2, \ldots, X_n$ be independent random variables whose pdf's $f_1, f_2, \ldots, f_n$ are elements of $\mathcal{G}$ and let $Y = \max\{X_1, X_2, \ldots, X_n\}$, then the set $\{f_1, f_2, \ldots, f_n\}$ is uniquely determined by the distribution of Y so that if $Y_1, Y_2, \ldots, Y_m$ is any collection of m random variables whose pdf's are in $\mathcal{G}$ and with $Y = \max\{Y_1, Y_2, \ldots, Y_m\}$ then $m = n$ and $f_1, f_2, \ldots, f_n$ must be the pdf's of $Y_1, Y_2, \ldots, Y_n$ (but not necessarily respectively).

Survival analysis and reliability theory are other areas in which parameter identification by the distiribution of the extreme variate is important. From survival analysis we consider a population in which each individual is subject to n causes of death. Let X_i be the time until death of an individual from cause $i, 1 \le i \le n$. Let $Y = \min\{X_1, X_2, \ldots, X_n\}$. The variable Y gives the time until death and is observable while the variables $X_1, X_2, \ldots, X_n$ are not observable but knowledge of their distributions is highly desirable. Problems of this type (finding the distributions of $X_1, X_2, \ldots, X_n$ from the distribution of the minimum variate) are called *competing risks* problems. A similar type problem from reliability theory considers a system composed of n components each of which is vital. When any component fails, the system fails. Here we let X_i be the time until failure of component $i, 1 \le i \le n$ and the observable $Y = \min\{X_1, X_2, \ldots, X_n\}$ gives the time until failure for the system. We now consider a different type n-component system. In this system the components are redundant (or at least not individually vital) so that the system fails only after all n of the components have failed. Here the variable representing the time until failure for the system is the observable $Y = \max\{X_1, X_2, \ldots, X_n\}$. Problems of this type (finding the distributions of $X_1, X_2, \ldots, X_n$ from the distribution of the maximum variate) are called *complementary risks* problems.

Certainly it is not always the case that the parameters can be determined by the distribution of the extreme variate. In general if $X_1, X_2, \ldots, X_n$ are independent random variables with distribution functions $F_1, F_2, \ldots, F_n$ respectively and $Y = \max\{X_1, X_2, \ldots, X_n\}$ is the maximum variate with distribution function F then

$$\begin{aligned} F = P(Y \le y) &= P(\max\{X_1, X_2, \ldots, X_n\} \le y) \\ &= P(X_1 \le y, X_2 \le y, \ldots, X_n \le y) \\ &= P(X_1 \le y)P(X_2 \le y)\cdots P(X_n \le y) \\ &= F_1 F_2 \cdots F_n \end{aligned}$$

Now suppose $Y_1, Y_2, \ldots, Y_m$ is a collection of independent, identically distributed random variables with distribution function $g = \sqrt[m]{F}$ and $m \ne n$. Observe that the distribution function of the maximum variate of this last collection of random variables is again F. However, this last collection of random variables are quite different in distribution from the former set and the distribution of the maximum variate does not identify the parameters.

Basu [5] gives another easy example: Let X_1, X_2, X_3, X_4 be independent random variables with $X_i \sim \exp(\lambda_i)$, $1 \le i \le 4$. Observe that distribution function for the minimum variate $U = \min\{X_1, X_2\}$ is given by

$$\begin{aligned} F_U(t) = P(U \le t) &= P(\min\{X_1, X_2\} \le t) \\ &= 1 - P(\min\{X_1, X_2\} > t) \\ &= 1 - P(X_1 > t)P(X_2 > t) \\ &= 1 - e^{-\lambda_1 t} e^{-\lambda_2 t} = 1 - e^{-(\lambda_1 + \lambda_2)t} \end{aligned}$$

So long as $\lambda_1 + \lambda_2 = \lambda_3 + \lambda_4$ the minimum variate $U = \min\{X_1, X_2\}$ and the minimum variate $V = \min\{X_3, X_4\}$ are identically distributed since the density functions $f_U(t) = (\lambda_1 + \lambda_2)e^{-(\lambda_1+\lambda_2)t} = (\lambda_3 + \lambda_4)e^{-(\lambda_3+\lambda_4)t} = f_V(t)$ for $0 \le t < \infty$. Therefore the distribution of the minimum variate does not identify the parameters when $\mathcal{G}$ is the family of pdf's for a collection of independent, exponentially distributed random variables.

The parameters for a finite collection of bivariate normal random vectors are identified by the distribution of a bivariate vector whose components are the maximum of the corresponding components of the vectors in the collection. That is, suppose $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ is a collection of independent bi-variate normal random vectors with $(X_i, Y_i) \sim N(\vec{\mu}_i, \Sigma_i)$ for $i = 1, 2, \ldots, n$. Now let $M_1 = \max\{X_1, X_2, \ldots, X_n\}$ and $M_2 = \max\{Y_1, Y_2, \ldots, Y_n\}$. The distribution of (M_1, M_2) determines the parameters $\vec{\mu}_i, \Sigma_i$ for $i = 1, 2, \ldots, n$. Anderson and Ghurye [1] proved this under the conditions that the vectors in the collection had means zero, non-negative correlations, and a non-singular covariance matrix. Mukherjea, Nakassis, and Miyashita [11] extended these results to include the conditions of negative correlations and means not necessarily zero.

Mukherjea, Nakassis, and Miyashita also showed that for the Cauchy distribution, the parameters for a collection of independent random variables could be identified by the distribution of the maximum variate. This is done by proving the following: Let

$$H_i(x) = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}(a_i x), \mathrm{i} = 1, 2, \ldots, \mathrm{n}$$

be the distribution functions for a collection of n Cauchy random variables. Let

$$L_j(x) = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}(b_j x), \mathrm{j} = 1, 2, \ldots, \mathrm{m}$$

be the distribution functions for a collection of m Cauchy random variables. Now suppose $H_1 H_2 \cdots H_n = L_1 L_2 \cdots L_m$. It follows that $m = n$ and that $a_1, a_2, \ldots, a_n$ is some permutation of $b_1, b_2, \ldots, b_n$.

Earlier we saw with the disequilibrium econometric model that the parameters of a pair of normal random variables could be identified from the distribution of the minimum variate. A key condition set there was that the pair of random variables be independent. Gilliland and Hannan [9] remove this condition and show that if the covariance matrix for the bivariate normal vector formed by the two random variables is non singular, then the five parameters for this vector are identified by the distribution of the minimum variate.

This is accomplished by first writing the pdf of the minimum variate as a function of the five parameters then using the asymptotic behavior of the pdf to find the value for each parameter. Since the vectors (V,U) and (U,V) have the same minimum variate, the parameters are said to be identified *up to switch.* This means that the mean and variance is identifiable for the two variables but assignment cannot be made to the specific variable. If the five parameters are $\mu_1,\mu_2,\sigma_1,\sigma_2,\rho_{12}$ we can identify μ_1,σ_1 as the mean and standard deviation of one of the variables (in fact we can identify these as belonging to the variable with the larger variance) and μ_2,σ_2 as the mean and standard deviation of the other variable, but we can not say which pair belongs to U and which pair belong to V.

Another type of identification problem occurs when given a collection of random variables, $X_1,X_2,\ldots,X_n$, and not only the value of the minimum variate is observable but also the index of the variable giving the minimum value is observable. That is, the random variable, *I*, defined by

$$I : I = i \text{ when } Y = \max\{X_1,X_2,\ldots,X_n\} = X_i$$

is also observable. This is the case in a competing risk survival analysis problem when an autopsy is performed to determine the specific cause after each death. The random variable I is called the identified minimum. Nadas [13] proved that the parameters of a bi-variate normal could be identified by the distribution of the minimum variate with the identified minimum provided $1-\rho_{ij}\sigma_i\sigma_j > 0$. Basu and Ghosh [3] proved the same for the tri-variate normal case. Elnaggar and Mukherjea [7] proved the tri-variate normal case without the $1-\rho_{ij}\sigma_i\sigma_j > 0$ restriction. They also proved the the tri-variate normal case without the identified minimum but with the restriction that all correlation have the same value and the means are all zero.

In this thesis we prove the tri-variate normal case where the correlations are all negative and the means are not necessarily zero.

Chapter 2

Lemmas and Corollaries

Domination: If $f(t)$ and $g(t)$ are two functions, we say that $g(t)$ *dominates* $f(t)$ as $t \to -\infty$ if and only if $\lim_{t\to-\infty} \frac{f(t)}{g(t)} = 0$. We will use the Landau symbol $o(g(t))$ and use the expression "$f(t)$ is $o(g(t))$" to mean "$g(t)$ *dominates* $f(t)$ ". We will refer to $f(t)$ and $g(t)$ as being of the *same dominating order* if $\lim_{t\to-\infty} \frac{f(t)}{g(t)} = \gamma$ where γ is some non-zero constant. We will refer to $f(t)$ and $g(t)$ as being of the *same order* and as being *equivalent* if $\lim_{t\to-\infty} \frac{f(t)}{g(t)} = 1$. We will write "$f(t) \sim g(t)$ as $t \to -\infty$" to express that $f(t)$ is equivalent to $g(t)$. Suppose $f(t) = A(t) + B(t) + C(t)$ and further suppose $A(t)$ is of the same dominating order as $f(t)$, then we say $A(t)$ is a *dominating term* of $f(t)$.

Lemma 2.1 Let A, B, a, b, c, d, e, and f be constants. Let m and n be integers. Define $f(t) = \frac{A}{|t|^m} \exp\left[\frac{-1}{2}\left(at^2 + bt + c\right)\right]$ $\quad g(t) = \frac{B}{|t|^n} \exp\left[\frac{-1}{2}\left(dt^2 + et + f\right)\right]$. The order of domination is given by

$a < d \Rightarrow f(t)$ dominates $g(t)$ as $t \to \pm\infty$

$d < a \Rightarrow g(t)$ dominates $f(t)$ as $t \to \pm\infty$

$a = d$ and $b < e \Rightarrow f(t)$ dominates $g(t)$ as $t \to +\infty$

$g(t)$ dominates $f(t)$ as $t \to -\infty$

$a = d$ and $e < b \Rightarrow f(t)$ dominates $g(t)$ as $t \to -\infty$

$g(t)$ dominates $f(t)$ as $t \to +\infty$

$a = d$ and $b = e$ and $\mathrm{m} < \mathrm{n} \Rightarrow f(t)$ dominates $g(t)$ as $t \to \pm\infty$

$a = d$ and $b = e$ and $\mathrm{n} < \mathrm{m} \Rightarrow g(t)$ dominates $f(t)$ as $t \to \pm\infty$

Proof

$$\frac{f(t)}{g(t)} = \frac{A}{B}|t|^{n-m} \ \exp\left[\frac{-1}{2}\left((a-d)t^2 + (b-e)t + (c-f)\right)\right]$$

By L'hộpital's rule $\lim\limits_{t\to\pm\infty} \frac{f(t)}{g(t)} = \infty$ if $(a-d) < 0$

$$\lim_{t\to\pm\infty} \frac{f(t)}{g(t)} = 0 \ \text{ if } (a-d) > 0$$

$$\lim_{t\to-\infty} \frac{f(t)}{g(t)} = \infty \ \text{ if } (a-d) = 0 \text{ and } (b-e) > 0$$

$$\lim_{t\to-\infty} \frac{f(t)}{g(t)} = 0 \ \text{ if } (a-d) = 0 \text{ and } (b-e) < 0$$

$$\lim_{t\to+\infty} \frac{f(t)}{g(t)} = \infty \ \text{ if } (a-d) = 0 \text{ and } (b-e) < 0$$

$$\lim_{t\to+\infty} \frac{f(t)}{g(t)} = 0 \ \text{ if } (a-d) = 0 \text{ and } (b-e) > 0$$

if $(a-d) = 0$ and $(b-e) = 0$ then $\frac{f(t)}{g(t)} = \frac{A}{B}|t|^{n-m}$ so that

$\lim\limits_{t\to\pm\infty} \frac{f(t)}{g(t)} = \infty$ if $(n-m) > 0$, $\lim\limits_{t\to\pm\infty} \frac{f(t)}{g(t)} = 0$ if $(n-m) < 0$

□

Lemma 2.2 Let $A(t), B(t), C(t)$ be functions in the single variable t and let $f(t) = A(t) + B(t)$. Further assume for large values of $|t|, C(t) \neq 0$ Then as $t \to -\infty$, the following are equivalent:

1. $f(t) \sim A(t)$
2. $A(t)$ dominates $B(t)$
3. $C(t)f(t) \sim C(t)A(t)$

Proof

$$f(t) \sim A(t) \Rightarrow \lim_{t\to-\infty} \frac{f(t)}{A(t)} = 1 \Rightarrow \lim_{t\to-\infty} \frac{A(t)+B(t)}{A(t)} = 1 + \lim_{t\to-\infty} \frac{B(t)}{A(t)} = 1$$

$\Rightarrow \lim\limits_{t\to-\infty} \dfrac{B(t)}{A(t)} = 0$,that is $A(t)$ dominates $B(t) \Rightarrow \lim\limits_{t\to-\infty} \dfrac{C(t)f(t)}{C(t)A(t)} = \lim\limits_{t\to-\infty} \dfrac{C(t)A(t)+C(t)B(t)}{C(t)A(t)}$

$= \lim\limits_{t\to-\infty} \dfrac{C(t)A(t)}{C(t)A(t)} + \lim\limits_{t\to-\infty} \dfrac{C(t)B(t)}{C(t)A(t)} = 1 + \lim\limits_{t\to-\infty} \dfrac{B(t)}{A(t)} = 1 + 0 = 1$; that is $C(t)f(t) \sim C(t)A(t)$

Finally, $C(t)f(t) \sim C(t)A(t) \Rightarrow 1 = \lim\limits_{t\to-\infty} \dfrac{C(t)f(t)}{C(t)A(t)} = \lim\limits_{t\to-\infty} \dfrac{f(t)}{A(t)} \Rightarrow f(t) \sim A(t)$

□

Corollary 2.2.1 Let $g(t), g_1(t), \cdots, g_n(t)$ be functions in the single variable t and let $g(t) = g_1(t) + \cdots + g_n(t)$ with $g_{i_{(1)}}(t), \cdots, g_{i_{(m)}}(t)$ dominating and of the same dominating order. Then as $t \to -\infty$, $g(t) \sim g_{i_{(1)}}(t) + \cdots + g_{i_{(m)}}(t)$.

Lemma 2.3 Suppose r, σ_i, σ_j are positive real numbers and $\rho_{ij} < 0$. Then

$$r = A_{ij} \equiv \frac{\sigma_i^2 + \sigma_j^2 - 2\rho_{ij}\sigma_i\sigma_j}{\sigma_i^2\sigma_j^2\left(1-\rho_{ij}^2\right)} \quad \text{if and only if} \quad \rho_{ij} = \frac{-1}{\sigma_i\sigma_j}\left[\sqrt{\left(\sigma_i^2 - \frac{1}{r}\right)\left(\sigma_j^2 - \frac{1}{r}\right)} - \frac{1}{r}\right]$$

Proof

(if)

$$\rho_{ij} = \frac{-1}{\sigma_i\sigma_j}\left[\sqrt{\left(\sigma_i^2-\frac{1}{r}\right)\left(\sigma_j^2-\frac{1}{r}\right)}-\frac{1}{r}\right] \Rightarrow \left(\sigma_i\sigma_j\rho_{ij}-\frac{1}{r}\right)^2 = \left(\sigma_i^2-\frac{1}{r}\right)\left(\sigma_j^2-\frac{1}{r}\right)$$

$$\Rightarrow \sigma_i^2\sigma_j^2\rho_{ij}^2 - 2\frac{\sigma_i\sigma_j\rho_{ij}}{r}+\frac{1}{r^2} = \sigma_i^2\sigma_j^2 - \frac{\sigma_i^2}{r} - \frac{\sigma_j^2}{r} + \frac{1}{r^2}$$

$$\Rightarrow r\sigma_i^2\sigma_j^2\rho_{ij}^2 - 2\sigma_i\sigma_j\rho_{ij} = r\sigma_i^2\sigma_j^2 - \sigma_i^2 - \sigma_j^2$$

$$\Rightarrow \sigma_i^2 + \sigma_j^2 - 2\sigma_i\sigma_j\rho_{ij} = r\left(\sigma_i^2\sigma_j^2 - \sigma_i^2\sigma_j^2\rho_{ij}^2\right)$$

$$\Rightarrow r = \frac{\sigma_i^2+\sigma_j^2-2\rho_{ij}\sigma_i\sigma_j}{\sigma_i^2\sigma_j^2\left(1-\rho_{ij}^2\right)}$$

(only if)

$$r = \frac{\sigma_i^2+\sigma_j^2-2\rho_{ij}\sigma_i\sigma_j}{\sigma_i^2\sigma_j^2\left(1-\rho_{ij}^2\right)} \Rightarrow r > \frac{\sigma_i^2+\sigma_j^2-2\rho_{ij}\sigma_i\sigma_j}{\sigma_i^2\sigma_j^2} \Rightarrow r > \frac{\sigma_i^2+\sigma_j^2}{\sigma_i^2\sigma_j^2} = \frac{1}{\sigma_i^2}+\frac{1}{\sigma_j^2} \text{ and}$$

$\sigma_i^2\sigma_j^2 > \frac{1}{r}\left(\sigma_i^2+\sigma_j^2\right) \Rightarrow \sqrt{\left(\sigma_i^2-\frac{1}{r}\right)\left(\sigma_j^2-\frac{1}{r}\right)}$ is real and positive and greater than $\frac{1}{r}$.

Therefore

$r = \frac{\sigma_i^2+\sigma_j^2-2\rho_{ij}\sigma_i\sigma_j}{\sigma_i^2\sigma_j^2\left(1-\rho_{ij}^2\right)} \Rightarrow \rho_{ij} = \frac{\pm 1}{\sigma_i\sigma_j}\sqrt{\left(\sigma_i^2-\frac{1}{r}\right)\left(\sigma_j^2-\frac{1}{r}\right)}+\frac{1}{\sigma_i\sigma_j r}$ with ρ_{ij} real, and

since $\rho_{ij} < 0$, $\rho_{ij} = \frac{-1}{\sigma_i\sigma_j}\left[\sqrt{\left(\sigma_i^2-\frac{1}{r}\right)\left(\sigma_j^2-\frac{1}{r}\right)}-\frac{1}{r}\right]$

□

Lemma 2.4 Define $\alpha_{ij} = \frac{\sigma_j - \sigma_i\rho_{ij}}{\sigma_i\sigma_j\sqrt{1-\rho_{ij}^2}}$, $\beta_{ij} = \frac{\sigma_i\mu_j\rho_{ij} - \sigma_j\mu_i}{\sigma_i\sigma_j\sqrt{1-\rho_{ij}^2}}$

For $1 \le i, j \le 3$ and $i \ne j$

1. $\dfrac{1}{\sigma_j^2}+\alpha_{ij}^2=\dfrac{1}{\sigma_i^2}+\alpha_{ji}^2\equiv A_{ij}$

2. $\alpha_{ij}\beta_{ij}-\dfrac{\mu_j}{\sigma_j^2}=\alpha_{ji}\beta_{ji}-\dfrac{\mu_i}{\sigma_i^2}\equiv B_{ij}$

3. $\dfrac{\mu_j^2}{\sigma_j^2}+\beta_{ij}^2=\dfrac{\mu_i^2}{\sigma_i^2}+\beta_{ji}^2\equiv C_{ij}$

We define A_{ij} , B_{ij} and C_{ij} by the above identities

Proof

1. $$\frac{1}{\sigma_j^2}+\alpha_{ij}^2=\frac{1}{\sigma_j^2}+\left(\frac{\sigma_j-\sigma_i\rho_{ij}}{\sigma_i\sigma_j\sqrt{1-\rho_{ij}^2}}\right)^2=\frac{\sigma_i^2(1-\rho_{ij}^2)+\sigma_j^2-2\sigma_i\sigma_j\rho_{ij}+\sigma_i^2\rho_{ij}^2}{\sigma_i^2\sigma_j^2(1-\rho_{ij}^2)}$$

$$=\frac{\sigma_i^2+\sigma_j^2-2\sigma_i\sigma_j\rho_{ij}}{\sigma_i^2\sigma_j^2(1-\rho_{ij}^2)}=\frac{\sigma_j^2(1-\rho_{ij}^2)+\sigma_i^2-2\sigma_i\sigma_j\rho_{ij}+\sigma_j^2\rho_{ij}^2}{\sigma_i^2\sigma_j^2(1-\rho_{ij}^2)}$$

$$\frac{1}{\sigma_i^2}+\left(\frac{\sigma_i-\sigma_j\rho_{ij}}{\sigma_i\sigma_j\sqrt{1-\rho_{ij}^2}}\right)^2=\frac{1}{\sigma_i^2}+\alpha_{ji}^2$$

2. $$\alpha_{ij}\beta_{ij}-\frac{\mu_j}{\sigma_j^2}=\left(\frac{\sigma_j-\sigma_i\rho_{ij}}{\sigma_i\sigma_j\sqrt{1-\rho_{ij}^2}}\right)\left(\frac{\sigma_i\mu_j\rho_{ij}-\sigma_j\mu_i}{\sigma_i\sigma_j\sqrt{1-\rho_{ij}^2}}\right)-\frac{\mu_j}{\sigma_j^2}=$$

$$\left(\frac{\sigma_j\sigma_i\mu_j\rho_{ij}-\sigma_j\sigma_j\mu_i-\sigma_i\rho_{ij}\sigma_i\mu_j\rho_{ij}+\sigma_i\rho_{ij}\sigma_j\mu_i}{\sigma_i^2\sigma_j^2(1-\rho_{ij}^2)}\right)-\frac{\mu_j}{\sigma_j^2}=$$

$$\frac{\sigma_j\sigma_i\mu_j\rho_{ij}-\sigma_j^2\mu_i-\sigma_i^2\mu_j\rho_{ij}^2+\sigma_i\sigma_j\mu_i\rho_{ij}-\mu_j\sigma_i^2(1-\rho_{ij}^2)}{\sigma_i^2\sigma_j^2(1-\rho_{ij}^2)}=$$

$$\frac{\sigma_j\sigma_i\mu_j\rho_{ij}-\sigma_j^2\mu_i+\sigma_i\sigma_j\mu_i\rho_{ij}-\mu_j\sigma_i^2}{\sigma_i^2\sigma_j^2(1-\rho_{ij}^2)}=$$

$$\frac{\sigma_i\sigma_j\mu_i\rho_{ij}-\sigma_i^2\mu_j-\sigma_j^2\mu_i\rho_{ij}^2+\sigma_j\sigma_i\mu_j\rho_{ij}-\mu_i\sigma_j^2(1-\rho_{ij}^2)}{\sigma_i^2\sigma_j^2(1-\rho_{ij}^2)}=$$

$$\left(\frac{\sigma_i\sigma_j\mu_i\rho_{ij}-\sigma_i\sigma_i\mu_j-\sigma_j\rho_{ij}\sigma_j\mu_i\rho_{ij}+\sigma_j\rho_{ij}\sigma_i\mu_j}{\sigma_i^2\sigma_j^2(1-\rho_{ij}^2)}\right)-\frac{\mu_i}{\sigma_i^2}=$$

$$\left(\frac{\sigma_i-\sigma_j\rho_{ij}}{\sigma_i\sigma_j\sqrt{1-\rho_{ij}^2}}\right)\left(\frac{\sigma_j\mu_i\rho_{ij}-\sigma_i\mu_j}{\sigma_i\sigma_j\sqrt{1-\rho_{ij}^2}}\right)-\frac{\mu_i}{\sigma_i^2}=\alpha_{ji}\beta_{ji}-\frac{\mu_i}{\sigma_i^2}$$

3. $$\frac{\mu_j^2}{\sigma_j^2}+\beta_{ij}^2=\frac{\mu_j^2}{\sigma_j^2}+\left(\frac{\sigma_i\mu_j\rho_{ij}-\sigma_j\mu_i}{\sigma_i\sigma_j\sqrt{1-\rho_{ij}^2}}\right)^2=$$

$$\frac{\mu_j^2\sigma_i^2\left(1-\rho_{ij}^2\right)+\sigma_i^2\mu_j^2\rho_{ij}^2-2\sigma_i\sigma_j\mu_i\mu_j\rho_{ij}+\sigma_j^2\mu_i^2}{\sigma_i^2\sigma_j^2\left(1-\rho_{ij}^2\right)}=$$

$$\frac{\mu_i^2\sigma_j^2\left(1-\rho_{ij}^2\right)+\sigma_i^2\mu_j^2\rho_{ij}^2-2\sigma_i\sigma_j\mu_i\mu_j\rho_{ij}+\sigma_i^2\mu_j^2}{\sigma_i^2\sigma_j^2\left(1-\rho_{ij}^2\right)}=$$

$$\frac{\mu_i^2}{\sigma_i^2}+\left(\frac{\sigma_j\mu_i\rho_{ij}-\sigma_i\mu_j}{\sigma_i\sigma_j\sqrt{1-\rho_{ij}^2}}\right)^2=\frac{\mu_i^2}{\sigma_i^2}+\beta_{ji}^2$$

□

Lemma 2.5 Suppose Σ is a n x n non-singular covariance matrix. Then Σ is positive definite and if $\Sigma_{ij}<0\ \forall i,j\,(i\neq j)$, then $\left(\Sigma^{-1}\right)_{ij}>0\ \forall i,j$

Proof

By induction for n = 2

$$\Sigma=\begin{pmatrix}\sigma_1^2 & \rho\sigma_1\sigma_2\\ \rho\sigma_1\sigma_2 & \sigma_2^2\end{pmatrix}\text{ with }\rho<0$$

Then $$\Sigma^{-1}=\frac{1}{\left(1-\rho^2\right)\sigma_1\sigma_2}\begin{pmatrix}\sigma_2^2 & -\rho\sigma_1\sigma_2\\ -\rho\sigma_1\sigma_2 & \sigma_1^2\end{pmatrix}$$

So that $\left(\Sigma^{-1}\right)_{ij}>0\ \forall i,j$

Now assume the assertion is true for $k=n$ for some integer $k\geq 2$

Let Σ be some $(k+1)\times(k+1)$ covariance matrix with $\Sigma_{ij}<0\ \forall i,j\,(i\neq j)$

Partition Σ as

$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \sigma^2_{k+1} \end{pmatrix}$ where Σ_{11} is $k \times k$, Σ_{12} is $k \times 1$, and Σ_{21} is $1 \times k$.

Now denote the partition of the inverse of the covariance matrix by

$\Sigma^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with corresponding sub-matrices being of the same dimensions.

Notice that since Σ is symmetric, positive definite, and non-singular, $\Sigma^{-1}, \Sigma_{11}, A, A^{-1}$ are also symmetric, positive definite, and non-singular.

Now since $\Sigma\Sigma^{-1} = I_{k+1}$, we have $\begin{pmatrix} \Sigma_{11}A + \Sigma_{12}C & \Sigma_{11}B + \Sigma_{12}D \\ \Sigma_{21}A + \sigma^2_{k+1}C & \Sigma_{21}B + \sigma^2_{k+1}D \end{pmatrix} = \begin{pmatrix} I_k & \underset{k\times 1}{0} \\ \underset{1\times k}{0} & 1 \end{pmatrix}$

which yields the following equations:

$$\Sigma_{11}A + \Sigma_{12}C = I_k \quad (2.1)$$
$$\Sigma_{11}B + \Sigma_{12}D = \underset{k\times 1}{0} \quad (2.2)$$
$$\Sigma_{21}A + \sigma^2_{k+1}C = \underset{1\times k}{0} \quad (2.3)$$
$$\Sigma_{21}B + \sigma^2_{k+1}D = 1 \quad (2.4)$$

From (2.3) we obtain

$$C = \frac{-1}{\sigma^2_{k+1}}\Sigma_{21}A \quad (2.5)$$

Now rewrite (2.1) as

$$\Sigma_{11}A - \frac{1}{\sigma^2_{k+1}}\Sigma_{12}\Sigma_{21}A = I_k \quad (2.6)$$

from which we obtain

$$A = \left(\Sigma_{11} - \frac{1}{\sigma^2_{k+1}}\Sigma_{12}\Sigma_{21}\right)^{-1} \quad (2.7)$$
$$A^{-1} = \left(\Sigma_{11} - \frac{1}{\sigma^2_{k+1}}\Sigma_{12}\Sigma_{21}\right) \quad (2.8)$$

Now A, A^{-1} are $k \times k$ so that

when $i \neq j (1 \leq i, j \leq k)$

$$\left(A^{-1}\right)_{ij} = \left(\Sigma_{11}\right)_{ij} - \frac{1}{\sigma_{k+1}^2}\left(\Sigma_{12}\Sigma_{21}\right)_{ij}$$
$$= \Sigma_{ij} - \frac{1}{\sigma_{k+1}^2}\left(\Sigma_{i,k+1}\right)\left(\Sigma_{k+1,j}\right) < 0 \quad (2.9)$$

since $\left(\Sigma_{11}\right)_{ij} = \Sigma_{ij} < 0$ when $i \neq j$, and both i and j are less than $k+1$ resulting in both $\left(\Sigma_{i,k+1}\right)$ and $\left(\Sigma_{k+1,j}\right)$ being negative.

when $i = j (1 \le i = j \le k)$

$$\left(A^{-1}\right)_{ij} = \left(A^{-1}\right)_{ii}$$
$$= \Sigma_{ii} - \frac{1}{\sigma_{k+1}^2}\left(\Sigma_{i,k+1}\right)\left(\Sigma_{k+1,i}\right)$$
$$= \Sigma_{ii} - \frac{1}{\sigma_{k+1}^2}\left(\Sigma_{i,k+1}\right)^2 > 0 \quad (2.10)$$

The inequality in (2.10) holds since Σ being full-rank $\Rightarrow \rho_{ij}^2 < 1$

$$\Rightarrow \frac{\sigma_{i,k+1}^2}{\sigma_i^2 \sigma_{k+1}^2} < 1 \Rightarrow \frac{\left(\Sigma_{i,k+1}\right)^2}{\sigma_{k+1}^2} < \left(\Sigma_{ii}\right)$$

Hence A^{-1} is a $k \times k$ full-rank covariance matrix with $\left(A^{-1}\right)_{ij} < 0$ when $i \neq j$.
By the induction hypothesis $A_{ij} > 0$ for $1 \le i, j \le k$

Each component of Σ_{21} is negative, therefore $\left(\Sigma_{21}A\right)_{1j} < 0, 1 \le j \le k$, so that each component of $C = \frac{-1}{\sigma_{k+1}^2}\Sigma_{21}A$ is positive.

Since Σ^{-1} is symmetric, $B = C^T$ so that each component of B is positive.

Finally, from (2.4) we have that $D = \frac{1}{\sigma_{k+1}^2} - \Sigma_{21}B > 0$ since $\Sigma_{21}B < 0$

Lemma 2.6 Let $(Y_1, Y_2, \ldots, Y_n) \sim N(\vec{0}, \Sigma)$ where $\Sigma_{ij} = \begin{Bmatrix} r_{ij} < 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{Bmatrix}$. Also, let $\delta = (\delta_1, \delta_2, \ldots, \delta_n)$ be a vector of constants and the n-vector $\mathbf{1} = (1, 1, \ldots, 1)$. Finally, define $\begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix} \equiv \Sigma^{-1}\mathbf{1}^T$ Then as $t \to \infty$,

$$P(Y_1 > t + \delta_1, Y_2 > t + \delta_2, \ldots, Y_n > t + \delta_n) \sim \frac{\exp\left\{-\frac{1}{2}\left[t^2\left(\mathbf{1}\Sigma^{-1}\mathbf{1}^T\right) + 2t\delta\Sigma^{-1}\mathbf{1}^T + \delta\Sigma^{-1}\delta^T\right]\right\}}{\left(\sqrt{2\pi}\right)^n \sqrt{|\Sigma|}\, t^n \prod_{i=1}^{n} \beta_i} \quad (2.11)$$

Proof

$P(Y_1 > t + \delta_1, Y_2 > t + \delta_2, \ldots, Y_n > t + \delta_n) \sim$

$$= \int_{t+\delta_n}^{\infty} \cdots \int_{t+\delta_1}^{\infty} \frac{1}{\left(\sqrt{2\pi}\right)^n \sqrt{|\Sigma|}} \exp\left\{-\frac{1}{2}\left(Y\Sigma^{-1}Y^T\right)\right\} dY$$

Put $X = Y - \delta$

$$= \int_{t}^{\infty} \cdots \int_{t}^{\infty} \frac{1}{\left(\sqrt{2\pi}\right)^n \sqrt{|\Sigma|}} \exp\left\{-\frac{1}{2}\left[(X+\delta)\Sigma^{-1}(X+\delta)^T\right]\right\} dX$$

Since $(X+\delta)\Sigma^{-1}(X+\delta)^T = \left(X\Sigma^{-1}X^T\right) + 2\left(X\Sigma^{-1}\delta^T\right) + \left(\delta\Sigma^{-1}\delta^T\right)$

$$= \frac{\exp\left\{-\frac{1}{2}\left[\delta\Sigma^{-1}\delta^T\right]\right\}}{\left(\sqrt{2\pi}\right)^n \sqrt{|\Sigma|}} \int_{t}^{\infty} \cdots \int_{t}^{\infty} \exp\left\{-\frac{1}{2}\left[\left(X\Sigma^{-1}X^T\right) + 2\left(X\Sigma^{-1}\delta^T\right)\right]\right\} dX$$

Put $Z = X - \vec{t}$ where $\vec{t} = t\mathbf{1}$

$$= \frac{\exp\left\{-\frac{1}{2}\left[\delta\Sigma^{-1}\delta^T\right]\right\}}{\left(\sqrt{2\pi}\right)^n \sqrt{|\Sigma|}} \int_{0}^{\infty} \cdots \int_{0}^{\infty} \exp\left\{-\frac{1}{2}\left[\left(Z+\vec{t}\right)\Sigma^{-1}\left(Z+\vec{t}\right)^T + 2\left(Z+\vec{t}\right)\Sigma^{-1}\delta^T\right]\right\} dZ$$

$$\left(Z+\vec{t}\right)\Sigma^{-1}\left(Z+\vec{t}\right)^T = Z\Sigma^{-1}Z^T + 2t\left(Z\Sigma^{-1}\mathbf{1}^T\right) + t^2\left(\mathbf{1}\Sigma^{-1}\mathbf{1}^T\right)$$

$$2\left(Z+\vec{t}\right)\Sigma^{-1}\delta^T = 2\left(Z\Sigma^{-1}\delta^T\right) + 2t\left(\delta\Sigma^{-1}\mathbf{1}^T\right)$$

$$= \frac{\exp\left\{-\frac{1}{2}\left[t^2\left(\mathbf{1}\Sigma^{-1}\mathbf{1}^T\right)+2t\left(\delta\Sigma^{-1}\mathbf{1}^T\right)+\left(\delta\Sigma^{-1}\delta^T\right)\right]\right\}}{\left(\sqrt{2\pi}\right)^n\sqrt{|\Sigma|}}\int_0^\infty\cdots\int_0^\infty \exp\left\{-\frac{1}{2}\left[\left(Z\Sigma^{-1}Z^T\right)+2t\left(Z\Sigma^{-1}\mathbf{1}^T\right)+2\left(Z\Sigma^{-1}\delta^T\right)\right]\right\}dZ$$

Let $W = tZ$ with $t > 0$, then

$$\int_0^\infty\cdots\int_0^\infty \exp\left\{-\frac{1}{2}\left[\left(Z\Sigma^{-1}Z^T\right)+2t\left(Z\Sigma^{-1}\mathbf{1}^T\right)+2\left(Z\Sigma^{-1}\delta^T\right)\right]\right\}dZ$$

$$= \frac{1}{t^n}\int_0^\infty\cdots\int_0^\infty \exp\left\{-\left[\frac{1}{2t^2}\left(W\Sigma^{-1}W^T\right)+\frac{1}{t}\left(W\Sigma^{-1}\delta^T\right)+\left(W\Sigma^{-1}\mathbf{1}^T\right)\right]\right\}dW$$

As $t\to\infty$, $\exp\left\{-\frac{1}{2t^2}\left(W\Sigma^{-1}W^T\right)\right\}\to 1$

$$\exp\left\{-\frac{1}{t}\left(W\Sigma^{-1}\delta^T\right)\right\}\to 1$$

$$\exp\left\{-\left(W\Sigma^{-1}\mathbf{1}^T\right)\right\} = \exp\left\{-\sum_{i=1}^n \beta_i W_i\right\}$$

$$\Rightarrow \int_0^\infty\cdots\int_0^\infty \exp\left\{-\left[\frac{1}{2t^2}\left(W\Sigma^{-1}W^T\right)+\frac{1}{t}\left(W\Sigma^{-1}\delta^T\right)+\left(W\Sigma^{-1}\mathbf{1}^T\right)\right]\right\}dW \to \prod_{i=1}^n\left(\frac{1}{\beta_i}\right)$$

Therefore, as $t\to\infty$,

$$P\left(Y_1 > t+\delta_1, Y_2 > t+\delta_2,\ldots,Y_n > t+\delta_n\right) \sim \frac{\exp\left\{-\frac{1}{2}\left[t^2\left(\mathbf{1}\Sigma^{-1}\mathbf{1}^T\right)+2t\delta\Sigma^{-1}\mathbf{1}^T+\delta\Sigma^{-1}\delta^T\right]\right\}}{\left(\sqrt{2\pi}\right)^n\sqrt{|\Sigma|}\,t^n\prod_{i=1}^n\beta_i}$$

□

Corollary 2.6.1 Let $\left(Y_1, Y_2\right) \sim N(0,0;1,1;\rho)$ where $-1 < \rho \le 0$. Also, let $\alpha > 0$, $\beta > 0$, while δ_1 and δ_2 are constants. Then as $t\to-\infty$,

$$P\left(Y_1 \le \alpha t+\delta_1, Y_2 \le \beta t+\delta_2\right) \sim \frac{D}{2\pi t^2}\exp\left[\frac{-1}{2}\left(at^2+2bt+c\right)\right] \quad (2.12)$$

Where $D = \frac{\left(1-\rho^2\right)^{\frac{3}{2}}}{(\alpha-\rho\beta)(\beta-\rho\alpha)}$, $a = \frac{\alpha^2 - 2\rho\alpha\beta + \beta^2}{1-\rho^2}$, $b = \frac{\alpha\delta_1 - \rho(\beta\delta_1 + \alpha\delta_2) + \beta\delta_2}{1-\rho^2}$,

$c = \frac{\delta_1^2 - 2\delta_1\delta_2\rho + \delta_2^2}{1-\rho^2}$

Therefore $P(Y_1 \le \alpha t + \delta_1, Y_2 \le \beta t + \delta_2)$ is both $o\left(\frac{\phi(\alpha t + \delta_1)}{|\alpha t + \delta_1|}\right)$ and $o\left(\frac{\phi(\beta t + \delta_2)}{|\beta t + \delta_2|}\right)$

Proof

Since $P(-Y_1 \le x, -Y_2 \le y) = P(Y_1 > -x, Y_2 > -y)$

$$= \int_{-y}^{\infty}\int_{-x}^{\infty} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left(r^2 - 2rs\rho + s^2\right)\right\} dr ds$$

Let $u = -r$ and $v = -s$

$$= \int_{-\infty}^{y}\int_{-\infty}^{x} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left(u^2 - 2uv\rho + v^2\right)\right\} du dv$$

$= P(Y_1 \le x, Y_2 \le y)$

$\Rightarrow (-Y_1, -Y_2) \overset{d}{=} (Y_1, Y_2)$. It follows that

$P(Y_1 \le \alpha t + \delta_1, Y_2 \le \beta t + \delta_2) = P(-Y_1 > \alpha(-t) - \delta_1, -Y_2 > \beta(-t) - \delta_2)$

$= P(Y_1 > \alpha(-t) - \delta_1, Y_2 > \beta(-t) - \delta_2) = P\left(\frac{Y_1}{\alpha} > (-t) - \frac{\delta_1}{\alpha}, \frac{Y_2}{\beta} > (-t) - \frac{\delta_2}{\beta}\right)$

. Then as $t \to -\infty$, $(-t) \to \infty$, and by Lemma 2.6

$$P\left(\frac{Y_1}{\alpha} > (-t) - \frac{\delta_1}{\alpha}, \frac{Y_2}{\beta} > (-t) - \frac{\delta_2}{\beta}\right) \sim \frac{\exp\left\{-\frac{1}{2}\left[t^2\left(\mathbf{1}\Sigma^{-1}\mathbf{1}^T\right) + 2(-t)\delta\Sigma^{-1}\mathbf{1}^T + \delta\Sigma^{-1}\delta^T\right]\right\}}{2\pi\sqrt{|\Sigma|}\, t^2 \prod_{i=1}^{2}\beta_i}$$

Where Σ is the covariance matrix for $\left(\frac{Y_1}{\alpha}, \frac{Y_2}{\beta}\right)$. Therefore

$$\Sigma = \begin{pmatrix} \frac{1}{\alpha^2} & \frac{\rho}{\alpha\beta} \\ \frac{\rho}{\alpha\beta} & \frac{1}{\beta^2} \end{pmatrix}, \qquad |\Sigma| = \frac{1-\rho^2}{\alpha^2\beta^2}, \quad \Sigma^{-1} = \begin{pmatrix} \frac{\alpha^2}{1-\rho^2} & \frac{-\rho\alpha\beta}{1-\rho^2} \\ \frac{-\rho\alpha\beta}{1-\rho^2} & \frac{\beta^2}{1-\rho^2} \end{pmatrix},$$

$$\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \equiv \Sigma^{-1}\mathbf{1}^T = \begin{pmatrix} \frac{\alpha^2 - \rho\alpha\beta}{1-\rho^2} \\ \frac{\beta^2 - \rho\alpha\beta}{1-\rho^2} \end{pmatrix}, \ \mathbf{1}\Sigma^{-1}\mathbf{1}^T = \frac{\alpha^2 - 2\rho\alpha\beta + \beta^2}{1-\rho^2}, \ \delta = \left(\frac{-\delta_1}{\alpha}, \frac{-\delta_2}{\beta} \right)$$

$\delta\Sigma^{-1}\delta^T = \frac{\delta_1^2 - 2\delta_1\delta_2\rho + \delta_2^2}{1-\rho^2}$ and $\delta\Sigma^{-1}\mathbf{1}^T = \frac{-\alpha\delta_1 + \rho(\beta\delta_1 + \alpha\delta_2) - \beta\delta_2}{1-\rho^2}$ The desired results follow: as $t \to -\infty$,

$$P(Y_1 \le \alpha t + \delta_1, Y_2 \le \beta t + \delta_2) \sim$$

$$\frac{(1-\rho^2)^{\frac{3}{2}} \exp\left\{ \frac{-1}{2} \left[\left(\frac{\alpha^2 - 2\rho\alpha\beta + \beta^2}{1-\rho^2} \right) t^2 + 2\left(\frac{\alpha\delta_1 - \rho(\beta\delta_1 + \alpha\delta_2) + \beta\delta_2}{1-\rho^2} \right) t + \left(\frac{\delta_1^2 - 2\delta_1\delta_2\rho + \delta_2^2}{1-\rho^2} \right) \right] \right\}}{2\pi t^2 (\alpha - \rho\beta)(\beta - \rho\alpha)}$$

Furthermore, since $\frac{\alpha^2 - 2\rho\alpha\beta + \beta^2}{1-\rho^2} = \frac{(\alpha-\rho\beta)^2}{1-\rho^2} + \beta^2 = \frac{(\beta-\rho\alpha)^2}{1-\rho^2} + \alpha^2 > \max\{\alpha^2, \beta^2\}$,

We have by Lemma 2.1 that $P(Y_1 \le \alpha t + \delta_1, Y_2 \le \beta t + \delta_2)$ is both $o\left(\frac{\phi(\alpha t + \delta_1)}{|\alpha t + \delta_1|} \right)$ and $o\left(\frac{\phi(\beta t + \delta_2)}{|\beta t + \delta_2|} \right)$ as $t \to -\infty$,

□

Corollary 2.6.2 Let $(Y_1, Y_2) \sim N(0,0;1,1;\rho)$ where $-1 < \rho \le 0$. Also, let $\alpha > 0$, $\beta > 0$, while δ_1 and δ_2 are constants. Then as $t \to -\infty$,

$$[1 - P(Y_1 > \alpha t + \delta_1, Y_2 > \beta t + \delta_2)] \sim \frac{C}{\sqrt{2\pi}\,|ut + \delta|} \exp\left[\frac{-1}{2}(ut + \delta)^2 \right] \quad (2.13)$$

where

$u = \min\{\alpha, \beta\}$

$C = \begin{Bmatrix} 1 \text{ if } \alpha \neq \beta \text{ or } \delta_1 \neq \delta_2 \\ 2 \text{ if } \alpha = \beta \text{ and } \delta_1 = \delta_2 \end{Bmatrix}$

$\delta = \begin{Bmatrix} \max\{\delta_1, \delta_2\} \text{ if } \alpha = \beta \\ \delta_1 \qquad \text{if } \alpha < \beta \\ \delta_2 \qquad \text{if } \alpha > \beta \end{Bmatrix}$

Proof

$$[1 - P(Y_1 > \alpha t + \delta_1, Y_2 > \beta t + \delta_2)] = P(Y_1 \leq \alpha t + \delta_1 \cup Y_2 \leq \beta t + \delta_2)$$

$$= P(Y_1 \leq \alpha t + \delta_1) + P(Y_2 \leq \beta t + \delta_2) - P(Y_1 \leq \alpha t + \delta_1, Y_2 \leq \beta t + \delta_2)$$

By Mill's Ratio, as $t \to -\infty$, $P(Y_1 \leq \alpha t + \delta_1) \sim \dfrac{\phi(\alpha t + \delta_1)}{|\alpha t + \delta_1|}$

$= \dfrac{1}{|\alpha t + \delta_1|\sqrt{2\pi}} \exp\left[\dfrac{-1}{2}\left(\alpha^2 t^2 + 2\alpha\delta_1 t + \delta_1^2\right)\right]$ and

$P(Y_2 \leq \beta t + \delta_2) \sim \dfrac{\phi(\beta t + \delta_2)}{|\beta t + \delta_2|} = \dfrac{1}{|\beta t + \delta_2|\sqrt{2\pi}} \exp\left[\dfrac{-1}{2}\left(\beta^2 t^2 + 2\beta\delta_2 t + \delta_2^2\right)\right]$ and by

Corollary 2.6.1, $P(Y_1 \leq \alpha t + \delta_1, Y_2 \leq \beta t + \delta_2)$ is both $o\left(\dfrac{\phi(\alpha t + \delta_1)}{|\alpha t + \delta_1|}\right)$ and

$o\left(\dfrac{\phi(\beta t + \delta_2)}{|\beta t + \delta_2|}\right)$.

It follows by Lemma 2.1 that

$P(Y_1 \leq \alpha t + \delta_1)$ dominates $P(Y_2 \leq \beta t + \delta_2)$ and $\alpha t + \delta_1 = ut + \delta$

when $\alpha < \beta$ or when $\alpha = \beta$ and $\delta_1 > \delta_2$ but

$P(Y_2 \leq \beta t + \delta_2)$ dominates $P(Y_1 \leq \alpha t + \delta_1)$ and $\beta t + \delta_2 = ut + \delta$

when $\alpha > \beta$ or when $\alpha = \beta$ and $\delta_1 < \delta_2$

Therefore by Lemma 2.2 , when $\alpha \neq \beta$ or $\delta_1 \neq \delta_2$

$$[1 - P(Y_1 \geq \alpha t + \delta_1, Y_2 \geq \beta t + \delta_2)] \sim \frac{1}{\sqrt{2\pi}\,|ut + \delta|} \exp\left[\frac{-1}{2}(ut + \delta)^2\right]$$

Clearly, when $\alpha = \beta$ and $\delta_1 = \delta_2$

$$[1 - P(Y_1 \geq \alpha t + \delta_1, Y_2 \geq \beta t + \delta_2)] \sim \frac{2}{\sqrt{2\pi}\,|ut + \delta|} \exp\left[\frac{-1}{2}(ut + \delta)^2\right]$$

□

Corollary 2.6.3 Let $(Y_1, Y_2) \sim N(0,0;1,1;\rho)$ where $-1 < \rho \leq 0$. Also, let α, β, δ_1, δ_2, δ and u be as in corollary 2.6.2. Then as $t \to -\infty$, (2.14)

$$[1 - P(Y_1 > \alpha t + \delta_1, Y_2 > \beta t + \delta_2) - P(Z \leq ut + \delta)] \sim \frac{1}{\sqrt{2\pi}\,|Mt + d|} \exp\left[\frac{-1}{2}(Mt + d)^2\right]$$

which is $o(P(Z \leq ut + \delta))$. Here $M = \begin{Bmatrix} \alpha \text{ if } u = \beta \\ \beta \text{ if } u = \alpha \end{Bmatrix}$ and $d = \begin{Bmatrix} \delta_1 \text{ if } \delta = \delta_2 \\ \delta_2 \text{ if } \delta = \delta_1 \end{Bmatrix}$

If it is additionally true that $\alpha = \beta$ and $\delta_1 = \delta_2$, then as $t \to -\infty$,

$$[1 - P(Y_1 > \alpha t + \delta_1, Y_2 > \beta t + \delta_2) - 2P(Z \leq ut + \delta)]$$

$$= P(Y_1 \leq \alpha t + \delta_1, Y_2 \leq \beta t + \delta_2) \sim \frac{D}{2\pi t^2} \exp\left[\frac{-1}{2}(at^2 + 2bt + c)\right]$$

Where $D = \dfrac{(1-\rho^2)^{\frac{3}{2}}}{(\alpha - \rho\beta)(\beta - \rho\alpha)}$, $a = \dfrac{\alpha^2 - 2\rho\alpha\beta + \beta^2}{1 - \rho^2}$, $b = \dfrac{\alpha\delta_1 - \rho(\beta\delta_1 + \alpha\delta_2) + \beta\delta_2}{1 - \rho^2}$,

$c = \dfrac{\delta_1^2 - 2\delta_1\delta_2\rho + \delta_2^2}{1 - \rho^2}$

Proof

As in the proof of corollary 2.6.2, we have

$$[1 - P(Y_1 > \alpha t + \delta_1, Y_2 > \beta t + \delta_2)] = P(Y_1 \leq \alpha t + \delta_1 \cup Y_2 \leq \beta t + \delta_2)$$

$$= P(Y_1 \leq \alpha t + \delta_1) + P(Y_2 \leq \beta t + \delta_2) - P(Y_1 \leq \alpha t + \delta_1, Y_2 \leq \beta t + \delta_2)$$

W.L.O.G. assume $\alpha < \beta$. Then

$$[1 - P(Y_1 > \alpha t + \delta_1, Y_2 > \beta t + \delta_2) - P(Z \leq ut + \delta)]$$

$$= [1 - P(Y_1 > \alpha t + \delta_1, Y_2 > \beta t + \delta_2) - P(Y_1 \leq \alpha t + \delta_1)]$$

$$= P(Y_2 \le \beta t + \delta_2) - P(Y_1 \le \alpha t + \delta_1, Y_2 \le \beta t + \delta_2)$$

And since by corollary 2.6.1 $P(Y_1 \le \alpha t + \delta_1, Y_2 \le \beta t + \delta_2)$ is $o\left(\frac{\phi(\beta t + \delta_2)}{|\beta t + \delta_2|}\right)$,

$$[1 - P(Y_1 > \alpha t + \delta_1, Y_2 > \beta t + \delta_2) - P(Y_1 \le \alpha t + \delta_1)]$$

$$\sim P(Y_1 \le \beta t + \delta_2) = P(Z \le Mt + d) \sim \frac{1}{\sqrt{2\pi}|Mt + d|} \exp\left[\frac{-1}{2}(Mt + d)^2\right]$$

Which is clearly $o(P(Z \le ut + \delta))$.

When $\alpha = \beta = u$ and $\delta_1 = \delta_2 = \delta$,

$$[1 - P(Y_1 > \alpha t + \delta_1, Y_2 > \beta t + \delta_2) - 2P(Y_1 \le \alpha t + \delta_1)]$$
$$= [1 - P(Y_1 > \alpha t + \delta_1, Y_2 > \beta t + \delta_2) - P(Y_1 \le \alpha t + \delta_1) - P(Y_1 \le \beta t + \delta_2)]$$

$$= P(Y_1 \le \alpha t + \delta_1, Y_2 \le \beta t + \delta_2) \sim \frac{D}{2\pi t^2} \exp\left[\frac{-1}{2}(at^2 + 2bt + c)\right]$$

This last line was proven in corollary 2.6.2

□

Lemma 2.7 Let $(X_1, X_2, \ldots, X_n)$ be a normally distributed n-vector with $X_i \sim N(\mu_i, \sigma_i^2)$ for $1 \le i \le n$ and with $\rho_{ij} < 0$ as the correlation coefficient for (X_i, X_j) for $1 \le i < j \le n$. We further assume $(\sigma_1 > \sigma_2 > \cdots > \sigma_n)$. Let $Y = \min(X_1, X_2, \ldots, X_n)$ Then the pdf for Y is given by

$$f_Y(t) = \sum_{j=1}^{n} \frac{1}{\sigma_j} \phi\left(\frac{t - \mu_j}{\sigma_j}\right) p\left[\bigcap_{\substack{i=1 \\ i \ne j}}^{n} (W_{ij} > \alpha_{ij} t + \beta_{ij})\right] \text{where}$$

$$\alpha_{ij} = \frac{\sigma_j - \sigma_i \rho_{ij}}{\sigma_i \sigma_j \sqrt{1 - \rho_{ij}^2}} \quad , \quad \beta_{ij} = \frac{\sigma_i \mu_j \rho_{ij} - \sigma_j \mu_i}{\sigma_i \sigma_j \sqrt{1 - \rho_{ij}^2}} \quad , \quad \phi(t) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} t^2\right)$$

For every triplet $i, j, k \in \{1, 2, \cdots, n\}$ where $i \ne j, k \ne j, i \ne k$,

$$\left(W_{ij}, W_{kj}\right) \sim N\left(0,0;1,1;\rho_{ik\bullet j}\right) \text{ with } \rho_{ik\bullet j} = \frac{\rho_{ik} - \rho_{ij}\rho_{jk}}{\sqrt{1-\rho_{ij}^2}\sqrt{1-\rho_{jk}^2}}$$

Proof

Let $F_Y(t)$ be the distribution function for $Y = \min\left(X_1, X_2, \ldots, X_n\right)$.

$$F_Y(t) = P\left(Y \le t\right)$$

$$= P\left(Y \le t, \bigcup_{j=1}^{n} Y = X_j\right)$$

$$= P\left[\bigcup_{j=1}^{n}\left(Y \le t, Y = X_j\right)\right] \text{and since when } i \ne k, P\left[\{Y = X_i\} \cap \{Y = X_k\}\right] = 0$$

$$= \sum_{j=1}^{n} P\left(Y \le t, Y = X_j\right)$$

$$= \sum_{j=1}^{n} P\left(X_j \le t, X_j \le X_i \;\; \forall i \ne j\right)$$

$$= \sum_{j=1}^{n} \int_{x_j=-\infty}^{t} \left[\left(\prod_{\substack{i=1 \\ i \ne j}}^{n} \int_{x_i=x_j}^{\infty}\right) \varphi_n\left(x_1, \cdots, x_n\right) \prod_{\substack{i=1 \\ i \ne j}}^{n} dx_i\right] dx_j \text{, where } \varphi_n \text{ is the pdf for } \left(X_1, X_2, \ldots, X_n\right)$$

$$= \sum_{j=1}^{n} \int_{x_j=-\infty}^{t} \varphi_1\left(x_j\right) \left[\left(\prod_{\substack{i=1 \\ i \ne j}}^{n} \int_{x_i=x_j}^{\infty}\right) \varphi_{n-1}\left(x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n \mid x_j\right) \prod_{\substack{i=1 \\ i \ne j}}^{n} dx_i\right] dx_j$$

Where $\varphi_1\left(x_j\right)$ is the univariate pdf of X_1 and $\varphi_{n-1}\left(x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n \mid x_j\right)$ is the conditional pdf of $\left(X_1, X_2, \ldots, X_{j-1}, X_{j+1}, \ldots, X_n\right)$ given $X_j = x_j$

$$= \sum_{j=1}^{n} \int_{x_j=-\infty}^{t} \frac{1}{\sigma_j} \phi\left(\frac{x_j - \mu_j}{\sigma_j}\right) \left\{ P\left[\left(\bigcap_{\substack{i=1 \\ i \ne j}}^{n} \left(X_i \ge x_j\right)\right) \mid X_j = x_j\right]\right\} dx_j$$

Where ϕ is the standard normal pdf.

We can now find the pdf of Y, $f_Y(t)$, by differentiating using the Fundamental Theorem of Calculus:

$$f_Y(t) = \frac{d}{dt}\left[F_Y(t)\right]$$

$$= \sum_{j=1}^{n} \frac{1}{\sigma_j}\phi\left(\frac{t-\mu_j}{\sigma_j}\right)\left\{P\left[\left(\bigcap_{\substack{i=1\\ i\neq j}}^{n}(X_i \geq t)\right) \mid X_j = t\right]\right\}$$

$$= \sum_{j=1}^{n} \frac{1}{\sigma_j}\phi\left(\frac{t-\mu_j}{\sigma_j}\right)\left\{P\left(\bigcap_{\substack{i=1\\ i\neq j}}^{n}\left(X_{ij}^* \geq t\right)\right)\right\}$$

Where $\left(X_{1j}^*, X_{2j}^*, \ldots, X_{j-1j}^*, X_{j+1j}^*, \ldots, X_{nj}^*\right)$ is (n-1)-variate normal with

$X_{ij}^* \sim N\left(\mu_i + \frac{\sigma_i \rho_{ij}\left(t-\mu_j\right)}{\sigma_j}, \sigma_i^2\left(1-\rho_{ij}^2\right)\right)$ and $corr\left(X_{ij}^*, X_{kj}^*\right) = \frac{\rho_{ik} - \rho_{ij}\rho_{jk}}{\sqrt{1-\rho_{ij}^2}\sqrt{1-\rho_{jk}^2}}$

This is a standard result from multivariate analysis (see Y. L. Tong [14])

Now define $W_{ij} = \frac{X_{ij}^* - \left(\mu_i + \frac{\sigma_i \rho_{ij}\left(t-\mu_j\right)}{\sigma_j}\right)}{\sigma_i^2\left(1-\rho_{ij}^2\right)}$ so that $W_{ij} \sim N(0,1)$ and

$corr\left(W_{ij}, W_{kj}\right) = \frac{\rho_{ik} - \rho_{ij}\rho_{jk}}{\sqrt{1-\rho_{ij}^2}\sqrt{1-\rho_{jk}^2}}$. We now have that

$f_Y(t) = \sum_{j=1}^{n} \frac{1}{\sigma_j}\phi\left(\frac{t-\mu_j}{\sigma_j}\right) p\left[\bigcap_{\substack{i=1\\ i\neq j}}^{n}\left(W_{ij} > \alpha_{ij} t + \beta_{ij}\right)\right]$ where

$$\alpha_{ij} = \frac{\sigma_j - \sigma_i \rho_{ij}}{\sigma_i \sigma_j \sqrt{1-\rho_{ij}^2}} \quad , \quad \beta_{ij} = \frac{\sigma_i \mu_j \rho_{ij} - \sigma_j \mu_i}{\sigma_i \sigma_j \sqrt{1-\rho_{ij}^2}}$$

Chapter 3

The Tri-variate Normal Case with Negative Correlations

Assume (X_1, X_2, X_3) is a normally distributed vector with $X_i \sim N(\mu_i, \sigma_i^2)$ for $1 \le i \le 3$ and with $\rho_{ij} < 0$ as the correlation coefficient for (X_i, X_j) for $1 \le i < j \le 3$. We further assume $\sigma_1 > \sigma_2 > \sigma_3$. Let $Y = \min\{X_1, X_2, X_3\}$ and $f(t)$ be the pdf of Y. In this chapter we will prove that the distribution of Y uniquely gives the distribution of (X_1, X_2, X_3). This will be done by showing that given $f(t)$, the parameters $\mu_1, \mu_2, \mu_3, \sigma_1, \sigma_2, \sigma_3, \rho_{12}, \rho_{13}, \rho_{23}$ are uniquely determined.

By Lemma 2.7 we have the pdf for Y is given by

$$f(t) = \sum_{j=1}^{3} \frac{1}{\sigma_j} \phi\left(\frac{t-\mu_j}{\sigma_j}\right) p\left[\bigcap_{\substack{i=1 \\ i \ne j}}^{3} \left(W_{ij} > \alpha_{ij} t + \beta_{ij}\right)\right] \quad (3.1)$$

where

$$\alpha_{ij} = \frac{\sigma_j - \sigma_i \rho_{ij}}{\sigma_i \sigma_j \sqrt{1-\rho_{ij}^2}} \quad , \quad \beta_{ij} = \frac{\sigma_i \mu_j \rho_{ij} - \sigma_j \mu_i}{\sigma_i \sigma_j \sqrt{1-\rho_{ij}^2}} \quad , \quad \phi(t) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} t^2\right)$$

$(W_{ij}, W_{kj}) \sim N(0,0;1,1;\rho_{ik\bullet j})$ with $\rho_{ik\bullet j} = \dfrac{\rho_{ik} - \rho_{ij}\rho_{jk}}{\sqrt{1-\rho_{ij}^2}\sqrt{1-\rho_{jk}^2}}$ and (i, j, k) some permutation of $(1,2,3)$.

Now. $\rho_{ij} < 0 \Rightarrow \alpha_{ij} > 0 \Rightarrow for\ \ j = 1,2,3,\ \lim_{t\to-\infty} p\left[\bigcap_{\substack{i=1\\ i\neq j}}^{3}\left(W_{ij} > \alpha_{ij}t + \beta_{ij}\right)\right] = 1$ Therefore

as $t \to -\infty$

$$\text{term } j \text{ of } \ f(t) \sim \frac{1}{\sigma_j}\phi\left(\frac{t-\mu_j}{\sigma_j}\right) = \frac{1}{\sigma_j\sqrt{2\pi}}\exp\left[\frac{-1}{2}\left(\frac{1}{\sigma_j^2}t^2 + 2t\left(\frac{-\mu_j}{\sigma_j^2}\right) + \frac{\mu_j^2}{\sigma_j^2}\right)\right] \quad (3.2)$$

Since $\frac{1}{\sigma_1^2} < \frac{1}{\sigma_j^2}$ for $j > 1$, term 1 dominates. It follows that

$f(t) \sim \frac{1}{\sigma_1\sqrt{2\pi}}\exp\left[\frac{-1}{2}\left(\frac{1}{\sigma_1^2}t^2 + 2t\left(\frac{-\mu_1}{\sigma_1^2}\right) + \frac{\mu_1^2}{\sigma_1^2}\right)\right]$ and

$$-2\lim_{t\to-\infty}\left(\frac{\ln|f(t)|}{t^2}\right) = \left(\frac{1}{\sigma_1^2}\right) \quad (3.3)$$

Thus σ_1 is identified.

Define $f_{(1)}(t) \ \equiv f(t)\exp\left[\frac{1}{2}\frac{t^2}{\sigma_1^2}\right]$ (3.4)

By lemma 2.2,

$$\text{term } j \text{ of } \ f_{(1)}(t) \sim \frac{1}{\sigma_j\sqrt{2\pi}}\exp\left[\frac{-1}{2}\left(\left(\frac{1}{\sigma_j^2} - \frac{1}{\sigma_1^2}\right)t^2 + 2t\left(\frac{-\mu_j}{\sigma_j^2}\right) + \frac{\mu_j^2}{\sigma_j^2}\right)\right] \quad (3.5)$$

Since $\frac{1}{\sigma_j^2} - \frac{1}{\sigma_1^2}$ is smallest when $j = 1$, term 1 dominates.

$$f_{(1)}(t) \sim \frac{1}{\sigma_1\sqrt{2\pi}}\ \exp\left[\frac{-1}{2}\left(2t\left(\frac{-\mu_1}{\sigma_1^2}\right) + \frac{\mu_1^2}{\sigma_1^2}\right)\right] \quad (3.6)$$

$$\lim_{t\to-\infty}\left(\frac{\ln|f_{(1)}(t)|}{t}\right) = \frac{\mu_1}{\sigma_1^2} \quad (3.7)$$

Thus μ_1 is identified

Define $f_1(t) \equiv f(t) - \frac{1}{\sigma_1}\phi\left(\frac{t-\mu_1}{\sigma_1}\right)$ (3.8)

$$= \frac{-1}{\sigma_1}\phi\left(\frac{t-\mu_1}{\sigma_1}\right)\left[1 - p(W_{21} > \alpha_{21}t + \beta_{21}, W_{31} > \alpha_{31}t + \beta_{31})\right]$$

$$+ \frac{1}{\sigma_2}\phi\left(\frac{t-\mu_2}{\sigma_2}\right)p(W_{12} > \alpha_{12}t + \beta_{12}, W_{32} > \alpha_{32}t + \beta_{32})$$

$$+ \frac{1}{\sigma_3}\phi\left(\frac{t-\mu_3}{\sigma_3}\right)p(W_{13} > \alpha_{13}t + \beta_{13}, W_{23} > \alpha_{23}t + \beta_{23})$$

By corollary 2.6.2, as $t \to -\infty$

1st term of $f_1(t) \sim \frac{-C_1}{\sigma_1|u_1t+\delta_1|2\pi} \exp\left\{-\frac{1}{2}\left[\left(\frac{t-\mu_1}{\sigma_1}\right)^2 + (u_1t+\delta_1)^2\right]\right\}$

$$= \frac{-C_1}{\sigma_1|u_1t+\delta_1|2\pi} \exp\left\{-\frac{1}{2}\left[\left(\frac{1}{\sigma_1^2} + u_1^2\right)t^2 + 2\left(u_1\delta_1 - \frac{\mu_1}{\sigma_1^2}\right)t + \frac{\mu_1^2}{\sigma_1^2} + \delta_1^2\right]\right\} \quad (3.9)$$

$u_1 = \alpha_{21}, \delta_1 = \beta_{21}, C_1 = 1$ when $\alpha_{21} < \alpha_{31}$

$u_1 = \alpha_{31}, \delta_1 = \beta_{31}, C_1 = 1$ when $\alpha_{31} < \alpha_{21}$

$u_1 = \alpha_{21} = \alpha_{31}, \delta_1 = \beta_{21}, C_1 = 1$ when $\alpha_{21} = \alpha_{31}, \beta_{21} > \beta_{31}$

$u_1 = \alpha_{21} = \alpha_{31}, \delta_1 = \beta_{31}, C_1 = 1$ when $\alpha_{21} = \alpha_{31}, \beta_{31} > \beta_{21}$

$u_1 = \alpha_{21} = \alpha_{31}, \delta_1 = \beta_{21} = \beta_{31}, C_1 = 2$ when $\alpha_{21} = \alpha_{31}, \beta_{21} = \beta_{31}$

2nd term of $f_1(t) \sim \frac{1}{\sigma_2\sqrt{2\pi}} \exp\left[\frac{-1}{2}\left(\left(\frac{1}{\sigma_2^2}\right)t^2 + 2\left(\frac{-\mu_2}{\sigma_2^2}\right)t + \frac{\mu_2^2}{\sigma_2^2}\right)\right]$

3rd term of $f_1(t) \sim \frac{1}{\sigma_3\sqrt{2\pi}} \exp\left[\frac{-1}{2}\left(\left(\frac{1}{\sigma_3^2}\right)t^2 + 2\left(\frac{-\mu_3}{\sigma_3^2}\right)t + \frac{\mu_3^2}{\sigma_3^2}\right)\right]$

Since $\frac{1}{\sigma_2^2} < \frac{1}{\sigma_3^2}$ the 2nd term dominates the 3rd term. Also

$$\left\{\begin{array}{l}\frac{1}{\sigma_2^2}<\frac{1}{\sigma_2^2}+\alpha_{12}^2=\frac{1}{\sigma_1^2}+\alpha_{21}^2 \\ \frac{1}{\sigma_2^2}<\frac{1}{\sigma_3^2}+\alpha_{13}^2=\frac{1}{\sigma_1^2}+\alpha_{31}^2\end{array}\right\} \Rightarrow \frac{1}{\sigma_2^2}<\frac{1}{\sigma_1^2}+u_1^2 \quad . \quad (3.10)$$

Consequently, the 2nd term dominates the 1st term.

$$f_1(t) \sim \frac{1}{\sigma_2\sqrt{2\pi}} \exp\left[\frac{-1}{2}\left(\left(\frac{1}{\sigma_2^2}\right)t^2+2\left(\frac{-\mu_2}{\sigma_2^2}\right)t+\frac{\mu_2^2}{\sigma_2^2}\right)\right] \quad (3.11)$$

$$-2\lim_{t\to-\infty}\left(\frac{\ln|f_1(t)|}{t^2}\right)=\left(\frac{1}{\sigma_2^2}\right) \quad (3.12)$$

Thus σ_2 is identified.

Define $f_{(2)}(t) \equiv f_1(t)\exp\left[\frac{1}{2}\frac{t^2}{\sigma_2^2}\right]$ (3.13)

By lemma 2.2, as $t \to -\infty$

$$1^{\text{st}} \text{ term of } f_{(2)}(t) \sim \frac{-1}{\sigma_1|u_1 t+\delta_1|2\pi} \exp\left\{-\frac{1}{2}\left[\left(\frac{t-\mu_1}{\sigma_1}\right)^2+(u_1 t+\delta_1)^2-\frac{t^2}{\sigma_2^2}\right]\right\}$$

$$=\frac{-1}{\sigma_1|u_1 t+\delta_1|2\pi} \exp\left\{-\frac{1}{2}\left[\left(\frac{1}{\sigma_1^2}+u_1^2-\frac{1}{\sigma_2^2}\right)t^2+2\left(u_1\delta_1-\frac{\mu_1}{\sigma_1^2}\right)t+\frac{\mu_1^2}{\sigma_1^2}+\delta_1^2\right]\right\} \quad (3.14)$$

$$2^{\text{nd}} \text{ term of } f_{(2)}(t) \sim \frac{1}{\sigma_2\sqrt{2\pi}} \exp\left[\frac{-1}{2}\left(2\left(\frac{-\mu_2}{\sigma_2^2}\right)t+\frac{\mu_2^2}{\sigma_2^2}\right)\right]$$

$$3^{\text{rd}} \text{ term of } f_{(2)}(t) \sim \frac{1}{\sigma_3\sqrt{2\pi}} \exp\left[\frac{-1}{2}\left(\left(\frac{1}{\sigma_3^2}-\frac{1}{\sigma_2^2}\right)t^2+2\left(\frac{-\mu_3}{\sigma_3^2}\right)t+\frac{\mu_3^2}{\sigma_3^2}\right)\right]$$

$\frac{1}{\sigma_1^2}+u_1^2-\frac{1}{\sigma_2^2}>0$ and $\frac{1}{\sigma_3^2}-\frac{1}{\sigma_2^2}>0$, so term 2 dominates.

$$\lim_{t\to-\infty}\left(\frac{\ln|f_{(2)}(t)|}{t}\right)=\frac{\mu_2}{\sigma_2^2} \quad (3.15)$$

Thus μ_2 is identified.

Define $f_2(t) \equiv f(t) - \frac{1}{\sigma_1}\phi\left(\frac{t-\mu_1}{\sigma_1}\right) - \frac{1}{\sigma_2}\phi\left(\frac{t-\mu_2}{\sigma_2}\right)$ (3.16)

$$= \frac{-1}{\sigma_1}\phi\left(\frac{t-\mu_1}{\sigma_1}\right)\left[1 - P\left(W_{21} > \alpha_{21}t + \beta_{21},\, W_{31} > \alpha_{31}t + \beta_{31}\right)\right]$$

$$\frac{-1}{\sigma_2}\phi\left(\frac{t-\mu_2}{\sigma_2}\right)\left[1 - P\left(W_{12} > \alpha_{12}t + \beta_{12},\, W_{32} > \alpha_{32}t + \beta_{32}\right)\right]$$

$$\frac{1}{\sigma_3}\phi\left(\frac{t-\mu_3}{\sigma_3}\right)P\left(W_{13} > \alpha_{13}t + \beta_{13},\, W_{23} > \alpha_{23}t + \beta_{23}\right)$$

Using Corollary 2.6.2, as $t \to -\infty$

1st term of $f_2(t) \sim \frac{-C_1}{\sigma_1|u_1t+\delta_1|2\pi}\ \exp\left\{-\frac{1}{2}\left[\left(\frac{1}{\sigma_1^2}+u_1^2\right)t^2 + 2\left(u_1\delta_1 - \frac{\mu_1}{\sigma_1^2}\right)t + \frac{\mu_1^2}{\sigma_1^2} + \delta_1^2\right]\right\}$

2nd term of $f_2(t) \sim$

$$\frac{-C_2}{\sigma_2|u_2t+\delta_2|2\pi}\ \exp\left\{-\frac{1}{2}\left[\left(\frac{1}{\sigma_2^2}+u_2^2\right)t^2 + 2\left(u_2\delta_2 - \frac{\mu_2}{\sigma_2^2}\right)t + \frac{\mu_2^2}{\sigma_2^2} + \delta_2^2\right]\right\} \quad (3.17)$$

$u_2 = \alpha_{12}, \delta_2 = \beta_{12}, C_2 = 1$ when $\alpha_{12} < \alpha_{32}$

$u_2 = \alpha_{32}, \delta_2 = \beta_{32}, C_2 = 1$ when $\alpha_{32} < \alpha_{12}$

$u_2 = \alpha_{12} = \alpha_{32}, \delta_2 = \beta_{12}, C_2 = 1$ when $\alpha_{12} = \alpha_{32}, \beta_{12} > \beta_{32}$

$u_2 = \alpha_{12} = \alpha_{32}, \delta_2 = \beta_{32}, C_2 = 1$ when $\alpha_{12} = \alpha_{32}, \beta_{32} > \beta_{12}$

$u_2 = \alpha_{12} = \alpha_{32}, \delta_2 = \beta_{12} = \beta_{32}, C_2 = 2$ when $\alpha_{12} = \alpha_{32}, \beta_{12} = \beta_{32}$

3rd term of $f_2(t) \sim \frac{1}{\sigma_3\sqrt{2\pi}}\exp\left[\frac{-1}{2}\left(\left(\frac{1}{\sigma_3^2}\right)t^2 + 2\left(\frac{-\mu_3}{\sigma_3^2}\right)t + \frac{\mu_3^2}{\sigma_3^2}\right)\right]$

By lemma 2.4 $\left\{\begin{array}{l} \frac{1}{\sigma_3^2} < \frac{1}{\sigma_3^2} + \alpha_{13}^2 = \frac{1}{\sigma_1^2} + \alpha_{31}^2 \\ \frac{1}{\sigma_3^2} < \frac{1}{\sigma_3^2} + \alpha_{23}^2 = \frac{1}{\sigma_2^2} + \alpha_{32}^2 \end{array}\right\}$ (3.18)

$$\Rightarrow \frac{1}{\sigma_3^2} < \min\left\{\frac{1}{\sigma_1^2} + u_1^2, \frac{1}{\sigma_2^2} + u_2^2\right\} \text{ if and only if } \frac{1}{\sigma_3^2} < \frac{1}{\sigma_1^2} + \alpha_{21}^2 = \frac{1}{\sigma_2^2} + \alpha_{12}^2 = A_{12} \quad (3.19)$$

Using lemma 2.1, we must consider the two cases:

Case 1: Term 1 and term 2 dominate when $\frac{1}{\sigma_3^2} > A_{12}$

(so that $u_1 = \alpha_{21}$, $u_2 = \alpha_{12}$, $\delta_1 = \beta_{21}$, and $\delta_2 = \beta_{12}$)

OR when $\frac{1}{\sigma_3^2} = A_{12}$ and $\frac{-\mu_3}{\sigma_3^2} < \alpha_{12}\beta_{12} - \frac{\mu_2}{\sigma_2^2}\left(= \alpha_{21}\beta_{21} - \frac{\mu_1}{\sigma_1^2}\right) = B_{12}$

$$\Rightarrow f_2(t) \sim \left(\sum_{j=1}^{2} \frac{-C_j}{\sigma_j \left|u_j t + \delta_j\right| 2\pi}\right) \exp\left\{-\frac{1}{2}\left[A_{12}t^2 + 2B_{12}t + C_{12}\right]\right\} (3.20)$$

Case 2: Term 3 dominates when $\frac{1}{\sigma_3^2} < A_{12}$

OR when $\frac{1}{\sigma_3^2} = A_{12}$ and $\frac{-\mu_3}{\sigma_3^2} \geq \alpha_{12}\beta_{12} - \frac{\mu_2}{\sigma_2^2}\left(= \alpha_{21}\beta_{21} - \frac{\mu_1}{\sigma_1^2}\right) = B_{12}$

$$\Rightarrow f_2(t) \sim \frac{1}{\sigma_3\sqrt{2\pi}} \exp\left[\frac{-1}{2}\left(\left(\frac{1}{\sigma_3^2}\right)t^2 + 2\left(\frac{-\mu_3}{\sigma_3^2}\right)t + \frac{\mu_3^2}{\sigma_3^2}\right)\right] (3.21)$$

To determine which case applies we define:

$$r_2^2 \equiv -2 \lim_{t\to-\infty}\left(\frac{\ln\left|f_2(t)\right|}{t^2}\right) \quad = \begin{pmatrix} A_{12} & \text{for } case1 \\ \frac{1}{\sigma_3^2} & \text{for } case2 \end{pmatrix} \quad (3.22)$$

$$s_2 \equiv -2 \lim_{t\to-\infty}\left(\frac{\ln\left|f_2(t)e^{\frac{1}{2}r_2^2t^2}\right|}{t}\right) \quad = \begin{pmatrix} 2\left(\alpha_{21}\beta_{21} - \frac{\mu_1}{\sigma_1^2}\right) & \text{for } case1 \\ 2\left(\frac{-\mu_3}{\sigma_3^2}\right) & \text{for } case2 \end{pmatrix} \quad (3.23)$$

Finally, we define

$$b_2 \equiv \lim_{t\to-\infty}\left(f_2(t)e^{\frac{1}{2}\left[r_2^2t^2+s_2t\right]}|t|\right) \quad = \begin{pmatrix} \text{negative number} & \text{for } case1 \\ \infty & \text{for } case2 \end{pmatrix} \quad (3.24)$$

We will determine case by use of b_2.

Case 1: $b_2 = \text{negative number}$

Since $r_2^2 = A_{12}$, by lemma 2.3, $\rho_{12} = \dfrac{-1}{\sigma_1\sigma_2}\left[\sqrt{\left(\sigma_1^2 - \dfrac{1}{r_2^2}\right)\left(\sigma_2^2 - \dfrac{1}{r_2^2}\right)} - \dfrac{1}{r_2^2}\right]$

ρ_{12} is now identified and since σ_1, σ_2, μ_1, μ_2 were already identified, we can now identify α_{12}, α_{21}, β_{12}, β_{21}.

Define

$$h(t) \equiv -f_2(t) - \frac{1}{\sigma_1}\phi\left(\frac{t-\mu_1}{\sigma_1}\right)P(W_{21} \le \alpha_{21}t + \beta_{21}) - \frac{1}{\sigma_2}\phi\left(\frac{t-\mu_2}{\sigma_2}\right)P(W_{12} \le \alpha_{12}t + \beta_{12}) \quad (3.25)$$

$$= \frac{1}{\sigma_1}\phi\left(\frac{t-\mu_1}{\sigma_1}\right)\left[1 - p(W_{21} > \alpha_{21}t + \beta_{21}, W_{31} > \alpha_{31}t + \beta_{31}) - P(W_{21} \le \alpha_{21}t + \beta_{21})\right]$$

$$+ \frac{1}{\sigma_2}\phi\left(\frac{t-\mu_2}{\sigma_2}\right)\left[1 - p(W_{12} > \alpha_{12}t + \beta_{12}, W_{32} > \alpha_{32}t + \beta_{32}) - P(W_{12} \le \alpha_{12}t + \beta_{12})\right]$$

$$+ \frac{-1}{\sigma_3}\phi\left(\frac{t-\mu_3}{\sigma_3}\right)p(W_{13} > \alpha_{13}t + \beta_{13}, W_{23} > \alpha_{23}t + \beta_{23})$$

In corollary 2.6.3 let $u_1 = \alpha_{21}, \delta_1 = \beta_{21}$ and $M_1 = \alpha_{31}, d_1 = \beta_{31}$ and also $u_2 = \alpha_{12}, \delta_2 = \beta_{12}$ and $M_2 = \alpha_{32}, d_2 = \beta_{32}$

then as $t \to -\infty$

1st term of $h(t)$

$$\sim \frac{1}{\sigma_1|\alpha_{31}t + \beta_{31}|2\pi}\exp\left\{-\frac{1}{2}\left[\left(\frac{1}{\sigma_1^2} + \alpha_{31}^2\right)t^2 + 2\left(\alpha_{31}\beta_{31} - \frac{\mu_1}{\sigma_1^2}\right)t + \frac{\mu_1^2}{\sigma_1^2} + \beta_{31}^2\right]\right\} \quad (3.26)$$

2^{nd} term of $h(t)$

$$\sim \frac{1}{\sigma_2|\alpha_{32}t+\beta_{32}|2\pi}\exp\left\{-\frac{1}{2}\left[\left(\frac{1}{\sigma_2^2}+\alpha_{32}^2\right)t^2+2\left(\alpha_{32}\beta_{32}-\frac{\mu_2}{\sigma_2^2}\right)t+\frac{\mu_2^2}{\sigma_2^2}+\beta_{32}^2\right]\right\} \text{(3.27)}$$

3^{rd} term of $h(t) \sim \dfrac{-1}{\sigma_3\sqrt{2\pi}}\exp\left[\dfrac{-1}{2}\left(\left(\dfrac{1}{\sigma_3^2}\right)t^2+2\left(\dfrac{-\mu_3}{\sigma_3^2}\right)t+\dfrac{\mu_3^2}{\sigma_3^2}\right)\right]$

Since $\dfrac{1}{\sigma_3^2}<\min\left\{\dfrac{1}{\sigma_1^2}+\alpha_{31}^2,\dfrac{1}{\sigma_2^2}+\alpha_{32}^2\right\}$, term 3 of $h(t)$ dominates

$-2\lim\limits_{t\to-\infty}\left(\dfrac{\ln|h(t)|}{t^2}\right)=\dfrac{1}{\sigma_3^2}$, σ_3 is identified for case 1.

Define $f_{(3)}(t) \equiv h(t)\exp\left[\dfrac{1}{2}\dfrac{t^2}{\sigma_3^2}\right]$ (3.28)

as $t\to-\infty$

1^{st} term of $f_{(3)}(t)$

$$\sim \frac{1}{\sigma_1|\alpha_{31}t+\beta_{31}|2\pi}\exp\left\{-\frac{1}{2}\left[\left(\frac{1}{\sigma_1^2}+\alpha_{31}^2-\frac{1}{\sigma_3^2}\right)t^2+2\left(\alpha_{31}\beta_{31}-\frac{\mu_1}{\sigma_1^2}\right)t+\frac{\mu_1^2}{\sigma_1^2}+\beta_{31}^2\right]\right\} \text{(3.29)}$$

2^{nd} term of $f_{(3)}(t)$

$$\sim \frac{1}{\sigma_2|\alpha_{32}t+\beta_{32}|2\pi}\exp\left\{-\frac{1}{2}\left[\left(\frac{1}{\sigma_2^2}+\alpha_{32}^2-\frac{1}{\sigma_3^2}\right)t^2+2\left(\alpha_{32}\beta_{32}-\frac{\mu_2}{\sigma_2^2}\right)t+\frac{\mu_2^2}{\sigma_2^2}+\beta_{32}^2\right]\right\} \text{(3.30)}$$

3^{rd} term of $f_{(3)}(t)$

$$\sim \frac{-1}{\sigma_3\sqrt{2\pi}} \exp\left[\frac{-1}{2}\left(2\left(\frac{-\mu_3}{\sigma_3^2}\right)t + \frac{\mu_3^2}{\sigma_3^2}\right)\right]$$

Since $\frac{1}{\sigma_3^2} < \min\left\{\frac{1}{\sigma_1^2} + \alpha_{31}^2, \frac{1}{\sigma_2^2} + \alpha_{32}^2\right\}$, term 3 of $f_{(3)}(t)$ dominates

$$\lim_{t\to-\infty}\left(\frac{\ln\left|f_{(3)}(t)\right|}{t^2}\right) = \frac{\mu_3}{\sigma_3^2}, \quad (3.31)$$

μ_3 is identified for case 1.

Case 2: $b_2 = \infty$

Since $r_2^2 = \frac{1}{\sigma_3^2}$, σ_3 is identified for case 2.

Since $s_2 = 2\left(\frac{-\mu_3}{\sigma_3^2}\right)$, μ_3 is identified for case 2.

For both cases we have identified $\mu_1, \mu_2, \mu_3, \sigma_1, \sigma_2, \sigma_3$.

Define $f_3(t) \equiv f(t) - \frac{1}{\sigma_1}\phi\left(\frac{t-\mu_1}{\sigma_1}\right) - \frac{1}{\sigma_2}\phi\left(\frac{t-\mu_2}{\sigma_2}\right) - \frac{1}{\sigma_3}\phi\left(\frac{t-\mu_3}{\sigma_3}\right)$ (3.32)

$$= \frac{-1}{\sigma_1}\phi\left(\frac{t-\mu_1}{\sigma_1}\right)\left[1 - P\left(W_{21} > \alpha_{21}t + \beta_{21},\, W_{31} > \alpha_{31}t + \beta_{31}\right)\right]$$

$$+ \frac{-1}{\sigma_2}\phi\left(\frac{t-\mu_2}{\sigma_2}\right)\left[1 - P\left(W_{12} > \alpha_{12}t + \beta_{12},\, W_{32} > \alpha_{32}t + \beta_{32}\right)\right]$$

$$+ \frac{-1}{\sigma_3}\phi\left(\frac{t-\mu_3}{\sigma_3}\right)\left[1 - P\left(W_{13} > \alpha_{13}t + \beta_{13},\, W_{23} > \alpha_{23}t + \beta_{23}\right)\right]$$

As usual, we consider each of the addends as a term of $f_3(t)$. By corollary 2.6.2, as $t \to -\infty$

$$1^{\text{st}} \text{ term of } f_3(t) \sim \frac{-C_1}{\sigma_1 |u_1 t + \delta_1| 2\pi} \exp\left\{ -\frac{1}{2}\left[\left(\frac{1}{\sigma_1^2} + u_1^2 \right) t^2 + 2\left(u_1 \delta_1 - \frac{\mu_1}{\sigma_1^2} \right) t + \frac{\mu_1^2}{\sigma_1^2} + \delta_1^2 \right] \right\}$$

$$2^{\text{nd}} \text{ term of } f_3(t) \sim \frac{-C_2}{\sigma_2 |u_2 t + \delta_2| 2\pi} \exp\left\{ -\frac{1}{2}\left[\left(\frac{1}{\sigma_2^2} + u_2^2 \right) t^2 + 2\left(u_2 \delta_2 - \frac{\mu_2}{\sigma_2^2} \right) t + \frac{\mu_2^2}{\sigma_2^2} + \delta_2^2 \right] \right\}$$

3^{rd} term of $f_3(t) \sim$

$$\frac{-C_3}{\sigma_3 |u_3 t + \delta_3| 2\pi} \exp\left\{ -\frac{1}{2}\left[\left(\frac{1}{\sigma_3^2} + u_3^2 \right) t^2 + 2\left(u_3 \delta_3 - \frac{\mu_3}{\sigma_3^2} \right) t + \frac{\mu_3^2}{\sigma_3^2} + \delta_3^2 \right] \right\} (3.33)$$

$u_3 = \alpha_{13}, \delta_3 = \beta_{13}, C_3 = 1$ when $\alpha_{13} < \alpha_{23}$

$u_3 = \alpha_{23}, \delta_3 = \beta_{23}, C_3 = 1$ when $\alpha_{23} < \alpha_{13}$

$u_3 = \alpha_{13} = \alpha_{23}, \delta_3 = \beta_{13}, C_3 = 1$ when $\alpha_{13} = \alpha_{23}, \beta_{13} > \beta_{23}$

$u_3 = \alpha_{13} = \alpha_{23}, \delta_3 = \beta_{23}, C_3 = 1$ when $\alpha_{13} = \alpha_{23}, \beta_{23} > \beta_{13}$

$u_3 = \alpha_{13} = \alpha_{23}, \delta_3 = \beta_{13} = \beta_{23}, C_3 = 2$ when $\alpha_{13} = \alpha_{23}, \beta_{13} = \beta_{23}$

For $j = 1,2,3$, define:

$J \equiv \{ j \mid \text{term } j \text{ is a dominating term for } f_3(t) \}$, $A^* \equiv \min\left\{ \frac{1}{\sigma_l^2} + u_l^2 \right\}_{l=1}^{3}$, and

$$J^* \equiv \left\{ j \mid \frac{1}{\sigma_j^2} + u_j^2 = A^* \right\} (3.34)$$

Lemma 3.1 There are at least two dominating terms for $f_3(t)$:

Proof

By lemma 2.1, there is some $j \in J^*$ such that term j is a dominating term. Also there is some $i, i \neq j$, such that $u_j = \alpha_{ij}$ and $\delta_j = \beta_{ij}$ Since $A^* = \frac{1}{\sigma_j^2} + u_j^2 = \frac{1}{\sigma_j^2} + \alpha_{ij}^2 = \frac{1}{\sigma_i^2} + \alpha_{ji}^2$.

It follows that $u_i = \alpha_{ji}$ Also, by lemma 4, $\alpha_{ij}\beta_{ij} - \frac{\mu_j}{\sigma_j^2} = \alpha_{ji}\beta_{ji} - \frac{\mu_i}{\sigma_i^2}$ which implies

$\delta_i = \beta_{ji}$ Thus term i and term j are of the same dominating order. Therefore $i \in J$ and $j \in J$ so we must conclude J has at least two elements meaning there are at least two dominating terms for $f_3(t)$.

□

By lemma 2.2.

$$f_3(t) \sim \sum_{j\in J} \left(\text{term } j \text{ of } f_3(t)\right)$$

$$\sim \sum_{j\in J} \left(\frac{-C_j \exp\left\{-\frac{1}{2}\left[\left(\frac{1}{\sigma_j^2} + u_j^2\right)t^2 + 2\left(u_j\delta_j - \frac{\mu_j}{\sigma_j^2}\right)t + \frac{\mu_j^2}{\sigma_j^2} + \delta_j^2\right]\right\}}{\sigma_j \left|u_j t + \delta_j\right| 2\pi} \right) \quad (3.35)$$

By lemma 2.1, for each $j \in J$ we have

$$\frac{1}{\sigma_j^2} + u_j^2 = \min\left\{\frac{1}{\sigma_l^2} + u_l^2\right\}_{l=1}^{3} \quad \text{and} \quad u_j\delta_j - \frac{\mu_j}{\sigma_j^2} = \max\left\{u_l\delta_l - \frac{\mu_l}{\sigma_l^2}\right\}_{l\in J^*}$$

$$\Rightarrow f_3(t)$$

$$\sim \left(\sum_{j\in J} \frac{-C_j c_j}{\sigma_j \left|u_j t + \delta_j\right| 2\pi}\right) \exp\left\{-\frac{1}{2}\left[\left(\min\left\{\frac{1}{\sigma_l^2} + u_l^2\right\}_{l=1}^{3}\right)t^2 + 2\left(\max\left\{u_l\delta_l - \frac{\mu_l}{\sigma_l^2}\right\}_{l\in J^*}\right)t\right]\right\} \quad (3.36)$$

where $c_j = \exp\left[\frac{-1}{2}\left(\frac{\mu_j^2}{\sigma_j^2} + \delta_j^2\right)\right]$.

Define $r_3^2 \equiv -2 \lim_{t\to-\infty}\left(\frac{\ln|f_3(t)|}{t^2}\right)$, $g(t) \equiv f_3(t)\exp\left[\frac{1}{2}r_3^2t^2\right]$, $s_3 \equiv -\lim_{t\to-\infty}\left(\frac{\ln|g(t)|}{t}\right)$ (3.37)

It now follows that

$$r_3^2 = \min\left\{\frac{1}{\sigma_l^2} + u_l^2\right\}_{l=1}^3 \text{ and } s_3 = \max\left\{u_l\delta_l - \frac{\mu_l}{\sigma_l^2}\right\}_{l\in J^*} \quad (3.38)$$

so we can write

$$f_3(t) \sim \left(\sum_{j\in J}\frac{-C_jc_j}{\sigma_j|u_jt+\delta_j|2\pi}\right)\exp\left\{-\frac{1}{2}\left[r_3^2t^2 + 2s_3t\right]\right\} \quad (3.39)$$

We can therefore give an equivalent definition for J:

$$J \equiv \left\{j \mid \frac{1}{\sigma_j^2} + u_j^2 = r_3^2 \text{ and } u_j\delta_j - \frac{\mu_j}{\sigma_j^2} = s_3\right\} \quad (3.40)$$

and now define

$$I_j \equiv \left\{m \mid \frac{1}{\sigma_j^2} + \alpha_{mj}^2 = r_3^2 \text{ and } \alpha_{mj}\beta_{mj} - \frac{\mu_j}{\sigma_j^2} = \mathrm{s}_3\right\} \quad (3.41)$$

and $D_j \equiv card\, I_j$. (3.42)

Notice that $D_j = C_j$ when term j is a dominating term, but $D_j = 0$ when term j is not a dominating term. Also notice that we always have $D_j \in \{0,1,2\}$

Lemma 3.2 $j \in I_i$ if and only if $i \in I_j$

Proof

$$j \in I_i \Leftrightarrow r_3^2 = \frac{1}{\sigma_i^2} + \alpha_{ji}^2 = \frac{1}{\sigma_j^2} + \alpha_{ij}^2$$

$$\text{and } s_3 = \alpha_{ji}\beta_{ji} - \frac{\mu_i}{\sigma_i^2} = \alpha_{ij}\beta_{ij} - \frac{\mu_j}{\sigma_j^2} \Leftrightarrow i \in I_j$$

□

Now define: $\overline{u_j} \equiv \sqrt{r_3^2 - \frac{1}{\sigma_j^2}}$, $\overline{\delta_j} = \dfrac{s_3 + \frac{\mu_j}{\sigma_j^2}}{\sqrt{r_3^2 - \frac{1}{\sigma_j^2}}}$ $\quad \bar{c}_j = \exp\left[\frac{-1}{2}\left(\frac{\mu_j^2}{\sigma_j^2} + \bar{\delta}_j^2\right)\right]$ (3.43)

When term j of $f_3(t)$ is a dominating term,

then, $\overline{u_j} = u_j$, $\overline{\delta_j} = \delta_j$, and $\bar{c}_j = c_j$. Again, $D_j = 0$ when term j is not a dominating term; therefore,

$$f_3(t) \sim \left(\sum_{j=1}^{3} \frac{-D_j \bar{c}_j}{\sigma_j \left| \overline{u_j} t + \overline{\delta_j} \right| 2\pi} \right) \exp\left\{ -\frac{1}{2}\left[r_3^2 t^2 + 2 s_3 t \right] \right\} \text{(3.44)}$$

Now define $b_3 \equiv \lim\limits_{t \to -\infty} \left(f_3(t) 2\pi |t| \exp \frac{1}{2}\left[r_3^2 t^2 + 2 s_3 t \right] \right)$ (3.45)

The following theorem will show there are exactly seven possible cases for the value of the vector (D_1, D_2, D_3). The values of $b_3, \bar{c}_1, \bar{c}_2,$ and $\bar{c}_3$ will be used to determine case.

Theorem 3.1: Let i, j, k be some permutation of $1,2,3$. Also, let (D_1, D_2, D_3) be the vector whose components are defined by (3.42). Then, the following three statements hold:

1. No more than one coordinate has the value of zero. If one of the coordinates has the value of zero, the other two coordinates must have the value of one. Then one of the correlation coefficients is identified. More specifically, $D_j = 0$ implies that $D_i = D_k = 1$ and $r_3^2 = A_{ik}$ so that ρ_{ik} is identified. Also, $\bar{c}_i = \bar{c}_k$
2. If no coordinate has the value zero, then at least one of the coordinates has the value two and two of the correlation coefficients are identified. More specifically, if $D_i = D_k = 1$ and $D_j \neq 0$, then $D_j = 2$ and $r_3^2 = A_{ij} = A_{jk}$ so that ρ_{ij} and ρ_{jk} are identified. Also, $\bar{c}_i = \bar{c}_j = \bar{c}_k$

3. If two of the coordinates have the value two, then the third coordinate must also have the value two and $r_3^2 = A_{ij} = A_{ik} = A_{jk}$ so that all three of the correlation coefficients are identified. Again we have $\bar{c}_i = \bar{c}_j = \bar{c}_k$

Proof of 1.:

Suppose $D_j = 0$. Then term j is not a dominating term. Since two of the terms must be dominating, term *i* and term *k* are both dominating terms. Therefore I_i and I_k are not empty while I_j is empty. Furthermore, from lemma 3.2 we have $j \notin I_i$ and $j \notin I_k$.

$\Rightarrow I_i = \{k\}$ and $I_k = \{i\}$ so that $D_i = D_k = 1$ and $u_i = \alpha_{ki}, \delta_i = \beta_{ki}, u_k = \alpha_{ik}, \delta_k = \beta_{ik}$

It follow that $r_3^2 = \dfrac{1}{\sigma_i^2} + \alpha_{ki}^2 = \dfrac{1}{\sigma_k^2} + \alpha_{ik}^2 = A_{ik} \Rightarrow$ (by lemma 3)

$$\rho_{ik} = \frac{-1}{\sigma_i \sigma_k}\left[\sqrt{\left(\sigma_i^2 - \frac{1}{r_3^2}\right)\left(\sigma_k^2 - \frac{1}{r_3^2}\right)} - \frac{1}{r_3^2}\right]$$

Also, by lemma 2.4, $c_i = \exp\left[\dfrac{-1}{2}\left(\dfrac{\mu_i^2}{\sigma_i^2} + \delta_i^2\right)\right] = \exp\left[\dfrac{-1}{2}\left(\dfrac{\mu_i^2}{\sigma_i^2} + \beta_{ki}^2\right)\right]$

$$= \exp\left[\frac{-1}{2}\left(\frac{\mu_k^2}{\sigma_k^2} + \beta_{ik}^2\right)\right] = \exp\left[\frac{-1}{2}\left(\frac{\mu_k^2}{\sigma_k^2} + \delta_k^2\right)\right] = c_k$$

And since term *i* and term *k* are dominating terms, $\bar{c}_i = \bar{c}_k$

Proof of 2.:

Let $D_i = D_k = 1$ and suppose $D_j \neq 0$. Then term j is a dominating term. Hence all three terms are dominating terms:

$$\frac{1}{\sigma_j^2} + u_j^2 = \frac{1}{\sigma_i^2} + u_i^2 = \frac{1}{\sigma_k^2} + u_k^2 = r_3^2 \text{ and } u_j\delta_j - \frac{\mu_j}{\sigma_j^2} = u_i\delta_i - \frac{\mu_i}{\sigma_i^2} = u_k\delta_k - \frac{\mu_k}{\sigma_k^2} = s_3$$

Since $D_i = 1$ either $I_i = \{j\}$ or $I_i = \{k\}$

Suppose $I_i = \{k\}$ then by lemma 3.2 $i \in I_k$ and $I_k = \{i\}$ since $D_k = 1$.

$\Rightarrow j \notin I_i$ and $j \notin I_k \Rightarrow i \notin I_j$ and $k \notin I_j \Rightarrow I_j = \varnothing$ which contradicts $D_j \neq 0$.

We must therefore have $I_i = \{j\}, I_k = \{j\},$ and $I_j = \{i,k\} \Rightarrow D_j = 2$

We now have $\delta_i = \beta_{ji}, \delta_j = \beta_{ij} = \beta_{kj}$, and $\delta_k = \beta_{jk}$

$$u_i = \alpha_{ji}, u_j = \alpha_{ij} = \alpha_{kj}, \text{and } u_k = \alpha_{jk}$$

Since $r_3^2 = \dfrac{1}{\sigma_j^2} + \alpha_{ij}^2 = \dfrac{1}{\sigma_j^2} + \alpha_{kj}^2 = \mathrm{A_{ij}} = \mathrm{A_{jk}}$ we have by lemma 2.3

$$\rho_{ij} = \frac{-1}{\sigma_i \sigma_j}\left[\sqrt{\left(\sigma_i^2 - \frac{1}{r_3^2}\right)\left(\sigma_j^2 - \frac{1}{r_3^2}\right)} - \frac{1}{r_3^2}\right] \text{ and } \rho_{jk} = \frac{-1}{\sigma_j \sigma_k}\left[\sqrt{\left(\sigma_j^2 - \frac{1}{r_3^2}\right)\left(\sigma_k^2 - \frac{1}{r_3^2}\right)} - \frac{1}{r_3^2}\right]$$

We now show that $\bar{c}_i = \bar{c}_j = \bar{c}_k$:

$$c_i = \exp\left[\frac{-1}{2}\left(\frac{\mu_i^2}{\sigma_i^2} + \delta_i^2\right)\right] = \exp\left[\frac{-1}{2}\left(\frac{\mu_i^2}{\sigma_i^2} + \beta_{ji}^2\right)\right]$$

$$= \exp\left[\frac{-1}{2}\left(\frac{\mu_j^2}{\sigma_j^2} + \beta_{ij}^2\right)\right] = \exp\left[\frac{-1}{2}\left(\frac{\mu_j^2}{\sigma_j^2} + \delta_j^2\right)\right] = c_j$$

Also

$$c_j = \exp\left[\frac{-1}{2}\left(\frac{\mu_j^2}{\sigma_j^2} + \delta_j^2\right)\right] = \exp\left[\frac{-1}{2}\left(\frac{\mu_j^2}{\sigma_j^2} + \beta_{kj}^2\right)\right]$$

$$= \exp\left[\frac{-1}{2}\left(\frac{\mu_k^2}{\sigma_k^2} + \beta_{jk}^2\right)\right] = \exp\left[\frac{-1}{2}\left(\frac{\mu_k^2}{\sigma_k^2} + \delta_k^2\right)\right] = c_k$$

All three terms are dominating terms so $\bar{c}_i = \bar{c}_j = \bar{c}_k$

Proof of 3.:

Let $D_i = D_k = 2 \Rightarrow I_i = \{j,k\}$ and $I_k = \{i,j\} \Rightarrow i,k \in I_j$ and $D_j = 2$

We now have $\delta_i = \beta_{ji} = \beta_{ki}, \delta_j = \beta_{ij} = \beta_{kj}$, and $\delta_k = \beta_{ik} = \beta_{jk}$

$$u_i = \alpha_{ji} = \alpha_{ki}, u_j = \alpha_{ij} = \alpha_{kj}, \text{and } u_k = \alpha_{ik} = \alpha_{jk}$$

It follows that $r_3^2 = \mathrm{A_{ij}} = \mathrm{A_{ik}} = \mathrm{A_{jk}}$. Thus, by lemma 2.3

$$\rho_{ij} = \frac{-1}{\sigma_i \sigma_j}\left[\sqrt{\left(\sigma_i^2 - \frac{1}{r_3^2}\right)\left(\sigma_j^2 - \frac{1}{r_3^2}\right)} - \frac{1}{r_3^2}\right] \quad \rho_{ik} = \frac{-1}{\sigma_i \sigma_k}\left[\sqrt{\left(\sigma_i^2 - \frac{1}{r_3^2}\right)\left(\sigma_k^2 - \frac{1}{r_3^2}\right)} - \frac{1}{r_3^2}\right]$$

$$\rho_{jk} = \frac{-1}{\sigma_j \sigma_k}\left[\sqrt{\left(\sigma_j^2 - \frac{1}{r_3^2}\right)\left(\sigma_k^2 - \frac{1}{r_3^2}\right)} - \frac{1}{r_3^2}\right]$$

Using the argument given above in the proof of 2, we find that $\bar{c}_i = \bar{c}_j = \bar{c}_k$:

□

It is now clear that there are seven possible values for (D_1, D_2, D_3) namely $(1,1,0),(1,0,1),(0,1,1),(2,1,1),(1,2,1),(1,1,2),(2,2,2)$. Thus there are seven possible cases. For each case either exactly two of the values $\bar{c}_1, \bar{c}_2, \bar{c}_3$ are equal (ie $\bar{c}_i = \bar{c}_j \neq \bar{c}_k$ where i,j,k is the appropriate permutation of $1,2,3$) or $\bar{c}_1 = \bar{c}_2 = \bar{c}_3$. Notice $\bar{c}_i = \bar{c}_j \neq \bar{c}_k$ only when $D_i = D_j = 1$ and $D_k = 0$. So here case is readily determined. When $\bar{c}_1 = \bar{c}_2 = \bar{c}_3$ we determine case by calculating b_3. By definition $b_3 \equiv \lim_{t \to -\infty}\left(f_3(t) 2\pi |t| \exp \frac{1}{2}\left[r_3^2 t^2 + 2 s_3 t\right]\right)$.

Now define $\bar{b}_3 \equiv \sum_{j=1}^{3} \frac{-D_j \bar{c}_j}{\sigma_j |\overline{u_j}|} = -c\left(\frac{D_1}{\sqrt{\sigma_1^2 r_3^2 - 1}} + \frac{D_2}{\sqrt{\sigma_2^2 r_3^2 - 1}} + \frac{D_3}{\sqrt{\sigma_3^2 r_3^2 - 1}} \right)$ (3.46) where $c = \bar{c}_1 = \bar{c}_2 = \bar{c}_3$. Observe that $\bar{b}_3$ has a different value for each of the seven cases. For the correct case $\bar{b}_3 = b_3$. Thus when $\bar{c}_1 = \bar{c}_2 = \bar{c}_3$, case can be determined using $\bar{b}_3$ and b_3.

Theorem 3.2 summarizes these findings.

Theorem 3.2: The value of (D_1, D_2, D_3) and the value for at least one of the correlation coefficients can be determined by the values of $b_3, \bar{c}_1, \bar{c}_2,$ and $\bar{c}_3$ as follows:

Case 1 $\quad \bar{c}_1 = \bar{c}_2 \neq \bar{c}_3$ or

$$\bar{c}_1 = \bar{c}_2 = \bar{c}_3 = c \text{ and } b_3 = \frac{-\bar{c}_1}{\sigma_1 |\overline{u_1}|} + \frac{-\bar{c}_2}{\sigma_2 |\overline{u_2}|} = -c\left(\frac{1}{\sqrt{\sigma_1^2 r_3^2 - 1}} + \frac{1}{\sqrt{\sigma_2^2 r_3^2 - 1}}\right)$$

$$\text{iff } (D_1, D_2, D_3) = (1,1,0) \text{ and } \rho_{12} = \frac{-1}{\sigma_1 \sigma_2}\left[\sqrt{\left(\sigma_1^2 - \frac{1}{r_3^2}\right)\left(\sigma_2^2 - \frac{1}{r_3^2}\right)} - \frac{1}{r_3^2}\right]$$

Case 2 $\quad \bar{c}_1 = \bar{c}_3 \neq \bar{c}_2$ or

$$\bar{c}_1 = \bar{c}_2 = \bar{c}_3 = c \text{ and } b_3 = \frac{-\bar{c}_1}{\sigma_1 |\overline{u_1}|} + \frac{-\bar{c}_3}{\sigma_3 |u_3|} = -c\left(\frac{1}{\sqrt{\sigma_1^2 r_3^2 - 1}} + \frac{1}{\sqrt{\sigma_3^2 r_3^2 - 1}}\right)$$

$$\text{iff } (D_1, D_2, D_3) = (1,0,1) \text{ and } \rho_{13} = \frac{-1}{\sigma_1 \sigma_3}\left[\sqrt{\left(\sigma_1^2 - \frac{1}{r_3^2}\right)\left(\sigma_3^2 - \frac{1}{r_3^2}\right)} - \frac{1}{r_3^2}\right]$$

Case 3 $\quad \bar{c}_2 = \bar{c}_3 \neq \bar{c}_1$ or

$$\bar{c}_1 = \bar{c}_2 = \bar{c}_3 = c \text{ and } b_3 = \frac{-\bar{c}_2}{\sigma_2 |\overline{u_2}|} + \frac{-\bar{c}_3}{\sigma_3 |\overline{u_3}|} = -c\left(\frac{1}{\sqrt{\sigma_2^2 r_3^2 - 1}} + \frac{1}{\sqrt{\sigma_3^2 r_3^2 - 1}}\right)$$

$$\text{iff } (D_1, D_2, D_3) = (0,1,1) \text{ and } \rho_{23} = \frac{-1}{\sigma_2 \sigma_3}\left[\sqrt{\left(\sigma_2^2 - \frac{1}{r_3^2}\right)\left(\sigma_3^2 - \frac{1}{r_3^2}\right)} - \frac{1}{r_3^2}\right]$$

Case 4 $\quad \bar{c}_1 = \bar{c}_2 = \bar{c}_3 = c$ and

$$b_3 = \frac{-2\bar{c}_1}{\sigma_1\left|\overline{u_1}\right|} + \frac{-\bar{c}_2}{\sigma_2\left|\overline{u_2}\right|} + \frac{-\bar{c}_3}{\sigma_3\left|\overline{u_3}\right|} = -c\left(\frac{2}{\sqrt{\sigma_1^2 r_3^2 - 1}} + \frac{1}{\sqrt{\sigma_2^2 r_3^2 - 1}} + \frac{1}{\sqrt{\sigma_3^2 r_3^2 - 1}}\right)$$

iff $(D_1, D_2, D_3) = (2,1,1)$ and $\rho_{12} = \frac{-1}{\sigma_1\sigma_2}\left[\sqrt{\left(\sigma_1^2 - \frac{1}{r_3^2}\right)\left(\sigma_2^2 - \frac{1}{r_3^2}\right)} - \frac{1}{r_3^2}\right]$ and

$$\rho_{13} = \frac{-1}{\sigma_1\sigma_3}\left[\sqrt{\left(\sigma_1^2 - \frac{1}{r_3^2}\right)\left(\sigma_3^2 - \frac{1}{r_3^2}\right)} - \frac{1}{r_3^2}\right]$$

Case 5 $\quad \bar{c}_1 = \bar{c}_2 = \bar{c}_3 = c$ and

$$b_3 = \frac{-\bar{c}_1}{\sigma_1\left|\overline{u_1}\right|} + \frac{-2\bar{c}_2}{\sigma_2\left|\overline{u_2}\right|} + \frac{-\bar{c}_3}{\sigma_3\left|\overline{u_3}\right|} = -c\left(\frac{1}{\sqrt{\sigma_1^2 r_3^2 - 1}} + \frac{2}{\sqrt{\sigma_2^2 r_3^2 - 1}} + \frac{1}{\sqrt{\sigma_3^2 r_3^2 - 1}}\right)$$

iff $(D_1, D_2, D_3) = (1,2,1)$ and $\rho_{12} = \frac{-1}{\sigma_1\sigma_2}\left[\sqrt{\left(\sigma_1^2 - \frac{1}{r_3^2}\right)\left(\sigma_2^2 - \frac{1}{r_3^2}\right)} - \frac{1}{r_3^2}\right]$ and

$$\rho_{23} = \frac{-1}{\sigma_2\sigma_3}\left[\sqrt{\left(\sigma_2^2 - \frac{1}{r_3^2}\right)\left(\sigma_3^2 - \frac{1}{r_3^2}\right)} - \frac{1}{r_3^2}\right]$$

Case 6 $\quad \bar{c}_1 = \bar{c}_2 = \bar{c}_3 = c$ and

$$b_3 = \frac{-\bar{c}_1}{\sigma_1\left|\overline{u_1}\right|} + \frac{-\bar{c}_2}{\sigma_2\left|\overline{u_2}\right|} + \frac{-2\bar{c}_3}{\sigma_3\left|\overline{u_3}\right|} = -c\left(\frac{1}{\sqrt{\sigma_1^2 r_3^2 - 1}} + \frac{1}{\sqrt{\sigma_2^2 r_3^2 - 1}} + \frac{2}{\sqrt{\sigma_3^2 r_3^2 - 1}}\right)$$

iff $(D_1, D_2, D_3) = (1,1,2)$ and $\rho_{13} = \frac{-1}{\sigma_1\sigma_3}\left[\sqrt{\left(\sigma_1^2 - \frac{1}{r_3^2}\right)\left(\sigma_3^2 - \frac{1}{r_3^2}\right)} - \frac{1}{r_3^2}\right]$ and

$$\rho_{23} = \frac{-1}{\sigma_2\sigma_3}\left[\sqrt{\left(\sigma_2^2 - \frac{1}{r_3^2}\right)\left(\sigma_3^2 - \frac{1}{r_3^2}\right)} - \frac{1}{r_3^2}\right]$$

Case 7 $\quad \overline{c}_1 = \overline{c}_2 = \overline{c}_3 = c$ and

$$b_3 = \frac{-2\overline{c}_1}{\sigma_1\left|\overline{u_1}\right|} + \frac{-2\overline{c}_2}{\sigma_2\left|\overline{u_2}\right|} + \frac{-2\overline{c}_3}{\sigma_3\left|\overline{u_3}\right|} = -c\left(\frac{2}{\sqrt{\sigma_1^2 r_3^2 - 1}} + \frac{2}{\sqrt{\sigma_2^2 r_3^2 - 1}} + \frac{2}{\sqrt{\sigma_3^2 r_3^2 - 1}}\right)$$

$$\text{iff } (D_1, D_2, D_3) = (2,2,2) \text{ and } \rho_{12} = \frac{-1}{\sigma_1\sigma_2}\left[\sqrt{\left(\sigma_1^2 - \frac{1}{r_3^2}\right)\left(\sigma_2^2 - \frac{1}{r_3^2}\right)} - \frac{1}{r_3^2}\right],$$

$$\rho_{13} = \frac{-1}{\sigma_1\sigma_3}\left[\sqrt{\left(\sigma_1^2 - \frac{1}{r_3^2}\right)\left(\sigma_3^2 - \frac{1}{r_3^2}\right)} - \frac{1}{r_3^2}\right] \text{and } \rho_{23} = \frac{-1}{\sigma_2\sigma_3}\left[\sqrt{\left(\sigma_2^2 - \frac{1}{r_3^2}\right)\left(\sigma_3^2 - \frac{1}{r_3^2}\right)} - \frac{1}{r_3^2}\right]$$

□

For each case we have identified at least one of the correlation coefficients. To identify the remaining correlation coefficients under each of the seven cases, we will use the following definitions:

$$g_1(t) \equiv \; -f_3(t) - \sum_{i=1}^{3} \frac{1}{\sigma_i}\phi\left(\frac{t-\mu_i}{\sigma_i}\right) D_i P\left(Z \le \overline{u_i}\, t + \overline{\delta_i}\right)$$

$$r_{3\bullet 1}^2 \equiv -2 \lim_{t\to-\infty}\left(\frac{\ln\left|g_1(t)\right|}{t^2}\right)$$

$$g_{(1)}(t) \equiv g_1(t) \;\; \exp\left[\frac{1}{2}t^2 r_{3\bullet 1}^2\right]$$

$$s_{3\bullet 1} \equiv -\lim_{t\to-\infty}\left(\frac{\ln\left|g_{(1)}(t)\right|}{t}\right)$$

$$\overline{u_{i\bullet 1}} \equiv \sqrt{r_{3\bullet 1}^2 - \frac{1}{\sigma_i^2}}\;,\; \overline{\delta_{i\bullet 1}} = \frac{s_{3\bullet 1} - \frac{\mu_i}{\sigma_i^2}}{\sqrt{r_{3\bullet 1}^2 - \frac{1}{\sigma_i^2}}} \;\; \overline{c}_{i\bullet 1} = \exp\left[\frac{-1}{2}\left(\frac{\mu_i^2}{\sigma_i^2} + \overline{\delta}_{i\bullet 1}^2\right)\right]$$

Case 1: $(D_1, D_2, D_3) = (1,1,0)$

Occurs when $\bar{c}_1 = \bar{c}_2 \neq \bar{c}_3$ or

$$\bar{c}_1 = \bar{c}_2 = \bar{c}_3 \text{ and } b_3 = \frac{-\bar{c}_1}{\sigma_1 |\overline{u_1}|} + \frac{-\bar{c}_2}{\sigma_2 |\overline{u_2}|}$$

From Theorem 3.2 we can conclude

$$\rho_{12} = \frac{-1}{\sigma_1 \sigma_2}\left[\sqrt{\left(\sigma_1^2 - \frac{1}{r_3^2}\right)\left(\sigma_2^2 - \frac{1}{r_3^2}\right)} - \frac{1}{r_3^2}\right]$$

By corollary 2.6.3 we find as $t \to -\infty$

1st term of

$$g_1(t) \sim \frac{1}{\sigma_1 |\alpha_{31} t + \beta_{31}| 2\pi} \exp\left\{-\frac{1}{2}\left[\left(\frac{1}{\sigma_1^2} + \alpha_{31}^2\right)t^2 + 2\left(\alpha_{31}\beta_{31} - \frac{\mu_1}{\sigma_1^2}\right)t + \frac{\mu_1^2}{\sigma_1^2} + \beta_{31}^2\right]\right\}$$

2nd term of

$$g_1(t) \sim \frac{1}{\sigma_2 |\alpha_{32} t + \beta_{32}| 2\pi} \exp\left\{-\frac{1}{2}\left[\left(\frac{1}{\sigma_2^2} + \alpha_{32}^2\right)t^2 + 2\left(\alpha_{32}\beta_{32} - \frac{\mu_2}{\sigma_2^2}\right)t + \frac{\mu_2^2}{\sigma_2^2} + \beta_{32}^2\right]\right\}$$

3rd term of $g_1(t) \sim \dfrac{-C_3}{\sigma_3 |u_3 t + \delta_3| 2\pi} \exp\left\{-\dfrac{1}{2}\left[\left(\dfrac{1}{\sigma_3^2} + u_3^2\right)t^2 + 2\left(u_3\delta_3 - \dfrac{\mu_3}{\sigma_3^2}\right)t + \dfrac{\mu_3^2}{\sigma_3^2} + \delta_3^2\right]\right\}$

where C_3 is as in (3.33).

We now form the vector $(D_{1\bullet 1}, D_{2\bullet 1}, D_{3\bullet 1})$ where

$$D_{1\bullet 1} = \begin{Bmatrix} 1 \text{ if term 1 is a dominating term} \\ 0 \text{ otherwise} \end{Bmatrix} \quad D_{2\bullet 1} = \begin{Bmatrix} 1 \text{ if term 2 is a dominating term} \\ 0 \text{ otherwise} \end{Bmatrix}$$

$$D_{3\bullet 1} = \begin{Bmatrix} 1 \text{ if } D_{1\bullet 1} \neq D_{2\bullet 1} \\ 2 \text{ otherwise} \end{Bmatrix}$$

and since

$u_3 = \min\{\alpha_{13}, \alpha_{23}\}$ and $\delta_3 = \begin{Bmatrix} \beta_{13} & \text{if term 1 is dominating} \\ \beta_{23} & \text{otherwise} \end{Bmatrix}$ we have

$$r_{3\bullet 1}^2 = \min\{A_{13}, A_{23}\}$$

$$s_{3\bullet 1} = \begin{Bmatrix} \alpha_{13}\beta_{13} - \dfrac{\mu_3}{\sigma_3^2} & \text{when term 1 is dominating} \\ \\ \alpha_{23}\beta_{23} - \dfrac{\mu_3}{\sigma_3^2} & \text{otherwise} \end{Bmatrix}$$ So, we can write

$$g_1(t) \sim \left(\sum_{j=1}^{3} \frac{D_{j\bullet 1}\bar{c}_{j\bullet 1}}{\sigma_j \left| u_{j\bullet 1} t + \delta_{j\bullet 1} \right| 2\pi} \right) \exp\left\{ -\frac{1}{2}\left[r_{3\bullet 1}^2 t^2 + 2 s_{3\bullet 1} t \right] \right\}$$

Observe that term 3 of $g_1(t)$ must be a dominating term and at least one of the other terms must be dominating. There are three possible outcomes:

1. Only term 1 and term 3 are dominating terms so that $(D_{1\bullet 1}, D_{2\bullet 1}, D_{3\bullet 1}) = (1,0,1)$ and $\bar{c}_{1\bullet 1} = \bar{c}_{3\bullet 1}$. We will call this outcome subcase 1A.
2. Only term 2 and term 3 are dominating terms so that $(D_{1\bullet 1}, D_{2\bullet 1}, D_{3\bullet 1}) = (0,1,1)$ and $\bar{c}_{2\bullet 1} = \bar{c}_{3\bullet 1}$. We will call this outcome subcase 1B
3. All three terms are dominating terms so that $(D_{1\bullet 1}, D_{2\bullet 1}, D_{3\bullet 1}) = (1,1,2)$ and $\bar{c}_{1\bullet 1} = \bar{c}_{2\bullet 1} = \bar{c}_{3\bullet 1}$. We will call this outcome subcase 1C

We now define $b_{3\bullet 1} \equiv \lim_{t\to -\infty}\left(\left| g_1(t) \right| 2\pi |t| \exp\frac{1}{2}\left[r_{3\bullet 1}^2 t^2 + 2 s_{3\bullet 1} t \right] \right) = \sum_{j=1}^{3} \frac{-D_{j\bullet 1}\bar{c}_{j\bullet 1}}{\sigma_j \left| u_{j\bullet 1} \right|}$

and when $\bar{c}_{1\bullet 1} = \bar{c}_{2\bullet 1} = \bar{c}_{3\bullet 1}$ we define

$$\bar{b}_{3\bullet 1} \equiv \sum_{j=1}^{3} \frac{D_j \bar{c}_j}{\sigma_j \left| u_j \right|} = \bar{c}\left(\frac{D_{1\bullet 1}}{\sqrt{\sigma_1^2 r_{3\bullet 1}^2 - 1}} + \frac{D_{2\bullet 1}}{\sqrt{\sigma_2^2 r_{3\bullet 1}^2 - 1}} + \frac{D_{3\bullet 1}}{\sqrt{\sigma_3^2 r_{3\bullet 1}^2 - 1}} \right)$$ where $\bar{c} = \bar{c}_{1\bullet 1} = \bar{c}_{2\bullet 1} = \bar{c}_{3\bullet 1}$.

Observe that $\bar{b}_{3\bullet 1}$ has a different value for each of the three sub-cases. For the correct sub-case $\bar{b}_{3\bullet 1} =. b_{3\bullet 1}$ Thus when $\bar{c}_{1\bullet 1} = \bar{c}_{2\bullet 1} = \bar{c}_{3\bullet 1}$, case can be determined using $\bar{b}_3$ and b_3.

Similar to Theorem 3.2 we have:

Subcase 1A : $(D_{1\bullet 1}, D_{2\bullet 1}, D_{3\bullet 1}) = (1,0,1)$ if and only if

$\bar{c}_{1\bullet 1} = \bar{c}_{3\bullet 1} \neq \bar{c}_{2\bullet 1}$ or

$\bar{c}_{1\bullet 1} = \bar{c}_{2\bullet 1} = \bar{c}_{3\bullet 1}$ and $b_{3\bullet 1} = \dfrac{\bar{c}_1}{\sigma_1 \left|\overline{u_{1\bullet 1}}\right|} + \dfrac{\bar{c}_3}{\sigma_3 \left|\overline{u_{3\bullet 1}}\right|}$

Here only terms 1 and 3 are dominating. so that $\overline{u_{1\bullet 1}} = \alpha_{31}$, $\overline{u_{3\bullet 1}} = \alpha_{13}$, $r_{3\bullet 1}^2 = A_{13}$

$$\Rightarrow \rho_{13} = \frac{-1}{\sigma_1 \sigma_3}\left[\sqrt{\left(\sigma_1^2 - \frac{1}{r_{3\bullet 1}^2}\right)\left(\sigma_3^2 - \frac{1}{r_{3\bullet 1}^2}\right)} - \frac{1}{r_{3\bullet 1}^2}\right]$$

To identify ρ_{23} , define $g_2(t) \equiv \; g_1(t) - \sum_{i=1}^{3} \dfrac{1}{\sigma_i} \phi\left(\dfrac{t-\mu_i}{\sigma_i}\right) D_{i\bullet 1} P\left(Z \leq \overline{u_{i\bullet 1}}\, t + \overline{\delta_{i\bullet 1}}\right)$

Again, by corollary 2.6.3 we find as $t \to -\infty$

1^{st} term of $g_2(t) \sim \dfrac{D_1}{\sigma_1 (2\pi)^{\frac{3}{2}} t^2} \exp\left\{-\dfrac{1}{2}\left[\left(\dfrac{1}{\sigma_1^2} + a_1^2\right)t^2 + 2\left(b_1 - \dfrac{\mu_1}{\sigma_1^2}\right)t + c_1 + \dfrac{\mu_1^2}{\sigma_1^2}\right]\right\}$

Where parameters D_1, a_1, b_1, c_1 are as defined in corollary 2.6.3.

2^{nd} term

of $g_2(t) \sim \dfrac{1}{\sigma_2 |\alpha_{32} t + \beta_{32}| 2\pi} \exp\left\{-\dfrac{1}{2}\left[\left(\dfrac{1}{\sigma_2^2} + \alpha_{32}^2\right)t^2 + 2\left(\alpha_{32}\beta_{32} - \dfrac{\mu_2}{\sigma_2^2}\right)t + \dfrac{\mu_2^2}{\sigma_2^2} + \beta_{32}^2\right]\right\}$

3^{rd} term

of $g_2(t) \sim \dfrac{1}{\sigma_3 |\alpha_{23} t + \beta_{23}| 2\pi} \exp\left\{-\dfrac{1}{2}\left[\left(\dfrac{1}{\sigma_3^2} + \alpha_{23}^2\right)t^2 + 2\left(\alpha_{23}\beta_{23} - \dfrac{\mu_3}{\sigma_3^2}\right)t + \dfrac{\mu_3^2}{\sigma_3^2} + \beta_{23}^2\right]\right\}$

Now define $r_{3\bullet 2}^2 \equiv -2 \lim_{t\to-\infty}\left(\frac{\ln g_2(t)}{t^2}\right)$, $g_{(2)}(t) \equiv g_2(t)\, e^{\left[\frac{1}{2}t^2 r_{3\bullet 2}^2\right]}$, $s_{3\bullet 2} \equiv -\lim_{t\to-\infty}\left(\frac{g_{(2)}(t)}{t}\right)$

$$b_{3\bullet 2} \equiv \lim_{t\to-\infty}\left(g_2(t) 2\pi t^2 \exp\frac{1}{2}\left[r_{3\bullet 2}^2 t^2 + 2s_{3\bullet 2}t\right]\right)$$

<u>*Case 1:*</u> $b_{3\bullet 2}$ is infinite

Terms 2 and 3 dominate: $r_{3\bullet 2}^2 = A_{23} \Rightarrow \rho_{23} = \frac{-1}{\sigma_2\sigma_3}\left[\sqrt{\left(\sigma_2^2 - \frac{1}{r_{3\bullet 2}^2}\right)\left(\sigma_3^2 - \frac{1}{r_{3\bullet 2}^2}\right)} - \frac{1}{r_{3\bullet 2}^2}\right]$

<u>*Case 2:*</u> $b_{3\bullet 2}$ is finite

Term 1 dominates: $r_{3\bullet 2}^2 = \frac{1}{\sigma_1^2} + a_1 = \frac{1}{\sigma_1^2} + \frac{\alpha_{21}^2 - 2\rho_{23\bullet 1}\alpha_{21}\alpha_{31} + \alpha_{31}^2}{1-\rho_{23\bullet 1}^2}$

Where $\rho_{23\bullet 1} = \frac{\rho_{23} - \rho_{12}\rho_{13}}{\sqrt{1-\rho_{21}^2}\sqrt{1-\rho_{31}^2}}$ hence ρ_{23} is identified"

Subcase 1B : $(D_{1\bullet 1}, D_{2\bullet 1}, D_{3\bullet 1}) = (0,1,1)$ if and only if

$\bar{c}_{2\bullet 1} = \bar{c}_{3\bullet 1} \neq \bar{c}_{1\bullet 1}$ or

$\bar{c}_{1\bullet 1} = \bar{c}_{2\bullet 1} = \bar{c}_{3\bullet 1}$ and $b_{3\bullet 1} = \frac{\bar{c}_2}{\sigma_2\left|\overline{u_{2\bullet 1}}\right|} + \frac{\bar{c}_3}{\sigma_3\left|\overline{u_{3\bullet 1}}\right|}$

ρ_{13} and ρ_{23} are found using similar procedures as in *Subcase 1A*

Subcase 1C : $(D_{1\bullet 1}, D_{2\bullet 1}, D_{3\bullet 1}) = (1,1,2)$ if and only if

$\bar{c}_{1\bullet 1} = \bar{c}_{2\bullet 1} = \bar{c}_{3\bullet 1}$ and $b_{3\bullet 1} = \frac{\bar{c}_1}{\sigma_1\left|\overline{u_{1\bullet 1}}\right|} + \frac{\bar{c}_2}{\sigma_2\left|\overline{u_{2\bullet 1}}\right|} + \frac{2\bar{c}_3}{\sigma_3\left|\overline{u_{3\bullet 1}}\right|}$

Here $\overline{u_{3\bullet1}} = \alpha_{13} = \alpha_{23}$. $r_{3\bullet1}^2 = A_{13} = A_{23}$

$$\Rightarrow \rho_{13} = \frac{-1}{\sigma_1\sigma_3}\left[\sqrt{\left(\sigma_1^2 - \frac{1}{r_{3\bullet1}^2}\right)\left(\sigma_3^2 - \frac{1}{r_{3\bullet1}^2}\right)} - \frac{1}{r_{3\bullet1}^2}\right]$$

and $\rho_{23} = \dfrac{-1}{\sigma_2\sigma_3}\left[\sqrt{\left(\sigma_2^2 - \frac{1}{r_{3\bullet1}^2}\right)\left(\sigma_3^2 - \frac{1}{r_{3\bullet1}^2}\right)} - \frac{1}{r_{3\bullet1}^2}\right]$

Case 2: $(D_1, D_2, D_3) = (1,0,1)$ and ***Case 3***: $(D_1, D_2, D_3) = (0,1,1)$

We use similar procedures to those used in ***Case 1*** .

Case 4: $(D_1, D_2, D_3) = (2,1,1)$

Occurs when $\bar{c}_1 = \bar{c}_2 = \bar{c}_3$ and $b_3 = \dfrac{-2\bar{c}_1}{\sigma_1|\overline{u_1}|} + \dfrac{-\bar{c}_2}{\sigma_2|\overline{u_2}|} + \dfrac{-\bar{c}_3}{\sigma_3|\overline{u_3}|}$

Here $u_1 = \alpha_{21} = \alpha_{31}$, $u_2 = \alpha_{12}$, $u_3 = \alpha_{13}$

$$r_{3\bullet1}^2 = A_{12} = A_{13},\ s_3 = B_{12} = B_{13} \Rightarrow \rho_{12} = \frac{-1}{\sigma_1\sigma_2}\left[\sqrt{\left(\sigma_1^2 - \frac{1}{r_3^2}\right)\left(\sigma_2^2 - \frac{1}{r_3^2}\right)} - \frac{1}{r_3^2}\right],$$

$$\rho_{13} = \frac{-1}{\sigma_1\sigma_3}\left[\sqrt{\left(\sigma_1^2 - \frac{1}{r_3^2}\right)\left(\sigma_3^2 - \frac{1}{r_3^2}\right)} - \frac{1}{r_3^2}\right],$$

as $t \to -\infty$

1st term of $g_1(t) \sim \dfrac{D_1}{\sigma_1(2\pi)^{\frac{3}{2}}t^2}\exp\left\{-\dfrac{1}{2}\left[\left(\dfrac{1}{\sigma_1^2} + a_1^2\right)t^2 + 2\left(b_1 - \dfrac{\mu_1}{\sigma_1^2}\right)t + c_1 + \dfrac{\mu_1^2}{\sigma_1^2}\right]\right\}$

Where parameters D_1, a_1, b_1, c_1 are as defined in corollary 2.6.3.

2^nd^ term of $g_1(t)$

$$\sim \frac{1}{\sigma_2|\alpha_{32}t+\beta_{32}|2\pi}\exp\left\{-\frac{1}{2}\left[\left(\frac{1}{\sigma_2^2}+\alpha_{32}^2\right)t^2+2\left(\alpha_{32}\beta_{32}-\frac{\mu_2}{\sigma_2^2}\right)t+\frac{\mu_2^2}{\sigma_2^2}+\beta_{32}^2\right]\right\}$$

3^rd^ term of $g_1(t)$

$$\sim \frac{1}{\sigma_3|\alpha_{23}t+\beta_{23}|2\pi}\exp\left\{-\frac{1}{2}\left[\left(\frac{1}{\sigma_3^2}+\alpha_{23}^2\right)t^2+2\left(\alpha_{23}\beta_{23}-\frac{\mu_3}{\sigma_3^2}\right)t+\frac{\mu_3^2}{\sigma_3^2}+\beta_{23}^2\right]\right\}$$

Continue as in subcase 1A to identify ρ_{23} since the terms of $g_1(t)$ here are the same as the terms of $g_2(t)$ in subcase 1A.

Case 5: $(D_1, D_2, D_3) = (1,2,1)$ and ***Case 6***: $(D_1, D_2, D_3) = (1,1,2)$

We use similar procedures to those used in ***Case 4*** .

***Case** 7*: $(D_1, D_2, D_3) = (2,2,2)$

$$u_1 = \alpha_{21} = \alpha_{31},\ u_2 = \alpha_{12} = \alpha_{32},\ u_3 = \alpha_{13} = \alpha_{23}$$

$$r_3^2 = A_{12} = A_{13} = A_{23},\ s_3 = B_{12} = B_{13} = B_{23} \Rightarrow$$

$$\rho_{12} = \frac{-1}{\sigma_1\sigma_2}\left[\sqrt{\left(\sigma_1^2-\frac{1}{r_3^2}\right)\left(\sigma_2^2-\frac{1}{r_3^2}\right)}-\frac{1}{r_3^2}\right],$$

$$\rho_{13} = \frac{-1}{\sigma_1\sigma_3}\left[\sqrt{\left(\sigma_1^2-\frac{1}{r_3^2}\right)\left(\sigma_3^2-\frac{1}{r_3^2}\right)}-\frac{1}{r_3^2}\right],$$

and $\rho_{23} = \frac{-1}{\sigma_2 \sigma_3} \left[\sqrt{\left(\sigma_2^2 - \frac{1}{r_3^2} \right)\left(\sigma_3^2 - \frac{1}{r_3^2} \right)} - \frac{1}{r_3^2} \right].$

Chapter 4
Examples

Example 1

Let X_1, X_2, X_3 be iid N(0,1) and let $a > 0,\ b > 0,\ c > 0,$ and $a > bc$

Define

$$
\begin{aligned}
Y_1 &= X_1 - aX_2 - bX_3 \\
Y_2 &= X_2 - cX_3 \\
Y_3 &= X_3
\end{aligned}
$$

Then

$$
\begin{aligned}
\operatorname{cov}(Y_1, Y_2) &= E[(X_1 - aX_2 - bX_3)(X_2 - cX_3)] \\
&= \mathrm{E}(X_1X_2) - c\mathrm{E}(X_1X_3) - a\mathrm{E}(X_2X_2) + ac\mathrm{E}(X_2X_3) - b\mathrm{E}(X_2X_3) + bc\mathrm{E}(X_3X_3) \\
&= -a + bc < 0 \\
\operatorname{cov}(Y_1, Y_3) &= E[(X_1 - aX_2 - bX_3)(X_3)] \\
&= \mathrm{E}(X_1X_3) - a\mathrm{E}(X_2X_3) - b\mathrm{E}(X_3X_3) \\
&= -b \\
\operatorname{cov}(Y_2, Y_3) &= E[(X_2 - cX_3)(X_3)] \\
&= \mathrm{E}(X_2X_3) - c\mathrm{E}(X_3X_3) \\
&= -c
\end{aligned}
$$

So that (Y_1, Y_2, Y_3) is tri-variate normal with negative correlation coefficients.

Example 2

Let (X, Y, Z) be non-singular tri-variate normal with zero means and co-variances all positive. Let $b_1, b_2, b_3, d_1, d_2, s^2$ be positive constants and $a_1, a_2, a_3, c_1, c_2, c_3$ be negative constants such that

$$\begin{aligned}
&d_1 < \min\{\text{all the entries of the co-variance matrix of } (\mathrm{X}, \mathrm{Y}, \mathrm{Z})\} \\
&d_2 > \max\{\text{all the entries of the co-variance matrix of } (\mathrm{X}, \mathrm{Y}, \mathrm{Z})\} \\
&-d_2(a_1 + a_2 + a_3) < d_1(b_1 + b_2 + b_3) \\
&d_2[(a_1 + a_2 + a_3)(c_1 + c_2 + c_3)] < s^2 < -d_1[(b_1 + b_2 + b_3)(c_1 + c_2 + c_3)]
\end{aligned} \tag{4.1}$$

Now define

$$\begin{aligned}
X' &= a_1 X - a_2 Y - a_3 Z \\
Y' &= b_1 X - b_2 Y - b_3 Z \\
Z' &= c_1 X - c_2 Y - c_3 Z
\end{aligned}$$

Let $W \sim N(0, s^2)$ and independent of (X, Y, Z). It follows that W is independent of (X', Y', Z'). Finally, define

$$\begin{aligned}
X'' &= X' + W \\
Y'' &= Y' - W \\
Z'' &= Z' - W
\end{aligned}$$

(X'', Y'', Z'') is tri-variate normal. We now show that the co-variances are negative.

Observe that

$$\begin{aligned}
\mathrm{E}(X'Z') &= E\left[(a_1c_1X^2 + a_2c_2Y^2 + a_3c_3Z^2) + (a_1c_2 + a_2c_1)XY + (a_1c_3 + a_3c_1)XZ + (a_2c_3 + a_3c_2)YZ\right] \\
&< d_2[(a_1c_1 + a_2c_2 + a_3c_3) + (a_1c_2 + a_2c_1) + (a_1c_3 + a_3c_1) + (a_2c_3 + a_3c_2)] \qquad (4.2) \\
&= d_2[(a_1 + a_2 + a_3)(c_1 + c_2 + c_3)] \\
&\text{since } d_2 > \max\{EX^2, EY^2, EZ^2, EXY, EXZ, EYZ\}
\end{aligned}$$

Also

$$
\begin{aligned}
\mathrm{E}(Y'Z') &= E\left[\left(b_1c_1X^2 + b_2c_2Y^2 + b_3c_3Z^2\right) + (b_1c_2 + b_2c_1)XY + (b_1c_3 + b_3c_1)XZ + (b_2c_3 + b_3c_2)YZ\right] \\
&< d_1\left[(a_1 + a_2 + a_3)(b_1 + b_2 + b_3)\right] \qquad (4.3) \\
&\text{since } d_1 < \min\left\{EX^2, EY^2, EZ^2, EXY, EXZ, EYZ\right\} \\
&\text{and } (a_1 + a_2 + a_3)(b_1 + b_2 + b_3) < 0
\end{aligned}
$$

Thus yielding the inequality

$$
\mathrm{E}(X'Z') < s^2 < -\mathrm{E}(Y'Z') \qquad (4.4)
$$

$$
\begin{aligned}
\operatorname{cov}(X'', Y'') &= E\left[(X' + W)(Y' - W)\right] \\
&= \mathrm{E}(X'Y') - \mathrm{E}X'\mathrm{E}W + \mathrm{E}W\mathrm{E}Y' - EW^2 \\
&= \left[a_1b_1EX^2 + a_2b_2EY^2 + a_3b_3EZ^2 + (a_1b_2 + a_2b_1)EXY + (a_1b_3 + a_3b_1)EXZ + (a_2b_3 + a_3b_2)EYZ\right] \\
&\quad - \left[(a_1 + a_2 + a_3)\cdot 0\right] + \left[0 \cdot (b_1 + b_2 + b_3)\right] - s^2 \\
&< 0 \qquad \text{since } a_ib_j < 0 \text{ when } 1 \le i, j \le 3
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{cov}(X'', Z'') &= E\left[(X' + W)(Z' - W)\right] \\
&= E(X'Z') - EX'EW + EWEZ' - EW^2 \\
&= E(X'Z') - s^2 \\
&< 0 \qquad \text{since } E(X'Z') < s^2 \text{ has been established}
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{cov}(Y'', Z'') &= E\left[(Y' - W)(Z' - W)\right] \\
&= E(Y'Z') + s^2 \\
&< 0 \qquad \text{since } \text{-}E(Y'Z') > s^2 \text{ has been established}
\end{aligned}
$$

Example 3

Let $(X_1, X_2, X_3) \sim TVN(\vec{0}, \Sigma)$ where $\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_2^2 & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_3^2 \end{pmatrix}$ is a full-rank covariance matrix with $1 < \sigma_{12} < \dfrac{\sigma_1 \sigma_2}{2}$. Now define

$$\begin{aligned} Y_1 &= aX_1 \\ Y_2 &= -X_1 + bX_2 \qquad (4.4) \\ Y_3 &= -X_1 + cX_2 + dX_3 \end{aligned}$$

where

$$\begin{aligned} & a > 0, b = \frac{\sigma_1^2}{2\sigma_{12}}, c = -\left(\frac{\sigma_1^2}{4\sigma_{12}} + \frac{\sigma_{12}}{\sigma_2^2}\right) \\ & and\ \ d = \min\left\{\left|\frac{-\sigma_1^2 + c\sigma_{12}}{2\sigma_{13}}\right|, \left|\frac{\sigma_1^2 + bc\sigma_2^2 - (b+c)\sigma_{12}}{2(b\sigma_{23} - \sigma_{13})}\right|\right\} \end{aligned}$$

Observe that

$$\operatorname{cov}(Y_1, Y_2) = -a\sigma_1^2 + a\left(\frac{\sigma_1^2}{2\sigma_{12}}\right)\sigma_{12} = -\frac{1}{2}\sigma_1^2 < 0 \qquad (4.5)$$

$$\operatorname{cov}(Y_1, Y_3) = -a\sigma_1^2 + ac\sigma_{12} + ad\sigma_{13} \qquad (4.6)$$

since $-\sigma_1^2 + c\sigma_{12} < 0$, when $\sigma_{13} \le 0$, $\operatorname{cov}(Y_1, Y_3) < 0$

Also when $\sigma_{13} > 0$,

$$\operatorname{cov}(Y_1, Y_3) \le a\left[\left(-\sigma_1^2 + c\sigma_{12}\right) + \left|\frac{-\sigma_1^2 + c\sigma_{12}}{2\sigma_{13}}\right|\sigma_{13}\right] < 0 \quad (4.7)$$

$$\operatorname{cov}(Y_2,Y_3) = \sigma_1^2 + bc\sigma_2^2 - c\sigma_{12} - b\sigma_{12} + d(-\sigma_{13} + b\sigma_{23}) \quad (4.8)$$

Notice $\sigma_1^2 - c\sigma_{12} > 0$ while $bc\sigma_2^2 - b\sigma_{12} < 0$

also since $\sigma_{12} < \dfrac{\sigma_1\sigma_2}{2}$, we have

a) $\dfrac{\sigma_{12}^2}{\sigma_2^2} < \dfrac{\sigma_1^2}{4}$, and b) $\dfrac{\sigma_1\sigma_2}{2\sigma_{12}} > 1$

so that

$$\begin{aligned}
&\sigma_1^2 + bc\sigma_2^2 - c\sigma_{12} - b\sigma_{12} \\
&= \left[\sigma_1^2 - c\sigma_{12}\right] + \left[bc\sigma_2^2 - b\sigma_{12}\right] \\
&= \left[\sigma_1^2 + \frac{\sigma_1^2}{4} + \frac{\sigma_{12}^2}{\sigma_2^2}\right] + \left[\frac{-\sigma_1^2}{2}\left(\frac{\sigma_1\sigma_2}{2\sigma_{12}}\right)^2 - \sigma_1^2\right] < 0
\end{aligned}$$

It now follows that when $-\sigma_{13} + b\sigma_{23} < 0$

$\operatorname{cov}(Y_2,Y_3) < 0$

When $-\sigma_{13} + b\sigma_{23} > 0$

$$\begin{aligned}
\operatorname{cov}(Y_2,Y_3) &= \sigma_1^2 + bc\sigma_2^2 - c\sigma_{12} - b\sigma_{12} + d(-\sigma_{13} + b\sigma_{23}) \\
&\le \sigma_1^2 + bc\sigma_2^2 - c\sigma_{12} - b\sigma_{12} + \left|\frac{\sigma_1^2 + bc\sigma_2^2 - (b+c)\sigma_{12}}{2(b\sigma_{23} - \sigma_{13})}\right|(-\sigma_{13} + b\sigma_{23}) \\
&< 0
\end{aligned}$$

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