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The Leray-Schauder Approach for the Degree of Perturbed Maximal Monotone

Ibrahimou Boubakari
University of South Florida

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The Leray-Schauder Approach for the Degree of Perturbed Maximal Monotone

Operators

by

Ibrahimou Boubakari

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
Department of Mathematics
College of Arts and Sciences
University of South Florida

Major Professor: Athanassios Kartsatos, Ph.D.
Marcus McWaters, Ph.D.
Arunava Mukherjea, Ph.D.
Boris Shekhtman, Ph.D.
Yuncheng You, Ph.D.

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DEDICATION

To the memory of my mother Madji Nassourou
To my father Yerima Boubakari Ahmadou

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TABLE OF CONTENTS

Abstract	iii
Introduction	1
1 Preliminaries	4
1.1 Mappings of Monotone Type	4
1.2 Definition of the Topological Degree	9
2 Construction of the Degree	11
2.1 Degree for Bounded Perturbations of Type (S_+)	11
2.2 Basic Properties of the Degree	24
2.3 Degree for Unbounded Perturbation of Type (S_+)	39
2.4 Degree for Quasimonotone Perturbation	42
3 Applications	46
3.1 The Subdifferential Operator	46
3.2 Application in Partial Differential Equations	48
3.3 Invariance of Domain	57
3.4 Eigenvalue Problem	68
4 Existence and Surjectivity Result	76
4.1 Noncoercive Mappings	76
4.2 Odd Mappings	84
References	88

THE LERAY-SCHAUDER APPROACH FOR THE DEGREE OF PERTURBED MAXIMAL
MONOTONE OPERATORS

IBRAHIMOU BOUBAKARI

ABSTRACT

In this work, we demonstrate that the Leray-Schauder topological degree theory can be used for the development of a topological degree theory for maximal monotone perturbations of demicontinuous operators of type (S_+) in separable reflexive Banach spaces. This is an extension of Berkovits' degree development for operators as the perturbations above.

Berkovits has developed a topological degree for demicontinuous mappings of type (S_+) , and has shown that the degree mapping is unique under the assumption that it satisfies certain general properties. He proved that if f is a bounded demicontinuous mapping of type (S_+) , G is an open bounded subset of X , and $0 \notin f(\partial G)$, then there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ we have $0 \notin (I + \frac{1}{\varepsilon}QQ^*(f))(\partial G)$. Here, Q is a compact linear injection from a Hilbert space H into X , such that $Q(H)$ is dense in X , and Q^* its adjoint. The map $I + \frac{1}{\varepsilon}QQ^*(f)$ is a compact displacement of the identity, for which the Leray-Schauder degree is well defined. The Berkovits degree is obtained as the limit of this Leray-Schauder degree as ε tends to zero. We utilize a demicontinuous (S_+) -approximation of the form $T_t + f$, where T_t is the Yosida approximant of T . Namely, we show that if G is an open bounded set in X and $0 \notin (T + f)(\partial G)$, then there exist $\varepsilon_0 > 0$, $t_0 > 0$, such that for every $\varepsilon \in (0, \varepsilon_0)$, $t \in (0, t_0)$, we have $0 \notin (I + \frac{1}{\varepsilon}QQ^*(T_t + f))(\partial G)$. Our degree is the limit of the Leray-Schauder degree

of the compact displacement of the identity $I + \frac{1}{\varepsilon}QQ^*(T_t + f)$ as $\varepsilon, t \rightarrow 0$. Various extension of the degree has been considered. Finally some properties and applications in invariance of domain, eigenvalue and surjectivity results have also been discussed.

INTRODUCTION

The solvability of many problems in mathematics and physics can be reduced to the study of the set of solutions of an equation of the form $F(x) = y$ where F is a mapping between some appropriate spaces X and Y , and y is a fixed element of Y . One of the powerful tools used for the solvability of such an equation is a topological degree theory, which is a fundamental concept in both Algebraic Topology and Analysis. Topological degree theories are often used in the study of problems of ordinary and partial differential equations involving general existence and multiplicity of solutions, bifurcation theory and fixed point theory. Degree theories are also used in other types of existence problems involving ranges of sums of nonlinear operators, invariance of domain and eigenvalues.

The solvability of such problems is often achieved by estimating values of the “degree” function $d(F, G, y)$, which is always an integer or zero. Here, G is an open bounded subset of X and $y \notin F(\partial G)$. The integer-valued function d is said to be a topological degree if it satisfies certain normalizing, additivity and homotopy properties. It turns out that if this degree is not zero, then the equation $F(x) = y$ has a solution $x \in G$. The most important property of a degree function is the homotopy invariance, which was originally developed by Henri Poincaré and which consists of embedding the problem in a parametrized family of problems and then showing that such parametric families of operators are, in a sense, homotopic to operators with known degrees.

The notion of “topological degree” was first introduced by Brouwer in 1912 [7] for continuous mappings between finite dimensional spaces. This theory was generalized in 1934 by Leray and Schauder for operators of the type $I - T$ in infinite dimensional

spaces. Such operators, with I the identity mapping and T compact, are called “compact displacements of the identity”. They developed the fundamental theory of degree in their paper [36], which also contains striking applications to nonlinear integral equations as well as Dirichlet problems. Although the theory was introduced early in the beginning of the century, its applicability did not become very obvious until 1950 when additional theoretical progress was made by Leray himself, Rothe, Tychonoff and Nagumo. A great number of applications of the Leray Schauder degree theory to various problems of existence, multiplicity and bifurcation of solutions to nonlinear equations in Banach spaces has already been made in the past five decades. One such application is by Krasnosel’skii, who initiated the topological approach to bifurcation theory. It was until fairly recently, in 1972 and 1973, when Führer [19], and Amann and Weiss [2] showed that the Brouwer degree function is uniquely determined by just a few conditions. These conditions provide a natural basis for the formal definition of a classical topological degree.

There have been various extensions and generalizations of the Leray-Schauder degree theory in different directions. For example, Skrypnik developed in [49] a topological degree for bounded demicontinuous mappings of class (S_+) , which map an open bounded subset of a Banach space X into its dual X^* . On the other hand, Browder [8] developed a degree theory for demicontinuous (S_+) -perturbations of maximal monotone operators. This Browder degree is obtained as the limit of the Skrypnik degrees for associated bounded demicontinuous (S_+) -mappings. Kartsatos and Skrypnik developed degree theories in [26] for densely defined mappings of monotone type.

In this paper we are concentrating on the demonstration of the fact that the Leray-Schauder topological degree theory can be used for the development of a topological degree theory for maximal monotone perturbations of demicontinuous operators of type (S_+) in separable reflexive Banach spaces. This is an extension of the corresponding Berkovits [5] degree development for just demicontinuous operators of type (S_+) . It is also a variant approach to the associated Browder degree described above. We also demonstrate the possible applicability of our results in the field of Partial Differential Equations and invariance of domain and eigenvalue problems. Since the

importance of a degree theory lies in the degree of generality of its associated homotopies, we discuss some admissible homotopies for our degree.

Our study is structured as follows. In chapter one we give some preliminaries and definitions needed for our study. Chapter two is devoted for the development of our degree theory for various perturbations of maximal monotone mapping and the study of some admissible homotopies. Chapter three deals with applications in the field of Partial Differential Equations, the discussion of problems of invariance of domain and eigenvalues. In chapter four we discuss further applications by looking at noncoercive as well as odd mappings.

1 PRELIMINARIES

1.1 Mappings of Monotone Type

In what follows, X stands for a real reflexive Banach space, and X^* its dual space. By a well-known renorming theorem due to Troyanski [50], given a reflexive Banach space, we can always renorm it equivalently so that both X and X^* are locally uniformly convex. Thus, without loss of generality, we suppose that X and X^* are locally uniformly convex. In what follows, the symbol $B_r(x_0)$ denotes the open ball of X or X^* with center at x_0 and radius $r > 0$. The symbol \mathbb{R} (\mathbb{R}_+) stands for the set $(-\infty, \infty)$ ($[0, \infty)$), and $\partial G, \overline{G}$ denote the strong boundary and closure of the set G , respectively.

For an operator $T : X \supset D(T) \rightarrow 2^{X^*}$ we denote by $D(T) = \{x \in X : Tx \neq \emptyset\}$, $R(T) = \{x^* \in X^* : x^* \in Tx \text{ for some } x \in D(T)\}$ the domain and the range of T , respectively. The mapping $J : X \rightarrow 2^{X^*}$, given by

$$Jx = \{x^* \in X^* \mid \langle x^*, x \rangle = \|x\|^2 = \|x^*\|^2\},$$

is the normalized "duality mapping". From the definition of J , it follows that J is odd ($J(-x) = -J(x)$), positive homogeneous ($J(\lambda x) = \lambda J(x)$ for all $\lambda > 0$) and bounded. In our setting, J is also single-valued, invertible and bicontinuous.

Definition 1.1.1 *An operator $T : X \supset D(T) \rightarrow 2^{X^*}$ is called "monotone" if for any $x, y \in D(T)$ and every $u \in Tx, v \in Ty$ we have*

$$\langle u - v, x - y \rangle \geq 0.$$

It is “strictly monotone” if equality holds above only for $x = y$. It is “strongly monotone” if there exist a positive constant α such that the inequality above holds with 0 replaced by $\alpha\|x - y\|^2$. A monotone operator T is “maximal monotone” if its graph $G(T) = \{(x, x^*) : x \in D(T), x^* \in Tx\}$ is a maximal monotone subset of $X \times X^*$ when partially ordered by inclusion. In our setting, a monotone operator T is “maximal monotone” if $R(T + \lambda J) = X^*$ for all $\lambda \in (0, \infty)$.

It is said to be “cyclically monotone” if

$$\langle x_0 - x_1, u_0 \rangle + \dots + \langle x_{n-1} - x_n, u_{n-1} \rangle + \langle x_n - x_0, u_n \rangle \geq 0$$

where $u_i \in Tx_i$, $i = 0, \dots, n$.

Definition 1.1.2 We say that an operator $T : X \supset D(T) \rightarrow 2^{X^*}$ satisfies condition “ (S_+) ” on $B \subset D(T)$ if $\{x_n\} \subset B$, $x_n \rightarrow x_0$ and

$$\limsup_{n \rightarrow \infty} \langle u_n, x_n - x_0 \rangle \leq 0,$$

for some $u_n \in Tx_n$, imply $x_n \rightarrow x_0$.

In particular, if $T : X \supset D(T) \rightarrow X^*$ is single-valued, then T satisfies condition “ (S_+) ” on $B \subset D(T)$ if $\{x_n\} \subset B$, $x_n \rightarrow x_0$ and

$$\limsup_{n \rightarrow \infty} \langle Tx_n, x_n - x_0 \rangle \leq 0,$$

imply $x_n \rightarrow x_0$.

Definition 1.1.3 We say that a map $T : X \supset D(T) \rightarrow X^*$ is “quasimonotone” on $B \subset D(T)$ if for every sequence $\{x_n\} \subset B$ such that $x_n \rightarrow x_0$ we have

$$\limsup_{n \rightarrow \infty} \langle Tx_n, x_n - x_0 \rangle \geq 0.$$

Definition 1.1.4 $T : X \supset D(T) \rightarrow X^*$ is “pseudomonotone” on $B \subset D(T)$ if for

every sequence $\{x_n\} \subset B$ such that $x_n \rightarrow x_0$ and

$$\limsup_{n \rightarrow \infty} \langle Tx_n, x_n - x_0 \rangle \leq 0,$$

we have

$$\limsup_{n \rightarrow \infty} \langle Tx_n, x_n - x_0 \rangle = 0,$$

and if $x_0 \in \overline{B}$, then $Tx_n \rightarrow Tx_0$.

Definition 1.1.5 $T : X \supset D(T) \rightarrow 2^{X^*}$ is “injective” if for every $x_1, x_2 \in D(T)$ with $Tx_1 \cap Tx_2 \neq \emptyset$ we have $x_1 = x_2$.

Definition 1.1.6 An operator $T : X \rightarrow X^*$, is “bounded” if it maps bounded subsets of $D(T)$ onto bounded sets. It is “locally bounded” if for each $x \in X$ there exists a neighborhood U of x such that $T(U)$ is bounded.

Definition 1.1.7 $T : X \supset D(T) \rightarrow X^*$ is said to be “continuous” if for every sequence $\{x_n\} \subset D(T)$ such that $x_n \rightarrow x \in D(T)$ we have $Tx_n \rightarrow Tx$. T is said to be “demicontinuous” if $Tx_n \rightarrow Tx$ for every sequence $\{x_n\} \subset D(T)$ such that $x_n \rightarrow x \in D(T)$. T is “completely continuous” if for every sequence $\{x_n\} \subset D(T)$ such that $x_n \rightarrow x \in D(T)$ we have $Tx_n \rightarrow Tx$.

It can be shown that every demicontinuous map of class (S_+) is pseudomonotone and every pseudomonotone map is quasimonotone see for example Pascali and Sburlan [40, p. 226] and Berkovits [5].

Definition 1.1.8 A normed linear space is “uniformly convex” if given $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that

$$\frac{\|x + y\|}{2} \leq 1 - \delta(\epsilon), \quad \text{whenever } \|x - y\| \geq \epsilon \quad \text{and } \|x\| = \|y\| = 1.$$

It is “locally uniformly convex” if given $\epsilon > 0$ and an element x with $\|x\| = 1$

there exists $\delta(\epsilon, x) > 0$, such that

$$\frac{\|x + y\|}{2} \leq 1 - \delta(\epsilon, x), \quad \text{whenever } \|x - y\| \geq \epsilon \quad \text{and } \|y\| = 1.$$

From Browder [13, Proposition 8] we have the following result:

Proposition 1.1.9 *In our setting, the duality mapping $J : X \rightarrow X^*$ is single-valued, injective, surjective, strictly monotone and of type $(S+)$. It is also a homeomorphism.*

Let $T : X \supset D(T) \rightarrow 2^{X^*}$ be a maximal monotone operator, the operator $T_t \equiv (T^{-1} + tJ^{-1})^{-1} : X \rightarrow X^*$ is called the Yosida approximant of T and the following is true, see for example Pascali and Sburlan [40, p. 128].

Proposition 1.1.10 *Let $T : X \supset D(T) \rightarrow 2^{X^*}$, be a maximal monotone operator with $0 \in D(T)$ and $0 \in T(0)$. Then*

- (i) T_t , $t \in (0, \infty)$, is a bounded maximal monotone mapping with $T_t(0) = 0$;
- (ii) $T_t x \rightarrow T^{\{0\}}x$ in X , as $t \rightarrow 0$, for all $x \in D(T)$, where $T^{\{0\}}x$ denotes the element $y^* \in Tx$ of minimum norm;
- (iii) $\|T_t x\| \rightarrow \infty$ as $t \rightarrow 0$ for all $x \notin \overline{D(T)}$.

Also, $T_t x \in T J_t x$, where $J_t = I - tJ^{-1}T_t : X \rightarrow X$ satisfies:

Proposition 1.1.11 *If $x \in \overline{\text{conv}D(T)}$, then $J_t x \rightarrow x$ in X as $t \rightarrow 0$.*

where $\text{conv}M$ denotes the convex hull of the set M .

The following result can be found in Kartsatos and Skrypnik [28].

Proposition 1.1.12 *Let $T : X \supset D(T) \rightarrow 2^{X^*}$ be a maximal monotone operator with $0 \in D(T)$ and $0 \in T(0)$. Then the mapping $(t, x) \rightarrow T_t x$ is continuous on $(0, \infty) \times X$;*

We are now ready to introduce the “admissible homotopies” for maps T and f . These homotopies were introduced by Browder.

Definition 1.1.13 A family of maximal monotone maps $\{T_s : s \in [0, 1]\}$ from X to 2^{X^*} is said to be a “pseudomonotone homotopy” of maximal monotone maps if it satisfies the mutually equivalent conditions:

- (i) Suppose that $s_n \rightarrow s$ in $[0, 1]$, $(x_n, x_n^*) \in G(T_{s_n})$ with $x_n \rightarrow x$ in X and $x_n^* \rightarrow x^*$ in X^* , and $\limsup_{n \rightarrow \infty} \langle x_n^*, x_n \rangle \leq \langle x^*, x \rangle$. Then $(x, x^*) \in G(T_s)$ and $\langle x_n^*, x_n \rangle \rightarrow \langle x^*, x \rangle$.
- (ii) $\psi(s, x^*) = (T_s + J)^{-1}(x^*)$ is continuous from $[0, 1] \times X^*$ into X , with both X and X^* furnished with their norm topologies.
- (iii) For each $x^* \in X^*$, $s \rightarrow \psi(s, x^*) = (T_s + J)^{-1}(x^*)$ is continuous from $[0, 1]$ into X endowed with the norm topology.
- (iv) Given $(x, x^*) \in G(T_s)$ and $s_n \rightarrow s$ in $[0, 1]$, then there exists a sequence $(x_n, x_n^*) \in G(T_{s_n})$ such that $x_n \rightarrow x$ in X and $x_n^* \rightarrow x^*$ in X^* .

The admissible homotopies for perturbations f of our mappings $T + f$ are given in the next definition.

Definition 1.1.14 Let $\{f_s : s \in [0, 1]\}$ be a one parameter family of maps from \overline{G} into X^* , where $G \subset X$. Then $\{f_s\}$ is said to be a “homotopy of class (S_+) ” if for every $\{x_n\} \subset \overline{G}$ for which $x_n \rightarrow x$ in X and every $\{s_n\} \subset [0, 1]$ such that $s_n \rightarrow s$ and

$$\limsup_{n \rightarrow \infty} \langle f_{s_n}(x_n), x_n - x \rangle \leq 0$$

we have $x_n \rightarrow x$ in X and $f_{s_n}(x_n) \rightarrow f_s(x)$ in X^* .

Furthermore, we say that a demicontinuous family $\{f_s : s \in [0, 1]\}$ is a “quasimonotone homotopy” if for every sequence $\{x_n\} \subset \overline{G}$ with $x_n \rightarrow x$ and every sequence $\{s_n\} \subset [0, 1]$ such that $s_n \rightarrow s$ we have

$$\limsup_{n \rightarrow \infty} \langle f_{s_n}(x_n), x_n - x \rangle \geq 0.$$

Remark 1.1.15 It is important to note that Browder has shown in [13] that if $f_i : \overline{G} \rightarrow X^*$ are demicontinuous and of type (S_+) , then the family $\{f_s\}$, $s \in [0, 1]$, is an

(S_+) -homotopy, where

$$f_s = sf_1(x) + (1 - s)f_2(x).$$

For facts about the theory of monotone operators the reader is referred to Barbu [4], Browder [13], Pascali and Sburlan [40], Zeidler [52], the books of Browder [9], Cioranescu [18], Petryshyn [41], [42], Simons [47], Skrypnik [48], [49] and Zeidler [52].

1.2 Definition of the Topological Degree

In this section we give the definition of the topological degree and establish some of its properties.

Let X and Y be topological spaces and let \mathcal{O} be a class of open subsets G of X . For each G in \mathcal{O} , we associate a class F_G of maps of \overline{G} into Y . For each G in \mathcal{O} we also associate a class H_G of maps $[0, 1] \times \overline{G}$ into Y (admissible homotopies); if $H \in H_G$, then we also let $f_s(x) = H(s, x)$ and speak about the homotopy f_s , $0 \leq s \leq 1$, where s is the “parameter” of the homotopy f_s . For any $f \in F_G$, $G \in \mathcal{O}$, and for any y in Y with $y \notin f(\partial G)$, we associate an integer $d(f, G, y)$.

Definition 1.2.1 *The integer-valued mapping d is said to be a “classical topological degree” if the following conditions are satisfied.*

(i) *If $d(f, G, y) \neq 0$, then there exists a solution of the equation $f(x) \ni y$ in G .*

(ii) *(Additivity) If $D \in \mathcal{O}$, $\overline{D} \subset G \in \mathcal{O}$ and $f \in F_G$, then the restriction $f|_{\overline{D}} \in F_D$. Let (G_1, G_2) be a pair of disjoint open subsets of $G \in \mathcal{O}$ and suppose that $y \notin f(\overline{G} \setminus (G_1 \cup G_2))$. Then*

$$d(f, G, y) = d(f, G_1, y) + d(f, G_2, y).$$

(iii) *(Invariance under homotopy) If f_s , $0 \leq s \leq 1$, is a homotopy in H_G , then $f_s \in F_G$ for any fixed s in $[0, 1]$, and if $\{y(s), 0 \leq s \leq 1\}$ is a continuous curve*

in Y with $y(s) \notin f_s(\partial G)$ for any $s \in [0, 1]$, then

$$d(f_s, G, y(s)) \text{ is constant on } [0, 1].$$

(iv) (Normalization) There exists a normalizing map $j : X \rightarrow Y$ such that $j|_{\overline{G}}$ lies in F_G for each $G \in \mathcal{O}$, and if $y \in j(G)$, then

$$d(j, G, y) = +1.$$

For information on various degree theories, the reader is referred, e.g., to Browder [9]-[14], Kartsatos and Skrypnik [26], [27], Berkovits [5], Kittilä [32], Kobayashi and Otani [33], Leray and Schauder [36], Lloyd [37], Nagumo [39], Petryshyn [41], [42], Rothe [46], and Skrypnik [48], [49] and the references therein.

Applications of degree theories in various problems of Nonlinear Analysis may be found, e.g., in Adhikari and Kartsatos [3], Berkovits [5], Browder [9], [14], Kartsatos [21], [22], Kartsatos et al. [23]-[30], Kittilä [32], Kobayashi and Otani [33], Pascali and Sburlan [40], Petryshyn [41], [42], Skrypnik [48], [49], and Zeidler [52] and the references therein. We now state the following theorem. It is assumed that all the homotopies in it are admissible and the degree mapping d is well defined.

Theorem 1.2.2 *Let $T : X \supset D(T) \rightarrow 2^Y$ and $f \in F_G$, suppose that $y \notin (T+f)(\partial G)$. Then $f - y \in F_G$, $0 \notin (T + f - y)(\partial G)$ and $d(T + f, G, y) = d(T + f - y, G, 0)$.*

Proof. It is easy to see that $f - y \in F_G$. Consider the homotopy $T + (1 - s)f + s(f - y)$, $0 \leq s \leq 1$, and the continuous curve $y(s) = (1 - s)y$, $0 \leq s \leq 1$. We can easily see that $y(s) \notin (T + (1 - s)f + s(f - y))(\partial G)$, for any $s \in [0, 1]$. Hence the conclusion follows from the homotopy invariance property (iii) of Definition 1.2.1. ■

2 CONSTRUCTION OF THE DEGREE

In this chapter we develop a topological degree theory for perturbations of maximal monotone mappings. In section 2.1 we consider mappings $T + f$, where T is a maximal monotone operator and f is a bounded demicontinuous map of type (S_+) . In Section 2.2 we introduce some admissible homotopies and discuss some basic properties of the degree. In Section 2.3 we weaken the boundedness condition on the mapping f . We assume instead that f is quasibounded and define the degree after a suitable reduction of the domain of f . Section 2.4 is devoted to the generalization of the degree to mappings of the form $T + f$, where f is assumed to be quasimonotone.

It should be noted that in the case of a bounded demicontinuous (S_+) -perturbation f , this degree coincides with the Browder degree. We should also mention that if, in addition, the operator T is bounded demicontinuous maximal monotone and defined on the space X , then this degree coincides with the Skrypnik-Browder-Berkovits degree because the mapping $T + f$ is then a bounded demicontinuous (S_+) -mapping.

2.1 Degree for Bounded Perturbations of Type (S_+)

Let X be an infinite dimensional real reflexive separable Banach space. We further assume that X and X^* are locally uniformly convex. Let G be a bounded open set in X . Let $T : X \supset D(T) \rightarrow 2^{X^*}$ be a maximal monotone operator and $f : \overline{G} \rightarrow X^*$ a bounded demicontinuous mapping of type (S_+)

We first recall the following embedding Proposition due to Browder and Ton.

Proposition 2.1.1 *Let X be a reflexive separable Banach space. Then there exists a*

separable Hilbert space H and a linear compact injection $Q : H \rightarrow X$ such that $Q(H)$ is dense in X .

We let $Q^* : X^* \rightarrow H$ be the adjoint operator of Q and observe that

$$\langle Q^*(w), v \rangle = \langle w, Q(v) \rangle \text{ for all } v \text{ in } H \text{ and } w \text{ in } X^* \quad (2.1.1)$$

The operator Q^* is also linear and compact, and since $Q(H)$ is dense in X , it follows that Q^* is injective.

For the construction of the degree we will need the following lemmas. The proof of the next one can be found in Zeidler [52, p. 915].

Lemma 2.1.2 *Let $T : X \supset D(T) \rightarrow 2^{X^*}$ be maximal monotone. Thus the following are true:*

- (i) $\{x_n\} \subset D(T)$, $x_n \rightarrow x_0$ and $Tx_n \ni y_n \rightarrow y_0$ imply $x_0 \in D(T)$, and $y_0 \in Tx_0$
- (ii) $\{x_n\} \subset D(T)$, $x_n \rightarrow x_0$ and $Tx_n \ni y_n \rightarrow y_0$ imply $x_0 \in D(T)$, and $y_0 \in Tx_0$

Remark 2.1.3 *It is easy to see that either (i) or (ii) implies that T is closed.*

Definition 2.1.4 *An operator $T : X \supset D(T) \rightarrow 2^{X^*}$ is said to be “strongly quasi-bounded” if for each $S > 0$ there exists $K(S) > 0$ such that*

$$\|x\| \leq S, \quad \langle u, x \rangle \leq S, \quad \text{for some } u \in Tx.$$

imply $\|u\| \leq K(S)$.

The following lemma, which is due to Browder and Hess [16], can be also found in Kartsatos and Quarcoo [24, Lemma D].

Lemma 2.1.5 *Let $T : X \supset D(T) \rightarrow 2^{X^*}$ be a strongly quasibounded maximal monotone operator such that $0 \in T(0)$. Let $\{t_n\} \subset (0, \infty)$ and $\{u_n\} \subset X$ be such that*

$$\|u_n\| \leq S, \quad \langle T_{t_n} u_n, u_n \rangle \leq S_1,$$

where S, S_1 are positive constants. Then there exists a number $K = K(S, S_1) > 0$ such that $\|T_{t_n} u_n\| \leq K$ for all $n = 1, 2, \dots$

Proof. Let

$$w_n = T_{t_n} u_n = (T^{-1} + t_n J^{-1})^{-1} u_n.$$

We have

$$w_n \in T J_{t_n} u_n = T x_n, \quad t_n w_n = J(u_n - x_n),$$

where $x_n = J_{t_n} u_n$. Thus,

$$\begin{aligned} \langle w_n, x_n \rangle &= \langle w_n, u_n - t_n J^{-1} w_n \rangle \\ &= \langle w_n, u_n \rangle - t_n \langle w_n, J^{-1} w_n \rangle \\ &= \langle w_n, u_n \rangle - t_n \|w_n\|^2 \\ &\leq \langle T_{t_n} u_n, u_n \rangle \\ &\leq S_1. \end{aligned} \tag{2.1.2}$$

From (2.1.2) we obtain

$$t_n \|w_n\|^2 = \langle w_n, u_n \rangle - \langle w_n, x_n \rangle.$$

Now, since $0 \in T(0)$ and $w_n \in T x_n$, we have $\langle w_n, x_n \rangle \geq 0$, which implies $t_n \|w_n\|^2 \leq S_1$. We claim that $\{w_n\}$ is bounded. If it is not, we may assume that $\|w_n\| \rightarrow \infty$ and $\|w_n\| \leq \|w_n\|^2$ for all n . Thus, $t_n \|w_n\| \leq S_1$ and

$$t_n \|w_n\| = \|J(u_n - x_n)\| = \|u_n - x_n\|,$$

which implies that $\{x_n\}$ is bounded. Now, since T is strongly quasibounded, the boundedness of $\{x_n\}$ and $\{\langle w_n, x_n \rangle\}$ implies the boundedness of $\{w_n\}$, i.e., a contradiction. It follows that $\{T_{t_n} u_n\}$ is bounded. ■

The following lemma was, essentially, first proved by Brézis, Crandall and Pazy in

[6, Lemmas 1.2 and 1.3]. For a proof of it in its present form, the reader is referred to [3, Lemma 1].

Lemma 2.1.6 *Assume that the operators $T : X \supset D(T) \rightarrow 2^{X^*}$, $S : X \supset D(S) \rightarrow X^*$ are maximal monotone, with $0 \in D(T) \cap D(S)$ and $0 \in T(0) \cap S(0)$. Assume, further, that $T + S$ is maximal monotone. Assume that there is a positive sequence $\{t_n\}$ such that $t_n \downarrow 0$, a sequence $\{x_n\} \subset D(S)$ and a sequence $w_n \in Sx_n$ such that $x_n \rightharpoonup x_0 \in X$ and $T_{t_n}x_n + w_n \rightharpoonup y_0^* \in X^*$. Then the following are true:*

(i) *the inequality*

$$\lim_{n \rightarrow \infty} \langle T_{t_n}x_n + w_n, x_n - x_0 \rangle < 0$$

is impossible;

(ii) *if*

$$\lim_{n \rightarrow \infty} \langle T_{t_n}x_n + w_n, x_n - x_0 \rangle = 0,$$

then $x_0 \in D(T + S)$ and $y_0^ \in (T + S)x_0$.*

Lemma 2.1.7 *Assume that $T : X \supset D(T) \rightarrow 2^{X^*}$ is maximal monotone and bounded (i.e. if M is a bounded set in X , then the set $\bigcup\{Tx ; x \in D(T) \cap M\}$ is bounded). Then if M is a bounded set in X the set*

$$\{T_t x : (t, x) \in (0, \infty) \times (D(T) \cap M)\}$$

is also bounded.

In what follows, we set

$$U(\varepsilon, t) = \left(I + \frac{1}{\varepsilon} Q Q^* (T_t + f) \right).$$

Lemma 2.1.8 *Let $T : X \supset D(T) \rightarrow 2^{X^*}$ be a strongly quasibounded maximal monotone operator with $0 \in T(0)$ and $f : \bar{G} \rightarrow X^*$ a bounded demicontinuous operator of type (S_+) . Let $T_t = (T^{-1} + tJ^{-1})^{-1}$ be the Yosida approximant of T . Let A be a closed*

subset of \overline{G} and assume that

$$0 \notin (T + f)(D(T) \cap A).$$

Then there exist $\varepsilon_0 > 0$, $t_0 > 0$ such that for $0 < \varepsilon \leq \varepsilon_0$ and $0 < t \leq t_0$ we have

$$0 \notin U(\varepsilon, t)(A).$$

Remark 2.1.9 *The compactness of QQ^* says that the map $I + \frac{1}{\varepsilon}QQ^*$ is of the Leray-Schauder type, i.e., a compact displacement of the identity. Such mappings are the basic admissible operator types on which the Leray-Schauder degree can be defined.*

Proof. Suppose that the conclusion is false. Then there exist sequences $\{\varepsilon_n\} \subset \mathbb{R}_+$, $\{t_n\} \subset \mathbb{R}_+$ and $\{x_n\} \subset A$ with $\varepsilon_n \downarrow 0$, $t_n \downarrow 0$ and

$$\left(I + \frac{1}{\varepsilon_n} QQ^*(T_{t_n} + f) \right) (x_n) = 0 \quad (2.1.3)$$

for all n .

This says

$$x_n = -\frac{1}{\varepsilon_n} QQ^*(T_{t_n} + f)(x_n) \quad (2.1.4)$$

and

$$\begin{aligned} \langle (T_{t_n} + f)(x_n), x_n \rangle &= \langle (T_{t_n} + f)(x_n), -\frac{1}{\varepsilon_n} QQ^*(T_{t_n} + f)(x_n) \rangle \\ &= -\frac{1}{\varepsilon_n} \langle (T_{t_n} + f)(x_n), QQ^*(T_{t_n} + f)(x_n) \rangle \\ &= -\frac{1}{\varepsilon_n} \|Q^*(T_{t_n} + f)(x_n)\|^2 \\ &\leq 0. \end{aligned}$$

This implies

$$\langle T_{t_n} x_n, x_n \rangle \leq -\langle f(x_n), x_n \rangle$$

and

$$|\langle T_{t_n} x_n, x_n \rangle| \leq \|f x_n\| \|x_n\|.$$

Thus, $\langle T_{t_n} x_n, x_n \rangle$ is bounded because of the boundedness of f and $\{x_n\}$. We may therefore assume that $\|x_n\| \leq S$ and $\langle T_{t_n} x_n, x_n \rangle \leq S$. By the strong quasiboundedness of T , it follows that $\|T_{t_n} x_n\| \leq K(S)$ (see Lemma 2.1.5). Hence, at least for subsequences, we may assume that $x_n \rightharpoonup x \in X$, $T_{t_n} x_n \rightharpoonup u$, and $f(x_n) \rightharpoonup v$ in X^* . Setting $w = u + v$ we have

$$T_{t_n} x_n + f(x_n) \rightharpoonup w.$$

Writing (2.1.4) as $QQ^*(T_{t_n} + f)(x_n) = -\varepsilon_n x_n$ and using the boundedness of $\{x_n\}$ and $\varepsilon_n \rightarrow 0$, we obtain

$$QQ^*(T_{t_n} + f)(x_n) \rightarrow 0.$$

Since QQ^* is compact and linear, it is completely continuous. Hence,

$$QQ^*(T_{t_n} + f)(x_n) \rightarrow QQ^*(w).$$

By the uniqueness of the limit, $QQ^*(w) = 0$. By the injectivity of QQ^* , we have $w = 0$, i.e.,

$$(T_{t_n} + f)(x_n) \rightharpoonup 0. \tag{2.1.5}$$

By (2.1.5)

$$\langle (T_{t_n} + f)(x_n), x \rangle \rightarrow 0$$

we find

$$\limsup_{n \rightarrow \infty} \langle (T_{t_n} + f)(x_n), x_n - x \rangle = \limsup_{n \rightarrow \infty} \langle (T_{t_n} + f)(x_n), x_n \rangle \leq 0.$$

We need the inequality

$$\limsup_{n \rightarrow \infty} \langle f(x_n), x_n - x \rangle \leq 0. \tag{2.1.6}$$

The proof of this inequality follows exactly as in Theorem 1 of Kartsatos and Skrypnik [28]. It is included herein for completeness. Assume that this inequality is not true.

Then there exists a subsequence of $\{x_n\}$, denoted again by $\{x_n\}$, such that

$$\lim_{n \rightarrow \infty} \langle f(x_n), x_n - x \rangle > 0.$$

This implies

$$\lim_{n \rightarrow \infty} \langle T_{t_n} x_n, x_n - x \rangle < 0.$$

Using the fact that $T_{t_n} x_n \rightharpoonup u$ along with

$$\langle T_{t_n} x_n, x_n \rangle = \langle T_{t_n} x_n, x_n - x \rangle + \langle T_{t_n} x_n, x \rangle,$$

we obtain

$$\limsup_{n \rightarrow \infty} \langle T_{t_n} x_n, x_n \rangle < \limsup_{n \rightarrow \infty} \langle T_{t_n} x_n, x \rangle = \langle u, x \rangle.$$

Now, let $y \in D(T)$ and $y^* \in Ty$. Since

$$\begin{aligned} \langle T_{t_n} x_n - y^*, x_n - y \rangle &= \langle T_{t_n} x_n - y^*, J_{t_n} x_n - y \rangle + \langle T_{t_n} x_n - y^*, t_n J^{-1} T_{t_n} x_n \rangle \\ &\geq \langle T_{t_n} x_n - y^*, t_n J^{-1} T_{t_n} x_n \rangle, \end{aligned}$$

we have

$$\langle T_{t_n} x_n, x_n \rangle \geq \langle T_{t_n} x_n, y \rangle + \langle y^*, x_n - y \rangle + \langle T_{t_n} x_n - y^*, t_n J^{-1} T_{t_n} x_n \rangle,$$

which implies

$$\begin{aligned} \liminf_{n \rightarrow \infty} \langle T_{t_n} x_n, x_n \rangle &\geq \liminf_{n \rightarrow \infty} \langle T_{t_n} x_n, y \rangle + \langle y^*, x - y \rangle \\ &= \langle u, y \rangle + \langle y^*, x - y \rangle. \end{aligned} \tag{2.1.7}$$

Thus,

$$\langle u, y \rangle + \langle y^*, x - y \rangle < \langle u, x \rangle$$

or

$$\langle u - y^*, x - y \rangle > 0. \tag{2.1.8}$$

Since (y, y^*) is arbitrary in $G(T)$ and T is maximal monotone, we have $x \in D(T)$ and $u \in Tx$. Taking $y = x$ and $y^* = u$ in (2.1.8), we obtain a contradiction. Consequently (2.1.6) is true. Now since f is of class (S_+) , it follows that $x_n \rightarrow x$. Repeating the same argument starting from (2.1.6) we get (2.1.8) where " $>$ " is replaced by " \geq ". By the maximal monotonicity of T , we have $x \in D(T)$ and $u \in Tx$. Hence,

$$T_{t_n}x_n + f(x_n) \rightarrow 0 = u + v \in Tx + f(x),$$

which shows that $0 \in (T + f)(x)$, $x \in A$. This is a contradiction to $0 \notin (T + f)(A)$. Thus, there exist $\varepsilon_0 > 0$, $t_0 > 0$, such that for $0 < \varepsilon < \varepsilon_0$ and $0 < t < t_0$ we have

$$0 \notin \left(I + \frac{1}{\varepsilon} QQ^*(T_t + f) \right) (A).$$

■

Before we prove our second basic lemma, Lemma 2.1.11 below, we need first a simple but basic relation in Lemma 2.1.10 below. The symbol d_{LS} stands for the Leray-Schauder degree.

Lemma 2.1.10 *Let $T : X \supset D(T) \rightarrow 2^{X^*}$, $t > 0$, $s > 0$. be maximal monotone. Let T_t be its Yoshida approximant. Let $T^s = sT$. Then $T_t^s x = sT_{st}x$, $x \in X$.*

Proof. Let $x \in X$, $s > 0$, and $t > 0$. Then

$$\begin{aligned} y = T_t^s x = ((sT)^{-1} + tJ^{-1})^{-1} x &\iff x \in (sT)^{-1}y + tJ^{-1}y \\ &\iff x \in T^{-1} \left(\frac{y}{s} \right) + tsJ^{-1} \left(\frac{y}{s} \right) \\ &\iff y = s (T^{-1} + stJ^{-1})^{-1} x = sT_{st}x. \end{aligned}$$

Thus, $T_t^s x = sT_{st}x$, $x \in X$.

■

We now give the full proof, for the sake of completeness, of our second basic lemma. It is part of the proof of Kartsatos and Skrypnik [28, Theorem 4.4, (iii)].

Lemma 2.1.11 *Let $T : X \supset D(T) \rightarrow 2^{X^*}$ be maximal monotone with $0 \in D(T)$ and $0 \in T(0)$. Then, for every $\delta > 0$, the mapping $(s, x) \rightarrow T_\delta^s x$ is continuous on $(0, \infty) \times X$.*

Proof. Define the mapping

$$J_\delta^s \equiv I - \delta J^{-1} T_\delta^s : X \rightarrow X,$$

Then

$$J(x - J_\delta^s x) = \delta T_\delta^s x \in \delta T^s J_\delta^s x = \delta s T J_\delta^s x \quad (2.1.9)$$

is true for $s \in (0, \infty)$, $x \in X$. Let $s_n \in (0, \infty)$, $x_n \in X$ be such that $s_n \rightarrow s_0 \in (0, \infty)$, $x_n \rightarrow x_0$. From (2.1.9) we obtain the existence of $y_n^* \in T J_\delta^{s_n} x_n$, $y_0^* \in T J_\delta^{s_0} x_0$ such that

$$\delta s_n y_n^* = J(x_n - J_\delta^{s_n} x_n), \quad \delta s_0 y_0^* = J(x_0 - J_\delta^{s_0} x_0). \quad (2.1.10)$$

Using this, the monotonicity of the operator T and the assumptions $0 \in D(T)$, $0 \in T(0)$, we have

$$\begin{aligned} \|x_n - J_\delta^{s_n} x_n\|^2 &= \langle J(x_n - J_\delta^{s_n} x_n), x_n - J_\delta^{s_n} x_n \rangle \\ &= \delta s_n \langle y_n^*, x_n - J_\delta^{s_n} x_n \rangle \\ &\leq \langle J(x_n - J_\delta^{s_n} x_n), x_n \rangle, \end{aligned}$$

which implies the boundedness of the sequence $\{J_\delta^{s_n} x_n\}$. By the monotonicity of the duality mapping J , we have

$$\langle J(x_n - J_\delta^{s_n} x_n) - J(x_0 - J_\delta^{s_0} x_0), (x_n - J_\delta^{s_n} x_n) - (x_0 - J_\delta^{s_0} x_0) \rangle \geq 0,$$

which implies

$$\begin{aligned} &\|J(x_n - J_\delta^{s_n} x_n) - J(x_0 - J_\delta^{s_0} x_0)\| \|x_n - x_0\| \\ &\geq \langle J(x_n - J_\delta^{s_n} x_n) - J(x_0 - J_\delta^{s_0} x_0), x_n - x_0 \rangle \\ &\geq \langle J(x_n - J_\delta^{s_n} x_n) - J(x_0 - J_\delta^{s_0} x_0), J_\delta^{s_n} x_n - J_\delta^{s_0} x_0 \rangle \end{aligned}$$

$$\begin{aligned}
&= \delta \langle s_n y_0^* - s_0 y_0^*, J_\delta^{s_n} x_n - J_\delta^{s_0} x_0 \rangle \\
&\quad + \delta \langle s_n y_n^* - s_n y_0^*, J_\delta^{s_n} x_n - J_\delta^{s_0} x_0 \rangle \\
&= \delta \langle s_n y_0^* - s_0 y_0^*, J_\delta^{s_n} x_n - J_\delta^{s_0} x_0 \rangle \\
&\quad + \delta s_n \langle y_n^* - y_0^*, J_\delta^{s_n} x_n - J_\delta^{s_0} x_0 \rangle \\
&\geq \delta (s_n - s_0) \langle y_0^*, J_\delta^{s_n} x_n - J_\delta^{s_0} x_0 \rangle.
\end{aligned}$$

From this inequality, the boundedness of the sequence $\{J_\delta^{s_n} x_n\}$, $s_n \rightarrow s_0$ and $x_n \rightarrow x_0$ we obtain

$$\lim_{n \rightarrow \infty} \langle J z_n - J z_0, z_n - z_0 \rangle = 0,$$

where $z_n = x_n - J_\delta^{s_n} x_n$, $z_0 = x_0 - J_\delta^{s_0} x_0$. This and the (S)-property of the duality mapping imply $z_n \rightarrow z_0$ and, consequently, $J_\delta^{s_n} x_n \rightarrow J_\delta^{s_0} x_0$. Then $T_\delta^{s_n} x_n \rightarrow T_\delta^{s_0} x_0$, by (2.1.9), and this finishes the proof of the continuity of the mapping $T_\delta^s x$ w.r.t. $(s, x) \in (0, \infty) \times X$. ■

A uniform boundedness type of result is contained in the following lemma.

Lemma 2.1.12 *Let $T : X \supset D(T) \rightarrow 2^{X^*}$ be maximal monotone and $G \subset X$ bounded. Let*

$$0 < s_1 < s_2, \quad 0 < t_1 < t_2.$$

Let $T^s := sT$. Then there exists a constant $K_1 > 0$, independent of s, t , such that

$$\|T_t^s u\| \leq K_1, \quad x \in \overline{G}, \quad s \in [s_1, s_2], \quad t \in [t_1, t_2].$$

Proof. For every $u \in X$, we have

$$T_t^s u = \frac{1}{t} J(u - J_t^s u).$$

Let $(v, h) \in G(T)$. Then $sh \in T^s v = sTv$, By the monotonicity of T , we have

$$\begin{aligned} \langle sh, J_t^s u - v \rangle &\leq \langle T_t^s u, J_t^s u - v \rangle = -\frac{1}{t} \langle J(J_t^s u - u), J_t^s u - v \rangle \\ &= -\frac{1}{t} \langle J(J_t^s u - u), J_t^s u - u \rangle - \frac{1}{t} \langle J(J_t^s u - u), J_t^s u - v \rangle. \end{aligned}$$

This implies

$$\begin{aligned} \|J_t^s u - u\|^2 &\leq -t \langle sh, J_t^s u - v \rangle - \langle J(J_t^s u - u), J_t^s u - v \rangle \\ &\leq t_2 s_2 \|h\| \|J_t^s u - v\| + \|J_t^s u - u\| \|u - v\| \\ &\leq t_2 s_2 \|h\| (\|J_t^s u - u\| + \|u - v\|) + \|J_t^s u - u\| \|u - v\|, \\ &\leq t_2 s_2 \|h\| (\|J_t^s u - u\| + B + \|v\|) + \|J_t^s u - u\| (B + \|v\|), \end{aligned}$$

where B is an upper bound for $u \in G$. It follows that

$$\|J_t^s u - u\| \leq K, \quad u \in \overline{G}, \quad s \in [s_1, s_2], \quad t \in [t_1, t_2],$$

where $K > 0$ does not depend on s and t . Consequently,

$$\|T_t^s u\| = \frac{1}{t} \|J_t^s u - u\| \leq \frac{1}{t_1} K = K_1, \quad u \in G, \quad s \in [s_1, s_2], \quad t \in [t_1, t_2],$$

where $K_1 > 0$ does not depend on s and t . ■

Many times, the Leray-Schauder degree d_{LS} below is actually the Nagumo degree from [39] (cf. also Rothe [46]). It has the four basic properties of the original Leray-Schauder degree, but it enjoys the following advantage. In the Leray-Schauder homotopy $x - H(t, x)$, $t \in [0, 1]$, $x \in \overline{G}$, the function $H(t, x)$ is supposed to satisfy the following two properties:

- (i) $H(t, x)$ be continuous in t uniformly w.r.t. $x \in \overline{G}$;
- (ii) the mapping $x \rightarrow H(t, x)$ is compact on \overline{G} for each $t \in [0, 1]$.

The Nagumo homotopy in [39] (or the Rothe homotopy in [46]), $x - H(t, x)$, is such

that

(iii) The mapping $H(t, x) : [0, 1] \times \overline{G} \rightarrow X$ is compact.

We know that (i) and (ii) together imply (iii), but (iii) does not imply (i).

For the relevant discussion and examples the reader is referred to Rothe [46, pp. 56-57]. Several times below, the homotopy $H(t, x)$ does satisfy (i) and (ii) above, and is a Leray-Schauder homotopy. The Nagumo degree is of course the Leray-Schauder degree, by the uniqueness of the Leray-Schauder degree, on homotopies satisfying (i) and (ii) above.

The next lemma contains a basic invariance property of the associated Leray-Schauder degree.

Theorem 2.1.13 *Let $T : X \supset D(T) \rightarrow 2^{X^*}$ be a strongly quasibounded maximal monotone operator with $0 \in D(T)$ and $0 \in T(0)$. Let $f : \overline{G} \rightarrow X^*$ be bounded demicontinuous and of type (S_+) , where G is a bounded subset of X . Let $T_t = (T^{-1} + tJ^{-1})^{-1}$ be the Yosida approximant of T . Assume that*

$$0 \notin (T + f)(D(T) \cap \partial G).$$

Then

(i) *there exists $t_0 > 0$ such that $0 < t \leq t_0$ implies*

$$0 \notin \left(I + \frac{1}{t} QQ^*(T_t + f) \right) (\partial G);$$

(ii) *the degree $d_{LS}(U_0(t), G, 0)$ is constant for $0 < t \leq t_0$, where $U_0(t) = U(t, t)$.*

Proof. (i) This is a particular case of Lemma 2.1.8, where $A = \partial G$. Note that T is assumed strongly quasibounded.

(ii) It suffices to show that for any two numbers $t_1, t_2 \in (0, t_0]$ we have

$$d_{LS}(U_0(t_1), G, 0) = d_{LS}(U_0(t_2), G, 0). \quad (2.1.11)$$

Obviously, the degree $d_{LS}(U_0(t), G, 0)$ here is well defined because of (i). Now, let $t_1, t_2 \in (0, t_0]$ be given with $t_1 < t_2$ and consider the curve

$$s(t) := tt_1 + (1 - t)t_2, \quad t \in [0, 1].$$

In order to show (2.1.11), all we need to show, according to the Leray-Schauder theory (cf. Nagumo [39, Theorem 7]), is that the operator

$$S(t, x) := \frac{1}{s(t)}QQ^*(T_{s(t)} + f)(x)$$

is continuous on $[0, 1] \times \overline{G}$ and the set

$$\overline{S([0, 1] \times \overline{G})}$$

is compact in X . In fact, the continuity of the operator S follows immediately from the continuity of the mapping $(t, x) \rightarrow T_t x$ (see Proposition 1.1.12), the demicontinuity of f and the complete continuity of the linear compact operator QQ^* . The relative compactness of the set $S([0, 1] \times \overline{G})$ follows from the compactness of the operator QQ^* and the boundedness of the set

$$\left\{ \frac{1}{s(t)}(T_{s(t)} + f)(x) ; (t, x) \in [0, 1] \times \overline{G} \right\}.$$

Here, the boundedness of the set

$$\{T_{s(t)}x ; (t, x) \in [0, 1] \times \overline{G}\}$$

follows from Lemma 2.1.12 for $s = 1$ there. The proof is complete. ■

We are now ready for the definition of our degree mapping.

Definition 2.1.14 *Let the operators T , f satisfy the assumptions of Theorem 2.1.13.*

In particular, assume that $0 \notin (T + f)(\partial G)$. We define

$$d(T + f, G, 0) = \lim_{t \rightarrow 0} d_{LS}(U_0(t), G, 0). \quad (2.1.12)$$

For $y \notin (T + f)(\partial G)$, we set

$$d(T + f, G, y) = d(T + f - y, G, 0). \quad (2.1.13)$$

2.2 Basic Properties of the Degree

Since the most important property of a degree theory is the invariance under suitable homotopies, we devote this section to the introduction of some admissible homotopies for the above constructed degree theory, and discuss some properties of the degree with respect to these homotopies. We have the following definitions.

Definition 2.2.1 Let $T^s : [0, 1] \times X \rightarrow 2^{X^*}$ be a family of maximal monotone operators. T^s is said to be uniformly strongly quasibounded if for every bounded sequence $\{x_n\}$ and every sequence $y_n \in T^{s_n}x_n$ such that $\langle y_n, x_n \rangle$ is bounded the sequence $\{y_n\}$ is bounded.

Definition 2.2.2 We say that the operators $T^{(0)} + f^{(0)}, T^{(1)} + f^{(1)}$ are "homotopic" with respect to the open bounded set $G \subset X$ if there exists a pseudomonotone homotopy $T^s : [0, 1] \times X \rightarrow 2^{X^*}$, and a $(S+)$ -homotopy $f^s : \bar{G} \rightarrow X^*$, $s \in [0, 1]$, such that

$$T^{(i)} = T^i, \quad f^{(i)} = f^i, \quad i = 0, 1$$

and

$$T^s x + f^s x \neq 0, \quad x \in \partial G, \quad s \in [0, 1]. \quad (2.2.14)$$

We give the following lemmas for completeness and future reference.

Lemma 2.2.3 Let $\{T^s, \quad s \in [0, 1]\}$ be a uniformly strongly quasibounded family of maximal monotone operators such that $0 \in T^s(0)$ and let $\{u_n\}$ be a bounded sequence. If $\langle T_{t_n}^{s_n} u_n, u_n \rangle$ is bounded, then $T_{t_n}^{s_n} u_n$ is bounded, where $T_{t_n}^{s_n} u_n = (T^{s_n-1} + t_n J^{-1})^{-1} u_n$.

Proof. Letting $w_n = T_{t_n}^{s_n} u_n = (T^{s_n-1} + t_n J^{-1})^{-1} u_n$, we have

$$w_n \in T^{s_n} J_{t_n} u_n = T^{s_n} x_n, \quad t_n w_n = J(u_n - x_n),$$

where $x_n = J_{t_n} u_n$. Thus,

$$\begin{aligned} \langle w_n, x_n \rangle &= \langle w_n, u_n - t_n J^{-1} w_n \rangle \\ &= \langle w_n, u_n \rangle - t_n \langle w_n, J^{-1} w_n \rangle \\ &= \langle w_n, u_n \rangle - t_n \|w_n\|^2 \\ &\leq \langle T_{t_n}^{s_n} u_n, u_n \rangle \\ &\leq S_1, \end{aligned} \tag{2.2.15}$$

where S_1 is an obvious upper bound. From (2.2.15), we also obtain

$$t_n \|w_n\|^2 = \langle w_n, u_n \rangle - \langle w_n, x_n \rangle.$$

Since $0 \in T^{s_n}(0)$ and $w_n \in T^{s_n} x_n$, we have $\langle w_n, x_n \rangle \geq 0$, which implies $t_n \|w_n\|^2 \leq S_1$. Now, if $\{w_n\}$ is not bounded, we may assume that $\|w_n\| \rightarrow \infty$ and $\|w_n\| \leq \|w_n\|^2$ for all n . Thus, $t_n \|w_n\| \leq S_1$ and

$$t_n \|w_n\| = \|J(u_n - x_n)\| = \|u_n - x_n\|$$

implies that $\{x_n\}$ is bounded. The boundedness of $\{x_n\}$ and $\{\langle w_n, x_n \rangle\}$, where $w_n \in T^{s_n} x_n$, and the uniform quasiboundedness of T^s , imply the boundedness of $\{w_n\}$, i.e., a contradiction. It follows that $T_{t_n}^{s_n} u_n$ is bounded. ■

Lemma 2.2.4 *Let $h^s : \overline{G} \rightarrow X^*$ be a demicontinuous quasimonotone homotopy and $f^s : \overline{G} \rightarrow X^*$ an (S_+) -homotopy. Then $h^s + f^s$ is a homotopy of type (S_+) .*

Proof. We first show that

$$\liminf_{n \rightarrow \infty} \langle h^{s_n}(x_n), x_n - x \rangle \geq 0 \tag{2.2.16}$$

for every sequence $\{x_n\}$ in \overline{G} , s_n in $[0, 1]$ with $x_n \rightarrow x$ in X and $s_n \rightarrow s$ in $[0, 1]$.

Suppose that it is not true. Then, for a subsequence of $\{n\}$ denoted again by $\{n\}$, we have

$$\liminf_{n \rightarrow \infty} \langle h^{s_n}(x_n), x_n - x \rangle = \lim \langle h^{s_n}(x_n), x_n - x \rangle < 0,$$

which contradict the quasimonotonicity of h , hence (2.2.16) is true.

Now, let $\{x_n\}$ in G be a sequence such that $x_n \rightarrow x$ and

$$\limsup_{n \rightarrow \infty} \langle h^{s_n}(x_n) + f^{s_n}(x_n), x_n - x \rangle \leq 0$$

then

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle f^{s_n}(x_n), x_n - x \rangle &\leq \limsup_{n \rightarrow \infty} \langle h^{s_n}(x_n) + f^{s_n}(x_n), x_n - x \rangle \\ &\quad - \liminf_{n \rightarrow \infty} \langle h^{s_n}(x_n), x_n - x \rangle \\ &\leq \limsup_{n \rightarrow \infty} \langle h^{s_n}(x_n) + f^{s_n}(x_n), x_n - x \rangle \\ &\leq 0. \end{aligned}$$

By the (S_+) -property of f^s , we have that $x_n \rightarrow x$ and $f^{s_n}(x_n) \rightarrow f^s(x)$. By the demicontinuity of h^s we have $h^{s_n}(x_n) \rightarrow h^s(x)$. Hence, $h^{s_n}(x_n) + f^{s_n}(x_n) \rightarrow h^s + f^s$, and the proof is complete. ■

Definition 2.2.5 Let the mapping $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be such that $\phi(0) = 0$ and if $r_n > 0$, $n = 1, 2, \dots$, with

$$\lim_{n \rightarrow \infty} \phi(r_n) = 0,$$

then we have $r_n \rightarrow 0^+$. We say that the operator $f : X \rightarrow X^*$ belong to the class Γ_ϕ if there exists a function ϕ , as above, such that $\langle f(x), x \rangle \geq \phi(\|x\|)$, $x \in X$.

The following theorem contains the main properties of our new degree mapping.

Theorem 2.2.6 Let $T : X \supset D(T) \rightarrow 2^{X^*}$ be a strongly quasibounded maximal monotone operator with $0 \in D(T)$ and $0 \in T(0)$. Let $f : \overline{G} \rightarrow X^*$ be bounded demicontinuous and of type (S_+) .

- (i) If $0 \in G$, then $d(J, G, 0) = 1$.
- (ii) if $y \notin (T + f)(D(T) \cap \partial G)$ and $d(T + f, G, y) \neq 0$ then there exists $x \in D(T) \cap G$ such that $(T + f)x \ni y$;
- (iii) If $0 \in G$, the degree $d(H(s, \cdot), G, 0)$ is invariant under homotopies of the type

$$H(s, x) \equiv (T + f)(x) - y^*(s), \quad s \in [0, 1],$$

where $y^* : [0, 1] \rightarrow X^*$ is a continuous curve. Here, $0 \notin H(s, \cdot)(\partial G)$, $s \in [0, 1]$;

- (iv) If G_1, G_2 are disjoint open and bounded sets in G such that $y \notin (T + f)(D(T) \cap (\overline{G} \setminus (G_1 \cup G_2)))$, then

$$d(T + f, G, y) = d(T + f, G_1, y) + d(T + f, G_2, y).$$

- (v) if $0 \in G$, then the degree $d(H(s, \cdot), G, 0)$ is well defined and invariant under homotopies of the type

$$H(s, x) \equiv s(T + f_1)(x) + (1 - s)f_2(x), \quad s \in (0, 1],$$

provided that $0 \notin H(s, \cdot)(\partial G)$, $s \in (0, 1]$. Here, f_1, f_2 are bounded demicontinuous of type (S_+) . Actually, $d(H(s, \cdot), G, 0) = d(T + f_1, G, 0)$;

- (vi) Let the assumptions of Part (v) be satisfied and assume, further, that the mapping f_2 is of class Γ_ϕ . Then $d(H(s, \cdot), G, 0)$ is well defined and invariant under homotopies of type $H(s, x)$ in (v), for $s \in [0, 1]$, provided that $0 \notin H(s, \cdot)(\partial G)$, $s \in [0, 1]$. In particular, $d(T + f_1, G, 0) = d(f_2, G, 0)$,

- (vii) if $0 \in G$, then the degree $d(H(s, \cdot), G, 0)$ is well defined and invariant under homotopies of the type

$$H(s, x) \equiv T + sf_1(x) + (1 - s)f_2(x), \quad s \in [0, 1],$$

provided that $0 \notin H(s, \cdot)(\partial G)$, $s \in [0, 1]$. Here, f_1, f_2 are as in (ii). In particu-

$$\text{lar, } d(T + f_1, G, 0) = d(T + f_2, G, 0);$$

Proof. (i) We note that

$$H(t, x) := tx + (1 - t) \left[I + \frac{1}{\varepsilon} QQ^*(J)(x) \right]$$

is an admissible homotopy for the Leray-Schauder degree. It is an affine combination of two displacements of the identity. We show that for small $\varepsilon > 0$ we have

$$0 \notin (tI + (1 - t) \left[I + \frac{1}{\varepsilon} QQ^*(J) \right]) (\partial G), \quad t \in [0, 1].$$

in fact

$$\begin{aligned} \langle J(x), tI(x) + (1 - t)(I + \frac{1}{\varepsilon} QQ^*(J)) \rangle &= \langle J(x), x \rangle + \langle J(x), \frac{1}{\varepsilon} QQ^*(J) \rangle \\ &= \|x\|^2 + \frac{1 - t}{\varepsilon} \|Q^* J(x)\|_W^2 \\ &> 0 \end{aligned}$$

for all $x \neq 0$. Thus, for small $\varepsilon > 0$,

$$\begin{aligned} d_{LS}(H(1, \cdot), G, 0) &= d_{LS}(I, G, 0) \\ &= d_{LS}(H(0, \cdot), G, 0) \\ &= d(J, G, 0); \end{aligned}$$

(ii) We show only the case $y = 0$. The case $y \neq 0$ is similar. Assume that $d(T + f, G, 0) \neq 0$. If we also have $0 \notin (T + f)(G)$, then, by Lemma 2.1.8,

$$0 \notin (U_0(t))(\overline{G})$$

for all small $t > 0$, which implies

$$d_{LS}(U_0(t), G, 0) = 0.$$

Hence, by the definition of d , $d(T + f, G, 0) = 0$, i.e., a contradiction. Thus $0 \in (T + f)(G)$.

(iii) We first show (a) that there exists $t_0 > 0$ such that

$$0 \notin (I + \frac{1}{t}QQ^*(T_t + f + y^*(s)))(\partial G), \quad s \in [0, 1], \quad t \in (0, t_0],$$

and then (b)

$$d(T + f + y^*(0), G, 0) = d_{LS}(I + \frac{1}{t}QQ^*(T_t + f + y^*(0)), G, 0) \quad t \in (0, t_0]. \quad (2.2.17)$$

(a) We have given an even more elaborate proof of such a situation in Part (vii) below, where $f^s = sf_1 + (1 - s)f_2$. The proof of Part (a) of (iii) is therefore omitted.

(b) We observe that the function

$$H_1(s, x) = \frac{1}{t}QQ^*(T_t + f + y^*(s))(x), \quad s \in [0, 1], \quad x \in \overline{G},$$

is continuous on the set $[0, 1] \times \overline{G}$, and the set

$$\overline{H_1([0, 1], \overline{G})}$$

is compact. Using (2.2.17), we find

$$d_{LS}(H_1(s, \cdot), G, 0) = d_{LS}(H_1(0, \cdot), G, 0) = d(T + f + y^*(0), G, 0), \quad s \in [0, 1].$$

Taking the limit as $t \rightarrow 0$, observing that the function $H(t, x)$ is now depending on t with s fixed, we obtain

$$d(T + f + y^*(s), G, 0) = d(T + f + y^*(0), G, 0), \quad s \in [0, 1],$$

and this completes the proof of this part.

(iv) We consider only the case $y = 0$. Let G_1, G_2 be as in (iv). By Lemma 2.1.8, where $A = \overline{G} \setminus G_1 \cup G_2$, we get $\epsilon_0 > 0$ and $t_0 > 0$ such that $0 \notin (U_0(t)(A))$ for all

$t \in (0, t_0]$. Since the additivity holds for the Leray-Schauder degree, we have

$$d_{LS}(U_0(t), G, 0) = d_{LS}(U_0(t), G_1, 0) + d_{LS}(U_0(t), G_2, 0), \quad t \in (0, t_0].$$

Letting $t \rightarrow 0$, we obtain our desired conclusion.

(v) We pick $s_0 \in (0, 1)$ and consider the function

$$H_0(s, t, x) := [s_0(1 - s) + s](T_t + f_1)(x) + (1 - s)(1 - s_0)f_2(x).$$

We also consider the operators $T_t^{q(s)}$, for $t \in (0, \infty)$, $q(s) = s_0(1 - s) + s$, $s \in [0, 1]$. We observe that $s_0 \leq q(s) \leq 1$, $1 - q(s) = (1 - s)(1 - s_0)$ and show that there exists $t_0 > 0$ such that

$$0 \notin \left(I + \frac{1}{t}QQ^*(T_t^{q(s)} + q(s)f_1 + (1 - q(s))f_2) \right) (\partial G), \quad (2.2.18)$$

for every $t \in (0, t_0]$, $s \in [0, 1]$. Assume that this not true. Then there exist sequences $t_n \downarrow 0$, $s_n \in [0, 1]$ with $s_n \rightarrow \tilde{s}$, and $x_n \in \partial G$ such that

$$x_n + \frac{1}{t_n}QQ^* \left(T_{t_n}^{q(s_n)} + q(s_n)f_1 + (1 - q(s_n))f_2 \right) (x_n) = 0.$$

or

$$x_n = -\frac{1}{t_n}QQ^* \left(T_{t_n}^{q(s_n)} + f^{q(s_n)} \right) (x_n),$$

where $f^{q(s_n)} = q(s_n)f_1 + (1 - q(s_n))f_2$. Then

$$\begin{aligned} & \langle T_{t_n}^{q(s_n)}x_n + f^{q(s_n)}(x_n), x_n \rangle \\ &= -\frac{1}{t_n} \left\langle T_{t_n}^{q(s_n)}x_n + f^{q(s_n)}(x_n), QQ^*(T_{t_n}^{q(s_n)}x_n + f^{q(s_n)}(x_n)) \right\rangle \\ &= -\frac{1}{t_n} \left\langle Q^*(T_{t_n}^{q(s_n)}x_n + f^{q(s_n)}(x_n)), Q^*(T_{t_n}^{q(s_n)}x_n + f^{q(s_n)}(x_n)) \right\rangle \quad (2.2.19) \\ &= -\frac{1}{t_n} \|Q^*(T_{t_n}^{q(s_n)}x_n + f^{q(s_n)}(x_n))\|^2 \\ &\leq 0 \end{aligned}$$

which implies

$$\langle T_{t_n}^{q(s_n)} x_n, x_n \rangle \leq -\langle f^{q(s_n)}(x_n), x_n \rangle \leq \|f^{q(s_n)}\| \|x_n\|.$$

From this and the strong quasiboundedness of T , we obtain

$$\langle T_{t_n}^{q(s_n)} x_n, x_n \rangle \leq K, \quad \text{for all } n \text{ and some } K > 0.$$

In fact, this is a consequence of the boundedness of the sequences $\{f^{q(s_n)}\}$, $\{x_n\}$ and the strong quasiboundedness of the sequence of operators $T_{t_n}^{q(s_n)} = q(s_n)T_{q(s_n)t_n}x_n$ (see Lemmas 2.1.5 and 2.1.10). Hence, we may assume that $T_{t_n}^{q(s_n)}x_n \rightharpoonup h^*$ and, by the boundedness of $f^{q(s)}$, we may also assume that $f^{s_n}(x_n) \rightharpoonup v^*$, so that $T_{t_n}^{q(s_n)}x_n + f^{s_n}(x_n) \rightharpoonup h^* + v^* \equiv w^*$. Using the properties of Q , Q^* , as in the proof Lemma 2.1.8, we obtain $w^* = 0$, which implies

$$\limsup_{n \rightarrow \infty} \langle T_{t_n}^{q(s_n)} x_n + f^{q(s_n)}(x_n), x_n - x_0 \rangle \leq 0. \quad (2.2.20)$$

We claim that

$$\limsup_{n \rightarrow \infty} \langle f^{q(s_n)}(x_n), x_n - x_0 \rangle \leq 0. \quad (2.2.21)$$

Assume that it is not true. Then there exist subsequences of $\{s_n\}$, $\{x_n\}$, respectively, denoted again by $\{s_n\}$, $\{x_n\}$, respectively, such that

$$\lim_{n \rightarrow \infty} \langle f^{q(s_n)}(x_n), x_n - x_0 \rangle > 0.$$

Using this with (2.2.20), we get, possibly at the expense of choosing new subsequences,

$$\begin{aligned} & q(\tilde{s}) \lim_{n \rightarrow \infty} \langle T_{q(s_n)t_n} x_n, x_n - x_0 \rangle \\ &= \lim_{n \rightarrow \infty} q(s_n) \langle T_{q(s_n)t_n} x_n, x_n - x_0 \rangle \\ &= \lim_{n \rightarrow \infty} \langle T_{t_n}^{q(s_n)} x_n, x_n - x_0 \rangle < 0. \end{aligned} \quad (2.2.22)$$

Since $q(\tilde{s}) \geq s_0 > 0$, we have

$$\lim_{n \rightarrow \infty} \langle T_{q(s_n)t_n} x_n, x_n - x_0 \rangle < 0. \quad (2.2.23)$$

Invoking Lemma 2.1.6, (i), with S equal to the zero operator, we see that (2.2.23) is impossible. Consequently, (2.2.21) is true and the fact that $f^{q(s_n)}$ is a (S_+) -homotopy implies that $x_n \rightarrow x_0 \in \partial G$. Starting now with

$$\limsup_{n \rightarrow \infty} \langle f^{q(s_n)}(x_n), x_n - x_0 \rangle = 0, \quad (2.2.24)$$

we obtain, possibly for suitable subsequences,

$$\lim_{n \rightarrow \infty} \langle T_{q(s_n)t_n} x_n, x_n - x_0 \rangle \leq 0. \quad (2.2.25)$$

This, however, and Lemma 2.1.6 imply $x_0 \in D(T^{q(\tilde{s})}) = D(T)$ and $h^* \in T^{q(\tilde{s})}x_0$. Thus,

$$w^* = 0 = h^* + v^* \in (T^{q(\tilde{s})} + f^{q(\tilde{s})})(x_0) \subset (T^{q(\tilde{s})} + f^{q(\tilde{s})})(D(T) \cap \partial G).$$

This is another contradiction. Thus, our assertion about (2.2.18) is true.

Now, we see that for each $t \in (0, t_0]$ the mapping $(s, x) \rightarrow T_t^{q(s)}x = q(s)T_{q(s)t}x$ is continuous by Proposition 1.1.12. This implies the continuity of the mapping

$$H_1 : (s, x) \rightarrow \frac{1}{t}QQ^* \left(T_t^{q(s)} + q(s)f_1 + (1 - q(s))f_2 \right) (x)$$

on $[0, 1] \times \overline{G}$. From Lemma 2.1.12, we see that since $s_0 \leq q(s) \leq 1$, we have the boundedness of the set

$$\left\{ \left(T_t^{q(s)} + q(s)f_1 + (1 - q(s))f_2 \right) (x) \mid (s, x) \in [0, 1] \times \overline{G} \right\}.$$

and the compactness of the set

$$\overline{H_1([0, 1], \overline{G})}.$$

Thus, the degree

$$d\left(I + \frac{1}{t}QQ^* \left(T_t^{q(s)} + q(s)f_1 + (1 - q(s))f_2, G, 0\right)\right) \quad (2.2.26)$$

is well defined and constant for all $s \in [0, 1]$. We also pick t_0 sufficiently small so that

$$d(T + f_1, G, 0) = d_{LS}\left(I + \frac{1}{t}QQ^* (T_t + f_1, G, 0)\right), \quad (2.2.27)$$

for all $t \in (0, t_0]$. We see now that the constant degree in (2.2.26) equals, for $s = 1$, the degree in (2.2.27), which is independent of the parameter t . It follows that for every $s \in [0, 1]$ and every $t \in (0, t_0]$ the degree in (2.2.26) is constant. Taking the limit in it as $t \rightarrow 0$, we obtain the desired conclusion

$$d(q(s)(T + f_1) + (1 - q(s))f_2, G, 0) = d(T + f_1, G, 0).$$

Since s_0 is an arbitrary number in $(0, 1)$, we have

$$d(H(s, \cdot), G, 0) = d(s(T + f_1) + (1 - s)f_2, G, 0) = d(T + f_1, G, 0), \quad s \in (0, 1].$$

This complete the proof of Part (v).

(vi) We know from (v) that

$$d(H(s, \cdot), G, 0) = d(T + f_1, G, 0), \quad s \in (0, 1].$$

We need to show that this is also true for $s = 0$. To this end, we show first that there exists $t_0 > 0$ such that

$$0 \notin \left(I + \frac{1}{t}QQ^*(s(T_t + f_1) + (1 - s)f_2)\right)(\partial G), \quad s \in [0, 1], t \in (0, t_0]. \quad (2.2.28)$$

Suppose that this is not true. Then there exist sequences $x_n \in \partial G$ with $x_n \rightharpoonup x$, $s_n \in [0, 1]$ with $s_n \rightarrow s_0$, and $t_n \in (0, \infty)$ with $t_n \downarrow 0$ such that

$$x_n + \frac{1}{t_n}QQ^*(s_n(T_{t_n} + f_1)x_n + (1 - s_n)f_2(x_n)) = 0. \quad (2.2.29)$$

We claim that, eventually, $s_n > 0$. Otherwise, from

$$x_n + \frac{1}{t_n}QQ^*(f_2(x_n)) = 0,$$

with a possible subsequence of $\{x_n\}$ denoted again by $\{x_n\}$, we obtain

$$\begin{aligned} \langle f_2(x_n), x_n \rangle &= -\frac{1}{t_n} \langle f_2(x_n), QQ^*f_2(x_n) \rangle \\ &= -\frac{1}{t_n} \|Q^*f_2(x_n)\|^2 \leq 0, \end{aligned}$$

which contradict the fact that f_2 is in class Γ_ϕ .

Now, we may assume that $s_n > 0$ for all n . We first see that

$$\begin{aligned} &\langle s_n(T_{t_n} + f_1)x_n + (1 - s_n)f_2(x_n), x_n \rangle \\ &= -\frac{1}{t_n} \langle s_n(T_{t_n} + f_1)x_n + (1 - s_n)f_2(x_n), \\ &\quad QQ^*(s_n(T_{t_n} + f_1)x_n + (1 - s_n)f_2(x_n)) \rangle \\ &= -\frac{1}{t_n} \|Q^*(s_n(T_{t_n} + f_1)x_n + (1 - s_n)f_2(x_n))\|^2 \\ &\leq 0 \end{aligned} \quad (2.2.30)$$

If $s_0 = 0$, then we get from (2.2.30)

$$\begin{aligned} (1 - s_n) \langle f_2(x_n), x_n \rangle &\leq -s_n \langle T_{t_n}x_n, x_n \rangle - s_n \langle f_1(x_n), x_n \rangle \\ &\leq -s_n \langle f_1(x_n), x_n \rangle, \end{aligned}$$

which implies

$$\limsup_{n \rightarrow \infty} \langle f_2x_n, x_n \rangle \leq 0,$$

i.e., a contradiction to the fact f_2 in Γ_ϕ because it implies that, for a subsequence $\{x_{n_k}\}$ of $\{x_n\}$, we have

$$\lim_{k \rightarrow \infty} \|x_{n_k}\| = 0.$$

Let $s_0 > 0$. Then from

$$\langle T_{t_n} x_n, x_n \rangle \leq -\langle f_1(x_n), x_n \rangle + \left(\frac{1}{s_n} - 1 \right) \langle f_2(x_n), x_n \rangle \leq -\langle f_1(x_n), x_n \rangle$$

and the strong quasiboundedness of T , we deduce that $T_{t_n} x_n \rightharpoonup y^* \in X^*$ and $s_n(T_{t_n} + f_1)x_n + (1 - s_n)f_2(x_n) \rightharpoonup w^* \in X^*$. Since x_n is bounded and $t_n \rightarrow 0$ it follows from (2.2.29) that

$$QQ^*(s_n(T_{t_n} + f_1)x_n + (1 - s_n)f_2(x_n)) \rightarrow 0.$$

Also, from the complete continuity of QQ^* we deduce that

$$QQ^*(s_n(T_{t_n} + f_1)x_n + (1 - s_n)f_2(x_n)) \rightarrow QQ^*(w^*).$$

By the uniqueness of the limit and the injectivity of QQ^* , we have $w^* = 0$ and, along with (2.2.30),

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \langle s_n(T_{t_n} + f_1)x_n + (1 - s_n)f_2(x_n), x_n - x \rangle \\ & \leq \limsup_{n \rightarrow \infty} \langle s_n(T_{t_n} + f_1)x_n + (1 - s_n)f_2(x_n), x_n \rangle \\ & \quad + \limsup_{n \rightarrow \infty} [-\langle s_n(T_{t_n} + f_1)x_n + (1 - s_n)f_2(x_n), x \rangle] \\ & \leq 0. \end{aligned}$$

This implies in turn, as in the proof of the part of (v) starting with (2.2.23),

$$\limsup_{n \rightarrow \infty} \langle s_n f_1(x_n) + (1 - s_n)f_2(x_n), x_n - x \rangle \leq 0.$$

Since $s f_1 + (1 - s)f_2$ is a (S_+) -homotopy, we obtain $x_n \rightarrow x \in \partial G$ and

$$\limsup_{n \rightarrow \infty} \langle T_{t_n} x_n, x_n - x \rangle = 0.$$

Applying Lemma 2.1.6, it follows that $x \in D(T)$ and $y^* \in Tx$. Thus, $0 \in H(s, \partial G)$, which contradicts our assumption on H . It follows that (2.2.28) is true.

We now pick t_0 sufficiently small so that besides the validity of (2.2.28) we also have the validity of

$$d(T + f_1, G, 0) = d_{LS} \left(I + \frac{1}{t} QQ^*(T_t + f_1), G, 0 \right), \quad (2.2.31)$$

$$d(f_2, G, 0) = d_{LS} \left(I + \frac{1}{t} QQ^*(f_1), G, 0 \right), \quad t \in (0, t_0]. \quad (2.2.32)$$

It is easy to see now, as before, that the mapping

$$H_1(s, x) = x + \frac{1}{t} QQ^*(s(T_t + f_1)x + (1 - s)f_2(x)), \quad (s, x) \in [0, 1] \times \overline{G}$$

is a Leray-Schauder homotopy for every, but fixed, $t \in (0, t_0]$. In fact, the mapping $H_1(s, x)$ is uniformly continuous in s w.r.t. $x \in \overline{G}$ (see Lemma 2.1.12 for $s = 1$). Also, it is compact in $x \in \overline{G}$. Consequently, the degree

$$d_{LS}(H_1(s, \cdot), G, 0)$$

is constant for $s \in [0, 1]$. Thus, for a fixed $t \in (0, t_0]$,

$$\begin{aligned} d_{LS}(H_1(1, \cdot), G, 0) &= d_{LS}(I + \frac{1}{t} QQ^*(T_t + f_1), G, 0) \\ &= d(T + f_1, G, 0) \\ &= d_{LS}(H_1(0, \cdot), G, 0) = d_{LS}(I + \frac{1}{t} QQ^*(f_2), G, 0) \\ &= d(f_2, G, 0). \end{aligned}$$

and the proof of this part is complete.

(vii) We first show that there exists $t_0 > 0$ such that

$$0 \notin \left(I + \frac{1}{t} QQ^*(T_t + sf_1 + (1 - s)f_2) \right) (\partial G), \quad s \in [0, 1], \quad t \in (0, t_0]$$

and

$$d(T + f_1, G, 0) = d_{LS}(I + \frac{1}{t}QQ^*(T_t + f_1, G, 0), \quad t \in (0, t_0].$$

Assume that the first equality above is not true. Then there exist sequences $t_n \downarrow 0$, $s_n \in [0, 1]$ with $s_n \rightarrow \tilde{s}$, and $x_n \in \partial G$ such that

$$x_n + \frac{1}{t_n}QQ^*(T_{t_n} + s_n f_1 + (1 - s_n)f_2)(x_n) = 0.$$

or

$$x_n = -\frac{1}{t_n}QQ^*(T_{t_n} + s_n f_1 + (1 - s_n)f_2)(x_n).$$

We observe that

$$\begin{aligned} \langle T_{t_n}x_n + f^{s_n}(x_n), x_n \rangle &= -\frac{1}{\epsilon_n} \langle T_{t_n}x_n + f^{s_n}(x_n), QQ^*(T_{t_n}x_n + f^{s_n}(x_n)) \rangle \\ &= -\frac{1}{\epsilon_n} \langle Q^*(T_{t_n}x_n + f^{s_n}(x_n)), Q^*(T_{t_n}x_n + f^{s_n}(x_n)) \rangle \\ &= -\frac{1}{\epsilon_n} \|Q^*(T_{t_n}x_n + f^{s_n}(x_n))\|^2 \\ &\leq 0, \end{aligned}$$

where $f^{s_n}(x_n) = s_n f_1 + (1 - s_n)f_2$. This implies the boundedness of $T_{t_n}x_n$ by the boundedness of $\{f^{s_n}(x_n)\}$ and the strong quasiboundedness of T (see Lemma 2.1.5). Thus, we may assume that $T_{t_n}x_n \rightharpoonup h^*$ and $f^{s_n}(x_n) \rightharpoonup v^*$. The desired contradiction follows now as in (v). It is therefore omitted. We observe that the degree

$$d(I + \frac{1}{t}QQ^*(T_t + s f_1 + (1 - s)f_2), G, 0)$$

is well defined and constant for $s \in [0, 1]$. This follows from the fact that the mapping

$$H_1(s, x) = \frac{1}{t}QQ^*(T_t + s f_1(x) + (1 - s)f_2(x))$$

is continuous on $[0, 1] \times \overline{G}$ and the set

$$\overline{H_1([0, 1], \overline{G})}$$

is compact. Consequently,

$$\begin{aligned} d_{LS}(I + \frac{1}{t}QQ^*(T_t + sf_1 + (1-s)f_2), G, 0) &= d_{LS}(I + \frac{1}{t}QQ^*(T_t + f_1, G, 0) \\ &= d(T + f_1, G, 0) \end{aligned}$$

for all $s \in [0, 1]$. Taking the limit above as $t \rightarrow 0$, we obtain

$$d(T + sf_1 + (1-s)f_2, G, 0) = d(T + f_1, G, 0),$$

and the proof of the theorem is complete. ■

Remark 2.2.7 *As we have already mentioned above, one of the reasons we have studied the homotopy H in Part (v) of Theorem 2.2.6 above is the interesting fact that when $T : X \rightarrow X^*$ is single-valued, maximal monotone, demicontinuous and bounded, this homotopy can be easily used to show that our degree $d(T + f, G, 0)$ is actually the same as that of Berkovits [5], Browder [13] (here, $T + f$, like f , is demicontinuous, bounded and of type (S_+)) and Skrypnik [48], [49].*

The homotopy statement of Theorem 2.2.6, (vi), is applicable in many existence problems on Nonlinear Analysis. In fact, such homotopies H can be defined as $H(t, x) = t(T + f + \varepsilon J) + (1-t)\varepsilon J$, where T is maximal monotone and f is demicontinuous, bounded and of type (S_+) . In many cases, homotopies like the one in (vii) of Theorem 1 may also be very useful in obtaining the solvability of various relevant problems in Nonlinear Analysis.

Let $T : X \supset D(T) \rightarrow 2^{X^*}$ be maximal monotone with $0 \in T(0)$ and let $f_1, f_2 : \overline{G} \rightarrow X^*$ be demicontinuous, bounded and of type (S_+) . Kobayashi and Otani have shown in [33] that the family of operators $\{T^s\}$, $s \in [0, 1]$, with $T^s x = sTx$, is not a pseudomonotone homotopy unless $\overline{D(T)} = X$. They have also shown in [33] that the mapping $H(t, x) := s(T + f_1) + (1-t)f_2$ is not, generally, an admissible homotopy for the Browder degree. As we have demonstrated in Theorem 2.2.6, (vi), this mapping $H(t, x)$ is an admissible homotopy for our degree when the mapping f_2 is of type Γ_ϕ . We believe that this is true for the Browder degree as well.

2.3 Degree for Unbounded Perturbation of Type (S_+)

In this section we deal with the extension of the definition of the topological degree theory to maps of the form $T + f$, where T is a strongly quasibounded maximal monotone operator with $0 \in T(0)$, and f is a possibly unbounded demicontinuous map of type (S_+) . We do assume that f is strongly quasibounded. The idea here is to suitably reduce $T + f$ to define a unique topological degree for the resulting bounded function. This was done by Berkovits in [5] for $T = 0$. We remind the reader that “demicontinuous, strongly quasibounded and (S_+) ” does not necessarily imply “bounded”. For example, $f(x) = \ln(x + 1)$, $x \in (-1, \infty)$, is demicontinuous, strongly quasibounded and of type (S_+) with constant $S = S(M) = M + 1$, where $|x| \leq M$ and $\langle f(x), x \rangle = xf(x) \leq M$. However, it is not bounded. The extension of the degree is a consequence of the following four Lemmas.

Lemma 2.3.1 *Let $T : X \supset D(T) \cap \overline{G} \rightarrow 2^{X^*}$ be a strongly quasibounded maximal monotone operator with $0 \in T(0)$. Let $f : \overline{G} \rightarrow X^*$ be demicontinuous, strongly quasibounded and of type (S_+) . Then the set*

$$K = \{x \in D(T) \cap \overline{G} / Tx + f(x) \ni 0\}$$

is a compact subset of X .

Proof. If K is empty or finite, we are done. Otherwise, let $\{x_n\} \subset K$ be an infinite sequence. By the definition of K , we have $Tx_n + fx_n \ni 0$. Since $x_n \in \overline{G}$ and G is bounded, we may assume that $x_n \rightharpoonup x \in X$ and $y_n + f(x_n) = 0$ for some $y_n \in Tx_n$.

From

$$\langle f(x_n), x_n \rangle = \langle y_n + f(x_n), x_n \rangle - \langle y_n, x_n \rangle \leq 0,$$

and the quasiboundedness of f , it follows that $f(x_n)$ is bounded and, since $y_n + f(x_n) = 0$, we also have that y_n is bounded. We may thus assume that $y_n \rightharpoonup y \in X^*$. Also,

$$\limsup_{n \rightarrow \infty} \langle y_n + f(x_n), x_n - x \rangle = 0.$$

We are going to show that

$$\limsup_{n \rightarrow \infty} \langle f(x_n), x_n - x \rangle \leq 0. \quad (2.3.33)$$

Assume that this is not true. Then there exist a subsequence of $\{x_n\}$, denoted again by $\{x_n\}$, such that

$$\lim_{n \rightarrow \infty} \langle f(x_n), x_n - x \rangle > 0.$$

This implies

$$\lim_{n \rightarrow \infty} \langle y_n, x_n - x \rangle < 0.$$

Since we also have $y_n \rightarrow y$, we use

$$\langle y_n, x_n \rangle = \langle y_n, x_n - x \rangle + \langle y_n, x \rangle \quad (2.3.34)$$

to obtain

$$\limsup_{n \rightarrow \infty} \langle y_n, x_n \rangle < \limsup_{n \rightarrow \infty} \langle y_n, x \rangle = \langle y, x \rangle.$$

Let $z \in D(T)$ and $z^* \in Tz$. Then, as in the proof of Lemma 2.1.8, we have

$$\begin{aligned} \liminf_{n \rightarrow \infty} \langle y_n, x_n \rangle &\geq \liminf_{n \rightarrow \infty} \langle y_n, z \rangle + \langle z^*, x - z \rangle \\ &= \langle y, z \rangle + \langle z^*, x - z \rangle. \end{aligned} \quad (2.3.35)$$

Thus, by (2.3.35),

$$\langle y, z \rangle + \langle z^*, x - z \rangle < \langle y, x \rangle,$$

or

$$\langle y - z^*, x - z \rangle > 0. \quad (2.3.36)$$

Since (z, z^*) is arbitrary in $G(T)$ and T is maximal monotone, we have $x \in D(T)$ and $y \in Tx$. Taking $z = x$ and $z^* = y$ in (2.3.36), we obtain a contradiction. Consequently, (2.3.33) is true. Since f is of type (S_+) , it follows that $x_n \rightarrow x \in \overline{G}$ and $f(x_n) \rightarrow f(x)$. Now, repeating the above argument starting from (2.3.34), we obtain (2.3.36), where " $>$ " is replaced by " \geq ". By the maximal monotonicity of T we

have $x \in D(T)$ and $y \in Tx$. Hence it follows that $Tx + f(x) \ni 0$ and the compactness of K is proved. ■

The proof of the following Lemma can be found in [5, p. 25].

Lemma 2.3.2 *Let $S \subset G$ be any fixed compact set of X and $f : \overline{G} \rightarrow X^*$ an unbounded map of type (S_+) . Then there exists an open set G' and $R > 0$ such that*

- (a) $S \subset G' \subset G$,
- (b) $\|f(x)\| \leq R$ for all $0 \leq s \leq 1$ and $x \in \overline{G'}$.

Remark 2.3.3 *Lemma 2.3.2 means that by restricting the domain of f to G' we get a bounded map. By applying Lemma 2.3.2 to f and to the compact set $S = (T + f)^{-1}(0)$ we get*

Lemma 2.3.4 *Let $T : X \supset D(T) \cap \overline{G} \rightarrow 2^{X^*}$ be a strongly quasibounded maximal monotone operator with $0 \in T(0)$ and $f : \overline{G} \rightarrow X^*$ demicontinuous strongly quasibounded of type (S_+) with $0 \notin (T + f)(\partial G)$. Then there exists an open set $G' \subset X$ such that*

- (a) $(T + f)^{-1}(0) \subset G' \subset G$,
- (b) $f(\overline{G'})$ is bounded.

In view of Lemma 2.3.4 the restriction of f to $\overline{G'}$ is bounded demicontinuous of type (S_+) and $0 \notin (T + f)(\partial G')$. Hence, the degree $d(T + f, G', 0)$ is well defined.

Lemma 2.3.5 *Let $T : X \supset D(T) \cap \overline{G} \rightarrow 2^{X^*}$ be a strongly quasibounded maximal monotone operator with $0 \in T(0)$ and $f : \overline{G} \rightarrow X^*$ be demicontinuous strongly quasibounded of type (S_+) with $0 \notin (T + f)(\partial G)$. Then the value $d(T + f, G', 0)$ does not depend on the choice of G' , provided that it satisfies the conditions (a) and (b) of Lemma 2.3.2.*

Proof. Let G_1 and G_2 be two open sets satisfying (a) and (b) of Lemma 2.3.2. It is easy to see that $G_1 \cap G_2$ also satisfies the same conditions, and by the additivity

property of d we have $d(T + f, G_1, 0) = d(T + f, G_1 \cap G_2, 0) = d(T + f, G_2, 0)$, which complete the proof. ■

Definition 2.3.6 *Let $T : X \supset D(T) \cap \overline{G} \rightarrow 2^{X^*}$ be a strongly quasibounded maximal monotone operator with $0 \in T(0)$ and $f : \overline{G} \rightarrow X^*$ demicontinuous strongly quasibounded of type (S_+) with $0 \notin (T + f)(\partial G)$, we define*

$$\hat{d}(T + f, G, 0) = d(T + f, G', 0) \quad (2.3.37)$$

where G' is any open set satisfying (a) and (b) of Lemma 2.3.2.

If $y \notin (T + f)(\partial G)$, we define

$$\hat{d}(T + f, G, y) = \hat{d}(T + f - y, G, 0). \quad (2.3.38)$$

In particular, if f is bounded, then we can choose G' to be G , and thus $\hat{d}(T + f, G, 0) = d(T + f, G, 0)$. This means that \hat{d} and d coincide on bounded demicontinuous type (S_+) -perturbations of maximal monotone operators.

If the operator T is demicontinuous with $\text{int}D(T) \supset \overline{G}$, then the operator $T + f$ is demicontinuous and of type (S_+) . If it is unbounded on \overline{G} , we may use Lemmas 2.6 and 2.7 in Berkovits [5] in order to show the existence of the set G' as above, and, subsequently, the existence of the associated degree mapping.

2.4 Degree for Quasimonotone Perturbation

In this section we extend the definition of the topological degree for $T + f$ to the case where $T : X \supset D(T) \rightarrow 2^{X^*}$ is a strongly quasibounded maximal monotone operator with $0 \in T(0)$ and $f : \overline{G} \rightarrow X^*$ is demicontinuous quasimonotone. Since, however the image $(T + f)(A)$ of a closed set $A \subset \overline{G}$ is not always closed, we encounter some difficulty if we wish the degree to verify conditions (i) - (iv) of Definition 1.2.1 For this reason, we will have to modify slightly the definition of the topological degree function.

Definition 2.4.1 Let $T : X \supset D(T) \rightarrow 2^{X^*}$ be a strongly quasibounded maximal monotone operator with $0 \in T(0)$, $f : \overline{G} \rightarrow X^*$ a demicontinuous quasimonotone map, and $y \notin \overline{(T + f)(\partial G)}$. We associate to each such triplet $(T + f, G, y)$ an integer valued function $d_q(T + f, G, y)$, which is said to be a degree function in the extended or weak sense, if the following conditions are satisfied see [5]:

- (i') If $d_q(T + f, G, y) \neq 0$, then $y \in \overline{(T + f)(G)}$.
- (ii') Let G_1 and G_2 be two open disjoint subsets of G and assume that y is a point of X^* such that $y \notin \overline{(T + f)(\overline{G} \setminus (G_1 \cup G_2))}$. Then $d_q(T + f, G, y) = d_q(T + f, G_1, y) + d_q(T + f, G_2, y)$.
- (iii') Let $H : [0, 1] \times X^* \rightarrow 2^{X^*}$ be a quasimonotone perturbation of maximal monotone homotopy, and let $\{y(s); 0 \leq s \leq 1\}$ be a continuous curve in X^* . Let $H(s, x) = T_s x + f_s x$ and assume that there exists $r > 0$ such that for the ball $B_r(y(s)) = \{z \in X^*; \|z - y(s)\| < r\}$, we have $((T_s + f_s)(\partial G)) \cap B_r(y(s)) = \emptyset$ for all $0 \leq s \leq 1$. Then $d_q(T_s + f_s, G, y(s))$ is constant in s on $[0, 1]$.
- (iv') $d_q(J, G, y) = +1$ if $y \in J(G)$.

The following two lemmas are needed before the definition of the topological degree

Lemma 2.4.2 Let $T : X \supset D(T) \rightarrow 2^{X^*}$ be a strongly quasibounded maximal monotone operator with $0 \in T(0)$ and $f : \overline{G} \rightarrow X^*$ be demicontinuous quasimonotone. Let A be a closed subset of \overline{G} , and y a point of X^* such that $y \notin \overline{(T + f)(A)}$, then there exists $\epsilon_0 > 0$ such that $y \notin \overline{(T + f + \epsilon J)(A)}$ for all $0 < \epsilon < \epsilon_0$.

Proof. Suppose that this is not true. Then there exist sequences $\{\epsilon_n\} \subset (0, \infty)$ with $\epsilon_n \downarrow 0$ and $\{x_n\} \subset A$ such that

$$(T + f + \epsilon_n J)(x_n) \ni y.$$

This implies that $y_n + f(x_n) = y - \epsilon_n J x_n \rightarrow y \in \overline{(T + f)(A)}$, for some $y_n \in T x_n$, which is a contradiction. ■

Lemma 2.4.3 *Let $T : X \supset D(T) \rightarrow 2^{X^*}$ be a strongly quasibounded maximal monotone operator with $0 \in T(0)$ and $f : \overline{G} \rightarrow X^*$ be demicontinuous quasimonotone, and y a point of X^* such that $y \notin \overline{(T + f)(\partial G)}$. Then $\hat{d}(T + f + \epsilon J, G, y)$ is constant for all $0 < \epsilon < \epsilon_0$, where $\epsilon_0 > 0$ is given in lemma 2.4.2 with $A = \partial G$.*

Proof. Let ϵ_1, ϵ_2 be such that $0 < \epsilon_1 < \epsilon_2 < \epsilon_0$ and consider the homotopy

$$T + (1 - s)(f + \epsilon_1 J) + s(f + \epsilon_2 J)$$

or

$$T + f + \epsilon_s J,$$

where $\epsilon_s = (1 - s)\epsilon_1 + s\epsilon_2$, $0 \leq s \leq 1$, lies between ϵ_1 and ϵ_2 . Then $y \notin (T + f + \epsilon_s J)(\partial G)$ for any s in $[0, 1]$ by Lemma 2.4.2, and the assertion follows from the homotopy invariance property of \hat{d} . ■

Now we are ready to give the following definition.

Definition 2.4.4 *For every strongly quasibounded maximal monotone operator $T : X \supset D(T) \rightarrow 2^{X^*}$ with $0 \in T(0)$, every demicontinuous quasimonotone $f : \overline{G} \rightarrow X^*$ and every $y \in X^* \setminus \overline{(T + f)(\partial G)}$, we define*

$$d_q(T + f, G, y) = \hat{d}(T + f + \epsilon J, G, y),$$

$0 < \epsilon < \epsilon_0$, where ϵ_0 is given by Lemma 2.4.2.

Theorem 2.4.5 *The integer valued function d_q given by the definition above is a degree function in the weak sense on the class of all demicontinuous quasimonotone perturbation of strongly quasibounded maximal monotone operators. It is invariant under quasimonotone perturbation homotopies and normalized by J .*

Proof. Let us now verify the condition (i') - (iv') given at the beginning of this section.

(i') Suppose $d_q(T + f, G, y) \neq 0$. By Lemma 2.4.2, there exists $\varepsilon_0 > 0$ such that $\hat{d}(T + f + \varepsilon J, G, y) \neq 0$ for $0 < \varepsilon < \varepsilon_0$. Thus, for each ε , $0 < \varepsilon < \varepsilon_0$, there exists a point $x_\varepsilon \in G$ such that

$$Tx_\varepsilon + f(x_\varepsilon) + \varepsilon Jx_\varepsilon \ni y.$$

This implies that $y_\varepsilon + f(x_\varepsilon) + \varepsilon Jx_\varepsilon = y$, for some $y_\varepsilon \in Tx_\varepsilon$. As $\varepsilon \rightarrow 0$, we get $y \in \overline{(T + f)(G)}$.

(ii') Let G_1 and G_2 be two disjoint open subsets of G and assume that $y \notin \overline{(T + f)(\overline{G} \setminus G_1 \cup G_2)}$. By applying Lemma 2.4.2 for $A = (\overline{G} \setminus G_1 \cup G_2)$ we find $\varepsilon_0 > 0$ such that $y \notin (T + f + \varepsilon J)(A)$ for $0 < \varepsilon < \varepsilon_0$. Thus, the additivity property of \hat{d} implies

$$\hat{d}(T + f + \varepsilon J, G, y) = \hat{d}(T + f + \varepsilon J, G_1, y) + \hat{d}(T + f + \varepsilon J, G_2, y),$$

for $0 < \varepsilon < \varepsilon_0$, and (ii') follows from the definition.

(iii') Let $H : [0, 1] \times X^* \rightarrow 2^{X^*}$ be a quasimonotone perturbation of maximal monotone homotopy, and let $\{y(s), 0 \leq s \leq 1\}$ be a continuous curve in X^* . Denote $H(s, \cdot)$ by $T_s(\cdot) + f_s(\cdot)$ and suppose that there exists $r > 0$ such that $(B_r(y(s))) \cap ((T_s + f_s)(\partial G)) = \emptyset$ for all $0 \leq s \leq 1$. Proceeding as in Lemma 2.4.2 we find $\varepsilon_0 > 0$ such that $y(s) \notin ((T_s + f_s) + \varepsilon J)(\partial G)$ for $0 \leq s \leq 1$ and $0 < \varepsilon < \varepsilon_0$. Hence, by Definition 2.4.4 and the homotopy invariance of \hat{d} , we get

$$d_q(H(s, \cdot), G, y(s)) = d_q(H(s, \cdot) + \varepsilon J, G, y(s)) = \text{constant}$$

for all $0 \leq s \leq 1$ and $0 < \varepsilon < \varepsilon_0$.

(iv') For every sufficiently small $\varepsilon > 0$, we have $d_q(J, G, y) = \hat{d}(J + \varepsilon J, G, y) = +1$ if $y \in J(G)$. The proof is now complete. ■

3 APPLICATIONS

3.1 The Subdifferential Operator

In this section we recall the definition and properties of the subdifferential operator.

Let C be a convex set of X , a function ϕ is proper on C if $\phi(x) > -\infty$ for all $x \in C$ and $\phi(x) < \infty$ at least in one point $x \in C$.

Let $\phi : X \rightarrow \overline{\mathbb{R}}$ be a proper convex function, we denote by $\partial\phi(x)$ the set of all $x^* \in X^*$ such that

$$\phi(y) - \phi(x) \geq \langle x^*, y - x \rangle \quad \forall y \in X.$$

$\partial\phi(x)$ is called the subdifferential of ϕ at x .

Remark 3.1.1 *If ϕ is Gâteaux differentiable at $x \in X$, then it is subdifferentiable at x and $\partial\phi = \{\phi'(x)\}$.*

The following results which proof can be found in [44] are true:

Theorem 3.1.2 *If ϕ is a lower semicontinuous proper convex function on X , the $\partial\phi$ is a maximal monotone operator from X to X^* .*

Theorem 3.1.3 *Let $T : X \rightarrow 2^{X^*}$ be a multivalued mapping. In order that there exist a lower semicontinuous proper convex function ϕ on X such that $T = \partial\phi$, it is necessary and sufficient that T be maximal cyclically monotone operator. Moreover, in this case T determines ϕ uniquely up to an additive constant.*

We collect now some elementary properties of the subdifferential:

- For any $x \in X$, the set $\partial\phi(x)$ is convex and weakly* closed in X^* .

- $D(\partial\phi) \subset \text{dom}\phi$, or equivalently, $x \notin \text{dom}\phi$ implies that $\phi(x) = \infty$.
- ϕ has a minimum value at $x \in D(\partial\phi)$ if and only if $0 \in \partial\phi(x)$.
- A subdifferentiable function ϕ is convex *l.s.c.* on any open convex set $C \subset \text{dom}\phi$.

Let $C \subset X$ a closed convex set and consider $\phi_C : X \rightarrow \mathbb{R}_+ \cup \{\infty\}$ defined by

$$\phi_C(x) = \begin{cases} 0, & \text{if } x \in C, \\ \infty, & \text{otherwise.} \end{cases} \quad (3.1.1)$$

The function ϕ_C is proper, convex and lower semicontinuous on X , and $x^* \in \partial\phi_C(x)$, for $x \in C$, if and only if

$$\langle x^*, y - x \rangle \leq 0, \quad y \in C.$$

We also have

$$\begin{cases} D(\partial\phi_C) = C \text{ and } 0 \in \partial\phi_C(x), \ x \in C, \\ \partial\phi_C(x) = \{0\}, \ x \in \text{int}C. \end{cases}$$

The operator $\partial\phi_C : X \rightarrow 2^{X^*}$ is maximal monotone with $0 \in \text{int}D(\partial\phi)$ and $0 \in \partial\phi_C(0)$. It is thus strongly quasibounded (see for example Kenmochi [31]). Now if $\text{int}C \neq \emptyset$ and we add to $\partial\phi$ a maximal monotone operator $\tilde{T} : X \rightarrow 2^{X^*}$, with $0 \in \tilde{T}(0)$ then we have an operator $T = \partial\phi_C + \tilde{T}$, which is a nontrivial example of an operator T that satisfies our assumptions, since T is maximal monotone, and $0 \in \text{int}D(T) = \text{int}D(\partial\phi_C)$.

In particular for $C = \overline{B}_r$ then see Kenmochi [31] the subdifferential operator is given by the following:

$$\partial\phi_{\overline{B}_r}(x) = \begin{cases} 0, & \|x\| < r, \\ \{\lambda Jx : \lambda \geq 0\}, & \|x\| = r, \\ \emptyset, & \|x\| > r. \end{cases}$$

3.2 Application in Partial Differential Equations

In this section we are interested in discussing an application of our theory in the field of differential equation. We first state and prove an existence result, which we apply in the example that follow.

Theorem 3.2.1 *Let $T : X \supset D(T) \rightarrow 2^{X^*}$ be strongly quasibounded and maximal monotone with $0 \in D(T)$ and $0 \in T(0)$. Let $f : \overline{G} \rightarrow X^*$ be demicontinuous, bounded and of type (S_+) , where $G \subset X$ is open, bounded and contains 0. Assume that for every $\lambda \geq 0$ we have*

$$Tx + f(x) + \lambda Jx \not\supset 0, \quad x \in \partial G. \quad (3.2.2)$$

Then there is a solution $x \in G$ of the inclusion $Tx + f(x) \ni 0$.

Proof. We show first that for all $t \in [0, 1]$ and all sufficiently small $\varepsilon > 0$ we have

$$0 \notin H([0, 1] \times \partial G), \quad (3.2.3)$$

where

$$H(t, x) := t(T + f + \varepsilon J)x + (1 - t)Jx, \quad (t, x) \in [0, 1] \times \overline{G}.$$

To this end, we assume that the contrary is true. Then there exist sequences $t_n \in [0, 1]$, $\varepsilon_n \in (0, \infty)$, $x_n \in \partial G$, $x_n^* \in Tx_n$ such that $t_n \rightarrow t_0$, $\varepsilon_n \rightarrow 0$, $x_n \rightharpoonup x_0$ and

$$t_n(x_n^* + f(x_n) + \varepsilon_n Jx_n) + (1 - t_n)Jx_n = 0. \quad (3.2.4)$$

We assume first that $t_n > 0$, $t_0 = 0$. We have

$$\begin{aligned} \langle f(x_n), x_n \rangle &\leq \langle x_n^* + f(x_n) + \varepsilon_n Jx_n, x_n \rangle \\ &= - \left(\frac{1}{t_n} - 1 \right) \langle Jx_n, x_n \rangle \\ &= - \left(\frac{1}{t_n} - 1 \right) \|x_n\|^2. \end{aligned}$$

This contradicts the boundedness of $\langle f(x_n), x_n \rangle$, because it implies

$$\lim_{n \rightarrow \infty} \langle f(x_n), x_n \rangle = -\infty.$$

Now, we assume that $t_0 \in (0, 1]$. Using the monotonicity of J , we obtain

$$\begin{aligned} \langle x_n^* + f(x_n), x_n - x_0 \rangle &= -\varepsilon_n \langle Jx_n, x_n - x_0 \rangle - \left(\frac{1}{t_n} - 1 \right) \langle Jx_n, x_n - x_0 \rangle \\ &\leq -\varepsilon_n \langle Jx_n, x_n - x_0 \rangle - \left(\frac{1}{t_n} - 1 \right) \langle Jx_0, x_n - x_0 \rangle \end{aligned} \quad (3.2.5)$$

which implies

$$\limsup_{n \rightarrow \infty} \langle x_n^* + f(x_n), x_n - x_0 \rangle \leq 0. \quad (3.2.6)$$

If we assume that

$$\limsup_{n \rightarrow \infty} \langle f(x_n), x_n - x_0 \rangle > 0, \quad (3.2.7)$$

we obtain a contradiction, for an appropriate subsequence of $\{x_n\}$, if necessary, from Lemma 2.1.6 for $T = 0$. Thus, we must have

$$\limsup_{n \rightarrow \infty} \langle f(x_n), x_n - x_0 \rangle \leq 0. \quad (3.2.8)$$

Since f is of type (S_+) , we must have $x_n \rightarrow x_0 \in \partial G$ and $f(x) \rightarrow f(x_0)$. Using this in 3.2.4, we get

$$x_n^* \rightarrow -f(x_0) - \left(\frac{1}{t_0} - 1 \right) Jx_0.$$

Using the demiclosedness of T , (Lemma 2.1.2, (i)), we have $x_0 \in D(T)$ and

$$Tx_0 + f(x_0) + \left(\frac{1}{t_0} - 1 \right) Jx_0 \ni 0.$$

This, however, is a contradiction to our assumption (3.2.2).

All the other possibilities for $\{t_n\}$ can be handled either trivially, or as above. They are therefore omitted. It follows that (3.2.3) is true for all sufficiently small $\varepsilon > 0$, say, for all $\varepsilon \in (0, \varepsilon_0]$.

At this point we invoke Theorem 2.2.6, (v), in order to obtain the invariance of our degree mapping on the homotopy function $H(t, x)$. In particular, we have

$$d(T + f + \varepsilon J, G, 0) = d(I, G, 0) = 1, \quad \varepsilon \in (0, \varepsilon_0].$$

By Theorem 2.2.6, (ii), we have the solvability in G of the problem

$$Tx + f(x) + \varepsilon Jx \ni 0,$$

for all $\varepsilon \in (0, \varepsilon_0]$. Letting $\varepsilon_n = 1/n$, we may assume that there exist $x_n \in D(T) \cap G$, $x_n^* \in T(x_n)$ such that

$$x_n^* + f(x_n) + (1/n)Jx_n = 0.$$

Thus,

$$\lim_{n \rightarrow \infty} \langle x_n^* + f(x_n), x_n - x_0 \rangle = \lim_{n \rightarrow \infty} (1/n) \langle Jx_n, x_n - x_0 \rangle = 0. \quad (3.2.9)$$

Working with subsequences, if necessary, we see that (3.2.9) implies that (3.2.7) is impossible, and that (3.2.8) implies that $x_n \rightarrow x_0 \in \overline{G}$, $f(x_n) \rightarrow f(x_0)$ and $x_n^* \rightarrow -f(x_0)$. Using again the demiclosedness of T (Lemma 2.1.2, (i)), we see that $x_0 \in D(T)$ and

$$Tx_0 + f(x_0) \ni 0.$$

By our assumption (3.2.2), $x_0 \notin \partial G$. This completes the proof. ■

We consider the space $X = W_0^{m,p}(\Omega)$ with the integer $m \geq 1$, the number $p \in (1, \infty)$, and the domain $\Omega \subset \mathbb{R}^N$. We let N_0 denote the number of all multi-indices $\alpha = (\alpha_1, \dots, \alpha_N)$ such that $|\alpha| = \alpha_1 + \dots + \alpha_N \leq m$. For every $\xi = (\xi_\alpha)_{|\alpha| \leq m} \in \mathbb{R}^{N_0}$ we have the representation $\xi = (\eta, \zeta)$, where $\eta = (\eta_\alpha)_{|\alpha| \leq m} \in \mathbb{R}^{N_1}$, $\zeta = (\zeta_\alpha)_{|\alpha|=m} \in \mathbb{R}^{N_2}$ and $N_0 = N_1 + N_2$. We let

$$\xi(u) = (D^\alpha u)_{|\alpha| \leq m}, \quad \eta(u) = (D^\alpha u)_{|\alpha| \leq m-1}, \quad \zeta(u) = (D^\alpha u)_{|\alpha|=m}$$

where

$$D^\alpha = \prod_{i=1}^N \left(\frac{\partial}{\partial x_i} \right)^{\alpha_i}.$$

We also set $q = p/(p-1)$.

We consider the partial differential operator in divergence form

$$(Au)(x) = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, u(x), \dots, D^m u(x)), \quad x \in \Omega.$$

The coefficients $A_\alpha : \Omega \times \mathbb{R}^{N_0} \rightarrow \mathbb{R}$ are assumed to be Carathéodory functions, i.e. each $A_\alpha(x, \xi)$ is measurable in x for fixed $\xi \in \mathbb{R}^{N_0}$ and continuous in ξ for almost all $x \in \Omega$. We consider the following conditions.

(A₁) There exist $p \in (1, \infty)$, $c_1 > 0$ and $\kappa_1 \in L^q(\Omega)$ such that

$$|A_\alpha(x, \xi)| \leq c_1 |\xi|^{p-1} + \kappa_1(x), \quad x \in \Omega, \quad \xi \in \mathbb{R}^{N_0}, \quad |\alpha| \leq m.$$

(A₂) The Leray-Lions condition

$$\sum_{|\alpha|=m} (A_\alpha(x, \eta, \zeta_1) - A_\alpha(x, \eta, \zeta_2)) (\zeta_{1\alpha} - \zeta_{2\alpha}) > 0$$

is satisfied for every $x \in \Omega$, $\eta \in \mathbb{R}^{N_1}$, $\zeta_1, \zeta_2 \in \mathbb{R}^{N_2}$ with $\zeta_1 \neq \zeta_2$.

(A₃)

$$\sum_{|\alpha| \leq m} (A_\alpha(x, \xi_1) - A_\alpha(x, \xi_2)) (\xi_{1\alpha} - \xi_{2\alpha}) \geq 0$$

is satisfied for every $x \in \Omega$, $\xi_1, \xi_2 \in \mathbb{R}^{N_0}$.

(A₄) There exist $c_2 > 0$, $\kappa_2 \in L^1(\Omega)$ such that

$$\sum_{|\alpha| \leq m} A_\alpha(x, \xi) \xi_\alpha \geq c_2 |\xi|^p - \kappa_2(x), \quad x \in \Omega, \quad \xi \in \mathbb{R}^{N_0}$$

If an operator $\tilde{T} : W_0^{m,p}(\Omega) \rightarrow W^{-m,q}$, is given by

$$\langle \tilde{T}u, v \rangle = \int_{\Omega} \sum_{|\alpha| \leq m} A_{\alpha}(x, \xi(u)) D^{\alpha} v, \quad u, v \in W_0^{m,p},$$

then conditions (A_1) , (A_3) imply that it is bounded, continuous and monotone (cf, e.g., Kittilä [32, pp. 25-26], Pascali and Sburlan [40, pp. 274-275],). Since it is continuous, it is also maximal monotone.

Similarly, condition $(A1)$, with A replaced by B , implies that the operator $f : W_0^{m,p}(\Omega) \rightarrow W^{-m,q}(\Omega)$, defined by

$$\langle f(u), v \rangle = \int_{\Omega} \sum_{|\alpha| \leq m} B_{\alpha}(x, \xi(u)) D^{\alpha} v, \quad u, v \in W_0^{m,p}(\Omega),$$

is a bounded continuous mapping. We also know that conditions $(A1)$, $(A2)$ and $(A4)$, with B in place of A everywhere, imply that the operator f is of type (S_+) (cf. Kittilä [32, p. 27]).

We consider a proper closed convex subset K of X such that $0 \in \text{int}K$. Let $\varphi_K : X \rightarrow \mathcal{R}_+ \cup \{\infty\}$ be defined by

$$\phi_K(x) = \begin{cases} 0 & \text{if } x \in K, \\ \infty & \text{otherwise.} \end{cases} \quad (3.2.10)$$

The function φ_K is proper, convex and lower semicontinuous on X , and $x^* \in \partial\varphi_K(x)$, for $x \in K$, if and only if

$$\langle x^*, y - x \rangle \leq 0, \quad y \in K.$$

Also,

$$\begin{cases} D(\partial\phi_K) = K \text{ and } 0 \in \partial\phi_K(x), \ x \in K, \\ \partial\phi_K(x) = \{0\}, \ x \in \text{int}K. \end{cases}$$

The operator $\partial\varphi_K : X \rightarrow 2^{X^*}$ is maximal monotone with $0 \in \text{int}D(\partial\varphi_K)$ and $0 \in \partial\varphi_K(0)$. It is thus strongly quasibounded. For these facts see, e.g., Kenmochi [31]. If we add to $\partial\varphi_K$ a nontrivial maximal monotone operator $T_0 : X \rightarrow 2^{X^*}$, $0 \in T_0(0)$, then we end up with an operator $\tilde{T} = \partial\varphi_K + T_0$, which is a nontrivial example of an operator T that may be covered by our present theory.

Theorem 3.2.2 *Assume that the operators T, f are defined as above with $T(0) = 0$. Assume, further, that T satisfies conditions (A1),(A3) and f satisfies conditions (A1),(A2),A(4). Let K be a proper closed convex subset of X with $0 \in \text{int}K$, and assume that*

$$Tx + \partial\varphi_K(x) + f(x) + \lambda Jx \not\supseteq 0, \quad (\lambda, x) \in [0, \infty) \times (K \cap \partial G), \quad (3.2.11)$$

where $G \subset X$ is open and bounded with $0 \in G$. Then the Dirichlet boundary value problem

$$\begin{aligned} (Au)(x) + (\partial\varphi_K(u))(x) + (Bu)(x) &= 0, & x \in \Omega, \\ (D^\alpha u)(x) &= 0, & x \in \partial\Omega, |\alpha| \leq m-1, \end{aligned} \quad (3.2.12)$$

has a “weak” solution $u \in K \cap G \subset X$ which satisfies the inclusion

$$Tu + (\partial\varphi_K)(u) + f(u) \ni 0. \quad (3.2.13)$$

If, moreover, the inclusion

$$Tu + ((\partial\varphi_K)(u) \setminus \{0\}) + f(u) \ni 0$$

has no solution $u \in \partial K$, then this weak solution $u \in K \cap G \subset X$ satisfies the equation

$$Tu + f(u) = 0.$$

Proof. This is an application of the existence result, Theorem 3.2.1, in the previous section. We note that the operator $\tilde{T} := T + \partial\varphi_K : K \rightarrow 2^{X^*}$ is maximal monotone,

strongly quasibounded, and such that $0 \in \text{int}D(\tilde{T}) = \text{int}D(\partial\varphi_K) = \text{int}K$. ■

We should note here that we cannot necessarily handle Theorem 3.2.2 without the term $\partial\varphi_K$ in the inclusion (3.2.13), although $\varphi_K(x) = \{0\}$, $x \in \text{int}K$. In fact, without this term, the boundary condition (3.2.11) does not necessarily hold when the set $K \cap \partial G$ is replaced by just the set ∂G . In other words, the set $K \cap \partial G$, which is used in order to place the desired solution inside K , is generally “smaller” than the set $D(T) \cap \partial G = X \cap \partial G = \partial G$.

In particular for $m = 2$, we consider the space $X = W_0^{2,2}(\Omega)$ where the domain $\Omega \subset \mathbb{R}^2$ is bounded and we can choose the operators A and B as below in order to define T and f .

$$\begin{aligned} A_{(0,0)}(x, y, \xi) &= x^2 + y^2 + \xi_0^{\frac{3}{5}} \\ A_{(1,0)}(x, y, \xi) &= x + \xi_1 \\ A_{(0,1)}(x, y, \xi) &= 0 \\ A_{(2,0)}(x, y, \xi) &= xy^3 + \xi_3 \\ A_{(1,1)}(x, y, \xi) &= 3\xi_4 \\ A_{(0,2)}(x, y, \xi) &= 0 \end{aligned}$$

The partial differential operator in divergence form

$$(Au)(x) = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, u(x), \dots, D^m u(x)), \quad x \in \Omega.$$

becomes

$$(Au)(x, y) = x^2 + y^2 - 1 + u^{\frac{3}{5}}(x, y) - \frac{\partial^2 u}{\partial x^2}(x, y) + \frac{\partial^4 u}{\partial x^4}(x, y) + 3 \frac{\partial^4 u}{\partial x^2 \partial y^2}(x, y) \quad (x, y) \in \Omega$$

Coefficients A_α are clearly Carathéodory functions.

Now we check conditions A_1 , A_2 , A_3 and A_4 .

(A₁)

$$\begin{aligned}
|A_{(0,0)}(x, y, \xi)| &= |x^2 + y^2 + \xi_0^{\frac{3}{5}}| \\
&\leq x^2 + y^2 + |\xi_0^{\frac{3}{5}}| \\
&\leq x^2 + y^2 + 1 + |\xi| \\
|A_{(1,0)}(x, y, \xi)| &= |x + \xi_1| \leq |x| + |\xi_1| \\
|A_{(0,1)}(x, y, \xi)| &= 0 \\
|A_{(2,0)}(x, y, \xi)| &= |xy^3 + \xi_3| \leq |xy^3| + |\xi| \\
|A_{(1,1)}(x, y, \xi)| &= |3\xi_4| \leq 3|\xi| \\
|A_{(0,2)}(x, y, \xi)| &= 0
\end{aligned}$$

If we choose $c_1 = 3$ and $\kappa_1(x, y) = x^2 + y^2 + 1 + |xy^3|$, condition A_1 is verified.

(A₂) Since

$$(\xi_3 - \eta_3)^2 + 3(\xi_4 - \eta_4)^2 > 0$$

for every $\xi, \eta \in \mathbb{R}^3$, with $\xi \neq \eta$

it follows that (A₂) is satisfied.

(A₃) From

$$(\xi_0^{\frac{3}{5}} - \eta_0^{\frac{3}{5}})(\xi_0 - \eta_0) + (\xi_1 - \eta_1)^2 + (\xi_3 - \eta_3)^2 + 3(\xi_4 - \eta_4)^2 \geq 0$$

we obtain condition (A₃)

(A₄)

$$\begin{aligned}
\sum_{|\alpha| \leq 2} A_\alpha(x, \xi) \xi_\alpha &= (x^2 + y^2 + \xi_0^{\frac{3}{5}}) \xi_0 + (x + \xi_1) \xi_1 + (xy^3 + \xi_3) \xi_3 + 3\xi_4^2 \\
&= (x^2 + y^2) \xi_0 + x \xi_1 + xy^3 \xi_3 + \xi_0^{\frac{8}{5}} + \xi_1^2 + \xi_3^2 + 3\xi_4^2 \\
&= |\xi|^2 - \xi_0^{\frac{2}{5}} - \xi_2^2 + 2\xi_4^2 + (x^2 + y^2) \xi_0 + x \xi_1 + xy^3 \xi_3
\end{aligned}$$

Since Ω is bounded, we can find a constant $c > 0$ such that

$$\sum_{|\alpha| \leq 2} A_\alpha(x, \xi) \xi_\alpha \geq |\xi|^2 - c(\xi_0^{\frac{2}{5}} + \xi_0 + \xi_1 + \xi_2^2 + \xi_3 + 2\xi_4^2)$$

Now if we choose

$$\kappa_2(x, y) = c \left(u^{\frac{2}{5}}(x, y) + u(x, y) + \frac{\partial u(x, y)}{\partial x} + \left(\frac{\partial u(x, y)}{\partial y} \right)^2 + \frac{\partial^2 u(x, y)}{\partial x^2} + 2 \left(\frac{\partial^2 u(x, y)}{\partial x \partial y} \right)^2 \right)$$

we have that condition (A_4) is also true.

Now we define an operator $\tilde{T} : W_0^{2,2}(\Omega) \rightarrow W^{-2,2}$ by

$$\begin{aligned} \langle \tilde{T}u, v \rangle &= \int_{\Omega} (x^2 + y^2 + u^{\frac{3}{5}})v + \left(x + \frac{\partial u}{\partial x}\right) \frac{\partial v}{\partial x} + \left(xy^3 + \frac{\partial^2 u}{\partial x^2}\right) \frac{\partial^2 v}{\partial x^2} \\ &\quad + 3 \frac{\partial^2 u}{\partial x \partial y} \frac{\partial^2 v}{\partial x \partial y} dx dy \quad u, v \in W_0^{2,2}. \end{aligned}$$

Then conditions (A_1) , (A_3) imply that it is bounded, continuous and monotone (cf, e.g., Kittilä[32, pp. 25-26], Pascali and Sburlan [40, pp. 274-275],). From

$$\begin{aligned} |\langle \tilde{T}u, v \rangle| &= \left| \int_{\Omega} (x^2 + y^2 + u^{\frac{3}{5}})v + \left(x + \frac{\partial u}{\partial x}\right) \frac{\partial v}{\partial x} + \left(xy^3 + \frac{\partial^2 u}{\partial x^2}\right) \frac{\partial^2 v}{\partial x^2} \right. \\ &\quad \left. + 3 \frac{\partial^2 u}{\partial x \partial y} \frac{\partial^2 v}{\partial x \partial y} dx dy \right| \\ &\leq \sum_{|\alpha| \leq 2} \int_{\Omega} |(\xi + \kappa_1(x, y))| \left| v + \frac{\partial v}{\partial x} + \dots + \frac{\partial^2 v}{\partial y^2} dx dy \right| \\ &\leq \sum_{|\alpha| \leq 2} \left(\int_{\Omega} (|\xi|^2)^{\frac{1}{2}} + \int_{\Omega} (|\kappa_1|^2)^{\frac{1}{2}} \right) \|v\|_{W_0^{2,2}} \\ &\leq C(\|u\|_{W_0^{2,2}}) \|v\|_{W_0^{2,2}} \end{aligned}$$

we deduce the boundedness of \tilde{T} .

Now for $u, v \in D(\tilde{T})$ we have

$$\langle \tilde{T}u - \tilde{T}v, u - v \rangle = \int_{\Omega} (u^{\frac{3}{5}} - v^{\frac{3}{5}})(u - v) + \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} \right)^2 + \left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 v}{\partial x^2} \right)^2 + 3 \left(\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 v}{\partial x \partial y} \right)^2 \geq 0$$

which show the monotonicity of \tilde{T} .

Proceeding as above we can also show that it is continuous.

Similarly, condition (A_1) , (A_2) and (A_4) , with A replaced by B , implies that the operator $f : W_0^{2,2}(\Omega) \rightarrow W^{-2,2}$, defined by

$$\begin{aligned} \langle f(u), v \rangle &= \int_{\Omega} (B_{(0,0)}((x, y), \xi(u))v + (B_{(1,0)}((x, y), \xi(u)))\frac{\partial v}{\partial x} + \\ &\quad \dots + (B_{(0,2)}((x, y), \xi(u)))\frac{\partial^2 v}{\partial y^2} dx dy \end{aligned}$$

is an operator of type (S_+) . The proof of this fact can be found in Kittilä, [32, pp 27-28].

If we choose,

$$\begin{aligned} B_{(0,0)}(x, y, \xi) &= y^2 \\ B_{(1,0)}(x, y, \xi) &= x + \xi_0 \\ B_{(0,1)}(x, y, \xi) &= \xi_1 \\ B_{(2,0)}(x, y, \xi) &= 0 \\ B_{(1,1)}(x, y, \xi) &= 0 \\ B_{(0,2)}(x, y, \xi) &= x^3 y + \xi_5 \end{aligned}$$

then,

$$(Bu)(x, y) = y^2 - \left(1 + \frac{\partial u}{\partial y}\right) - \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^4 u}{\partial y^4}$$

3.3 Invariance of Domain

In this section we demonstrate the applicability of our degree to an invariance of domain problem. The relevant result below is a special case of a result of Kartsatos and Skrypnik [30]. We include it here in order to show how exactly our theory applies to this problem.

Theorem 3.3.1 *Let $T : X \supset D(T) \rightarrow 2^{X^*}$ be maximal monotone with $D(T)$ open and $0 \in T(0)$, and $f : \overline{G} \rightarrow X^*$ be bounded demicontinuous of type (S_+) , where $G \subset X$ is open and bounded. Assume that $T + f + \delta J$ is locally injective on G for all $\delta \geq 0$. Then $(T + f)(D(T) \cap G)$ is open.*

The proof is based on the following Lemma.

Lemma 3.3.2 *Let $T : X \supset D(T) \rightarrow 2^{X^*}$ be maximal monotone with $D(T)$ open and $0 \in T(0)$, and $f : \overline{G} \rightarrow X^*$ be bounded demicontinuous of type (S_+) , where $G \subset X$ is open and bounded. Assume that there exist an open ball $B_q(0)$ such that $\overline{B_q(0)} \subset G$ and $T + f$ is injective on $\overline{B_q(0)}$. Then there exist an open ball $B_r(0)$ such that*

$$\left((T + f)(D(T) \cap \partial B_q(0)) \right) \cap B_r(0) = \emptyset.$$

Proof. Without loss of generality, we assume that $f(0) = 0$, otherwise we pick $x_0 \in (D(T) \cap G)$ and consider instead $\tilde{f}(x) = f(x + x_0) - f(x_0)$ and $\tilde{G} = G - x_0$.

To prove the Lemma, let assume the contrary and let $r_n \downarrow 0$, $p_n^* \in B_{r_n}(0)$, $\{x_n\} \subset (D(T) \cap \partial B_q(0))$, $u_n^* \in Tx_n$ be such that

$$u_n^* + f(x_n) = p_n^*. \tag{3.3.14}$$

We may assume that $x_n \rightharpoonup x_0$ and $f(x_n) \rightharpoonup f^*$. We are going to show that

$$\limsup_{n \rightarrow \infty} \langle f(x_n), x_n - x_0 \rangle \leq 0. \tag{3.3.15}$$

We assume instead that

$$\limsup_{n \rightarrow \infty} \langle f(x_n), x_n - x_0 \rangle > 0,$$

for a subsequence of $\{x_n\}$ which is denoted again by $\{x_n\}$ and, further

$$\lim_{n \rightarrow \infty} \langle f(x_n), x_n - x_0 \rangle > 0, \tag{3.3.16}$$

for another subsequence which we also denote by $\{x_n\}$. From

$$\langle u_n^*, x_n - x_0 \rangle = -\langle f(x_n), x_n - x_0 \rangle + \langle p_n^*, x_n - x_0 \rangle$$

we obtain

$$\limsup_{n \rightarrow \infty} \langle u_n^*, x_n - x_0 \rangle < 0. \quad (3.3.17)$$

Also, (3.3.14) implies

$$u_n^* \rightharpoonup -f^*$$

and

$$\limsup_{n \rightarrow \infty} \langle u_n^*, x_n \rangle \leq \langle -f^*, x_0 \rangle.$$

Now, fix $x \in D(T)$, $x^* \in Tx$. As in the proof of Lemma 2.1.8, we obtain

$$\begin{aligned} \liminf_{n \rightarrow \infty} \langle u_n^*, x_n \rangle &\geq \liminf_{n \rightarrow \infty} \langle u_n^*, x \rangle + \langle x^*, x_0 - x \rangle \\ &= \langle -f^*, x \rangle + \langle x^*, x_0 - x \rangle. \end{aligned}$$

This and the display above it imply

$$\langle f^* - x^*, x_0 - x \rangle \geq 0,$$

which, by the maximal monotonicity of T , implies $x_0 \in D(T)$. Letting $u_0^* \in Tx_0$ and using the monotonicity of T , we obtain

$$\langle u_n^*, x_n - x_0 \rangle \geq \langle u_0^*, x_n - x_0 \rangle$$

and

$$\limsup_{n \rightarrow \infty} \langle u_n^*, x_n - x_0 \rangle \geq 0,$$

ie a contradiction with (3.3.17). It follows that (3.3.15) is true. Since f satisfies condition (S_+) , we have $x_n \rightarrow x_0 \in \partial B_q(0)$, $f(x_n) \rightarrow f(x_0)$. Thus $u_n^* \rightharpoonup -f(x_0)$ and the demiclosedness of T (see Lemma 2.1.2) imply $x_0 \in D(T)$ and $u_0^* \in Tx_0$. However, since $T + f$ is injective on $\overline{B}_q(0)$ and $0 \notin (T + f)(D(T) \cap \partial B_q(0))$, we have a contra-

diction.

Proof of Theorem 3.3.1. Let $p^* \in (T + f)(D(T) \cap G)$. We show that there is a neighborhood of p^* lying in $(T + f)(D(T) \cap G)$. Without loss of generality, we may assume that $p^* = 0$, $0 \in D(T) \cap G$, $0 \in T(0)$, and $f(0) = 0$. Since $T + f$ is locally injective on G , there exists an open ball $B_q(0)$ such that $\overline{B_q(0)} \subset G$ and $T + f$ is injective on $\overline{B_q(0)}$. It suffices to show that there is an open ball with center at 0 lying in the set $(T + f)(D(T) \cap \overline{B_q(0)})$.

By Lemma 3.3.2 we know that there exist a ball $B_r(0)$ such that

$$\left((T + f)(D(T) \cap \partial B_q(0)) \right) \cap B_r(0) = \emptyset.$$

We now fix $p^* \in B_r(0)$ with $p^* \neq 0$ and consider the path $h(t)$, $t \in [0, 1]$, with $h(0) = 0$ and $h(1) = p^*$. We note that $h(t)$ lies in $B_r(0)$ for all $t \in [0, 1]$. We claim that there exist an $\epsilon_0 > 0$, $s_0 > 0$ such that

$$x + \frac{1}{\epsilon} QQ^*(T_s + f - h(t))(x) = 0 \tag{3.3.18}$$

has no solution $x \in \partial B_q(0)$ for any $0 < \epsilon < \epsilon_0$, $0 < s < s_0$, $t \in [0, 1]$, where $h(t) = tp^*$, $t \in [0, 1]$, and T_s is the Yosida approximant of T . Note that $T + f - h(t)$ is an admissible homotopy of our degree by Theorem 2.2.6.

We assume that the conclusion is not true. Then, without loss of generality, for subsequences $\{x_n\} \subset \partial B_q(0)$ (with $x_n \rightarrow x_0$), $\{s_n\} \subset (0, \infty)$ (with $s_n \downarrow 0$), $\{\epsilon_n\} \subset (0, \infty)$ (with $\epsilon_n \downarrow 0$) and $\{t_n\} \subset [0, 1]$ (with $t_n \rightarrow t_0$) we have

$$x_n + \frac{1}{\epsilon_n} QQ^*(T_{s_n} x_n + f(x_n) - h(t_n)) = 0$$

we may also assume that $T_{s_n} x_n \rightarrow u$, $f(x_n) \rightarrow f^*$ and $w = u + f^* - h(t_0)$.

We are going to show that

$$\limsup_{n \rightarrow \infty} \langle T_{s_n} x_n + f(x_n) - h(t_n), x_n - x_0 \rangle \leq 0.$$

We have that

$$T_{s_n}x_n + f(x_n) - h(t_n) \rightharpoonup w,$$

and since $x_n = -\frac{1}{\epsilon_n}QQ^*(T_{s_n}x_n + f(x_n) - h(t_n))$ and $\{x_n\}$ bounded, it follows that

$$QQ^*(T_{s_n}x_n + f(x_n) - h(t_n)) \rightarrow 0.$$

Now QQ^* is compact and linear, hence

$$QQ^*(T_{s_n}x_n + f(x_n) - h(t_n)) \rightarrow QQ^*(w).$$

By the uniqueness of the limit $QQ^*(w) = 0$ and by the injectivity of QQ^* , it follows that $w = 0$; ie $T_{s_n}x_n + f(x_n) - h(t_n) \rightharpoonup 0$.

Hence

$$\begin{aligned} \langle T_{s_n}x_n + f(x_n) - h(t_n), x_n \rangle &= -\frac{1}{\epsilon_n} \langle T_{s_n}x_n + f(x_n) - h(t_n), QQ^*(T_{s_n}x_n + f(x_n) - h(t_n)) \rangle \\ &= -\frac{1}{\epsilon_n} \|Q^*(T_{s_n}x_n + f(x_n) - h(t_n))\|^2 \\ &\leq 0 \end{aligned}$$

and

$$\lim_{n \rightarrow \infty} \langle T_{s_n}x_n + f(x_n) - h(t_n), x_0 \rangle = 0$$

imply

$$\limsup_{n \rightarrow \infty} \langle T_{s_n}x_n + f(x_n) - h(t_n), x_n - x_0 \rangle \leq 0.$$

Further, we are going to show that

$$\limsup_{n \rightarrow \infty} \langle f(x_n), x_n - x \rangle \leq 0. \quad (3.3.19)$$

Assume that it is not true. Then there exist a subsequence of $\{x_n\}$, denoted again by $\{x_n\}$, such that

$$\lim_{n \rightarrow \infty} \langle f(x_n), x_n - x \rangle > 0.$$

From

$$\langle T_{s_n}x_n + f(x_n) - h(t_n), x_n - x_0 \rangle = \langle T_{s_n}x_n - h(t_n), x_n - x_0 \rangle + \langle f(x_n), x_n - x_0 \rangle$$

we obtain

$$\limsup_{n \rightarrow \infty} \langle T_{s_n}x_n - h(t_n), x_n - x_0 \rangle < 0.$$

and further

$$\limsup_{n \rightarrow \infty} \langle T_{s_n}x_n, x_n - x_0 \rangle < 0.$$

Also we have

$$T_{s_n}x_n \rightharpoonup u.$$

Consequently, along with

$$\langle T_{s_n}x_n, x_n \rangle = \langle T_{s_n}x_n, x_n - x_0 \rangle + \langle T_{s_n}x_n, x_0 \rangle$$

we obtain

$$\limsup_{n \rightarrow \infty} \langle T_{s_n}x_n, x_n \rangle < \limsup_{n \rightarrow \infty} \langle T_{s_n}x_n, x_0 \rangle = \langle u, x_0 \rangle.$$

Let now $y \in D(T)$ and $y^* \in Ty$, then as in the proof of Lemma 2.1.8, we have

$$\liminf_{n \rightarrow \infty} \langle T_{s_n}x_n, x_n \rangle \geq \liminf_{n \rightarrow \infty} \langle T_{s_n}x_n, y \rangle + \langle y^*, x_0 - y \rangle = \langle u, y \rangle + \langle y^*, x_0 - y \rangle. \quad (3.3.20)$$

Then, by (3.3.20)

$$\langle u, y \rangle + \langle y^*, x_0 - y \rangle < \langle u, x_0 \rangle$$

or

$$\langle u - y^*, x_0 - y \rangle > 0. \quad (3.3.21)$$

Since (y, y^*) is arbitrary in $G(T)$ and T is maximal monotone, we have $x_0 \in D(T)$ and $u \in Tx_0$. Taking $y = x_0$ and $y^* = u$ in (3.3.21), we obtain a contradiction. Consequently (3.3.19) is true.

Now since f is of type (S_+) , it follows that $x_n \rightarrow x_0 \in \partial B_q(0)$. Repeating the same argument starting from (3.3.19) where " \leq " is replaced by "=", we obtain (3.3.21)

where " $>$ " is replaced by " \geq ". The demiclosedness of T (see Lemma 2.1.2) imply $x_0 \in D(T)$, $u \in Tx_0$ and $w \in Tx_0 + f(x_0) - h(t_0)$, hence

$$T_{s_n}x_n + f(x_n) - h(t_n) \rightharpoonup 0 \in Tx_0 + f(x_0) - h(t_0)$$

which implies that

$$h(t_0) \in (T + f)(x_0).$$

Since $x_0 \in D(T) \cap \partial B_q(0)$ and $T + f$ is injective on $D(T) \cap \overline{B}_q(0)$ we get a contradiction to $((T + f)(D(T) \cap \partial B_q(0))) \cap B_r(0) = \emptyset$. Consequently, our assertion about (3.3.18) is true.

We consider the homotopy

$$H(t, s, x) = t(T_s + f)(x) + (1 - t)Jx$$

from which we consider the homotopy equation

$$x + \frac{1}{\epsilon}QQ^*(t(T_s + f) + (1 - t)J)(x) = 0. \quad (3.3.22)$$

We are going to show that there exist $\epsilon_1 > 0$ and $s_1 > 0$ such that (3.3.22) has no solution $x \in \partial B_q(0)$ for $0 < s < s_1$, $0 < \epsilon < \epsilon_1$, and $t \in [0, 1]$. To this end, assume that the contrary is true. Then, without loss of generality, we may assume that there exist sequences $\{x_n\} \in \partial B_q(0)$, $\{s_n\} \subset \mathbb{R}_+$, and $\{\epsilon_n\} \subset \mathbb{R}_+$ and $\{t_n\} \subset [0, 1]$ such that $x_n \rightharpoonup x_0$, $T_{s_n}x_n \rightharpoonup u$, $f(x_n) \rightharpoonup f^*$, $Jx_n \rightarrow j^*$, $s_n \rightarrow 0$, $t_n \rightarrow t_0$ and

$$x_n + \frac{1}{\epsilon_n}QQ^*(\eta_n) = 0. \quad (3.3.23)$$

where

$$\eta_n = t_n(T_{s_n} + f)(x_n) + (1 - t_n)Jx_n.$$

Since $\{x_n\}$ is bounded we have that

$$QQ^*(\eta_n) \rightarrow 0,$$

we also have $\eta_n \rightharpoonup w$. QQ^* is compact and linear, hence

$$QQ^*(\eta_n) \rightarrow QQ^*(w).$$

By the uniqueness of the limit we have that $QQ^*(w) = 0$ and by the injectivity of QQ^* , it follows that $w = 0$, ie

$$\eta_n \rightharpoonup 0. \tag{3.3.24}$$

Now using (3.3.23) we have

$$\begin{aligned} \langle \eta_n, x_n \rangle &= \langle \eta_n, -\frac{1}{\epsilon_n} QQ^*(\eta_n) \rangle \\ &= -\frac{1}{\epsilon_n} \langle \eta_n, QQ^*(\eta_n) \rangle \\ &= -\frac{1}{\epsilon_n} \|Q^*(\eta_n)\|^2 \\ &\leq 0 \end{aligned}$$

and

$$\lim_{n \rightarrow \infty} \langle \eta_n, x_0 \rangle = 0$$

imply

$$\limsup_{n \rightarrow \infty} \langle \eta_n, x_n - x_0 \rangle \leq 0.$$

We distinguish two cases:

(i) $t_0 = 0$,

(ii) $t_0 > 0$.

(i) If $t_0 = 0$, then we have

$$\lim_{n \rightarrow \infty} \langle t_n(T_{s_n} + f)(x_n), x_n - x_0 \rangle = 0$$

because $(T_{s_n} + f)(x_n)$ is bounded and $t_n \rightarrow t_0 = 0$

Hence

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle Jx_n, x_n - x_0 \rangle &= \limsup_{n \rightarrow \infty} \langle (1 - t_n)Jx_n, x_n - x_0 \rangle \\ &= \limsup_{n \rightarrow \infty} \langle t_n(T_{s_n} + f)(x_n) + (1 - t_n)Jx_n, x_n - x_0 \rangle \leq 0 \end{aligned}$$

and since J is of type (S_+) , we have $x_n \rightarrow x_0 \in \partial B_q(0)$ and by Lemma 2.1.2 and the demicontinuity of f we have that $x_0 \in D(T)$ and

$$\eta_n \rightarrow 0 = Jx_0.$$

which implies $x_0 = 0$ which is a contradiction since $x_0 \in \partial B_q(0)$.

(ii) If $t_0 > 0$,

We are going to show that

$$\limsup_{n \rightarrow \infty} \langle f(x_n), x_n - x \rangle \leq 0. \quad (3.3.25)$$

Assume that it is not true. Then there exist a subsequence of $\{x_n\}$, denoted again by $\{x_n\}$, such that

$$\lim_{n \rightarrow \infty} \langle f(x_n), x_n - x \rangle > 0.$$

From

$$\langle \eta_n, x_n - x_0 \rangle = \langle t_n T_{s_n} x_n + (1 - t_n)Jx_n, x_n - x_0 \rangle + \langle t_n f(x_n), x_n - x_0 \rangle$$

we obtain

$$\limsup_{n \rightarrow \infty} \langle t_n T_{s_n} x_n + (1 - t_n)Jx_n, x_n - x_0 \rangle < 0.$$

and further

$$\limsup_{n \rightarrow \infty} \langle T_{s_n} x_n, x_n - x_0 \rangle < 0.$$

Also we have

$$T_{s_n} x_n \rightharpoonup u.$$

Consequently, along with

$$\langle T_{s_n} x_n, x_n \rangle = \langle T_{s_n} x_n, x_n - x_0 \rangle + \langle T_{s_n} x_n, x_0 \rangle$$

we obtain

$$\limsup_{n \rightarrow \infty} \langle T_{s_n} x_n, x_n \rangle < \limsup_{n \rightarrow \infty} \langle T_{s_n} x_n, x_0 \rangle = \langle u, x_0 \rangle.$$

Let now $y \in D(T)$ and $y^* \in Ty$, then as in the proof of Lemma 2.1.8, we have

$$\liminf_{n \rightarrow \infty} \langle T_{s_n} x_n, x_n \rangle \geq \liminf_{n \rightarrow \infty} \langle T_{s_n} x_n, y \rangle + \langle y^*, x_0 - y \rangle = \langle u, y \rangle + \langle y^*, x_0 - y \rangle. \quad (3.3.26)$$

Then, by (3.3.26)

$$\langle u, y \rangle + \langle y^*, x_0 - y \rangle < \langle u, x_0 \rangle$$

or

$$\langle u - y^*, x_0 - y \rangle > 0. \quad (3.3.27)$$

Since (y, y^*) is arbitrary in $G(T)$ and T is maximal monotone, we have $x_0 \in D(T)$ and $u \in Tx_0$. Taking $y = x_0$ and $y^* = u$ in (3.3.27), we obtain a contradiction. Consequently (3.3.25) is true.

Now since f is of type (S_+) , it follows that $x_n \rightarrow x_0 \in \partial B_q(0)$ and further by the demiclosedness of T , we arrive at

$$\eta_n \rightarrow 0 \in t_0(T + f)(x_0) + (1 - t_0)Jx_0$$

or

$$0 \in (T + f)(x_0) + \frac{1 - t_0}{t_0} Jx_0.$$

This contradicts the fact that $0 \in (T + f + \frac{1-t_0}{t_0} J)(B_q(0))$ and the operator $T + f + \frac{1-t_0}{t_0} J$ is injective in $D(T) \cap \overline{B}_q(0)$.

It follows that the homotopy equation (3.3.22) has no solution $x \in \partial B_q(0)$ for all $0 < s < s_1$ for some $s_1 > 0$, $0 < \epsilon < \epsilon_1$, for some $\epsilon_1 > 0$, and all $t \in [0, 1]$. We may assume without loss of generality that $s_0 = s_1$ and $\epsilon_0 = \epsilon_1$.

It follows that the Berkovits degree d_B is well defined and we have

$$\begin{aligned}
d_B(H(t, s, \cdot), B_q(0), 0) &= d_B(H(1, s, \cdot), B_q(0), 0) \\
&= d_B(H(0, s, \cdot), B_q(0), 0) \\
&= d_B(J, B_q(0), 0) \\
&= 1.
\end{aligned}$$

Now we consider the homotopy

$$H_1(t, s, x) = T_s x + f(x) - h(t)$$

from which we consider the homotopy equation

$$x + \frac{1}{\epsilon} Q Q^* (T_s x + f(x) - h(t)) = 0.$$

We already know that this equation has no solution $x \in \partial B_q(0)$ and $H_1(t, s, x)$ is admissible for Berkovits degree. We thus have

$$\begin{aligned}
d_B(H_1(t, s, \cdot), B_q(0), 0) &= d_B(H_1(0, s, \cdot), B_q(0), 0) \\
&= d_B(T_s + f, B_q(0), 0) \\
&= d_B(H(1, s, \cdot), B_q(0), 0) \\
&= d_B(H(t, s, \cdot), B_q(0), 0) \\
&= 1.
\end{aligned}$$

Since $d(T + f - p^*, B_q(0), 0) = \lim_{s \rightarrow 0} d_B(T_s + f - p^*, B_q(0), 0) = 1$, we have that $p^* \in (T + f)B_q(0)$. Since p^* is arbitrary in $B_r(0)$, the proof is now complete.

3.4 Eigenvalue Problem

In this section we demonstrate the applicability of our degree to the eigenvalue problem:

$$Tx + f(\lambda, x) \ni 0.$$

The relevant result below is a special case of a result of Kartsatos and Skrypnik [29]. We include it here in order to show how exactly our theory applies to this problem.

We say that the operator $T : X \supset D(T) \rightarrow 2^{X^*}$ satisfies condition (S_q) on a set $A \subset D(T)$ if for every sequence $\{x_n\} \subset A$ such that $x_n \rightarrow x_0 \in X$ and any $y_n^* \in Tx_n$, with $y_n^* \rightarrow$ (some) $y^* \in X^*$, we have $x_n \rightarrow x_0$. If $A = D(T)$, then we say that T satisfies (S_q) .

Let $G \subset X$ be open and bounded, $\Lambda > 0$. An operator $f : [0, \Lambda] \times \overline{G} \rightarrow X^*$ is "demicontinuous" if $[0, \Lambda] \times \overline{G} \ni (t_n, x_n) \rightarrow (t_0, x_0)$ implies $f(t_n, x_n) \rightarrow f(t_0, x_0)$. A demicontinuous operator $f(t, x)$ as above is continuous in t , "uniformly" w.r.t. $x \in \overline{G}$ if $[0, \Lambda] \ni t_n \rightarrow t_0$ implies $f(t_n, x) \rightarrow f(t_0, x)$ uniformly w.r.t. $x \in \overline{G}$. A demicontinuous operator f as above is said to satisfy condition " (S_+) " if for every $\lambda \in (0, \Lambda]$ and every sequence $\{x_n\} \subset \overline{G}$ with $x_n \rightarrow x_0$ and

$$\limsup_{n \rightarrow \infty} \langle f(\lambda, x_n), x_n - x_0 \rangle \leq 0$$

we have $x_n \rightarrow x_0$.

Theorem 3.4.1 *Let $G \subset X$ be an open and bounded set. Let $T : X \supset D(T) \rightarrow 2^{X^*}$ be strongly quasibounded maximal monotone operator with $0 \in T(0)$. Let $f : [0, \Lambda] \times \overline{G} \rightarrow X^*$ be bounded demicontinuous of type (S_+) and such that $f(0, x) = 0$, $x \in \overline{G}$, and $f(t, x)$ is continuous in t uniformly w.r.t $x \in \overline{G}$. Let ϵ, ϵ_0 be positive numbers. Assume that*

(P) there exist $\lambda \in (0, \Lambda]$ such that the inclusion

$$Tx + f(\lambda, x) + \epsilon Jx \ni 0 \tag{3.4.28}$$

has no solution $x \in D(T) \cap G$.

Then

(i) there exists $(\lambda_0, x_0) \in (0, \Lambda] \times (D(T) \cap \partial G)$ such that

$$Tx_0 + f(\lambda_0, x_0) + \epsilon Jx_0 \ni 0; \quad (3.4.29)$$

(ii) if $0 \notin T(D(T) \cap \partial G)$, T satisfies condition (S_q) on ∂G , and property (P) is satisfied for every $\epsilon \in (0, \epsilon_0]$, then there exists $(\lambda_0, x_0) \in (0, \Lambda] \times (D(T) \cap \partial G)$ such that $Tx_0 + f(\lambda_0, x_0) \ni 0$.

Proof. (i) Assume that (3.4.29) is not true. Then for every $\lambda \in (0, \Lambda]$, the equation

$$Tx + f(\lambda, x) + \epsilon Jx \ni 0$$

has no solution $x \in D(T) \cap \partial G$. It is also true for $\lambda = 0$ because of the injectivity of $T + \epsilon J$ by the strict monotonicity of the duality mapping.

We set $H(\lambda, x) \equiv Tx + f(\lambda, x) + \epsilon J$ and observe that

$$H(\lambda, D(T) \cap \partial G) \not\ni 0, \quad \lambda \in [0, \Lambda]. \quad (3.4.30)$$

We are going to show that there exist $\eta_0 > 0$, $t_0 > 0$, $\lambda_0 \in (0, \Lambda]$ such that for every $0 < \eta \leq \eta_0$, $0 < s \leq s_0$, $0 < \lambda \leq \lambda_0$ we have

$$\left(I + \frac{1}{\eta} QQ^*(H_1(t, \lambda, \cdot)) \right) (\partial G) \not\ni 0 \quad (3.4.31)$$

where

$$H_1(t, \lambda, x) \equiv T_t x + f(\lambda, x) + \epsilon Jx.$$

Assume that it is not true, then there exist sequences $\{\eta_n\} \subset (0, \infty)$ with $\eta_n \downarrow 0$, $\{t_n\} \subset (0, \infty)$ with $t_n \downarrow 0$, $\{\lambda_n\} \subset [0, \Lambda]$ with $\lambda_n \rightarrow \lambda_0$, $\{x_n\} \subset \partial G$ with $x_n \rightharpoonup x$, $Jx_n \rightharpoonup j^*$, for some $x \in X$ and $j^* \in X^*$, and such that

$$x_n + \frac{1}{\eta_n} QQ^*(T_{t_n} x_n + f(\lambda_n, x_n) + \epsilon Jx_n) = 0. \quad (3.4.32)$$

or

$$x_n = -\frac{1}{\eta_n} QQ^*(T_{t_n}x_n + f(\lambda_n, x_n) + \epsilon Jx_n). \quad (3.4.33)$$

As in the proof of Lemma 2.1.8, $T_{t_n}x_n$ is bounded, and we may assume that $T_{t_n}x_n \rightharpoonup u$, $f(\lambda_n, x_n) \rightharpoonup v$ and setting $w = u + v + \epsilon J^*$ we have $T_{t_n}x_n + f(\lambda_n, x_n) + \epsilon Jx_n \rightharpoonup w$. From (3.4.33) we have that $QQ^*(T_{t_n}x_n + f(\lambda_n, x_n) + \epsilon Jx_n) = -\eta_n x_n$, and since $\{x_n\}$ is bounded, and $\eta_n \rightarrow 0$, we have that

$$QQ^*(T_{t_n}x_n + f(\lambda_n, x_n) + \epsilon Jx_n) \rightarrow 0.$$

QQ^* is compact and linear, hence

$$QQ^*(T_{t_n}x_n + f(\lambda_n, x_n) + \epsilon Jx_n) \rightarrow QQ^*(w).$$

By the uniqueness of the limit, $QQ^*(w) = 0$ and by the injectivity of QQ^* , it follows that $w = 0$; ie

$$T_{t_n}x_n + f(\lambda_n, x_n) + \epsilon Jx_n \rightharpoonup 0. \quad (3.4.34)$$

Now as in the proof of Lemma 2.1.8 we have,

$$\langle T_{t_n}x_n + f(\lambda_n, x_n) + \epsilon Jx_n, x_n \rangle \leq 0$$

and since by (3.4.34) we also have

$$\langle T_{t_n}x_n + f(\lambda_n, x_n) + \epsilon Jx_n, x \rangle \rightarrow 0.$$

It follows that

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \langle T_{t_n}x_n + f(\lambda_n, x_n) + \epsilon Jx_n, x_n - x \rangle \\ &= \limsup_{n \rightarrow \infty} \langle T_{t_n}x_n + f(\lambda_n, x_n) + \epsilon Jx_n, x_n \rangle \leq 0. \end{aligned}$$

We are going to show that

$$\limsup_{n \rightarrow \infty} \langle f(\lambda_n, x_n) + \epsilon Jx_n, x_n - x \rangle \leq 0. \quad (3.4.35)$$

Assume that it is not true. Then there exist a subsequence of $\{x_n\}$, denoted again by $\{x_n\}$, such that

$$\lim_{n \rightarrow \infty} \langle f(\lambda_n, x_n) + \epsilon Jx_n, x_n - x \rangle > 0.$$

This implies

$$\lim_{n \rightarrow \infty} \langle T_{t_n} x_n, x_n - x \rangle < 0.$$

Since we have $T_{t_n} x_n \rightharpoonup u$, consequently, along with

$$\langle T_{t_n} x_n, x_n \rangle = \langle T_{t_n} x_n, x_n - x \rangle + \langle T_{t_n} x_n, x \rangle$$

we obtain

$$\limsup_{n \rightarrow \infty} \langle T_{t_n} x_n, x_n \rangle < \limsup_{n \rightarrow \infty} \langle T_{t_n} x_n, x \rangle = \langle u, x \rangle.$$

Let now $y \in D(T)$ and $y^* \in Ty$, then, as in the proof of Lemma 2.1.8, we have

$$\begin{aligned} \liminf_{n \rightarrow \infty} \langle T_{t_n} x_n, x_n \rangle &\geq \liminf_{n \rightarrow \infty} \langle T_{t_n} x_n, y \rangle + \langle y^*, x - y \rangle \\ &= \langle u, y \rangle + \langle y^*, x - y \rangle. \end{aligned} \quad (3.4.36)$$

Then, by (3.4.36)

$$\langle u, y \rangle + \langle y^*, x - y \rangle < \langle u, x \rangle$$

or

$$\langle u - y^*, x - y \rangle > 0 \quad (3.4.37)$$

since (y, y^*) is arbitrary in $G(T)$ and T is maximal monotone, we have $x \in D(T)$ and $u \in Tx$. Taking $y = x$ and $y^* = u$ in (3.4.37), we obtain a contradiction. Consequently (3.4.35) is true.

From (3.4.35) it follows that

$$\limsup_{n \rightarrow \infty} \langle f(\lambda_n, x_n), x_n - x \rangle \leq 0, \quad (3.4.38)$$

or

$$\limsup_{n \rightarrow \infty} \langle Jx_n, x_n - x \rangle \leq 0.$$

From

$$\langle f(\lambda_n, x_n), x_n - x \rangle = \langle f(\lambda_n, x_n) - f(\lambda_0, x_n), x_n - x \rangle + \langle f(\lambda_0, x_n), x_n - x \rangle.$$

and using the fact that $f(\lambda_n, x_n) - f(\lambda_0, x_n) \rightarrow 0$, we obtain by (3.4.38) that

$$\limsup_{n \rightarrow \infty} \langle f(\lambda_0, x_n), x_n - x \rangle \leq 0$$

Now since f and J are of type (S_+) , it follows that $x_n \rightarrow x$. Repeating the same argument starting from (3.4.35) we get (3.4.37) where " $>$ " is replaced by " \geq ". By the maximal monotonicity of T we have that $x \in D(T)$ and $u \in Tx$. Hence we get

$$T_{t_n} x_n + f(\lambda_n, x_n) + \epsilon J x_n \rightarrow 0 = u + \epsilon j^* \in Tx + \epsilon Jx,$$

which show that $0 \in (T + \epsilon J)(x)$, $x \in \partial G$. Which is a contradiction with $0 \in (T + \epsilon J)G$ and $T + \epsilon J$ is injective by the strict monotonicity of the duality mapping. Thus our assertion is true.

Now, we fix $s \in (0, s_0]$, $\lambda \in (0, \Lambda]$, $\eta \in (0, \eta]$ and consider the homotopy function

$$x + \frac{1}{\eta} Q Q^*(H_2(t, \lambda, x))$$

where

$$H_2(s, x) \equiv T_t x + f(s\lambda, x) + \epsilon Jx \quad (3.4.39)$$

(3.4.39) is a homotopy of type (S_+) see Kartsatos and Skrypnik [29]. And using the fact that $(T_t + \epsilon J)(0) = 0$, we note that $0 \notin H_2(s, \partial G)$ for any $s \in [0, 1]$ and therefore

$$\begin{aligned} d_B(H_2(s, \cdot), G, 0) &= d_B(H_2(1, \cdot), G, 0) \\ &= d_B(H_2(0, \cdot), G, 0) = d_B(T_t + \epsilon J, G, 0) = 1. \end{aligned}$$

Hence

$$d(H(\lambda, \cdot), G, 0) = \lim_{s \rightarrow 0} d_B(H_1(s, \lambda, \cdot), G, 0)$$

$$= \lim_{s \rightarrow 0} d_B(H_2(1, \cdot), G, 0) = 1$$

because $H_1(s, \lambda, x) = H_2(1, x)$. Thus

$$0 \in (T + f(\lambda, \cdot) + \epsilon J)(D(T) \cap G),$$

which contradict property (P). Therefore (3.4.29) is true.

(ii) The Proof goes true exactly as in Kartsatos and Skrypnik in [29] it is repeated here for completeness.

Let the sequences $\{x_n\} \subset D(T) \cap \partial G$, $u_n^* \in Tx_n$, $\lambda_n \in (0, 1]$ be such that

$$u_n^* + f(\lambda_n, x_n) + (1/n)Jx_n = 0 \quad (3.4.40)$$

We may assume that $\lambda_n \rightarrow \lambda_0 \in [0, \Lambda]$, $x_n \rightarrow x_0$, $f(\lambda_n, x_n) \rightarrow f^*$ and $Jx_n \rightarrow j^*$. We consider two cases:

(j) $\lambda_0 = 0$;

(jj) $\lambda_0 > 0$.

(j). Since for some $u_n^* \in Tx_n$ we have $u_n^* = -f(\lambda_n, x_n) - (1/n)Jx_n \rightarrow 0$ and T satisfies condition (Sq), we have $x_n \rightarrow x_0 \in \partial G$. The closedness of T (see Lemma 2.1.2) implies now that $0 \in Tx_0$, which contradict $0 \notin T(D(T) \cap \partial G)$.

(jj). We are going to show first that

$$\limsup_{n \rightarrow \infty} \langle f(\lambda_n, x_n), x_n - x_0 \rangle \leq 0. \quad (3.4.41)$$

Assume the contrary. Then we may also choose $\{x_n\}$, or a subsequence of it denoted again by $\{x_n\}$, so that

$$\lim_{n \rightarrow \infty} \langle f(\lambda_n, x_n), x_n - x_0 \rangle > 0. \quad (3.4.42)$$

We have

$$\langle u_n^*, x_n - x_0 \rangle = -\langle f(\lambda_n, x_n), x_n - x_0 \rangle - \langle (1/n)Jx_n, x_n - x_0 \rangle,$$

which says

$$\limsup_{n \rightarrow \infty} \langle u_n^*, x_n - x_0 \rangle < 0. \quad (3.4.43)$$

Since, by (3.4.40), $u_n^* \rightharpoonup -c^*$, we also have

$$\langle u_n^*, x_n \rangle = \langle u_n^*, x_n - x_0 \rangle + \langle u_n^*, x_0 \rangle,$$

and

$$\limsup_{n \rightarrow \infty} \langle u_n^*, x_n \rangle < \langle -c^*, x_0 \rangle. \quad (3.4.44)$$

Now we fix $(x, x^*) \in G(T)$ and examine

$$\langle u_n^* - x^*, x_n - x \rangle \geq 0.$$

We obtain

$$\langle u_n^*, x_n \rangle \geq \langle u_n^*, x \rangle + \langle x^*, x_n - x \rangle,$$

which implies

$$\liminf_{n \rightarrow \infty} \langle u_n^*, x_n \rangle \geq \langle -c^*, x \rangle + \langle x^*, x_0 - x \rangle.$$

Combining this and (3.4.44), we find

$$\langle -c^* - x^*, x_0 - x \rangle > 0. \quad (3.4.45)$$

Since T is maximal monotone and $(x, x^*) \in G(T)$ is arbitrary, we get $x_0 \in D(T)$ and $-c^* \in Tx_0$. However, letting $x = x_0$ in (3.4.45) we get a contradiction. Thus, (3.4.44) is true. We observe that

$$\langle f(\lambda_n, x_n), x_n - x_0 \rangle = \langle f(\lambda_n, x_n) - f(\lambda_0, x_n), x_n - x_0 \rangle + \langle f(\lambda_0, x_n), x_n - x_0 \rangle.$$

Using the fact that $f(\lambda_n, x_n) - f(\lambda_0, x_n) \rightarrow 0$, we obtain

$$\lim_{n \rightarrow \infty} \langle f(\lambda_0, x_n), x_n - x_0 \rangle \leq 0.$$

Since f is of type (S_+) , we have $x_n \rightarrow x_0 \in \partial G$, $f(\lambda_n, x_n) \rightarrow f(\lambda_0, x_0) = c^*$ and $u_n^* \rightarrow -f(\lambda_0, x_0)$. The demiclosedness of T (see Lemma 2.1.2) implies $Tx_0 + f(\lambda_0, x_0) \ni 0$, and the proof of the theorem is complete.

4 EXISTENCE AND SURJECTIVITY RESULT

4.1 Noncoercive Mappings

In this section we will discuss some surjectivity results based on our new degree theory. Similiar results have been carry out for example in [40] without using degree theory.

Lemma 4.1.1 *Let $T : X \supset D(T) \rightarrow 2^{X^*}$ be strongly quasibounded maximal monotone with $0 \in T(0)$ and $f : \bar{G} \rightarrow X^*$ be demicontinuous of type (S_+) . Assume that there exists $\bar{x} \in G$ such that*

$$\langle y + f(x), x - \bar{x} \rangle > -\|Tx + f(x)\| \|x - \bar{x}\|, \quad \text{for all } x \in \partial G \text{ and } y \in Tx. \quad (4.1.1)$$

Then $d(T+f, G, 0) = +1$, and there exists at least one $x_0 \in G$ such that $0 \in (T+f)(x_0)$

Proof. It follows from (4.1.1) that $0 \notin (T+f)(\partial G)$ and therefore $d(T+f, G, 0)$ is well defined. Let \bar{J} be the map defined by $\bar{J}(x) = J(x - \bar{x})$, for all $x \in X$. It is easy to see that \bar{J} is an (S_+) mapping. We are going to show that

$$d(T+f, G, 0) = d(\bar{J}, G, 0) = d(J, G, J(\bar{x})). \quad (4.1.2)$$

Since $J(\bar{x}) \in J(G)$, then $d(J, G, J(\bar{x})) = +1$ and the Lemma will follows from (4.1.2).

Consider first an affine homotopy $H(s, x)$ between $T+f$ and \bar{J} , that is

$$\begin{aligned} H(s, x) &= (1-s)(T+f)(x) + s\bar{J}x \\ &= (1-s)Tx + (1-s)f(x) + s\bar{J}x \end{aligned}$$

We claim that $0 \notin H(s, x)$ for every $x \in \partial G$ and $s \in [0, 1]$. Suppose the contrary, that is there exists $x_1 \in \partial G$ and $s_1 \in [0, 1]$ such that $0 \in H(s_1, x_1)$ which means that

$$0 \in (1 - s_1)Tx_1 + (1 - s_1)f(x_1) + s_1\bar{J}(x_1)$$

this imply that for some $y_1 \in Tx_1$ we have

$$(1 - s_1)y_1 + (1 - s_1)f(x_1) + s_1\bar{J}(x_1) = 0. \quad (4.1.3)$$

If $s_1 = 0$, then (4.1.3) is equivalent to $y_1 + f(x_1) = 0$, which contradict (4.1.1).

If $s_1 = 1$, (4.1.3) gives $\bar{J}x_1 = 0$ which implies that $x_1 = \bar{x}$, and again this is a contradiction to (4.1.1). Hence $s_1 \neq 0$ and $s_1 \neq 1$ and (4.1.3) is equivalent to

$$y_1 + f(x_1) = -\frac{s_1}{1 - s_1}\bar{J}(x_1),$$

and

$$\begin{aligned} \langle y_1 + f(x_1), x_1 - \bar{x} \rangle &= \left\langle -\frac{s_1}{1 - s_1}\bar{J}(x_1), x_1 - \bar{x} \right\rangle \\ &= -\frac{s_1}{1 - s_1} \langle J(x_1 - \bar{x}), x_1 - \bar{x} \rangle \\ &= -\frac{s_1}{1 - s_1} \|x_1 - \bar{x}\|^2 \\ &= -\frac{s_1}{1 - s_1} \|x_1 - \bar{x}\| \|x_1 - \bar{x}\| \\ &= -\frac{s_1}{1 - s_1} \|\bar{J}(x_1)\| \|x_1 - \bar{x}\| \\ &= -\|y_1 + f(x_1)\| \|x_1 - \bar{x}\| \\ &\leq -\|Tx_1 + f(x_1)\| \|x_1 - \bar{x}\|, \end{aligned}$$

which is a contradiction to (4.1.1). Hence $d(H(s, \cdot), G, 0)$ is constant and it follows that

$$d(T + f, G, 0) = d(\bar{J}, G, 0). \quad (4.1.4)$$

We consider next an affine homotopy $H_1(s, x)$ between \bar{J} and $J - J(\bar{x})$, that is

$$H_1(s, x) = s\bar{J}(x) + (1 - s)(J(x) - J(\bar{x})), \quad x \in \bar{G}.$$

We claim that $0 \notin H_1(s, x)$ for every $x \in \partial G$ and $s \in [0, 1]$. Indeed if it is not true then there exists $x_1 \in \partial G$ and $s_1 \in [0, 1]$ such that $0 = H_1(s_1, x_1)$ which means that

$$0 = s_1\bar{J}(x_1) + (1 - s_1)(J(x_1) - J(\bar{x})) \quad (4.1.5)$$

If $s_1 = 0$ or $s_1 = 1$ we get by the injectivity of J that $x_1 = \bar{x}$ which contradict (4.1.1). Hence assume that $s_1 \neq 0$ and $s_1 \neq 1$ then (4.1.5) become

$$J(x_1) - J(\bar{x}) = -\frac{s_1}{1 - s_1}J(x_1 - \bar{x}).$$

Now

$$\begin{aligned} \langle J(x_1) - J(\bar{x}), x - \bar{x} \rangle &= \|x\|^2 - \langle J(x), \bar{x} \rangle - \langle J(\bar{x}), x \rangle + \|\bar{x}\|^2 \\ &\geq \|x\|^2 - 2\|x\|\|\bar{x}\| + \|\bar{x}\|^2 \\ &= (\|x\| - \|\bar{x}\|)^2 \\ &\geq 0 \end{aligned}$$

but on the other hand we have

$$\begin{aligned} \left\langle -\frac{s_1}{1 - s_1}J(x - \bar{x}), x - \bar{x} \right\rangle &= -\frac{s_1}{1 - s_1}\|x - \bar{x}\|^2 \\ &\leq 0. \end{aligned}$$

From the two inequalities, it follows that $x_1 = \bar{x}$ which again contradict (4.1.1). We conclude, that $H_1(s, x) \neq 0$ for all $s \in [0, 1]$ and $x \in \partial G$. By the homotopy invariance

we have

$$d(\bar{J}, G, 0) = d(J - J(\bar{x}), G, 0) = d(J, G, J(\bar{x})). \quad (4.1.6)$$

Finally the assertion (4.1.2) follows from (4.1.4) and (4.1.6). ■

Remark 4.1.2 *It is actually possible to show that condition (4.1.1) is equivalent to the following:*

$$-\alpha J(x - \bar{x}) \notin Tx + f(x) \text{ for all } x \in \partial G \text{ and } \alpha \geq 0,$$

which means that there is a homotopy between $T + f$ and \bar{J} that does not vanish on ∂G

Theorem 4.1.3 *Let $T : X \supset D(T) \rightarrow 2^{X^*}$ be strongly quasibounded maximal monotone with $0 \in T(0)$ and $f : \bar{G} \rightarrow X^*$ be demicontinuous and quasimonotone such that $(T + f)(\bar{G})$ is closed in X^* . Assume that there exists $\bar{x} \in G$ such that*

$$\langle y + f(x), x - \bar{x} \rangle > -\|Tx + f(x)\| \|x - \bar{x}\| \text{ for all } x \in \partial G \text{ and } y \in Tx. \quad (4.1.7)$$

Then there exists at least one $x_0 \in G$ such that $0 \in (T + f)(x_0)$ and $d(T + f, G, 0) = +1$ whenever defined.

Proof. Assume first that $0 \in \overline{(T + f)(\partial G)}$. Since $(T + f)(\bar{G})$ is closed, then $0 \in (T + f)(\bar{G})$. Clearly by (4.1.7) $0 \notin (T + f)(\partial G)$ and thus necessarily $0 \in (T + f)(G)$.

Assume next that $0 \notin \overline{(T + f)(\partial G)}$, then the degree $d(T + f, G, 0)$ is well defined. For any $\epsilon > 0$ consider a perturbation map

$$Tx + f_\epsilon(x) = Tx + f(x) + \epsilon J(x - \bar{x})$$

It is easy to see that f_ϵ is demicontinuous and of type (S_+) , and for any $y \in Tx$ we have

$$\langle y + f_\epsilon(x), x - \bar{x} \rangle = \langle y + f(x) + \epsilon J(x - \bar{x}), x - \bar{x} \rangle$$

$$\begin{aligned}
&= \langle y + f(x), x - \bar{x} \rangle + \epsilon \langle J(x - \bar{x}), x - \bar{x} \rangle \\
&> -\|Tx + f(x)\| \|x - \bar{x}\| - \epsilon \|J(x - \bar{x})\| \|x - \bar{x}\| \\
&> -(\|Tx + f(x)\| + \|\epsilon J(x - \bar{x})\|) \|x - \bar{x}\| \\
&> -\|Tx + f(x) + \epsilon J(x - \bar{x})\| \|x - \bar{x}\| \\
&> -\|Tx + f_\epsilon(x)\| \|x - \bar{x}\|
\end{aligned}$$

which say that $T + f_\epsilon$ satisfies the hypothesis of Lemma 4.1.1 for all $\epsilon > 0$ and, consequently we have that $d(T + f_\epsilon, G, 0) = +1$ for all $\epsilon > 0$.

We claim that there exist $\epsilon_1 > 0$ such that

$$0 \notin Tx + (1 - s)(f(x) + \epsilon J(x)) + sf_\epsilon(x)$$

for all $0 \leq s \leq 1$, $0 < \epsilon < \epsilon_1$ and $x \in \partial G$. In fact, if this is not true then there exist sequences $\{s_n\} \subset [0, 1]$, $\{\epsilon_n\}$ with $\epsilon_n \downarrow 0$ and $\{x_n\} \subset \partial G$ such that

$$0 \in Tx_n + (1 - s_n)f(x_n) + \epsilon_n J(x_n) + s_n f_{\epsilon_n}(x_n)$$

which further implies that there exist $y_n \in Tx_n$ such that

$$y_n + (1 - s_n)f(x_n) + \epsilon_n J(x_n) + s_n f_{\epsilon_n}(x_n) = 0$$

or

$$y_n + f(x_n) = -\epsilon_n Jx_n - \epsilon_n J(x_n - \bar{x})$$

and since J is bounded and $\epsilon_n \downarrow 0$, we have that

$$y_n + f(x_n) \rightarrow 0 \in \overline{(T + f)(\partial G)}$$

which is a contradiction. Hence, by Definition 2.4.4

$$d(T + f, G, 0) = d(T + f + \epsilon J, G, 0)$$

$$\begin{aligned}
&= d(T + f_\epsilon, G, 0) \\
&= +1
\end{aligned}$$

for any $\epsilon > 0$, with $0 < \epsilon < \epsilon_1$. Thus $0 \in \overline{(T + f)(\overline{G})}$, and since $(T + f)(\overline{G})$ is closed and $0 \notin (T + f)(\partial G)$, we see that there exist $x_0 \in G$ such that $0 \in (T + f)(x_0)$. ■

As application of the above Theorem, we look at the following Corollary.

Corollary 4.1.4 *Let $T : X \supset D(T) \rightarrow 2^{X^*}$ be strongly quasibounded maximal monotone with $0 \in T(0)$ and $f : \overline{G} \rightarrow X^*$ be bounded demicontinuous and pseudomonotone. Assume that G is open bounded and convex and that (4.1.7) is verified, then there exist at least one $x_0 \in \overline{G}$ such that $0 \in (T + f)(x_0)$*

Proof. We show that $(T + f)(\overline{G})$ is closed. In fact let p_n be a sequence in $(T + f)(\overline{G})$ such that $p_n \rightarrow p$ in X^* , then $p_n = y_n + f(x_n)$ for some $y_n \in Tx_n$ and $x_n \in \overline{G}$. Since G is bounded, there exist a subsequence of $\{x_n\}$ denoted again by $\{x_n\}$ such that $x_n \rightharpoonup x$ and since \overline{G} is weakly closed then $x \in \overline{G}$. More over we have that

$$\langle y_n + f(x_n), x_n - x \rangle \rightarrow 0. \quad (4.1.8)$$

We claim that

$$\limsup_{n \rightarrow \infty} \langle f(x_n), x_n - x \rangle \leq 0. \quad (4.1.9)$$

In fact if it is not true then there exist a subsequence of $\{x_n\}$, denoted again by $\{x_n\}$, such that

$$\lim_{n \rightarrow \infty} \langle f(x_n), x_n - x \rangle > 0.$$

This implies

$$\lim_{n \rightarrow \infty} \langle y_n, x_n - x \rangle < 0.$$

Since $y_n = p_n - f(x_n)$ and f is bounded we can assume that $y_n \rightharpoonup y$ for some y in X^* . Consequently, along with

$$\langle y_n, x_n \rangle = \langle y_n, x_n - x \rangle + \langle y_n, x \rangle$$

we obtain

$$\limsup_{n \rightarrow \infty} \langle y_n, x_n \rangle < \limsup_{n \rightarrow \infty} \langle y_n, x \rangle = \langle y, x \rangle.$$

Now let $z \in D(T)$ and $z^* \in Tz$, then, as in the proof of Lemma 2.1.8, we obtain

$$\begin{aligned} \liminf_{n \rightarrow \infty} \langle y_n, x_n \rangle &\geq \liminf_{n \rightarrow \infty} \langle y_n, z \rangle + \langle z^*, x - z \rangle \\ &= \langle y, z \rangle + \langle z^*, x - z \rangle. \end{aligned} \tag{4.1.10}$$

Then, by (4.1.10)

$$\langle y, z \rangle + \langle z^*, x - z \rangle < \langle y, x \rangle$$

or

$$\langle y - z^*, x - z \rangle > 0 \tag{4.1.11}$$

since (z, z^*) are arbitrary in $G(T)$ and T is maximal monotone, we have $x \in D(T)$ and $y \in Tx$. Taking $z = x$ and $z^* = y$ in (4.1.11), we obtain a contradiction. Consequently (4.1.9) is true.

Now by the pseudomonotonicity of f , we have $\lim_{n \rightarrow \infty} \langle f(x_n), x_n - x \rangle = 0$ and since $x \in \overline{G}$, $f(x_n) \rightharpoonup f(x)$ in X^* . Repeating the proof starting from (4.1.8) we have $\langle y_n, x_n - x \rangle \rightarrow 0$ and we obtain again (4.1.11) where " $>$ " is replaced by " \geq ". Since (z, z^*) are arbitrary in $G(T)$ and T is maximal monotone, we have $x \in D(T)$ and $y \in Tx$. Hence $p_n = y_n + f(x_n) \rightharpoonup p = y + f(x) \in (T + f)(x)$. Hence $p \in (T + f)(\overline{G})$ and then $(T + f)(\overline{G})$ is closed and the assertion follows from Theorem 4.1.3. ■

Let $F : X \rightarrow 2^{X^*}$ be a given operator. We say that F satisfy the property (B) if: (B) for any $z \in X^*$ there exists a neighborhood U of z such that $F^{-1}(U)$ is bounded.

It is easy to see that if F satisfy the property (B) then for every $z_n \in Fx_n$, if $z_n \rightarrow z$, then $\{x_n\}$ is bounded. In fact let $z_n \in Fx_n$ such that $z_n \rightarrow z$. By property (B) there exists a neighborhood U of z such that $F^{-1}(U)$ is bounded. Also since $z_n \rightarrow z$, there exist $N \in \mathbb{N}$ such that $n > N$ implies $z_n \in U$ and now $x_n = F^{-1}(z_n) \in F^{-1}(U)$, which shows the boundedness of $\{x_n\}$.

We can derive the following Theorem.

Theorem 4.1.5 *Let $T : X \supset D(T) \rightarrow 2^{X^*}$ be strongly quasibounded maximal monotone operator with $0 \in T(0)$ and $f : X \rightarrow X^*$ be demicontinuous and quasimonotone map. Assume that $(T+f)(\overline{B_R})$ is closed for each ball $B_R = \{x \in X/\|x\| < R\}$, $R > 0$, and that $T + f$ satisfy the property (B). If there exists $R > 0$ such that*

$$\frac{\langle y + f(x), x \rangle}{\|x\|} + \|Tx + f(x)\| > 0 \text{ for all } \|x\| \geq R, y \in Tx \quad (4.1.12)$$

then $(T + f)(X) = X^*$ ie, the equation $Tx + f(x) \ni p$ admits a solution for any $p \in X^*$.

Proof. Let $p \in X^*$ be fixed, we can choose $R' \geq R$ and $k > 0$ such that

$$\|y + f(x) - tp\| \geq k \text{ for all } t \in [0, 1] \text{ and } \|x\| \geq R' \quad (4.1.13)$$

Indeed, If it is not true, then there exists sequences $\{x_n\} \subset X$ with $\|x_n\| \rightarrow \infty$, $y_n \in Tx_n$ and $\{t_n\} \subset [0, 1]$ such that $\|y_n + f(x_n) - t_n p\| \rightarrow 0$ as $n \rightarrow \infty$. We can assume that $t_n \rightarrow t_0$, which implies that $y_n + f(x_n) \rightarrow t_0 p$. By the property (B), $\{x_n\}$ is bounded, which is a contradiction with our assumption. Thus by the invariance under homotopy we can conclude that $d(T + f, B_{R'}, p) = d(T + f, B_{R'}, 0)$. By (4.1.12) we have

$$\langle y + f(x), x \rangle > -\|Tx + f(x)\|\|x\| \text{ for all } \|x\| = R'.$$

And hence the assumption of Theorem 4.1.3 are satisfied with $\bar{x} = 0$. Thus $d(T + f, B_{R'}, p) = d(T + f, B_{R'}, 0) = +1$, which implies that $p \in \overline{(T + f)(B_{R'})}$, and since $(T + f)(\overline{B_{R'}})$ is closed and $p \notin (T + f)(\partial B_{R'})$, we have $p \in (T + f)(B_{R'})$ and the proof

is complete. ■

4.2 Odd Mappings

In this section we will consider some generalization of Berkovits results concerning odd mappings of type (S_+) which was a generalization of Borsuk's theorem of Leray Schauder theory of odd mappings. We will recall those results in the following propositions.

Proposition 4.2.1 (*Borsuk's Theorem*). *Let G be an open, bounded set of X containing the origin and symmetric, i.e., $-x \in G$ whenever $x \in G$ and let $f : \overline{G} \rightarrow X^*$ be a map of the Leray Schauder type, that is $I - f$ is compact. If $0 \notin f(\partial G)$ and $f(-x) = -f(x)$ for all $x \in \partial G$, Then $d_{LS}(f, G, 0)$ is odd.*

Proposition 4.2.2 . *Let G be an open bounded set of X containing the origin and symmetric, and let $f : \overline{G} \rightarrow X^*$ be demicontinuous of type (S_+) and odd on ∂G , i.e.,*

$$f(-x) = -f(x) \text{ for all } x \in \partial G.$$

Then there exist in \overline{G} a solution of the equation $f(x) = 0$ and, moreover, $d_B(f, G, 0)$ is odd whenever defined.

Proposition 4.2.3 . *Let G be an open bounded set of X containing the origin and symmetric, and let $f : \overline{G} \rightarrow X^*$ be demicontinuous quasimonotone map such that $f(\overline{G})$ is closed. If f is odd on ∂G , i.e.,*

$$f(-x) = -f(x) \text{ for all } x \in \partial G.$$

Then there exist in \overline{G} a solution of the equation $f(x) = 0$ and, moreover, $d_B(f, G, 0)$ is odd whenever defined.

For our next theorem we will need the following

Lemma 4.2.4 . Let $T : X \rightarrow 2^{X^*}$ be a maximal monotone operator, then if T is odd that is

$$-y \in T(-x) \text{ provided that } y \in T(x)$$

then the Yosida approximant T_t is odd.

Proof. Obvious ■

Lemma 4.2.5 Let G be an open bounded symmetric set of X with $0 \in G$, let $T : X \supset D(T) \rightarrow 2^{X^*}$ be maximal monotone and $f : \bar{G} \rightarrow X^*$ be demicontinuous of type (S_+) with $0 \notin (T + f)(\partial G)$. Assume that T and f are odd for all $x \in \bar{G}$ then there exist a symmetric open set \tilde{G} such that

- (i) $(T + f)^{-1}(0) \subset \tilde{G} \subset G$
- (ii) $f(\bar{\tilde{G}})$ is bounded.

The Proof is a consequence of Lemma 2.3.2 and can be found in Berkovits [5] with obvious modification.

We can state now the following theorem

Theorem 4.2.6 Let G be an open bounded symmetric set of X with $0 \in G$, and let $T : X \supset D(T) \rightarrow 2^{X^*}$ be maximal monotone with $0 \in \text{int}D(T)$ and $f : \bar{G} \rightarrow X^*$ be demicontinuous of type (S_+) . Assume that T and f are odd on ∂G that is

$$-y \in T(-x) \text{ when } y \in Tx$$

and

$$f(-x) = -f(x)$$

for all $x \in \partial G$. Then there exists in \bar{G} a solution of the inclusion $(T + f)(x) \ni 0$ and, more over, $d(T + f, G, 0)$ is an odd integer.

Proof. If $0 \in (T + f)(\partial G)$, we are done. If not assume that $0 \notin (T + f)(\partial G)$. So $d(T + f, G, 0)$ is well defined. Define maps:

$$\tilde{T}(x) = \frac{1}{2}(T(x) - T(-x))$$

and

$$\tilde{f}(x) = \frac{1}{2}(f(x) - f(-x))$$

clearly \tilde{T} is maximal monotone with $0 \in \text{int}D(\tilde{T})$, \tilde{f} is demicontinuous of type (S_+) , \tilde{T} and \tilde{f} are odd on \overline{G} and $\tilde{T} + \tilde{f}$ coincide with $T + f$ on ∂G . Hence

$$d(T + f, G, 0) = d(\tilde{T} + \tilde{f}, G, 0). \quad (4.2.14)$$

Now by Lemma (4.2.5) there exists an open symmetric subset \tilde{G} of G such that $(\tilde{T} + \tilde{f})^{-1}(0) \subset \tilde{G}$ and the restriction $\tilde{f} : \overline{\tilde{G}} \rightarrow X^*$ is bounded. By the additivity property of the degree

$$d(\tilde{T} + \tilde{f}, G, 0) = d(\tilde{T} + \tilde{f}, \tilde{G}, 0) = d_B(\tilde{T}_t + \tilde{f}, \tilde{G}, 0) \quad (4.2.15)$$

for all $0 < t < t_0$, where \tilde{T}_t is the Yosida approximant of \tilde{T} . By Lemma (4.2.4) $\tilde{T}_t + \tilde{f}$ is odd on $\overline{\tilde{G}}$. Consequently using Proposition (4.2.2) we have:

$$d_B(\tilde{T}_t + \tilde{f}, \tilde{G}, 0) = \text{odd}, \quad 0 < t < t_0. \quad (4.2.16)$$

Combining (4.2.14), (4.2.15), and (4.2.16) we deduce that $d(T + f, G, 0)$ is odd number, which implies that $0 \in (T + f)(G)$. ■

Theorem 4.2.7 *Let $G \subset X$ be an open bounded symmetric set with $0 \in G$ and let $T : X \supset D(T) \rightarrow 2^{X^*}$ be maximal monotone operator with $0 \in \text{int}D(T)$ and $f : \overline{G} \rightarrow X^*$ be demicontinuous quasimonotone map such that $(T + f)(\overline{G})$ is closed. If T, f are odd on ∂G , then there exist in \overline{G} a solution of the equation $(T + f)(x) = 0$ and $d(T + f, G, 0) = \text{odd}$ whenever defined.*

Proof. If $0 \in \overline{(T+f)(\partial G)} \subset (T+f)(\overline{G})$, the assertion follows. Now suppose that $0 \notin \overline{(T+f)(\partial G)}$, then $d(T+f, G, 0)$ is well defined. Since f , J and T are odd, we have that $f + \epsilon J$ is odd, hence by Definition (2.4.4) we can conclude that

$$d(T+f, G, 0) = d(T+f+\epsilon J, G, 0) = \text{odd}$$

for all $0 < \epsilon < \epsilon'$ for some $\epsilon' > 0$. Hence $0 \in \overline{(T+f)(G)} \subset (T+f)(\overline{G})$ and complete the proof. ■

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