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Applications of Degree Theories to Nonlinear Operator Equations in Banach Spaces

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by

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Dedication

To the memory of my father.

To my mother, wife Bhagabati and daughter Shreya.

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Dhruba R. Adhikari

ABSTRACT

Let X be a real Banach space and G_1, G_2 two nonempty, open and bounded subsets of X such that $0 \in G_2$ and $\overline{G_2} \subset G_1$. The problem $(*)$ $Tx + Cx = 0$ is considered, where $T : X \supset D(T) \to X$ is an accretive or monotone operator with $0 \in D(T)$ and $T(0) = 0$, while $C : X \supset D(C) \to X$ can be, e.g., one of the following types: (a) compact; (b) continuous and bounded with the resolvents of T compact; (c) demicontinuous, bounded and of type (S_+) with T positively homogeneous of degree one; (d) quasi-bounded and satisfies a generalized (S_+)-condition w.r.t. the operator T, while T is positively homogeneous of degree one. Solutions are sought for the problem $(*)$ lying in the set $D(T + C) \cap (G_1 \setminus G_2)$. Nontrivial solutions of $(*)$ exist even when $C(0) = 0$. The degree theories of Leray and Schauder, Browder, and Skrypnik as well as the degree theory by Kartsatos and Skrypnik for densely defined operators T, C are used. The last three degree theories do not assume any compactness conditions on the operator C. The excision and additivity properties of these degree theories are employed, and the main results are significant extensions or generalizations of previous results by Krasnoselskii, Guo, Ding and Kartsatos involving the relaxation of compactness conditions and/or conditions on the boundedness of the operator T. Moreover, a new degree theory developed by Kartsatos and Skrypnik has been used to prove a similar result for operators of type $T + C$, where $T : X \supset D(T) \to 2^{X^*}$ is a multi-valued maximal monotone operator, with $0 \in D(T)$ and $0 \in T(0)$, and $C : X \supset D(C) \to X^*$ is a densely defined quasi-bounded and finitely continuous operator of type $(\tilde{S}_+)$. The problem of existence of nonzero solutions for $Tx + Cx + Gx \ni 0$ is also considered. Here, T is maximal monotone, C is bounded demicontinuous of type (S_+), and G is of class (P). Eigenvalue and invariance of domain results have also been established for the sum $L + T + C : \overline{G} \cap D(L) \to 2^{X^*}$, where $G \subset X$ is open and bounded, $L : X \supset D(L) \to X^*$ densely defined linear maximal monotone, $T : X \to 2^{X^*}$ bounded maximal monotone, and $C : \overline{G} \to X^*$ bounded demicontinuous of type (S_+) w. r. t. $D(L)$.

Chapter 1

Introduction-Preliminaries

In this chapter we give notations, definitions, and some basic results in nonlinear functional analysis that we need in the sequel. We give a description of the organization of other chapters in the last three paragraphs of this chapter.

1.1 Banach Spaces and Operators of Monotone Type

Let X be a real Banach space with norm $\|\cdot\|$, X^* its dual and $J : X \to 2^{X^*}$ the normalized duality mapping. We denote by $[x, x^*]$ an element of $X \times X^*$ whenever $x \in X$, $x^* \in X^*$. We denote by $\langle x, x^* \rangle$ and $\langle x^*, x \rangle$ the value of the functional $x^* \in X^*$ at $x \in X$. Let $K \subset X$ be a closed and convex set which is closed under multiplication by nonnegative scalars and $K \cap (-K) = \{0\}$. Such a set K is called a "cone" and it induces a partial ordering " $\le$ " in X defined by $x \le y$ if $y - x \in K$, $x, y \in X$. If $\{x_n\}$ is a sequence in X, we denote its strong convergence to x_0 in X by $x_n \to x_0$ and its weak convergence in X by $x_n \rightharpoonup x_0$. The symbols $\mathcal{R}$ and $\mathcal{R}_+$ denote $(-\infty, \infty)$ and $[0, \infty)$ respectively. An operator $T : X \supset D(T) \to Y$, with Y another Banach space, is said to be "bounded" if it maps bounded subsets of $D(T)$ onto bounded subsets of Y. Here, $D(T)$ is the domain of T. The operator T is said to be "compact" if it maps bounded subsets of $D(T)$ onto relatively compact subsets in Y. It is called "demicontinuous" if it is strong-weak continuous on $D(T)$. Let $\alpha > 0$. An operator $T : X \supset D(T) \to Y$ is said to be "positively homogeneous of degree α" if, for all $t \in \mathcal{R}_+$, $x \in D(T)$ implies $tx \in D(T)$ and $T(tx) = t^\alpha Tx$. We denote by I the identity operator on X.

Definition 1.1 *The duality mapping $J : X \to X^*$ is defined as*

$$Jx = \{x^* \in X^* : \langle x^*, x\rangle = \|x\|^2 = \|x^*\|^2\}, \quad x \in X.$$

By the Hahn-Banach theorem, Jx is nonempty for every $x \in X$. If J is single-valued, we omit the braces in the above set.

Definition 1.2 *An operator $T : X \supset D(T) \to 2^X$ is said to be "accretive" if for every $x, y \in D(T)$ there exists $j \in J(x-y)$ such that*

$$\langle u - v, j\rangle \geq 0 \quad \text{for every } u \in Tx, v \in Ty.$$

An accretive operator T is said to be "strongly accretive" if 0 in the right side of the above inequality can be replaced by $\alpha\|x-y\|^2$ for some fixed $\alpha > 0$. An acrretive operator T is said to be "m-accretive" if $R(T+\lambda I) = X$ for every $\lambda > 0$.

An obvious definition of strong accretiveness of T can be made on any subset of $D(T)$ instead of $D(T)$ itself.

Definition 1.3 *An operator $T : X \supset D(T) \to 2^{X^*}$ is said to be "monotone" if for every $x, y \in D(T)$ we have*

$$\langle u - v, x - y\rangle \geq 0, \quad \text{for all } u \in Tx, v \in Ty, \tag{1.1.1}$$

where $\langle x^*, x\rangle$ is the value of the functional $x^* \in X^*$ at $x \in X$. T is "strictly monotone" if "$\geq$" in (1.1.1) can be replaced by "$>$" for $x \neq y$ and it is "strongly monotone" if there exists a positive constant α such that (1.1.1) holds with 0 replaced by $\alpha\|x-y\|^2$. A monotone operator $T : X \supset D(T) \to 2^{X^*}$ is "maximal monotone" if and only if

$$\langle u - u_0^*, x - x_0\rangle \geq 0$$

for some $(x_0, u_0^*) \in X \times X^*$ and for every $x \in D(T)$, $u \in Tx$ implies $x_0 \in D(T)$ and $u_0^* \in Tx_0$.

We denote by $D(T)$, $R(T)$ and $G(T)$ the domain, the range and the graph of a mapping $T : X \supset D(T) \to 2^{X^*}$. We have $D(T) = \{x \in X : Tx \neq \emptyset\}$, $R(T) = \cup_{x \in D(T)} Tx$ and $G(T) = \{[x, y] : x \in D(T), y \in Tx\}$.

The graph $G(T)$ of a monotone operator T is said to be monotone set in $X \times X^*$. In terms of the graph of T, T is maximal monotone if and only if $G(T)$ is a maximal monotone set in $X \times X^*$, where $X \times X^*$ is partially ordered by inclusion.

If X is reflexive, a monotone operator $T : X \supset D(T) \to 2^{X^*}$ is maximal monotone if and only if $R(T + \lambda J) = X^*$ for every $\lambda > 0$.

For facts involving monotone and accretive operators, and other related concepts, the reader is referred to Barbu [3], Browder [7], Cioranescu [11], and Zeidler [35]. For a survey paper on compactness and accretivity, we cite the paper by Kartsatos [19].

The following lemma can be found in Zeidler ([35], p. 915).

Lemma 1.4 *Let $T : X \supset D(T) \to 2^{X^*}$ be maximal monotone. Then the following are true:*

(i) *$\{x_n\} \subset D(T)$, $x_n \to x_0$ and $Tx_n \ni y_n \rightharpoonup y_0$ imply $x_0 \in D(T)$ and $y_0 \in Tx_0$.*

(ii) *$\{x_n\} \subset D(T)$, $x_n \rightharpoonup x_0$ and $Tx_n \ni y_n \to y_0$ imply $x_0 \in D(T)$ and $y_0 \in Tx_0$.*

From Lemma 1.4 we see that either one of (i), (ii) implies that the graph $G(T)$ of the operator T is closed, i.e., $G(T)$ is a closed subset of $X \times X^*$.

Definition 1.5 *An operator $T : X \supset D(T) \to X^*$ is said to be of type "(S_+)" if for every sequence $\{x_n\} \subset D(T)$ with $x_n \rightharpoonup x_0$ in X and*

$$\lim_{n \to \infty} \langle Tx_n, x_n - x_0 \rangle \leq 0,$$

we have $x_n \to x_0$.

The following definition is from the paper ([21], p. 3852) by Kartsatos and Skrypnik.

Definition 1.6 *An operator $T : X \supset D(T) \to 2^{X^*}$ satisfies condition (S_q) on a set $A \subset D(T)$ if for every sequence $\{x_n\} \subset A$ such that $x_n \rightharpoonup x_0$ in X and any $y_n^* \in Tx_n$ with $y_n^* \to y^*$ for some $y^* \in X^*$, we have $x_n \to x_0$. If $A = D(T)$, we simply say that T satisfies (S_q).*

Definition 1.7 *A Banach space X is said to be locally uniformly convex if for every $\epsilon > 0$ and $x \in X$ with $\|x\| = 1$ there exists $\delta > 0$ such that $\|x - y\| \geq \epsilon$ implies*

$$\left\| \frac{x+y}{2} \right\| \leq 1 - \delta \quad \text{for all} \quad y \in X, \ \|y\| = 1.$$

We have the following theorem from Pascali and Sburlan ([30], p. 5).

Theorem 1.8 (Trojanski) *Let X be a reflexive Banach space. Then there exist equivalent norms on X and X^* such that both spaces, which are still dual to each other, are locally uniformly convex.*

We have the following theorem from Browder [6].

Theorem 1.9 *Let X be a reflexive Banach space which is renormed so that both X and X^* are locally uniformly convex. Then the duality mapping J is single-valued, bicontinuous, and of type (S_+).*

1.2 Topological Degree Theories

In this section we give the classical definition of topological degree and some examples in concrete settings.

Definition 1.10 *Let X and Y be topological spaces and let $\mathcal{O}$ be a class of open subsets G of X. For each $G \in \mathcal{O}$, we associate a class F_G of maps from $\overline{G}$ into Y and a class H_G of maps $[0,1] \times \overline{G}$ into Y (admissible homotopies). For any $f \in F_G$; $G \in \mathcal{O}$, and for any $y \in Y \setminus f(\partial G)$, we associate an integer $d(f, G, y)$.*

The integer-valued function d is said to be a classical topological degree if the following conditions are satisfied:

(a) **(Existence of Solution)** *If $d(f, G, y) \neq 0$, there exists an $x \in G$ such that $f(x) = y$.*

(b) **(Additivity)** *If $D \subset G \in \mathcal{O}$; $D \in \mathcal{O}$ and $f \in F_G$, then the restriction $f|_{\overline{D}} \in F_D$ (the restricted map is usually denoted by the same symbol). Let G_1, G_2 be a pair* *of disjoint subsets of G belonging to $\mathcal{O}$ and suppose that $y \notin f(\overline{G} \setminus (G_1 \cup G_2))$. Then*

$$d(f, G, y) = d(f, G_1, y) + d(f, G_2, y).$$

(c) **(Invariance under Homotopy)** *If f_t, $0 \le t \le 1$, is a homotopy in H_G, then $f_t \in F_G$ for each fixed $t \in [0,1]$, and if $\{y(t) : t \in [0,1]\}$ is a continuous curve in Y with $y(t) \notin f_t(\partial G)$ for any $t \in [0,1]$, then $d(f_t, G, y(t))$ is constant in $t \in [0,1]$.*

(d) **(Normalization)** *There exists a map $j \colon X \to Y$ called "normalizing map" such that $j|_{\overline{G}} \in F_G$ for each $G \in \mathcal{O}$, and if $y \in j(G)$, then*

$$d(j, G, y) = 1.$$

Let $X = Y = \mathcal{R}^n$, $\mathcal{O}$ all open bounded sets, F_G all continuous maps of $\overline{G}$ into $\mathcal{R}^n$, H_G all continuous homotopies, i.e., continuous maps of $[0,1] \times \overline{G}$ into $\mathcal{R}^n$, and $j = I$, the identity map, then we have the uniqueness of the Brouwer degree.

Let $X = Y$ be a real infinite dimensional Banach space, $\mathcal{O}$ all open bounded sets, F_G all continuous maps $f : \overline{G} \to X$ such that $I - f$ is compact, (i.e., it takes bounded sets into relatively compact sets), H_G the class of continuous homotopies of the form $I - T$, where $T : [0,1] \times \overline{G} \to X$ is compact, and $j = I$ the identity map. Then we have the uniqueness of the Leray-Schauder degree.

Let X be a real reflexive Banach space, $Y = X^*$, $\mathcal{O}$ all open bounded sets, F_G all bounded demicontinuous mappings of type (S_+) from $\overline{G}$ into X^*, H_G the class of homotopies of type (S_+) (defined below), and $j = J : X \to X^*$ the duality mapping corresponding to an equivalent norm on X in which both X and X^* are locally uniformly convex. Then we have the degree for bounded demicontinuous mappings of type (S_+) developed by Browder [8] and Skrypnik [33].

Definition 1.11 (Homotopy of type (S_+)) *Let $G \subset X$ be an open and bounded set and $H : [0,1] \times \overline{G} \to X^*$. Then $H(t,x)$ is said to be a homotopy of type (S_+) if the following condition holds: For every $\{x_n\} \subset \overline{G}$ and $\{t_n\} \subset [0,1]$ with $x_n \rightharpoonup x_0$ in X*

and $t_n \to t_0 \in [0,1]$ *such that*

$$\limsup_{n\to\infty} \langle H(t_n, x_n), x_n - x_0\rangle \le 0,$$

we have $x_n \to x_0$ *and* $H(t_n, x_n) \rightharpoonup H(t_0, x_0)$.

A degree theory for (S_+)-perturbation of maximal monotone operator is developed by Browder [6]. Kartsatos and Skrypnik [26] have developed a degree theory for densely defined maximal monotone operator perturbed by generalized (S_+) mappings. A number of degree theories for various combinations of nonlinear operators have been developed by various authors.

The following excision property of topological degree (cf. Brown [10], Theorem 11.5), which also holds for Brouwer degree, will be used in Chapter 2 and Chapter 3.

Theorem 1.12 (Excision) *Let* X *be a Banach space,* $G \subset X$ *open and bounded, and* $T : \overline{G} \to Y$ *with* $Y = X$ *or* X^* *with* X *reflexive. Let* $p \in Y \setminus T(\partial G)$ *and* $U \subset G$ *an open set containing all solutions of* $Tx = p$*. Assume that* $I - T$ *is compact whenever* $Y = X$ *and* T *is bounded demicontinuous of type* (S_+) *whenever* $Y = X^*$*. Then we have*

$$d(T, G, p) = d(T, U, p).$$

In Chapter 2, we consider the problem of the existence of nonzero solutions of operator equations of the form $Tx + Cx \ni 0$, where $T : X \supset D(T) \to 2^{X^*}$ is accretive or maximal monotone and $C : \overline{D(T)} \to X^*$ is either compact or of monotone type. The degree theories of Leray-Schauder, Browder [8], and Skrypnik [33] have been used.

In Chapter 3, we consider the problem of the existence of nonzero solutions of $Tx + Cx = 0$ for densely defined operators T, C. Degree theories developed by Kartsatos and Skrypnik ([22], [24]) have been employed.

Finally, in Chapter 4, we consider eigenvalue problems and invariance of domain results for the operators of the form $L + T + C$, where $L : X \supset D(L) \to X^*$ is densely defined linear maximal monotone, $T : X \to 2^{X^*}$ is bounded maximal monotone, and C is operator of monotone type w.r.t. $D(L)$. The operators of the form $L + C$ were considered by Berkovits and Mustonen [4] whereas the operators of the form $L+T+C$ were considered by Addou and Mermri [1].

Chapter 2

Accretive and Monotone Operator Equations

This chapter deals with the existence of nonzero solutions of operator equations involving perturbed accretive and maximal monotone operators in Banach spaces. Here, we are motivated by the results of Krasnoselskii [28] (e.g., see Theorem 2.1 below) about the existence of nonzero fixed points of compact operators that expand or compress a positive cone K in a Banach space X. Guo and Lakshmikantham in [16] improved the results of Krasnoselskii by assuming that the operators satisfy conditions only on the boundaries of two open sets intersecting a positive cone. The most recent results in this direction are by Ding and Kartsatos in [12] where the authors replaced the positive cone K by the Banach space X itself and the compact operators by operators of the form $T + C$ with T bounded accretive or maximal monotone and C compact.

In Section 2.1 we apply the Leray-Schauder degree theory to generalize results in [12] for operators of the form $T + C$, where T is accretive or maximal monotone and C is compact. We do not assume the boundedness of T.

In Section 2.2, we apply the Browder and Skrypnik degree theory to give extensions of the results in Section 2.1 to maximal monotone operators T and bounded demicontinuous operator C of type (S_+). The properties such as existence of solution, invariance under homotopy, normalization and excision of underlying degree theories have been employed.

Theorem 2.1 (Krasnoselskii) *Let X be a real Banach space and K a positive cone of X. Let $C : K \to K$ be compact with $C(0) = 0$ and suppose that there are numbers $r,\ R$ such that $0 < r < R$ and the following two conditions are satisfied.*

(i) *For every* $\epsilon > 0$, *we have* $Cx \not\geq (1+\epsilon)x$ *whenever* $x \in K \cap \overline{B_r(0)} \setminus \{0\}$*;*

(ii) $Cx \not\leq x$ *whenever* $x \in K$ *and* $\|x\| \geq R$.

Then C *has a nonzero fixed point in the cone* K.

D. Guo and V. Lakshmikantham have shown in [16] that conditions (i), (ii) of the above theorem can be replaced by one of the following statements.

For G_1, G_2 open subsets of X, G_1 bounded and $\overline{G_2} \subset G_1$,

(i) $Cx \not\geq x$ for $x \in K \cap \partial G_1$, and $Cx \not\leq x$ for $x \in K \cap \partial G_2$.

(ii) $Cx \not\geq x$ for $x \in K \cap \partial G_2$, and $Cx \not\leq x$ for $x \in K \cap \partial G_1$.

A nonzero fixed point of C exists in $K \cap (G_1 \setminus G_2)$.

2.1 Accretive Operators T and Leray-Schauder Degree

In this section we establish the results that improve Theorem 1 and Theorem 3 in [12] by Ding and Kartsatos concerning the existence of nonzero solutions of operator equations. The first existence result is contained in the following theorem. We start with a definition.

Definition 2.2 *Let* X *and* Y *be Banach spaces. The operator* $T : X \supset D(T) \to Y$ *is said to satisfy condition* (A_∞) *on bounded set* F *of* X *if there is no* $h \in Y$ *such that: for some* $\{x_n\} \subset D(T) \cap F$ *with* $\|Tx_n\| \to \infty$ *we have*

$$\lim_{n\to\infty} \frac{Tx_n}{\|Tx_n\|} = h. \tag{2.1.1}$$

Condition (A_∞) was first used by Kartsatos and Skrypnik in [25] in order to show the existence of eigenvalues for various problems involving perturbations of monotone and accretive operators. Closed positively α-homogeneous operators (e.g., closed linear operators), satisfy condition (A_∞) on every bounded subset of their domains. Indeed, let $F \subset X$ be bounded and let $\{x_n\} \subset D(T) \cap F$ be such that $\|Tx_n\| \to \infty$ and (2.1.1) holds. Then

$$\frac{Tx_n}{\|Tx_n\|} = T\left(\frac{x_n}{\|Tx_n\|^{\frac{1}{\alpha}}}\right) \to h \quad \text{as} \quad n \to \infty,$$

with $\|h\| = 1$. Since $x_n/\|Tx_n\|^{1/\alpha} \to 0$, the closedness of T implies that $(0, h) \in G(T)$, i.e. $T(0) = h$. Since T is homogeneous, $T(0) = 0$, i.e., a contradiction to $\|h\| = 1$.

Theorem 2.3 *Let the following conditions be satisfied, where G_1, G_2 are open subsets of X such that G_1 is bounded, $0 \in G_2$ and $\overline{G_2} \subset G_1$.*

- (H_1) *$T : X \supset D(T) \to X$ is bijective and has a continuous inverse $T^{-1} : X \to X$. Moreover, T satisfies the condition (A_∞) with $Y = X$, $F = G_1$ and is such that $0 \in D(T)$ and $T(0) = 0$;*
- (H_2) *$C : \overline{D(T)} \to X$ and (a) C is compact, or (b) C is continuous, bounded and T^{-1} is compact.*
- (H_3) *There exists $v_0 \in X \setminus \{0\}$ such that $Tx + Cx \neq \lambda v_0$, $(\lambda, x) \in \mathcal{R}_+ \times (D(T) \cap \partial G_1)$;*
- (H_4) *$Tx + tCx \neq 0$, $(t, x) \in [0, 1] \times (D(T) \cap \partial G_2)$.*

Then $Tx + Cx = 0$ has a nonzero solution $x \in D(T) \cap (G_1 \setminus G_2)$.

Proof: We first consider the case that C is compact. The rest is similar and therefore omitted. Let G be an open set in X. Since $T^{-1} : X \to D(T)$ is continuous, the set $T(D(T) \cap G)$ is open in X. For a similar reason, the set $T(D(T) \cap \overline{G})$ is closed in X. We observe that

$$\begin{aligned} \overline{T(D(T) \cap \overline{G})} &= T(D(T) \cap \overline{G}) = T(D(T) \cap G) \cup T(D(T) \cap \partial G) \\ &\supset \overline{T(D(T) \cap G)} = T(D(T) \cap G) \cup \partial(T(D(T) \cap G)). \end{aligned}$$

From this, we obtain the boundary inclusion

$$\partial(T(D(T) \cap G)) \subset T(D(T) \cap \partial G). \tag{2.1.2}$$

We now show the existence of a $\tau_0 > 0$ such that the equation

$$Tx + Cx = \tau v_0 \tag{2.1.3}$$

has no solution $x \in G_1$ for $\tau \geq \tau_0$. The vector v_0 is the one given in (H_3). Suppose that this is not true. Then, for every $n = 1, 2, \ldots$, there exists $\tau_n > 0$ such that $\tau_n \to \infty$

and, for some $x_n \in G_1$,

$$Tx_n + Cx_n = \tau_n v_0 \tag{2.1.4}$$

Since Cx_n is bounded, $v_0 \neq 0$ and $\tau_n \to \infty$, we must have $\|Tx_n\| \to \infty$. Consequently, in view of (2.1.4), we have

$$\frac{Tx_n}{\|Tx_n\|} + \frac{Cx_n}{\|Tx_n\|} = \frac{\tau_n}{\|Tx_n\|} v_0, \tag{2.1.5}$$

which implies

$$\frac{\tau_n \|v_0\|}{\|Tx_n\|} \to 1 \quad \text{and} \quad \frac{\tau_n}{\|Tx_n\|} \to \frac{1}{\|v_0\|} \quad \text{as } n \to \infty. \tag{2.1.6}$$

From (2.1.5) and (2.1.6), we obtain

$$\frac{Tx_n}{\|Tx_n\|} \to \frac{v_0}{\|v_0\|} \quad \text{as } n \to \infty. \tag{2.1.7}$$

This contradicts the fact that T satisfies condition (A_∞) on G_1. Now, we fix such a number τ_0 and consider the homotopy function

$$H_1(t,u) = (I + CT^{-1})u - t\tau_0 v_0 \tag{2.1.8}$$

defined on the set $[0,1] \times \overline{T(D(T) \cap G_1)}$, where v_0 is given by (H_3). We claim that all the solutions $u_t \in \overline{T(D(T) \cap G_1)} \subset T(D(T) \cap \overline{G}_1)$ of the equation $H_1(t,u) = 0$ are uniformly bounded independently of $t \in [0,1]$. In fact, assume that u_t is such a solution. Then we have

$$u_t = -CT^{-1}u_t + t\tau_0 v_0.$$

Then, for some $x_t \in D(T) \cap \overline{G_1}$, we have $u_t = Tx_t$ and therefore,

$$u_t = -CT^{-1}Tx_t + t\tau_0 v_0 = Cx_t + t\tau_0 v_0.$$

This implies that

$$\|u_t\| \leq \|Cx_t\| + \tau_0 \|v_0\| \leq s_0,$$

where

$$s_0 = \sup_{x \in D(T) \cap \overline{G_1}} \{\|Cx\|\} + \tau_0 \|v_0\|.$$

This proves the uniform boundedness of the solutions u_t. We fix a number $s > s_0$ and set $U_1 = T(D(T) \cap G_1) \cap B_s(0)$. Clearly, U_1 is open and bounded and $H_1 : [0,1] \times \overline{U_1} \to X$ is a compact displacement of identity.

We next show that $H_1(t, u) = 0$ has no solution on the boundary ∂U_1 for any $t \in [0,1]$. In fact, if this is not true then there are $t_0 \in [0,1]$ and

$$\begin{aligned} u_0 \in \partial U_1 &= \partial(T(D(T) \cap G_1) \cap B_s(0)) \\ &\subset \partial(T(D(T) \cap G_1)) \cup \partial B_s(0) \\ &\subset T(D(T) \cap \partial G_1) \cup \partial B_s(0) \end{aligned}$$

such that

$$(I + CT^{-1})u_0 = t_0 \tau_0 v_0.$$

By our choice of s, we have $u_0 \notin \partial B_s(0)$. Thus, $u_0 \in T(D(T) \cap \partial G_1)$. Choose $x_0 \in D(T) \cap \partial G_1$ such that $u_0 = Tx_0$. Then

$$Tx_0 + Cx_0 = t_0 \tau_0 v_0,$$

i.e., a contradiction to our assumption (H_3).

Thus, the Leray-Schauder degree function $d(H_1(t, \cdot), U_1, 0)$ is well-defined for all $t \in [0,1]$. Then we have

$$d(I + CT^{-1}, U_1, 0) = d(I + CT^{-1} - \tau_0 v_0, U_1, 0).$$

If $d(I + CT^{-1}, U_1, 0) \neq 0$, then there exists $u_0 \in U_1 = T(D(T) \cap G_1) \cap B_s(0)$ such that

$$u_0 + CT^{-1} u_0 = \tau_0 v_0.$$

This implies

$$Tx_0 + Cx_0 = \tau_0 v_0, \tag{2.1.9}$$

for some $x_0 \in D(T) \cap G_1 \cap T^{-1}(B_s(0))$ with $u_0 = Tx_0$. In view of (2.1.9), we have a contradiction to the choice of the number τ_0. Consequently, we have

$$d(I + CT^{-1}, U_1, 0) = 0. \tag{2.1.10}$$

Now, we consider another homotopy function

$$H_2(t, u) = u + tCT^{-1}u \tag{2.1.11}$$

defined on the set $[0,1] \times \overline{T(D(T) \cap G_2)}$. We also show that all the solutions $u_t \in \overline{T(D(T) \cap G_2)}$ of the equation $H_2(t, u) = 0$ are uniformly bounded independently of $t \in [0, 1]$. To show this, let $u_t \in \overline{T(D(T) \cap G_2)} \subset T(D(T) \cap \overline{G_2})$ be such a solution. Then for some $x_t \in D(T) \cap \overline{G_2}$ such that $u_t = Tx_t$, we have

$$\|u_t\| = t\|CT^{-1}u_t\| = t\|CT^{-1}Tx_t\| = t\|Cx_t\| \leq s_1,$$

where

$$s_1 = \sup_{x \in D(T) \cap \overline{G_2}} \{\|Cx\|\}.$$

We fix $s > s_1$ and we note that we may adjust it so that it is the same bound for the solutions of the homotopy equations $H_1(t, u) = 0$, $H_2(t, u) = 0$. We set $U_2 = T(D(T) \cap G_2) \cap B_s(0)$. Then U_2 is open and bounded subset of X. We then define the homotopy function $H_2(t, u)$ on the set $[0, 1] \times \overline{U_2}$. Thus, $H_2(t, u)$ is a compact displacement of identity with the Leray-Schauder degree $d(H_2(t, \cdot), U_2, 0)$ well-defined if we show that the equation $H_2(t, u) = 0$ has no solution on the boundary ∂U_2. Since $T(0) = 0$ and $0 \in \overline{G}_2$ and T is injective, we have $0 \notin \partial(T(D(T) \cap G_2)) \subset T(D(T) \cap \partial G_2)$. Also, $0 \notin \partial B_s(0)$. Thus, $H(0, u) = u = 0$ has no solution on ∂U_2. Next, assume that for some $t_0 \in (0, 1]$ and for some $u_0 \in \partial U_2 \subset T(D(T) \cap \partial G_2) \cup \partial B_s(0)$ we have $H_2(t_0, u_0) = 0$. By our choice of $s > 0$, we know that $u_0 \notin B_s(0)$. It then follows that $u_0 \in T(D(T) \cap \partial G_2)$. Choose $x_0 \in D(T) \cap \partial G_2$ such that $u_0 = Tx_0$. Consequently,

$$Tx_0 + t_0 Cx_0 = 0,$$

i.e., a contradiction to our assumption (H_4). Since $H_2(t,u)$ is homotopic to the identity, we have

$$d(I + CT^{-1}, U_2, 0) = d(I, U_2, 0) = 1. \tag{2.1.12}$$

By (2.1.10) and (2.1.12), we have

$$d(I + CT^{-1}, U_1, 0) \neq d(I + CT^{-1}, U_2, 0), \tag{2.1.13}$$

with

$$U_2 = T(D(T) \cap G_2) \cap B_s(0) \subset T(D(T) \cap G_1) \cap B_s(0) = U_1.$$

We conclude that $I + CT^{-1}$ must have a zero in $U_1 \setminus U_2$. If this is not true, then all the fixed points of the compact mapping $-CT^{-1}$ lie in the set U_2. By the excision property of the Leray-Schauder degree (cf. Theorem 1.12), we have

$$d(I + CT^{-1}, U_1, 0) = d(I + CT^{-1}, U_2, 0),$$

i.e., a contradiction to (2.1.13). It follows that there exists a point

$$u \in [T(D(T) \cap G_1) \cap B_s(0)] \setminus [T(D(T) \cap G_2) \cap B_s(0)]$$

such that $u + CT^{-1}u = 0$. Let $x = T^{-1}u$. Then

$$\begin{aligned} x &\in T^{-1}[(T(D(T) \cap G_1) \cap B_s(0)) \setminus (T(D(T) \cap G_2) \cap B_s(0))] \\ &= (D(T) \cap G_1) \cap T^{-1}(B_s(0)) \setminus (D(T) \cap G_2) \cap T^{-1}(B_s(0)) \qquad (2.1.14) \\ &= D(T) \cap T^{-1}(B_s(0)) \cap (G_1 \setminus G_2). \end{aligned}$$

Since $0 \notin G_1 \setminus G_2$, we have $x \neq 0$. Thus, we have $x \in G_1 \setminus G_2$ and $Tx + Cx = 0$. This completes the proof of the theorem. ■

The following lemma by Ding and Kartsatos (cf. [12], p. 1336) shows that there are several types of operators which map relatively open sets onto open sets and relatively closed sets onto closed sets.

Lemma A *Let $G \subset X$ be open and assume that one of the following statements are true.*

(i) $T : X \supset D(T) \to X$ is strongly accretive, closed and surjective;

(ii) X^* is uniformly convex and $T : D(T) = \overline{G} \to X$ is demicontinuous and strongly accretive;

(iii) $T : D(T) = \overline{G} \to X$ is continuous and strongly accretive.

Then (a) in Case (i), for every open $M \subset X$, the set $T(D(T) \cap M)$ is open. If $M \subset X$ is closed, then $T(D(T) \cap M)$ is closed. (b) In Case (ii), (iii), for every open $M \subset G$ the set TM is open and for every closed $M \subset G$, the set TM is closed.

It is then obvious that the results analogous to Theorem 2.3 are possible for the cases (i), (ii), (iii) of Lemma A.

Now we turn our attention to the results involving m-accretive operators T. It is a standard argument that the desired inclusion may be achieved by solving the approximate problems of the type

$$Tx + Cx + \frac{1}{n}x = 0. \tag{2.1.15}$$

The following theorem reflects a situation of this type.

Theorem 2.4 *Let $T : X \supset D(T) \to X$ be m-accretive, with $0 \in D(T)$ and $T(0) = 0$, satisfying condition (A_∞) with $Y = X$ and $F = G_1$. Assume that the conditions $(H_2) - (H_4)$ of Theorem 2.3 are satisfied with the compactness of T^{-1} replaced by the compactness of the resolvent $(T + I)^{-1}$ of the operator T. Assume, further, that under $((H_2), (a))$ T is strongly accretive on $D(T) \cap \overline{G_1}$. Then the problem $Tx + Cx = 0$ has a nonzero solution $x \in D(T) \cap (G_1 \setminus G_2)$.*

Proof: (a) Assume that C is compact and that T is strongly accretive on $D(T) \cap \overline{G_1}$. We solve first the problem (2.1.15). In order to do this, we show that all the assumptions of Theorem 2.3 are satisfied, for large n, with the operator T_n in place of

T, where

$$T_n(x) = Tx + \frac{1}{n}x.$$

Since T is m-accretive, $T_n : X \supset D(T_n) = D(T) \to X$ has continuous inverse $(T + \frac{1}{n}I)^{-1} : X \to D(T)$ for all n. Obviously, $T_n(0) = 0$ and T_n is bijective for all n. We now verify that each T_n satisfies the condition (A_∞) with $Y = X$ and $F = G_1$. Fix n and suppose that there exists a sequence $\{x_j\} \subset D(T) \cap G_1$ such that $\|T_n x_j\| \to \infty$ as $j \to \infty$ and

$$\lim_{j\to\infty} \frac{T_n x_j}{\|T_n x_j\|} = h \tag{2.1.16}$$

for some $h \in X$ with $\|h\| = 1$. It is then clear that

$$\lim_{j\to\infty} \frac{T x_j}{\|T_n x_j\|} = h. \tag{2.1.17}$$

Now, by (2.1.17), we obtain

$$\frac{Tx_j}{\|Tx_j\|} = \frac{\|T_n x_j\|}{\|Tx_j\|}\left(\frac{Tx_j}{\|T_n x_j\|}\right) \to h \quad \text{ as } j \to \infty,$$

i.e., a contradiction to the assumption that T satisfies the condition (A_∞). This verifies that each T_n satisfies (H_1) of Theorem 2.3. Condition (H_2) is obvious in this case. To show that (H_3) holds for large n, let us assume that it doesn't. Then we may assume that there exist sequences $\{\lambda_n\} \subset \mathcal{R}_+$, $\{x_n\} \subset D(T) \cap \partial G_1$ such that

$$Tx_n + Cx_n + \frac{1}{n}x_n = \lambda_n v_0, \quad n = 1, 2, \ldots. \tag{2.1.18}$$

As in the proof of Theorem 2.3, it follows that the condition (A_∞) implies the boundedness of the sequence $\{\lambda_n\}$. We may assume that $\lambda_n \to \lambda_0 \in \mathcal{R}_+$. Since $\{x_n\}$ is bounded, we may assume, for a subsequence of $\{x_n\}$ again denoted by $\{x_n\}$, that $Cx_n \to y$ for some $y \in X$. Since $\frac{1}{n}x_n \to 0$, we conclude from (2.1.18) that the sequence $\{Tx_n\}$ is Cauchy. We note that

$$\|Tx_n - Tx_m\| \, \|x_n - x_m\| \ge \langle Tx_n - Tx_m, x^* \rangle \ge \alpha_1 \|x_n - x_m\|^2, \tag{2.1.19}$$

where α_1 is a constant appearing in the strong accretivity of T and $x^* \in J(x_n - x_m)$ is an appropriate functional for the accretivity. From (2.1.19), it follows that $\{x_n\}$ is Cauchy and so there exists $x_0 \in X$ such that $x_n \to x_0 \in \overline{D(T) \cap \partial G_1} \subset \overline{D(T)} \cap \partial G_1$. Since C is continuous, we have $Cx_n \to Cx_0$ so that $Tx_n \to -Cx_0 + \lambda_0 v_0$. Since T is closed (being m-accretive), we have that $x_0 \in D(T) \cap \partial G_1$ and $Tx_0 + Cx_0 = \lambda_0 v_0$, i.e., a contradiction to (H_3).

In order to show that (H_4) holds for large n with T_n in place of T, we suppose that it doesn't. Then we may assume that there exist sequences $\{t_n\} \subset [0,1]$ with $t_n \to t_0 \in [0,1]$ and $\{x_n\} \subset D(T) \cap \partial G_2$ such that

$$T_n x_n + t_n C x_n = 0, \quad n = 1, 2, \ldots. \tag{2.1.20}$$

Since C is compact and $\{x_n\}$ is bounded, we may assume, as usual, that $Cx_n \to y$ for some $y \in X$. Then the strong accretivity of T and a relation like (2.1.19) imply that $\{x_n\}$ is Cauchy and so it converges to some point $x_0 \in \overline{D(T) \cap \partial G_2} \subset \overline{D(T)} \cap \partial G_2$. Again, by the closedness of T and the continuity of C, it follows that $x_0 \in D(T) \cap \partial G_2$ and $Tx_0 + t_0 C x_0 = 0$, i.e., a contradiction to (H_4).

Before we apply Theorem 2.3, we need to show that the number τ_0 in that proof, which is defined so that the equation

$$Tx + Cx + \frac{1}{n} x = \tau v_0 \tag{2.1.21}$$

has no solution $x \in G_1$ for all $\tau \geq \tau_0$, is actually independent of n for all sufficiently large n. We need this fact because we will use a uniform bound $s > 0$ for the solutions of the two homotopy equations $H_1(t,u) = 0$ and $H_2(t,u) = 0$, where

$$H_1(t,u) = u + C\left(T + \frac{1}{n} I\right)^{-1} - t\tau_0 v_0 \quad \text{and} \quad H_2(t,u) = u + tC\left(T + \frac{1}{n} I\right)^{-1}. \tag{2.1.22}$$

These solutions initially lie in the closures of the sets

$$\left(T + \frac{1}{n} I\right)(D(T) \cap G_1) \quad \text{and} \quad \left(T + \frac{1}{n} I\right)(T(D(T) \cap G_2),$$

respectively.

To show that τ_0 is independent of n for all sufficiently large n, let us assume that there exist sequences $\{\tau_n\} \subset (0,\infty)$ and $\{x_n\} \subset G_1$ such that $\tau_n \to \infty$ and

$$Tx_n + Cx_n + \frac{1}{n}x_n = \tau_n v_0.$$

Arguing exactly the same as before, we can arrive at a contradiction to condition (A_∞) satisfied by T.

Thus, by Theorem 2.3, there exist $n_0 \geq 1$ and $s > 0$ independent of n, such that the equation (2.1.15) holds for every $n \geq n_0$ with a solution

$$\begin{aligned} x_n &\in \left[D(T) \cap G_1 \cap \left(T + \frac{1}{n}I\right)^{-1}(B_s(0))\right] \setminus \left[D(T) \cap G_2 \cap \left(T + \frac{1}{n}I\right)^{-1}(B_s(0))\right] \\ &= D(T) \cap \left[\left(T + \frac{1}{n}I\right)^{-1}(B_s(0))\right] \cap (G_1 \setminus G_2). \end{aligned} \tag{2.1.23}$$

Since $\{x_n\}$ is bounded, C is compact and T is m-accretive and strongly accretive on $D(T) \cap \overline{G_1}$, it follows that $x_n \to$ (some)$x_0 \in D(T)$ and $Tx_0 + Cx_0 = 0$. Also,

$$x_0 \in \overline{G_1 \setminus G_2} = (G_1 \setminus G_2) \cup \partial(G_1 \setminus G_2) \subset (G_1 \setminus G_2) \cup (\partial G_1 \cup \partial G_2). \tag{2.1.24}$$

Note that $x_0 \notin \partial G_1 \cup \partial G_2$. Therefore, $x_0 \in D(T) \cap (G_1 \setminus G_2)$ and we have proved the theorem for this case.

(b) We now assume that C is bounded, continuous and the resolvent $(T + I)^{-1}$ is compact. We note that the compactness of the resolvent $(T + I)^{-1}$ implies the compactness of any resolvent $(\lambda T + \mu I)^{-1}$, $\lambda > 0$, $\mu > 0$, by the well-known resolvent identity for m-accretive operators. We consider the same homotopy mappings H_1, H_2 and maintain the same constants s, τ_0 as in the first case.

To prove that (H_3) holds for the operators T_n in place of T, for all large n, we assume that it doesn't. Then there exist sequences $\{\lambda_n\} \subset (0,\infty)$, $\{x_n\} \subset D(T) \cap \partial G_1$

such that

$$Tx_n + Cx_n + \frac{1}{n}x_n = \lambda_n v_0, \quad n = 1, 2, \ldots. \tag{2.1.25}$$

If $\lambda_n \to \infty$, as in the proof of Theorem 2.3, we obtain a contradiction to the condition (A_∞). Thus, the sequences $\{\lambda_n\}$ and $\{Tx_n\}$ are bounded. We may now assume that $\lambda_n \to \lambda_0 \in \mathcal{R}_+$. Also, from (2.1.25), we see that

$$Tx_n + x_n = -Cx_n + x_n - \frac{1}{n}x_n + \lambda_n v_0 = -Cx_n + \left(1 - \frac{1}{n}\right)x_n + \lambda_n v_0,$$

which implies that

$$x_n = (T + I)^{-1}\left[- Cx_n + \left(1 - \frac{1}{n}\right)x_n + \lambda_n v_0\right]. \tag{2.1.26}$$

Since $(T + I)^{-1}$ is compact and $\{Cx_n\}$ is bounded, we see from (2.1.26) that $\{x_n\}$ lies in a relatively compact set and so $\{x_n\}$ has a convergent subsequence which we again denote by itself. Letting $x_n \to x_0 \in \overline{D(T)} \cap \partial G_1$, we note that $Cx_n \to Cx_0$ and $Tx_n \to -Cx_0 + \lambda_0 v_0$. Since T is closed, we conclude that $x_0 \in D(T) \cap \partial G_1$ and $Tx_0 + Cx_0 = \lambda_0 v_0$, i.e., a contradiction to (H_3).

We now show that (H_4) holds for T_n in place of T for all large n. Assume that it doesn't. We may then choose sequences $\{t_n\} \subset [0, 1]$, $\{x_n\} \subset D(T) \cap \partial G_2$ such that

$$T_n x_n + t_n C x_n = 0, \quad n = 1, 2, \ldots, \tag{2.1.27}$$

and $t_n \to t_0 \in [0, 1]$. From (2.1.27), we see that

$$x_n = (T + I)^{-1}\left[- t_n Cx_n + \left(1 - \frac{1}{n}\right)x_n\right]. \tag{2.1.28}$$

Since $(T + I)^{-1}$ is compact and C is bounded, it follows that $\{x_n\}$ lies in a relatively compact set. We may assume, without loss of generality, that $x_n \to x_0 \in \overline{D(T)} \cap \partial G_2$. The continuity of C implies that $Cx_n \to Cx_0$ and the closedness of T implies that $x_0 \in D(T) \cap \partial G_2$ and $Tx_0 = -t_0 Cx_0$, i.e., $Tx_0 + t_0 Cx_0 = 0$, which is a contradiction to (H_4).

Finally, we obtain a solution x_n as in (2.1.23) satisfying

$$Tx_n + Cx_n + \frac{1}{n}x_n = 0,$$

which may be rewritten as

$$x_n = (T+I)^{-1}\Big[-Cx_n + \Big(1-\frac{1}{n}\Big)x_n\Big].$$

Since $\{x_n\}$ is bounded, $(T+I)^{-1}$ is compact and C is bounded, we may assume that $x_n \to x_0 \in \overline{D(T)} \cap \overline{G_1 \setminus G_2}$. Since T is closed and C is continuous, it follows that $x_0 \in D(T) \cap \overline{G_1 \setminus G_2}$, $Cx_n \to Cx_0$ and $Tx_0 + Cx_0 = 0$. Again, since $x_0 \notin \partial G_1 \cup \partial G_2$, we must have $x_0 \in D(T) \cap (G_1 \setminus G_2)$. This completes the proof. ■

The following condition $(N1)$ was introduced in [12] in order to avoid the assumption of boundedness of the operators T considered there.

$(N1)$ There exists $v_0 \in X$ such that $v_0 \notin T(D(T) \cap \overline{G_1})$ and

$$Tx + tCx \neq (1-t)v_0, \quad t \in [0,1], \quad x \in D(T) \cap \partial G_1.$$

This condition was used in [12] along with Nagumo's degree (cf. [29]) in order to show the existence of solutions in $G_1 \setminus G_2$. We show here that this condition may be used here without condition (A_∞) and (H_3) and without resorting to the Nagumo degree, which is for continuous perturbations taking closure of the (possibly unbounded) open set $T(D(T) \cap G_1)$ into a relatively compact set. Thus, we have the following extension of Theorem 2.3. Extensions to results similar to those of Theorem 2.4 are obviously possible.

Theorem 2.5 *Let the assumptions of Theorem 2.3 be satisfied with conditions (A_∞) and (H_3) replaced by $(N1)$. Then the conclusion of Theorem 2.3 holds true.*

Proof: We consider the homotopy function

$$H_1(t,u) = u + tCT^{-1}u - (1-t)v_0, \tag{2.1.29}$$

which is initially defined on the set $[0,1] \times \overline{T(D(T) \cap G_1)}$. We can easily see that for every solution u_t of the equation $H_1(t,u) = 0$, we have

$$\|u_t\| \le t\|CT^{-1}u_t\|u + (1-t)\|v_0\| \le s_1, \tag{2.1.30}$$

where

$$s_1 = \sup_{x \in D(T) \cap \overline{G_1}} \{\|Cx\|\} + \|v_0\|.$$

Let $s > s_1$. We see that it suffices to consider the homotopy function $H_1(t,u)$, with $H_1(t,\cdot)$ defined only on the closure of the open and bounded set

$$U_1 := T(D(T) \cap G_1) \cap B_s(0).$$

In order to show that the equation $H_1(t,u) = 0$ has no solution on the set $[0,1] \times \partial U_1$, assume that the contrary is true. Then, for some $(t, u_t) \in [0,1] \times \partial U_1$, we have $u_t + tCT^{-1}u_t - (1-t)v_0 = 0$. By the choice of s, we have that $u_t \notin \partial B_s(0)$, which implies $u_t \in \partial(T(T(D) \cap G_1)) \subset T(D(T) \cap \partial G_1)$. Thus, $Tx + tCx = (1-t)v_0$, where $x = T^{-1}u_t \in D(T) \cap \partial G_1$, i.e., a contradiction to the condition $(N1)$. It follows that the Leray-Schauder degree

$$d(H_1(t,\cdot), U_1, 0)$$

is well-defined and constant on $[0,1]$. Consequently, we have

$$d(I + CT^{-1}, U_1, 0) = d(I - v_0, U_1, 0) = d(I, U_1, v_0) = 0,$$

because

$$\overline{U_1} = \overline{T(D(T) \cap G_1) \cap B_s(0)} \subset \overline{T(D(T) \cap G_1)} \subset T(D(T) \cap \overline{G_1})$$

and

$$v_0 \notin T(D(T) \cap \overline{G_1}).$$

The rest of the proof follows as in the proof of Theorem 2.3. ■

Remark 2.6 Condition (H_4) is implied by the following boundary condition $(B1)$.
$(B1)$: For each $x \in D(T) \cap \partial G_2$ and each $x^* \in Jx$ we have

$$\langle Tx, x^* \rangle > 0 \quad \text{and} \quad \langle Tx + Cx, x^* \rangle > 0. \tag{2.1.31}$$

In fact, let (B_1) be true and fix $t \in [0,1]$, $x \in D(T) \cap \partial G_2$ such that $Tx + tCx = 0$. We pick $x^* \in Jx$ and consider two cases: (a) $\langle Cx, x^* \rangle < 0$; (b) $\langle Cx, x^* \rangle \geq 0$. In case (a), we get

$$\begin{aligned} 0 &< \langle Tx + Cx, x^* \rangle = \langle Tx, x^* \rangle + \langle Cx, x^* \rangle \leq \langle Tx, x^* \rangle + t\langle Cx, x^* \rangle \\ &= \langle Tx, x^* \rangle + \langle tCx, x^* \rangle = \langle Tx + tCx, x^* \rangle = 0, \end{aligned}$$

i.e., a contradiction. In case (b), we have

$$0 < \langle Tx, x^* \rangle \leq \langle Tx, x^* \rangle + \langle tCx, x^* \rangle = \langle Tx + tCx, x^* \rangle = 0,$$

i.e., a contradiction.

2.2 Monotone Operators and Browder and Skrypnik Degrees

In what follows the spaces X, X^* will be assumed to be locally uniformly convex with X reflexive.

In this section we give extensions of Theorem 2.3 and Theorem 2.5 to maximal monotone operators T. If T is maximal monotone, then the resolvent $(T + \mu J)^{-1}$ is continuous and bounded for all $\mu > 0$. It is also known that the duality mapping J is a bounded, single-valued and bicontinuous operator. Our first two results in this section generalize Theorem 2.3 and Theorem 2.5 to maximal monotone operators $T : X \supset D(T) \to X^*$ and operators $C : \overline{D(T)} \to X^*$.

Theorem 2.7 *Let the following conditions be satisfied, where G_1, G_2 are bounded open subsets of X such that $0 \in G_2$ and $\overline{G_2} \subset G_1$.*

- (H_1) *$T : X \supset D(T) \to X^*$ is bijective and has a continuous inverse $T^{-1} : X \to D(T)$. Moreover, T satisfies condition (A_∞) with $Y = X^*$, $F = G_1$, and is such that $0 \in D(T)$ and $T(0) = 0$;*
- (H_2) *$C : \overline{D(T)} \to X^*$ and (a) C is compact, or (b) C is continuous, bounded and T^{-1} is compact;*
- (H_3) *There exists a nonzero vector $v_0^* \in X^*$ such that $Tx + Cx \neq \lambda v_0^*$ for every $(\lambda, x) \in \mathcal{R}_+ \times (D(T) \cap \partial G_1)$;*
- (H_4) *$Tx + tCx \neq 0$ for every $(t, x) \in [0, 1] \times (D(T) \cap \partial G_2)$.*

Then there exists a nonzero solution $x \in D(T) \cap (G_1 \setminus G_2)$ of the equation $Tx + Cx = 0$.

Proof: The proof follows as in Theorem 2.3 because the mappings

$$H_1(t, u^*) = (I + CT^{-1})u^* - t\tau_0 v_0^*, \quad (t, u^*) \in [0, 1] \times \overline{T(D(T) \cap G_1) \cap B_s(0)},$$

and

$$H_2(t, u^*) = (I + tCT^{-1})u^*, \quad (t, u^*) \in [0, 1] \times \overline{T(D(D) \cap G_2) \cap B_s(0)}$$

are well-defined compact displacements of identity on the closure of open and bounded sets $U_1 = T(D(T) \cap G_1) \cap B_s(0)$ and $U_2 = T(D(T) \cap G_2) \cap B_s(0)$ respectively. The numbers τ_0 and s are chosen in a manner similar to that of Theorem 2.3. ■

Theorem 2.8 *Let $T : X \supset D(T) \to X^*$ be maximal monotone with $0 \in D(T)$, $T(0) = 0$ and satisfy the condition (A_∞) with $Y = X^*$ and $F = G_1$. Assume that the conditions $(H_2) - (H_4)$ of Theorem 2.7 are satisfied with the compactness of T^{-1} replaced by the compactness of the resolvent $(T + J)^{-1}$ operator. Assume, further, that under $((H_2), (a))$ T is strongly monotone on $D(T) \cap \overline{G_1}$. Then the problem $Tx + Cx = 0$ has a nonzero solution $x \in D(T) \cap (G_1 \setminus G_2)$.*

Proof: We now use the approximate problem

$$Tx + Cx + \frac{1}{n} Jx = 0$$

to prove the theorem. We note that the mapping $(T+\frac{1}{n}J)^{-1}: X^* \to D(T)$ is always continuous and bounded. Also, the compactness of the resolvent $(T+J)^{-1}$ implies that compactness of all the resolvents $(\nu T+\mu J)^{-1}$, $\nu>0$, $\mu>0$, via the resolvent identity. Thus, the proof of Theorem 2.4 goes through with necessary modifications by using the homotopy functions

$$H_1(t,u^*) = u^* + C\left(T+\frac{1}{n}J\right)^{-1}u^* - t\tau_0 v_0^*, \quad H_2(t,u^*) = u^* + tC\left(T+\frac{1}{n}J\right)^{-1}u^*$$

which are initially defined on the closures of the sets

$$\left(T+\frac{1}{n}J\right)(D(T)\cap G_1), \quad \left(T+\frac{1}{n}J\right)(D(T)\cap G_2)$$

respectively. Therefore, we omit the details of the proof. ■

It would be interesting to study the problem of nonzero solutions under the absence of compactness condition on the operator C or on the resolvents of the operator T. In this case, we may study the problems more directly, without using the inverse T^{-1} or the resolvents $(T+\frac{1}{n}J)^{-1}$ of the operator T in the manner they were used in Theorem 2.3 and Theorem 2.5. Such a method requires, of course, the validity of the excision property of the topological degree under consideration.

If $T: X \supset D(T) \to X^*$ is maximal monotone, then, for $t>0$, the operator $T_t := (T^{-1}+tJ^{-1})^{-1} : X \to X^*$ called the "Yosida Approximant" of T is single-valued, bounded demicontinuous and maximal monotone. Kartsatos and Skrypnik in ([22], Lemma 3.1, p. 127) showed that the mapping $(t,x)\mapsto T_t x$ is continuous on $(0,\infty)\times X$. In addition, $T_t x \rightharpoonup Tx$ as $t\to 0^+$ for every $x\in D(T)$. Also, $T_t x = TJ_t x$ where $J_t := I - tJ^{-1}T_t : X \to X$ and satisfies $\lim_{t\to 0^+} J_t x = x$ for all $x \in \overline{\mathrm{co}D(T)}$, where co$A$ denotes the convex hull of the set A. The operators T_t and J_t were first introduced by Breźis, Crandall and Pazy in [5]. For their basic properties, we refer the reader to [5] as well as Pascali and Sburlan ([30], p. 128-130)]. We also refer the reader to Simons [32] for many monotonicity properties.

Browder developed in [6] a degree theory for operators of the type $T+C$, where

T is (possibly multi-valued) maximal monotone and C is slightly more general than demicontinuous, bounded and of type (S_+) on the closure of a bounded open set $G \subset X$. When C is demicontinuous, bounded and of type (S_+), Browder's degree mapping $d_B(T+C,G,0)$ in [6] is the limit, as $t \downarrow 0$, of the Skrypnik degree $d_S(T_t+C,G,0)$ in [34].

In the following result, we show the existence of a nonzero solution to the problem $Tx+Cx=0$ by using the Browder and Skrypnik degrees. Our operator T is maximal monotone and homogeneous of degree 1. In particular, T could be a linear maximal monotone operator. We need the following lemma.

Lemma 2.9 *Assume that the operators $T : X \supset D(T) \to 2^{X^*}$ and $S : X \supset D(S) \to 2^{X^*}$ are maximal monotone, with $0 \in D(T) \cap D(S)$ and $0 \in S(0) \cap T(0)$. Assume, further, that $T+S$ is maximal monotone and that there is a sequence $\{t_n\} \subset (0,\infty)$ such that $t_n \downarrow 0$, and a sequence $\{x_n\} \subset D(S)$ such that $x_n \rightharpoonup x_0 \in X$ and $T_{t_n}x_n + w_n^* \rightharpoonup y_0^* \in X^*$, where $w_n^* \in Sx_n$. Then the following are true:*

(i) The inequality

$$\lim_{n\to\infty} \langle T_{t_n}x_n + w_n^*, x_n - x_0\rangle < 0 \tag{2.2.32}$$

is impossible;

(ii) If

$$\lim_{n\to\infty} \langle T_{t_n}x_n + w_n^*, x_n - x_0\rangle = 0, \tag{2.2.33}$$

then $x_0 \in D(T+S)$ and $y_0^* \in (T+S)x_0$.

Proof: Assume that (2.2.32) is true. Then

$$\lim_{n\to\infty} \langle T_{t_n}x_n + w_n^*, x_n\rangle = \lim_{n\to\infty} \langle T_{t_n}x_n + w_n^*, x_n - x_0\rangle + \langle y_0^*, x_0\rangle < \langle y_0^*, x_0\rangle. \tag{2.2.34}$$

Let $v_n^* = T_{t_n}x_n \in T(x_n - t_nJ^{-1}v_n^*)$. Then

$$\begin{aligned} \langle v_n^*, x_n\rangle &= \langle v_n^*, x_n - t_nJ^{-1}v_n^* + t_nJ^{-1}v_n^*\rangle \\ &= \langle v_n^*, x_n - t_nJ^{-1}v_n^*\rangle + \langle v_n^*, t_nJ^{-1}v_n^*\rangle \\ &\geq t_n\|v_n^*\|^2. \end{aligned} \tag{2.2.35}$$

Since $\langle w_n^*, x_n \rangle \geq 0$, (2.2.34) implies

$$\limsup_{n\to\infty} \langle v_n^*, x_n \rangle < \langle y_0^*, x_0 \rangle,$$

which, by (2.2.35), implies

$$t_n \|v_n^*\|^2 \leq K,$$

where K is a positive constant. This implies that $t_n v_n^* \to 0$.

Next, we fix $[x, x^*] \in G(T+S)$. Then $x^* = x_1^* + x_2^*$ with $x_1^* \in Tx$ and $x_2^* \in Sx$. Thus,

$$\langle v_n^* - x_1^*,\ x_n - t_n J^{-1} v_n^* - x \rangle \geq 0$$

gives

$$\begin{aligned} \langle v_n^* - x_1^*,\ x_n - x \rangle &\geq \langle v_n^* - x_1^*,\ t_n J^{-1} v_n^* \rangle \\ &= t_n \langle v_n^*,\ J^{-1} v_n^* \rangle - \langle x_1^*,\ t_n J^{-1} v_n^* \rangle \\ &\geq -t_n \|v_n^*\| \|x_1^*\| \end{aligned} \tag{2.2.36}$$

From the monotonicity of S we have

$$\langle w_n^* - x_2^*,\ x_n - x \rangle \geq 0.$$

Using this with (2.2.36) we have

$$\langle v_n^* + w_n^* - x^*,\ x_n - x \rangle = \langle v_n^* + w_n^* - (x_1^* + x_2^*),\ x_n - x \rangle \geq -t_n \|v_n^*\| \|x_1^*\|$$

and

$$\langle v_n^* + w_n^*, x_n \rangle \geq \langle v_n^* + w_n^*, x \rangle - \langle x^*, x \rangle + \langle x^*, x_n \rangle - t_n \|v_n^*\| \|x_1^*\|.$$

This implies

$$\liminf_{n\to\infty} \langle v_n^* + w_n^*, x_n \rangle \geq \langle y_0^*, x \rangle - \langle x^*, x \rangle + \langle x^*, x_0 \rangle. \tag{2.2.37}$$

Combining (2.2.34) and (2.2.37), we get

$$\langle y_0^*, x\rangle - \langle x^*, x\rangle + \langle x^*, x_0\rangle < \langle y_0^*, x_0\rangle,$$

which gives

$$\langle y_0^* - x^*,\ x_0 - x\rangle > 0, \quad [x, x^*] \in G(T+S). \tag{2.2.38}$$

Since $T+S$ is maximal monotone, we conclude that $x_0 \in D(T+S)$ and $y_0^* \in (T+S)x_0$. This, however, contradicts (2.3.62) because we may now take $x = x_0$.

If (2.2.33) is true, then we can repeat the above argument to arrive at (2.3.62), but with " $>$ " replaced by " $\geq$ ". In this case we conclude again that $x_0 \in D(T+S)$ and $y_0^* \in (T+S)x_0$ and we are done. ■

The following simpler lemma is an interesting and easy version of the above lemma without using the Yosida approximants of the maximal monotone operator and has been reserved here for using it in Chapter 3 and Chapter 4.

Lemma 2.10 *Assume that the operators $T : X \supset D(T) \to 2^{X^*}$, $S : X \supset D(S) \to 2^{X^*}$ are maximal monotone, with $0 \in D(T) \cap D(S)$ and $0 \in S(0) \cap T(0)$. Assume, further, that $T+S$ is maximal monotone. Assume that there is a sequence $\{x_n\} \subset D(T) \cap D(S)$ such that $x_n \rightharpoonup x_0 \in X$ and $y_n^* + w_n^* \rightharpoonup y_0^* \in X^*$, where $y_n^* \in Tx_n$ and $w_n^* \in Sx_n$. Then the following are true:*

(i) *the inequality*

$$\lim_{n\to\infty} \langle y_n^* + w_n^*, x_n - x_0\rangle < 0 \tag{2.2.39}$$

is impossible;

(ii) *if*

$$\lim_{n\to\infty} \langle y_n^* + w_n^*, x_n - x_0\rangle = 0, \tag{2.2.40}$$

then $x_0 \in D(T+S)$ and $y_0^* \in (T+S)x_0$.

Proof: Assume that (2.2.39) is true. Then

$$\lim_{n\to\infty}\langle y_n^* + w_n^*, x_n\rangle = \lim_{n\to\infty}\langle y_n^* + w_n^*, x_n - x_0\rangle + \langle y_0^*, x_0\rangle < \langle y_0^*, x_0\rangle. \tag{2.2.41}$$

Fix $[x, x^*] \in G(T+S)$. Then $x^* = x_1^* + x_2^*$ with $x_1^* \in Tx$ and $x_2^* \in Sx$. Using the monotonicity of $T+S$, we have

$$\langle y_n^* + w_n^* - x^*, x_n - x\rangle \geq 0.$$

This implies that

$$\langle y_n^* + w_n^*, x_n\rangle \geq \langle y_n^* + w_n^*, x\rangle + \langle x^*, x_n - x\rangle$$

and then

$$\liminf_{n\to\infty}\langle y_n^* + w_n^*, x_n\rangle \geq \langle y_0^*, x\rangle + \langle x^*, x_0 - x\rangle.$$

This with (2.2.41) gives

$$\langle y_0^*, x_0\rangle > \langle y_0^*, x\rangle + \langle x^*, x_0 - x\rangle.$$

Thus

$$\langle y_0^* - x^*, x_0 - x\rangle > 0 \quad \text{for all} \quad [x, x^*] \in G(T+S). \tag{2.2.42}$$

Since $T+S$ is maximal monotone, we have that $x_0 \in D(T+S)$ and $y_0^* \in (T+S)x_0$. This is a contradiction to (2.2.42) because we may take $x = x_0$.

If (2.2.40) is true, we can then repeat all the argument above to arrive at (2.2.42) with "$>$" replaced by "$\geq$". Thus, we conclude that $x_0 \in D(T+S)$ and $y_0^* \in (T+S)$. ■

Theorem 2.11 *Assume that G_1, $G_2 \subset X$ are open, bounded with $0 \in G_2$ and $\overline{G}_2 \subset G_1$. Assume that $T : X \supset D(T) \to X^*$ is maximal monotone and positively homogeneous of degree 1. Assume, further, that the operator $C : \overline{G_1} \to X^*$ is bounded, demicontinuous and of type (S_+). Moreover, assume the following:*

- (H_3) *There is $v_0^* \in X^* \setminus \{0\}$ s.t. $Tx + Cx \neq \lambda v_0^*$ for every $(\lambda, x) \in \mathcal{R}_+ \times (D(T) \cap \partial G_1)$;*

(H_4) *$Tx + Cx + \lambda Jx \neq 0$ for every $(\lambda, x) \in \mathcal{R}_+ \times (D(T) \cap \partial G_2)$.*

Then there exists $x \in D(T) \cap (G_1 \setminus G_2)$ such that $Tx + Cx = 0$.

Proof: We consider the equation

$$Tx + Cx = 0, \tag{2.2.43}$$

and the associated equation

$$T_t x + Cx = 0, \quad t \in (0, \infty), \;\; x \in \overline{G_1}. \tag{2.2.44}$$

We show that the equation (2.2.44) has a solution $x_t \in G_1 \setminus G_2$ for all sufficiently small t. To this end, we show first that there exist $\tau_0 > 0$, $t_0 > 0$ such that the equation

$$T_t x + Cx = \tau v_0^*, \tag{2.2.45}$$

has no solution in G_1 for every $\tau \geq \tau_0$, $t \in (0, t_0]$. Assume that this is not the case, and let $\{\tau_n\} \subset (0, \infty)$, $\{t_n\} \subset (0, 1]$, $\{x_n\} \subset G_1$ be such that $\tau_n \to \infty$, $t_n \downarrow 0$ and

$$T_{t_n} x_n + Cx_n = \tau_n v_0^*. \tag{2.2.46}$$

Since $\|\tau_n v_0^*\| \to \infty$ and $\{Cx_n\}$ is bounded, we must have $\|T_{t_n} x_n\| \to \infty$. We also note that T_t is positively homogeneous of degree 1 for each $t > 0$. In fact, let

$$y = T_t(sx) = (T^{-1} + tJ^{-1})^{-1}(sx),$$

for $s > 0$, $x \in X$. Then

$$sx \in (T^{-1} + tJ^{-1})y = T^{-1}y + tJ^{-1}y,$$

implies

$$\begin{aligned} y = T(-tJ^{-1}y + sx) &= T\left(s\Big(-\frac{t}{s}J^{-1}y + x\Big)\right) \\ &= sT\left(-\frac{t}{s}J^{-1}y + x\right) \\ &= sT\left(-tJ^{-1}\left(\frac{y}{s}\right) + x\right), \end{aligned}$$

where we have used the homogeneity of the duality mapping J. This implies

$$x \in T^{-1}\left(\frac{y}{s}\right) + tJ^{-1}\left(\frac{y}{s}\right)$$

and

$$y = s(T^{-1} + tJ^{-1})^{-1}x = sT_t x.$$

Now that T_t is homogeneous of degree 1, from (2.2.46) we get

$$T_{t_n}\left(\frac{x_n}{\|T_{t_n}x_n\|}\right) + \frac{Cx_n}{\|T_{t_n}x_n\|} = \frac{\tau_n}{\|T_{t_n}x_n\|}v_0^*. \tag{2.2.47}$$

It can be easily seen from (2.2.47) that

$$\frac{\tau_n}{\|T_{t_n}x_n\|} \to \frac{1}{\|v_0^*\|}.$$

Letting

$$u_n = \frac{x_n}{\|T_{t_n}x_n\|},$$

we have $T_{t_n}u_n \to h$, where

$$h = \frac{v_0^*}{\|v_0^*\|}.$$

Since $u_n \to 0$, we see that

$$\lim_{n\to\infty}\langle T_{t_n}u_n, u_n\rangle = \langle h, 0\rangle = 0.$$

Applying (ii) of Lemma 2.9 with $S = 0$, we obtain $T(0) = h$, i.e., a contradiction to $T(0) = 0$ because $\|h\| = 1$. We have used here $0 \in D(T)$ and $T(0) = 0$ since T is homogeneous of degree 1.

Now, we consider the homotopy function

$$H_1(s,x) = T_t x + Cx - s\tau_0 v_0^*, \quad (s,x) \in [0,1] \times \overline{G_1}, \tag{2.2.48}$$

where $t \in (0, t_0]$ is fixed. It is easy to see that, for every $s \in [0,1]$, the operator $x \mapsto Cx - s\tau_0 v_0^*$ is demicontinuous and bounded on $\overline{G_1}$. To see that it is also of type (S_+), assume that $\{x_n\} \subset \overline{G_1}$ is such that $x_n \rightharpoonup x_0 \in X$ and

$$\limsup_{n\to\infty} \langle Cx_n - s\tau_0 v_0^*, x_n - x_0 \rangle \le 0.$$

Then

$$\limsup_{n\to\infty} \langle Cx_n, x_n - x_0 \rangle \le 0,$$

which, by the (S_+)-property of C, implies that $x_n \to x_0$. Before we consider the Skrypnik degree of the homotopy function $H_1(s,\cdot)$, we show that the equation $H_1(s,x) = 0$ has no solution on $[0,1] \times \partial G_1$ for all sufficiently small $t \in (0, t_0]$. To this end, assume that the contrary is true. Then there exist sequences $\{x_n\} \subset \partial G_1$, $\{t_n\} \subset (0, t_0]$, $\{s_n\} \subset [0,1]$ such that $t_n \downarrow 0$, $s_n \to s_0 \in [0,1]$, $x_n \rightharpoonup x_0 \in X$ and

$$T_{t_n} x_n + Cx_n = s_n \tau_0 v_0^*.$$

Since $\{Cx_n\}$ is bounded, we may assume that $Cx_n \rightharpoonup y_0^* \in X$. Then we have $T_{t_n} x_n \rightharpoonup -y_0^* + s_0 \tau_0 v_0^*$. This and

$$\langle T_{t_n} x_n, x_n - x_0 \rangle + \langle Cx_n, x_n - x_0 \rangle = \langle s_n \tau_0 v_0^*, x_n - x_0 \rangle$$

imply

$$\lim_{n\to\infty} [\langle T_{t_n} x_n, x_n - x_0 \rangle + \langle Cx_n, x_n - x_0 \rangle] = 0. \tag{2.2.49}$$

If

$$\limsup_{n\to\infty} \langle Cx_n, x_n - x_0 \rangle > 0,$$

then there exists a subsequence of $\{n\}$, which we still denote by $\{n\}$, such that

$$\lim_{n\to\infty}\langle Cx_n, x_n - x_0\rangle = q, \tag{2.2.50}$$

for some constant $q > 0$. This combined with (2.2.49) gives

$$\lim_{n\to\infty}\langle T_{t_n}x_n, x_n - x_0\rangle = -q < 0.$$

Applying (i) of Lemma 2.9 with $S = 0$, we obtain a contradiction. Thus, we must have

$$\limsup_{n\to\infty}\langle Cx_n, x_n - x_0\rangle \le 0.$$

Since C is of type (S_+), we have $x_n \to x_0 \in \partial G_1$. Since C is demicontinuous, $Cx_n \rightharpoonup Cx_0 = y_0^*$. This implies that $T_{t_n}x_n \rightharpoonup -Cx_0 + s_0\tau_0 v_0^*$. Applying (ii) of Lemma 2.9 with $S = 0$, we obtain $x_0 \in D(T) \cap \partial G_1$ and $Tx_0 = -Cx_0 + s_0\tau_0 v_0^*$. This, however, contradicts the condition (H_3).

We may now choose the number t_0 further smaller if necessary so that the equation $H_1(s,x) = 0$ has no solution on $[0,1] \times \partial G_1$ for all $t \in (0, t_0]$. The mapping $H_1(s,x)$ is an admissible homotopy for the Skrypnik degree. Thus, the Skrypnik degree $d_S(H_1(s,\cdot), G_1, 0)$ is well-defined and constant for $s \in [0,1]$. Here, we note that the Browder degree $d_B(T + C - \tau_0 v_0^*, G_1, 0)$ satisfies

$$d_B(T + C - \tau_0 v_0^*, G_1, 0) = d_S(T_t + C - \tau_0 v_0^*, G_1, 0), \tag{2.2.51}$$

for all sufficiently small $t \in (0, t_0]$. Assume that $d_S(H_1(1,\cdot), G_1, 0) \neq 0$ for some $t_1 \in (0, t_0]$. We see from (2.2.51) and the basic property of the Skrypnik degree that there exists $x \in G_1$ such that

$$T_{t_1}x + Cx = \tau_0 v_0^*,$$

which contradicts the choice of τ_0 as in (2.2.45). Consequently,

$$d_S(T_t + C - \tau_0 v_0^*, G_1, 0) = 0, \quad t \in (0, t_0]$$

and then, by the homotopy invariance of the Skrypnik degree, we obtain

$$d_S(T_t + C, G_1, 0) = 0, \quad t \in (0, t_0]. \tag{2.2.52}$$

We next consider the homotopy function

$$H_2(s, x) = s(T_t + C)x + (1 - s)Jx, \quad (s, x) \in [0, 1] \times \overline{G_2}. \tag{2.2.53}$$

We first show that there exists $t_1 \in (0, t_0]$ such that the equation $H_2(s, x) = 0$ has no solution on $[0, 1] \times \partial G_2$ for any $t \in (0, t_1]$. If we assume the contrary, then there exist sequences $\{t_n\} \subset (0, t_0]$, $\{s_n\} \subset [0, 1]$, $\{x_n\} \subset \partial G_2$ such that $t_n \downarrow 0$, $s_n \to s_0 \in [0, 1]$, $x_n \rightharpoonup x_0 \in X$, $Cx_n \rightharpoonup c_0^* \in X^*$, $Jx_n \rightharpoonup z_0^* \in X^*$, and

$$s_n(T_{t_n} x_n + Cx_n) + (1 - s_n)Jx_n = 0. \tag{2.2.54}$$

Since $s_n = 0$ is impossible as $J(0) = 0$ and J is injective, we may assume that $s_n > 0$ for all n. If $s_n \to 0$, then

$$\langle T_{t_n} x_n + Cx_n, x_n \rangle = \left(\frac{1}{s_n} - 1 \right) \langle Jx_n, x_n \rangle = \left(\frac{1}{s_n} - 1 \right) \|x_n\|^2 \to -\infty \tag{2.2.55}$$

because $\{\|x_n\|\}$ is bounded away from zero. Since

$$\langle T_{t_n} x_n, x_n \rangle \geq 0$$

and $\{\langle Cx_n, x_n \rangle\}$ is bounded, we see that (2.2.55) is impossible. This means that $s_0 \in (0, 1]$ and then (2.2.54) implies

$$T_{t_n} x_n \rightharpoonup -c_0^* - \left(\frac{1}{s_0} - 1 \right) z_0^*.$$

From (2.2.55) along with the monotonicity of the duality mapping J, we obtain

$$\begin{aligned}\langle T_{t_n}x_n + Cx_n, x_n - x_0\rangle &= -\left(\frac{1}{s_n} - 1\right)\langle Jx_n, x_n - x_0\rangle \\ &= -\left(\frac{1}{s_n} - 1\right)[\langle Jx_n - Jx_0, x_n - x_0\rangle + \langle Jx_0, x_n - x_0\rangle] \\ &\leq -\left(\frac{1}{s_n} - 1\right)\langle Jx_0, x_n - x_0\rangle,\end{aligned}$$

which implies

$$\limsup_{n\to\infty}\langle T_{t_n}x_n + Cx_n, x_n - x_0\rangle \leq 0.$$

Assume that

$$\limsup_{n\to\infty}\langle Cx_n, x_n - x_0\rangle > 0. \tag{2.2.56}$$

Then, for some subsequence of $\{n\}$, denoted by $\{n\}$ again, we have

$$\lim_{n\to\infty}\langle Cx_n, x_n - x_0\rangle = q > 0.$$

Thus,

$$\limsup_{n\to\infty}\langle T_{t_n}x_n, x_n - x_0\rangle \leq \limsup_{n\to\infty} a_n + \lim_{n\to\infty}[-\langle Cx_n, x_n - x_0\rangle] \leq -q < 0, \tag{2.2.57}$$

where

$$a_n = \langle T_{t_n}x_n + Cx_n, x_n - x_0\rangle.$$

This is a contradiction to (i) of Lemma 2.9 and so (2.2.56) doesn't hold. Thus, we must have

$$\limsup_{n\to\infty}\langle Cx_n, x_n - x_0\rangle \leq 0.$$

Since C is of type (S_+), we get $x_n \to x_0$ which implies that $Cx_n \rightharpoonup Cx_0 = c_0^*$, $Jx_n \to Jx_0 = z_0^*$ and

$$T_{t_n}x_n \rightharpoonup -Cx_0 - \left(\frac{1}{s_0} - 1\right)Jx_0.$$

Moreover, since $x_n \to x_0$, we have

$$\lim_{n\to\infty} \langle T_{t_n} x_n, x_n - x_0 \rangle = 0.$$

Using (ii) of Lemma 2.9, we get $x_0 \in D(T) \cap \partial G_2$ and

$$Tx_0 + Cx_0 + \left(\frac{1}{s_0} - 1\right) Jx_0 = 0,$$

i.e., a contradiction to assumed condition (H_4). For the sake of our convenience, we may choose t_0 further smaller (if necessary) so that we may take $t_1 = t_0$. Since our homotopy function $H_2(s, x)$ is admissible for the Skrypnik degree, then, by invariance of homotopy property of the Skrypnik degree, we have

$$\begin{aligned} d_S(T_t + C, G_2, 0) &= d_S(H_2(1, \cdot), G_2, 0) \\ &= d_S(H_2(0, \cdot), G_2, 0) \\ &= d_S(J, G_2, 0) \\ &= 1. \end{aligned}$$

Thus, for all $t \in (0, t_0]$, we have

$$d_S(T_t + C, G_1, 0) \neq d_S(T_t + C, G_2, 0).$$

From the excision property of the Skrypnik degree which is an easy consequence of its finite-dimensional counterpart, we obtain a solution $x_t \in G_1 \setminus G_2$ of the equation $T_t x + Cx = 0$ for every $t \in (0, t_0]$. Let $\{t_n\} \subset (0, t_0]$ be such that $t_n \downarrow 0$, and let x_n be the corresponding solution of $T_t x + Cx = 0$. We have

$$T_{t_n} x_n + Cx_n = 0.$$

We may assume that $x_n \rightharpoonup x_0 \in X$. If

$$\limsup_{n\to\infty} \langle Cx_n, x_n - x_0 \rangle > 0,$$

then for a subsequence of $\{n\}$, denoted by $\{n\}$ again we have

$$\lim_{n\to\infty}\langle Cx_n, x_n - x_0\rangle = q$$

for some number $q > 0$. For this subsequence of $\{n\}$, we then have

$$\lim_{n\to\infty}\langle T_{t_n}x_n, x_n - x_0\rangle = -\lim_{n\to\infty}\langle Cx_n, x_n - x_0\rangle = -q < 0,$$

which contradicts (i) of Lemma 2.9. Therefore, we must have

$$\limsup_{n\to\infty}\langle Cx_n, x_n - x_0\rangle \leq 0.$$

The (S_+)-property of C implies that $x_n \to x_0 \in \overline{G_1 \setminus G_2}$. Since C is demicontinuous, $Cx_n \rightharpoonup Cx_0$ and so $T_{t_n}x_n \rightharpoonup -Cx_0$. Since

$$\lim_{n\to\infty}\langle T_{t_n}x_n, x_n - x_0\rangle = 0,$$

by (ii) of Lemma 2.9, we obtain $x_0 \in D(T)$ and $Tx_0 + Cx_0 = 0$.

Finally, since $x_0 \notin \partial G_1 \cup \partial G_2$ in view of the conditions (H_3) and (H_4) and

$$x_0 \in \overline{G_1 \setminus G_2} = (G_1 \setminus G_2) \cup \partial(G_1 \setminus G_2) \subset (G_1 \setminus G_2) \cup \partial G_1 \cup \partial G_2,$$

we conclude that $x_0 \in D(T) \cap (G_1 \setminus G_2)$. This completes the proof. ■

Remark 2.12 In the above theorem, we can replace the positively homogeneous operator T of degree 1 by a positively homogeneous operator of degree $\alpha \in (0, 1]$.

2.3 Operators of the form $T + C + G$

Hu and papageorgiou [17] generalized the degree theory of Browder [6] to the mappings of the form $T + C + G$, where T is maximal monotone, C bounded demicontinuous of type (S_+) and G upper semicontinuous compact multifunction. In this section, an existence of nonzero solution of $Tx + Cx + Gx \ni 0$ has been established.

Definition 2.13 *An operator $G : X \supset D(G) \to 2^{X^*}$ is said to belong to class (P) if it maps bounded sets to relatively compact sets, for every $x \in B$, $G(x)$ is closed and convex subsets of X^* and $G(\cdot)$ is upper-semicontinuous (usc), i.e., for every closed set $F \subset X^*$, the set $G^-(F) = \{x \in D(G) : G(x) \cap F \neq \emptyset\}$ is closed in X.*

An important fact about a compact-set valued usc operator G is that it is closed. Furthermore, for every sequence $\{[x_n, y_n]\} \subset Gr(G)$ such that $x_n \to x \in D(G)$, the sequence $\{y_n\}$ has a cluster point in $G(x)$. Here, $Gr(G)$ is the graph of G.

Definition 2.14 *An operator $T : X \supset D(T) \to 2^{X^*}$ is said to be positively homogeneous of degree $\alpha > 0$ if, for a fixed $\alpha > 0$, $x \in D(T)$ implies $tx \in D(T)$ for all $t \in \mathcal{R}_+$ and $T(tx) = t^\alpha Tx$.*

Theorem 2.15 *Assume that $G_1,\ G_2 \subset X$ are open, bounded with $0 \in G_2$ and $\overline{G_2} \subset G_1$. Assume that $T : X \supset D(T) \to 2^{X^*}$ is maximal monotone, and positively homogeneous of degree $\alpha \in (0, 1]$, and that $C : \overline{G}_1 \to X^*$ is bounded, demicontinuous and of type (S_+). Let $G : \overline{G}_1 \to 2^{X^*}$ be of class (P). Moreover, assume the following:*

- (H_3) *There exists $v_0^* \in X^* \setminus \{0\}$ such that $Tx + Cx + Gx \not\ni \lambda v_0^*$ for every $(\lambda, x) \in \mathcal{R}_+ \times (D(T) \cap \partial G_1)$;*
- (H_4) *$Tx + Cx + Gx + \lambda Jx \not\ni 0$ for every $(\lambda, x) \in\ \mathcal{R}_+ \times (D(T) \cap \partial G_2)$.*

Then the inclusion $Tx + Cx + Gx \ni 0$ has a nonzero solution $x \in D(T) \cap (G_1 \setminus G_2)$.

Proof: We consider the equation

$$Tx + Cx + Gx \ni 0$$

and then the associated equation

$$T_t x + Cx + g_\epsilon x = 0. \tag{2.3.58}$$

Here, $\epsilon > 0$ and $g_\epsilon : \overline{G_1} \to 2^{X^*}$ is an approximate continuous Cellina-selection (cf. [17], p. 136, Lemma 6) satisfying

$$g_\epsilon x \in G(B_\epsilon(x) \cap \overline{G_1}) + B_\epsilon(0)$$

for all $x \in \overline{G_1}$ and $g_\epsilon(\overline{G_1}) \subset \overline{\text{conv}}G(\overline{G_1})$.

We show that equation (2.3.58) has a solution $x_{t,\epsilon}$ for all sufficiently small t and ϵ. To this end, we first show that there exist $\tau_0 > 0$, $t_0 > 0$ and $\epsilon_0 > 0$ such that the equation

$$T_t x + Cx + g_\epsilon x = \tau v_0^* \tag{2.3.59}$$

has no solution in G_1 for every $\tau \geq \tau_0$, $t \in (0, t_0]$ and $\epsilon \in (0, \epsilon_0]$.

Assuming the contrary, let $\{\tau_n\} \subset (0, \infty)$, $\{t_n\} \subset (0, \infty)$, $\{\epsilon_n\} \subset (0, \infty)$ and $\{x_n\} \subset G_1$ be such that $\tau_n \to \infty$, $t_n \downarrow 0$, $\epsilon_n \downarrow 0$ and

$$T_{t_n} x_n + Cx_n + g_{\epsilon_n} x_n = \tau_n v_0^*. \tag{2.3.60}$$

We may assume that $g_{\epsilon_n} x_n \to g^* \in X^*$ in view of the properties of G. So, $\|T_{t_n} x_n\| \to \infty$ as $\|\tau_n v_0^*\| \to \infty$ and $\{Cx_n\}$ is bounded.

Thus, from (2.3.60), we get

$$\frac{T_{t_n} x_n}{\|T_{t_n} x_n\|} + \frac{Cx_n}{\|T_{t_n} x_n\|} + \frac{g_{\epsilon_n} x_n}{\|T_{t_n} x_n\|} = \frac{\tau_n}{\|T_{t_n} x_n\|} v_0^*, \tag{2.3.61}$$

We claim that

$$T_t(sx) = s^\alpha T_{ts^{\alpha-1}}(x) \text{ for all } t, s > 0. \tag{2.3.62}$$

In fact, let

$$y = T_t(sx) = (T^{-1} + tJ^{-1})^{-1}(sx),$$

for $t, s > 0$, $x \in X$. This and the homogeneity of the duality mapping J imply

$$\begin{aligned} y = T(-tJ^{-1}y + sx) &= T\left(s\left(-\frac{t}{s}J^{-1}y + x\right)\right) \\ &= s^\alpha T\left(-\frac{t}{s}J^{-1}y + x\right) \\ &= s^\alpha T\left(-\frac{t}{s^{1-\alpha}}J^{-1}\left(\frac{y}{s^\alpha}\right) + x\right). \end{aligned}$$

This implies

$$x \in T^{-1}\left(\frac{y}{s^\alpha}\right) + ts^{\alpha-1}J^{-1}\left(\frac{y}{s^\alpha}\right)$$

and

$$y = s^{\alpha}(T^{-1} + ts^{\alpha-1}J^{-1})^{-1}x = s^{\alpha}T_{ts^{\alpha-1}}(x).$$

This proves (2.3.62).

In view of (2.3.62), we obtain,

$$\frac{T_{t_n}x_n}{\|T_{t_n}x_n\|} = T_{t_n\lambda_n}\left(\frac{x_n}{\|T_{t_n}x_n\|^{\frac{1}{\alpha}}}\right), \tag{2.3.63}$$

where

$$\lambda_n = \|T_{t_n}x_n\|^{(\alpha-1)/\alpha}.$$

It clear that $\lambda_n \to 0$ for $\alpha \in (0,1)$ and $\lambda_n = 1$ for $\alpha = 1$. Now, (2.3.61) implies

$$1 - \left\|\frac{Cx_n}{\|T_{t_n}x_n\|} + \frac{g_{\epsilon_n}x_n}{\|T_{t_n}x_n\|}\right\| \leq \frac{\tau_n\|v_0^*\|}{\|T_{t_n}x_n\|} \leq 1 + \left\|\frac{Cx_n}{\|T_{t_n}x_n\|} + \frac{g_{\epsilon_n}x_n}{\|T_{t_n}x_n\|}\right\|.$$

Thus,

$$\frac{\tau_n\|v_0^*\|}{\|T_{t_n}x_n\|} \to 1 \quad \text{and} \quad \frac{\tau_n}{\|T_{t_n}x_n\|} \to \frac{1}{\|v_0^*\|} \quad \text{as } n \to \infty. \tag{2.3.64}$$

Let

$$u_n = \frac{x_n}{\|T_{t_n}x_n\|^{\frac{1}{\alpha}}}.$$

Then we have $u_n \to 0$. By (2.3.61), (2.3.63) and (2.3.64), we obtain $T_{t_n\lambda_n}u_n \to h$ with

$$h = \frac{v_0^*}{\|v_0^*\|}.$$

Therefore,

$$\lim_{n\to\infty}\langle T_{t_n\lambda_n}u_n, u_n\rangle = \langle h, 0\rangle = 0.$$

Since $t_n\lambda_n \to 0$, by (ii) of Lemma 2.9 with $S = 0$ we obtain, $0 \in D(T)$ and $h = T(0)$. Since $T(0) = 0$, this is a contradiction to $\|h\| = 1$.

We now consider the homotopy mapping

$$H_1(s, x, t, \epsilon) = T_tx + Cx + g_\epsilon x - s\tau_0 v_0^*, \quad s \in [0,1],\ x \in \overline{G_1}, \tag{2.3.65}$$

where $t \in (0, t_0]$ and $\epsilon \in (0, \epsilon_0]$ are fixed. For every $s \in [0,1]$ the operator $x \mapsto Cx - s\tau_0 v_0^*$ is demicontinuous and bounded on $\overline{G_1}$. In order to see that it is of type (S_+), assume that $\{x_n\} \subset \overline{G_1}$ is such that $x_n \rightharpoonup x_0 \in X$ and

$$\limsup_{n\to\infty} \langle Cx_n - s\tau_0 v_0^*, x_n - x_0 \rangle \le 0.$$

Then

$$\limsup_{n\to\infty} \langle Cx_n, x_n - x_0 \rangle \le 0,$$

which by the (S_+)-property of f, implies that $x_n \to x_0$. Before we consider the Skrypnik degree of this homotopy on the set G_1, we show that the equation $H_1(s, x, t, \epsilon) = 0$ has no solution on the boundary of G_1 for all sufficiently small $t \in (0, t_0]$, $\epsilon \in (0, \epsilon_0]$ and all $s \in [0,1]$. To this end, assume the contrary and let $\{x_n\} \subset \partial G_1$, $\{t_n\} \subset (0, t_0]$, $\{s_n\} \subset [0,1]$ and $\{\epsilon_n\} \subset (0, \epsilon_0]$ such that $t_n \downarrow 0$, $s_n \to s_0$ for some $s_0 \in [0,1]$, $\epsilon_n \downarrow 0$ and

$$T_{t_n} x_n + Cx_n + g_{\epsilon_n} x_n = s_n \tau_0 v_0^*.$$

We may assume that $x_n \rightharpoonup x_0 \in X$. Since $\{Cx_n\}$ is bounded, we may assume that $Cx_n \rightharpoonup y_0^* \in X^*$. We may also assume that $g_{\epsilon_n} x_n \to g^*$. Then we have $T_{t_n} x_n \rightharpoonup -y_0^* - g^* + s_0 \tau_0 v_0^*$. From

$$\langle T_{t_n} x_n, x_n - x_0 \rangle + \langle Cx_n, x_n - x_0 \rangle = \langle g_{\epsilon_n} x_n + s_n \tau_0 v_0^*, x_n - x_0 \rangle,$$

we obtain

$$\lim_{n\to\infty} [\langle T_{t_n} x_n, x_n - x_0 \rangle + \langle Cx_n, x_n - x_0 \rangle] = 0. \tag{2.3.66}$$

Let us assume that

$$\limsup_{n\to\infty} \langle Cx_n, x_n - x_0 \rangle > 0. \tag{2.3.67}$$

Then there exists a subsequence of $\{x_n\}$, which we still denote by $\{x_n\}$, such that

$$\lim_{n\to\infty} \langle Cx_n, x_n - x_0 \rangle = q, \tag{2.3.68}$$

for some constant $q > 0$. By (2.3.66) and (2.3.68), we obtain

$$\lim_{n\to\infty} \langle T_{t_n} x_n, x_n - x_0 \rangle = -q < 0.$$

Applying (i) of Lemma 2.9 with $S = 0$, we obtain a contradiction. Therefore, (2.3.67) is false. Let us assume that

$$\limsup_{n\to\infty} \langle C x_n, x_n - x_0 \rangle \le 0.$$

Since C is of type (S_+), we have $x_n \to x_0 \in \partial G_1$. Since C is also demicontinuous, $C x_n \rightharpoonup C x_0$. This implies

$$T_{t_n} x_n \rightharpoonup -C x_0 - g^* + s_0 \tau_0 v_0^*.$$

Applying (ii) of Lemma 2.9 with $S = 0$, we obtain $x_0 \in D(T) \cap \partial G_1$ and

$$T x_0 + C x_0 + G x_0 \ni s_0 \tau_0 v_0^*,$$

which is a contradiction to our hypothesis (H_3). Thus, we may now choose t_0 and ϵ_0 further so that we also have that $H_1(s, x, t, \epsilon) = 0$ has no solution $x \in \partial G_1$ for all $t \in (0, t_0]$, $\epsilon \in (0, \epsilon_0]$ and all $s \in [0, 1]$. It is clear that the mapping $H_1(s, x, t, \epsilon)$ is an admissible homotopy for Skrypnik's degree and the Skrypnik degree $d_S(H_1(s, \cdot, t, \epsilon), G_1, 0)$ is well-defined and is constant for all $s \in [0, 1]$ and for all $t \in (0, t_0]$, $\epsilon \in (0, \epsilon_0]$. Consequently, the Browder's degree generalized by Hu and Papageorgiou [17] $d_{HP}(T + C + G, G_1, 0)$ is well-defined and satisfies

$$d_{HP}(T + C + G - \tau_0 v_0^*, G_1, 0) = d_S(T_t + C + g_\epsilon - \tau_0 v_0^*, G_1, 0) \tag{2.3.69}$$

for $t \in (0, t_0], \epsilon \in (0, \epsilon_0]$.

Assume that

$$d_S(H_1(1, \cdot, t_1, \epsilon_1), G_1, 0) \ne 0,$$

for some sufficiently small $t_1 \in (0, t_0]$ and $\epsilon_1 \in (0, \epsilon_0]$. Then, the equation

$$T_{t_1}x + Cx + g_{\epsilon_1}x = \tau_0 v_0^*$$

has a solution in the set G_1. However, this contradicts our choice of the number τ_0 in (2.3.59). Consequently,

$$d_S(T_t + C + g_\epsilon, G_1, 0) = d_S(H_1(0, \cdot, t_1, \epsilon_1), G_1, 0) = 0, \quad t \in (0, t_0],\ \epsilon \in (0, \epsilon_0].$$

We next consider the homophony mapping

$$H_2(s, x, t, \epsilon) = s(T_t x + Cx + g_\epsilon x) + (1 - s)Jx, \quad (s, x) \in [0, 1] \times \overline{G_2}. \tag{2.3.70}$$

We first show that there exist $t_1 \in (0, t_0]$, $\epsilon_1 \in (0, \epsilon_0]$ such that the equation $H_2(s, x, t, \epsilon) = 0$ has no solution on ∂G_2 for any $s \in [0, 1]$, any $t \in (0, t_1]$ and any $\epsilon \in (0, \epsilon_1]$.

Let us assume the contrary. Then there exist sequences $t_n \in (0, t_0]$, $\epsilon_n \in (0, \epsilon_1]$, $s_n \in [0, 1]$, and $x_n \in \partial G_2$ such that $t_n \downarrow 0$, $\epsilon_n \downarrow 0$, $s_n \to s_0 \in [0, 1]$, $x_n \rightharpoonup x_0 \in X$, $Cx_n \rightharpoonup y_0^* \in X^*$, $g_{\epsilon_n} x_n \to g^* \in X^*$, $Jx_n \rightharpoonup z_0^* \in X^*$, and

$$s_n(T_{t_n} x_n + Cx_n + g_{\epsilon_n} x_n) + (1 - s_n)Jx_n = 0. \tag{2.3.71}$$

$s_n = 0$ is impossible because $J(0) = 0$ and J is injective, we may assume that $s_n > 0$, for all n. If $s_n \to 0$,

$$\langle T_{t_n} x_n + Cx_n, x_n \rangle = -\left(\frac{1}{s_n} - 1\right) \langle Jx_n, x_n \rangle - \langle g_{\epsilon_n} x_n, x_n \rangle \to -\infty \tag{2.3.72}$$

because $\{\|x_n\|\}$ is bounded below away from zero. Since $\langle T_{x_n} x_n, x_n \rangle \geq 0$ and $\{\langle Cx_n, x_n \rangle\}$ is bounded, we see that (2.3.72) is impossible. Thus $s_0 \in (0, 1]$ and (2.3.71) implies that

$$T_{t_n} \rightharpoonup -y_0^* - g^* - \left(\frac{1}{s_0} - 1\right) z_0^*.$$

Also, from (2.3.71),

$$\begin{aligned}\langle T_{t_n}x_n + Cx_n, x_n - x_0\rangle &= -\left(\frac{1}{s_n} - 1\right)\langle g_{\epsilon_n}x_n + Jx_n, x_n - x_0\rangle \\ &= -\left(\frac{1}{s_n} - 1\right)\Big[\langle Jx_n - Jx_0, x_n - x_0\rangle \\ &\quad +\langle g_{\epsilon_n}x_n + Jx_0, x_n - x_0\rangle\Big] \\ &\le -\left(\frac{1}{s_n} - 1\right)\langle g_{\epsilon_n}x_n + Jx_0, x_n - x_0\rangle,\end{aligned} \tag{2.3.73}$$

by the monotonicity of the duality mapping J. Since $s_0 \in (0, 1]$ and $x_n \rightharpoonup x_0$, we see from (2.3.73) that

$$\limsup_{n\to\infty}\{q_n := \langle T_{t_n}x_n + Cx_n, x_n - x_0\rangle\} \le 0.$$

Let

$$\limsup_{n\to\infty}\langle Cx_n, x_n - x_0\rangle > 0. \tag{2.3.74}$$

Then, for some subsequence of $\{n\}$ denoted by $\{n\}$ again, we have

$$\lim_{n\to\infty}\langle Cx_n, x_n - x_0\rangle = q > 0. \tag{2.3.75}$$

From

$$\langle T_{t_n}x_n, x_n - x_0\rangle = q_n - \langle Cx_n, x_n - x_0\rangle,$$

we see that

$$\limsup_{n\to\infty}\langle T_{t_n}x_n, x_n - x_0\rangle \le \limsup_{n\to\infty} q_n + \lim_{n\to\infty}[-\langle Cx_n, x_n - x_0\rangle] \le -q < 0.$$

This says

$$\limsup_{n\to\infty}\langle T_{t_n}x_n, x_n - x_0\rangle < 0.$$

Using (i) of Lemma 2.9, we conclude that (2.3.74) is impossible. Thus, (2.3.74) holds with "$\le$" in place of "$>$". Since C is of type (S_+), we have $x_n \to x_0 \in \partial G_2$. This implies $Cx_n \rightharpoonup Cx_0,\ Jx_n \to Jx_0$ and

$$T_{t_n}x_n \rightharpoonup -Cx_0 - g^* - \left(\frac{1}{s_0} - 1\right) Jx_0.$$

Since $x_n \to x_0$, $g^* \in G(x_0)$ and

$$\lim_{n\to\infty} \langle T_{t_n}x_n, x_n - x_0\rangle = 0.$$

Using (ii) of Lemma 2.9, we have that $x_0 \in D(T)$ and

$$-Cx_0 - g^* - \left(\frac{1}{s_0} - 1\right) Jx_0 \in Tx_0,$$

which implies that

$$Tx_0 + Cx_0 + Gx_0 + \left(\frac{1}{s_0} - 1\right) Jx_0 \ni 0.$$

Thus we arrived at a contradiction to our hypothesis (H_4) because $x_0 \in D(T) \cap \partial G_2$. For the sake of convenience, we assume that t_0 and ϵ_0 are sufficiently small so that we may take $t_1 = t_0$ and $\epsilon_1 = \epsilon_0$.

It is therefore obvious that the mapping $H_2(s, x, t, \epsilon)$ is an admissible homotopy for Skrypnik's degree and so the Skrypnik degree $d_S(H_2(s, \cdot, t, \epsilon), G_2, 0)$ is well-defined and constant for all $s \in [0, 1]$, all $t \in (0, t_0]$ and all $\epsilon \in (0, \epsilon_0]$.
By the invariance of the Skrypnik degree, for all $t \in (0, t_0]$, $\epsilon \in (0, \epsilon_0]$, we have

$$\begin{aligned} d_S(H_2(1, \cdot, t, \epsilon), G_2, 0) &= d_S(T_t + C + g_\epsilon, G_2, 0) \\ &= d_S(H_2(0, \cdot, t, \epsilon), G_2, 0) \\ &= d_S(J, G_2, 0) \\ &= 1. \end{aligned}$$

Thus, for all $t \in (0, t_0]$, $\epsilon \in (0, \epsilon_0]$, we have

$$d_S(T_t + C + g_\epsilon, G_1, 0) \neq d_S(T_t + C + g_\epsilon, G_2, 0).$$

From the excision property of the Skrypnik degree, which is an easy consequence of its finite-dimensional counterpart, we obtain a solution $x_{t,\epsilon} \in G_1 \setminus G_2$ of $T_t x + Cx + g_\epsilon x = 0$ for every $t \in (0, t_0]$ and every $\epsilon \in (0, \epsilon_0]$. We let $t_n \in (0, t_0]$ and $\epsilon_n \in (0, \epsilon_0]$ be such that $t_n \downarrow 0$, $\epsilon_n \downarrow 0$ and let $x_n \in G_1 \setminus G_2$ be the corresponding solutions of $T_t x + Cx + g_\epsilon x = 0$. We have

$$T_{t_n} x_n + C x_n + g_{\epsilon_n} x_n = 0.$$

We may assume that $x_n \rightharpoonup x_0$ and $g_{\epsilon_n} x_n \to g^* \in X^*$. We have

$$\langle T_{t_n} x_n, x_n, x_0 \rangle = -\langle C x_n + g_{\epsilon_n} x_n, x_n - x_0 \rangle.$$

If

$$\limsup_{n \to \infty} \langle C x_n + g_{\epsilon_n} x_n, x_n - x_0 \rangle > 0,$$

then we obtain a contradiction from (i) of Lemma 2.9. Consequently

$$\limsup_{n \to \infty} \langle C x_n + g_{\epsilon_n} x_n, x_n - x_0 \rangle \leq 0,$$

and hence

$$\limsup_{n \to \infty} \langle C x_n, x_n - x_0 \rangle \leq 0.$$

By the (S_+)-property of C, we obtain $x_n \to x_0 \in \overline{G_1 \setminus G_2}$. Then $C x_n \rightharpoonup C x_0$ and $T_{t_n} x_n \rightharpoonup -C x_0 - g^*$. Using this in (ii) of Lemma 2.9, we get $x_0 \in D(T)$ and $-C x_0 - g^* \in T x_0$. By a property of the selection $g_{\epsilon_n} x_n$ (cf. [17], p. 238), we have $g^* \in G(x_0)$ and therefore $T x_0 + C x_0 + G x_0 \ni 0$. We also have

$$x_0 \in \overline{G_1 \setminus G_2} = (G_1 \setminus G_2) \cup \partial(G_1 \setminus G_2) \subset (G_1 \setminus G_2) \cup \partial G_1 \cup \partial G_2.$$

But, by conditions (H_3) and (H_4), $x_0 \notin \partial G_1 \cup \partial G_2$. Thus $x_0 \in D(T) \cap (G_1 \setminus G_2)$ and the proof is complete. ■

2.4 Application

We consider the space $X = W_0^{m,p}(\Omega)$ with the integer $m \geq 1$, the number $p \in (1, \infty)$, and the domain $\Omega \subset \mathcal{R}^N$. We let N_0 denote the number of all multi-indices $\alpha = (\alpha_1, \ldots, \alpha_N)$ such that $|\alpha| = \alpha_1 + \cdots + \alpha_N \leq m$. For $\xi = (\xi_\alpha)_{|\alpha| \leq m} \in \mathcal{R}^{N_0}$, we have a representation $\xi = (\eta, \zeta)$, where $\eta = (\eta_\alpha)_{|\alpha| \leq m-1} \in \mathcal{R}^{N_1}$, $\zeta = (\zeta_\alpha)_{|\alpha|=m} \in \mathcal{R}^{N_2}$ and $N_0 = N_1 + N_2$. We let

$$\xi(u) = (D^\alpha u)_{|\alpha| \leq m}, \quad \eta(u) = (D^\alpha u)_{|\alpha| \leq m-1}, \quad \zeta(u) = (D^\alpha u)_{|\alpha|=m},$$

where

$$D^\alpha u = \prod_{i=1}^{N} \left(\frac{\partial}{\partial x_i} \right)^{\alpha_i}.$$

Also, let $q = p/(p-1)$.

We now consider the partial differential operator in divergence form

$$(Au)(x) = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, u(x), \ldots, D^m u(x)), \quad x \in \Omega.$$

The coefficients $A_\alpha : \Omega \times \mathcal{R}^{N_0} \to \mathcal{R}$ are assumed to be Carathéodory functions, i.e., each $A_\alpha(x, \xi)$ is measurable in x for fixed $\xi \in \mathcal{R}^{N_0}$ and continuous in ξ for almost all $x \in \Omega$. We consider the following conditions:

$(A1)$ There exist $p \in (1, \infty)$, $c_1 > 0$ and $\kappa_1 \in L^q(\Omega)$ such that

$$|A_\alpha(x, \xi)| \leq c_1 |\xi|^{p-1} + \kappa_1(x), \quad x \in \Omega, \ \ \xi \in \mathcal{R}^{N_0}, \ \ |\alpha| \leq m.$$

$(A2)$ The Leray-Lions Condition

$$\sum_{|\alpha|=m} (A_\alpha(x, \eta, \zeta_1) - A_\alpha(x, \eta, \zeta_2)(\zeta_{1_\alpha} - \zeta_{2_\alpha})) > 0$$

is satisfied for every $x \in \Omega$, $\eta \in \mathcal{R}^{N_1}$, $\zeta_1, \zeta_2 \in \mathcal{R}^{N_2}$ with $\zeta_1 \neq \zeta_2$.

$(A3)$

$$\sum_{|\alpha|\leq m} (A_\alpha(x,\xi_1) - A_\alpha(x,\xi_2)(\xi_{1_\alpha} - \xi_{2_\alpha})) \geq 0$$

is satisfied for every $x \in \Omega$, $\xi_1, \xi_2 \in \mathcal{R}^{N_0}$.

$(A4)$ There exist $c_2 > 0$, $\kappa_2 \in L^1(\Omega)$ such that

$$\sum_{|\alpha|\leq m} A_\alpha(x,\xi)\xi_\alpha \geq c_2|\xi|^p - \kappa_2(x), \quad x \in \Omega,\ \xi \in \mathcal{R}^{N_0}.$$

If an operator $T : W_0^{m,p}(\Omega) \to W^{-m,q}(\Omega)$ is given by

$$\langle Tu, v\rangle = \int_\Omega \sum_{|\alpha|\leq m} A_\alpha(x,\xi(u))D^\alpha v, \qquad u,\ v \in W_0^{m,p}(\Omega), \tag{2.4.76}$$

then conditions $(A1)$, $(A3)$ imply that it is bounded, continuous and monotone (cf. e. g. Kittila [27, pp. 25-26], Pascali and Sburlan [30, pp. 274-275]). Since it is continuous, it is maximal monotone. Similarly, condition $(A1)$, with A replaced by B, implies that the operator

$$\langle Cu, v\rangle = \int_\Omega \sum_{|\alpha|\leq m} B_\alpha(x,\xi(u))D^\alpha v, \qquad u,\ v \in W_0^{m,p}(\Omega), \tag{2.4.77}$$

is a bounded continuous mapping. We also know that conditions $(A1)$, $(A2)$ and $(A4)$, with B in place of A everywhere, imply that the operator C is of type (S_+) (cf. Kittila [27, p. 27]). We then have the following theorem.

Theorem 2.16 *Assume that the operators T and C defined as above with $T(0) = 0$, $C(0) = 0$ are such that T satisfies the conditions $(A1)$, $(A3)$ and C satisfies conditions $(A1)$, $(A2)$, $(A4)$. Assume, further, that the rest of the conditions of Theorem 2.11 are satisfied for two balls $G_1 = B_r(0)$ and $G_2 = B_q(0)$, where $0 < q < r$. Then the Dirichlet boundary value problems*

$$(Au)(x) + (Bu)(x) = 0, \quad x \in \Omega,$$
$$(D^\alpha u)(x) = 0, \quad x \in \partial\Omega, \quad |\alpha| \leq m-1,$$

has a "weak" nonzero solution $u \in B_r(0) \setminus B_q(0) \subset W_0^{m,p}(\Omega)$*, which satisfies the equation* $Tu + Cu = 0$.

Remark 2.17 This application is not covered by the theory developed in the paper [12] of Ding and Kartsatos because all the results there make compactness assumption.

Chapter 3

Densely Defined and Perturbed Maximal Monotone Operators

In this chapter we apply the degree theories developed by Kartsatos and Skrypnik in [22] and [24] for densely defined operators to show the existence of nonzero solutions of operator equations in Banach spaces. The degree theory in [24] is developed via the degree in finite dimension, and the degree theory in [22] is developed via the degree in [24]. For various results about the operator equations, the reader is referred to, for example, Browder and Hess [9], Guan [13], Guan, Kartsatos and Skrypnik [15], and the references therein.

In Section 3.1 we apply the degree theory in [24] to give new results about existence of nonzero solutions of operator equations of the form $(*)$ $Tx + Cx = 0$ with densely defined operators T, C.

In Section 3.2 we apply the degree theory in [22] to give similar results for the operator equations of the form $(*)$ for densely defined quasi-bounded and finitely continuous $(\tilde{S}_+)$–perturbation C of multivalued maximal monotone operator T. The domain of T need not have now a dense linear subspace in it. Since this new degree is also based on the degree theory developed by the above authors in [24], the excision property of this degree still holds.

3.1 Densely Defined Operators T, C

Let X be a reflexive Banach space with a norm in which both X and X^* are locally uniformly convex. Let L be a subspace of X. Let $T : X \supset D(T) \to X^*$ and $C : X \supset D(C) \to X^*$. Let $\mathcal{F}(L)$ be the set of all finite-dimensional subspaces of L. For the operator T, we consider the following assumptions:

$(T1)$ T is monotone, i.e.,

$$\langle Tu - Tv, u - v \rangle \geq 0, \tag{3.1.1}$$

for every $u,\ v \in D(T)$. Moreover,

$$L \subset D(T) \quad \text{and} \quad \overline{L} = X; \tag{3.1.2}$$

$(T2)$ if for every $(u_0, h_0) \in X \times X^*$ with

$$\langle Tu - h_0, u - u_0 \rangle \geq 0, \quad \text{for} \quad u \in L, \tag{3.1.3}$$

then we have $u_0 \in D(T)$ and $Tu_0 = h_0$;

$(T3)$ for any $u_0 \in D(T)$ we have

$$\inf\{\langle Tu - Tu_0, u - u_0 \rangle : u \in L\} = 0; \tag{3.1.4}$$

$(T4)$ for every $F \in \mathcal{F}(L)$, $v \in L$ the mapping $t(F, v) : F \to \mathcal{R}$, defined by

$$t(F, v)u = \langle Tu, v \rangle,$$

is continuous.

For the operator C, we consider the following assumptions:

$(C1)$

$$L \subset D(C) \tag{3.1.5}$$

and C is quasi-bounded with respect to T, i.e., for every number $S > 0$ there exists a number $K(S) > 0$ such that from the inequalities

$$\langle Tu + Cu, u \rangle \leq 0, \quad \|u\| \leq S, \quad u \in L, \tag{3.1.6}$$

we have $\|Cu\| \leq K(S)$;

$(C2)$ the operator C satisfies the following generalized (S_+) condition with respect to T: for every sequence $\{u_n\} \subset L$ such that $u_n \rightharpoonup u_0$, $Cu_n \rightharpoonup h_0$ and

$$\limsup_{n\to\infty}\langle Cu_n, u_n - u_0\rangle \le 0, \quad \langle Tu_n + Cu_n, u_n\rangle \le 0, \tag{3.1.7}$$

for some $u_0 \in X$, $h_0 \in X^*$, we have $u_n \to u_0$, $u_0 \in D(C)$ and $Cu_0 = h_0$;

$(C3)$ for every $F \in \mathcal{F}(L)$, $v \in L$ the mapping $t(F, v) : F \to \mathcal{R}$, defined by

$$t(F, v)u = \langle Cu, v\rangle,$$

is continuous.

We note that the conditions $(T2)$ and $(T3)$ are satisfied for a maximal monotone operator T when $D(T)$ is a subspace of X. In this case, we may take $D(T) = L$.

The following theorem defines the degree for the mapping $T+C$ as in ([24], p. 427).

Theorem A *Let X be a real reflexive Banach space and $T : X \supset D(T) \to X^*$ satisfies $(T1)$ – $(T4)$ and $C : X \supset D(C) \to X^*$ satisfies (C_1) – (C_3). Let $G \subset X$ be open and bounded such that*

$$Tx + Cx \neq 0, \quad x \in \partial G \cap D(T + C).$$

Then there exists a space $F_0 \in \mathcal{F}(L)$ such that for every space $F \in \mathcal{F}(L)$ with $F_0 \subset F$ the set

$$Z(F_0, F) := \{x \in \partial G_F \cap D(T+C) : \langle Tx+Cx, x\rangle \le 0, \langle Tx+Cx, v\rangle = 0, \text{ for all } v \in F_0\}$$

is empty, where $G_F = G \cap F$. Moreover, for every $F \in \mathcal{F}(L)$ with $F_0 \subset F$ we have

$$deg((T + C)_F, G_F, 0) = deg((T + C)_{F_0}, G_{F_0}, 0),$$

where $(T + C)_F : F \to F$ is defined by

$$(T + C)_F(x) = \sum_{i=1}^{k}\langle Tx + Cx, v_i\rangle v_i,$$

$\{v_1, \ldots, v_k\}$ *being a basis of* F *and* $deg(\cdot, \cdot, \cdot)$ *is the Brouwer degree.*

Definition 3.1 *Let the hypotheses of Theorem A be satisfied for the operators* T, C. *Then the degree* $d(T + C, G, 0)$ *in* [24] *is defined by*

$$d(T + C, G, 0) = deg((T + C)_F, G_F, 0),$$

where $F \in \mathcal{F}(L)$ *is such that* $F \supset F_0$ *with* F_0 *given by Theorem A.*

It turns out that $d(T + C, G, 0)$ is independent of the choice of the space F_0 and its basis.

Next, we consider admissible homotopy for the above degree (cf. [24], p. 432). Let X be a real reflexive Banach space, L be a subspace of X such that $\overline{L} = X$.

Consider the one-parameter family of operators $T_t : X \supset D(T_t) \to X^*$, $t \in [0, 1]$, satisfying the following conditions:

$t^{(1)}$: for every $t \in [0, 1]$, the operator T_t satisfies $(T1)$ – $(T3)$ with the space L independent of t;

$t^{(2)}$: for every $v \in L$, the mapping $\mu(v) : [0, 1] \to X^*$ defined by $\mu(v)(t) = T_t(v)$ is continuous;

$t^{(3)}$: for every $F \in \mathcal{F}(L)$ and $v \in L$, the mapping $\tilde{m}(F, v) : F \times [0, 1] \to \mathcal{R}$ defined by $\tilde{m}(F, v)(x, t) = \langle T_t(x), v \rangle$ is continuous.

Consider the second one-parameter family of operators $C_t : X \supset D(C_t) \to X^*$, $t \in [0, 1]$ satisfying the following conditions:

$c^{(1)}$: the family C_t is uniformly quasi-bounded w.r.t. T_t, i.e., for every $S > 0$ there exists $K(S) > 0$ such that

$$\langle T_t x + C_t x, x \rangle \leq S, \quad \|u\| \leq S, \quad x \in L,\ t \in [0, 1] \tag{3.1.8}$$

implies that $\|C_t u\| \leq K(S)$ for all $t \in [0, 1]$;

$c^{(2)}$: for every pair of sequences $\{t_n\} \subset [0,1]$, $\{x_n\} \subset L$ such that $x_n \rightharpoonup x_0$, $C_{t_n}x_n \rightharpoonup h_0$, $t_n \to t_0$ and

$$\limsup_{n\to\infty}\langle C_{t_n}x_n, x_n - x_0\rangle \le 0, \quad \langle (T_{t_n} + C_{t_n})x_n, x_n\rangle \le 0, \tag{3.1.9}$$

for some $x_0 \in X$, $h_0 \in X^*$, $t_0 \in [0,1]$, we have $x_n \to x_0$, $x_0 \in D(C_{t_0})$ and $C_{t_0}x_0 = h_0$;

$c^{(3)}$: for every $F \in \mathcal{F}(L)$ and $v \in L$, the mapping $\tilde{c}(F,v) : F \times [0,1] \to \mathcal{R}$ defined by $\tilde{c}(F,v)(x,t) = \langle C_t(x), v\rangle$ is continuous.

Definition 3.2 *Let $T^{(i)} : X \supset D(T^{(i)}) \to X^*$, $C^{(i)} : X \supset D(C^{(i)}) \to X^*$, $i = 0,1$, satisfy conditions $(T1)$ – $(T4)$ and $(C1)$ – $(C3)$ respectively, with a common space L. We say that the operators $T^{(0)} + C^{(0)}$ and $T^{(1)} + C^{(1)}$ are homotopic with respect to the open and bounded set $G \subset X$ if there exist one-parameter families of operators $T_t : X \supset D(T_t) \to X^*$ and $C_t : X \supset D(C_t) \to X^*$ satisfying conditions $t^{(1)}$ – $t^{(3)}$ and $c^{(1)}$ – $c^{(3)}$ respectively, and such that*

$$T^{(i)} = T_i, \quad C^{(i)} = C_i, \quad i = 0,1,$$

and

$$T_tx + C_tx \ne 0, \quad x \in \partial G \cap D(T_t + C_t),\ t \in [0,1].$$

The degree function $d(T + C, G, 0)$ is invariant under this homotopy.

Remark 3.3 We should note here that the above degree theory as developed in [24] used the stronger condition $(C1)$ with the number S in place of 0 in the first inequality of (3.1.6). A careful study on the development in [24] shows that all we need there is our present assumption. The same remark applies to the homotopy assumption $c^{(1)}$ (cf. [24], $a_t^{(1)}$, p. 432): we can replace S in the first inequality there by 0.

We also need the following condition on the operator C which is stronger than $(C2)$ meaning that the class of operators satisfying the condition $(\tilde{C}2)$ below is smaller than the class of operators satisfying the condition $(C2)$.

$(\tilde{C}2)$ for every number $S > 0$ and every sequence $\{u_n\} \subset L$ such that $u_n \rightharpoonup u_0$, $Cu_n \rightharpoonup h_0$ and

$$\limsup_{n\to\infty} \langle Cu_n, u_n - u_0\rangle \le 0, \quad \langle Tu_n + Cu_n, u_n\rangle \le S$$

for some $u_0 \in X$, $h_0 \in X^*$, we have $u_n \to u_0$, $u_0 \in D(C)$ and $Cu_0 = h_0$.

Before we use the above degree theory by Kartsatos and Skrypnik (cf. [24]) to establish a result on the nonzero solution, we need the following lemma.

Lemma 3.4 *Assume that the operator $T : X \supset D(T) \to X^*$, $D(T) = L$, is maximal monotone and that the operator $C : X \supset D(C) \to X^*$ satisfies $(C1)$ (with 0 in the first inequality of (3.1.6) replaced by S), $(\tilde{C}2)$, $(C3)$. Assume that, for some $p^* \in X^*$, the equation*

$$Tx + Cx = p^* \tag{3.1.10}$$

has no solution $x \in L \cap \overline{G_1}$, where G_1 is an open and bounded set in X with $0 \in G_1$. Then there exists $\epsilon_0 > 0$ such that the equation

$$Tx + Cx + \epsilon Jx = p^* \tag{3.1.11}$$

has no solution $x \in L \cap \overline{G_1}$ for any $\epsilon \in (0, \epsilon_0]$.

Proof: Assume that the conclusion of the lemma is false and let $\epsilon_n > 0$, $\{x_n\} \subset L \cap \overline{G_1}$ be such that $\epsilon_n \downarrow 0$ and

$$Tx_n + Cx_n + \epsilon_n Jx_n = p^*. \tag{3.1.12}$$

Since $\{x_n\}$ and $\{Jx_n\}$ are bounded and

$$\langle Tx_n + Cx_n, x_n\rangle = -\epsilon_n \langle Jx_n, x_n\rangle + \langle p^*, x_n\rangle \le \|p^*\|\|x_n\|,$$

there exists a constant $S > 0$ such that

$$\|x_n\| \le S \quad \text{and} \quad \langle Tx_n + Cx_n, x_n\rangle \le S.$$

Since C is quasi-bounded w.r.t. T, there exists $K(S) > 0$ such that $\|Cx_n\| \leq K(S)$. We may assume that $x_n \rightharpoonup x_0 \in X$ and $Cx_n \rightharpoonup c_0^* \in X^*$. In order to show that

$$\limsup_{n\to\infty} \langle Cx_n, x_n - x_0\rangle \leq 0, \tag{3.1.13}$$

assume that this is not true. Then for a suitable subsequence of $\{n\}$, denoted by $\{n\}$ again, we have

$$\lim_{n\to\infty} \langle Cx_n, x_n - x_0\rangle = q,$$

for some $q > 0$. This combined with (3.1.12) gives

$$\begin{aligned} \lim_{n\to\infty} \langle Tx_n, x_n - x_0\rangle &= -\lim_{n\to\infty} [\langle Cx_n, x_n - x_0\rangle + \langle \epsilon_n Jx_n - p^*, x_n - x_0\rangle] \\ &= -q < 0. \end{aligned} \tag{3.1.14}$$

Since $Tx_n \rightharpoonup -c_0^* + p^*$, (3.1.14) implies

$$\limsup_{n\to\infty} \langle Tx_n, x_n\rangle < \langle -c_0^* + p^*, x_0\rangle. \tag{3.1.15}$$

Let $\tilde{x} \in L$. Then, by the monotonicity of T, we have

$$\langle Tx_n - T\tilde{x}, x_n - \tilde{x}\rangle \geq 0,$$

which implies

$$\langle Tx_n, x_n\rangle \geq \langle Tx_n, \tilde{x}\rangle + \langle T\tilde{x}, x_n - \tilde{x}\rangle.$$

Thus,

$$\liminf_{n\to\infty} \langle Tx_n, x_n\rangle \geq \langle -c_0^* + p^*, \tilde{x}\rangle + \langle T\tilde{x}, x_0 - \tilde{x}\rangle.$$

Combining this with (3.1.15), we get

$$\langle -c_0^* + p^*, x_0\rangle > \langle -c_0^* + p^*, \tilde{x}\rangle + \langle T\tilde{x}, x_0 - \tilde{x}\rangle,$$

which implies

$$\langle -c_0^* + p^* - T\tilde{x}, x_0 - \tilde{x}\rangle > 0. \tag{3.1.16}$$

Since T is maximal monotone, it follows that $x_0 \in D(T) = L$. This is a contradiction to (3.1.16) because we can now take $\tilde{x} = x_0$. Thus, (3.1.13) is true. Since C satisfies condition $(\tilde{C}2)$, we have $x_n \to x_0 \in D(C) \cap \overline{G_1}$ and $Cx_0 = c_0^*$.

Let $\tilde{x} \in L$ be arbitrary. The monotonicity of T implies

$$\langle T\tilde{x}, \tilde{x} - x_n \rangle + \langle Tx_n, x_n \rangle - \langle Tx_n, \tilde{x} \rangle = \langle T\tilde{x} - Tx_n, \tilde{x} - x_n \rangle \geq 0.$$

Taking the limit, we get

$$\langle T\tilde{x}, \tilde{x} - x_0 \rangle + \langle -Cx_0 + p^*, x_0 \rangle - \langle -Cx_0 + p^*, \tilde{x} \rangle \geq 0,$$

which gives

$$\langle T\tilde{x}, \tilde{x} - x_0 \rangle + \langle Cx_0 - p^*, -x_0 \rangle + \langle Cx_0 - p^*, \tilde{x} \rangle \geq 0$$

so that

$$\langle T\tilde{x} + Cx_0 - p^*, \tilde{x} - x_0 \rangle \geq 0.$$

By the maximal monotonicity of T, we obtain $x_0 \in D(T) = L$ and $-Cx_0 + p^* = Tx_0$ which implies that $Tx_0 + Cx_0 = p^*$. This contradicts our assumption on the equation (3.1.10) because $x_0 \in L \cap \overline{G_1}$. This completes the proof. ■

Theorem 3.5 *Assume that the operator $T : X \supset D(T) \to X^*$, $D(T) = L$, is maximal monotone and satisfies $(T4)$ and $T(0) = 0$ and that the operator $C : X \supset D(C) \to X^*$ satisfies $(C1)$ (with 0 in the first inequality of (3.1.6) replaced by S), $(\tilde{C}2)$, $(C3)$ and $C(0) = 0$. Let G_1, G_2 be open subsets of X such that $0 \in G_2$, $\overline{G_2} \subset G_1$ and G_1 is bounded. Assume further that the following conditions are satisfied:*

- (H_3) *there exists $v_0^* \in X^* \setminus \{0\}$ such that $Tx + Cx \neq \lambda v_0^*$, $(\lambda, x) \in \mathcal{R}_+ \times (L \cap \partial G_1)$;*
- (H_4) *for some $\tau_0 > 0$, $Tx + Cx \neq \tau_0 v_0^*$, $x \in L \cap G_1$;*
- (H_5) *$Tx + Cx + \lambda Jx \neq 0$, $(\lambda, x) \in \mathcal{R}_+ \times (L \cap \partial G_2)$.*

Then there is a solution of the problem $Tx + Cx = 0$ in $L \cap (G_1 \setminus G_2)$.

Proof: In view of (H_3) and (H_4), we see that

$$Tx + Cx \neq \tau_0 v_0^*, \quad x \in L \cap \overline{G_1}.$$

Then from Lemma 3.4 we know that there exists $\epsilon_0 > 0$ such that the equation

$$Tx + Cx + \epsilon Jx = \tau_0 v_0^*$$

has no solution in $L \cap \overline{G_1}$ for any $\epsilon \in (0, \epsilon_0]$. We now consider the homotopy function

$$H_1(t, x) = Tx + Cx + \epsilon Jx - t\tau_0 v_0^*, \quad (t, x) \in [0, 1] \times (L \cap \overline{G_1}).$$

This is an admissible homotopy for the degree in [24]. In order to see this, we set $T^t = T$ and $C^t = C + \epsilon J - t\tau_0 v_0^*$. We only have to check conditions $c^{(1)}$ with 0 in first inequality of (3.1.8) and $c^{(2)}$ because $c^{(3)}$ follows immediately. To verify the uniform quasiboundedness condition $c^{(1)}$, let $S > 0$ be given, and let $\|x\| \leq S$ and

$$\langle T^t x + C^t x, x\rangle = \langle Tx + Cx + \epsilon Jx - t\tau_0 v_0^*, x\rangle \leq S.$$

Then

$$\langle Tx + Cx, x\rangle \leq S_1,$$

where

$$S_1 = S + \tau_0 \|v_0^*\| S = S(1 + \tau_0 \|v_0^*\|).$$

So, by the quasiboundedness of C, there exists a number $K(S_1)$ such that $\|Cx\| \leq K(S_1)$. Thus,

$$\|C^t x\| \leq K_1(S),$$

where

$$K_1(S) = K(S_1) + \epsilon_0 S + \tau_0 \|v_0^*\|.$$

To see the uniform generalized (S_+)-condition $c^{(2)}$ of C^t with respect to T^t, consider the sequences $\{t_n\} \subset [0, 1]$, $\{x_n\} \subset L$ such that $t_n \to t_0 \in [0, 1]$, $x_n \rightharpoonup x_0 \in X$,

$C^{t_n}x_n \rightharpoonup c_0^*$ and

$$\limsup_{n\to\infty}\langle C^{t_n}x_n, x_n - x_0\rangle \le 0, \quad \langle T^{t_n}x_n + C^{t_n}x_n, x_n\rangle \le 0.$$

Then, using the monotonicity of J and the fact that $J(0) = 0$, we obtain

$$\limsup_{n\to\infty}\langle Cx_n - t_n\tau_0 v_0^*, x_n - x_0\rangle \le 0, \quad \langle Tx_n + Cx_n - t_n\tau_0 v_0^*, x_n\rangle \le 0. \tag{3.1.17}$$

The first inequality of (3.1.17) implies

$$\limsup_{n\to\infty}\langle Cx_n, x_n - x_0\rangle \le 0 \tag{3.1.18}$$

and the second inequality of (3.1.17) implies

$$\langle Tx_n + Cx_n, x_n\rangle \le S, \tag{3.1.19}$$

where $S = \sup\{\tau_0\|v_0^*\|\|x_n\| : n \ge 1\}$. Using condition $(\tilde{C}2)$, we obtain $x_n \to x_0 \in D(C)$ and $Cx_n \rightharpoonup Cx_0$. Thus $x_0 \in D(C^{t_0}) = D(C)$ and

$$C^{t_n}x_n = Cx_n + \epsilon Jx_n - t_n\tau_0 v_0^* \rightharpoonup Cx_0 + \epsilon Jx_0 - t_0\tau_0 v_0^* = C^{t_0}x_0.$$

This proves the fact that $H_1(t,x)$ is an admissible homotopy for the degree in [24]. Our condition (H_3) implies that the homotopy equation $H(t,x) = 0$ has no solution on $[0,1]\times(L\cap\partial G_1)$ and therefore the degree $d(H_1(t,\cdot), G_1, 0)$ is well-defined and constant. If $d(H_1(t,\cdot), G_1, 0) \ne 0$ for some $t \in [0,1]$, the equation $Tx + Cx + \epsilon Jx - \tau_0 v_0^* = 0$ would have a solution $x \in L \cap G_1$. This, however, contradicts the property of the constant τ_0. Consequently,

$$d(T + C + \epsilon J, G_1, 0) = d(H_1(0,\cdot), G_1, 0) = 0. \tag{3.1.20}$$

We next consider the homotopy function

$$H_2(t,x) = t(Tx + Cx + \epsilon J) + (1-t)Jx, \quad (t,x) \in [0,1]\times(L\cap\overline{G_2}),$$

where $\epsilon \in (0, \epsilon_0]$. Then this is an admissible homotopy for the degree in [24]. The proof of this is similar as in [20, Theorem 7].

We first show that $0 \notin H_2(t, x)$ for all $(t, x) \in [0, 1] \times (L \cap \partial G_2)$ for each $\epsilon \in (0, \epsilon_0]$. Otherwise, there exists a point $(t, x) \in [0, 1] \times (L \cap \partial G_2)$ for some $\epsilon \in (0, \epsilon_0]$, i.e.,

$$t(Tx + Cx + \epsilon J) + (1 - t)Jx = 0.$$

Obviously, $t \neq 0$ since $x \in \partial G_2$, $0 \in G_2$ and J is injective. We may assume that $t \in (0, 1]$. Then we have

$$Tx + Cx + \left(\epsilon + \frac{1-t}{t}\right) J = 0,$$

which contradicts condition (H_5). Next, we set $T^t = tT$ and $C^t = t(C + \epsilon J) + (1 - t)J$. We first show that C^t satisfies $c^{(1)}$, $c^{(2)}$, and $c^{(3)}$. It is obvious that $c^{(3)}$ is satisfied. Since $t \in (0, 1]$, we have that

$$\langle T^t x + C^t x, x\rangle = \langle tTx + tCx + ((\epsilon - 1)t + 1)Jx, x\rangle \leq 0$$

implies

$$\langle Tx + Cx, x\rangle \leq -\left(\epsilon + \frac{1}{t} - 1\right) \langle Jx, x\rangle \leq 0,$$

which, by the quasi-boundedness of C w.r.t. T, implies, for x bounded, the boundedness of $\{\|tCx\| : t \in (0, 1]\}$. The case $t = 0$ is trivial. Hence, C^t is uniformly quasi-bounded w.r.t. T^t. This verifies the condition $c^{(1)}$.

To verify condition $c^{(2)}$, let us consider the sequences $\{t_n\} \subset [0, 1]$, $\{x_n\} \subset L$ such that $t_n \to t_0 \in [0, 1]$, $x_n \rightharpoonup x_0 \in X$, $C^{t_n} x_n \rightharpoonup c_0^*$ and

$$\limsup_{n\to\infty} \langle C^{t_n} x_n, x_n - x_0\rangle \leq 0, \quad \langle T^{t_n} x_n + C^{t_n} x_n, x_n\rangle \leq 0. \tag{3.1.21}$$

Assume that $t_0 = 0$. If there is a subsequence of $\{t_k\}$ of t_n such that $t_k = 0$, $k = 1, 2, \ldots$, then

$$C^{t_k} x_k = t_k[Cx_k + \epsilon Jx_k] + (1 - t_k)Jx_k = Jx_k$$

and the second inequality in $c^{(2)}$ implies that

$$\|x_k\|^2 = \langle Jx_k, x_k\rangle \le 0.$$

This means that $x_k = 0$ so that $C^{t_k}x_k = Jx_k = J(0) = 0$. Consequently, $0 = C^{t_k}x_k \rightharpoonup c_0^* = 0$. The second inequality of $c^{(2)}$ implies that

$$\langle Tx_n + Cx_n, x_n\rangle \le 0 \tag{3.1.22}$$

because for $t_n = 0$, we have $x_n = 0$ as above. For $t_n > 0$, the inequality follows in an obvious manner. By the quasi-boundedness of C w.r.t. T, it follows that $\{Cx_n\}$ is bounded. Since $\{Jx_n\}$ is also bounded and $t_0 = 0$, the first inequality in $c^{(2)}$ now implies

$$\limsup_{n\to\infty}\langle Jx_n, x_n - x_0\rangle \le 0. \tag{3.1.23}$$

The (S_+)-property of J implies that $x_n \to x_0 = 0$. Also, $x_0 = 0 \in X = D(J) = D(C^0) = D(C^{t_0})$ and $C^{t_0}(x_0) = h^* = 0$. Thus, $c^{(2)}$ is satisfied in this case. We now assume that $t_n > 0$ for all n. Then the inequalities in (3.1.22) and (3.1.23) are true for the same reason as above and we obtain $x_n \to x_0$. Also, since $t_0 = 0$, we have

$$\lim_{n\to\infty} C^{t_n}x_n = \lim_{n\to\infty}[t_n(Cx_n + \epsilon Jx_n) + (1-t_n)Jx_n] = Jx_0 = c_0^*.$$

As above $x_0 \in X = D(J) = D(C^0) = D(C^{t_0})$ and $C^{t_0}x_0 = Jx_0 = c_0^*$. This shows that $c^{(2)}$ is again satisfied. We next assume that $t_0 > 0$. The second inequality of (3.1.21) implies again

$$\langle Tx_n + Cx_n, x_n\rangle \le 0,$$

which, by the quasi-boundedness of C w.r.t. T, implies $\{Cx_n\}$ is bounded. The first inequality of (3.1.21) gives

$$\limsup_{n\to\infty}\langle t_nCx_n + \epsilon t_nJx_n + (1-t_n)Jx_n, x_n - x_0\rangle \le 0. \tag{3.1.24}$$

If

$$\limsup_{n\to\infty} \langle Cx_n, x_n - x_0 \rangle > 0,$$

then there exists a subsequence of $\{x_n\}$ which we again denote by $\{x_n\}$ such that

$$\lim_{n\to\infty} \langle Cx_n, x_n - x_0 \rangle = q \tag{3.1.25}$$

for some positive number $q > 0$. Then from (3.1.24) we obtain

$$\limsup_{n\to\infty} \langle (\epsilon t_n + 1 - t_n) Jx_n, x_n - x_0 \rangle \leq - \lim_{n\to\infty} t_n \langle Cx_n, x_n - x_0 \rangle = -t_0 q < 0.$$

Consequently,

$$\limsup_{n\to\infty} \langle Jx_n, x_n - x_0 \rangle \leq 0,$$

which, by the (S_+)-property of J, implies $x_n \to x_0$. This is contradiction to (3.1.25) because $\{Cx_n\}$ is bounded and $q > 0$. Therefore, we must have

$$\limsup_{n\to\infty} \langle Cx_n, x_n - x_0 \rangle \leq 0,$$

and, by the generalized (S_+)-property of C w.r.t. T, we obtain $x_n \to x_0$, $x_0 \in D(C) = D(C^{t_0})$ and $t_0 C x_0 = c_0^* - \epsilon t_0 J x_0 - (1 - t_0) J x_0$. This means that $C^{t_0} x_0 = c_0^*$. This completely verifies condition $c^{(2)}$. For the rest of the proof of the admissibility of the homotopy $H_2(t, x)$, the reader is referred to ([24], pp. 139–140).

It now follows that

$$d(H_2(t, \cdot), G_2, 0) = d(H(1, \cdot), G_2, 0) = d(T + C + \epsilon J, G_2, 0) = 1$$

for sufficiently small $\epsilon > 0$. Thus, there exists ϵ_0 such that

$$0 = d(T + C + \epsilon J, G_1, 0) \neq d(T + C + \epsilon J, G_2, 0) = 1$$

for every $\epsilon \in (0, \epsilon_0]$. Hence, by the excision property of the degree, there exists a solution $x_\epsilon \in (G_1 \setminus G_2)$ of the equation

$$Tx + Cx + \epsilon Jx = 0$$

for every $\epsilon \in (0, \epsilon_0]$. Letting $\epsilon_n = \frac{1}{n}$ and $x_\epsilon = x_n$, we have

$$Tx_n + Cx_n + \frac{1}{n}Jx_n = 0. \tag{3.1.26}$$

Since $\{x_n\}$ is bounded and

$$\langle Tx_n + Cx_n, x_n\rangle = -\frac{1}{n}\langle Jx_n, x_n\rangle < 0, \tag{3.1.27}$$

by condition $(C1)$ on C we obtain $\{Cx_n\}$ is also bounded. We may now assume that $x_n \rightharpoonup x_0$, $Cx_n \rightharpoonup c_0^*$. In view of (3.2.36), we also obtain

$$\langle Tx_n + Cx_n, x_n - x_0\rangle = -\frac{1}{n}\langle Jx_n, x_n - x_0\rangle,$$

which implies

$$\limsup_{n\to\infty}\langle Tx_n + Cx_n, x_n - x_0\rangle \le 0.$$

By the same argument as in Lemma 3.4, we can see that

$$\limsup_{n\to\infty}\langle Cx_n, x_n - x_0\rangle > 0$$

leads to a contradiction. We then have

$$\limsup_{n\to\infty}\langle Cx_n, x_n - x_0\rangle \le 0.$$

This and (3.2.37) along with the generalized (S_+)-property of C imply $x_n \to x_0 \in D(C)$ and $Cx_n \rightharpoonup Cx_0 = c_0^*$. This with (3.2.36) gives $Tx_n \rightharpoonup -Cx_0$. Since T is closed, $x_0 \in D(T) = L$ and $Tx_0 + Cx_0 = 0$. By (H_3) and (H_5), $x_0 \notin \partial G_1 \cup \partial G_2$. Since $\partial(G_1 \setminus G_2) \subset \partial G_1 \cup \partial G_2$, we obtain $x_0 \in L \cap (G_1 \setminus G_2)$. This completes the proof. ■

3.2 $(\tilde{S}_+)$–Perturbations of Maximal Monotone Operators

We first define operators of the type $(\tilde{S}_+)$.

Definition 3.6 *An operator $C : X \supset D(C) \to X^*$ satisfies condition $(\tilde{S}_+)$ if for every $\{x_n\} \subset D(C)$ such that $x_n \rightharpoonup x_0 \in X$, $Cx_n \rightharpoonup h_0^* \in X^*$ and*

$$\limsup_{n\to\infty} \langle Cx_n, x_n - x_0 \rangle \le 0,$$

we have $x_n \to x_0$, $x_0 \in D(C)$ and $Cx_0 = h_0^*$.

Obviously, a bounded demicontinuous mapping $T : \overline{D} \to X^*$, $\overline{D} \subset D(T) \subset X$, of type (S_+) is of type $(\tilde{S}_+)$.

Let X be a real reflexive infinite dimensional Banach space with both X and X^* locally uniformly convex and L a dense linear subspace of X. Let $\mathcal{F}(L)$ be the set of all finite-dimensional subspaces of L. We assume that

t1: $T : X \supset D(T) \to 2^{X^*}$ is maximal monotone with $0 \in D(T)$ and $0 \in T(0)$;

c1: $C : X \supset D(C) \to X^*$, $L \subset D(C)$, is quasibounded, i.e., for every $S > 0$ there exists $K(S) > 0$ such that $u \in D(C)$ with

$$\|u\| \le S \quad \text{and} \quad \langle Cu, u \rangle \le S$$

implies $\|Cu\| \le K(S)$;

c2: the operator C satisfies condition $(\tilde{S}_+)$;

c3: for every $F \in \mathcal{F}(L)$, $v \in L$ the mapping $c(F, v) : F \to \mathcal{R}$, defined by $c(F, v)(u) = \langle Cu, v \rangle$, is continuous.

Condition c3 here is the same as condition $C3$. We need the following theorem which is a combination Theorem 1 and Theorem 2 in [22].

Theorem A *Assume that the operator T satisfies* t1 *and the operator C satisfies* c1-c3. *Assume that $G \subset X$ is open and bounded subset of X and that*

$$0 \notin (T+C)(D(T) \cap D(C) \cap \partial G).$$

Then there exists $t_1 \in (0,\infty)$ such that $0 \notin (T_t + C)(D(C) \cap \partial G)$ for all $t \in (0, t_1]$. Moreover, the degree $d(T_t + C, G, 0)$ is well-defined in the sense of Definition 3.1 *and is constant for every $t \in (0, t_1]$*

Definition 3.7 *Assume that the operator T, C and the set G are as in Theorem A. Assume that $0 \notin (T+C)(D(T) \cap D(C) \cap \partial G)$. Then degree $d(T+C, G, 0)$ is defined by*

$$d(T+C, G, 0) = d(T_t + C, G, 0), \quad t \in (0, t_1]$$

and

$$d(T+C, G, p^*) = d(T+C-p^*, G, 0)$$

for every $p^* \in X^*$ with $p^* \notin (T+C)(D(T) \cap D(C) \cap \partial G)$.

Next, we consider admissible homotopy for this degree as in [22]. Let $T^\tau : X \supset D(T^\tau) \to 2^{X^*}$ and $C^\tau : X \supset D(C^\tau) \to X^*$, $\tau \in [0,1]$, satisfy the following conditions:

- $t_\tau^{(1)}$: for each τ the operator T^τ is maximal monotone, $0 \in D(T^\tau)$, $0 \in T^\tau(0)$;
- $t_\tau^{(2)}$: given sequences $\{\tau_n\} \subset [0,1]$, $\{u_n\}$, $\{v_n^*\}$ such that $u_n \in D(T^{\tau_n})$, $v_n^* \in T^{\tau_n} u_n$, $\tau_n \to \tau_0$, $u_n \rightharpoonup u_0$, $v_n^* \rightharpoonup v_0^*$ and

$$\limsup_{n\to\infty} \langle v_n^*, u_n \rangle \le \langle v_0^*, u_0 \rangle,$$

for some $u_0 \in X$, $v_0^* \in X^*$, we have

$$u_0 \in D(T^{\tau_0}), \quad v_0^* \in T^{\tau_0} u_0 \quad \text{and} \quad \langle v_n^*, u_n \rangle \to \langle v_0^*, u_0 \rangle;$$

- $c_\tau^{(1)}$: the family C^τ is "uniformly quasibounded", i.e., for every $S > 0$ there exists

$K(S) > 0$ such that

$$\langle C^\tau u, u\rangle \le 0, \quad \|u\| \le S, \quad \tau \in [0,1], \quad u \in D(C^\tau),$$

imply $\|C^\tau u\| \le K(S)$;

$c_\tau^{(2)}$: for every pair of sequences $\{\tau_n\} \subset [0,1]$, $\{u_n\} \subset L$ such that $u_n \rightharpoonup u_0$, $C^{\tau_n} u_n \rightharpoonup h^*$, $\tau_n \to \tau_0$ and

$$\limsup_{n\to\infty} \langle C^{\tau_n} u_n, u_n - u_0\rangle \le 0, \quad \langle C^{\tau_n} u_n, u_n\rangle \le 0,$$

for some $\tau_0 \in [0,1]$, $u_0 \in X$, $h^* \in X^*$, we have $u_n \to u_0$, $u_0 \in D(C^{\tau_0})$ and $C^{\tau_0} u_0 = h^*$;

$c_\tau^{(3)}$: for every $F \in \mathcal{F}(L)$, $v \in L$, the mapping $\tilde{c}(F,v) : F \times [0,1] \to \mathcal{R}$, defined by $\tilde{c}(F,v)(u,\tau) = \langle C^\tau u, v\rangle$, is continuous.

When the operators T^τ and C^τ are as above, we say that the mapping $H(\tau,x) = (T^\tau + C^\tau)x$ is an "admissible homotopy" for the degree in Definition 3.7.

Lemma 3.8 *Assume that an operator $T : X \supset D(T) \to 2^{X^*}$ satisfies* t1 *and an operator $C : X \supset D(C) \to X^*$ satisfies* c1–c2. *Assume that, for some $p^* \in X^*$, the equation*

$$Tx + Cx \ni p^* \tag{3.2.28}$$

has no solution $x \in D(T) \cap D(C) \cap \overline{G_1}$, where G_1 is an open and bounded set in X with $0 \in G_1$. Then there exists $\epsilon_0 > 0$ such that the equation

$$Tx + Cx + \epsilon Jx \ni p^* \tag{3.2.29}$$

has no solution $x \in D(T) \cap D(C) \cap \overline{G_1}$ for any $\epsilon \in (0, \epsilon_0]$.

Proof: Assume that the conclusion of the lemma is false and let $\epsilon_n > 0$, $\{x_n\} \subset D(T) \cap D(C) \cap \overline{G_1}$ and $y_n^* \in Tx_n$ be such that $\epsilon_n \downarrow 0$ and

$$y_n^* + Cx_n + \epsilon_n Jx_n = p^*. \tag{3.2.30}$$

Since $\{x_n\}$ and $\{Jx_n\}$ are bounded and

$$\langle y_n^* + Cx_n, x_n \rangle = -\epsilon_n \langle Jx_n, x_n \rangle + \langle p^*, x_n \rangle \leq \|p^*\| \|x_n\|,$$

there exists a constant $S > 0$ such that

$$\|x_n\| \leq S \quad \text{and} \quad \langle Cx_n, x_n \rangle \leq \langle y_n^* + Cx_n, x_n \rangle \leq S$$

because $0 \in T(0)$. Since C is quasi-bounded w.r.t. T, there exists $K(S) > 0$ such that $\|Cx_n\| \leq K(S)$. We may assume that $x_n \rightharpoonup x_0 \in X$ and $Cx_n \rightharpoonup c_0^* \in X^*$. In order to show that

$$\limsup_{n \to \infty} \langle Cx_n, x_n - x_0 \rangle \leq 0, \tag{3.2.31}$$

assume that this is not true. Then, for a suitable subsequence of $\{n\}$ denoted by $\{n\}$ again, we have

$$\lim_{n \to \infty} \langle Cx_n, x_n - x_0 \rangle = q,$$

for some $q > 0$. This along with (3.2.30) gives

$$\begin{aligned} \lim_{n \to \infty} \langle y_n^*, x_n - x_0 \rangle &= -\lim_{n \to \infty} [\langle Cx_n, x_n - x_0 \rangle + \langle \epsilon_n Jx_n - p^*, x_n - x_0 \rangle] \\ &= -q < 0. \end{aligned} \tag{3.2.32}$$

Since $y_n^* \rightharpoonup -c_0^* + p^*$, (3.2.32) implies

$$\limsup_{n \to \infty} \langle y_n^*, x_n \rangle < \langle -c_0^* + p^*, x_0 \rangle. \tag{3.2.33}$$

Let $\tilde{x} \in D(T)$ and let $y^* \in T\tilde{x}$. Then by the monotonicity of T, we have

$$\langle y_n^* - y^*, x_n - \tilde{x} \rangle \geq 0,$$

which implies

$$\langle y_n^*, x_n \rangle \geq \langle y_n^*, \tilde{x} \rangle + \langle y^*, x_n - \tilde{x} \rangle$$

and

$$\liminf_{n\to\infty}\langle y_n^*, x_n\rangle \geq \langle -c_0^* + p^*, \tilde{x}\rangle + \langle y^*, x_0 - \tilde{x}\rangle. \tag{3.2.34}$$

Combining this with (3.2.33), we get

$$\langle -c_0^* + p^*, x_0\rangle > \langle -c_0^* + p^*, \tilde{x}\rangle + \langle y^*, x_0 - \tilde{x}\rangle,$$

which implies

$$\langle -c_0^* + p^* - y^*, x_0 - \tilde{x}\rangle > 0. \tag{3.2.35}$$

Since T is maximal monotone, it follows that $x_0 \in D(T)$. This is a contradiction to (3.2.35) because we can now take $\tilde{x} = x_0$. Thus, (3.2.31) is true. Since C satisfies condition c2, we have $x_n \to x_0 \in D(C) \cap \overline{G_1}$ and $Cx_0 = c_0^*$. Let $\tilde{x} \in D(T)$ be arbitrary and let $y^* \in T\tilde{x}$. The monotonicity of T implies

$$\langle y^*, \tilde{x} - x_n\rangle + \langle y_n^*, x_n\rangle - \langle y_n^*, \tilde{x}\rangle = \langle y^* - y_n^*, \tilde{x} - x_n\rangle \geq 0.$$

Taking the limit, we get

$$\langle y^*, \tilde{x} - x_0\rangle + \langle -Cx_0 + p^*, x_0\rangle - \langle -Cx_0 + p^*, \tilde{x}\rangle \geq 0,$$

which gives

$$\langle y^*, \tilde{x} - x_0\rangle + \langle Cx_0 - p^*, -x_0\rangle + \langle Cx_0 - p^*, \tilde{x}\rangle \geq 0.$$

Consequently,

$$\langle y^* + Cx_0 - p^*, \tilde{x} - x_0\rangle \geq 0.$$

By the maximal monotonicity of T, we obtain $x_0 \in D(T)$ and $-Cx_0 + p^* \ni Tx_0$ which implies $Tx_0 + Cx_0 \ni p^*$. This contradicts our assumption on the equation (3.2.28) because $x_0 \in D(T) \cap D(C) \cap \overline{G_1}$. The proof is complete. ■

Theorem 3.9 *Assume that the operator T satisfies* t1 *and the operator C satisfies* c1–c3. *Let G_1, G_2 be open subsets of X such that $0 \in G_2$, $\overline{G_2} \subset G_1$ and G_1 is bounded. Assume further that the following conditions are satisfied:*

(H_3) *there exists* $v_0^* \in X^* \setminus \{0\}$ *s.t.* $Tx + Cx \not\ni \lambda v_0^*$, $(\lambda, x) \in \mathcal{R}_+ \times (D(T) \cap D(C) \cap \partial G_1)$;

(H_4) *for some* $\tau_0 > 0$, $Tx + Cx \not\ni \tau_0 v_0^*, \quad x \in D(T) \cap D(C) \cap G_1$;

(H_5) $Tx + Cx + \lambda Jx \not\ni 0, \quad (\lambda, x) \in \mathcal{R}_+ \times (D(T) \cap D(C) \cap \partial G_2)$.

Then there is a solution of the problem $Tx + Cx \ni 0$ *in* $D(T) \cap D(C) \cap (G_1 \setminus G_2)$.

Proof: In view of (H_3) and (H_4), we see that

$$Tx + Cx \not\ni \tau_0 v_0^*, \quad x \in D(T) \cap D(C) \cap \overline{G_1}.$$

Then from Lemma 3.8 we know that there exists $\epsilon_0 > 0$ such that the equation

$$Tx + Cx + \epsilon Jx \ni \tau_0 v_0^*$$

has no solution in $D(T) \cap D(C) \cap \overline{G_1}$ for any $\epsilon \in (0, \epsilon_0]$.

We now consider the homotopy function

$$H_1(\tau, x) = Tx + Cx + \epsilon Jx - \tau\tau_0 v_0^*, \quad (\tau, x) \in [0,1] \times (D(T) \cap D(C) \cap \overline{G_1}).$$

This is an admissible homotopy for the degree in [22]. In order to see this, we set $T^\tau = T$ and $C^\tau = C + \epsilon J - \tau\tau_0 v_0^*$. It is not hard to prove that $C + \epsilon J$ satisfies conditions c1–c3.

We next consider the homotopy function

$$H_2(\tau, x) = \tau(Tx + Cx + \epsilon J) + (1 - \tau)Jx, \quad (\tau, x) \in [0,1] \times (D(T) \cap D(C) \cap \overline{G_2}),$$

where $\epsilon \in (0, \epsilon_0]$. This is an admissible homotopy for the degree in [22] and the proof follows as in [22, Theorem 4] by setting $T^\tau = \tau T$ and $C^\tau = \tau C + (1 - \tau)J$.

It now follows that

$$d(H_2(\tau, \cdot), G_2, 0) = d(H(1, \cdot), G_2, 0) = d(T + C + \epsilon J, G_2, 0) = 1$$

for sufficiently small $\epsilon > 0$. Thus, there exists ϵ_0 such that

$$0 = d(T + C + \epsilon J, G_1, 0) \neq d(T + C + \epsilon J, G_2, 0) = 1$$

for every $\epsilon \in (0, \epsilon_0]$. Hence, by the excision property of the degree, for every $\epsilon \in (0, \epsilon_0]$ there exists a solution $x_\epsilon \in (G_1 \setminus G_2)$ of the equation

$$Tx + Cx + \epsilon Jx \ni 0.$$

Letting $\epsilon_n = \frac{1}{n}$ and $x_\epsilon = x_n$, we have

$$Tx_n + Cx_n + \frac{1}{n} Jx_n \ni 0. \tag{3.2.36}$$

Let $y_n^* \in Tx_n$. Since $\{x_n\}$ is bounded, $0 \in T(0)$ and

$$\langle Cx_n, x_n \rangle \leq \langle y_n^* + Cx_n, x_n \rangle = -\frac{1}{n} \langle Jx_n, x_n \rangle < 0, \tag{3.2.37}$$

by condition c1 on C we obtain $\{Cx_n\}$ is also bounded. We may now assume that $x_n \rightharpoonup x_0$, $Cx_n \rightharpoonup c_0^*$. In view of (3.2.36), we also obtain

$$\langle y_n^* + Cx_n, x_n - x_0 \rangle = -\frac{1}{n} \langle Jx_n, x_n - x_0 \rangle,$$

which implies

$$\limsup_{n\to\infty} \langle y_n^* + Cx_n, x_n - x_0 \rangle \leq 0.$$

By the same argument as in Lemma 3.8, we can see that

$$\limsup_{n\to\infty} \langle Cx_n, x_n - x_0 \rangle > 0$$

leads to a contradiction. We then have

$$\limsup_{n\to\infty} \langle Cx_n, x_n - x_0 \rangle \leq 0.$$

This and (3.2.37) along with the $(\tilde{S}_+)$-property of C imply $x_n \to x_0 \in D(C)$ and $Cx_n \rightharpoonup Cx_0 = c_0^*$. This along with (3.2.36) gives $y_n^* \rightharpoonup -Cx_0$. Since T is closed, $x_0 \in D(T)$ and $Tx_0 + Cx_0 \ni 0$. By (H_3) and (H_5), $x_0 \notin \partial G_1 \cup \partial G_2$. Since $\partial(G_1 \setminus G_2) \subset \partial G_1 \cup \partial G_2$, we obtain $x_0 \in D(T) \cap D(C) \cap (G_1 \setminus G_2)$. This completes the proof. ∎

Remark 3.10 We have already noted that Kartsatos and Skrypnik in [22] developed a new degree theory for densely defined quasibounded $(\tilde{S}_+)$–perturbations of maximal monotone operators in reflexive Banach space. J. Quarcoo in his Ph.D. dissertation [31] developed a degree theory for mappings of the form $T + C$ in a real reflexive separable Banach space with a dense linear subspace L. Here, $T : X \supset D(T) \to 2^{X^*}$ is maximal monotone with $0 \in \mathrm{Int}D(T)$ and $0 \in T(0)$ and $C : X \supset D(C) \to X^*$, $L \subset D(C)$, is of class $(S_+)_L$ and $\langle Cx, x\rangle \geq -\psi(\|x\|)$, $x \in D(C)$, where $\psi : \mathcal{R}_+ \to \mathcal{R}_+$ is nondecreasing. We note that Browder and Hess in [9] have remarked that such an operator T is strongly quasibounded. The transfer of the quasiboundedness condition from C to T seems to be a motivation from [9] because in the solvability of nonlinear operator equations involving T and C [9, Theorem 7], it is assumed that either T is strongly quasibounded or C is quasibounded in addition to some other assumptions. Since this degree is defined via the degree developed by Kartsatos and Skrypnik [24] for mappings of class $(S_+)_{0,L}$, the excision property of degree is still valid. Therefore, an analogue of Theorem 3.9 for the operator inclusion $Tx + Cx \ni 0$ can be given by using this degree.

Chapter 4

Nonlinear Perturbations of Linear Maximal Monotone Operators

In this chapter we consider eigenvalue problems and invariance of domain results for nonlinear perturbations of linear densely defined maximal monotone operators using the methodology of the construction of the degree theories by Berkovits and Mustonen [4] and by Addou and Mermri [1]. One of the problems Kartsatos and Skrypnik [21] considered is the implicit eigenvalue problem $T(x) + C(\lambda, x) = 0$ with applications of various topological degree theories. Here, $T : X \supset D(T) \to 2^{X^*}$ is maximal monotone with $0 \in D(T)$ and $0 \in T(0)$, and $C : [0, \Lambda] \times \overline{G} \to X^*$, $G \subset X$ open bounded and $0 \in G$, is bounded demicontinuous of type (S_+). Kartsatos [18] established invariance of domain theorems for maximal monotone operators whose domain do not necessarily contain any open sets. Kartsatos and Skrypnik [20] have extended the well-known invariance of domain theorem of Schauder about injective operators of the type $I + C$ with C compact to the operators of the form $T + C$ with T maximal monotone and C bounded demicontinuous of type (S_+) using the topological degrees of Browder and Skrypnik. In addition, the authors [20] gave invariance of domain theorems for the operators of the form $T + C$ with both $T,\ C$ densely defined and T single-valued. These results make use of the topological degree theory developed by the authors for the sum $T+C$, where T is single-valued maximal monotone T and C satisfies conditions like quasiboundedness and (S_+) w.r.t T.

In Section 4.2 we extend the eigenvalue problem in [21] to the operators of the form $L + T + C$, where $L : X \supset D(L) \to X^*$ is densely defined linear maximal monotone, $T : X \to 2^{X^*}$ bounded maximal monotone, and $C : X \supset D(C) \to X^*$ is bounded demicontinuous of type (S_+) w.r.t. $D(L)$.

In Section 4.3 we extend the invariance of domain theorem in [20] to the operators of the form $L+T+C$ where L, T, and C are as in Section 4.2.

4.1 Introduction

Let X be a real reflexive Banach space. Assume that both X and its dual X^* are locally uniformly convex.

Let $L : X \supset D(L) \to X^*$ be a densely defined linear maximal monotone operator and $C : X \to X^*$ a bounded demicontinuous operator. We say that $C : X \to X^*$ is of type (S_+) w.r.t. to $D(L)$ if for every sequence $\{x_n\} \subset D(L)$ with $x_n \rightharpoonup x_0$ in X, $Lx_n \rightharpoonup Lx_0$ in X^* and

$$\limsup_{n\to\infty} \langle Cx_n, x_n - x_0\rangle \le 0,$$

we have $x_n \to x_0 \in$ in $\overline{D(L)}$.

Since the graph $G(L) = \{[x, Lx] : x \in D(L)\}$ of L is closed in $X \times X^*$, we can equip $Y = D(L)$ with the graph norm

$$\|x\|_Y = \|x\|_X + \|Lx\|_{X^*}, \quad x \in Y$$

to make Y a real reflexive Banach space. We assume that Y and its dual Y^* are locally uniformly convex.

Let $j : Y \to X$ be the natural embedding and $j^* : X^* \to Y^*$ its adjoint. Let $G \subset X$ be an open bounded set. Let $\mathcal{F}_G(L : T : S_+)$ denote the class of operators of the form $L + T + C : \overline{G} \cap Y \to 2^{X^*}$, where $L : X \supset D(L) \to X^*$ is densely defined linear maximal monotone operator, $T : X \to 2^{X^*}$ a bounded maximal monotone operator and $C : \overline{G} \to X^*$ a bounded demicontinuous operator of type (S_+) w.r.t. Y. Also, let $\mathcal{H}_G(L : T : S_+)$ denote the class of the operators of the form $L+T_t+C(t) : \overline{G}\cap Y \to X^*$, where $L : X \supset D(L) \to X^*$ is densely defined linear maximal monotone operator, $T_t : X \to 2^{X^*}, t \in [0, 1]$, a bounded pointwise graph-continuous homotopy of maximal monotone operators and $C(t) : \overline{G} \to X^*, t \in [0, 1]$ a homotopy of class (S_+) w.r.t. Y (for definitions, see below).

Definition 4.1 *([2]) A family $\{T_t : X \to 2^{X^*}, t \in [0,1]\}$ of maximal monotone operators is said to be pointwise graph-continuous if for every sequence $\{t_n\} \subset [0,1]$ with $t_n \to t_0 \in [0,1]$ and every $[u,v] \in G(T_{t_0})$ there exists a sequence $v_n \in T_{t_n}(u)$ such that $v_n \to v$ in X^*.*

Definition 4.2 *A family $C(t), t \in [0,1]$, of operators from $\overline{G}$ to X^* is called a "homotopy of type (S_+) w.r.t. Y" if for every $\{x_n\} \subset Y$ and $\{t_n\} \subset [0,1]$ with $x_n \rightharpoonup x_0$ in X, $Lx_n \rightharpoonup Lx$ in X^*, $t_n \to t$ and*

$$\limsup_{n\to\infty} \langle C(t_n)x_n, x_n - x_0\rangle \le 0,$$

we have $x_n \to x$ in X and $C(t_n)x_n \rightharpoonup C(t)x_0$ in X^*.

We define

$$\hat{L} = j^* \circ L \circ j : Y \to Y^*, \qquad \hat{C}(t) = j^* \circ C(t) \circ j : j^{-1}(\overline{G}) \to Y^*,$$

and for every $s > 0$

$$\hat{T}_{t,s} = j^* \circ T_{t,s} \circ j : Y \to Y^*,$$

where $T_{t,s}$ is Yosida approximation of the operator T_t. Note that

$$\overline{j^{-1}(G)} \subset j^{-1}(\overline{G}), \quad \text{and} \quad \partial(j^{-1}(G)) \subset j^{-1}(\partial G).$$

We also define $M : Y \to Y^*$ by

$$(Mx, y) = \langle Ly, J^{-1}(Lx)\rangle, \quad x, y \in Y. \tag{4.1.1}$$

Here, the duality pair $(\cdot\ ,\cdot)$ is in $Y \times Y^*$ and J^{-1} is the inverse of the duality map $J : X \to X^*$ and is identified with the duality map from X^* to X^{**}. In fact, for every $x \in Y$ such that $Mx \in j^*(X^*)$, we have $J^{-1}(Lx) \in D(L^*)$ and by (4.1.1)

$$Mx = j^* \circ L^* \circ J^{-1}(Lx).$$

Definition 4.3 *([2]) A sequence $\{T_n : X \to 2^{X^*}\}$ of maximal monotone operators is* *said to converge in the graph sense to a maximal monotone operator* $T : X \supset D(T) \to 2^{X^*}$ *if for every* $[u, v] \in G(T)$ *there exists a sequence* $[u_n, v_n] \in G(T_n)$ *such that* $u_n \to u$ *in* X *and* $v_n \to v$ *in* X^**. We write* $T_n \xrightarrow{G} T$*.*

Definition 4.4 *(*[2]*) A family* $\{T_t : X \to 2^{X^*}, t \in [0, 1]\}$ *of maximal monotone operators is said to be graph-continuous if for every sequence* $\{t_n\} \subset [0, 1]$ *with* $t_n \to t_0 \in [0, 1]$*, we have* $T_{t_n} \xrightarrow{G} T_{t_0}$*, i.e., for every sequence* $\{t_n\} \subset [0, 1]$ *with* $t_n \to t_0$ *and every* $[u, v] \in G(T_{t_0})$ *there exists a sequence* $[u_n, v_n] \in G(T_{t_0})$ *such that* $u_n \to u$ *in* X *and* $v_n \to v$ *in* X^**.*

Remark 4.5 A graph-continuous family $\{T_t : X \to 2^{X^*}, t \in [0, 1]\}$ of maximal monotone operators is a pseudomonotone homotopy introduced by Browder in [8].

Obviously, pointwise graph-continuity implies graph-continuity.

For a maximal monotone operator $T : X \supset D(T) \to 2^{X^*}$ with Yosida approximants T_s, $s > 0$, the following is true.

Proposition 4.6 ([2], p. 354) *Let* X *be a reflexive Banach space and* $T : X \supset D(T) \to 2^{X^*}$ *a maximal monotone operator. Then, for every sequence* $\{s_n\} \subset (0, \infty)$ *with* $s_n \to 0$*, we have* $T_{s_n} \xrightarrow{G} T$*.*

We need the following lemmas.

Lemma 4.7 ([1], Lemma 3.1, p. 277) *Let* $\{T_n : X \to 2^{X^*}\}$ *be a sequence of maximal monotone operators with the Yosida approximants* $T_{n,s}$*,* $s > 0$*, and let* $T : X \supset D(T) \to 2^{X^*}$ *be maximal monotone. Then the following properties hold:*

(a) If $T_n \xrightarrow{G} T$*, then for every sequence* $\{x_n\} \subset X$ *such that* $x_n \to x$ *in* X*, we have* $T_{n,s}(x_n) \to T_s(x)$ *in* X^* *for every* $s > 0$*. Moreover, if the sequence* $\{T_n\}$ *is bounded, we have* $T_{n,s_n}(x_n) \rightharpoonup w \in T(x)$ *for a subsequence* $\{x_n\}$ *and any sequence* $\{s_n\} \subset (0, \infty)$ *with* $s_n \to 0$*;*

(b) If the sequence T_n *is pointwise graph-continuous to* T*, then we have* $T_{n,s_n}(x) \to T^{\circ}(x)$ *for every sequence* $\{s_n\} \subset (0, \infty)$ *with* $s_n \to 0$*, where* $T^{\circ}(x)$ *is the element of minimum norm of the closed convex subset* Tx *of* X^**.*

Lemma 4.8 [[1], Lemma 3.3, p. 279] *If* $F(t) \in \mathcal{H}_G(L : T : S_+)$ *and* $s > 0$, *then* $\hat{F}_s(t) = \hat{L} + \hat{T}_{t,s} + \hat{C}(t) + sM$ *is a bounded homotopy of type* (S_+) *from* $j^{-1}(\overline{G}) \subset Y$ *to* Y^*. *In particular, for each* $s > 0$, $\hat{F}_s = \hat{L} + \hat{T}_s + \hat{C} + sM$ *is a bounded demicontinuous operator of type* (S_+) *from* $j^{-1}(\overline{G}) \subset Y$ *to* Y^*. *Moreover, for every continuous* $h : [0,1] \to X^*$, *the set* $\{u \in j^{-1}(\overline{G}) : \hat{F}_s(u) = j^*h(t)\}$ *is bounded in* Y.

4.2 The Eigenvalue Problem

The following definition is a variant of one in [21] by Kartsatos and Skrypnik , p. 3854.

Definition 4.9 *Let* $G \subset X$ *be open and bounded,* $\Lambda > 0$. *An operator* $C : [0,\Lambda] \times \overline{G} \to X^*$ *is said to be demicontinuous if* $\{(t_n, x_n)\} \subset [0,\Lambda] \times \overline{G}$ *such that* $(t_n, x_n) \to (t_0, x_0) \in [0,\lambda] \times \overline{G}$ *implies* $C(t_n, x_n) \to C(t_0, x_0)$. *A demicontinuous operator* $C(t,x)$ *is said to be continuous in* t *uniformly w.r.t.* $x \in \overline{G}$ *if* $\{t_n\} \subset [0,\Lambda]$ *with* $t_n \to t_0 \in [0,\lambda]$ *implies* $C(t_n, x) \to C(t_0, x)$ *for all* $x \in \overline{G}$. *A demicontinuous operator* $C : [0,\Lambda] \times \overline{G} \to X^*$ *is said to be of type* (S_+) *w.r.t* $D(L)$ *if for every sequence* $\{x_n\} \in D(L)$ *and every* $\lambda \in (0,\Lambda]$ *with* $x_n \rightharpoonup x_0$ *in* X, $Lx_n \rightharpoonup Lx_0$ *in* X^* *and*

$$\limsup_{n\to\infty} \langle C(\lambda, x_n), x_n - x_0 \rangle \leq 0,$$

we have $x_n \to x_0$ *in* X.

The following eigenvalue result is a variant of Theorem 1 in [21], p. 3854.

Theorem 4.10 *Let* $G \subset X$ *be open, bounded and* $0 \in G$. *Let* $L : X \supset D(L) \to X^*$ *be a densely defined maximal monotone operator and* $T : X \to 2^{X^*}$ *a bounded maximal monotone operator with* $0 \in D(T)$ *and* $0 \in T(0)$. *Let* $C : [0,\Lambda] \times \overline{G} \to X^*$ *be a bounded demicontinuous operator of type* (S_+) *w.r.t. to* $D(L)$ *and such that* $C(0,x) = 0, x \in \overline{G}$, *and* $C(t,x)$ *is continuous in* t *uniformly w.r.t.* $x \in \overline{G}$. *Let* ϵ, ϵ_0 *be positive numbers. Assume that*

$(\mathcal{P})$ *there exists* $\lambda \in (0,\Lambda]$ *such that the inclusion*

$$Lx + Tx + C(\lambda, x) + \epsilon Jx \ni 0$$

has no solution in $x \in D(L) \cap G$. *Then*

(i) there exists $(\lambda_0, x_0) \in (0, \Lambda] \times (D(L) \cap \partial G)$ *such that*

$$Lx_0 + Tx_0 + C(\lambda_0, x_0) + \epsilon J x_0 \ni 0; \tag{4.2.2}$$

(ii) if $0 \notin (L+T)(D(L) \cap \partial G)$, $L+T$ *satisfies condition* (S_q), *and property* $(\mathcal{P})$ *is satisfied for every* $\epsilon \in (0, \epsilon_0]$, *there exists* $(\lambda_0, x_0) \in (0, \Lambda] \times (D(L) \cap \partial G)$ *such that* $Lx_0 + Tx_0 + C(\lambda_0, x_0) = 0$.

Proof: Assume that (4.2.2) is not true. Then the equation

$$H(\lambda, x) \equiv Lx + Tx + C(\lambda, x) + \epsilon J x \ni 0$$

has no solution on $D(L) \cap \partial G$ for every $\lambda \in [0, \Lambda]$. Here, $L + T + \epsilon J$ is strictly monotone and so the assumption is obviously true for $\lambda = 0$. Thus,

$$H(\lambda, D(L) \cap \partial G) \not\ni 0, \quad \lambda \in [0, \Lambda]. \tag{4.2.3}$$

Let $Y = D(L)$ be equipped with the graph norm. We are now going to show that there exist $s_0 > 0$, $\lambda_0 \in (0, \Lambda]$ such that for every $s \in (0, s_0]$ and $\lambda \in (0, \lambda_0]$, the equation

$$H_1(s, \lambda, x) \equiv \hat{L}x + \hat{T}_s x + \hat{C}(\lambda, x) + \epsilon \hat{J} x + sMx = 0 \tag{4.2.4}$$

has no solution $x \in \partial G_R(Y)$, where $G_R(Y) = j^{-1}(G) \cap B_Y(0, R)$. Here, $B_Y(0, R) = \{y \in Y : \|y\|_Y < R\}$. By Lemma 4.8, the set of solutions of (4.2.4) in $j^{-1}(\overline{G})$ is bounded in Y and therefore such an $R > 0$ exists. We also note that $\partial(j^{-1}(G)) \subset j^{-1}(\partial G)$.

Assume that the assertion about (4.2.4) is not true. Then there exist $s_n \downarrow 0$, $\lambda_n \downarrow 0$, $x_n \in \partial(j^{-1}(G))$, $x_0 \in Y$ with $x_n \rightharpoonup x_0$ in Y such that

$$\hat{L}x_n + \hat{T}_{s_n} x_n + \hat{C}(\lambda_n, x_n) + \epsilon \hat{J} x_n + s_n M x_n = 0. \tag{4.2.5}$$

This and the definitions of $\hat{L}$, $\hat{T}_s$, $\hat{C}$, $\hat{J}$, M and monotonicity of J imply

$$\begin{aligned}
(\hat{L}x_n, x_n - x_0) &= -(\hat{C}(\lambda_n, x_n), x_n - x_0) - (\hat{T}_{s_n}x_n, x_n - x_0) \\
&\quad -\epsilon(\hat{J}x_n, x_n - x_0) - s_n(Mx_n, x_n - x_0) \\
&= -\langle C(\lambda_n, x_n), x_n - x_0\rangle - \langle T_{s_n}x_n, x_n - x_0\rangle \qquad (4.2.6) \\
&\quad -\epsilon\langle Jx_n, x_n - x_0\rangle - s_n\langle Lx_n - Lx_0, J^{-1}(Lx_n)\rangle \\
&\leq \|C(\lambda_n, x_n)\|\|x_n - x_0\| - \langle T_{s_n}x_0, x_n - x_0\rangle - \epsilon\langle Jx_0, x_n - x_0\rangle \\
&\quad -s_n\langle Lx_n - Lx_0, J^{-1}(Lx_n)\rangle.
\end{aligned}$$

Since $x_n \rightharpoonup x_0$ in Y implies $x_n \rightharpoonup x_0$ in X and $Lx_n \rightharpoonup Lx_0$ in X^* and since $T_{s_n}x_0 \to T^0(x_0)$ where $T^0(x_0)$ is the element of minimum norm in the closed convex set Tx_0, we obtain

$$\begin{aligned}
\limsup_{n\to\infty}(\hat{L}x_n, x_n - x_0) &\leq \lim_{n\to\infty}[\|C(\lambda_n, x_n)\|\|x_n - x_0\|] - \epsilon\lim_{n\to\infty}\langle Jx_0, x_n - x_0\rangle \\
&\quad - \lim_{n\to\infty}\langle T_{s_n}x_0, x_n - x_0\rangle - \lim_{n\to\infty} s_n\langle Lx_n - Lx_0, J^{-1}(Lx_n)\rangle \\
&= 0.
\end{aligned}$$

Here, we have also used the continuity of $C(\lambda, x)$ in λ uniformly w.r.t. $x \in \overline{G}$, the fact that $\partial(j^{-1}(G)) \subset j^{-1}(\partial G) = \partial G \cap Y \subset \overline{G}$ and the boundedness of J^{-1}. Also, the monotonicity of $\hat{L}$ implies

$$\liminf_{n\to\infty}(\hat{L}x_n, x_n - x_0) \geq \lim_{n\to\infty}(\hat{L}x_0, x_n - x_0) \geq 0.$$

Therefore,

$$\lim_{n\to\infty}(\hat{L}x_n, x_n - x_0) = 0.$$

This along with (4.2.6) gives

$$\limsup_{n\to\infty}\langle T_{s_n}x_n + \epsilon Jx_n, x_n - x_0\rangle = 0.$$

If

$$\limsup_{n\to\infty}\langle Jx_n, x_n - x_0\rangle > 0,$$

then it follows, for a subsequence $\{x_n\}$ which we again denote by $\{x_n\}$, that

$$\limsup_{n\to\infty}\langle T_{s_n}x_n, x_n - x_0\rangle < 0.$$

This is impossible by Lemma 2.9(i). So we must have

$$\limsup_{n\to\infty}\langle Jx_n, x_n - x_0\rangle \le 0.$$

Since J is of type (S_+), we have $x_n \to x_0$ in X and $x_0 \in \partial G \cap Y = j^{-1}(\partial G)$. The continuity of $\hat{J}$ implies $\hat{J}x_n \to \hat{J}x_0$.

Since T_n with $T_n \equiv T$ converges to T in the graph sense and T is bounded, by Lemma 4.7(a) we have $T_{s_n}x_n \rightharpoonup w \in Tx_0$ for a subsequence of $\{x_n\}$ which we again denote by $\{x_n\}$. Then (4.2.5) implies $\hat{L}x_n + \hat{T}_{s_n}x_n + \epsilon\hat{J}x_n \rightharpoonup \hat{L}x_0 + j^*w + \epsilon\hat{J}x_0 = 0$.

Let $y \in Y$. Then

$$(\hat{L}x_0 + j^*w + \epsilon\hat{J}x_0, y) = 0,$$

which implies

$$\begin{aligned}\langle Lx_0 + w + \epsilon Jx_0, y\rangle &= (j^*(Lx_0) + j^*w + \epsilon j^*(Jx_0), y)\\ &= ((j^*\circ L\circ j)x_0 + j^*w + \epsilon(j^*\circ J\circ j)x_0, y)\\ &= (\hat{L}x_0 + j^*w + \epsilon\hat{J}x_0, y)\\ &= 0.\end{aligned}$$

Since Y is dense in X, we have that $Lx_0 + w + \epsilon Jx_0 = 0$ which is a contradiction because $L+T+\epsilon J$ is strictly monotone and $0 \in G\cap Y$. Therefore , the assertion about (4.2.4) is true.

Now, we fix $s \in (0, s_0]$ and $\lambda \in (0, \lambda_0]$ and consider the homotopy function

$$H_2(t,x) \equiv \hat{L}x + \hat{T}_s x + \hat{C}(t\lambda, x) + \epsilon\hat{J}x + sMx, \quad (t,x) \in [0,1]\times\overline{j^{-1}(G)}. \tag{4.2.7}$$

By a similar argument as above, we can show that $0 \notin H_2(t, \partial G_R(Y))$ for all $t \in [0,1]$ and for possibly a bigger $R > 0$. Here, we need Lemma 4.7. Obviously, we can use this R hereafter.

Set $S(t) = C(t\lambda, \cdot) + \epsilon J$ and $T_t = T_s$. Then $\hat{S}(t) = \hat{C}(t\lambda, \cdot) + \epsilon \hat{J}$ and $\hat{T}_t = \hat{T}_s$. In order to show that $H_2(t,x)$ is an admissible homotopy for the Browder and Skrypnik degree, in view of Lemma 4.8, it suffices to show that $S(t)$ is a bounded homotopy of type (S_+) with respect to $D(L)$ and T_t is a bounded pointwise graph-continuous homotopy of maximal monotone operators. The latter follows immediately because T_s is bounded whenever T is bounded. So, let $\{x_n\} \subset D(L)$ be such that $x_n \rightharpoonup x_0$ in X, $Lx_n \rightharpoonup Lx_0$ in X^*, $t_n \to t \in [0,1]$ and

$$\limsup_{n\to\infty} \langle S(t_n)x_n, x_n - x_0\rangle \le 0. \tag{4.2.8}$$

We observe that

$$\begin{aligned} \langle S(t_n)x_n, x_n - x_0\rangle &= \langle C(t_n\lambda, x_n), x_n - x_0\rangle + \epsilon\langle Jx_n, x_n - x_0\rangle \\ &= \langle C(t_n\lambda, x_n), x_n - x_0\rangle + \epsilon\langle Jx_n - Jx_0, x_n - x_0\rangle \\ &\quad +\epsilon\langle Jx_0, x_n - x_0\rangle \\ &\ge \langle C(t_n\lambda, x_n), x_n - x_0\rangle + \epsilon\langle Jx_0, x_n - x_0\rangle. \end{aligned} \tag{4.2.9}$$

Using this with (4.2.8) we get

$$\limsup_{n\to\infty} \langle C(t_n\lambda, x_n), x_n - x_0\rangle \le 0. \tag{4.2.10}$$

If $t = 0$, then $C(t_n\lambda, x_n) \to 0$ and

$$\lim_{n\to\infty} \langle C(t_n\lambda, x_n), x_n - x_0\rangle = 0.$$

Using this in (4.2.10), we obtain

$$\limsup_{n\to\infty} \langle Jx_n, x_n - x_0\rangle \le 0. \tag{4.2.11}$$

Since J is of type (S_+), we obtain $x_n \to x_0$. This implies $C(t_n\lambda, x_n) \to C(0, x_0) = 0$ and $Jx_n \to Jx_0$. It follows that $S(t_n)x_n = C(t_n\lambda, x_n) + \epsilon Jx_n \to \epsilon Jx_0 = S(0, x_0)$.

We next consider the case $t > 0$ and observe that

$$\langle C(t_n\lambda, x_n), x_n - x_0\rangle = \langle C(t_n\lambda, x_n) - C(t\lambda, x_n), x_n - x_0\rangle + \langle C(t\lambda, x_n), x_n - x_0\rangle.$$

Since

$$\lim_{n\to\infty}[C(t_n\lambda, x_n) - C(t\lambda, x_n)] = 0,$$

we obtain, in view of (4.2.10), that

$$\limsup_{n\to\infty}\langle C(t\lambda, x_n), x_n - x_0\rangle \le 0.$$

The (S_+) property of C w.r.t. $D(L)$ implies $x_n \to x_0$ in X. So, $Jx_n \to Jx_0$ and by the demicontinuity of C we have

$$S(t_n)x_n = C(t_n\lambda, x_n) + \epsilon Jx_n \rightharpoonup C(t\lambda, x_0) + \epsilon Jx_0 = S(t)x_0.$$

This establishes the admissibility of the homotopy $H_2(t, x)$ according to the Skrypnik [33] degree d_S. Therefore, by the homotopy invariance of this degree, we have

$$\begin{aligned} d_S(H_2(t,\cdot), G_R(Y), 0) &= d_S(H_2(1,\cdot), G_R(Y), 0) \\ &= d_S(H_2(0,\cdot), G_R(Y), 0) \\ &= d_S(\hat{L} + \hat{T}_s + \epsilon\hat{J} + sM, G_R(Y), 0). \end{aligned} \tag{4.2.12}$$

Consider another homotopy function

$$H_0(t, x) = t(\hat{L} + \hat{T}_s + \epsilon\hat{J} + sM) + (1-t)(\hat{L} + \epsilon\hat{J} + sM), \quad (t, x) \in [0, 1] \times \overline{j^{-1}(G)}.$$

It is obvious that $H_0(t, x)$ is a homotopy of type (S_+) from $j^{-1}(\overline{G}) \subset Y$ to Y^*. Also, the set of solutions of $H_0(t, x) = 0$ is bounded in Y. Choose the number $R > 0$ bigger enough so that all the solutions of $H_0(t, x) = 0$ are contained in $B_Y(0, R)$ so that $H_0(t, x) = 0$ has no solution $(t, x) \in [0, 1] \times \partial G_R(Y)$. Otherwise, for some

$(t_0, x_0) \in [0,1] \times \partial G_R(Y)$, we have

$$\hat{L} + t\hat{T}_s + \epsilon\hat{J} + sM = 0.$$

Consequently,

$$(\hat{L}x_0, x_0) + t_0(\hat{T}_s x_0, x_0) + \epsilon(\hat{J}x_0, x_0) + s(Mx_0, x_0) = 0$$

which implies $x_0 = 0$. But $x_0 \in \partial(j^{-1}(G))$ which implies $x_0 \in G$. This is a contradiction. Thus, by the invariance under homotopy of the degree, we have

$$d_{S_+}(\hat{L} + \hat{T}_s + \epsilon\hat{J} + sM, G_R(Y), 0) = d_{S_+}(\hat{L} + \epsilon\hat{J} + sM, G_R(Y), 0). \tag{4.2.13}$$

The topological degree developed in [2] is based on the methodology of degree developed in [4] by Berkovits and Mustonen and the degree is the limit

$$\begin{aligned} d(H(\lambda,\cdot), G, 0) &= \lim_{s\downarrow 0} d_S(H_1(s,\lambda,\cdot), G_R(Y), 0) \\ &= \lim_{s\downarrow 0} d_S(H_2(1,\cdot), G_R(Y), 0) \\ &= \lim_{s\downarrow 0} d_S(\hat{L} + \hat{T}_s + \epsilon\hat{J} + sM, G_R(Y), 0) \\ &= \lim_{s\downarrow 0} d_S(\hat{L} + \epsilon\hat{J} + sM, G_R(Y), 0) \\ &= d(L + \epsilon J, G, 0) \\ &= 1. \end{aligned}$$

Here, we have used (4.2.12) and (4.2.13) and Corollary 1, p. 611 in [4]. So, there exists $x \in G \cap D(L)$ such that

$$Lx + Tx + C(\lambda, x) + \epsilon Jx \ni 0.$$

This contradicts our assumption $(\mathcal{P})$.

(ii) In view of (i), for each positive integer n, there exist $\{x_n\} \subset G \cap D(L)$, $x_n^* \in Tx_n$, $\lambda_n \in (0, \Lambda]$ such that

$$Lx_n + x_n^* + C(\lambda_n, x_n) + \frac{1}{n}Jx_n = 0. \tag{4.2.14}$$

We may assume that $\lambda_n \to \lambda_0 \in [0,1]$, $C(\lambda_n, x_n) \rightharpoonup c^*$ and $Jx_n \rightharpoonup p$. Since T is bounded, (4.2.14) implies $\{Lx_n\}$ is bounded in X^*. Since $\{x_n\}$ is bounded in X, it follows that $\{x_n\}$ is bounded in $Y = D(L)$ with the graph norm. Since Y is reflexive, we may assume that $x_n \rightharpoonup x_0$ in Y. Therefore, we have $x_n \rightharpoonup x$ in X^* and $Lx_n \rightharpoonup Lx_0$ in X^*.

We now consider two cases: (a) $\lambda_0 = 0$; (b) $\lambda_0 > 0$.

(a) Since

$$Lx_n + x_n^* = -C(\lambda_n, x_n) - \frac{1}{n} Jx_n \to 0$$

and $L+T$ satisfies (S_q), we have $x_n \to x_0 \in \partial G$ in X. Since the sum $L+T$ is maximal monotone, its closedness implies $x_0 \in D(L)$ and $0 \in Lx_0 + Tx_0$, which contradict $0 \notin (L+T)(\partial G \cap D(L))$.

(b) We first assert that

$$\limsup_{n\to\infty} \langle C(\lambda_n, x_n), x_n - x_0 \rangle \le 0. \tag{4.2.15}$$

Assume that it is not true. Then there is a subsequence of $\{x_n\}$, which we again denote by $\{x_n\}$, such that

$$\lim_{n\to\infty} \langle C(\lambda_n, x_n), x_n - x_0 \rangle = q > 0. \tag{4.2.16}$$

Since $Lx_n + x_n^* \rightharpoonup -c^*$, we invoke (4.2.14) and (4.2.16) to obtain

$$\lim_{n\to\infty} \langle Lx_n + x_n^*, x_n - x_0 \rangle < 0,$$

which is impossible by Lemma 2.10. Therefore, (4.2.15) is true. Using (4.2.15), $C(\lambda_n, x_n) - C(\lambda_0, x_n) \to 0$ and

$$\langle C(\lambda_0, x_n), x_n - x_0 \rangle = \langle C(\lambda_0, x_n) - C(\lambda_n, x_n), x_n - x_0 \rangle + \langle C(\lambda_n, x_n), x_n - x_0 \rangle,$$

we obtain

$$\limsup_{n\to\infty} \langle C(\lambda_0, x_n), x_n - x_0 \rangle \le 0.$$

Since C is a homotopy of class (S_+) w.r.t. Y, we obtain $x_n \to x_0$ in X. Since C is demicontinuous, $C(\lambda_n, x_n) \rightharpoonup C(\lambda_0, x_0) = c^*$. Thus, $Lx_n + x_n^* \rightharpoonup -C(\lambda_0, x_0)$. Since $L+T$ is maximal monotone, it is demiclosed. This implies $Lx_0 + Tx_0 + C(\lambda_0, x_0) \ni 0$ and the proof of the theorem is now complete. ■

4.3 Invariance of Domain

We start with a definition.

Definition 4.11 *An operator $T : X \subset D(T) \to 2^{X^*}$ is said to be "injective" on a set $G \subset D(T)$ if, for $x, y \in D(T)$, $Tx \cap Ty \neq \emptyset$ implies $x = y$. T is said to be "locally injective" on a set $G \subset D(T)$ if for each $x \in G$ there exists $q > 0$ such that T is locally injective on $G \cap B_q(0)$.*

We have the following invariance of result by Kartsatos and Skrypnik [20].

Theorem 4.12 (Invariance of Domain) *Let $T : X \supset D(T) \to 2^{X^*}$ be maximal monotone and $C : \overline{G} \to X^*$ demicontinuous, bounded and locally of type (S_+), where $G \subset X$ is open and bounded. Assume that $T + C + \epsilon J_1$ is locally injective on G for all $\epsilon \geq 0$ and for $J_1(\cdot) = J(\cdot - x_0)$ corresponding to every $x_0 \in D(T) \cap G$, J being the normalized duality map of X into X^*. Then $(T + C)(D(T) \cap G)$ is open.*

Theorem 4.13 *Let $L : X \supset D(L) \to X^*$ be a densely defined linear maximal monotone operator, $T : X \to 2^{X^*}$ a bounded maximal monotone operator and $C : \overline{G} \to X^*$ a bounded demicontinuous operator of type (S_+) with respect to $D(L)$, where $G \subset X$ is an open bounded subset of X. Assume that $L+T+C+\epsilon J_1$ is locally injective on G for all $\epsilon \geq 0$ and for $J_1(\cdot) = J(\cdot - x_0)$ corresponding to every $x_0 \in D(L) \cap G$, J being the normalized duality map of X into X^*. Then $(L+T+C)(D(L)\cap G)$ is open in X^*.*

Proof: Let $p^* \in (L+T+C)(D(L)\cap G)$. We may assume without loss of generality that $p^* = 0$, $0 \in D(L) \cap G$, $0 \in T(0)$, $0 \in C(0)$. Since $L + T + C$ is locally injective on G, choose $q > 0$ such that $\overline{B_q(0)} \subset G$ and $L + T + C$ is locally injective on $\overline{B_q(0)}$. It is sufficient to show the existence an $r > 0$ such that $B_r(0) \subset (L+T+C)(D(L)\cap B_q(0))$.

We claim that there is $r > 0$ such that $(L + T + C)(D(L) \cap \partial B_q(0)) \cap B_r(0) = \emptyset$. Suppose that the contrary is true. Then there exists a sequence $\{r_n\}$, $r_n \downarrow 0$ and

$\{x_n\} \subset (L+T+C)(D(L)\cap\overline{B_q(0)})$ and $p_n^* \in B_{r_n}(0)$, $v_n^* \in Tx_n$ such that

$$Lx_n + v_n^* + Cx_n = p_n^*. \tag{4.3.17}$$

Let $Y = D(L)$ with the graph norm. Since T and C are bounded, it follows that $\{\|Lx_n\|\}$ is bounded and hence $\{\|x_n\|_Y\}$ is bounded. Since Y is reflexive, we may assume that $x_n \rightharpoonup x_0$ in X and $Lx_n \rightharpoonup Lx_0$ in X^*. We are now going to show that

$$\limsup_{n\to\infty}\langle Cx_n, x_n - x_0\rangle \le 0. \tag{4.3.18}$$

If (4.3.18) is not true, we may assume that

$$\lim_{n\to\infty}\langle Cx_n, x_n - x_0\rangle > 0. \tag{4.3.19}$$

In view of (4.3.17) and (4.3.19), we obtain

$$\lim_{n\to\infty}\langle Lx_n + v_n^*, x_n - x_0\rangle < 0,$$

which is impossible by Lemma 2.10(i) since $L+T$ is maximal monotone. Thus (4.3.18) is true. Since C is of type (S_+) w.r.t. L, we obtain $x_n \to x_0 \in \partial B_q(0)$ in X. By the demicontinuity of C, we get $Cx_n \rightharpoonup Cx_0$ and therefore $Lx_n + v_n^* \rightharpoonup -Cx_0$. Since $L+T$ is demiclosed, we obtain $0 \in (L+T+C)(x_0)$ which is a contradiction to the injectivity of $L+T+C$ on $\overline{B_q(0)}$. Thus our claim is proved.

We now fix $p^* \in B_r(0)$ and define $f(t) = tp^*$, $t \in [0,1]$. Clearly, $f(t)$ lies in $B_r(0)$ for all $t \in [0,1]$. We next claim that there exist an integer $n_0 > 0$ and $s_0 > 0$ such that

$$\hat{L}x + \hat{T}_s x + \hat{C}x + sMx + \frac{1}{n}\hat{J}x = j^*(f(t)) \tag{4.3.20}$$

has no solution $x \in \partial G_R(Y)$, where $G_R(Y) = j^{-1}(B_q(0)) \cap B_Y(0,R)$. Here, $B_Y(0,R) = \{y \in Y : \|y\|_Y < R\}$. By Lemma 4.8, the set of solutions of (4.3.20) in $j^{-1}(\overline{G})$ is bounded in Y and so such a number $R > 0$ exists. We note that $\partial(j^{-1}(B_q(0))) \subset j^{-1}(\partial B_q(0))$.

Assume that our claim is not true. Then there exist sequences, $\{t_n\} \subset [0,1]$, $s_n \downarrow 0$, $x_n \in \partial(j^{-1}(B_q(0)))$, $x_0 \in Y$, $t_0 \in [0,1]$ with $t_n \to t_0$ and $x_n \rightharpoonup x_0$ in Y such that

$$\hat{L}x_n + \hat{T}_{s_n}x_n + \hat{C}x_n + s_n M x_n + \frac{1}{n}\hat{J}x_n = j^*(f(t_n)). \tag{4.3.21}$$

Since $x_n \rightharpoonup x_0$ in Y implies $x_n \rightharpoonup x_0$ in X and $Lx_n \rightharpoonup Lx_0$ in x^*, it follows that $\{Mx_n\}$ is bounded and hence, from (4.3.21), $\{\hat{T}_{s_n}x_n\}$ is bounded. We are going to show that

$$\limsup_{n\to\infty}\langle Cx_n, x_n - x_0\rangle \le 0. \tag{4.3.22}$$

Suppose that this is not true. Then we may assume that

$$\lim_{n\to\infty}\langle Cx_n, x_n - x_0\rangle > 0. \tag{4.3.23}$$

We observe that

$$\begin{aligned}
\langle Lx_n + T_{s_n}x_n, x_n - x_0\rangle &= (\hat{L}x_n + \hat{T}_{s_n}x_n, x_n - x_0)\\
&= -(\hat{C}x_n, x_n - x_0) - s_n(Mx_n, x_n - x_0) - \frac{1}{n}(\hat{J}x_n, x_n - x_0)\\
&\quad +(j^*(f(t_n)), x_n - x_0) \qquad (4.3.24)\\
&= \langle Cx_n, x_n - x_0\rangle - s_n\langle Lx_n - Lx_0, J^{-1}(Lx_n)\rangle\\
&\quad -\frac{1}{n}\langle Jx_n, x_n - x_0\rangle + \langle (f(t_n)), x_n - x_0\rangle,
\end{aligned}$$

which implies

$$\lim_{n\to\infty}\langle Lx_n + T_{s_n}x_n, x_n - x_0\rangle < 0.$$

This is impossible by Lemma 2.9(i). Thus (4.3.22) is true. Since C is of type (S_+) w.r.t. L, we get $x_n \to x_0 \in \partial B_q(0)$ in X. Since C is demicontinuous, we have $Cx_n \rightharpoonup Cx_0$ in X^*.

Since T_n with $T_n \equiv T$ converges to T in the graph sense and T is bounded, by Lemma 4.7(a) we have $T_{s_n}x_n \rightharpoonup w \in Tx_0$ for a subsequence of $\{x_n\}$ which we again denote by $\{x_n\}$. Then (4.3.21) implies $\hat{L}x_n + \hat{T}_{s_n}x_n + \hat{C}x_n \rightharpoonup \hat{L}x_0 + j^*w + \hat{C}x_0 = j^*(f(t_0))$.

For all $v \in Y$, we have

$$\langle Lx_0 + w + Cx_0, v\rangle = (\hat{L}x_0 + j^*w + \hat{C}x_0, v) = (j^*(f(t_0)), v) = \langle f(t_0), v\rangle.$$

Since Y is dense in X, we obtain $Lx_0 + w + Cx_0 = f(t_0)$ which implies $f(t_0) \in (L+T+C)(D(L)\cap \partial B_q(0))$. Since $x_0 \in D(L)\cap\partial B_q(0)$ and $f(t_0)\in B_r(0)$, we have a contradiction to $(L+T+C)(D(L)\cap\partial B_q(0))\cap B_r(0) = \emptyset$.

We now consider the homotopy function

$$H(s,t,x,n) = t\left(\hat{L}x + \hat{T}_s x + \hat{C}x + \frac{1}{n}\hat{J}x\right) + sMx + (1-t)\hat{J}x, \tag{4.3.25}$$

where $(t,x)\in[0,1]\times j^{-1}(\overline{B_q(0)})$. Let $G_R(Y) = j^{-1}(B_q(0))\cap B_Y(0,R)$ with $B_Y(0,R) = \{y \in Y : \|y\|_Y < R\}$. By Lemma 4.8, the set of solutions of $H(s,t,x,n) = 0$ in $j^{-1}(\overline{B_q(0)})$ is bounded in Y and such a number $R > 0$ exists.

We are going to show that there exist an integer $n_1 > 0$ and a number $s_1 > 0$ such that (4.3.25) has no solution $x \in \partial G_R(Y)$ for any $s \in (0, s_1]$, $n \geq n_1$ and $t\in[0,1]$. Assuming that the contrary is true, let there be sequences $\{x_n\}\subset \partial G_R(Y)$, $\{s_n\}\subset(0,\infty)$, and $\{t_n\}\subset[0,1]$ such that $x_n \rightharpoonup x_0$ in Y, $s_n \to 0$, $t_n\to t_0$ and

$$t_n\left(\hat{L}x_n + \hat{T}_{s_n}x_n + \hat{C}x_n + \frac{1}{n}\hat{J}x_n\right) + s_nMx_n + (1-t_n)\hat{J}x_n = 0. \tag{4.3.26}$$

If $t_n = 0$ for all n, then

$$s_nMx_n + \hat{J}x_n = 0,$$

which implies $x_n = 0$ for all n, and this is a contradiction to the choice of $\{x_n\}$. Also, if $t_n = 1$ for all n, we again have a contradiction by the argument as in the previous part with $j^*(f(t)) = 0$. Thus, we may assume that $t_n \in (0,1)$. Consider the cases: (a) $t_0 = 0$; (b) $t_0 > 0$.

Case (a): Since $x_n \rightharpoonup x_0$ in Y, it follows that $x_n \rightharpoonup x_0$ in X and $Lx_n \rightharpoonup Lx_0$ in X^*. In particular, $\{\|x_n\|\}$ is bounded. By the boundedness of C, $\{Cx_n\}$ is also bounded. Now,

$$t_n\hat{L}x_n + t_n\hat{T}_{s_n}x_n + \hat{J}x_n = -t_n\hat{C}x_n - t_n\left(\frac{1}{n}-1\right)\hat{J}x_n - s_nMx_n$$

implies

$$t_n(\hat{L}x_n, x_n)+t_n(\hat{T}_{s_n}x_n, x_n)+(\hat{J}x_n, x_n) = -t_n(\hat{C}x_n, x_n)-t_n\left(\frac{1}{n}-1\right)(\hat{J}x_n, x_n)-s_n(Mx_n, x_n).$$

By the monotonicity of L and T_s, we get

$$\langle Jx_n, x_n\rangle \le -t_n\langle Cx_n, x_n\rangle - t_n\left(\frac{1}{n}-1\right)\langle Jx_n, x_n\rangle - s_n\langle Lx_n, J^{-1}(Lx_n)\rangle \to 0.$$

This shows that $x_0 = 0$, i.e., a contradiction because $\{x_n\} \subset \partial B_q(0)$. Therefore, (a) is true.

Case (b): If $t_0 = 1$, let $d_n = \dfrac{1}{t_n} - 1$. Clearly, $d_n > 0$ and $d_n Jx_n \to 0$. Also, from (4.3.26), we have

$$\hat{L}x_n + \hat{T}_{s_n}x_n + \hat{C}x_n + \left(\frac{1}{n} + d_n\right)\hat{J}x_n + s_n Mx_n = 0.$$

This equation is similar to (4.3.21) with $f(t) \equiv 0$. This shows that the case $t_0 = 1$ is also impossible. Assume now that $t_0 \in (0, 1)$. Put

$$e_n = \frac{1}{t_n} + \frac{1}{n} - 1.$$

We may assume that $e_n > 0$ for all n. From (4.3.26), we have

$$\hat{L}x_n + \hat{T}_{s_n}x_n + \hat{C}x_n + e_n\hat{J}x_n + \frac{s_n}{t_n}Mx_n = 0. \tag{4.3.27}$$

We are now going to show that (4.3.22) is true. Assuming the contrary, suppose that (4.3.23) holds true. We observe that

$$\begin{aligned}
\langle Lx_n + T_{s_n}x_n, x_n - x_0\rangle &= (\hat{L}x_n + \hat{T}_{s_n}x_n, x_n - x_0)\\
&= -(\hat{C}x_n, x_n - x_0) - \frac{s_n}{t_n}(Mx_n, x_n - x_0) - \frac{e_n}{n}(\hat{J}x_n, x_n - x_0)\\
&\quad +(j^*(f(t_n)), x_n - x_0) \qquad (4.3.28)\\
&= \langle Cx_n, x_n - x_0\rangle - \frac{s_n}{t_n}\langle Lx_n - Lx_0, J^{-1}(Lx_n)\rangle\\
&\quad -\frac{e_n}{n}\langle Jx_n, x_n - x_0\rangle,
\end{aligned}$$

which implies

$$\lim_{n\to\infty} \langle Lx_n + T_{s_n}x_n, x_n - x_0\rangle < 0,$$

and this is impossible by Lemma 2.9(i). Thus (4.3.22) is true. Since C is of type (S_+) w.r.t. L, we get $x_n \to x_0 \in \partial B_q(0)$ in X. Since C is demicontinuous, we have $Cx_n \rightharpoonup Cx_0$ in X^*.

Since T_n with $T_n \equiv T$ converges to T in the graph sense and T is bounded, by Lemma 4.7(a) we have $T_{s_n}x_n \rightharpoonup w \in Tx_0$ for a subsequence of $\{x_n\}$ which we again denote by $\{x_n\}$. Then (4.3.27) implies

$$\hat{L}x_n + \hat{T}_{s_n}x_n + \hat{C}x_n \rightharpoonup \hat{L}x_0 + j^*w + \hat{C}x_0 = -\frac{1-t_0}{t_0}\hat{J}x_0.$$

For all $v \in Y$, we have

$$\left\langle Lx_0 + w + Cx_0 + \frac{1-t_0}{t_0}Jx_0, v\right\rangle = \left(\hat{L}x_0 + j^*w + \hat{C}x_0 + \frac{1-t_0}{t_0}\hat{J}x_0, v\right) = 0.$$

Since Y is dense in X, we obtain

$$Lx_0 + w + Cx_0 + \frac{1-t_0}{t_0}Jx_0 = 0$$

which implies

$$0 \in \left(L + T + C + \frac{1-t_0}{t_0}J\right)(D(L) \cap \partial B_q(0)).$$

Since $x_0 \in D(L) \cap \partial B_q(0)$ and $0 \in B_q(0)$, we have a contradiction to the injectivity of

$$L + T + C + \frac{1-t_0}{t_0}J$$

on $D(L) \cap \overline{B_q(0)}$. This shows that the homotopy equation (4.3.25) has no solution on $\partial G_R(Y)$ for all large n, and for all $s \in (0, s_1]$, for some $s_1 > 0$ and for all $t \in [0,1]$. We may thus take $s_0 = s_1$ and consider only $n \geq n_0$.

Since $H(s,t,x,n)$ is an affine homotopies of bounded demicontinuous operators of type (S_+) from $j^{-1}(\overline{B_q(0)}) \subset Y$ to Y^*, it is a bounded homotopy of type (S_+) from $j^{-1}(\overline{B_q(0)}) \subset Y$ to Y^*. So, by the homotopy invariance of the degree for (S_+), we obtain

$$\begin{aligned} d_S(H(s,1,\cdot,n),G_R(Y),0) &= d_S(H(s,0,\cdot,n),G_Y(R),0) \\ &= d_S(\hat{J}+sM,G_Y(R),0) \\ &= 1. \end{aligned}$$

Next, we consider the homotopy equation

$$H_1(s,t,x,n) = \hat{L}x + \hat{T}_s x + \hat{C}x + sMx + \frac{1}{n}\hat{J}x - j^*(f(t)).$$

We have already seen that the equation $H_1(s,t,x,n) = 0$ has no solution $x \in \partial G_Y(R)$. We notice that $H_1(s,t,x,n)$ is admissible for the Skrypnik degree for (S_+) mappings. By the invariance property of that degree, we obtain

$$\begin{aligned} d_S(H_1(s,t,\cdot,n),G_Y(R),0) &= d_S(H_1(s,0,\cdot,n),G_Y(R),0) \\ &= d_S\left(\hat{L}+\hat{T}_s+\hat{C}+\frac{1}{n}\hat{J}+sM,G_Y(R),0\right) \\ &= d_S(H(s,1,\cdot,n),G_Y(R),0) \\ &= 1. \end{aligned}$$

Since

$$d_S(H_1(s,1,\cdot,n),G_Y(R),0) = d_S(\hat{L}+\hat{T}_s+\hat{C}+sM+\frac{1}{n}\hat{J}-j^*(f(t)),G_Y(R),0),$$

we have that the degree of $L+T+C+\frac{1}{n}J$ as in [1] satisfies

$$\begin{aligned} & d(L+T+C+\frac{1}{n}J-f(t),B_q(0),0) \\ =\ & \lim_{s\to 0} d_S(\hat{L}+\hat{T}_s+\hat{C}+sM+\frac{1}{n}\hat{J}-j^*(f(t)),G_Y(R),0) \\ =\ & 1, \end{aligned}$$

for all $t \in [0,1]$ and for all $n \geq n_0$. Thus, for all $n \geq n_0$, we have

$$B_r(0) \subset \left(L + T + C + \frac{1}{n}J\right)(B_q(0) \cap D(L))$$

so that, for each $n \geq n_0$, there exists $x_n \in B_q(0) \cap D(L)$ such that

$$p^* = Lx_n + w_n + Cx_n + \frac{1}{n}Jx_n$$

for some $w_n \in Tx_n$. Since T and C are bounded, we have that $\{Lx_n\}$ is bounded and hence we may assume that $x_n \rightharpoonup x_0$ in Y. Since

$$\langle Lx_n + w_n, x_n - x_0 \rangle \geq 0$$

whenever it exists by Lemma 2.10(i), we conclude that

$$\limsup_{n \to \infty} \langle Cx_n, x_n - x_0 \rangle \leq 0.$$

By the (S_+)-property of C with respect to L, we get $x_n \to x_0$ in X and hence by the demicontinuity of C, we get $Cx_n \rightharpoonup Cx_0$ in X^*. Therefore, $w_n \rightharpoonup p^* - Lx_0 - Cx_0$ in X^*. Since T is demiclosed , we have that $p^* - Lx_0 - Cx_0 \in Tx_0$ and so by injectivity of $L + T + C$ on $D(L) \cap \overline{B_q(0)}$ we obtain

$$p^* \in (L + T + C)(D(L) \cap B_q(0)).$$

Since $p^* \in B_r(0)$ arbitrary, we have that

$$B_r(0) \subset (L + T + C)(D(L) \cap B_q(0)),$$

and this completes the proof. ■

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