

6-19-2008

Maxwell's Problem on Point Charges in the Plane

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Maxwell's Problem on Point Charges in the Plane

by

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A thesis submitted in partial fulfillment
of the requirements for the degree of
Master of Arts
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Date of Approval:
June 19, 2008

Keywords: Potential Theory, Electrostatics, Bezout's Theorem, Algebraic Curves,
Harmonic Functions

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Acknowledgments

I would like to thank all those who have helped and supported me through the research and development of this paper. My first thanks go to my wife, who has encouraged me to continue when it seemed there was no light at the end of the tunnel. I would like to extend thanks to my thesis committee members for all of their help and guidance. Special acknowledgment and thanks go to my advisor Dr. Khavinson for presenting me with this most interesting problem, helping me through so many problems, and responding to e-mails beginning with “It’s 12:30 am and I just had a great idea.” I would also like to thank my family for all of their prayers and support.

I greatly appreciate the support by the award USF Thrust Area: Computational tools for discovery. Grant for Graduate education and Research in Computer Vision and Pattern recognition, 2007-2009.

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ABSTRACT

This paper deals with approximating an upper bound for the number of equilibrium points of a potential field produced by point charges in the plane. This is a simplified form of a problem posed by Maxwell [4], who considered spatial configurations of the point charges. Using algebraic techniques, we will give an upper bound for planar charges that is sharper than the bound given in [6] for most general configurations of charges. Then we will study an example of a configuration of charges that has exactly the number of equilibrium points that Maxwell's conjecture predicts, and we will look into the nature of the extremal points in this case. We will conclude with a solution to the twin problem for the logarithmic potential, followed by a discussion of the conditions necessary for a degenerate case in the plane.

1. INTRODUCTION

In his book *A Treatise on Electricity and Magnetism* [4], J. C. Maxwell suggested that given any non-degenerate spatial configuration of n point charges, the maximum number of places that the electrostatic force can vanish is $(n - 1)^2$. However, he offered no proof of this conjecture. Very little progress has been made to verify this. In 2007, Shapiro, Gabrielov, and Novikov [6] were able to establish an upper bound of $4^{n^2}(3n)^{2n}$ for any non-degenerate spatial configuration of charges. For the case of three charges, this bound can be sharpened to 12.

In order to investigate this conjecture, we need an understanding of electrostatic potential and electrostatic force.

Definition. *Given n point charges in space, each of which has charge q_k and is located at some point $p_k = (x_k, y_k, z_k)$, the potential P generated by these point charges is*

$$(1.1) \quad P = c \sum_{k=1}^n \frac{q_k}{|p - p_k|}$$

where c is Coulomb's constant [7], p. 750.

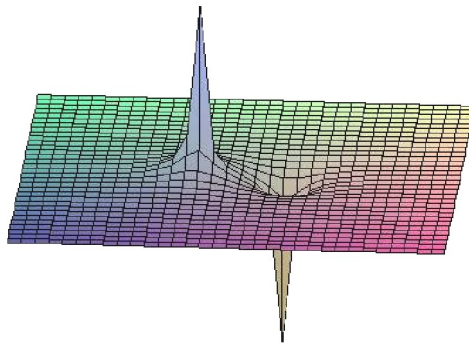


FIGURE 1. Potential field generated by a positive and a negative charge. The positive charge is at $(-1,0)$, and the negative charge is at $(1,0)$.

For our purposes, we will let the constant c equal 1 since it is not relevant to our study.

Definition. The electrostatic force F generated by the potential P is given by

$$F = -\nabla P = F_x \vec{i} + F_y \vec{j} + F_z \vec{k}$$

where

$$(1.2) \quad F_x = -\frac{\partial P}{\partial x} = \sum_{k=1}^n \frac{q_k(x - x_k)}{[(x - x_k)^2 + (y - y_k)^2 + (z - z_k)^2]^{3/2}}$$

$$(1.3) \quad F_y = -\frac{\partial P}{\partial y} = \sum_{k=1}^n \frac{q_k(y - y_k)}{[(x - x_k)^2 + (y - y_k)^2 + (z - z_k)^2]^{3/2}}$$

and

$$(1.4) \quad F_z = -\frac{\partial P}{\partial z} = \sum_{k=1}^n \frac{q_k(z - z_k)^2}{[(x - x_k)^2 + (y - y_k)^2 + (z - z_k)^2]^{3/2}}$$

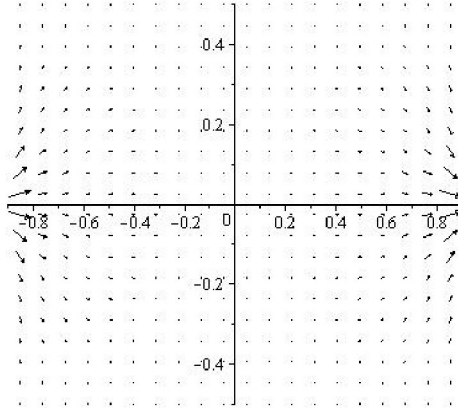


FIGURE 2. This is the force field generated by the potential represented in Figure 1

It is important to note that the equations $F_x = 0$ and $F_y = 0$ are algebraic curves (i.e. both are sums of rational functions whose numerators and denominators can be expressed as polynomials). This important fact will be used in establishing an upper bound on the number of equilibrium points in Section 3.

We will only consider configurations of point charges in the xy -plane. Any place at which we have $F = 0$ must have $F_x = 0$ and $F_y = 0$ (obviously, $F_z = 0$ since all of the point charges are in the plane $z = 0$). Our goal is to find the maximum number of places at which the force will vanish in the plane given n point charges. This is a simplified form of Maxwell's problem which may provide some insight on the more general non-degenerate cases. Note that a non-degenerate configuration refers

to a situation in which there are finitely many equilibrium points (points where the electrostatic force vanishes).

Before continuing onto the 2-charge case, first recall some well-known results from classical potential theory.

Definition. A smooth function $f(x, y, z)$ is said to be harmonic if $\Delta f = 0$; that is: $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$. [1], p. 73.

Proposition 1.1. In \mathbb{R}^3 , the potential P is a harmonic function away from the charges.

Definition. A smooth function $f(x, y)$ is said to be subharmonic if $\Delta f \geq 0$; similarly, we say that f is superharmonic if $-f$ is subharmonic. [5], pp. 28, 36

Proposition 1.2. In \mathbb{R}^2 , the potential P is subharmonic away from the charges if all of the charges are positive, and is superharmonic if all of the charges are negative.

In \mathbb{R}^3 , we have that P is always harmonic away from the charges, regardless of the signs of the charges. Unfortunately, in \mathbb{R}^2 , nothing of the sort can be said about P if there is a combination of positive and negative charges. This will prove to be one of the greatest hindrances to showing that a non-degenerate case cannot exist in \mathbb{R}^2 . The following theorem is one of the basic results about harmonic functions. See [5], p. 29, or [1], p. 248 for a proof.

Theorem 1.3. The Maximum Principle: *If f is a real-valued harmonic function in a domain D , then f has no local local maximum or local minimum in D , unless f is a constant everywhere.*

The maximum principle shows that zeros of the force are actually saddle points (i.e. critical points that are neither maximum nor minimum points) in \mathbb{R}^3 . Unfortunately, when we restrict ourselves to the plane, we lose all of the harmonic properties of the potential. In the special cases mentioned in Proposition 1.2 we have at best that the potential is subharmonic or superharmonic. The maximum principle does apply to subharmonic functions in that they cannot attain a local maximum. However, there is no restriction on a subharmonic function having a local minimum inside of the domain. Similarly, superharmonic functions cannot attain a minimum but can have a local maximum.

The properties of harmonic functions and algebraic curves will be invaluable tools as we proceed with our investigation of the validity of Maxwell's conjecture. We will begin with the simplest case: the two-charge configurations. Then we will establish an upper bound on the number of equilibrium points. This will be followed with a discussion of three-charge configurations, as well as other versions of this problem and some conjectures. Section 5 also contains a discussion about the existence of degenerate configurations of point charges in the plane.

2. TWO POINT CHARGE CASE

Maxwell's conjecture predicts that for a configuration of two point charges, there is at most one place at which the force will vanish. The following theorem shows this is true.

Theorem 2.1. *Given two point charges in the plane, there is at most one place at which the force will vanish.*

Remark: It is interesting to note that the proof (given in Section 6) shows that every configuration of two point charges will vanish at exactly one point, except when the two charges are of equal magnitude but opposite sign.

It will be useful to observe what type of extrema this point is. In order to do this, we need the Second Partial Test [3], p. 889.

Theorem 2.2. *Let f have continuous second partial derivatives on an open region containing a point (a, b) for which $f_x(a, b) = 0$ and $f_y(a, b) = 0$. To test for relative extrema of f consider the quantity*

$$d = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2.$$

1. *If $d > 0$ and $f_{xx}(a, b) > 0$, then f has a relative minimum at (a, b) .*
2. *If $d > 0$ and $f_{xx}(a, b) < 0$, then f has a relative maximum at (a, b) .*
3. *If $d < 0$, then $(a, b, f(a, b))$ is a saddle point.*
4. *The test is inconclusive if $d = 0$.*

Applying the Second Partial Test to the potential function of two point charges gives the following result.

Corollary 2.3. *Any point from a two point-charge configuration where the force vanishes is a saddle point for the potential.*

3. AN UPPER BOUND VIA BEZOUT'S THEOREM

We now wish to find an effective upper bound for point charges in the plane. One might wish to use the previous result as a basis step for an inductive proof to show Maxwell's conjecture holds, but an examination of the structure of $F_x = 0$ and $F_y = 0$ will be more useful. Recall that in Section 1 we noted that these equations are algebraic curves. The order of an algebraic curve is the highest degree of the polynomial. Before establishing this upper bound, however, we need the following well-known result from algebra whose proof can be found in [8], p. 54.

Theorem 3.1. (*Bezout's Theorem*) *Two plane algebraic curves of orders m and n with no common components have at most mn intersections.*

The algebraic curves we are considering are defined by $F_x = 0$ and $F_y = 0$. Therefore, to apply Bezout's Theorem, we must assume that we have a non-degenerate configuration of point charges. That is, if $F_x = 0$ and $F_y = 0$ share a common factor, then there exists a curve through the plane along which every point is a place where the force vanishes. In Section 5 we will consider this problem in more depth.

With Bezout's Theorem, we are now ready to give an upper bound on the number of points that the force vanishes.

Theorem 3.2. *Given n point charges in the plane, the maximum number of equilibrium points in the plane is $[2^{n-1}[3n - 2]]^2$, assuming a non-degenerate case.*

TABLE 1. Comparing Upper Bounds

n	Theorem 3.2	Upper Bound from [6]
2	64	331,776
3	784	1.393×10^{11}
4	6,400	1.847×10^{18}
5	43,264	6.493×10^{26}
6	262,144	5.463×10^{36}

Remark: Table 1 shows that the upper bound given by Theorem 3.2 provides a much smaller bound on the number of places where the force vanishes. The drawback is that the upper bound given in [6] considers spatial configurations of point charges, and the upper bound from Theorem 3.2 is restricted to configurations in the plane.

4. A LOOK AT THE THREE CHARGE SYSTEM

Maxwell's conjecture applied to the three charge system predicts that there are at most four points where the force will vanish. Theorem 1.5b of [6] gives an upper bound of 12 points. Though no computer simulations have been able to produce a configuration that yields 12 points (or even 5), it is possible to have 4 points where the force vanishes. We find that one of the simplest configurations of 3 points gives this result.

Proposition 4.1. *Given three equal point charges on the plane arranged in an equilateral triangle, there are exactly four places at which the electrostatic force vanishes.*

Remark: Note that this proof shows that one of the zeros of the force occurs at the “center of mass” (replace the point-charges with masses, and the gravitational potential and force remain the same) of this system. This may lead one to conclude that a center of mass of the system is always one of the locations of zero force. However, an obvious counter-example to this is the earth-sun system. The center of mass of the system is inside of the sun, but the point where the gravitational pulls cancel each other is close to earth.

The existence of a configuration that produces four points of zero force begs the question: does there exist a configuration in which we have more than four points where the force vanishes? An examination of the potential of this three charge configuration will give some insight into this problem, as well as shed some light on the existence of a curve in the plane on which the force vanishes.

Figure 3 shows the potential field generated by three positive, equal point charges placed in an equilateral triangle. We see that the four zeros of the force occur at the one minimum and the three saddle points. Obviously, in order to have additional zeros, either additional saddle points or minimum points must be generated. The following theorem gives some insight into the behavior of the potential.

Theorem 4.2. *Given a configuration of three positive charges, if \vec{v} is any vector originating on the boundary of the triangle, perpendicular to the boundary, and is directed into the interior of the triangle, then the potential takes at most one minimum*

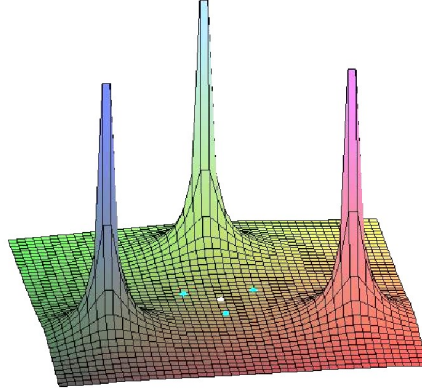


FIGURE 3. Potential of three equal charges in equilateral triangle. The charges are placed at $(-1, 0)$, $(1, 0)$, and $(0, \frac{\sqrt{3}}{3})$. There is one min and three saddle points.

on \vec{v} if \vec{v} is an altitude of the triangle and at most two minima for all other \vec{v} . The same holds for three negative charges using maximum points.

One interesting fallout of this proof is that the inward normal derivative of the potential is always positive. Though Theorem 4.2 does not prohibit the existence of multiple extremal points in the interior of the triangle, it does show that they cannot lie on the same \vec{v} .

5. OTHER CASES AND CONJECTURES

Thus far, we have only considered the so-called Newtonian kernel of the potential, which defines the potential by an inverse distance rule. We could change the definition of the potential and create new kernels having the same properties as the Newtonian kernel, but with some surprising results in the force equations. One such example is logarithmic potential [2], pp. 62-63, which in complex notation is defined by $P = \sum_{i=1}^n kq_i \log \left| \frac{1}{z-z_i} \right|$. Logarithmic potential is extremely useful when studying infinite cylinders or wires that carry charge. See [2], pp. 62-63, for more details.

Maxwell's conjecture in the context of logarithmic potential gives the following result:

Theorem 5.1. *The force F produced by $P = \sum_{i=1}^n kq_i \log \left| \frac{1}{z-z_i} \right|$ has precisely $n - 1$ equilibrium points, where $F = -\nabla P$.*

Another case we wish to consider is the existence of non-degenerate configuration of point charges in the plane. That is, does there exist a configurations of point charges in \mathbb{R}^2 that produces a curve in \mathbb{R}^2 on which the force vanishes at every point? It is possible for charges in \mathbb{R}^2 to produce such a curve in \mathbb{R}^3 . For example, consider the potential produced by placing an even number of equal in magnitude but alternating charges in a regular n -gon centered at the origin. Then every point of the z -axis not only has zero potential but also zero force.

Clearly, if the force is to vanish on a curve, then the potential must be a constant on that curve. In \mathbb{R}^3 , we know that the potential is harmonic away from the charges. Notice that in both \mathbb{R}^2 and \mathbb{R}^3 , the limit of the potential as x , y , or z goes to infinity is zero. If the force is to vanish on a curve that goes to infinity, then the potential must be zero on such a curve. However, when we restrict ourselves to \mathbb{R}^2 , we no longer have the potential being a harmonic function, and have by Proposition 1.2 that the potential is subharmonic or superharmonic only if all of the charges have the same sign. The proof of Theorem 5.2 will handle the possibility of a curve through the plane when all charges have the same sign. The only other possibility of a degenerate configuration is the existence of a closed loop on which the force vanishes.

The following theorem addresses the existence of a degenerate configuration of three charges of the same sign.

Theorem 5.2. *Given three charges in the plane having the same sign, there is no equilibrium curve in the plane.*

The proof of this theorem can be applied to any number of charges of the same sign; that is, the three possible positions of the supposed curve cannot exist regardless of the number of charges. However, in configurations of four or more charges, an additional case arises that is not handled by this proof. Consider a configuration of four charges in which one of the charges (call it q_4) is located in the interior of the triangle formed by the other three. One could then argue that there could exist an equipotential curve around q_4 on which the force vanishes. By the maximum principle, it is clear that every point of the curve would have to be a minimum (assuming all of the charges are positive).

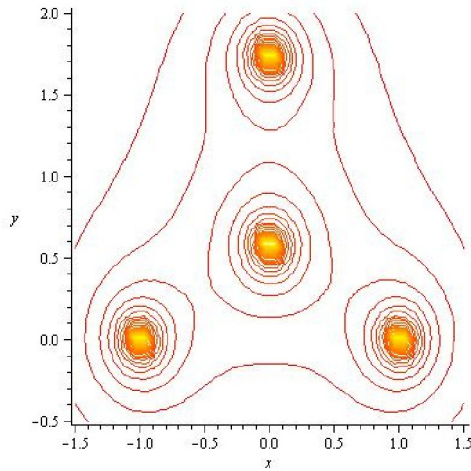


FIGURE 4. The contour plot of four equal charges. If a minimum curve existed, it would have to be on one of the level curves. In this particular case, the computer simulations can only find three equilibrium points.

Though it is still an open problem whether such a curve can exist if all of the charges have the same sign, the following theorem gives a necessary condition for every point on the curve, if it exists.

Theorem 5.3. *Define $X = \sum_{k=1}^n \frac{q_k(x-x_k)^2}{[(x-x_k)^2+(y-y_k)^2]^{5/2}}$ and $Y = \sum_{k=1}^n \frac{q_k(y-y_k)^2}{[(x-x_k)^2+(y-y_k)^2]^{5/2}}$. Given finitely many same sign charges in the plane, the necessary conditions for force vanish on a curve S are*

- 1.) $P_{xx}P_{yy} = (P_{xy})^2$ at every point of S ;
- 2.) either both $P_{xx} > 0$ and $P_{yy} > 0$, and the inequality $\frac{1}{2}X \leq Y \leq 2X$ must hold; or at most one of P_{xx} or P_{yy} can equal zero.

In computer simulations, the force will only vanish on a loop if there is a charged wire that forms a closed loop, with an additional charge in the interior of the loop. This leads us to the following conjecture:

Conjecture 5.1. *Given finitely many point charges in the plane, the electrostatic force will never vanish on a curve.*

This conjecture also applies to configurations of mixed charges. For example, consider a positive and a negative charge of equal magnitude placed at $(-1,0)$ and $(1,0)$, respectively. We know that one of the basic criteria for the force to vanish on a curve is that the potential must be constant on that curve. In this case, the entire y-axis has zero potential. However, the x -component of the force is always positive on the y-axis.

When we looked at the equilateral triangle, we saw that there are four equilibrium points: three saddle points and one minimum. Intuitively, this seems to be the only way to have four equilibrium points. For instance, consider a configuration of three positive charges in which we know that a minimum point exists somewhere in the triangle, but not on the boundary. Let \vec{v} be any vector originating at that minimum point, and consider the behavior of the potential along \vec{v} . Since \vec{v} originates at a minimum point, the potential must initially increase. However, we know that far enough away from the charges the potential will be decreasing. This implies that the potential takes a maximum on \vec{v} . However, we know that the potential is subharmonic, and subharmonic functions cannot have maximum points except on the boundary of the domain. So this maximum we see on \vec{v} must be a saddle point, or is some point on the ‘ridge’ joining two of the charges. This leads to the following conjecture:

Conjecture 5.2. *Given three positive charges, a minimum exists in the interior of the triangle if and only if there are three saddle points.*

This conjecture would seem to verify Maxwell’s conjecture for the case of three charges. The only missing component is the following conjecture:

Conjecture 5.3. *For any configuration of three positive charges, the potential can take at most one minimum in the interior of the triangle.*

If the minimum of the potential occurs on the boundary of the triangle, the force will not vanish at the minimum point. This follows from the fact that the inward normal derivative of the potential is always positive. When we looked at the equilateral triangle, we saw that there were three saddle points, each located “near” one of the legs of the triangle (i.e. the saddle points do not occur on the legs of the triangle, but one can identify a leg that each saddle point is “generated” by, in the context of the proof of Theorem 5.2). In computer simulations, we see that if all of the charges have the same sign and are equal in magnitude, then the potential of all right and obtuse configurations of the charges will have a minimum on the longest leg of the triangle. However, the two saddle points “near” the shorter legs remain.

Conjecture 5.4. *Given three equal charges arranged in a right or obtuse triangle, the force will only vanish at two saddle points near the shorter legs of the triangle.*

For a configuration of three mixed charges it would seem that the bound for Maxwell’s conjecture is not even reached. For instance, given two positive charges and one negative, the potential is strictly decreasing on the lines originating at the positive charges and directed towards the negative charge. The two saddle points that might have existed near these lines in the case of all positive charges cannot occur. However, the saddle point between the two positive charges may still exist, but would be located outside of the triangle.

Computer simulations on an equilibrium triangle with two +1 charges at $(-1,0)$ and $(1,0)$ and a negative charge on the y -axis at $(0, \sqrt{3})$ show two saddle points in the plane, both on the y -axis. One is at approximately $y = -.14629$, and the other at $y = 6.20448$. One might conjecture the existence of two saddle points not on the plane. However, computer simulations fail to find any equilibrium points not on the plane. This leads to the following conjecture:

Conjecture 5.5. *Given three point charges in the plane, the number of equilibrium points for charges of the same strength at the vertices of the equilibrium triangle is maximized when all charges have the same sign.*

6. PROOFS

Proof of Proposition 1.1: We need to show that $\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} + \frac{\partial^2 P}{\partial z^2} = 0$. By definition, we know that $F_x = -\frac{\partial P}{\partial x}$. Differentiating $-F_x$ with respect to x yields $\frac{\partial^2 P}{\partial x^2}$.

$$\begin{aligned} \frac{\partial^2 P}{\partial x^2} &= \frac{\partial}{\partial x} \left(- \sum_{k=1}^n \frac{q_k(x - x_k)}{[(x - x_k)^2 + (y - y_k)^2 + (z - z_k)^2]^{3/2}} \right) \\ &= - \sum_{k=1}^n \frac{q_k[-2(x - x_k)^2 + (y - y_k)^2 + (z - z_k)^2]}{[(x - x_k)^2 + (y - y_k)^2 + (z - z_k)^2]^{5/2}} \end{aligned}$$

Similarly, we see that

$$\frac{\partial^2 P}{\partial y^2} = - \sum_{k=1}^n \frac{q_k[(x - x_k)^2 - 2(y - y_k)^2 + (z - z_k)^2]}{[(x - x_k)^2 + (y - y_k)^2 + (z - z_k)^2]^{5/2}}$$

and

$$\frac{\partial^2 P}{\partial z^2} = - \sum_{k=1}^n \frac{q_k[(x - x_k)^2 + (y - y_k)^2 - 2(z - z_k)^2]}{[(x - x_k)^2 + (y - y_k)^2 + (z - z_k)^2]^{5/2}}.$$

Adding these together gives $\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} + \frac{\partial^2 P}{\partial z^2} = 0$.

Proof of Proposition 1.2: We need to show that $\Delta P \geq 0$, assuming all $q_k > 0$. Using the calculations of the previous proof, we have that in \mathbb{R}^2

$$\frac{\partial^2 P}{\partial x^2} = \sum_{k=1}^n \frac{q_k[2(x - x_k)^2 - (y - y_k)^2]}{[(x - x_k)^2 + (y - y_k)^2]^{5/2}}$$

and

$$\frac{\partial^2 P}{\partial y^2} = \sum_{k=1}^n \frac{q_k[-(x - x_k)^2 + 2(y - y_k)^2]}{[(x - x_k)^2 + (y - y_k)^2]^{5/2}}.$$

Adding these, we get

$$\begin{aligned} \frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} &= \sum_{k=1}^n \frac{q_k}{[(x - x_k)^2 + (y - y_k)^2]^{3/2}} \\ &> 0. \end{aligned}$$

Similarly, we see that if all $q_k < 0$, then $-P$ will be subharmonic, and therefore P is superharmonic.

Proof of Theorem 2.1: There are two possible cases to consider. In the first case, we have both point charges with the same sign (see Figure 4), and in the second case we have one positive and one negative charge (see Figure 1).

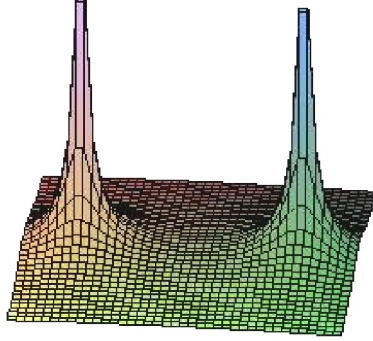


FIGURE 5. The potential field from 2 point charges with the same sign.

Case 1: Let q_1 and q_2 be positive and located at points (x_1, y_1) and (x_2, y_2) , respectively. Through a translation and rotation, these points can be moved so that one point is at $(0, 0)$, and the other is on the x -axis. With a dilation, the second point can be made $(1, 0)$ without affecting the number of zeros of the force.

Notice that at any point $p = (x, y)$ on the plane, the y -component of the force is given by

$$(6.1) \quad F_y = F_1 \sin \theta + F_2 \sin \varphi$$

where F_1 is the magnitude of the force generated by the charge q_1 located at the origin, F_2 is the magnitude of the force generated by the second charge, θ is the angle between F_1 and the x -axis, and φ is the angle between F_2 and the x -axis. Notice that F_1 and F_2 have the same sign from the definition of force (1.2). If the point p is not on the x -axis, then $\theta \neq 0$ and $\varphi \neq 0$, so $F_y \neq 0$. Therefore, any points at which the force vanishes must be on the x -axis.

For any point $p = (x, 0)$, the x -component of the force is given by

$$(6.2) \quad F_x = F_1 \cos \theta + F_2 \cos \varphi.$$

If $p \in (-\infty, 0) \cup (1, \infty)$, then both θ and φ will be 0 or π , and F_x cannot vanish. So we need to consider $p \in (0, 1)$.

Notice that if $p \in (0, 1)$ is a point where the force vanishes, we have $\theta = 0$ and $\varphi = \pi$, so: $F_x = F_1 - F_2 = 0 \Rightarrow F_1 = F_2$. Since the force is inversely proportional to

the square of the distance, we get the following result:

$$\begin{aligned}\frac{q_1}{x^2} &= \frac{q_2}{(1-x)^2} \\ q_1x^2 - 2q_1x + q_1 &= q_2x^2 \\ (q_1 - q_2)x^2 - q_1x + q_1 &= 0 \\ x &= \frac{q_1 \pm \sqrt{q_1q_2}}{q_1 - q_2}.\end{aligned}$$

Since both q_1 and q_2 are positive, the discriminant is non-zero, and we have two distinct roots to this equation.

If $q_1 > q_2$, then

$$q_1 + \sqrt{q_1q_2} > q_1 + \sqrt{q_2q_2} = q_1 + q_2 > q_1 - q_2$$

so

$$\frac{q_1 + \sqrt{q_1q_2}}{q_1 - q_2} > 1$$

Since $x \in (0, 1)$, this root can be ignored. Consider the remaining root:

$$\frac{q_1 - \sqrt{q_1q_2}}{q_1 - q_2} = \frac{\sqrt{q_1}(\sqrt{q_1} - \sqrt{q_2})}{(\sqrt{q_1} + \sqrt{q_2})(\sqrt{q_1} - \sqrt{q_2})} = \frac{\sqrt{q_1}}{\sqrt{q_1} + \sqrt{q_2}} < 1$$

Therefore,

$$x = \frac{q_1 - \sqrt{q_1q_2}}{q_1 - q_2}$$

is the only solution in $(0, 1)$ if $q_1 > q_2$.

Similarly, if $q_2 > q_1$, then $q_1 + \sqrt{q_1q_2} > 0$, and $q_1 - q_2 < 0$, so we have $\frac{q_1 + \sqrt{q_1q_2}}{q_1 - q_2} < 0$ and this root is not in $(0, 1)$. Also,

$$\frac{q_1 - \sqrt{q_1q_2}}{q_1 - q_2} < \frac{q_1 - \sqrt{q_2q_2}}{q_1 - q_2} = \frac{q_1 - q_2}{q_1 - q_2} = 1.$$

Therefore $\frac{q_1 - \sqrt{q_1q_2}}{q_1 - q_2}$ is the only solution in $(0, 1)$ if $q_2 > q_1$.

Also, note that if $q_1 = q_2$, then $F_1 = F_2$ has only one solution at $x = \frac{1}{2}$.

If both q_1 and q_2 are negative, then the new potential field will be identical to the first, just negated. The result is that the signs of the force equation are reversed, but the same results are obtained. Thus, Case 1 has been proven.

Case 2: Let q_1 be positive and q_2 be negative (see Figure 1 for a graph of the potential). Again, through a rigid motion and dilation, these charges can be placed at the origin and $(1, 0)$ without affecting the number of zeros. Assume that q_1 is

positive and that q_2 is negative. Then we have again that

$$F_x = F_1 \cos \theta - F_2 \cos \varphi$$

$$F_y = F_1 \sin \theta - F_2 \sin \varphi$$

The reason we write the F_2 terms with a negative sign is because q_2 is a negative charge. By writing the force this way, we can consider q_2 positive. Take $p = (x, y)$, where $x \in (0, 1)$, and consider the forces acting on this point. Since $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $\varphi \in (\frac{\pi}{2}, \frac{3\pi}{2})$, then the horizontal force will not vanish. If there exists a point $p = (x, y)$ where the force vanishes, then $x \in (-\infty, 0) \cup (1, \infty)$. Consider what happens if $y > 0$ (the same argument holds for $y < 0$) and $x \in (1, \infty)$. Obviously we have both $\theta, \varphi > 0$. Notice that in magnitude, we have

$$F_1 = \frac{q_1}{x^2 + y^2}$$

and

$$F_2 = \frac{q_2}{(x - 1)^2 + y^2}$$

Since $x > 1$, then $x^2 + y^2 > (x - 1)^2 + y^2$. If this point p is a point at which the electrostatic force vanishes, then we must have that $|q_1| > |q_2|$ and therefore $F_1 > F_2$. Notice that $\varphi > \theta$, so $\sin \varphi > \sin \theta$. Combining these inequalities gives

$$F_y = F_1 \sin \theta - F_2 \sin \varphi > F_1 \sin \theta - F_2 \sin \theta > F_2 \sin \theta - F_2 \sin \theta = 0$$

Therefore the vertical force will not vanish, hence $y = 0$ is the only way $F_y = 0$ for $x \in (-\infty, 0) \cup (1, \infty)$.

Take $p = (x, 0)$, $x > 1$. Then both $\theta, \varphi = 0$, so $F_x = F_1 - F_2 = 0$ gives us

$$\begin{aligned} F_1 &= F_2 \\ \frac{q_1}{x^2} &= \frac{q_2}{(x - 1)^2}. \end{aligned}$$

By the proof of Case 1, there will be one solution in $(0, 1)$ and one solution in $(1, \infty)$. The same argument holds for $x \in (-\infty, 0)$. Notice that if $|q_1| = |q_2|$, then the only solution to $F_1 = F_2$ is $x = \frac{1}{2}$, and therefore there is no solution. This completes the proof.

Proof of Corollary 2.3: In order to apply the Second Partial Test to the potential, we need P_{xx} , P_{yy} , and P_{xy} . A direct calculation shows that for n point charges

we have

$$P_{xx} = \sum_{k=1}^n \frac{q_k [2(x - x_k)^2 - (y - y_k)^2]}{[(x - x_k)^2 + (y - y_k)^2]^{5/2}}$$

$$P_{yy} = \sum_{k=1}^n \frac{q_k [-(x - x_k)^2 + 2(y - y_k)^2]}{[(x - x_k)^2 + (y - y_k)^2]^{5/2}}$$

and

$$P_{xy} = \sum_{k=1}^n \frac{3q_k(x - x_k)(y - y_k)}{[(x - x_k)^2 + (y - y_k)^2]^{3/2}}.$$

Using the simplifications from the proof of Theorem 2.1, let $(x_1, y_1) = (0, 0)$ and $(x_2, y_2) = (1, 0)$, so the only point p where the force can vanish must be of the form $p = (x, 0)$. Then the above equations simplify to

$$P_{xx} = 2 \sum_{k=1}^2 \frac{q_k}{(x - x_k)^3}$$

$$P_{yy} = - \sum_{k=1}^2 \frac{q_k}{(x - x_k)^3}$$

$$P_{xy} = 0.$$

Now apply the Second Partial Test by calculating the quantity d :

$$\begin{aligned} d &= P_{xx}P_{yy} - [P_{xy}]^2 \\ &= \left[2 \sum_{k=1}^2 \frac{q_k}{(x - x_k)^3}\right] \left[- \sum_{k=1}^2 \frac{q_k}{(x - x_k)^3}\right] - 0 \\ &= -2 \left[\sum_{k=1}^2 \frac{q_k}{(x - x_k)^3}\right]^2 \\ &\leq 0. \end{aligned}$$

All that remains is to check that $\sum_{k=1}^2 \frac{q_k}{(x - x_k)^3} \neq 0$ when $x = \frac{q_1 \pm \sqrt{q_1 q_2}}{q_1 - q_2}$, which was the only point where the force vanished. Recall that $x = \frac{q_1 - \sqrt{q_1 q_2}}{q_1 - q_2}$ when both charges had the same sign and $x = \frac{q_1 + \sqrt{q_1 q_2}}{q_1 - q_2}$ when the charges have different signs. First, assume

that $q_1 \neq q_2$. Then substituting gives

$$\begin{aligned}
\sum_{k=1}^2 \frac{q_k}{(x - x_k)^3} &= \frac{q_1}{\left[\frac{q_1 - \sqrt{q_1 q_2}}{q_1 - q_2}\right]^3} + \frac{q_2}{\left[\frac{q_1 - \sqrt{q_1 q_2}}{q_1 - q_2} - 1\right]^3} \\
&= \frac{q_1(q_1 - q_2)^3}{(q_1 - \sqrt{q_1 q_2})^3} + \frac{q_2(q_1 - q_2)^3}{(q_2 - \sqrt{q_1 q_2})^3} \\
&= (q_1 - q_2)^3 \left[\frac{q_1(q_2 - \sqrt{q_1 q_2})^3 + q_2(q_1 - \sqrt{q_1 q_2})^3}{(q_1 - \sqrt{q_1 q_2})^3(q_2 - \sqrt{q_1 q_2})^3} \right] \\
&= (q_1 - q_2)^3 \left[\frac{q_1 q_2^3 - 4q_1 q_2^2 \sqrt{q_1 q_2} + 6q_1^2 q_2^2 - 4q_1^2 q_2 \sqrt{q_1 q_2} + q_1^3 q_2}{(q_1 - \sqrt{q_1 q_2})^3(q_2 - \sqrt{q_1 q_2})^3} \right] \\
&= \frac{q_1 q_2 (q_1 - q_2)^3 (\sqrt{q_1} - \sqrt{q_2})^4}{(q_1 - \sqrt{q_1 q_2})^3 (q_2 - \sqrt{q_1 q_2})^3}.
\end{aligned}$$

Since we assumed that $q_1 \neq q_2$, then this quantity can never equal zero, and therefore $x = \frac{q_1 - \sqrt{q_1 q_2}}{q_1 - q_2}$ is a saddle point if both charges have the same sign. A similar computation shows that when $x = \frac{q_1 + \sqrt{q_1 q_2}}{q_1 - q_2}$, then

$$\sum_{k=1}^2 \frac{q_k}{(x - x_k)^3} = \frac{q_1 q_2 (q_1 - q_2)^3 (\sqrt{q_1} + \sqrt{q_2})^4}{(q_1 + \sqrt{q_1 q_2})^3 (q_2 + \sqrt{q_1 q_2})^3}.$$

Again, this only equals zero if $q_1 = q_2$. Therefore, when $q_1 \neq q_2$, the point where the force vanishes is a saddle point.

Consider what happens when $q_1 = q_2$. Obviously, the Second Partial Test cannot be applied because $\sum_{k=1}^2 \frac{q_k}{(x - x_k)^3} = \frac{0}{0}$. By Theorem 2.1, we know that this point $p = (x', 0)$ where the force vanishes must be on the line connecting the two point charges. Since both charges are positive, then p cannot be a maximum (the potential is subharmonic), and can therefore only be a saddle point or a minimum. Assume that p is a min. Then for all (x, y) in a ball around p , we must have $P(x, y) > P(p)$. Fix x' to be the x -coordinate where the force vanishes, and let $y > 0$. Then

$$\begin{aligned}
P(x', y) &= \sum_{k=1}^2 \frac{q_k}{\sqrt{(x' - x_k)^2 + (y - y_k)^2}} \\
&< \sum_{k=1}^2 \frac{q_k}{(x' - x_k)} \\
&= P(x', 0) \\
&= P(p).
\end{aligned}$$

Therefore, p cannot be a min and must be a saddle point. The same argument holds when both charges are negative by simply reversing the inequality and assuming that p is a max.

Therefore, the point where the force vanishes in the two-charge case is a saddle point.

Proof of Theorem 3.2: Before proving this theorem, we need the following Lemma and its Corollary.

Lemma 6.1. *An equation of the form $\sqrt{p_1} \pm \sqrt{p_2} \pm \dots \pm \sqrt{p_n} = k, k \neq 0$, and $p_n > 0$ for all n can be rationalized (in the sense that it contains no radical terms) by squaring n times.*

Proof: Without loss of generality, consider the case $\sqrt{p_1} + \sqrt{p_2} + \dots + \sqrt{p_n} = k$. By induction, it is trivial to see this for $n = 1$. However, checking this result for $n = 2$ will give some insight into the method of proving this for a general number of terms.

We have $\sqrt{p_1} + \sqrt{p_2} = k$. Squaring once gives $p_1 + p_2 + 2\sqrt{p_1 p_2} = k^2$. Note that the remaining radical can be isolated by moving it to the right side of the equation and subtracting k^2 to have

$$p_1 + p_2 - k^2 = -2\sqrt{p_1 p_2}.$$

Squaring a second time gives

$$p_1^2 + p_2^2 + k^4 + 2(p_1 p_2 - p_1 k^2 - p_2 k^2) = 4p_1 p_2.$$

Observe that this can be re-written as

$$p_1^2 + p_2^2 - 2p_1 p_2 = 2p_1 k^2 + 2p_2 k^2 - k^4.$$

Inductive Step: assume that $\sqrt{p_1} + \sqrt{p_2} + \dots + \sqrt{p_n} = k$ can be rationalized by squaring n times.

Need to show that $\sqrt{p_1} + \dots + \sqrt{p_n} + \sqrt{p_{n+1}} = k$ can be rationalized by squaring $n + 1$ times. Subtract $\sqrt{p_{n+1}}$ from both sides, and let $j = k - \sqrt{p_{n+1}}$, giving

$$\sqrt{p_1} + \sqrt{p_2} + \dots + \sqrt{p_n} = j.$$

Assume that $j \neq 0$ so the inductive step still applies (the case where $j = 0$ is handled in the Corollary). By the inductive step, this can be rationalized by squaring n times. As the above example for $n = 2$ shows, the equation resulting from these n

‘squarings’ can be written so that all terms with j can be isolated to the right-hand side of the equation. Since j is a binomial, any terms containing j raised to some power can be expanded. Subtract from the right-hand side all terms not containing $\sqrt{p_{n+1}}$, and then factor $\sqrt{p_{n+1}}$ from the remaining terms. Squaring both sides once more eliminates all radicals.

Corollary 6.2. *An equation of the form $\sqrt{p_1} \pm \sqrt{p_2} \pm \dots \pm \sqrt{p_n} = 0$, $p_i > 0$ for all $1 \leq i \leq n$, can be rationalized by squaring $n - 1$ times.*

Proof: Subtract $\sqrt{p_n}$ from both sides and let $k = -\sqrt{p_n}$. Then apply the Lemma.

Proof of Theorem 3.2: We have seen that F_x and F_y are identical except for terms in the numerator, which all have the form $q_i(x - x_i)$ or $q_i(y - y_i)$, so consider only $F_x = 0$. We have

$$\sum_{i=1}^n \frac{q_i(x - x_i)}{[(x - x_i)^2 + (y - y_i)^2]^{3/2}} = 0.$$

The LCM of the left hand side of the equation is $\prod_{j=1}^n [(x - x_i)^2 + (y - y_i)^2]^{3/2}$, so multiplying the equation through by the LCM transforms $F_x = 0$ into:

$$\sum_{i=1}^n q_i(x - x_i) \left[\prod_{j \neq i} [(x - x_j)^2 + (y - y_j)^2] \right]^{3/2} = 0$$

The terms $(x - x_i)^2 + (y - y_i)^2$ each have order two, so $\prod_{i \neq j} [(x - x_i)^2 + (y - y_i)^2]$ has order $2(n - 1)$ since there are $n - 1$ terms. Cubing this product brings the order to $6(n - 1)$. Distributing the term $q_i(x - x_i)$ into the radical requires squaring, so it adds two to the order of each term in the radical.

Now we can express F_x as:

$$\sqrt{p_1} \pm \sqrt{p_2} \pm \dots \pm \sqrt{p_n} = 0.$$

Since p_1, p_2, \dots, p_n are of order $6(n - 1) + 2 = 2[3(n - 1) + 1]$, then each term in this equation is of order $3(n - 1) + 1 = 3n - 2$ after applying the square root. By the Corollary, this equation must be squared $n - 1$ times to rid the equation of all radicals. Each time it is squared, the order of the equation is doubled, so the resulting polynomial has order $2^{n-1}[3n - 2]$. Notice that F_y has the same order and number of terms as F_x . By Bezout’s Theorem the number of solutions cannot exceed the product of the orders of F_x and F_y , assuming that there is no common factor between F_x and F_y .

Therefore, the maximum number of solutions is $[2^{n-1}[3n - 2]]^2$.

Proof of Proposition 4.1: Without loss of generality, by applying a rigid transformation together with dilation, assume that the three point charges $q_1 = q_2 = q_3 = +1$ are located at $(-1, 0)$, $(1, 0)$, and $(0, \sqrt{3})$. With this configuration, we have an equilateral triangle that has symmetry about the y-axis. Because of this symmetry, it is obvious that there is no horizontal force acting on any point of the y-axis (i.e for all $p = (0, y), F_x = 0$). We need to find any points y such that $F_y = 0$.

Call the forces generated by the charges on the x-axis F_1 and F_2 , and let F_3 be the force from the charge at $(0, \sqrt{3})$. By symmetry, for any point p on the y-axis, we have $|F_1| = |F_2|$ in both the horizontal and vertical directions.

If θ is the angle between F_1 and the x-axis (and by symmetry, F_2 as well), then we have that $F_y = F_1 \sin \theta + F_2 \sin \theta - F_3$ is the vertical force acting on a point on the y-axis. Notice that since $F_1 = F_2$, then at a point in which the force vanishes this becomes $2F_1 \sin \theta - F_3 = 0 \Rightarrow 2F_1 \sin \theta = F_3$. By construction, we have that $\sin \theta = \frac{y}{\sqrt{1+y^2}}$. Also, applying the definition of the forces, we get that $F_1 = \frac{1}{1+y^2}$ and $F_3 = \frac{1}{(\sqrt{3}-y)^2}$. Combining these yields

$$\begin{aligned} \frac{2y}{(1+y^2)^{3/2}} &= \frac{1}{(\sqrt{3}-y)^2} \\ 2y(\sqrt{3}-y)^2 &= (1+y^2)^{3/2} \\ 4y^2(\sqrt{3}-y)^4 &= (1+y^2)^3 \\ 4y^6 - 16\sqrt{3}y^5 + 72y^4 - 48\sqrt{3}y^3 + 36y^2 &= y^6 + 3y^4 + 3y^2 + 1 \\ 3y^6 - 16\sqrt{3}y^5 + 69y^4 - 48\sqrt{3}y^3 + 33y^2 - 1 &= 0 \\ (y - \frac{\sqrt{3}}{3})(3y^5 - 15\sqrt{3}y^4 + 54y^3 - 30\sqrt{3}y^2 + 3y + \sqrt{3}) &= 0. \end{aligned}$$

Clearly, $(0, \frac{\sqrt{3}}{3})$ is a zero of the force. Computer software is unable to find any factors of the polynomial $3y^5 - 15\sqrt{3}y^4 + 54y^3 - 30\sqrt{3}y^2 + 3y + \sqrt{3}$. However, if we set $f(y) = 3y^5 - 15\sqrt{3}y^4 + 54y^3 - 30\sqrt{3}y^2 + 3y + \sqrt{3}$, notice that $f(0) = \sqrt{3}$ and $f(\sqrt{3}) = -32\sqrt{3}$, so $f(y)$ has at least one zero on $[0, \sqrt{3}]$. Differentiating gives $f'(y) = 3(5y^4 - 20\sqrt{3}y^3 + 54y^2 - 20\sqrt{3}y + 1)$ and $f''(y) = 12(5y^3 - 15\sqrt{3}y^2 + 27y - 5\sqrt{3})$. Notice that $f'(0) = 3$ and $f'(\sqrt{3}) = -96$, so $f(y)$ assumes at least one maximum on $[0, \sqrt{3}]$. However, since $f''(0) = -60\sqrt{3}$ and $f''(\sqrt{3}) = -96\sqrt{3}$, we see that $f(y)$ is concave down at the endpoints. In order for $f(y)$ to have additional zeros on $[0, \sqrt{3}]$, there must be additional maximum and minimum points, implying that $f(y)$ must

change concavity on $[0, \sqrt{3}]$. But the only real root of $f''(y) = 0$ is

$$\frac{3^{5/6}(30^{1/3}(10 + \sqrt{10})^{2/3} + 3(10)^{2/3} + 5(3)^{2/3}(10 + \sqrt{10})^{1/3})}{15(10 + \sqrt{10})^{1/3}} \approx 3.94 > \sqrt{3}.$$

We see that $f(y)$ does not change concavity on $[0, \sqrt{3}]$. Therefore, there are no additional extrema, and $f(y)$ has only one root on $[0, \sqrt{3}]$.

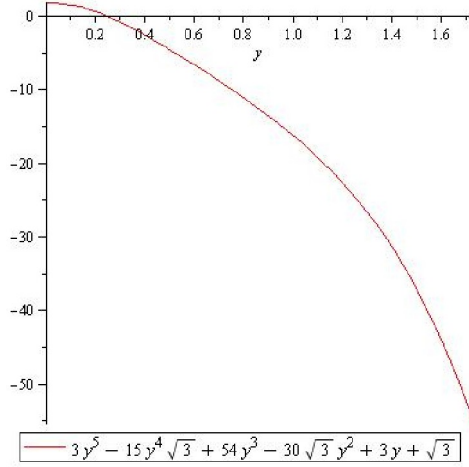


FIGURE 6. Graph of $3y^5 - 15\sqrt{3}y^4 + 54y^3 - 30\sqrt{3}y^2 + 3y + \sqrt{3}$

Figure 5 shows the only zero of $f(y)$ for $y \in (0, \sqrt{3})$ at approximately 0.2485. Since $(0, \frac{\sqrt{3}}{3})$ is the center of the triangle, two rotations of 120 degrees about $(0, \frac{\sqrt{3}}{3})$ will give two other points at which the force vanishes, for a total of four.

Observe that the three altitudes of the equilateral triangle partition it into six congruent right triangles. Now assume that the force vanishes at some point p in the interior of one of these right triangles. By reflecting p about the y -axis, and then applying the two rotations, we see that assuming the existence of one point p actually gives us six more equilibrium points in the force, for a total of ten. By [6], we know that any configuration of three charges cannot have more than twelve equilibrium points; therefore, there is at most one p in each of these right triangles. Consider the x -component of the forces at p in the triangle determined by $0 < x < 1$ and $0 < y < -\frac{\sqrt{3}}{3}x + \frac{\sqrt{3}}{3}$.

$$F_x = \frac{x}{[x^2 + (y - \sqrt{3})^2]^{3/2}} + \frac{x+1}{[(x+1)^2 + y^2]^{3/2}} + \frac{x-1}{[(x-1)^2 + y^2]^{3/2}}$$

Note that the bound $0 < y < -\frac{\sqrt{3}}{3}x + \frac{\sqrt{3}}{3}$ implies $0 > -y > \frac{\sqrt{3}}{3}x - \frac{\sqrt{3}}{3}$. Observe then that

$$\begin{aligned}
x^2 + (y - \sqrt{3})^2 &= x^2 + y^2 - 2\sqrt{3}y + 3 \\
&= x^2 + y^2 + 2\sqrt{3}(-y) + 3 \\
&> x^2 + y^2 + 2\sqrt{3}\left(\frac{\sqrt{3}}{3}x - \frac{\sqrt{3}}{3}\right) + 3 \\
&= x^2 + y^2 + 2x + 1 \\
&= (x + 1)^2 + y^2.
\end{aligned}$$

So $x^2 + (y - \sqrt{3})^2 > (x + 1)^2 + y^2$, and therefore $\frac{x}{[x^2 + (y - \sqrt{3})^2]^{3/2}} < \frac{x}{(x + 1)^2 + y^2}$. Substituting this result yields

$$(6.3) \quad F_x < \frac{2x + 1}{(x + 1)^2 + y^2} + \frac{x - 1}{[(x - 1)^2 + y^2]^{3/2}}.$$

Claim: $\frac{2x+1}{(x+1)^2+y^2} < \frac{1-x}{[(x-1)^2+y^2]^{3/2}}$.

Proof of Claim: Notice that along the line $x = 0$, equality holds. However, we are only considering $0 < x < 1$; also, notice that this bound on x gives that $0 < y < \frac{1}{\sqrt{3}}$. It is trivial to see that $(x + 1)^2 + y^2 > (x - 1)^2 + y^2$. Applying these inequalities gives

$$\begin{aligned}
\frac{\partial}{\partial x} \left(\frac{2x + 1}{[(x + 1)^2 + y^2]^{3/2}} \right) &= \frac{-4x^2 - 5x - 1 + 2y^2}{[(x + 1)^2 + y^2]^{5/2}} \\
&< \frac{-4x^2 - 5x - \frac{1}{3}}{[(x + 1)^2 + y^2]^{5/2}} \\
&< \frac{2x^2 - 4x + 2 - \frac{1}{3}}{[(x + 1)^2 + y^2]^{5/2}} \\
&< \frac{2x^2 - 4x + 2 - y^2}{[(x + 1)^2 + y^2]^{5/2}} \\
&< \frac{2(x - 1)^2 - y^2}{[(x - 1)^2 + y^2]^{5/2}} \\
&= \frac{\partial}{\partial x} \left(\frac{1 - x}{[(x - 1)^2 + y^2]^{3/2}} \right).
\end{aligned}$$

Since $\frac{\partial}{\partial x} \left(\frac{2x+1}{[(x+1)^2+y^2]^{3/2}} \right) < \frac{\partial}{\partial x} \left(\frac{1-x}{[(x-1)^2+y^2]^{3/2}} \right)$, and $\frac{2x+1}{[(x+1)^2+y^2]^{3/2}} = \frac{1-x}{[(x-1)^2+y^2]^{3/2}}$ when $x = 0$, then at no point inside the triangle can equality hold. Therefore $\frac{2x+1}{[(x+1)^2+y^2]^{3/2}} < \frac{1-x}{[(x-1)^2+y^2]^{3/2}}$, and the Claim is proven.

Substituting this result into (6.3) gives $F_x < 0$ at every point inside of the triangle determined by $0 < x < 1$ and $0 < y < -\frac{\sqrt{3}}{3}x + \frac{\sqrt{3}}{3}$. Therefore the force cannot vanish

inside this triangle; by symmetry, none of these six triangles can have additional points where the force vanishes.

Therefore, there are exactly four point equilibrium points.

Proof of Theorem 4.2: Without loss of generality, assume that q_1 , q_2 , and q_3 are three positive charges. Also, we can through re-scaling and translation place q_1 at the origin, q_2 at $(1, 0)$, and q_3 in the upper half-plane. Let \vec{v} be a vector having the following properties:

- 1.) The origin of \vec{v} is on the line joining q_1 and q_2 .
- 2.) \vec{v} is perpendicular to the line joining q_1 and q_2 .
- 3.) \vec{v} passes through the interior of the triangle.

Recall that the potential is defined by $P(x, y) = \sum_{k=1}^3 \frac{q_k}{\sqrt{(x-x_k)^2+(y-y_k)^2}}$. The parametric equations that describe the vector \vec{v} are

$$x(t) = c$$

$$y(t) = t$$

where $0 < c < 1$, and c is a constant. By the chain rule we have that

$$\frac{\partial P}{\partial t} = \frac{\partial P}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial P}{\partial y} \frac{\partial y}{\partial t}.$$

We have already calculated that

$$\frac{\partial P}{\partial x} = \sum_{k=1}^3 \frac{-q_k(x-x_k)}{[(x-x_k)^2+(y-y_k)^2]^{3/2}}$$

and

$$\frac{\partial P}{\partial y} = \sum_{k=1}^3 \frac{-q_k(y-y_k)}{[(x-x_k)^2+(y-y_k)^2]^{3/2}}$$

By our choice of \vec{v} , we have that $\frac{\partial x}{\partial t} = 0$ and $\frac{\partial y}{\partial t} = 1$ for all (x, y) on \vec{v} . With this construction, we have that t is the distance between (x, y) on \vec{v} and the line joining q_1 and q_2 .

Substitution gives

$$\begin{aligned}
\frac{\partial P}{\partial t} &= \sum_{k=1}^3 \frac{-q_k(y - y_k)}{[(x - x_k)^2 + (y - y_k)^2]^{3/2}} \\
&= -\frac{q_1 y}{[x^2 + y^2]^{3/2}} - \frac{q_2 y}{[(x - 1)^2 + y^2]^{3/2}} - \frac{q_3(y - y_3)}{[(x - x_3)^2 + (y - y_3)^2]^{3/2}} \\
&= -\frac{q_1 t}{[c^2 + t^2]^{3/2}} - \frac{q_2 t}{[(c - 1)^2 + t^2]^{3/2}} - \frac{q_3(t - y_3)}{[(c - x_3)^2 + (t - y_3)^2]^{3/2}}.
\end{aligned}$$

For $t > 0$, the first two terms of this equation are negative, but are increasing to zero. This implies that the first two terms of the potential are decreasing to zero. For (x, y) inside of the triangle, $t - y_3 < 0$, so the third term of the potential is strictly increasing.

Let $P(t) = f(t) + g(t)$, where $f(t)$ corresponds to the first two terms of the potential and $g(t)$ is the q_3 term of the potential. Then $f'(t) = -\frac{q_1 t}{[c^2 + t^2]^{3/2}} - \frac{q_2 t}{[(c-1)^2 + t^2]^{3/2}}$. In order to prove this theorem, we first want to show that $f'(t)$ has only one min. A direct calculation gives that

$$(6.4) \quad f''(t) = \frac{q_1(2t^2 - c^2)}{(t^2 + c^2)^{5/2}} + \frac{q_2(2t^2 - (c - 1)^2)}{(t^2 + (c - 1)^2)^{5/2}}.$$

Case 1: Let $c = \frac{1}{2}$. Then setting (6.4) equal to zero becomes

$$\begin{aligned}
\frac{q_1(2t^2 - \frac{1}{4})}{(t^2 + \frac{1}{4})^{5/2}} + \frac{q_2(2t^2 - \frac{1}{4})}{(t^2 + \frac{1}{4})^{5/2}} &= 0 \\
\frac{(q_1 + q_2)(2t^2 - \frac{1}{4})}{(t^2 + \frac{1}{4})^{5/2}} &= 0 \\
2t^2 - \frac{1}{4} &= 0 \\
t^2 &= \frac{1}{8} \\
t &= \sqrt{\frac{1}{8}}
\end{aligned}$$

So for $c = \frac{1}{2}$, which is the perpendicular bisector of the line joining q_1 and q_2 , we see that $f'(t)$ has only one min.

Case 2: Assume that $c < \frac{1}{2}$. Setting (6.4) equal to zero and rearranging terms gives

$$\left(\frac{q_1}{q_2}\right) \frac{2t^2 - c^2}{2t^2 - (c - 1)^2} = -\left[\frac{t^2 + c^2}{t^2 + (c - 1)^2}\right]^{5/2}.$$

Let $h(t) = \frac{q_1}{q_2} \frac{2t^2 - c^2}{2t^2 - (c-1)^2}$, and $j(t) = -[\frac{t^2 + c^2}{t^2 + (c-1)^2}]^{5/2}$. Observe some obvious properties of these functions:

$$\lim_{t \rightarrow \infty} h(t) = \frac{q_1}{q_2}.$$

$$\lim_{t \rightarrow \infty} j(t) = -1.$$

$$j(t) < 0 \text{ for all } t > 0.$$

$$h(t) = 0 \text{ at } t = \frac{c}{\sqrt{2}}.$$

$$h(t) \text{ has a vertical asymptote at } t^* = \sqrt{\frac{(c-1)^2}{2}}.$$

If $t > t^*$ then $\frac{2t^2 - c^2}{2t^2 - (c-1)^2} > \frac{(c-1)^2 - c^2}{2t^2 - (c-1)^2} = \frac{1-2c}{2t^2 - (c-1)^2} > 0$; so for $t > t^*$, we have that $h(t) > 0$. Also, we see that $\lim_{t \rightarrow t^{*+}} h(t) = +\infty$.

If $t < \frac{c}{\sqrt{2}}$, then $\frac{2t^2 - c^2}{2t^2 - (c-1)^2} > \frac{2t^2 - c^2}{c^2 - (c-1)^2} = \frac{2t^2 - c^2}{2c-1} > 0$; so for $0 < t < \frac{c}{\sqrt{2}}$ we have that $h(t) > 0$.

If $\frac{c}{\sqrt{2}} < t < t^*$, then by applying the lower bound we get $\frac{2t^2 - c^2}{2t^2 - (c-1)^2} < \frac{2t^2 - c^2}{c^2 - (c-1)^2} = \frac{2t^2 - c^2}{2c-1} < 0$; or similarly using the upper bound we get $\frac{2t^2 - c^2}{2t^2 - (c-1)^2} < \frac{(c-1)^2 - c^2}{2t^2 - (c-1)^2} = \frac{1-2c}{2t^2 - (c-1)^2} < 0$; so for $\frac{c}{\sqrt{2}} < t < t^*$, we have $h(t) < 0$. Also, we have that $\lim_{t \rightarrow t^{*-}} h(t) = -\infty$.

Since $j(t) < 0$ for all t , then any point where $h(t)$ and $j(t)$ are equal must have $t^2 \in (\frac{c^2}{2}, \frac{(c-1)^2}{2})$.

A direct calculation shows that

$$h'(t) = \frac{q_1}{q_2} \frac{-4t(1-2c)}{[2t^2 - (c-1)^2]^2}$$

and

$$j'(t) = [\frac{t^2 + c^2}{t^2 + (c-1)^2}]^{3/2} [\frac{-5t(1-2c)}{[t^2 + (c-1)^2]^2}].$$

Clearly $h'(t) < 0$ and $j'(t) < 0$, so both functions are monotonically decreasing. If $h'(t) < j'(t)$ on some interval, then there is at most one point of intersection of the graphs of $h(t)$ and $j(t)$.

Notice that for $c < \frac{1}{2}$ that $t^2 + c^2 < t^2 + \frac{1}{4}$ and $t^2 + (c-1)^2 > t^2 + \frac{1}{4}$, so $\frac{t^2 + c^2}{t^2 + (c-1)^2} < 1$, and therefore $(\frac{t^2 + c^2}{t^2 + (c-1)^2})^{3/2} < 1$. Therefore, $h'(t) = \frac{q_1}{q_2} \frac{-4t(1-2c)}{(2t^2 - (c-1)^2)^2} <$

$\frac{q_1}{q_2} \frac{-4t(1-2c)}{(2t^2-(c-1)^2)^2} \left(\frac{t^2+c^2}{t^2+(c-1)^2}\right)^{3/2}$. In order to have $h'(t) < j'(t)$ on some interval, we need

$$\begin{aligned} h'(t) &= \frac{q_1}{q_2} \frac{-4t(1-2c)}{(2t^2-(c-1)^2)^2} \\ &< \frac{q_1}{q_2} \frac{-4t(1-2c)}{(2t^2-(c-1)^2)^2} \left(\frac{t^2+c^2}{t^2+(c-1)^2}\right)^{3/2} \\ &< \left[\frac{t^2+c^2}{t^2+(c-1)^2}\right]^{3/2} \left[\frac{-5t(1-2c)}{(t^2+(c-1)^2)^2}\right] \\ &= j'(t) \end{aligned}$$

For that purpose we must investigate for which t we have $\frac{q_1}{q_2} \frac{-4t(1-2c)}{(2t^2-(c-1)^2)^2} < \frac{-5t(1-2c)}{(t^2+(c-1)^2)^2}$.

$$\begin{aligned} \frac{q_1}{q_2} \frac{-4t(1-2c)}{(2t^2-(c-1)^2)^2} &< \frac{-5t(1-2c)}{(t^2+(c-1)^2)^2} \\ -4\frac{q_1}{q_2}(t^2+(c-1)^2)^2 &< -5(2t^2-(c-1)^2)^2 \\ 5(2t^2-(c-1)^2)^2 &< 4\frac{q_1}{q_2}(t^2+(c-1)^2)^2 \\ \sqrt{5}(2t^2-(c-1)^2) &< 2\sqrt{\frac{q_1}{q_2}}(t^2+(c-1)^2) \\ 2\sqrt{5}t^2 - \sqrt{5}(c-1)^2 &< 2\sqrt{\frac{q_1}{q_2}}t^2 + 2\sqrt{\frac{q_1}{q_2}}(c-1)^2 \\ 2(\sqrt{5} - \sqrt{\frac{q_1}{q_2}})t^2 &< 2(\sqrt{5} + \sqrt{\frac{q_1}{q_2}})(c-1)^2 \\ (\sqrt{5} - \sqrt{\frac{q_1}{q_2}})t^2 &< (\sqrt{5} + \sqrt{\frac{q_1}{q_2}})(c-1)^2 \end{aligned}$$

If $\sqrt{5} - \sqrt{\frac{q_1}{q_2}} < 0$ then $t^2 > \left(\frac{\sqrt{5} + \sqrt{\frac{q_1}{q_2}}}{\sqrt{5} - \sqrt{\frac{q_1}{q_2}}}\right)(c-1)^2$, which holds for all t since the right hand side is negative. If $\sqrt{5} - \sqrt{\frac{q_1}{q_2}} > 0$, then this gives $t^2 < \left(\frac{\sqrt{5} + \sqrt{\frac{q_1}{q_2}}}{\sqrt{5} - \sqrt{\frac{q_1}{q_2}}}\right)(c-1)^2$. Since $\frac{\sqrt{5} + \sqrt{\frac{q_1}{q_2}}}{\sqrt{5} - \sqrt{\frac{q_1}{q_2}}} > 1$, then we see that $\left(\frac{\sqrt{5} + \sqrt{\frac{q_1}{q_2}}}{\sqrt{5} - \sqrt{\frac{q_1}{q_2}}}\right)(c-1)^2 > (c-1)^2 > \frac{(c-1)^2}{2}$.

So for all t such that $t^2 < \left(\frac{\sqrt{5} + \sqrt{\frac{q_1}{q_2}}}{\sqrt{5} - \sqrt{\frac{q_1}{q_2}}}\right)(c-1)^2$, we have that $h'(t) < j'(t)$. But we only needed to consider $t^2 \in \left(\frac{c^2}{2}, \frac{(c-1)^2}{2}\right)$. Therefore, there is at most one point of intersection of $h(t)$ and $j(t)$, and, therefore, (6.4) has only one zero, so $f'(t)$ has only one minimum when $c < \frac{1}{2}$.

This shows that for all $c < 1/2$, the potential on the segment of the line defined by vector \vec{v} assumes at most one minimum (excluding endpoints). However, by reflecting the charges about the line $x = \frac{1}{2}$, we see that this result holds for all c .

A direct calculation gives us

$$f'''(t) = -\frac{3q_1t(2t^2 - 3c^2)}{(c^2 + t^2)^{7/2}} - \frac{3q_2t(2t^2 - 3(c-1)^2)}{[(c-1)^2 + t^2]^{7/2}}$$

and we see that although $f'''(0) = 0$, for $t > 0$ sufficiently small we have $f'''(t) > 0$, so initially $f'(t)$ is concave up.

Also, note that $g''(t) = \frac{q_3[2(t-y_3)^2 - (c-x_3)^2]}{[(t-y_3)^2 + (c-x_3)^2]^{5/2}}$, so the maximum of $g'(t)$ must have

$$\begin{aligned} q_3[2(t-y_3)^2 - (c-x_3)^2] &= 0 \\ (t-y_3)^2 &= \frac{(c-x_3)^2}{2} \\ t &= y_3 \pm \frac{c-x_3}{\sqrt{2}} \end{aligned}$$

Clearly, one of these solutions is greater than y_3 , so $g'(t)$ has at most one maximum on \vec{v} .

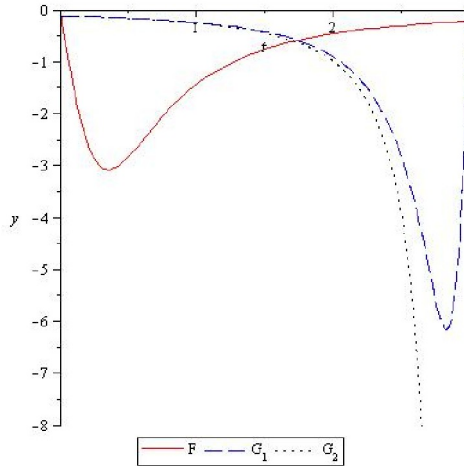


FIGURE 7. The graphs of f' and $-g'$. For this particular example, all charges are equal, and q_3 is located at $(\frac{1}{4}, 3)$. G_1 uses $c = \frac{1}{2}$, while G_2 uses $c = \frac{1}{4}$, the altitude of the triangle.

We now need the following lemmas.

Lemma 6.3. *Let $F(x) < 0$ for all $x > 0$, $F(0) = 0$, $\lim_{x \rightarrow \infty} F(x) = 0$, $F(x)$ has exactly one min, initially $F(x)$ is convex up, $G(0) < 0$, $G(x)$ is concave down and decreasing for all $x > 0$, $F(x)$ and $G(x)$ are both smooth functions. Then the graphs of $F(x)$ and $G(x)$ intersect at most twice.*

Proof of Lemma 6.3: Let x_0 be the min of F , and let x_1 be the first intersection of F and G . It is clear that if F and G intersect at some $x > x_0$, then because F is increasing and G is decreasing we must have $G > F$ approaching this intersection point. But by the initial conditions we have $G < F$, implying that F and G intersect at some $x < x_0$.

Observe that the Taylor series of F and G about an arbitrary point c are

$$F(c) + F'(c)(x - c) + F''(c)(x - c)^2$$

and

$$G(c) + G'(c)(x - c) + G''(c)(x - c)^2.$$

Let $x_1 < x_0$ be the first such point of intersection. If G is tangent to F at this point, then observe that since G is concave down, that $G'' < 0$, and since F is convex up we have that $F'' > 0$. It follows then that for $x > x_1$, even though both $F' < 0$ and $G' < 0$, we have F' is increasing to 0. Therefore this is the only point of intersection.

Next, assume that G is transversal to F . Because $F' \rightarrow 0$ and since G is decreasing, it follows from the Intermediate Value Theorem that F and G will intersect at least once more. Call this point x_2 . Assume that $x_2 > x_0$. Then it is trivial that x_2 is the only point of intersection because F is increasing and G is decreasing.

Next, assume that $x_2 \leq x_0$. Again, by the Taylor series we have that $G'' < 0$, $F'' > 0$, F' is increasing to 0, so x_2 is the only point of intersection.

Therefore there is at most two points of intersection.

Lemma 6.4. *Let $F(x) < 0$ for all $x > 0$, $F(0) = 0$, $\lim_{x \rightarrow \infty} F(x) = 0$, $F(x)$ has exactly one min, initially $F(x)$ is convex up, $F(x)$ switches concavity once after its min, $G(0) < 0$ for all $0 < x < p$, $G(x)$ has at most one minimum, $G(p) = 0$ for some $p > 0$, $F(x)$ and $G(x)$ are both smooth functions. Then the graphs of $F(x)$ and $G(x)$ intersect at most twice.*

Proof of Lemma 6.4: By the previous lemma, F and G can intersect at most two times before F takes its minimum at x_0 . Assume this is the case, and therefore $G < F$ after two intersections. Since $G(p) = 0$, G must take its minimum and therefore is concave up. By the same arguments of the previous lemma there can be at most two intersections of F and G before F switches concavity. But again this leaves $G < F$, and since $G(p) = 0$ and $\lim_{x \rightarrow \infty} F(x) = 0$, we must have $G = f$ once more. Q.E.D.

At $t = 0$, we have $f'(t) = 0$ and $g'(t) > 0$, so initially the potential will increase. Since \vec{v} is orthogonal to the boundary, we see that the inward normal derivative of the potential is always positive.

If $g'(t) > |f'(t)|$ for all t , then the potential is monotonically increasing on \vec{v} . However, if there exists a t^* such that $|f'(t^*)| > g'(t^*)$, since $f'(t)$ takes only one minimum we see that there are at most two intervals on which the potential will decrease. This follows from the lemmas. If we let $F = f'$ and $G = -g'$, then if we choose \vec{v} to be an altitude of the triangle we have the conditions of Lemma 6.3. For any other \vec{v} we have the conditions of Lemma 6.4.

Since the potential is initially increasing, the first point such that $f'(t) = -g'(t)$ must be a maximum. If \vec{v} was chosen to be an altitude of the triangle, then by Lemma 6.3 there is exactly one minimum. For any other \vec{v} consider the following: since the potential is decreasing on \vec{v} at y_3 , we must have that the last critical point is a maximum. If there are three critical points, we see that there are two maximum and one minimum. There is no way that we can have four critical points because the second and third would have to be minimums, which is impossible. Five critical points implies three maximums and two minimums.

The same proof applies to negative charges.

Proof of Theorem 5.1: Without loss of generality, let $k = 1$. Also, since $\log \left| \frac{1}{z-z_k} \right| = -\log |z - z_k|$, we only need to consider the zeros of $\log |z - z_k|$.

Note that $\log |z - z_0| = \frac{1}{2} \log (z - z_0) + \frac{1}{2} \log (\bar{z} - \bar{z}_0) = \frac{1}{2} \log (x + iy - z_0) + \frac{1}{2} \log (x - iy - \bar{z}_0)$. So $P(x, y) = \sum_{k=1}^n -\frac{q_k}{2} \log (x + iy - z_k) - \frac{q_k}{2} \log (x - iy - \bar{z}_k)$. Since $F = -\nabla P = -\frac{\partial P}{\partial x} - i \frac{\partial P}{\partial y}$, and

$$\frac{\partial P}{\partial x} = -\frac{1}{2} \sum_{k=1}^2 \frac{q_k(2x - z_k - \bar{z}_k)}{(x + iy - z_k)(x - iy - \bar{z}_k)}$$

and

$$\frac{\partial P}{\partial y} = -\frac{i}{2} \sum_{i=1}^n \frac{(q_k(2iy - z_k - \bar{z}_k))}{(x + iy - z_k)(x - iy - \bar{z}_k)},$$

then we see that $F = \sum_{i=1}^n \frac{q_i}{\bar{z} - \bar{z}_k}$.

So if $P = \sum_{i=1}^n q_i \log \left| \frac{1}{z-z_i} \right|$, then $F = -\nabla P$ reduces to

$$\begin{aligned} \sum_{k=1}^n \frac{q_k}{\bar{z} - \bar{z}_k} &= \frac{q_1}{\bar{z} - \bar{z}_1} + \frac{q_2}{\bar{z} - \bar{z}_2} + \dots + \frac{q_n}{\bar{z} - \bar{z}_n} \\ &= \frac{q_1(\bar{z} - \bar{z}_2)(\bar{z} - \bar{z}_3) \cdots (\bar{z} - \bar{z}_n) + q_2(\bar{z} - \bar{z}_1)(\bar{z} - \bar{z}_3) \cdots (\bar{z} - \bar{z}_n)}{(\bar{z} - \bar{z}_1)(\bar{z} - \bar{z}_2) \cdots (\bar{z} - \bar{z}_n)} \\ &\quad + \dots + \frac{q_n(\bar{z} - \bar{z}_1)(\bar{z} - \bar{z}_2) \cdots (\bar{z} - \bar{z}_{n-1})}{(\bar{z} - \bar{z}_1)(\bar{z} - \bar{z}_2) \cdots (\bar{z} - \bar{z}_n)}. \end{aligned}$$

Since the numerator is the conjugate of a polynomial of degree $n - 1$, then by the Fundamental Theorem of Algebra, F has exactly $n - 1$ solutions.

Proof of Theorem 5.2: Before proving this theorem, we need the following lemmas and definition.

Lemma 6.5. *Green's Theorem: Let D be a simply connected region with a piecewise smooth boundary C , oriented counterclockwise. If $F(x, y) = M(x, y)\vec{i} + N(x, y)\vec{j}$, and M and N have continuous partial derivatives in D , then $\int_C F \cdot \vec{n} ds = \int \int_D \nabla \cdot F dA$, where \vec{n} is the outward unit normal vector.*

For a proof, see [3], p. 1025. Notice that $\nabla \cdot F = \nabla \cdot \left(-\frac{\partial P}{\partial x} \vec{i} - \frac{\partial P}{\partial y} \vec{j} \right) = -\Delta P$.

Definition. *A singular point of a function f is a point (a, b) where the gradient of f (and possibly higher order derivatives) vanish; that is, $f_x(a, b) = f_y(a, b) = 0$ [8].*

It is clear that if the force vanishes on a curve, then every point of this curve is a singular point.

The following theorem of Puiseux found in [8], pp. 95, 111, shows that in a neighborhood of a singular point, the curve, through a proper change of coordinates, has a parametrization that is homeomorphic in some (possibly smaller) neighborhood to $y^p = x^q, p, q \in \mathbb{Z}$.

Lemma 6.6. *In a suitable coordinate system any given parametrization is equivalent to one of the type*

$$x = t^n, y = a_1 t^{n_1} + a_2 t^{n_2} + \dots$$

where $0 < n, 0 < n_1 < n_2 < \dots$

For a proof, see [8]. This theorem shows that the algebraic curve on which the force vanishes may only be of the form $y^p = x^q$ at a singular point (which is every point of the curve). If $p/q \in \mathbb{Z}$, then the curve is smooth. Otherwise, if $p/q \notin \mathbb{Z}$,

then the curve has a cusp at the singular point, but does not terminate. The singular point may be a point where the function branches (consider the origin in the graph of $(x^2 + y^2)^3 - 4x^2y^2 = 0$), but the branches form loops or go to infinity. It is not possible to have the function terminate at a singular point. Therefore, the curves where the force vanishes can either go to ∞ or are closed loops.

Proof of Theorem 5.2: Without loss of generality, assume that all of the charges are positive. Assume that C is an equipotential curve on which the force vanishes. There are three cases to consider:

Case 1: C is a curve that goes to infinity.

Since the $\lim_{(x,y) \rightarrow \infty} P = 0$, then by the maximum principle we must have $P = 0$ at every point on C . Otherwise, assume that $P = k > 0$ on C , and choose an $p = (x, y)$ on C sufficiently far from the three charges. Fix $\varepsilon > 0$, and let $B_\varepsilon(p)$ be a ball of radius ε centered at p . Let p' be any other point in $B_\varepsilon(p)$. If p' is on C , then $P(p') = k$. Otherwise, since $\lim_{(x,y) \rightarrow \infty} P = 0$ we have that $P(p') \rightarrow 0$ for all p' not on C , so $P(p') < k$. Therefore p is a local maximum, which violates the maximum principle. In order to not have a contradiction with the maximum principle, we need $P = 0$ on C .

But recall that by definition, the potential is the sum of three non-zero terms. Therefore $P \neq 0$ anywhere in the plane. Thus C cannot be a curve through the plane.

Case 2: C is a loop containing some or all of the charges.

It is obvious that if C contains all of the charges that the force cannot vanish on C because the outward normal derivative of the potential is negative. So we are left with the possibility that C is an equipotential loop that contains one or two of the charges. The portion of C outside of the triangle will not have vanishing force. This leaves a curve of finite length inside of the triangle on which the force vanishes. However, by Lemma 6.6, the parametrization of the portion of C on which the force vanishes must be a closed loop. Therefore, C cannot contain any of the charges.

Case 3: C is a closed loop contained in the interior of the triangle.

By Proposition 1.2, we know that $-\Delta P < 0$ what all charges are positive. If C is a closed curve on which the electrostatic force vanishes at every point, then $F \cdot \vec{n} = 0$

at every point of C . Applying Green's Theorem, we get

$$\begin{aligned} 0 &= \int_C F \cdot \vec{n} ds \\ &= \int \int_D \nabla \cdot F dA \\ &= \int \int_D -\Delta P dA \\ &< 0. \end{aligned}$$

This contradiction shows that C cannot be a closed loop in the interior of the triangle.

Proof of Theorem 5.3: Assume force vanishes along a curve S in the plane, then at every point on S we would have $F_x = 0$ and $F_y = 0$. Consider y to be a function of x , and recall that $F_x = -\frac{\partial P}{\partial x}$, and so implicit differentiation on $F_x = 0$ we get

$$-P_{xx} - P_{xy}y'_1 = 0.$$

Similarly, implicit differentiation on $F_y = 0$ gives

$$-P_{yx} - P_{yy}y'_2 = 0.$$

Solving each of these for y' gives

$$y'_1 = -\frac{P_{xx}}{P_{xy}}$$

and

$$y'_2 = -\frac{P_{yx}}{P_{yy}}.$$

Along a curve where the force vanishes, we must have that $y'_1 = y'_2$, which, after re-arranging terms and using the fact that $P_{xy} = P_{yx}$, becomes

$$(6.5) \quad P_{xx}P_{yy} = (P_{xy})^2.$$

Notice then that along S , we get the relation that the product $P_{xx}P_{yy}$ must be non-negative. Obviously, one way this condition can be satisfied is if $P_{xy} = 0$ at every point of S .

We have already seen that a direct calculation gives

$$P_{xx} = \sum_{k=1}^n \frac{q_k [2(x - x_k)^2 - (y - y_k)^2]}{[(x - x_k)^2 + (y - y_k)^2]^{5/2}}$$

$$P_{yy} = \sum_{k=1}^n \frac{q_k [-(x-x_k)^2 + 2(y-y_k)^2]}{[(x-x_k)^2 + (y-y_k)^2]^{5/2}}$$

and

$$P_{xy} = 3 \sum_{k=1}^n \frac{q_k (x-x_k)(y-y_k)}{[(x-x_k)^2 + (y-y_k)^2]^{5/2}}.$$

Define

$$X = \sum_{k=1}^n \frac{q_k (x-x_k)^2}{[(x-x_k)^2 + (y-y_k)^2]^{5/2}}$$

and

$$Y = \sum_{k=1}^n \frac{q_k (y-y_k)^2}{[(x-x_k)^2 + (y-y_k)^2]^{5/2}}.$$

We can write $P_{xx} = 2X - Y$ and $P_{yy} = -X + 2Y$. Notice that both X and Y are strictly positive.

We wish to find under what conditions $P_{xx}P_{yy} \geq 0$.

Case 1: When all charges are positive we know that the potential function is subharmonic; that is,

$$(6.6) \quad \Delta P = P_{xx} + P_{yy} \geq 0.$$

It is trivial then to see that $P_{xx} > 0$ and $P_{yy} > 0$ will satisfy the subharmonic condition. Notice, though, that if we write $P_{xx} = 2X - Y > 0$, then we get that $Y < 2X$. Also, $P_{yy} = -X + 2Y > 0$ gives that $Y > \frac{1}{2}X$. So we see that $\frac{1}{2}X < Y < 2X$.

Case 2: It is trivial that we cannot have both $P_{xx} < 0$ and $P_{yy} < 0$, so assume that $P_{xx} > 0$ and $P_{yy} < 0$, but $|P_{xx}| \geq P_{yy}$ to maintain the subharmonic inequality. But (6.5) says that the product $P_{xx}P_{yy} \geq 0$. Therefore, at no point of S can we have that $P_{xx} > 0$ and $P_{yy} < 0$, or vice versa. If $P_{xx} < 0$ at some point of S , then at that point we must have that both $P_{yy} = P_{xy} = 0$ in order for (6.5) to hold. However, if $P_{xx} < 0$ and $P_{yy} = 0$, then (6.6) is violated. Therefore, at no point of S can either $P_{xx} > 0$ or $P_{yy} > 0$.

Case 3: Assume that $P_{xx} = P_{yy} = 0$. Then we have the system of equations

$$\begin{aligned} 2X - Y &= 0 \\ -X + 2Y &= 0. \end{aligned}$$

The only solution to this system is $X = Y = 0$, which is not possible because X and Y were defined to be strictly positive. Therefore, P_{xx} and P_{yy} cannot both equal zero at any point of S .

Numerical method for finding zeros of the potential: The calculations for this paper were done using Maple 11. Here we discuss the program which finds zeros in the force of the equilateral triangle. With slight modification, this program can be used to find zeros of any configuration of any number of point charges in the plane or in space.

In order to plot and analyze a potential field, the positions of the point charges and their corresponding values must be established. For the equilateral triangle, the charges all had value +1, and were located at $(-1, 0)$, $(1, 0)$, and $(0, \sqrt{3})$. To place these charges on the plane using Maple, use the following command:

```
q[1] := 1; q[2] := 1; q[3] := 1;
x[1] := -1; y[1] := 0;
x[2] := 1; y[2] := 0;
x[3] := 0; y[3] := sqrt(3);
```

Note: semicolons can be replaced with colons to suppress output.

To define the potential as a function, use the command:

```
P := (x, y) -> sum(q[i]/sqrt((x - x[i])2 + (y - y[i])2), i = 1..3);
```

The force equations are defined by taking the derivative of $P(x, y)$ with respect to the two variables, and then negating.

```
Fx := -diff(P(x, y), x);
Fy := -diff(P(x, y), y);
```

Now we need the program to solve the system of equations

$$Fx = 0$$

$$Fy = 0.$$

Unfortunately, Maple's built-in **solve** command is sometimes unable to solve this system for all solutions. In some cases, it may return some solutions, but often it will generate a warning that solutions may have been lost. However, **fsolve** is a procedure that will attempt to find a solution over a specified range of the variable(s). The drawback is that **fsolve** will many times only return the first solution it finds. If **fsolve** is unable to find any solutions, it will return the calling command.

In order to have **fsolve** find all solutions, we need to partition the portion of the plane we wish to study into smaller squares to which **fsolve** can be applied. Given that

our charges are located at $(-1, 0)$, $(1, 0)$, and $(0, \sqrt{3})$, the square region $[-1, 1] \times [0, 2]$ contains the entire triangle and therefore all possible solutions.

Choose n to be the number of partitions. Let $d = 2/n$ be the length of the partitioned squares (the 2 comes from the length of the square region that is being partitioned). The following code will have **fsolve** check each of the squares for solutions and output them in a set. The following example uses $n = 20$ for the number of partitions.

```

n:=20:
SolutionSet:={}:
d:=2/n:
for i from 0 to n-1 do
for j from 0 to n-1 do
p:=fsolve([Fx=0, Fy=0], x=-1+i*d..-1+(i+1)*d, y=j*d..(j+1)*d):
if member({x=-1+i*d..-1+(i+1)*d, y=j*d..(j+1)*d}, p) then
next:
else
SolutionSet:= SolutionSet union {p}:
end if:
end do:
end do:
SolutionSet;

```

Line 7 is necessary for suppressing unwanted output. The Boolean procedure **member** checks if the output contained a portion of the calling command. If it does, then **fsolve** was unable to find a solution, and Line 8 iterates the loop. If Line 7 returned “false”, then Line 10 adds the solution to the *SolutionSet*.

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