Investigating Proportional Reasoning in a University Quantitative Literacy Course

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Abstract

The ability to reason with proportions is known to take a long time to develop and to be difficult to learn. We regard proportional reasoning (the ability to reason about quantities in relative terms) as a threshold concept for academic quantitative literacy. Our study of the teaching and learning of proportional reasoning in a university quantitative literacy course for law students consisted of iterative action research, in which we introduced various teaching interventions and analysed students' written responses to assessment questions requiring students to explain their reasoning in situations that call for proportional reasoning. For this analysis we used a modified phenomenographic method to develop and refine a framework to code the responses. This enabled us to broadly describe the responses in terms of the concept of the liminal space that a student must traverse in coming to a full understanding of a threshold concept, and to further define the liminal space to facilitate finer description of students' responses. Our latest analysis confirmed that many university students cannot reason with proportions, that this kind of thinking is difficult to learn, and that it takes more time than is available in a one-semester course. The context and structure of the questions have a marked effect on students' ability to apply proportional reasoning successfully. The fraction of students who were classified as 'at or over the threshold' (i.e., fairly competent at proportional reasoning) after instruction ranged between 8% for the most difficult question and 48% for the easiest.

Keywords

proportional reasoning, quantitative literacy, numeracy, threshold concepts, verbal reasoning

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Cover Page Footnote

Vera Frith is the coordinator of the Numeracy Centre, a unit within the Centre for Higher Education Development at the University of Cape Town. Her primary interests are the quantitative literacy development of university students and the appropriate curriculum for this purpose.

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Introduction

Research into the development of proportional reasoning of children and adolescents has been carried out over the last half-century, especially since Piaget's theory established proportional reasoning as a hallmark of the formal operations stage of development of thinking (Inhelder and Piaget 1958). Tourniaire and Pulos (1985) reviewed of the literature of the previous 25 years, noting that the body of research has many gaps, lacks cohesiveness and is difficult to apply to mathematics education. In the late 1980s and early 1990s the Rational Number Project resulted in the publication of numerous papers on proportional reasoning and related topics (for example, Lesh, Post and Behr 1988, Harel et al. 1991, Cramer, Post and Currier 1993). However, in looking back fifteen years and reviewing the work done by some of those researchers, Lamon (2007) lamented the small number of researchers engaged in long-term research agendas in the field and, in proposing a theoretical framework for research into rational numbers and proportional reasoning, encouraged further research. In her work on the challenges in the transition from whole number to rational number concepts, Long (2009) was interested in children's learning and the provision of insights and strategies to inform teaching. No major developments seem to have taken place since then. However the common thread amongst all this research is that

> fractions, ratios and proportions are the most protracted in terms of development, the most difficult to teach, the most mathematically complex, the most cognitively challenging, the most essential to success in higher mathematics and science. (Lamon 2007, 629)

Most of the research has involved children and young adolescents; we have found little evidence that mathematics education researchers have paid much attention to the development of the proportional reasoning abilities of students in higher education.

In this paper, we report on an ongoing research project focused on the development of proportional reasoning in university law students who attend a one-semester quantitative literacy course as part of their undergraduate degree programme at a South African university (Frith 2012). We have come to use the term 'quantitative literacy' in preference to 'numeracy' to emphasise our view that the abilities required to critically engage with quantitative data in society are firmly rooted in the domain of academic literacy (although in this paper we will use the two terms interchangeably).

The course was introduced by the Law Faculty at our university almost 15 years ago, as a result of concern expressed by the South African Law Society

about the lack of numeracy skills among candidate attorneys. Students are required to register for the course if their performance on a nationally administered quantitative literacy test for applicants to higher education institutions indicates that support will be needed for them to cope with the quantitative literacy demands of their programme of study (Frith and Prince 2006). On average, there are about 50 students who take each semester course. The student body is diverse in terms of school education background (students will have completed their schooling in schools across the spectrum of public and private, urban and rural, well-resourced and severely under-resourced schools), home language and age (some students have only a school-leaving certificate, others already have an undergraduate degree). The course has an overarching social justice theme and is context-based, drawing on issues and contexts that are relevant to a society in transition and, wherever possible, to the discipline of law.

A recent focus of attention in the course has been on enabling students to develop their proportional reasoning ability, as we assert that this ability is indispensable in enabling a critical understanding of data used to describe society. We have adopted the view that proportional reasoning is a *threshold concept* (Meyer and Land 2003) for quantitative literacy.

Over a period of four years, in cycles of action research, we have collected data that have enabled us to reflect on the teaching and learning of proportional reasoning: we have identified the elements that are involved in reasoning about qualitative comparison of fractions, rates and percentages; we have become aware of the importance of making explicit these elements in our teaching as well as focusing on the language involved in comparing proportions. Our research indicates that, even with directed teaching and learning interventions at intervals over time, proportional reasoning remains difficult for many young adults in higher education.

In this paper we start by outlining our view of quantitative literacy and reviewing some of the literature on proportional reasoning and threshold concepts. We then summarise the work we have done in the first few cycles of research and present in more detail the our most recent research: in particular, the framework we have created to analyse students' responses to a range of proportional reasoning situations and how this work has enabled an understanding of the progress of students in mastering this threshold concept.

Broad Context

Quantitative Literacy in Higher Education

There are many different definitions of quantitative literacy (or numeracy) in the literature which emphasise various aspects of this complex concept, but the core of all of them is the idea that quantitative literacy is concerned mainly with

mathematics and statistics used in context (e.g., Chapman and Lee 1990, Jablonka 2003, Steen 2004, Johnston 2007). We use the following definition, which is most strongly influenced by the definition of numerate behaviour underlying the assessment of numeracy in the Adult Literacy and Lifeskills (ALL) Survey (Gal et al. 2005) and the view of academic literacy and numeracy as social practice:

> Quantitative literacy (numeracy) is the ability to manage situations or solve problems in practice, and involves responding to quantitative (mathematical and statistical) information that may be presented verbally, graphically, in tabular or symbolic form; it requires the activation of a range of enabling knowledge, behaviours and processes and it can be observed when it is expressed in the form of a communication, in written, oral or visual mode (Frith and Prince 2006, 30).

The approach of the New Literacy Studies, which conceptualises literacy and numeracy as social practice (Street 2005, Street and Baker 2006, Kelly, Johnston and Baynham 2007), rests heavily on Gee's notion of *secondary Discourse*. Gee (1990, xvii) described "Discourses" as demanding "certain ways of using language, certain ways of acting and interacting, and the display of certain values and attitudes". There are different Discourses associated with different academic disciplines; he characterised them as examples of "secondary Discourses" (Gee 1990, 151) and defined literacy as "mastery of, or fluent control over, a secondary Discourse" (Gee 1990, 153). Given that in higher education there are many disciplinary Discourses requiring different types of literacy, there will also be different quantitative literacy practices associated with different academic disciplines. The implication is that academic quantitative literacy will be best developed within the particular disciplinary curriculum.

This view of quantitative literacy practice as a component of an academic Discourse, in which language is necessarily an integral part, leads to the conclusion that quantitative literacy and language are inextricably linked. This deep connection was also stressed by Chapman and Lee (1990), who even argued that numeracy should be seen as a component of literacy, rather than something separate. The language used for expressing quantitative concepts and reasoning often uses precise terminology and forms of expression. It also frequently uses everyday words with very specific meanings (consider, for example, the word 'rate' in the phrase 'crime rate' or the word 'relative' in the phrase 'relative sizes'). In order to be numerate within a particular discipline, a student will have to interpret or use this kind of expression within the language of the particular disciplinary Discourse.

In our definition, the statement 'it requires the activation of a range of enabling knowledge, behaviours and processes' refers to the full range of competencies necessary for quantitative literacy practice, including number sense, mathematical abilities, logical thinking and quantitative reasoning in context. Our definition also emphasises that responding appropriately to quantitative

information in a text and communicating quantitative ideas and reasoning are both essential components of quantitative literacy. The quantitative literacy VALUE rubric for assessing numeracy outcomes created by the Association of American Colleges and Universities strongly reinforces this view (Rhodes 2010). Lutsky (2007) and Madison (2014) also argued for the importance of learning how to use numerical information to support written arguments in the development of students' quantitative reasoning.

This emphasis on argument in teaching quantitative reasoning is particularly relevant to the Law students taking the course in which the research for this paper is situated. Thus, when we studied students' proportional reasoning, we looked at their written responses to questions in which they were asked to interpret quantitative information presented in the question text and a graphical chart or table. We consider students' written arguments provided in response to the question, as well as their interpretation of the question text and data provided, to be an essential element of numerate behaviour. We are not focussing narrowly on a student's understanding of, or ability to work algorithmically with, the concept of proportion, but more broadly on the quality of their reasoning and their ability to communicate this reasoning.

What Do We Mean by 'Proportional Reasoning'?

Before we discuss the concept of proportional reasoning and what we mean by the term in our research, a note about terminology is appropriate. There is considerable debate about the meanings of the terms "ratio", "fraction", "proportion", and "rate" (Lamon 2007), but for our purposes here we will use the terms "rate" or "fraction" to refer to any number that is of the form a/b, where a and b can be any numbers or measurements (with $b \neq 0$). This number may be represented as a decimal fraction, a percentage, or in some other conventional way. Some examples from contexts we use in our course would be birth rate (per 1 000), crime rate (per 100 000), inflation rate (and other examples of percentage change), and interest rate.

According to Lamon, in her review of research on rational numbers and proportional reasoning, the term 'proportional reasoning' has become an ill-defined umbrella term "referring to anything and everything related to ratio and proportion" (2007, 637). However, in general, the research on ratio and proportion has implicitly defined the domain in terms of two problem types, namely 'comparison problems' and 'missing value problems'. Comparison problems are ones where four values (a, b, c and d) are given, and the problem is to determine which of a/b and c/d is larger or whether they are the same. In a missing value problem three of the four values in a proportion $a/b = c/d$ are given and the problem is to determine the fourth value.

Lamon provided a more useful definition for proportional reasoning as:

> supplying reasons in support of claims made about the structural relationships among four quantities, (say *a, b, c, d*) in a context simultaneously involving covariance of quantities and invariance of ratios or products; this would consist of the ability to discern a multiplicative relationship between two quantities as well as the ability to extend the same relationship to other pairs of quantities. (Lamon 2007, 638)

Supplying reasons is stressed because many students can provide a correct numerical answer to a proportion problem using mechanical knowledge or algorithmic procedures, but this does not mean that 'proportional reasoning' has been employed.

The questions we have used in this research can be seen as examples of the comparison type (although some of the questions have structural similarities with the missing value type), but, in most cases, they are more complex than determining only the order of two fractions and all are more in line with Lamon's definition of proportional reasoning. They are examples of what Harel et al. (1991, 127) describe as "advanced multiplicative reasoning in which ratios and products are compared in terms of changes and compensations".

The structure of the questions can be summarised in the following way: Given two rates (fractions) of the form $r_1 = n_1/d_1$ and $r_2 = n_2/d_2$, with the two values for r (or n or d) not specified, deduce the relative sizes of these unspecified values. The reasoning involves comparing the relative sizes of the given quantities (n_1 vs. n_2 and d_1 vs. d_2, say) in order to describe the relationship between the other quantities (r_1 and r_2, say). In some cases, the comparison involves only saying which is bigger, but in other cases the question is of the form "How many times bigger or smaller …". Thus these questions do not only require determining the order relationship, but also quantifying the relationship (by estimation). In the case where the two given rates are the same and the student is required to determine the relative sizes of either the numerators (*n*) or the denominators (*d*), the questions are structurally very similar to missing value problems, but without the requirement to evaluate the missing value. More crucially, our questions require students to explain their reasoning without doing any calculations, because we wish to determine whether proportional reasoning has been employed and to what extent. Given that the questions we are studying are authentic assessment questions within the course, they are also structured in this way because of our intention to promote students' ability to express quantitative reasoning through verbal argument.

Proportional Reasoning Abilities of University Students

One of the graduate attributes valued by our university is that of quantitative literacy appropriate to the disciplines. We understand this to mean that all graduates, including those in law, should be able to engage confidently with data in an informed and critical way, and also be able to effectively communicate their reasoning. In noting "the increasing demand for a workforce that can think,

analyze and compute", Brakke (2003, 168) asserts that "quantitative reasoning in the disciplines and professional programs is essential if we are to move to increasing levels of sophistication in application."

Statistical indicators, such as infection rates, poverty rates, and lifestyle risks, and data showing government spending on social grants, are examples of measures used to describe aspects of society. Making comparisons between these indicators and measuring change over time in social data are some of the mechanisms by which progress, especially in a society in transition, can be judged. Comprehensive reasoning about this type of data requires comparisons in both absolute and relative terms; this analysis often involves reasoning about proportions.

Being able to communicate clearly about such reasoning is critical to making arguments using data. We believe that writing about proportional reasoning is an important ability for law students: as a way of practising and demonstrating lucid and logical reasoning and then expressing this reasoning using clear, coherent and economical language. Precision (even when using everyday language) is essential in describing the comparison of ratios, rates and percentages. Schield (2008, 94) succinctly notes, when arguing for this precision, that "Small changes in syntax can produce large changes in semantics."

However, in her work on college students' communication about percentages, Polito (2014, 4) observes that "The language … is often imprecise and confused, and fails to clearly communicate the relevant details to the reader" and calls for students to be taught to write effectively. This observation applies equally well to our students.

It is widely acknowledged in the literature that reasoning involving fractions, proportions, ratios is difficult for many people, both children and adults. In an early review of the literature on proportional reasoning, Tourniaire and Pulos (1985) introduce the topic by saying that "Despite its importance in everyday situations, in the sciences and in the educational system, the concept of proportions is difficult. It is acquired late … Moreover, many adults do not exhibit mastery of the concept ..." They go on to say that it is only in late adolescence that we could expect more than 50% of learners to be able to successfully solve proportion problems. Lamon (2007, 637) makes the startling claim that her "own estimate is that more than 90% of adults do not reason proportionally". According to Lamon (2007, 633) "Many adults, including middle school teachers … and preservice teachers … struggle with the same concepts and hold the same primitive ideas and misconceptions as students do." In a study of pre-service teachers in Namibia, Courtney-Clarke and Wessels (2014) found that only 25% of them could recognise the relative size of two common fractions (a 'comparison' problem). We have little reason to believe that teachers are any

better educated in South Africa, which puts our students' difficulties with this kind of reasoning in perspective.

Clearly the fraction of tertiary education students who can reason proportionally should be greater than in the general population, and one might even be tempted to assume that most should all be able to do this; however, "proportional reasoning remains problematic for many college students" (Lawton 1993, 460). Thornton and Fuller (1981) found in a study at U.S. colleges that only three quarters of science students displayed a good grasp of the ratio concept and Lawton (1993) reported that only about half of the undergraduate psychology students she studied could solve simple proportion problems (of the 'missing value' type). In South Africa, a study by Harries and Botha (2013) of medical students' ability to perform proportional dosage calculations found that only 23% were fully competent at the beginning of their third year of medical study.

Even though much of the research on the learning of proportional reasoning has been focussed on younger learners, there are several observations that have emerged (summarised by Lamon 2007) which are also relevant to the context of teaching proportional reasoning in a university quantitative literacy course.

Firstly, there are no 'quick fixes' for students who have not developed a proportional reasoning ability. Even amongst younger children, short-term teaching interventions "have been largely ineffective" and "indicate that building fraction, ratio and proportion knowledge will involve a long-term learning process" (Lamon 2007, 645). This means that we should have realistic expectations of the success of our teaching interventions in a single-semester course, and should be encouraged by even modest improvements.

Secondly, algorithmic methods learned in mathematics classes make it more difficult for students to reason intuitively about proportions. Lamon (2007) reported that studies in young children revealed that they had powerful intuitive reasoning strategies, but that five or more years of traditional mathematics instruction undermined this ability and replaced it with rules and algorithms, which often fail students. This finding supports our own observations about the dependence of our students on applying learned methods (often inappropriately) and the challenges that dependence creates in teaching for understanding.

Thirdly, the context of the problem and its structure influence how difficult it is for a student to solve it. There are numerous studies of factors that influence the difficulty of proportion problems (Lamon 2007). Two important factors are the context of the problem and how familiar students are with thinking about proportions in that particular context. This is especially relevant in our course, where students are expected to apply their reasoning in a range of unfamiliar contexts. The difficulty of a problem is also affected by the kinds of numbers involved and how easy it is to recognise the relationships between the numbers.

Threshold Concepts

We have already said that we regard proportional reasoning to be a threshold concept for quantitative literacy, so a brief summary of this theory is appropriate at this point. The notion of threshold concepts advanced by Meyer and Land (2003) as a way of "transforming the internal view of subject matter" (Meyer and Land 2005, 373) enables the identification of concepts that are the building blocks of disciplines. A threshold concept can be conceived of as a gateway, "opening up a new and previously inaccessible way of thinking about something" (Meyer and Land 2003, 1). These are concepts that are not only troublesome to students, but that are *transformative* – once fully understood, the result is a transformed perception of the concept (and the subject matter and perhaps even the self) and a shift in the use of language associated with it; *irreversible* – in that the new perspective is not easily undone; and *integrative* – it enables a view of linkages to other concepts in the discipline. The time taken for the process of internalising a threshold concept (and thus effecting a transition from one way of thinking to another) will vary depending on how troublesome the concept is. In this transitional space, described by Meyer and Land (2003) as the 'liminal space', a student experiences uncertainty and perhaps a sense of being stuck between a limited, superficial understanding of the concept and a full understanding. Students may also oscillate between stages of understanding.

As discussed above, many researchers have reported on the difficulty that children experience in mastering fractions, ratio and percentage, the time taken to learn them, and the fact that many people never achieve an understanding of them. Long, in her work on describing the learning challenges in the transition that school children undergo in moving from an understanding of whole number to rational number, has hypothesised that ratio is a threshold concept to higher order mathematical concepts (Long 2009). Building on this, we hold the view that proportional reasoning is a threshold concept for academic numeracy: opening up new ways of thinking about quantities as they arise in society and in academic disciplines.

This Study

Preceding Work

An objective of our quantitative literacy course has, from the outset, been that the focus of engagement with numbers and quantities in context should be on the interpretation of the result of calculations, rather than on calculations themselves. When we started teaching this course, we assumed that, because all school-leavers in South Africa have completed some form of mathematics to grade-12 level, they would have reasonable understanding of the basic mathematical concepts

commonly used in describing the quantitative aspects of everyday life, for example, fractions, ratios, rates and percentages. Our quantitative literacy course could then focus on the interpretation and use of these concepts in reaching conclusions, making arguments, or evaluating statements made in everyday and disciplinary contexts. We soon realised, however, that many of these concepts are not well understood by our students, with a memorised formula used to calculate an answer being a proxy for the concept itself.

Although students are generally comfortable with straightforward, algorithmic-type calculations (almost always accomplished using a calculator, regardless of the type or simplicity of numbers involved), even a slight variation in the presentation of a problem results in confusion or blind insertion of numbers into a formula (Frith and Lloyd 2014). Even after repeated exposure to the concept of percentage change in different contexts, using what Madison (2014, 12) describes as "spaced practice" rather than "massed practice", we remained unconvinced that students had truly mastered the concept. We decided to test this assumption by assessing students' ability to reason qualitatively about percentage change – i.e., that students understand percentage change as a relative measure, that it is described by a fraction, and how a change in the numerator or denominator influences the size of the fraction.

So, for example, by considering the information given in the chart in Figure 1 below, we would want our students to be able to assess the progress made by the provinces in enabling poor and vulnerable children to take up the social grants (Child Support Grant, CSG) to which they are entitled. One of the ways of assessing the progress made in improving the CSG take-up rate is to consider the percentage change in the take-up rate from 2005 to 2006.
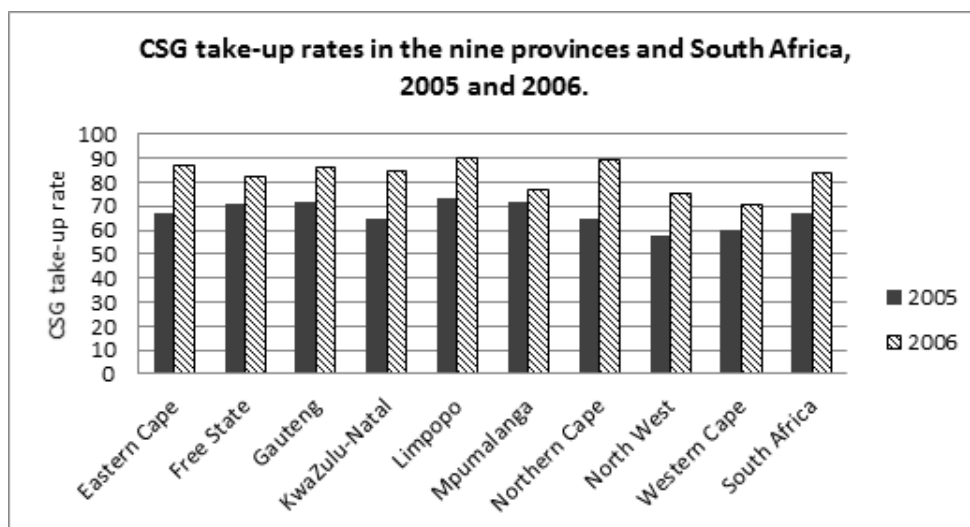

Figure 1. Example of a context where proportional reasoning is required for interpretation of the data. Chart created using data from South African Child Gauge 2006 (Leatt et al. 2006).

Similarly, if it is known that HIV infection rates are similar in two areas, but one area has a population that is three times the size of the other, then we would want our students to conclude that the number of people who are HIV+ in the area with the bigger population can be expected to be three times that of the other area.

To put our current research into context, we will give a brief outline of our study so far. In 2011 we assessed students' ability to reason qualitatively about fractions by asking, in a written assessment question and with reference to the chart in Figure 1, which of the two provinces, Limpopo or North West, experienced the greater percentage change in CSG take-up rate. Students were told not to perform any calculations, but to explain the reasoning behind their conclusion. Students had already been exposed to the context of the recently enacted Children's Act in working through materials in the classroom and had focused on budget allocations and expenditure on the social services envisaged by this Act. The students were thus familiar with the overall children's rights context, including the provision of the CSG, but not with the 'micro' context of the take-up rate of the CSG.

The *qualitative* comparison of the two provinces' percentage changes over the period is enabled if it is recognised that the absolute change in take-up rate from 2005 to 2006 was the same for both provinces, but in the case of North West this change came off a lower base. We called this kind of reasoning 'proportional comparison' and treated it as a threshold concept.

Having obtained ethics clearance from the Research Ethics Committee of the Centre for Higher Educational Development at the university and informed consent from the students, we recorded the students' written responses for analysis. We used an adapted phenomenonographic method of analysis (Marton and Booth 1997) to describe the variety of ways in which students experienced the notion of proportional comparison. A framework for identifying and describing the elements that are required in the reasoning about proportional comparison emerged from an iterative process of repeated examination of the students' responses. This enabled us to categorise the elements involved in the reasoning and to code students' responses accordingly. We were able to determine the proportion of students who were reasoning by using only absolute quantities rather than by using fractions and were also able to determine the proportion who were using some kind of proportional reasoning. The proportion of students who were reasoning with absolute quantities was disappointingly high. For a detailed description of this initial process, see Lloyd and Frith (2013).

The fact that many students had not realised that in thinking about percentage change they needed to consider a relative measure highlighted fractions, ratio, proportion and percentage as problematic concepts for students.

Meyer and Land's notion of threshold concepts and their proposal for a "conceptual framework within which teachers may advance their own reflective

practice" (Meyer and Land 2005, 373) have given us ways to think about our students' learning and our teaching. Their notion of the liminal space that is traversed en route to a full understanding of a concept was particularly useful in being able to categorise students' experiences of proportional reasoning as being pre-liminal, liminal, or at the threshold. Strategies used by students to reason about change or make comparisons that use absolute quantities only are regarded as pre-liminal. However, the concept "comes into view" (Meyer and Land 2005, 384) when a student 'sees' that reasoning must make use of ratios or fractions. The threshold is reached when proportional reasoning is used automatically and the logical process of this reasoning can be expressed coherently and concisely, making use of appropriate language. We were thus able to conclude that most of our students, in reasoning with absolute numbers rather than relative numbers, had not reached the threshold of reasoning qualitatively about quantities involving fractions – in fact, many were still at the pre-liminal stage of understanding proportional comparison (Lloyd and Frith 2013).

We realised that it was necessary to make explicit to students that the process of comparing percentage change in two quantities requires the comparison of two fractions: noticing any relationships that may exist between the numerators and denominators and how these affect the size of the fractions. Polito (2014, 15) comments that "Remarkably, the simple skill of describing these comparisons is rarely explicitly taught."

During 2012 and 2013, we made an effort to focus on making explicit the reasoning about fractions, percentages and proportions. This effort included directed classroom activities and on-line quizzes that were marked, and for which students received written feedback, as well as a continual emphasis on the language used to express this kind of reasoning.

Using the suggestions of authors working with threshold concepts (Land and Meyer 2010, Orsini-Jones 2010, Kabo and Baillie 2009), we also attempted to raise students' meta-cognitive awareness of the experience of learning a threshold concept by introducing them to the idea of threshold concepts and then having them code a previous cohort's responses to the CSG question, using our analytical framework. For the cohort whose responses are used in the research reported in this paper, this exercise was carried out after they had themselves attempted the CSG question in the first assessment.

Wishing to gauge the effect of our interventions, we repeated the process of analysing students' responses to the CSG question in the first assessment, using the framework. Again, we saw that a very small proportion of students could be said to be at the threshold, with one-third of the students still being at the pre-liminal stage. Wanting to give students additional exposure to qualitative reasoning about fractions, we introduced questions about comparisons of rates, such as mortality rates, into classroom materials and in later assessments. We then

created a similar framework for describing the variations in students' experiences of reasoning about these types of questions.

At the end of this period we concurred with previous authors (Tourniaire and Pulos 1985, Lamon 2007) that proportional reasoning is difficult, even for young adults, and takes a long time to master. Even though the proportional reasoning that we are trying to encourage in our students is perhaps more sophisticated than that described in many of the studies which are discussed in the literature, the concept has proved far more troublesome than we expected. It was clear that the interventions we introduced had only a very modest effect on students' learning of the proportional comparison concept: we found that, at best, less than a quarter of the students had reached the threshold; and, depending on the type of question, up to half of the students had not yet entered the liminal space. In addition, it was clear that, on the whole, students still did not have access to the appropriate language in the exposition of their reasoning and lacked clarity of expression. (For a more detailed description of this process, see Frith and Lloyd 2014).

Despite these somewhat disappointing results, we continued our attempts to improve students' proportional reasoning abilities as elaborated in the next section. In addition to the classroom interventions already mentioned, we also emphasised the difference between absolute and relative measures, and exercises that differentiated between them were introduced into the existing materials and tutorials. Graphics were used frequently in lecture slides to highlight absolute and relative measures. More opportunities were given to allow for reasoning about rates. We found Noelting's (1980) orange juice analogy for thinking about the comparison of rates to be a helpful aid for students in providing a concrete way to think about how a change in the numerator of a rate can be compensated for by a change in the denominator in order to maintain the rate. In this analogy the students are encouraged to consider an amount of orange concentrate (the numerator) and an amount of water (the denominator), with the resulting intensity of orange flavour representing the value of the rate.

To facilitate the analysis of the different types of questions we are interested in, we produced a single, refined framework that not only caters for both types of proportional reasoning questions (the comparison of percentage changes and comparisons involving rates), but can also be used to differentiate between responses within the liminal space.

The Current Study (2014 Cohort)

As we did previously, in the current study we have analysed student responses to questions requiring proportional reasoning. These questions were authentic in that they were used in the three course assessments (including the final examination). The number of responses to each question varied, because not all students

answered every question and we did not include answers that did not include a comprehensible explanation.

All questions[1] were based on real social data from various contexts in South Africa and had the following structure: given $r_1 = n_1/n_1$ and $r_2 = n_2/n_2$, with the two values for r (or n or d) not specified, deduce the relative sizes of these unspecified values. In questions [1], [2] and [3] (relating to percentage change) the information was given in a chart and the quantity that changed was itself measured in terms of a rate per 1000 or a percentage. Simplified versions of these questions are shown in Figure 2.
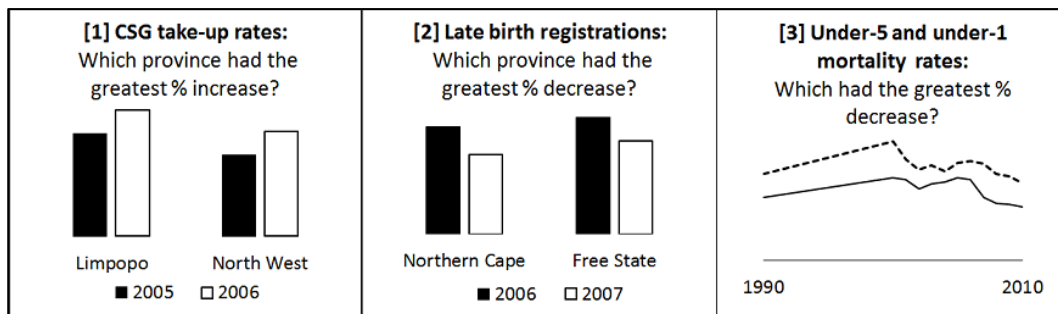

Figure 2. Simplified versions of questions [1] to [3].

Questions [1], [2] and [3] were all similar in structure, requiring students to reason about the relative sizes of two percentage changes, given the values for a quantity in two categories for two different years. Question [1] is the one we have always used in the first assessment each year and refers to two of the provinces (Limpopo and North West) in the chart in Figure 1. Students needed only to recognise that on the chart the two absolute changes between the two years were very similar in size and that one of the categories had overall smaller values, meaning that in this case the change was calculated as a percentage of a smaller base (denominator) and would thus be bigger for that category (that is, $n_1 \approx n_2$ and $d_1 > d_2 \rightarrow r_1 < r_2$). This question is one of the situations described by Lamon (2007) as easily solved intuitively (if presented in a familiar context), because it does not require quantification to determine which rate is bigger. However, our contexts were not familiar everyday ones and were complicated somewhat by the fact that the quantities (n and d) in these three questions were themselves measured as a percentage, or as a rate per 1000. This may have misled some students into interpreting "percentage change" as a difference, rather than a fraction. They had, however, seen numerical examples using this kind of data in class.

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[1] Original versions of the questions are in the appendix, following References.

In questions [4] to [8] the data were provided in a table, with some values deleted from the original table if necessary. Abbreviated versions of these questions are shown for convenience in Table 1.

Table 1.
Summarised Versions of Questions [4] to [8]

Question number	Question	Prov.	Data provided		
			r	n	d
[4]	How many times bigger is the rate in KZN than in L?	KZN L	Murder rate	No. of murders 3 625	Population 10 694 400
				729	5 630 500
[5]	How many times more deaths in EC than M?	EC M	Mortality rate 105 101	No. of deaths	No. of births 180 453 89 325
[6]	Which province had more deaths?	NW G	Mortality rate 105 86	No. of deaths	No. of births 72 640 228 370
[7]	Which province had more murders?	M NW	Murder rate 19.6 37.7	No. of murders	Population 4 229 300 3 676 300
[8]	How many times bigger is the population of KZN than of FS?	KZN FS	Murder rate 34.7 34.4	No. of murders 3 625 946	Population

As for the first three questions, question [4] required students to compare two rates; but here they had to quantify the relationship, not just say which rate was bigger. In this case, the necessary data — number of murders (n) and population (d) — was provided numerically in a table and the rate was given as the number of murders per 100 000. In this case the reasoning required can be summarised as follows: $n_1 \approx 5n_2$ and $d_1 \approx 2d_2 \rightarrow r_1 \approx 2.5r_2$.

For questions [5] to [8] the quantities to be compared were either the numerators (n) or the denominators (d). Questions [5] and [6] were about comparing numbers of infant deaths (n) given infant mortality rates (r) and populations (d) in two provinces. In question [5] students had to quantify the relationship, but in question [6] only say which was bigger. The reasoning required was $r_1 \approx r_2$ and $d_1 \approx 2d_2 \rightarrow n_1 \approx 2n_2$ for question [5] and $r_1 \approx r_2$ and $d_1 \approx 3d_2 \rightarrow n_1 > n_2$ for question [6]. Question [7] required students to compare numbers of murders (n) in two provinces given data about their respective murder rates (r) and population sizes (d), reasoning as follows: $r_1 \approx 2r_2$ and $d_1 \approx d_2 \rightarrow n_1 > n_2$. In question [8] students had to quantify the relationship between the population sizes (d) given values for murder rates (r) and number of murders (n) thus: $r_1 \approx r_2$ and $n_1 \approx 4n_2 \rightarrow d_1 \approx 4d_2$.

Student responses were coded using a refinement and synthesis of the frameworks used previously (Lloyd and Frith 2013, Frith and Lloyd 2014), which is shown in Table 2. Unlike those used earlier in our study, this more generic

framework can be used for analysing responses to questions where the fractions describe percentage change or rates, such as birth rates or mortality rates.

Table 2.
Framework for Analysing Proportional Reasoning Questions

Question structure: Given two rates (fractions) of the form $r_1 = \frac{n_1}{d_1}$ and $r_2 = \frac{n_2}{d_2}$, with the two values for r (or n or d) not specified, deduce the relative sizes of these unspecified values.

Code	Description	Notes	Example of student response from question [5]	Position relative to liminal scale
A_1	Compare the sizes of the Q_1s	Q_1 refers to the first given quantity, which is r (if r is given), otherwise n.	… both provinces had *similar under-five mortality rates* …	Responses with A or B only are pre-liminal (because reasoning involving fractions is absent)
A_2	Quantify the comparison of Q_1s if necessary			
B_1	Compare the sizes of the Q_2s	Q_2 refers to the second given quantity.		
B_2	Quantify the comparison of Q_2s if necessary		… the *number of births* in the Eastern Cape *is more than double* that of Mpumalanga.	
C	Recognise that rates are relative and involve fractions	Used to indicate that response has entered bottom of liminal scale, not used if D, E, F or G are present.	… more deaths in the Eastern Cape because the under-five mortality rate is higher and compared two rates *with the same base* therefore 'equivalent' comparison.	
D	Reasoning along the right lines, but not complete, for example not comparing the ratio of the Q_1s and the ratio of the Q_2s when necessary.	Steps in argument are missing, or in simpler questions, linking language is absent	The Eastern Cape deaths are twice as much as the Mpumalanga deaths *as the number of live births in the EC are more than Mpumalanga but the EC has a higher mortality rate*	Any responses with C, D, E and/or F are in the liminal space.
E	Comparing ratio of the Q_1s and the ratio of the Q_2s and quantifying comparison if necessary		… in Eastern Cape; they have *similar per 1000 but* Mp. has *roughly half the population*.	
F	False reasoning	For example: smaller denominator implies smaller rate, or greater rate implies greater numerator	Eastern Cape and Mpumalanga have roughly the same mortality rate. EC's population is roughly double Mpumalanga's *therefore there were twice as less deaths of under 5 children in the EC.*	
G	Correct conclusion, reasoning correct and complete.	Implies presence of A_1 (or A_2), B_1 (or B_2), C and E (if E is appropriate).	*E Cape had will have almost double the no. of under 5 deaths because the no. of live births is almost double while the mortality rate is relatively close.*	At (or over) the threshold
H	Question attempted, but no comprehensible explanation provided.			Not considered in the analysis

Italics in examples are for emphasis, not in the original.

This coding enabled us to place responses systematically in terms of whether they were pre-liminal, in the liminal space, or at the threshold, as we did before in Frith and Lloyd (2014).

The coding also enabled us to grade responses in the liminal space according to a five-point scale. For example, the second of the two responses to question [7] quoted below (coded B_1C) is much lower on the liminal scale than the first quoted response (coded A_1B_1D), which is near the top:

> There are more murders in the North West. Although Mpumalanga's population is larger in proportion the difference is small, thus the murder rate in North West, 37.7 shows there were more murders than in Mpumalanga which had 19.6 murder rate.

> North West. There were more murders in North West because the murder rate of North West comes from a smaller base compared to that of Mpumalanga.

The response in the second quote is only regarded as being in the liminal space because the phrase "comes from a smaller base" indicates there is some recognition that a fraction is involved, while the student who wrote the first response probably was reasoning correctly, but gave an incomplete explanation.

In addition, we used our subjective judgement (taking accuracy, economy and coherence in language use into account) to fine-tune our placement of the responses into one of five positions along the liminal scale. So, for example, although both the following responses to question [8] were coded A_1B_1D, the first is regarded as higher on the liminal scale on the grounds of better use of words such as "however" and "therefore" indicating logical connections between statements. The second is also lower on the scale because of the incorrect quantification of the relationship between the population sizes.

> Both provinces have almost the same murder rate sitting at around about 34 murders per 100 000. However, the Free State only had 946 murders while KwaZulu-Natal had 3625 murders. KwaZulu Natal must therefore have a larger population owing to its larger amount of murders despite having the same number per 100 000 as the Free State.

> The population of KZN is twice as big as the population of FS as the Free State has 3625 murders and 34,7 murders per 100 000 of the population and FS has 34,4 murders per 100 000 of the population which is similar to that of KZN. (sic) The number of murders in FS is also smaller to KZN which would insinuate that the population would be smaller than KZN taking into account the rate of murder.

We also made a distinction between those responses coded G that were 'at the threshold' and those that we felt were securely 'over the threshold'. This distinction was usually done on the basis of the economy, coherence and clarity of the language used in the explanation. Those students whose responses were over the threshold should ideally be those who had fully mastered the (threshold) concept of proportional reasoning as defined by Lamon (2007). Thus the first of the following two responses (also to question [8]) was considered over the threshold while the second was at the threshold:

> Free State and KwaZulu-Natal have the same murder rate of about 34, however the number of murders in KZN is about 4 times bigger than Free State. This suggests that KZN's population is around 3-4 times bigger than Free State's population.

> Looking at the graph we can see that in KwaZulu-Natal there are 3 625 murders and a murder rate of 34.7 per 100 000. However when looking at the Free State the murder rate is almost exactly the same at 34.4 murders per 100 000. However the key factor is that there were only 946 murders in the Free State as apose (sic) to the 3625 murders in KZN. Therefore we have 9 = Free State and 36 = KZN. It would seem therefore that the population in KZN is 4 times bigger than the population of the Free State.

We hoped that placing students' responses more precisely on a liminal scale would allow us to track individual students' performance over the semester and show the development of their proportional reasoning ability. It soon became clear, however, that performance on the questions we studied was most dramatically affected by the context and structure of the data provided in the question rather than chronology (as will be shown in the following section, under the heading 'Results and discussion'), so we did not proceed with this approach. This limitation is a consequence of the fact that our research is situated within the authentic course, and the questions we studied were actual assessment questions. Because we believed that students must experience the same mathematical content in a large variety of contexts in order to transfer their knowledge, we did not standardise the contexts of the questions for the benefit of the research. However we did not anticipate how great the effect of context and structure of questions would be.

Results and discussion

Table 3 shows a summary of the classification of the responses to the eight questions studied:

In general, the students' performance on the first three questions deteriorated as the semester progressed, with 31% at or over the threshold in the first assessment and only 19% in the third. Only two students who were at or over the threshold in assessment 1 maintained that position in assessments 2 and 3. However, the second question was more difficult than the first in that the differences between the values for the two years were small and so it was not as easy to see that the absolute changes were the same for both provinces. For example, a student who gave excellent explanations in assessments 1 and 3 wrote the following incorrect argument in assessment 2:

> The Northern Cape had the greater percentage decrease. This can be seen since the value of the 2007 figure is lower than the 2007 figure of the Free State and the value of the 2006s figures are similar. A smaller denominator will result in a greater figure.

Table 3.
Classification of Responses to Proportional Reasoning Questions According to Position Relative to Liminal Scale

Question structure: Given two rates (fractions) of the form $r_1 = \frac{n_1}{d_1}$ and $r_2 = \frac{n_2}{d_2}$, with the two values for r (or n or d) not specified, deduce the relative sizes of these unspecified values.

| | Question number and reasoning required | Assessment number | Percentage of responses that were analysed | | | | | | | |
			Pre-liminal	1	2	Liminal 3	4	5	At thres-hold	Over thres-hold
Compare fractions (% changes)	[1] (N = 36) $n_1 \approx n_2$ and $d_1 > d_2 \rightarrow r_1 < r_2$	1	36	3	3	22	3	3	14	17
	[2] (N = 26) $n_1 \approx n_2$ and $d_1 < d_2 \rightarrow r_1 > r_2$	2	62	8	4	15	4	0	0	8
	[3] (N = 27) $n_1 \approx n_2$ and $d_1 < d_2 \rightarrow r_1 > r_2$	3	63	15	4	0	0	0	4	15
Compare fractions (rates per 100 000)	[4] (N = 26) $n_1 \approx 5n_2$ and $d_1 \approx 2d_2 \rightarrow r_1 \approx 2.5r_2$	3	23	23	8	4	19	0	8	15
Compare values of n	[5] (N = 30) $r_1 \approx r_2$ and $d_1 \approx 2d_2 \rightarrow n_1 \approx 2n_2$	2	13	7	3	7	20	7	7	37
	[6] (N = 29) $r_1 \approx r_2$ and $d_1 \approx 3d_2 \rightarrow n_1 > n_2$	2	28	17	3	24	10	7	7	3
	[7] (N = 34) $r_1 \approx 2r_2$ and $d_1 \approx d_2 \rightarrow n_1 > n_2$	3	18	3	0	6	32	15	6	21
Compare values of d	[8] (N = 25) $r_1 \approx r_2$ and $n_1 \approx 4n_2 \rightarrow d_1 \approx 4d_2$	3	8	4	4	8	16	12	16	32

In the third question, many students seem to have been distracted by the fact that the data were given in a line chart, and 26% gave arguments that had to do with rate of change in sections of the line between the endpoints, or that depended on the fact that one graph had a more prominent peak than the other, all of which was irrelevant, but similar to the kind of description that from experience they would have associated with the description of trends in line charts. For example:

> The under-5 mortality rate had a greater percentage decrease as its gradient was steeper than that of under-1 mortality rate, indicating a greater dip in numbers.

> The under 5 mortality rate showed a greater percentage decrease between 1990 and 2010 as there seems to be a bigger decrease from the peak of the under 5 years to 2010 than there is from the peak of the under 1 year to 2010 relating to the mortality rates.

This question may also have been interpreted differently by the students because the question used the phrase "… *between* 1990 *and* 2010" rather than "… *from* 2005 *to* 2006" as in question [1]. The word "between" may have misdirected them to focus on the period between the endpoints rather than just the change from one endpoint to the other. We consider that this possibility is likely, as we have often observed that seemingly innocuous prepositions in English can provide barriers to understanding quantitative language for students, especially those who are not first-language English speakers.

It is remarkable that over 60% of responses were classified as pre-liminal for both questions [2] and [3], as compared to only 36% in question [1], indicating that even students who are capable of proportional reasoning could be completely unable to recognise that it was required in more challenging contexts. The following responses provide an example of how the changes in the contexts of the questions might have resulted in an unsuccessful trajectory over the semester, where the same student provided responses that we classified as over the threshold, liminal and pre-liminal in assessments 1, 2 and 3 respectively:

> Percentage increase is calculated as change between the years 2005 and 2006 over the initial value in 2005. Where the value in 2005 is greater, the overall percentage change is likely to be smaller. Limpopo has a starting value of ~72 while North West only 62. Both provinces have similar change between 2005 and 2006 (~20) hence only denominator value (initial 2005 value) is relevant. Since North West has a smaller initial value it will have a larger percentage increase.

> NC had a smaller base than FS/initial, and larger numerator than FS. Since % change is (final-initial)/initial it stands to reason that NC had the larger % change.

> Under 5 years had a mortality of 62/1000 and final of 56/1000 (in 2010). Under 1 had a start of 46/1000 and final of 47/1000 (in 2010). Since the change in the start and final values of under 5 is greater than that of under 1; one can conclude under 5 had a greater change.

The results for the first three questions show that many students cannot transfer what proportional reasoning abilities they have to unfamiliar situations, which reflects the observation reported by Lamon (2007) that both context and familiarity of thinking proportionally in that particular context affect a problem's difficulty. This effect of context and structure of questions on our students' performance will be further investigated in another paper. The relatively weaker performance on these similar questions in assessments 2 and 3 could also reflect the fact that most of the emphasis on teaching proportional reasoning took place in the earlier part of the course. This timing would imply that the learning of this concept for many students was still unstable, which is consistent with the threshold concept theory.

Questions [3] and [4] were both in assessment 3 and both involved comparing rates. The results were better for question [4], particularly with respect to the proportion of the students whose responses were pre-liminal. Question [4]

was a much more demanding question in terms of the reasoning required, because the relationship between the rates had to be quantified – and both the numerators and the denominators were different (numerators by a factor of 5 and denominator by a factor of 2). This result emphasises how strong the negative effect of the unfamiliar context (particularly the chart type used) in question [3] could be.

In both question [5] and question [8], the performance was much better than in other questions, with between 40% and 50% of the students at or over the threshold. Neither of these questions involved reasoning about inequalities. It seems that reasoning is easier when the given quantities are very clearly small multiples of each other, rather than just some indeterminate amount bigger or smaller. This effect is related to the observation that "the presence of integer ratios makes the problem easier", which was reported in the early review by Tourniaire and Pulos (1985, 188). However, that observation was based on research done in schools, and we see this effect even amongst university students, which supports our observation that our students generally have very weak number sense, probably resulting from over-dependence on calculators from an early age.

Related to this idea of the effect of the presence of integer ratios on difficulty is the difference in performance observed between questions [6] and [7]. Superficially, the reasoning involved in both questions appears to be very similar, but the performance on question [6] was much worse than on question [7]. It was easy in question [7] for students to recognise the relationship between the rates (that is, 37.7 is about double 19.6), but in question [6] it was difficult for students to recognise that 86 is fairly similar to 105 for the purposes of the reasoning, when juxtaposed with a four-fold difference in the denominators. So students struggled to express the idea that although the mortality rate for Gauteng was somewhat smaller (and many could not quantify this relationship) it was not small enough to compensate for the very much larger number of births in that province, and so the number of deaths would still be greater there. The following is an example of one student's struggle to express this notion:

> NW rate: 105: 1000   G rate: 86: 1000 but G had more than 150 000 more births (32%), so if G had 32% more deaths then it would be equal but because they have far more births and their death rate is only 19: 1000 less they have more deaths.

Another interesting effect (which is not, however, reflected in the figures in Table 3), was that 31% and 38% of the students concluded incorrectly in questions [5] and [6] respectively. In both these questions, the data for the denominators and the sizes of the fractions were given, and the students were required to reason about the relative sizes of the numerators. When the denominators differ and the effect of the difference has to be predicted, about a third of the students inverted the relationship (that is, they reasoned that a smaller value in the denominator implies a bigger numerator). Perhaps they were

generalising the fact that decreasing the denominator increases the size of the fraction, and concluding that a smaller denominator will mean that any of the other variables must be bigger. In some cases, the answers were well written and the reasoning sounded deceptively plausible, such as in these examples from question [5]:

> Mpumalanga had double (2 times more) the amount of deaths in 2007 than the Eastern cape because the Eastern cape had double the amount of live births that Mpumalanga yet their IMR was similar in 2007.

> The Eastern cape had ½ the amount of deaths of under 5-year olds as they had a very similar mortality rate as Mpumalanga, but approximately double the base.

Conclusion

We believe that proportional reasoning is essential for a critical understanding of data concerning changes and differences in society, as it allows meaningful comparisons to be made. Our analysis of the students' responses to questions requiring relatively sophisticated proportional reasoning (as defined by Lamon 2007) shows that this reasoning displays the characteristics of a threshold concept as described by Meyer and Land (2003, 2005). In particular, we have seen that the acquisition of proportional reasoning ability takes a long time, during which students find themselves in a liminal space where their ability shows variations. In addition, the development of this ability is accompanied by the acquisition of new forms of language.

In the latest iteration of our study, we have refined our framework for analysing students' responses and generalised it to be applied to the various kinds of questions that we are studying (unlike previously, where we treated questions about percentage change as qualitatively different from questions about rates, and used separate frameworks for these). This refinement of the framework has both resulted from and resulted in a clearer understanding of the structures of our questions and what they require of students. It has also allowed us to make comparisons between students' performance on questions that were previously seen as being of different types. In refining the framework, we have also introduced a grading system for the responses in the liminal space, which allows us to distinguish between responses higher and lower in the liminal space. This grading has taken into account the quality of students' use of language as well as the elements of reasoning which are present.

Iterative research cycles, which included development of teaching innovations alongside the refinement of analysis tools, have allowed us to develop a better understanding of what is needed in teaching this concept. Our main finding is that teaching interventions have only modest effects in the time-frame of a one-semester course (as expected from reading the literature) and that it takes

time for students to master this concept and learn to express their reasoning using appropriate language and terms. As pointed out by Madison (2014) "In a one-semester QR course, significant spacing of retrieval is not possible. Consequently, there is more need for continued practice at retrieval beyond the course." The implication is that if students are going to acquire, as a graduate attribute, the ability to reason with proportions in a transferable way, teaching proportional reasoning, including providing practice in using it, will have to be integrated into their disciplinary curriculum. A one-semester, first-year quantitative literacy course cannot provide a 'quick-fix', much as our colleagues in the disciplines would like it to.

That said, we are fairly satisfied that we have had some success in improving students' abilities in this area, bearing in mind that the students in the course are selected on the basis of having done poorly on a quantitative literacy test on application to university, so our sample represents those whose quantitative reasoning abilities are under-developed to begin with. In addition, the questions we have studied are more demanding than many used in other studies which have reported that low percentages of adults can use proportional reasoning. For example, Lawton (1993) reported that only about half of the undergraduate psychology students she studied could solve simple proportion problems (of the 'missing value' type). Our questions are more sophisticated than simple 'solve for the missing value' questions and our analysis takes into account the quality of students' explanations of their reasoning, rather than whether they can calculate correctly. Nevertheless, we have seen a similar proportion (40% to 50%) of our students coping well with questions that have reasonably easily recognisable ratios between the quantities to be reasoned with. These results encourage us to continue to focus on teaching proportional reasoning and to try to find new creative ways to facilitate the students' acquisition of this concept, within the constraints of the time available in the course.

Our data has revealed that the context and structure of the questions has a marked effect on students' ability to transfer their proportional reasoning abilities (more so than we had anticipated), which means that we were not able to track students' development over the time-span of the course. In fact, many of them appeared to regress. The extent of this effect of question context on performance, and how to address it in teaching, will require further study.

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Appendix: proportional reasoning questions

Question [1]

Eligibility for the CSG is determined by a means test. However, it has been found that not all children who are eligible actually take up the CSG. The chart below shows how the proportion of children who take up the CSG in the different provinces changed from 2005 to 2006.
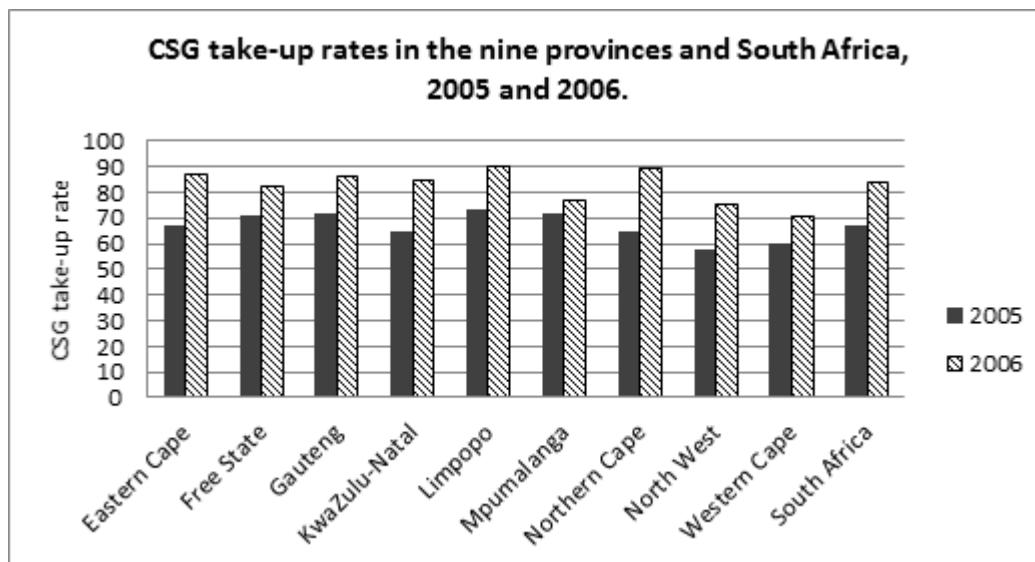

Figure 3: Chart used in question [1].

Question [1]:
Consider Limpopo and North West provinces. Without doing any calculations say which province experienced the larger percentage increase in CSG take-up rate from 2005 to 2006. Explain your reasoning.

Answer:
For both provinces the absolute change is about the same, but for North West the % change is calculated off a smaller base, therefore North West has the biggest % change.
[$n_1 \approx n_2$ and $d_1 > d_2 \rightarrow r_1 < r_2$, where n is the absolute change, d is the 2005 value and r is the % change]

Question [2]

As mentioned in the article,* there are difficulties in calculating statistics because of under-registration of births and deaths. The relevant authorities have now begun tracking late registration of births, as shown in the chart below.

(* Students were required to read a short article on the attainment of one of the Millennium Development Goals, that of reducing child mortality)
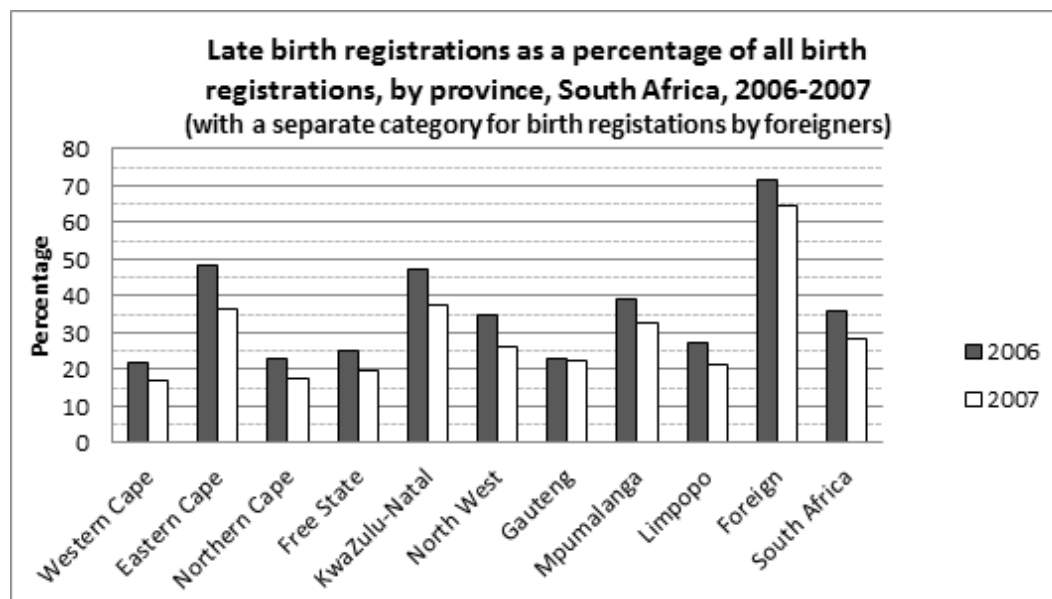

Figure 4: Chart used in question [2].

Question [2]:
Consider Northern Cape and Free State. By reasoning and without doing any calculations say which of these two provinces had the greater percentage decrease in proportion of birth registrations that are late.

Answer:
For both provinces the absolute change is about the same, but for Northern Cape the % change is calculated off a smaller base, therefore Northern Cape has the biggest % change.
[$n_1 \approx n_2$ and $d_1 > d_2 \rightarrow r_1 < r_2$, where n is the absolute change, d is the 2006 value and r is the % change]

Question [3]

Mortality rates of young children in a country are commonly used as a measure of child well-being. These rates are measured in terms of the number of deaths that occur for every 1 000 children born alive in that year. The following chart shows information about the under-five and infant (under-one) mortality rates of children in South Africa over time.
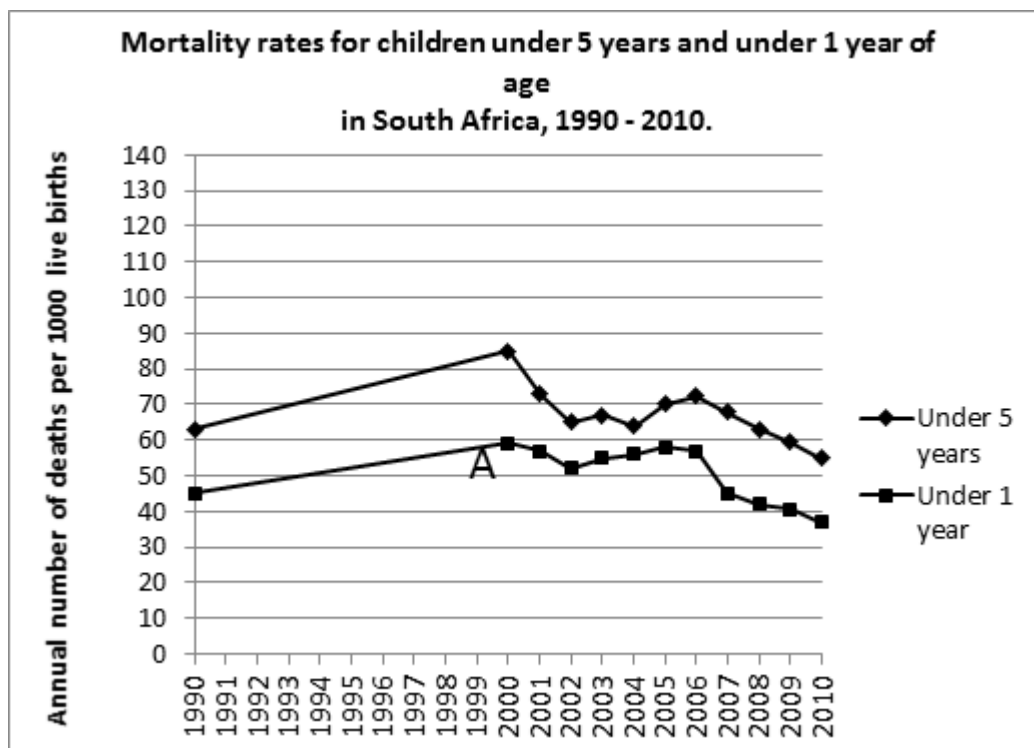

Figure 5: Chart used in question [3].

Question:
Without doing any calculations say which of the two mortality rates showed the greater percentage decrease between 1990 and 2010. Explain your reasoning.

Answer:
For both the 'under 1 year' and 'under 5 year' mortality rates the absolute change is about the same, but "under 1 year" is calculated off a smaller base, therefore "under 1 year" has the biggest % change.
[$n_1 \approx n_2$ and $d_1 < d_2 \rightarrow r_1 > r_2$, where n is the absolute change, d is the 1990 value and r is the % change]

Questions [4], [7] and [8]

The table below shows data about the murders in South Africa in 2014. Read it carefully and answer the questions that follow.

Populations and murder statistics in the provinces of South Africa, 2014					
Province	Population estimate	% of total population	Number of murders	Murder rate (number of murders per 100 000)	% change in murder rate 2013–2014
Eastern Cape	6 786 900	12.6		52.1	2.6
Free State			946	34.4	-7.5
Gauteng	12 914 800	23.9		26.2	9.2
KwaZulu-Natal	10 694 400	19.8	3 625	34.7	−1.1
Limpopo	5 630 500	10.4	729		2.3
Mpumalanga	4 229 300			19.6	**A**
Northern Cape	1 166 700	2.2		22.9	-7.3
North West	3 676 300	6.8		37.7	5.6
Western Cape	6 116 300	11.3		48.3	10.5
Total	54 002 000	100.0		32.2	3.5

Question [4]:
Use data from the table to estimate how many times bigger (or smaller) the murder rate is in KwaZulu-Natal than in Limpopo. Explain your reasoning.

Answer:
There were 5 times more murders in KwaZulu-Natal than in Limpopo and only 2 times the population, therefore the murder rate was 2.5 times bigger in KwaZulu-Natal.

[$n_1 \approx 5n_2$ and $d_1 \approx 2d_2 \rightarrow r_1 \approx 2.5r_2$, where n is the number of murders, d is the population and r is the murder rate]

Question [7]:
Consider Mpumalanga and North West. Without doing any calculations, say in which of the two provinces there were more murders in 2013/14. Explain your reasoning.

Answer:
The murder rate in North West was double that of Mpumalanga and the population was only a little smaller, therefore there were more murders in North West.

[$r_1 \approx 2r_2$ and $d_1 \approx d_2 \rightarrow n_1 > n_2$, where n is the number of murders, d is the population and r is the murder rate]

Question [8]:
By reasoning, and without doing any calculations, estimate how many times bigger or smaller is the population of KwaZulu-Natal than Free State.

Answer:
The murder rate in KwaZulu-Natal was about the same as in Free State, but the number of murders was 3 to 4 times more, therefore the population was 3 to 4 times bigger in KwaZulu-Natal.
[$r_1 \approx r_2$ and $n_1 \approx 4n_2 \rightarrow d_1 \approx 4d_2$, where n is the number of murders, d is the population and r is the murder rate]

Questions [5] and [6]

The table below shows the under-five mortality rates in the nine provinces in 2007.

Under-five mortality rates* and deaths by province, 2007			
Province	Under-five mortality rate	Number of deaths under 5 years	Number of live births
Eastern Cape	105		180 453
Free State	97		
Gauteng	86		228 370
KwaZulu-Natal	98		284 581
Limpopo	110	14 818	
Mpumalanga	101		89 325
Northern Cape	85		25 694
North West	105		72 640
Western Cape	78		112 751
Total	**104**		**1 051 038**

* Number of deaths before the fifth birthday per 1 000 live births

Question [5];
By reasoning, and without doing any calculations, estimate how many times more (or fewer) deaths of under-five children there were in Eastern Cape than in Mpumalanga in 2007. Explain your reasoning.

Answer:
The mortality rate in Eastern Cape was about the same as in Mpumalanga, and the number of live births was about double in Eastern Cape, therefore the number of deaths was about double in Eastern Cape.
[$r_1 \approx r_2$ and $d_1 \approx 2d_2 \rightarrow n_1 \approx 2n_2$, where n is the number of deaths, d is the number of live births and r is the mortality rate.]

Question [6]:
Explain, without doing any calculations, which of North West and Gauteng had the greater number of under-five deaths in 2007.

Answer:
The mortality rate in Gauteng was about the same as in North West, but the number of births was 3 times bigger in Gauteng, therefore the number of deaths was bigger in Gauteng.
[$r_1 \approx r_2$ and $d_1 \approx 3d_2 \rightarrow n_1 > n_2$, where n is the number of deaths, d is the number of live births and r is the mortality rate.]