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Subconstituent Algebras of Latin Squares

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by

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Dedication

To my loving parents,
my supportive husband,
and my three lovely daughters, Leen, Dana, and Yasmeen.

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Above all I must thank "Allah" who encouraged and comforted me in the whole process of this dissertation.

Table of Contents

List of Tables

List of Figures

Subconstituent algebra of Latin Squares

Ibtisam Daqqa

Abstract

Let n be a positive integer. A *Latin square of order* n is an $n \times n$ array L such that each element of some n-set occurs in each row and in each column of L exactly once. It is well-known that one may construct a 4-class association scheme on the positions of a Latin square, where the relations are the identity, being in the same row, being in the same column, having the same entry, and everything else. We describe the subconstituent (Terwilliger) algebras of such an association scheme. One also may construct several strongly regular graphs on the positions of a Latin square, where adjacency corresponds to any subset of the nonidentity relations described above. We describe the local spectrum and subconstituent algebras of such strongly regular graphs. Finally, we study various notions of isomorphism for subconstituent algebras using Latin squares as examples.

1 Introduction

Let n be a positive integer. A *Latin square of order* n is an $n \times n$ array L such that each element of some n-set occurs in each row and in each column of L exactly once. The simplest examples are the Cayley table of a group, as in Figure 1.1. Sudoku puzzles, when completed, form a Latin square.

$$\underset{Z_5}{\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & 2 \\ 4 & 5 & 1 & 2 & 3 \\ 5 & 1 & 2 & 3 & 4 \end{pmatrix}} \qquad \underset{Z_2 \times Z_2}{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{pmatrix}}$$

Figure 1.1: Two Cayley tables

Latin squares were introduced by Euler in 1779 in the context of a puzzle in recreational mathematics. Since that time Latin squares have found applications in a variety of mathematical areas. Such fields include group theory, graph theory, finite geometries, coding theory, design theory, cryptography, and statistics. These connections are developed in the general references [42, 43, 69].

Let L denote a Latin square of order $n \geq 3$, and encode L with the set $X = \{(i, j, L(i, j)) \,|\, 1 \leq i, j \leq n\}$. Define five relations on X: For all $x = (i, j, L(i, j))$, $x' = (i', j', L(i', j')) \in X$,

R_0 (identity): $\quad xR_0x'$ if $x = x'$,

R_1 (same row): $\quad xR_1x'$ if $i = i'$ and $x \neq x'$,

R_2 (same column): xR_2x' if $j = j'$ and $x \neq x'$,

R_3 (same entry): xR_3x' if $L(i,j) = L(i',j')$ and $x \neq x'$,

R_4 (everything else): xR_4x' if $i \neq i'$, $j \neq j'$, and $L(i,j) \neq L(i',j')$.

It is well-known that $(X, \{R_i\}_{i=0}^4)$ is a symmetric association scheme [5, 41], which we refer to as *the association scheme of* L. Hence the characteristic matrices of the five relations comprise the basis of Hadamard idempotents of a Bose-Mesner algebra $\mathcal{M}$, which we refer to as the *Bose-Mesner algebra of* L (see [5]).

Bose-Mesner algebras first arose in statistical designs [15, 16], in centralizer algebras of permutation groups [90], and in connection with distance-transitive graphs [12]. A period of growth in the subject occurred in the 1980's after Delsarte demonstrated applications to codes and designs [41] and as the classification of finite simple groups motivated and aided the study of distance-transitive graphs. Details can be found in [5, 10, 13, 17, 51]. Bose-Mesner algebras have been generalized in several directions, such as coherent algebras [53, 54], table algebras [1, 2, 3, 4], group-like association schemes [91]. More recently, connections have been developed to link invariants [22, 61, 62], quantum groups [38, 60], fusion algebras [8], maximal abelian subalgebras [52, 75], commuting squares [6], and subfactors [31, 63, 64].

In this work we study the subconsitutent (or Terwilliger) algebras of the association scheme of a Latin square. The subconstituent algebra refines the Bose-Mesner algebra by encoding additional combinatorial information concerning the relation between each point and the fixed base point. It is known that the (algebraic) isomorphism class of the Bose-Mesner algebra of a Latin square depends only upon its order and no other property of the Latin square. One of our motivations is to better distinguish Latin squares using subconstituent algebras.

Subconstitutent algebras have been studied in several papers. Themes of these papers include complete description of the subconstituent algebra of some class of association scheme [7, 11, 19, 20, 21, 30, 44, 49, 83, 88], partial description of the subconstituent algebra of some class of association scheme [23, 24, 29, 36, 45, 55, 58, 59, 84, 86, 87], descriptions of the association schemes whose subconstituent algebra satisfies some condition [25, 39], correspondences between certain combinatorial (often local) and algebraic conditions

[27, 32, 39, 50, 72, 73], algebraic connections [33, 34, 38, 89, 84], and generalizations [46, 47, 60, 81].

Fix a base point $p = (r_p, c_p, e_p) \in X$. Let $\mathcal{T}$ denote the subconstituent algebra of the association scheme of L with respect to p. We describe the action on the irreducible $\mathcal{T}$-modules in terms of a substructure of the Latin square which we now describe.

By the definition of a Latin square any two coordinates of an element of X uniquely determine the third: We refer to this as the *Latin square property.* Form a sequence of points as follows. Pick $x_1 \in X$ such that $x_1 R_1 p$–write $x_1 = (r_p, c_1, e_1)$. Once x_1 is chosen, all subsequent points are uniquely determined by the Latin square property. Given $x_i = (r_p, c_i, e_i)$, let $y_i \in X$ be the unique point such that $y_i R_2 p$ and $y_i R_3 x_i$–write $y_i = (r_i, c_p, e_i)$. Let z_i be the unique $z_i \in X$ such that $z_i R_3 p$ and $z_i R_1 y_i$–write $z_i = (r_i, c_{i+1}, e_p)$. Finally, let $x_{i+1} \in X$ be the unique point such that $x_{i+1} R_1 p$ and $x_{i+1} R_2 z_i$–write $x_{i+1} = (r_p, c_{i+1}, e_{i+1})$. Repeat this process until $x_{k+1} = x_1$. View x_1, x_2, $\ldots$,x_k as a cycle in a permutation of the $n-1$ points in relation R_1 with p, where the other cycles constructed similarly. The permutation constructed depends upon the base point p.

Figure 1.2 interprets two sequences of points in a Latin square with respect to $(1,1,1)$: $x_1 = (1,2,2)$, $y_1 = (2,1,2)$, $z_1 = (2,3,1)$, $x_2 = (1,3,3)$, $y_2 = (3,1,3)$, $z_2 = (3,2,1)$ and $x'_1 = (1,4,4)$, $y'_1 = (4,1,4)$, $z'_1 = (4,4,1)$. This gives rise to a permutation with one 1-cycle and one 2-cycle, namely $(x_1 x_2)(x'_1)$.

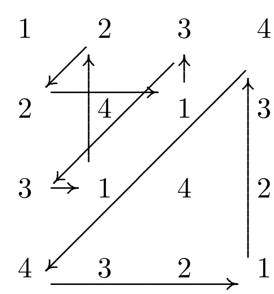

Figure 1.2: Two point sequences with respect to $(1,1,1)$.

In Chapter 3, we describe the irreducible $\mathcal{T}$-modules. We summarize the results now. There is always a unique 5-dimensional irreducible $\mathcal{T}$-module, and there are $n^2 - 6n + 7$ many mutually isomorphic 1-dimensional $\mathcal{T}$-modules in the standard module. The remaining $\mathcal{T}$-modules are related to the cycles of the permutation constructed above. If there is just one cycle (with $k = n - 1$), then there is a 6-dimensional irreducible $\mathcal{T}$-module associated with each k^{th} root of unity except 1 itself. If there is more than one cycle, then for each cycle of length k there is a 6-dimensional irreducible $\mathcal{T}$-module associated with each k^{th} root of unity. However, in this case one of the irreducible $\mathcal{T}$-modules associated with 1 must be dropped to obtain linear independence. The $\mathcal{T}$-action on each of these $(n-2)$-many 6-dimensional irreducible $\mathcal{T}$-modules is entirely determined by the associated root of unity: Two such modules are isomorphic if and only if they are associated with the same root of unity.

One may define a strongly regular graph with vertex set X for each subset C of coordinates 1, 2, 3 by declaring distinct $x = (i, j, L(i, j))$ and $x' = (i', j', L(i', j'))$ to be adjacent if for some coordinate $c \in C$, $x(c) = x'(c)$. In Chapter 4 we describe the local spectrum and subconstituent algebra of these strongly regular graphs using the cycles described above. The key idea is that the irreducible $\mathcal{T}$-modules remain modules (no longer irreducible) for the subconstituent algebras of these strongly regular graphs. Thus we show how to decompose these already small irreducible $\mathcal{T}$-modules into irreducible modules for the strongly regular graphs' subconstituent algebras. Then the local spectrum is determined by acting on these modules by an appropriate matrix.

In Chapter 5, we discuss isomorphisms of subconstituent algebras. The subconstituent algebra of a Latin square with respect to some base point is isomorphic to $\mathbb{M}_5 \oplus \mathbb{M}_6^{\ell} \oplus \mathbb{M}_1$, where ℓ is the number of mutually nonisomorphic irreducible $\mathcal{T}$-modules of dimension 6. Many distinct cycle structures may give rise to the same value of ℓ. However, the action of the subconstituent algebra on each irreducible $\mathcal{T}$-module is uniquely determined by the cycle structure of the permutation constructed above.

In Appendix A we gives tables relating cycle structure in small examples to possible irreducible modules, and in Appendix B we give *Mathematica* code which produces the irreducible modules of the subconstituent algebra of a Latin square and the related strongly regular graphs.

2 Algebraic Preliminaries

2.1 The Bose-Mesner Algebra

In this section we recall some background material. We first recall Bose-Mesner algebras and some of their basic properties. General references for the subject include [10, 17, 51].

Let X denote a finite, nonempty set, and let $\mathbb{M}_X$ denote the complex algebra of matrices with complex entries whose rows and columns are indexed by X. For $A \in \mathbb{M}_X$ and for x, $y \in X$, let $A(x, y)$ denote the (x, y)-entry of A. For A, $B \in \mathbb{M}_X$, let $A \circ B$ denote the Hadamard product of A and B: $(A \circ B)(x, y) = A(x, y)B(x, y)$. The ordinary matrix product of A and B will be denoted by juxtaposition: AB. For $A \in \mathbb{M}_X$, let ${}^{\mathrm{t}}A$ denote the transpose of A.

A *Bose-Mesner algebra on* X is a commutative subalgebra of $\mathbb{M}_X$ which is closed under Hadamard product, which is closed under transposition, and which contains the identity matrix I and the all-ones matrix J.

Let $\mathcal{M}$ denote a $(d+1)$-dimensional Bose-Mesner algebra on X. The *basis of Hadamard idempotents* of $\mathcal{M}$ is the unique basis $\{A_i\}_{i=0}^{d}$ such that

$$A_0 = I, \tag{2.1.1}$$

$$A_i \circ A_j = \delta_{ij} A_i \qquad (0 \leq i,\, j \leq d), \tag{2.1.2}$$

$$\sum_{i=0}^{d} A_i = J, \tag{2.1.3}$$

where δ_{ij} denotes the Kronecker symbol. Let A_0, A_1, $\ldots$, A_d be an ordering of the Hadamard idempotents of $\mathcal{M}$. Then relative to this ordering, the *intersection numbers*

p_{ij}^h of $\mathcal{M}$ are defined by

$$A_i A_j = \sum_{h=0}^{d} p_{ij}^h A_h \quad (0 \leq i, j \leq d). \tag{2.1.4}$$

The *basis of primitive idempotents* of $\mathcal{M}$ is the unique basis $\{E_i\}_{i=0}^d$ such that

$$E_0 = n^{-1} J, \tag{2.1.5}$$

$$E_i E_j = \delta_{ij} E_i, \tag{2.1.6}$$

$$\sum_{i=0}^{d} E_i = I. \tag{2.1.7}$$

Let E_0, E_1, ..., E_d be an ordering of the primitive idempotents of $\mathcal{M}$. Then relative to this ordering, the *Krein parameters* q_{ij}^h of $\mathcal{M}$ are defined by

$$E_i \circ E_j = n^{-1} \sum_{h=0}^{d} q_{ij}^h E_h \quad (0 \leq i, j \leq d). \tag{2.1.8}$$

Relative to the orderings of the Hadamard and primitive idempotents, the *eigenvalues* $P(j,i)$ and the *dual eigenvalues* $Q(j,i)$ of $\mathcal{M}$ are defined respectively by

$$A_i = \sum_{j=0}^{d} P(j,i) E_j \qquad (0 \leq i \leq d), \tag{2.1.9}$$

$$E_i = n^{-1} \sum_{j=0}^{d} Q(j,i) A_j \qquad (0 \leq i \leq d). \tag{2.1.10}$$

The $(d+1) \times (d+1)$ matrices P and Q with (j,i)-entries $P(j,i)$ and $Q(j,i)$ are called the *eigenmatrix* and *dual eigenmatrix* of $\mathcal{M}$, respectively.

It is known that the each of these sets of parameters (the intersection numbers, the Krein parameters, the eigenmatrix, and the dual eigenmatrix) determines the others. See, for example, [10] for precise details.

2.2 The Dual Bose-Mesner Algebra

We now recall from [84] the dual Bose-Mesner algebra of a Bose-Mesner algebra $\mathcal{M}$ on X. Fix $p \in X$. For each $A \in \mathcal{M}$, let $\rho(A) \in \mathbb{M}_X$ denote the diagonal matrix with (x,x)-entry $\rho(A)(x,x) = A(p,x)$ $(x \in X)$. Let $\mathcal{M}^* = \rho(\mathcal{M})$. We refer to $\mathcal{M}^*$ as *the* *dual Bose-Mesner algebra of $\mathcal{M}$ with respect to p.* Observe that $\rho : \mathcal{M} \to \mathcal{M}^*$ is a linear bijection and $\rho(A \circ B) = \rho(A)\rho(B)$.

Set $E_i^* = \rho(A_i)$. Then $\{E_i^*\}_{i=0}^d$ is a basis of $\mathcal{M}^*$. We refer to $\{E_i^*\}_{i=0}^d$ as the *basis of dual idempotents*. Applying ρ to (2.1.2) and (2.1.3) gives

$$E_i^* E_j^* = \delta_{ij} E_i^* \qquad (0 \leq i, j \leq d), \tag{2.2.11}$$

$$\sum_{i=0}^{d} E_i^* = I. \tag{2.2.12}$$

Set $A_i^* = \rho(nE_i)$ $(0 \leq i \leq d)$. Then $\{A_i^*\}_{i=0}^d$ is a basis of $\mathcal{M}^*$. We refer to $\{A_i^*\}_{i=0}^d$ as the *basis of dual Hadamard idempotents* of $\mathcal{M}^*$. Applying ρ to (2.1.5), (2.1.7), and (2.1.8) gives

$$A_0^* = I, \tag{2.2.13}$$

$$\sum_{i=0}^{d} A_i^* = nE_0^*, \tag{2.2.14}$$

$$A_i^* A_j^* = \sum\nolimits_{h=0}^{d} q_{ij}^h A_h^*, \qquad (0 \leq i, j \leq d). \tag{2.2.15}$$

Applying ρ to (2.1.9) and (2.1.10) gives

$$A_i^* = \sum\nolimits_{j=0}^{d} Q(j,i) E_j^* \quad (0 \leq i \leq d), \tag{2.2.16}$$

$$E_i^* = n^{-1} \sum\nolimits_{j=0}^{d} P(j,i) A_j^* \quad (0 \leq i \leq d). \tag{2.2.17}$$

2.3 The Subconstituent Algebra and its Modules

We recall from [84] some facts concerning the subconstituent algebra of a Bose-Mesner algebra $\mathcal{M}$ on X. Fix $p \in X$. The *subconstituent* (or *Terwilliger*) *algebra of $\mathcal{M}$ with respect to p* is the subalgebra of $\mathbb{M}_X$ generated by $\mathcal{M} \cup \mathcal{M}^*$. By (2.1.1), (2.1.4), (2.2.11), and (2.2.12), $\mathcal{T}$ is also generated by $\{E_i^* A_j E_k^* \,|\, 0 \leq i, j, k \leq d\}$.

Lemma 2.3.1 *[84] For all h, i, j $(0 \leq h, i, j \leq d)$,*

$$E_i^* A_h E_j^* = 0 \quad \textit{if and only if} \quad p_{ij}^h = 0. \tag{2.3.18}$$

Theorem 2.3.2 *[84] Subconstituent algebras are semisimple.*

We may appeal to Wedderburn theory [40] to describe subconstituent algebras by their irreducible modules.

Let $V = C^X$ denote the column vector space with entries indexed by X. Endow V with the Hermitian inner product defined by $\langle u, v\rangle = {}^t u\bar{v}$. Observe that $\mathbb{M}_X$ acts on V by left-multiplication. We refer to V as the *standard module* for $\mathcal{T}$. By a $\mathcal{T}$*-module* we mean a linear subspace U of V which is closed under the action of $\mathcal{T}$: $Au \in U$ for all $A \in \mathcal{T}$ and for all $u \in U$.

Let Λ be an index set for the isomorphism classes of irreducible $\mathcal{T}$-modules. Let V_λ be the sum of all irreducible $\mathcal{T}$-modules in the isomorphism class of irreducible $\mathcal{T}$-modules indexed by $\lambda \in \Lambda$. Each V_λ is an orthogonal direct sum of mutually isomorphic irreducible $\mathcal{T}$-modules. While the direct summands of V_λ are not necessarily unique, their number is. Write $\mathrm{mult}(\lambda)$ to denote this number, and for any irreducible $\mathcal{T}$-module W contained in V_λ, set $\mathrm{mult}(W) = \mathrm{mult}(\lambda)$. We refer to $\mathrm{mult}(W)$ as the *multiplicity* of W. Note that $V = \oplus_{\lambda\in\Lambda} V_\lambda$ (orthogonal direct sum). We note the following consequence of this discussion.

Lemma 2.3.3 *Suppose W and W' are non-isomorphic irreducible $\mathcal{T}$-modules. Then W and W' are orthogonal to one another.*

The primitive central idempotents of $\mathcal{T}$ are also indexed by Λ as they are in bijective correspondence with the V_λ. For each subspace V_λ there is a unique primitive central idempotent φ_λ such that $V_\lambda = \varphi_\lambda V$, and conversely for each primitive central idempotent φ, $\varphi V = V_\lambda$ for some $\lambda \in \Lambda$. Let $W \subseteq V_\lambda$ be an irreducible $\mathcal{T}$-module. Then the map taking $L \in \varphi_\lambda\mathcal{T}$ to the endomorphism $w \mapsto Lw$ $(w \in W)$ is an isomorphism. Thus $\varphi_\lambda\mathcal{T} \cong \mathrm{End}_{\mathbb{C}}(W)$. However, $\mathrm{End}_{\mathbb{C}}(W)$ is isomorphic to the $k \times k$ complex matrix algebra, where k is the dimension of W. Now $\mathcal{T} = \oplus_{\lambda\in\Lambda}\varphi_\lambda\mathcal{T}$ (direct sum). Thus $\mathcal{T}$ is isomorphic to a direct sum of complex matrix algebras.

For all elements x of X, define $[\![x]\!] \in V$ to be the characteristic vector of x, ie, the vector with a one in the row indexed by x and zeros everywhere else. Observe that the set $\{[\![x]\!] \,|\, x \in X\}$ is the standard basis of V. If S is a multi-set of elements of X, define

$[\![S]\!] = \sum_{x \in S} [\![x]\!]$, where each summand occurs once for each occurrence in S. For all $x \in X$, define $\Gamma_i(x) = \{y \in X \mid xR_iy\}$.

Lemma 2.3.4 *[84] For all $x \in X$,*

$$
\begin{aligned}
E_i^*[\![x]\!] &= \begin{cases} [\![x]\!] & \text{if } pR_ix, \\ 0 & \text{otherwise,} \end{cases} \\
A_i[\![x]\!] &= [\![\Gamma_i(x)]\!].
\end{aligned}
$$

In particular,

$$
E_i^*A_jE_k^*[\![x]\!] = \begin{cases} [\![\Gamma_i(p) \cap \Gamma_j(x)]\!] & \text{if } xR_kp, \\ 0 & \text{otherwise.} \end{cases}
$$

Proof. From the definition of A_i, $0 \le i \le d$, we have $A_i[\![x]\!] = \sum_{yR_ix} [\![y]\!] = [\![\Gamma_i(x)]\!]$. Now since E_i^* is diagonal where $E_i^*(y, y) = 1$ if yR_ip and zero any where else, we have $E_i^*[\![x]\!] = \delta_{hi}[\![x]\!]$, where h such that pR_hx. For $E_i^*A_jE_k^*$, we have

$$
\begin{aligned}
E_i^*A_jE_k^*[\![x]\!] &= \begin{cases} E_i^*A_j[\![x]\!] & \text{if } xR_kp, \\ 0 & \text{otherwise,} \end{cases} \\
&= \begin{cases} [\![\Gamma_j(x) \cap \Gamma_i(p)]\!] & \text{if } xR_kp, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
$$

□

The *support* of a vector $v \in V$ is the set $\mathrm{supp}(v) = \{x \in X \mid v(x) \neq 0\}$.

Lemma 2.3.5 *Let $U \subset V$ be a subset of nonzero vectors. Suppose that for all subsets $S \subset U$ and all $u \in U \backslash S$, the symmetric difference of $\cup_{s \in S} \mathrm{supp}(s)$ and $\mathrm{supp}(u)$ is nonempty. Then U is linearly independent.*

Proof. Suppose that $\sum_{i=1}^{k} \alpha_i u_i = 0$, where $u_i \in U$. Let $S = \{u_2, u_3, \ldots, u_k\}$ and $u = u_1$. Then by assumption the symmetric difference of $\cup_{s \in S} \mathrm{supp}\, s$ and $\mathrm{supp}(u)$ is nonempty. In particular, the sum is nonzero unless $\alpha_1 = 0$. Proceeding by induction, we find that $\alpha_i = 0$ for all i, so U is linearly independent. □

We recall some facts about a special irreducible $\mathcal{T}$-module.

Lemma 2.3.6 *[84] For all i, j, k $(0 \le i, j, k \le d)$*

$$E_j^* A_k E_i^* [\![\Gamma_i(p)]\!] = p_{ij}^k [\![\Gamma_j(p)]\!].$$

Lemma 2.3.7 *[84] For all i, j, k $(0 \le i, j, k \le d)$ and for all $x \in X$*

$$E_i^* J E_k^* [\![x]\!] = \begin{cases} [\![\Gamma_i(p)]\!] & \text{if } xR_kp, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By Lemma 2.3.4,

$$E_j^* A_k E_i^* [\![\Gamma_i(p)]\!] = \sum_{xR_ip} E_j^* A_k E_i^* [\![x]\!] = \sum_{xR_ip} [\![\Gamma_k(x) \cap \Gamma_j(p)]\!] = \sum_{\substack{xR_ip \\ yR_kx \\ yR_jp}} p_{ij}^k [\![y]\!] = p_{ij}^k [\![\Gamma_j(p)]\!].$$

□

Theorem 2.3.8 *[84] There is an irreducible $\mathcal{T}$-module with basis $\{[\![\Gamma_i(p)]\!] \,|\, 0 \le i \le d\}$. We refer to as the* primary $\mathcal{T}$-module *and denote by $\mathcal{P}$.*

Proof. Observe that $\{[\![\Gamma_i(p)]\!] \,|\, 0 \le i \le d\}$ spans a $\mathcal{T}$-module by Lemma 2.3.6. This lemma also implies that this module is irreducible. These vectors are linearly independent by Lemma 2.3.5, so the result follows. □

3 The Subconstituent Algebra of a Latin Square

3.1 Latin Squares and Bose-Mesner Algebras

Let L denote a Latin square of order $n \geq 3$. Encode L with the set $X = \{(i, j, L(i, j)) \,|\, 1 \leq i, j \leq n\}$. (This encoding is sometimes referred to as the *orthogonal array representation* of L). To describe the associated Bose-Mesner algebra we first define five relations on X as follows: For all $x = (i, j, L(i, j))$, $x' = (i', j', L(i', j')) \in X$,

$$R_0 \text{ (identity):} \qquad xR_0x' \text{ if } x = x', \tag{3.1.1}$$

$$R_1 \text{ (same row):} \qquad xR_1x' \text{ if } i = i' \text{ and } x \neq x', \tag{3.1.2}$$

$$R_2 \text{ (same column):} \qquad xR_2x' \text{ if } j = j' \text{ and } x \neq x', \tag{3.1.3}$$

$$R_3 \text{ (same entry):} \qquad xR_3x' \text{ if } L(i, j) = L(i', j') \text{ and } x \neq x', \tag{3.1.4}$$

$$R_4 \text{ (everything else):} \qquad xR_4x' \text{ if } i \neq i',\ j \neq j', \text{ and } L(i, j) \neq L(i', j'). \tag{3.1.5}$$

Now $(X, \{R_i\}_{i=0}^{4})$ is a commutative association scheme, and hence the characteristic matrices of the five relations comprise the basis of Hadamard idempotents of a Bose-Mesner algebra $\mathcal{M}$, which we refer to as the *Bose-Mesner algebra of* L (see [5]).

The equivalence of Bose-Mesner algebras and commutative association schemes is well-known [10, 17]. Given a Latin square L and its association scheme $(X, \{R_i\}_{i=0}^{d})$, define for $i = 0, 1, 2, 3, 4$, matrices $A_i \in \mathbb{M}_X$ by $A_i(x, y) = 1$ if xR_iy and 0 otherwise $(x, y \in X)$. Then $\{A_i\}_{i=0}^{4}$ is the basis of Hadamard idempotents of a Bose-Mesner algebra, which we refer to as the *Bose-Mesner algebra of* L. We now recall some facts concerning the Bose-Mesner algebra of a Latin square.

For completeness we provide brief proofs of these facts. Their intersection numbers are well-known.

Theorem 3.1.1 *[5] Let L denote a Latin square of order n, and let $\mathcal{M}$ denote the Bose-Mesner algebra of $\mathcal{M}$.*

(i) *The intersection numbers of $\mathcal{M}$ are given by*

$$(p^0_{ij}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & n-1 & 0 & 0 & 0 \\ 0 & 0 & n-1 & 0 & 0 \\ 0 & 0 & 0 & n-1 & 0 \\ 0 & 0 & 0 & 0 & n^2-3n+2 \end{pmatrix},$$

$$(p^1_{ij}) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & n-2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & n-2 \\ 0 & 0 & 1 & 0 & n-2 \\ 0 & 0 & n-2 & n-2 & n^2-5n+6 \end{pmatrix},$$

$$(p^2_{ij}) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & n-2 \\ 1 & 0 & n-2 & 0 & 0 \\ 0 & 1 & 0 & 0 & n-2 \\ 0 & n-2 & 0 & n-2 & n^2-5n+6 \end{pmatrix},$$

$$(p^3_{ij}) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & n-2 \\ 0 & 1 & 0 & 0 & n-2 \\ 1 & 0 & 0 & n-2 & 0 \\ 0 & n-2 & n-2 & 0 & n^2-5n+6 \end{pmatrix},$$

$$(p_{ij}^4) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & n-3 \\ 0 & 1 & 0 & 1 & n-3 \\ 0 & 1 & 1 & 0 & n-3 \\ 1 & n-3 & n-3 & n-3 & n^2-6n+10 \end{pmatrix}.$$

Proof. Let $A_r = A_1 + I$, $A_c = A_2 + I$, and $A_e = A_3 + I$ where A_1, A_2, A_3 are the respective adjacency matrices of the relations R_1, R_2, R_3 on X. Then A_r is the matrix of the union of relations R_0 and R_1, etc. To compute p_{ij}^k we will compute A_iA_j for $0 \leq i, j \leq 3$. To do so, let $R_r = R_0 \cup R_1$, $R_c = R_0 \cup R_2$, $R_e = R_0 \cup R_3$. Note that

$$\begin{aligned} (A_rA_r)(x,y) &= \sum_{\gamma \in X} A_r(x,\gamma)A_r(\gamma,y) \\ &= \mid \{\gamma : \gamma R_r x, \gamma R_r y\} \mid \\ &= nA_r(x,y). \end{aligned}$$

i.e $A_rA_r = nA_r$. Similarly $A_c^2 = nA_c$ and $A_e^2 = nA_e$. Also note that

$$(A_rA_c)(x,y) = \mid \{\gamma : \gamma R_r x, \gamma R_c y\} \mid = 1.$$

Thus $A_rA_c = J$. Similarly $A_cA_r, A_rA_e, A_eA_r, A_cA_e$, and A_eA_c are equal to J. Now using the above computations we compute p_{ij}^h for $i \geq j$, $0 \leq i, j \leq 3$ and $0 \leq h \leq 4$:

$$\begin{aligned} A_1A_1 &= (A_r - I)(A_r - I) = A_r^2 - 2A_r + I = (n-2)(A_r - I) + (n-1)I \\ &= (n-2)A_1 + (n-1)A_0. \end{aligned}$$

Thus $p_{11}^0 = n-1$, $p_{11}^1 = n-2$, and $p_{11}^2 = p_{11}^3 = p_{11}^4 = 0$. Similarly we compute all other intersection numbers p_{ij}^h, for $0 \leq i, j \leq 3$, $i \geq j$, and $0 \leq h \leq 4$. Using the symmetry we deduce p_{ij}^h, for $0 \leq i, j \leq 3$, $i < j$. Now for the p_{ij}^h with at least one of i, j equal to 4, note that for $0 \leq h \leq d$, $\sum_{i=0}^d p_{ij}^h = k_j$ where k_j is the sum of each row in A_j (the valency). In the Latin square case $k_0 = 1$,$k_1 = k_2 = k_3 = n-1$, and $k_4 = n^2 - 3n + 2$. $\square$

3.2 Some Permutations

We now begin our study of the subconstituent algebra of the Bose-Mesner algebra of a Latin square. In this section we show that certain elements of the subcontsituent algebra induce permutations on $\Gamma_1(p)$, on $\Gamma_2(p)$, and on $\Gamma_3(p)$.

Notation 3.2.1 Let L denote a Latin square of order $n \geq 3$ and with symbol set $\{1, 2, \ldots, n\}$. Let X denote the set $\{(i, j, L(i, j)) \,|\, 1 \leq i, j \leq n\}$. Let $\mathcal{M}$ denote the Bose-Mesner algebra of L. Fix $p = (r_p, c_p, e_p) \in X$, and let $\mathcal{T}$ denote the subconstitutent algebra of $\mathcal{M}$ with respect to p.

Lemma 3.2.2 *With Notation 3.2.1, fix a permutation i, j, k of 1, 2, 3. For each $x \in \Gamma_i(p)$, the row of $E_i^* A_j E_k^*$ indexed by x has a unique entry equal to one and all other entries are equal to zero. For each $y \in \Gamma_k(p)$, the column of $E_i^* A_j E_k^*$ indexed by y has a unique entry equal to one and all other entries are equal to zero. All other entries of $E_i^* A_j E_k^*$ are zero.*

Proof. Immediate from the definitions of E_i^*, A_j, and E_k^*, and the fact that $p^j_{ik} = 1$ by Theorem 3.1.1. □

We note the action of $E_i^* A_j E_k^*$ on the standard basis of the standard module in the Latin square case.

Lemma 3.2.3 *With Notation 3.2.1, fix a permutation i, j, k of 1, 2, 3. Let $x \in X$. Then $E_i^* A_j E_k^* [\![x]\!] = \delta_{x(k),p(k)} [\![y]\!]$, where $y(i) = p(i)$, $y(j) = x(j)$, and $y(k)$ is uniquely determined by the Latin square property.*

Proof. The sum in Lemma 2.3.4 runs over all y such that $y(i) = p(i)$ and $y(j) = x(j)$ by the definitions of the relations. There is exactly one such y by the Latin square property. □

Lemma 3.2.4 *With Notation 3.2.1, for all permutations i, j, k of 1, 2, 3, the principal minor of $E_i^* A_j E_k^* A_i E_j^* A_k E_i^*$ indexed by $\Gamma_i(p)$ is a permutation matrix and every other entry of $E_i^* A_j E_k^* A_i E_j^* A_k E_i^*$ is zero.*

Proof. In light of Lemma 5.3.2, it is enough to treat just $E_1^* A_2 E_3^* A_1 E_2^* A_3 E_1^*$ acting on an element of $\Gamma_1(p)$. By Lemma 3.2.3, for all $(r_p, c, e) \in \Gamma_1(p)$

$$\begin{aligned} E_1^* A_2 E_3^* A_1 E_2^* A_3 E_1^* [\![r_p, c, e]\!] &= E_1^* A_2 E_3^* A_1 E_2^* [\![r, c_p, e]\!] \\ &= E_1^* A_2 E_3^* [\![r, c', e_p]\!] \\ &= [\![r_p, c', e']\!], \end{aligned}$$

where c' and e' are uniquely determined by the Latin square property. Note that $(r_p, c', e') \in \Gamma_1(p)$. Thus the principal minor of $E_1^* A_2 E_3^* A_1 E_2^* A_3 E_1^*$ indexed by $\Gamma_1(p)$ is a permutation matrix. All other entries are zero by Lemma 3.2.2. $\square$

Lemma 3.2.5 *With reference to Lemma 3.2.4, the following are equivalent.*

(i) $E_1^* A_2 E_3^* A_1 E_2^* A_3 E_1^*$ *induces a* k*-cycle on* $\Gamma_1(p)$ *of the form* $(r_p, c_1, e_1),\ (r_p, c_2, e_2),\ \ldots,\ (r_p, c_k, e_k)$.

(ii) $E_2^* A_3 E_1^* A_2 E_3^* A_1 E_2^*$ *induces a* k*-cycle on* $\Gamma_2(p)$ *of the form* $(r_1, c_p, e_1),\ (r_2, c_p, e_2),\ \ldots,\ (r_k, c_p, e_k)$.

(iii) $E_3^* A_1 E_2^* A_3 E_1^* A_2 E_3^*$ *induces a* k*-cycle on* $\Gamma_3(p)$ *of the form* $(r_1, c_2, e_p),\ (r_2, c_3, e_p),\ \ldots,\ (r_k, c_1, e_p)$.

Proof. (i)$\Rightarrow$(ii): By Lemma 3.2.3,

$$\begin{aligned} E_2^* A_3 E_1^* [\![r_p, c_i, e_i]\!] &= [\![r_i, c_p, e_i]\!], \\ E_2^* A_3 E_1^* [\![r_p, c_{i+1}, e_{i+1}]\!] &= [\![r_{i+1}, c_p, e_{i+1}]\!]. \end{aligned}$$

By (i), $E_1^* A_2 E_3^* A_1 E_2^* A_3 E_1^* [\![r_p, c_i, e_i]\!] = [\![r_p, c_{i+1}, e_{i+1}]\!]$. Hence

$$\begin{aligned} E_2^* A_3 E_1^* A_2 E_3^* A_1 E_2^* [\![r_i, c_p, e_i]\!] &= E_2^* A_3 E_1^* \cdot E_1^* A_2 E_3^* A_1 E_2^* A_3 E_1^* [\![r_p, c_i, e_i]\!] \\ &= E_2^* A_3 E_1^* [\![r_p, c_{i+1}, e_{i+1}]\!] \\ &= [\![r_{i+1}, c_p, e_{i+1}]\!]. \end{aligned}$$

$$[\![r_{i-1}, c_i, e_p]\!] \underset{321}{\overset{123}{\rightleftarrows}} [\![r_p, c_i, e_i]\!] \underset{132}{\overset{231}{\rightleftarrows}} [\![r_i, c_p, e_i]\!] \underset{213}{\overset{312}{\rightleftarrows}} [\![r_i, c_{i+1}, e_p]\!] \underset{321}{\overset{123}{\rightleftarrows}} [\![r_p, c_{i+1}, e_{i+1}]\!]$$

Figure 3.1: The interleaving of cycles (Lemma 3.2.5, Definition 3.2.7)

Thus (i) implies (ii). The other consequences are proven in a similar manner. See Figure 3.1, where we abbreviate $ijk = E_i^* A_j E_k^*$. □

Corollary 3.2.6 *With reference to Lemma 3.2.4, the cycle structure of the permutation on $\Gamma_i(p)$ induced by $E_i^* A_j E_k^* A_i E_j^* A_k E_i^*$ is independent of i, j, k for all permutations i, j, k of 1, 2, 3. We refer to the common cycle structure as the* cycle structure of L with respect to p.

Proof. Clear from Lemma 3.2.5. □

Definition 3.2.7 With reference to Lemma 3.2.5, we refer to the triple of k-cycles

$$\begin{aligned}
\mathcal{C}_1 &= ((r_p, c_1, e_1), (r_p, c_2, e_2), \ldots, (r_p, c_k, e_k)), \\
\mathcal{C}_2 &= ((r_1, c_p, e_1), (r_2, c_p, e_2), \ldots, (r_k, c_p, e_k)), \\
\mathcal{C}_3 &= ((r_1, c_2, e_p), (r_2, c_3, e_p), \ldots, (r_k, c_1, e_p))
\end{aligned}$$

as an *interleaved triple of k-cycles.*

3.3 Cycle Modules

We produce a $\mathcal{T}$-module for each interleaved triple of cycles. We will need to sum over elements of X with one of the three coordinates fixed. Since no triples other than those in X are considered here, we shall not explicitly write this condition. We shall place dots over the two coordinates which vary in the summation, e.g. $(r_i, \dot{c}, \dot{e})$, to remind ourselves that the triple must be an element of X. Thus although the two coordinates vary, they do not do so independently.

Theorem 3.3.1 *With Notation 3.2.1, fix an interleaved triple $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ of k-cycles as in Definition 3.2.7.*

(i) *For $1 \leq h \leq k$ and $0 \leq j \leq 4$, the vectors*

$$\begin{aligned}
u_{1,h} &:= [\![r_p, c_h, e_h]\!], \\
u_{2,h} &:= [\![r_h, c_p, e_h]\!], \\
u_{3,h} &:= [\![r_h, c_{h+1}, e_p]\!], \\
v_{1,h} &:= \sum_{(r_h,\dot{c},\dot{e})\in\Gamma_4(p)} [\![r_h, \dot{c}, \dot{e}]\!], \\
v_{2,h} &:= \sum_{(\dot{r},c_h,\dot{e})\in\Gamma_4(p)} [\![\dot{r}, c_h, \dot{e}]\!], \\
v_{3,h} &:= \sum_{(\dot{r},\dot{c},e_h)\in\Gamma_4(p)} [\![\dot{r}, \dot{c}, e_h]\!], \\
&[\![\Gamma_j(p)]\!]
\end{aligned}$$

span a $\mathcal{T}$-module. We refer to this $\mathcal{T}$-module as the cycle module *of $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ and denote it $W(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$.*

(ii) *The action of the generators $E_i^* A_j E_k^*$ of $\mathcal{T}$ is as shown in Figures 3.2–3.5, where the subscripts are taken modulo k. In Figures 3.2–3.5 we abbreviate $ijk = E_i^* A_j E_k^*$ and $\overline{ijk} = [\![\Gamma_i]\!] - E_i^* A_j E_k^*$. The action of $E_i^* A_4 E_k^*$ and E_i^* are omitted as they can be deduced from (2.1.3) and (2.2.11). All other omitted actions are zero.*

Proof. In light of Lemma 5.3.2, it is enough to prove the theorem for one of each type of vector. By Lemma 2.3.1, Theorem 3.1.1, and (2.2.11), the generators $E_i^* A_j E_k^*$ of $\mathcal{T}$ that act on $u_{1,h}$ in a nonzero manner have $ijk \in \{011, 101, 111, 23, 321, 421, 431, 341, 241, 441\}$. We do not derive formulae for $ijk \in \{341, 241, 441\}$ since their action is deduced from (2.1.3), and we do not do so for $ijk = 101$ since $E_1^* = E_1^* A_0 E_1^*$ acts as the identity on $u_{1,h}$ by (2.2.11). By Lemma 3.2.3, $E_2^* A_3 E_1^* u_{1,h} = u_{2,h}$ and $E_3^* A_2 E_1^* u_{1,h} = u_{3,h}$. The remaining actions are deduced using Lemma 2.3.4. Since $\Gamma_0(p) = \{p\}$ and $(r_p, c_h, e_h) R_1 p$, we find $E_0^* A_1 E_1^* u_{1,h} = [\![r_p, c_p, e_p]\!]$. Also,

$$E_1^* A_1 E_1^* u_{1,h} = \sum_{(r_p,\dot{c},\dot{e})\neq p,(r_p,c_h,e_h)} [\![r_p, \dot{c}, \dot{e}]\!] = [\![\Gamma_1(p)]\!] - u_{1,h},$$

$$\begin{aligned}
E_4^* A_2 E_1^* u_{1,h} &= \sum_{(\dot r, c_h, \dot e)\in\Gamma_4(p)} [\![\dot r, c_h, \dot e]\!] = v_{2,h},\\
E_4^* A_3 E_1^* u_{1,h} &= \sum_{(\dot r, \dot c, e_h)\in\Gamma_4(p)} [\![\dot r, \dot c, e_h]\!] = v_{3,h}.
\end{aligned}$$

By Lemma 2.3.1, Theorem 3.1.1, and (2.2.11), the generators $E_i^* A_j E_k^*$ of $\mathcal{T}$ which act on $v_{1,h}$ in a nonzero manner have $ijk \in \{044,\ 404,\ 124,\ 134,\ 144,\ 214,\ 234,\ 244,\ 314,\ 324,\ 344,\ 414,\ 424,\ 434,\ 444\}$. As above, we needn't derive formulae for $ijk \in \{144, 244, 344, 444, 044, 404\}$. By Lemma 2.3.4,

$$\begin{aligned}
E_3^* A_1 E_4^* v_{1,h} &= \sum_{(r_h, \dot c, \dot e)\in\Gamma_4(p)} E_3^* A_1 E_4^* [\![r_h, \dot c, \dot e]\!]\\
&= \sum_{(r_h, \dot c, \dot e)\in\Gamma_4(p)} [\![r_h, c_{h+1}, e_p]\!] = \sum_{\substack{(r_h,\dot c,\dot e)\\ \dot c\neq c_h, c_p}} u_{3,h}\\
&= (n-2)u_{3,h},\\
E_2^* A_1 E_4^* v_{1,h} &= (n-2)u_{2,h},\\
E_2^* A_3 E_4^* v_{1,h} &= \sum_{(r_h, \dot c, \dot e)\in\Gamma_4(p)} E_2^* A_3 E_4^* [\![r_h, \dot c, \dot e]\!]\\
&= \sum_{(r_h,\dot c,\dot e)\neq(r_h,c_p,e_h),(r_h,c_{h+1},e_p)} [\![r_h, \dot c, \dot e]\!]\\
&= [\![\Gamma_2(p)]\!] - u_{2,h},\\
E_1^* A_3 E_4^* v_{1,h} &= [\![\Gamma_1(p)]\!] - u_{1,h},\\
E_3^* A_2 E_4^* v_{1,h} &= [\![\Gamma_3(p)]\!] - u_{3,h},\\
E_1^* A_2 E_4^* v_{1,h} &= [\![\Gamma_1(p)]\!] - u_{1,h+1},\\
E_4^* A_1 E_4^* v_{1,h} &= \sum_{(r_h, \dot c, \dot e)\in\Gamma_4(p)} E_4^* A_1 E_4^* [\![r_h, \dot c, \dot e]\!]\\
&= \sum_{(r_h, \dot c, \dot e)\in\Gamma_4(p)} \sum_{(r_h,\dot c',\dot e')\neq(r_h,\dot c,\dot e),(r_h,c_{h+1},e_p),(r_h,c_p,e_h)} [\![r_h, \dot c', \dot e']\!]\\
&= (n-3)v_{1,h},
\end{aligned}$$

$$
\begin{aligned}
E_4^*A_2E_4^*v_{1,h} &= \sum_{(r_h,\dot{c},\dot{e})\in\Gamma_4(p)} E_4^*A_2E_4^*[\![r_h,\dot{c},\dot{e}]\!] \\
&= \sum_{(r_h,\dot{c},\dot{e})\in\Gamma_4(p)} \sum_{(\dot{r}',\dot{c},\dot{e}')\neq(r_h,\dot{c},\dot{e}),(r_p,\dot{c},\cdot),(\cdot,\dot{c},e_p)} [\![\dot{r}',\dot{c},\dot{e}']\!] \\
&= [\![\Gamma_4(p)]\!] - v_{2,h+1} - v_{1,h}, \\
E_4^*A_3E_4^*v_{1,h} &= [\![\Gamma_4(p)]\!] - v_{3,h} - v_{1,h}.
\end{aligned}
$$
□

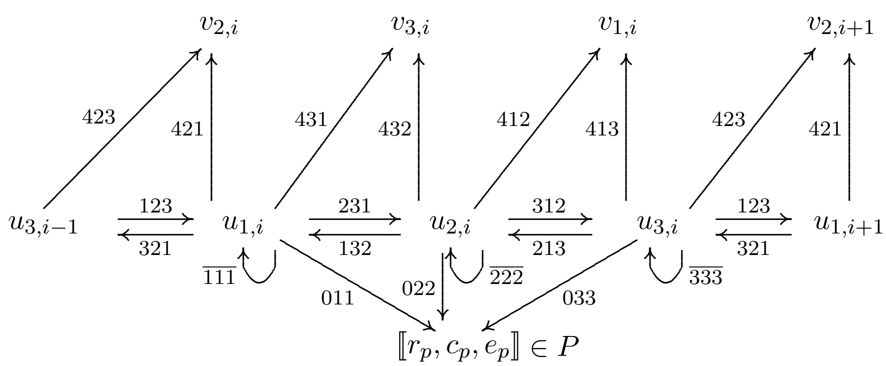

Figure 3.2: The action on $u_{1,i}$, $u_{2,i}$, $u_{3,i}\in W(\mathcal{C}_1,\mathcal{C}_2,\mathcal{C}_3)$

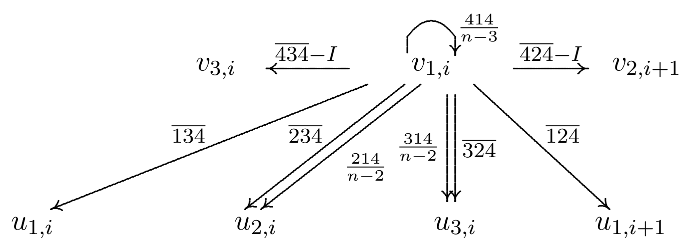

Figure 3.3: The action on $v_{1,i}\in W(\mathcal{C}_1,\mathcal{C}_2,\mathcal{C}_3)$

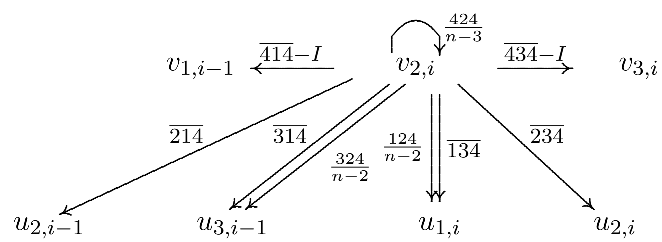

Figure 3.4: The action on $v_{2,i}\in W(\mathcal{C}_1,\mathcal{C}_2,\mathcal{C}_3)$

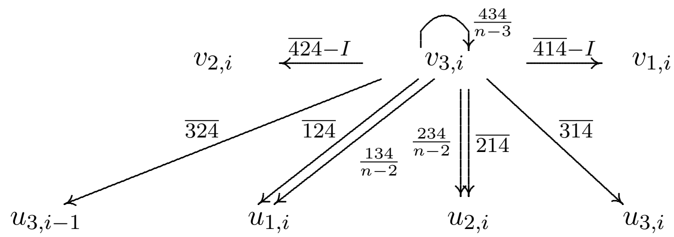

Figure 3.5: The action on $v_{3,i} \in W(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$

3.4 Decomposition into Irreducible Modules

In this section we describe the decomposition of each cycle module into irreducible $\mathcal{T}$-modules.

Lemma 3.4.1 *The primary module $\mathcal{P}$ is an irreducible $\mathcal{T}$-submodule of each cycle module.*

Proof. Clear from Theorems 2.3.8 and 3.3.1. $\square$

Lemma 3.4.2 *With Notation 3.2.1, fix an interleaved triple of k-cycles $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ as in Definition 3.2.7. Assume $1 \leq k < n-1$.*

(i) *There is an irreducible $\mathcal{T}$-submodule of $W(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$ spanned by*

$$\begin{aligned}
u_1^1 &:= \sum_{j=1}^{k} u_{1,j} - \frac{k}{n-1}[\![\Gamma_1(p)]\!] = \sum_{j=1}^{k} [\![r_p, c_j, e_j]\!] - \frac{k}{n-1}[\![\Gamma_1(p)]\!], \\
u_2^1 &:= \sum_{j=1}^{k} u_{2,j} - \frac{k}{n-1}[\![\Gamma_2(p)]\!] = \sum_{j=1}^{k} [\![r_j, c_p, e_j]\!] - \frac{k}{n-1}[\![\Gamma_2(p)]\!], \\
u_3^1 &:= \sum_{j=1}^{k} u_{3,j} - \frac{k}{n-1}[\![\Gamma_3(p)]\!] = \sum_{j=1}^{k} [\![r_j, c_{j+1}, e_p]\!] - \frac{k}{n-1}[\![\Gamma_3(p)]\!], \\
v_1^1 &:= \sum_{j=1}^{k} v_{1,j} - \frac{k}{n-1}[\![\Gamma_4(p)]\!] = \sum_{j=1}^{k} \sum_{(r_j,\dot{c},\dot{e})\in\Gamma_4(p)} [\![r_j, \dot{c}, \dot{e}]\!] - \frac{k}{n-1}[\![\Gamma_4(p)]\!], \\
v_2^1 &:= \sum_{j=1}^{k} v_{2,j} - \frac{k}{n-1}[\![\Gamma_4(p)]\!] = \sum_{j=1}^{k} \sum_{(\dot{r},c_j,\dot{e})\in\Gamma_4(p)} [\![\dot{r}, c_j, \dot{e}]\!] - \frac{k}{n-1}[\![\Gamma_4(p)]\!], \\
v_3^1 &:= \sum_{j=1}^{k} v_{3,j} - \frac{k}{n-1}[\![\Gamma_4(p)]\!] = \sum_{j=1}^{k} \sum_{(\dot{r},\dot{c},e_j)\in\Gamma_4(p)} [\![\dot{r}, \dot{c}, e_j]\!] - \frac{k}{n-1}[\![\Gamma_4(p)]\!].
\end{aligned}$$

We denote this $\mathcal{T}$-module $W^1(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$.

(ii) *The action of the generators $E_i^* A_j E_k^*$ on u_1^1, u_2^1, u_3^1, v_1^1, v_2^1, and v_3^1 is as shown in Figures 3.6–3.9 with $\epsilon = 1$, where the action of $E_i^* A_4 E_k^*$ and E_i^* are omitted as they can be deduced from (2.1.3) and (2.2.11). All other omitted actions are zero.*

(iii) *If $n \geq 5$, then u_1^1, u_2^1, u_3^1, v_1^1, v_2^1, and v_3^1 are linearly independent.*

Proof. (ii): In light of Lemma 5.3.2, it suffices to show that the generators $E_i^* A_j E_k^*$ of $\mathcal{T}$ act on u_1^1 and v_1^1 as claimed. As in the proof of Theorem 3.3.1, we need only consider the action of $E_i^* A_j E_k^*$ with $ijk \in \{011, 111, 231, 321, 421, 431\}$ on u_1^1. By Lemma 2.3.6 and Theorem 3.3.1,

$$
\begin{aligned}
E_0^* A_1 E_1^* u_1^1 &= k[\![r_p, c_p, e_p]\!] - \frac{k}{(n-1)}(n-1)[\![r_p, c_p, e_p]\!] = 0, \\
E_1^* A_1 E_1^* u_1^1 &= \sum_{j=1}^{k} \sum_{(r_p,\dot{c},\dot{e}) \neq p, (r_p, c_j, e_j)} [\![r_p, \dot{c}, \dot{e}]\!] - \frac{k}{(n-1)} E_1^* A_1 E_1^* [\![\Gamma_1(p)]\!] \\
&= k[\![\Gamma_1(p)]\!] - \sum_{j=1}^{k} [\![r_p, c_j, e_j]\!] - \frac{k}{(n-1)}(n-2)[\![\Gamma_1(p)]\!] = -u_1^1, \\
E_2^* A_3 E_1^* u_1^1 &= \sum_{j=1}^{k} [\![r_j, c_p, e_j]\!] - \frac{k}{(n-1)}[\![\Gamma_2(p)]\!] = u_2^1, \\
E_4^* A_3 E_1^* u_1^1 &= \sum_{j=1}^{k} \sum_{(\dot{r},\dot{c},e_j) \in \Gamma_4(p)} [\![\dot{r}, \dot{c}, e_j]\!] - \frac{k}{(n-1)} E_4^* A_3 E_1^* [\![\Gamma_1(p)]\!] \\
&= \sum_{j=1}^{k} \sum_{(\dot{r},\dot{c},e_j) \in \Gamma_4(p)} [\![\dot{r}, \dot{c}, e_j]\!] - \frac{k}{(n-1)}[\![\Gamma_4(p)]\!] = v_3^1, \\
E_3^* A_2 E_1^* u_1^1 &= u_3^1, \\
E_4^* A_2 E_1^* u_1^1 &= v_2^1, \\
E_0^* A_4 E_4^* v_1^1 &= \sum_{j=1}^{k} \sum_{(r_j,\dot{c},\dot{e}) \in \Gamma_4(p)} [\![r_p, c_p, e_p]\!] - \frac{k}{(n-1)} E_0^* A_4 E_4^* [\![\Gamma_4(p)]\!] \\
&= k(n-2)[\![r_p, c_p, e_p]\!] - \frac{k}{(n-1)}(n-1)(n-2)[\![r_p, c_p, e_p]\!] = 0,
\end{aligned}
$$

$$
\begin{aligned}
E_4^*A_2E_4^*v_1^1 &= \sum_{j=1}^{k}\left(\llbracket\Gamma_4(p)\rrbracket - \sum_{(\dot{r},c_j,\dot{e})\in\llbracket\Gamma_4(p)\rrbracket}\llbracket\dot{r},c_j,\dot{e}\rrbracket\right) - \frac{k(n-3)}{(n-1)}\llbracket\Gamma_4(p)\rrbracket \\
&= k\llbracket\Gamma_4(p)\rrbracket - \sum_{j=1}^{k}\sum_{(\dot{r},c_j,\dot{e})\in\Gamma_4(p)}\llbracket\dot{r},c_j,\dot{e}\rrbracket - \frac{k(n-3)}{(n-1)}\llbracket\Gamma_4(p)\rrbracket \\
&= -v_1^1 - v_2^1, \\
E_1^*A_2E_4^*v_1^1 &= \sum_{j=1}^{k}\left(\llbracket\Gamma_1(p)\rrbracket - \sum_{j=1}^{k}\llbracket r_p, c_{j+1}, e_{j+1}\rrbracket\right) - \frac{k(n-2)}{(n-1)}\llbracket\Gamma_1(p)\rrbracket \\
&= -\sum_{j=1}^{k}\llbracket r_p, c_{j+1}, e_{j+1}\rrbracket + \frac{k}{(n-1)}\llbracket\Gamma_1(p)\rrbracket \\
&= -u_1^1.
\end{aligned}
$$

The other actions are similarly verified.

(i): By (ii), $W^1(\mathcal{C}_1,\mathcal{C}_2,\mathcal{C}_3)$ is a $\mathcal{T}$-module. To show that $W^1(\mathcal{C}_1,\mathcal{C}_2,\mathcal{C}_3)$ is irreducible, we show that $W^1(\mathcal{C}_1,\mathcal{C}_2,\mathcal{C}_3) \subseteq \mathcal{T}u$ for any nonzero $u \in W^1(\mathcal{C}_1,\mathcal{C}_2,\mathcal{C}_3)$. First suppose $E_i^*u \neq 0$ for some i with $1 \leq i \leq 3$: Say $i = 3$. Then $E_3^*u = \alpha u_3^1 \in E_3^*W^1(\mathcal{C}_1,\mathcal{C}_2,\mathcal{C}_3)$, and $E_1^*A_2E_3^*u = \alpha u_1^1$, are nonzero elements of $E_1^*W^1(\mathcal{C}_1,\mathcal{C}_2,\mathcal{C}_3)$. Now $W^1(\mathcal{C}_1,\mathcal{C}_2,\mathcal{C}_3) \subseteq \mathcal{T}u$ by Theorem 3.3.1.

Now suppose $E_i^*u = 0$ for $1 \leq i \leq 3$, so $u = \alpha_1v_1^1 + \alpha_2v_2^1 + \alpha_3v_3^1$ for some scalars α_i ($i = 1,2,3$) which are not all zero. Applying $E_2^*A_1E_4^*$, $E_1^*A_2E_4^*$, and $E_1^*A_3E_4^*$ to u we get (3.4.6)–(3.4.8). At least one of these coefficients is nonzero since (3.4.9) has no nonzero solution. Now $W^1(\mathcal{C}_1,\mathcal{C}_2,\mathcal{C}_3) \subseteq \mathcal{T}u$.

(iii): We now show that u_1^1, u_2^1, u_3^1, v_1^1, v_2^1, and v_3^1 are linearly independent whenever $n \geq 5$. We note that if $n \geq 5$, the supports are not distinct so the following argument wouldn't work. Suppose $u = \beta_1u_1^1 + \beta_2u_2^1 + \beta_3u_3^1 + \alpha_1v_1^1 + \alpha_2v_2^1 + \alpha_3v_3^1 = 0$. Then $\beta_1 = \beta_2 = \beta_3 = 0$ by Lemma 2.3.5. Now the following (3.4.6)–(3.4.8) are zero:

$$
\begin{aligned}
E_2^*A_1E_4^*u &= (n-2)\alpha_1u_2^1 - \alpha_2u_2^1 - \alpha_3u_2^1 \\
&= ((n-2)\alpha_1 - \alpha_2 - \alpha_3)u_2^1, && (3.4.6) \\
E_1^*A_2E_4^*u &= ((n-2)\alpha_2 - \alpha_1 - \alpha_3)u_1^1, && (3.4.7) \\
E_1^*A_3E_4^*u &= ((n-2)\alpha_3 - \alpha_1 - \alpha_2)u_1^1. && (3.4.8)
\end{aligned}
$$

Since u_1^1 and u_2^1 are linearly independent, their coefficients in (3.4.6)–(3.4.8) are zero. Thus

$$(n-2)\alpha_1 - \alpha_2 - \alpha_3 = (n-2)\alpha_2 - \alpha_1 - \alpha_3 = (n-2)\alpha_3 - \alpha_1 - \alpha_2 = 0. \tag{3.4.9}$$

Equations (3.4.9) have no non-zero solutions. Hence u_1^1, u_2^1, u_3^1, v_1^1, v_2^1, and v_3^1 are linearly independent. □

The case $k = n-1$ which was excluded from Lemma 3.4.2 behaves differently.

Lemma 3.4.3 *With reference to Theorem 3.3.1, suppose $k = n-1$. Then*

$$\begin{aligned}
\sum_{i=1}^{k} u_{j,i} &= [\![\Gamma_j(p)]\!] \qquad (j = 1,2,3), \\
\sum_{i=1}^{k} v_{j,i} &= [\![\Gamma_4(p)]\!] \qquad (j = 1,2,3).
\end{aligned}$$

Proof. Clear. □

Lemma 3.4.4 *With Notation 3.2.1, fix an interleaved triple of k-cycles $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ as in Definition 3.2.7. Assume $1 < k \leq n-1$. Let $\epsilon \neq 1$ be a k^{th} root of unity.*

(i) *There is an irreducible $\mathcal{T}$-submodule of $W(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$ spanned by*

$$\begin{aligned}
u_1^\epsilon &:= \sum_{j=1}^{k} \epsilon^j u_{1,j} = \sum_{j=1}^{k} \epsilon^j [\![r_p, c_j, e_j]\!], \\
u_2^\epsilon &:= \sum_{j=1}^{k} \epsilon^j u_{2,j} = \sum_{j=1}^{k} \epsilon^j [\![r_j, c_p, e_j]\!], \\
u_3^\epsilon &:= \sum_{j=1}^{k} \epsilon^j u_{3,j} = \sum_{j=1}^{k} \epsilon^j [\![r_j, c_{j+1}, e_p]\!], \\
v_1^\epsilon &:= \sum_{j=1}^{k} \epsilon^j v_{1,j} = \sum_{j=1}^{k} \sum_{(r_j, \dot{c}, \dot{e}) \in \Gamma_4(p)} \epsilon^j [\![r_j, \dot{c}, \dot{e}]\!], \\
v_2^\epsilon &:= \sum_{j=1}^{k} \epsilon^j v_{2,j} = \sum_{j=1}^{k} \sum_{(\dot{r}, c_j, \dot{e}) \in \Gamma_4(p)} \epsilon^j [\![\dot{r}, c_j, \dot{e}]\!],
\end{aligned}$$

$$v_3^\epsilon := \sum_{j=1}^{k} \epsilon^j v_{3,j} = \sum_{j=1}^{k} \sum_{(\dot{r},\dot{c},e_j)\in\Gamma_4(p)} \epsilon^j [\![\dot{r},\dot{c},e_j]\!].$$

We denote this $\mathcal{T}$-module $W^\epsilon(\mathcal{C}_1,\mathcal{C}_2,\mathcal{C}_3)$.

(ii) *The action of the generators $E_i^* A_j E_k^*$ on these vectors is as shown in Figures 3.6–3.9, where the action of $E_i^* A_4 E_k^*$ and E_i^* are omitted as they can be deduced from (2.1.3) and (2.2.11). All other omitted actions are zero.*

(iii) *If $n \geq 5$, then u_1^ϵ, u_2^ϵ, u_3^ϵ, v_1^ϵ, v_2^ϵ, and v_3^ϵ are linearly independent.*

Proof. (ii): The action follows from Theorem 3.3.1. Note that $k > 1$, so the sum of all k^{th} roots of unity is zero. Thus for example,

$$\begin{aligned}
E_1^* A_2 E_4^* v_1^\epsilon &= \sum_{j=1}^{k} \epsilon^j E_1^* A_2 E_4^* v_{1,j} \\
&= \sum_{j=1}^{k} \epsilon^j ([\![\Gamma_1(p)]\!] - u_{1,j+1}) \\
&= (\sum_{j=1}^{k} \epsilon^j)[\![\Gamma_1(p)]\!] - \epsilon^{-1} \sum_{j=1}^{k} \epsilon^{j+1} u_{1,j+1} \\
&= -\epsilon^{-1} u_1^\epsilon, \\
E_4^* A_3 E_4^* v_1^\epsilon &= \sum_{j=1}^{k} \epsilon^j E_2^* A_3 E_4^* v_{1,j} \\
&= \sum_{j=1}^{k} \epsilon^j ([\![\Gamma_4(p)]\!] - v_{3,j} - v_{1,j}) \\
&= -v_3^\epsilon - v_1^\epsilon \\
E_4^* A_2 E_1^* u_1^\epsilon &= \sum_{j=1}^{k} \epsilon^j E_4^* A_2 E_1^* u_{1,j} \\
&= \sum_{j=1}^{k} \epsilon^j v_{2,j} = v_2^\epsilon.
\end{aligned}$$

The remaining actions are computed similarly.

(i): Arguing as in Lemma 3.4.4 gives that $\mathrm{span}(u_1^\epsilon, u_2^\epsilon, u_3^\epsilon v_1^\epsilon, v_2^\epsilon, v_3^\epsilon)$ is closed under the action of the generators $E_i^* A_j E_k^*$ of $\mathcal{T}$ and that $W^\epsilon(\mathcal{C}_1,\mathcal{C}_2,\mathcal{C}_3)$ is irreducible.

(iii): To show linearly independence, let $v = \beta_1 u_1^\epsilon + \beta_2 u_2^\epsilon + \beta_3 u_3^\epsilon + \alpha_1 v_1^\epsilon + \alpha_2 v_2^\epsilon + \alpha_3 v_3^\epsilon = 0$. By Lemma 2.3.5 we have $\beta_1 = \beta_2 = \beta_3 = 0$. To show that $\alpha_1 = \alpha_2 = \alpha_3 = 0$, apply $E_1^* A_2 E_4^*$, $E_1^* A_3 E_4^*$, and $E_2^* A_1 E_4^*$ to v to find

$$\begin{aligned}
(-\epsilon\alpha_1 + (n-2)\alpha_2 - \alpha_3)u_1^\epsilon &= 0, \\
(-\alpha_1 - \epsilon\alpha_2 + (n-2)\alpha_3)u_1^\epsilon &= 0, \\
((n-2)\alpha_1 - \alpha_2 - \alpha_3)u_2^\epsilon &= 0.
\end{aligned}$$

Thus

$$-\alpha_1\epsilon + \alpha_2(n-2) - \alpha_3 = -\alpha_1 - \alpha_2\epsilon + (n-2)\alpha_3 = (n-2)\alpha_1 - \alpha_2 - \alpha_3 = 0.$$

Solving these three equations gives $\alpha_1 = \alpha_2 = \alpha_3 = 0$. Hence u_1^ϵ, u_2^ϵ, u_3^ϵ, v_1^ϵ, v_2^ϵ, and v_3^ϵ are linearly independent. $\square$

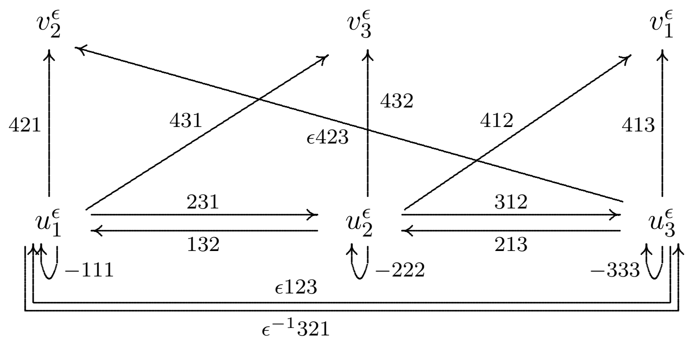

Figure 3.6: The action on $u_i^\epsilon \in W^\epsilon(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$

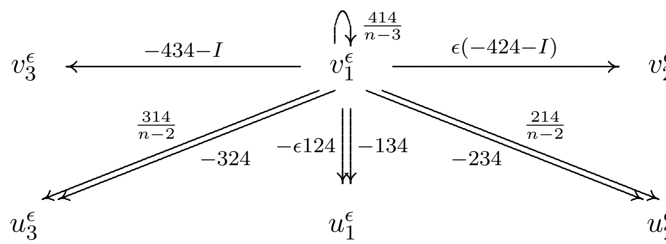

Figure 3.7: The action on $v_1^\epsilon \in W^\epsilon(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$

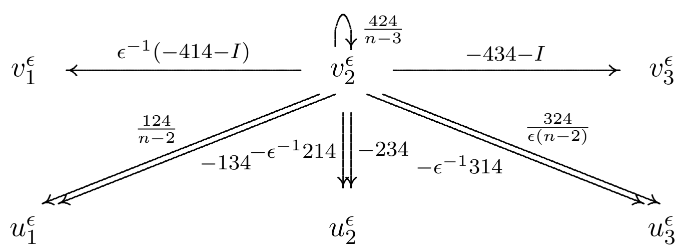

Figure 3.8: The action on $v_2^\epsilon \in W^\epsilon(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$

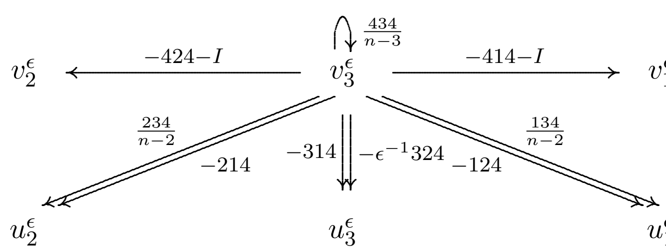

Figure 3.9: The action on $v_3^\epsilon \in W^\epsilon(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$

Lemma 3.4.5 *With Notation 3.2.1, fix an interleaved triple of k-cycles $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ as in Definition 3.2.7. Let ϵ and δ be distinct k^{th} roots of unity. Then $W^\epsilon(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$ and $W^\delta(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$ are non-isomorphic $\mathcal{T}$-modules. Moreover, $W^\epsilon(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$ and $W^\delta(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$ are orthogonal.*

Proof. Suppose $\epsilon \neq 1$. Observe that

$$\begin{aligned} E_1^* A_2 E_3^* A_1 E_2^* A_3 E_1^* u_1^\epsilon &= E_1^* A_2 E_3^* A_1 E_2^* A_3 E_1^* \sum_{j=1}^{k} \epsilon^j [\![r_p, c_j, e_j]\!] \\ &= \sum_{j=1}^{k} \epsilon^j [\![r_p, c_{j+1}, e_{j+1}]\!] \\ &= \epsilon^{-1} \sum_{j=1}^{k} \epsilon^{j+1} [\![r_p, c_{j+}, e_{j+1}]\!] \\ &= \epsilon^{-1} u_1^\epsilon. \end{aligned}$$

Also $E_1^* A_2 E_3^* A_1 E_2^* A_3 E_1^*$ acts as zero on u_2^ϵ, u_3^ϵ, v_1^ϵ, v_2^ϵ, v_3^ϵ. Similarly, if $\delta \neq 1$, then $E_1^* A_2 E_3^* A_1 E_2^* A_3 E_1^* u_1^\delta = \delta^{-1} u_1^\delta$, and $E_1^* A_2 E_3^* A_1 E_2^* A_3 E_1^*$ acts as zero on u_2^δ, u_3^δ, v_1^δ, v_2^δ, v_3^δ. Thus the result holds in this case. Suppose $\delta = 1$. Now by Lemma 2.3.6 and Theorem 3.1.1,

$$\begin{aligned} E_1^* A_2 E_3^* A_1 E_2^* A_3 E_1^* u_1^1 &= E_1^* A_2 E_3^* A_1 E_2^* A_3 E_1^* \Big(\sum_{j=1}^{k} [\![r_p, c_j, e_j]\!] - \frac{k}{n-1} [\![\Gamma_1(p)]\!]\Big) \\ &= \sum_{j=1}^{k} [\![r_p, c_{j+1}, e_{j+1}]\!] - \frac{k}{n-1} [\![\Gamma_1(p)]\!] \\ &= u_1^1. \end{aligned}$$

Also $E_1^* A_2 E_3^* A_1 E_2^* A_3 E_1^*$ acts as zero on u_2^1, u_3^1, v_1^1, v_2^1, v_3^1. It follows that $W^\epsilon(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$ and $W^\delta(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$ are non-isomorphic. The orthogonality of these two modules follows from Lemma 2.3.3. □

Theorem 3.4.6 *With Notation 3.2.1, fix an interleaved triple of k-cycles $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$.*

(i) *If $k \neq n-1$, then $W(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$ has orthogonal direct decomposition into irreducible*

$\mathcal{T}$*-modules*

$$W(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3) = P \oplus \bigoplus_{\substack{\epsilon \in \mathbb{C} \\ \epsilon^k = 1}} W^\epsilon(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3).$$

(ii) *If* $k = n - 1$*, then* $W(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$ *has orthogonal direct decomposition into irreducible* $\mathcal{T}$*-modules*

$$W(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3) = P \oplus \bigoplus_{\substack{\epsilon \in \mathbb{C} \\ \epsilon^k = 1, \epsilon \neq 1}} W^\epsilon(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3).$$

Proof. It is clear from Lemmas 2.3.3 and 3.4.5 that the sums in (i) and (ii) are orthogonal (and hence direct). First suppose $k < n - 1$. Then by orthogonality, the sum in (i) spans a subspace of dimension $6k+5$. But by construction, $\dim W(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3) \leq 6k+5$. Thus (i) holds. Next suppose $k = n - 1$. Then by orthogonality, the sum in (ii) spans a subspace of dimension $6k - 1$. Also by Lemma 3.4.3 $\dim W(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3) \leq 6k - 1$. Thus (ii) holds. $\square$

Note that Theorem 3.4.6 holds with no restriction on n, but for $n \leq 2$, the only irreducible $\mathcal{T}$-module is the primary module. For $n = 3$, there is just one main class of Latin squares, that of the Cayley table of $\mathbb{Z}_3$, which has cycle structure 2^1 with respect to all points (see Section 3.9).

3.5 Some Intermediate Modules

We give a common generalization of Theorem 3.3.1 and Lemmas 3.4.2 and 3.4.4 which allows us to produce some nice submodules of a cycle module.

Lemma 3.5.1 *With Notation 3.2.1, fix an interleaved triple of* k*-cycles* $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ *as in Definition 3.2.7. Pick nonnegative integers* ℓ *and* m *such that* $k = \ell m$*. Let* ϵ *be any* *complex m^{th} root of 1. For $i = 1, \ldots, \ell$ set*

$$\begin{aligned}
u_{1,i}^{\epsilon,\ell} &= \sum_{j=1}^{m} \epsilon^j [\![r_p, c_{j\ell+i}, e_{j\ell+i}]\!], \\
u_{2,i}^{\epsilon,\ell} &= \sum_{j=1}^{m} \epsilon^j [\![r_{j\ell+i}, c_p, e_{j\ell+i}]\!], \\
u_{3,i}^{\epsilon,\ell} &= \sum_{j=1}^{m} \epsilon^j [\![r_{j\ell+i-1}, c_{j\ell+i}, e_p]\!], \\
v_{1,i}^{\epsilon,\ell} &= \sum_{j=1}^{m} \sum_{(r_{j\ell+i},\dot{c},\dot{e})\in\Gamma_4(p)} \epsilon^j [\![r_{j\ell+i}, \dot{c}, \dot{e}]\!], \\
v_{2,i}^{\epsilon,\ell} &= \sum_{j=1}^{m} \sum_{(\dot{r},c_{j\ell+i},\dot{e})\in\Gamma_4(p)} \epsilon^j [\![\dot{r}, c_{j\ell+i}, \dot{e}]\!], \\
v_{3,i}^{\epsilon,\ell} &= \sum_{j=1}^{m} \sum_{(\dot{r},\dot{c},e_{j\ell+i})\in\Gamma_4(p)} \epsilon^j [\![\dot{r}, \dot{c}, e_{j\ell+i}]\!].
\end{aligned}$$

For $t = 1, 2, 3$, let $\mathcal{U}_t = \{u_{t,i}^{\epsilon,\ell}\}_{i=1}^{\ell}$ and $\mathcal{V}_t = \{v_{t,i}^{\epsilon,\ell}\}_{i=1}^{\ell}$. Then $(\cup_{t=1}^{3}\mathcal{U}_t) \cup (\cup_{t=1}^{3}\mathcal{V}_t) \cup \{[\![\Gamma_t]\!]\}_{t=0}^{4}$ spans a $\mathcal{T}$-module. Moreover if $n \geq 5$, then the following hold.

- **(i)** *If $k < n-1$ or $\epsilon \neq 1$, then the $6\ell + 5$ vectors in $(\cup_{t=1}^{3}\mathcal{U}_t) \cup (\cup_{t=1}^{3}\mathcal{V}_t) \cup \{[\![\Gamma_t]\!]\}_{t=0}^{4}$ are linearly independent.*
- **(ii)** *If $k = n-1$ and $\epsilon = 1$, then $(\cup_{t=1}^{3}\mathcal{U}_t) \cup (\cup_{t=1}^{3}\mathcal{V}_t) \cup \{[\![\Gamma_t]\!]\}_{j=0}^{4}$ span a $(6\ell-1)$-dimensional $\mathcal{T}$-module.*

Proof. It is easy to see from Lemma 2.3.6 and Theorem 3.3.1 that span$((\cup_{t=1}^{3}u_t) \cup (\cup_{t=1}^{3}v_t) \cup \{[\![\Gamma_t]\!]\}_{t=0}^{4})$ is closed under the action of the generators $E_i^* A_j E_k^*$ of $\mathcal{T}$, and hence is $\mathcal{T}$-module.

(i): The dimension of this module is at most $6\ell + 5$ by construction, and it is at least this large since it contains the primary module and ℓ-many 6-dimensional orthogonal irreducible submodules by Lemmas 3.4.2 and 3.4.4. Hence equality holds. It follows that the given vectors are linearly independent.

(ii): Suppose $\epsilon = 1$ and $k = n - 1$. The dimension is at most $6\ell - 1$ by construction and Lemma 3.4.3. It is at least this large by Lemma 3.4.4. Hence equality holds. □

Parts (i) and (ii) of Lemma 3.5.1 fail if $n \leq 5$ since the dimension of E_4^*V is too small.

We note that the cycle module $W(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$ appears in Lemma 3.5.1 in the case $m = 1$, $\ell = k$ (in which case $\epsilon = 1$). The irreducible submodule $W^\epsilon(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$ appears in Lemma 3.5.1 in the case $m = k$, $\ell = 1$.

3.6 Collecting Cycle Modules

We begin by extending Notation 3.2.1. To avoid degenerate situations, only consider Latin squares with order at least 5.

Notation 3.6.1 Let L denote a Latin square of order $n \geq 5$ and with symbol set $\{1, 2, \dots, n\}$. Let X denote the set $\{(i, j, L(i, j)) \,|\, 1 \leq i, j \leq n\}$. Let $\mathcal{M}$ denote the Bose-Mesner algebra of L. Fix $p = (r_p, c_p, e_p) \in X$, and let $\mathcal{T}$ denote the subconstitutent algebra of $\mathcal{M}$ with respect to (r_p, c_p, e_p). Let $\mathcal{I}^1$, $\mathcal{I}^2$, …, $\mathcal{I}^m$ denote the interleaved cycles of L with respect to p. Denote the elements of $\mathcal{I}^j$ as $\mathcal{C}_1^j$, $\mathcal{C}_2^j$, $\mathcal{C}_3^j$. Use $X(j)$ to refer to object X associated with $W(\mathcal{C}_1^j, \mathcal{C}_2^j, \mathcal{C}_3^j)$; for example $u_1^\epsilon(j)$.

Lemma 3.6.2 *With Notation 3.6.1. Fix two distinct interleaved triples of cycles $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ and $\mathcal{C}_1'$, $\mathcal{C}_2'$, $\mathcal{C}_3'$. Suppose the $\mathcal{C}_i$ are k-cycles and the $\mathcal{C}_i'$ are k'-cycles. Let ℓ be a positive integer such that $\ell | k$ and $\ell | k'$, and let ϵ be an ℓ^{th} root of unity. Then $W^\epsilon(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$ and $W^\epsilon(\mathcal{C}_1', \mathcal{C}_2', \mathcal{C}_3')$ are isomorphic $\mathcal{T}$-modules.*

Proof. Note that neither k nor k' is $n-1$ since there are two distinct interleaved triples of cycles. Thus $W^1(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$ and $W^1(\mathcal{C}_1', \mathcal{C}_2', \mathcal{C}_3')$ are both defined. However, the case $\epsilon = 1$ need not be treated separately. Suppose the modules $W^\epsilon(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$ and $W^\epsilon(\mathcal{C}_1', \mathcal{C}_2', \mathcal{C}_3')$ have respective bases $\{u_1^\epsilon, u_2^\epsilon, u_3^\epsilon, v_1^\epsilon, v_2^\epsilon, v_3^\epsilon\}$ and $\{s_1^\epsilon, s_2^\epsilon, s_3^\epsilon, t_1^\epsilon, t_2^\epsilon, t_3^\epsilon\}$. Define a linear map $\phi : W^\epsilon(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3) \to W^\epsilon(\mathcal{C}_1', \mathcal{C}_2', \mathcal{C}_3')$ by $\phi(u_i^\epsilon) = s_i^\epsilon$ and $\phi(v_i^\epsilon) = t_i^\epsilon$ $(1 \leq i \leq 3)$. It is clear that ϕ is a bijection. To show that for all $A \in \mathcal{T}$ and $v \in V$, $\phi(Av) = A\phi(v)$ it is enough to treat the case where A is of the form $E_i^* A_j E_k^*$ and v is one of the basis elements u_i^ϵ or v_i^ϵ. For example,

$$\phi(E_3^* A_2 E_1^* u_1^\epsilon) = \phi(\epsilon v_2^\epsilon) = \epsilon\phi(u_3^\epsilon) = \epsilon v_3^\epsilon = E_3^* A_2 E_1^* s_1^\epsilon = E_3^* A_2 E_1^* \phi(u_1^\epsilon).$$

The result follows from Theorem 3.3.1 (Figures 3.6–3.9). □

Lemma 3.6.3 *With Notation 3.6.1. If ϵ and δ are both primitive ℓ^{th} roots of unity, then*

$$\operatorname{mult}(W^{\epsilon}(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)) = \operatorname{mult}(W^{\delta}(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)).$$

Proof. Clear since the multiplicity of each is the number of interleaved cycles with length divisible by ℓ. □

In contrast to the situation for non-isomorphic irreducible $\mathcal{T}$-modules, the isomorphic irreducible $\mathcal{T}$-modules contained in distinct cycle modules are generally not orthogonal to one another. It turns out that with the exception of the dependencies noted in the following lemma, isomorphic irreducible $\mathcal{T}$-modules constructed so far are linearly independent.

Lemma 3.6.4 *With Notation 3.6.1, suppose $m \geq 1$ for $i = 1$, 2, 3,*

$$\sum_{j=1}^{m} u_i^1(j) = 0 \quad \textit{and} \quad \sum_{j=1}^{m} v_i^1(j) = 0.$$

Proof. Suppose that $\mathcal{C}_1^j$ has length $k(j)$ $(1 \leq h \leq m)$. Recall that

$$u_1^1(j) = \sum_{(r_p,\dot{c},\dot{e}) \in \mathcal{C}_1^j} [\![r_p, \dot{c}, \dot{e}]\!] - \frac{k(j)}{(n-1)} [\![\Gamma_1(p)]\!],$$

so

$$\begin{aligned} \sum_{j=1}^{m} u_1^1(j) &= \sum_{j=1}^{m} \sum_{(r_p,\dot{c},\dot{e}) \in \mathcal{C}_1^j} [\![r_p, \dot{c}, \dot{e}]\!] - \sum_{j=1}^{m} \frac{k(j)}{(n-1)} [\![\Gamma_1(p)]\!] \\ &= [\![\Gamma_1(p)]\!] - [\![\Gamma_1(p)]\!] = 0. \end{aligned}$$

Similarly, $\sum_{j=1}^{m} u_2^1(j) = \sum_{j=1}^{m} u_3^1(j) = 0$. Note that

$$v_1^1(j) = \sum_{h=1}^{k(j)} \sum_{(r_h(j),\dot{c},\dot{e}) \in [\![\Gamma_4(p)]\!]} [\![r_h(j), \dot{e}, \dot{c}]\!] - \frac{k(j)}{(n-1)} [\![\Gamma_4(p)]\!].$$

Thus

$$\begin{aligned}\sum_{j=1}^{m} v_1^1(j) &= \sum_{j=1}^{m}\sum_{h=1}^{k(j)} \sum_{(r_h(j),\dot{c},\dot{e})\in[\![\Gamma_4(p)]\!]} [\![r_h(j),\dot{c},\dot{e}]\!] - [\![\Gamma_4(p)]\!] \\ &= [\![\Gamma_4(p)]\!] - [\![\Gamma_4(p)]\!] = 0.\end{aligned}$$

Similarly, $\sum_{j=1}^{m} v_2^1(j) = \sum_{j=1}^{m} v_3^1(j) = 0$. □

When $m = 1$, Lemma 3.6.4 restates Lemma 3.4.3.

Lemma 3.6.5 *With Notation 3.6.1, the following set is linearly independent:*

$$\cup_{j=1}^{m-1}\{u_1^1(j), u_2^1(j), u_3^1(j), v_1^1(j), v_2^1(j), v_3^1(j)\}.$$

Proof. For $1 \leq j \leq m$, let

$$w_1^1(j) = u_1^1(j) + \frac{k(j)}{n-1}[\![\Gamma_1(p)]\!] = \sum_{h=1}^{k(j)} [\![r_p, c_h(j), e_h(j)]\!].$$

Since the first coordinate of the support of each $w_1^1(j)$ are disjoint, the set $\{w_1^1(j)\}_{j=1}^{m}$ is linearly independent by Lemma 2.3.5. It follows that $\{u_1^1(j)\}_{j=1}^{m-1}$ is linearly independent. Similarly $\{u_2^1(j)\}_{j=1}^{m-1}$ and $\{u_3^1(j)\}_{j=1}^{m-1}$ are linearly independent sets.

Let

$$u = \sum_{h=1}^{3}\sum_{j=1}^{m-1} \alpha_h^{(j)} u_h^1(j) + \sum_{h=1}^{3}\sum_{j=1}^{m-1} \beta_h^{(j)} v_h^1(j),$$

and suppose $u = 0$. Then $\alpha_1^{(j)} = \alpha_2^{(j)} = \alpha_3^{(j)} = 0$ by Lemma 2.3.5 and the above. Applying $E_1^* A_2 E_4^*$, $E_1^* A_3 E_4^*$, and $E_2^* A_1 E_4^*$ to u gives

$$\begin{aligned}
0 &= (-\beta_1^{(1)} + (n-2)\beta_2^{(1)} - \beta_3^{(1)})u_1^1(1) + \cdots \\
&\qquad + (-\beta_1^{(m-1)} + (n-2)\beta_2^{(m-1)} - \beta_3^{(m-1)})u_1^1(m-1), \\
0 &= (-\beta_1^{(1)} - \beta_2^{(1)} + (n-2)\beta_3^{(1)})u_1^1(1) + \cdots \\
&\qquad + (-\beta_1^{(m-1)} - \beta_2^{(m-1)} + (n-2)\beta_3^{(m-1)})u_1^1(m-1), \\
0 &= ((n-2)\beta_1^{(1)} - \beta_2^{(1)} - \beta_3^{(1)})u_2^1(1) + \cdots \\
&\qquad + ((n-2)\beta_1^{(m-1)} - \beta_2^{(m-1)} - \beta_3^{(m-1)})u_2^1(m-1).
\end{aligned}$$

Hence for $1 \leq j \leq m$,

$$\begin{aligned}
-\beta_1^{(j)} + (n-2)\beta_2^{(j)} - \beta_3^{(j)} &= 0, \\
-\beta_1^{(j)} - \beta_2^{(j)} + (n-2)\beta_3^{(j)} &= 0, \\
(n-2)\beta_1^{(j)} - \beta_2^{(j)} - \beta_3^{(j)} &= 0.
\end{aligned}$$

As in the proof of Lemma 3.4.2, it follows that $\beta_1^{(j)} = \beta_2^{(j)} = \beta_3^{(j)} = 0$ for all j $(1 \leq j \leq m-1)$. $\square$

The choice of omission in Lemma 3.6.5 is was arbitrary.

Lemma 3.6.6 *With Notation 3.6.1, let ϵ be a primitive ℓ^{th} root of unity other than 1. Let $\mathcal{I}^{i_1}$, $\mathcal{I}^{i_2}$, $\ldots$, $\mathcal{I}^{i_p}$ be all interleaved cycles with order divisible by ℓ. Then the following set is linearly independent:*

$$\cup_{j=1}^{p}\{u_1^\epsilon(i_j), u_2^\epsilon(i_j), u_3^\epsilon(i_j), v_1^\epsilon(i_j), v_2^\epsilon(i_j), v_3^\epsilon(i_j)\}.$$

Proof. Let

$$u = \sum_{h=1}^{3}\sum_{j=1}^{p}\alpha_h^{(j)}u_h^\epsilon(i_j) + \sum_{h=1}^{3}\sum_{j=1}^{p}\beta_h^{(j)}v_h^\epsilon(i_j),$$

and suppose $u = 0$. Then $\alpha_1^{(j)} = \alpha_2^{(j)} = \alpha_3^{(j)} = 0$ $(1 \leq j \leq p)$ by Lemma 2.3.5. Applying $E_1^* A_2 E_4^*$, $E_1^* A_3 E_4^*$, and $E_2^* A_1 E_4^*$ to u gives

$$\begin{aligned}
0 &= (-\beta_1^{(1)}\epsilon + (n-2)\beta_2^{(1)} - \beta_3^{(1)})u_1^\epsilon(i_1) + \cdots \\
&\qquad + (-\beta_1^{(p)}\epsilon + (n-2)\beta_2^{(p)} - \beta_3^{(p)})u_1^\epsilon(i_p), \\
0 &= (-\beta_1^{(1)} - \beta_2^{(1)}\epsilon + (n-2)\beta_3^{(1)})u_1^\epsilon(i_1) + \cdots \\
&\qquad + (-\beta_1^{(p)} - \beta_2^{(p)}\epsilon + (n-2)\beta_3^{(p)})u_1^\epsilon(i_p), \\
0 &= ((n-2)\beta_1^{(1)} - \beta_2^{(1)} - \beta_3^{(1)})u_2^\epsilon(i_1) + \cdots \\
&\qquad + ((n-2)\beta_1^{(p)} - \beta_2^{(p)} - \beta_3^{(p)})u_2^\epsilon(i_p).
\end{aligned}$$

But the set $\{u_h^\epsilon(i_j)\}_{j=1}^p$ is linearly independent by Lemma 2.3.5. Hence,

$$\begin{aligned}
-\beta_1^{(j)}\epsilon + (n-2)\beta_2^{(j)} - \beta_3(j) &= 0, \\
-\beta_1^{(j)} - \beta_2^{(j)}\epsilon + (n-2)\beta_3^{(j)} &= 0, \\
(n-2)\beta_1^{(j)} - \beta_2^{(j)} - \beta_3^{(j)} &= 0.
\end{aligned}$$

Thus $\beta_1(j) = \beta_2(j) = \beta_3(j) = 0$ for all j $(1 \leq j \leq p)$, as in the proof of Lemma 3.4.4. $\square$

The $u_i^\epsilon(j)$ are always pairwise orthogonal and orthogonal to all $v_i^\epsilon(j)$ by Lemma 2.3.5. One may apply Gram-Schmidt orthonormalization procedure to the basis of the above lemma if orthogonality is required. The resulting basis will no longer be so nicely related to the cycle modules.

Corollary 3.6.7 *With Notation 3.6.1,*

$$\dim \sum_{j=1}^{m} W(\mathcal{C}_1^j, \mathcal{C}_2^j, \mathcal{C}_3^j) = 6n - 7.$$

Proof. Recall that

$$W(\mathcal{C}_1^j, \mathcal{C}_2^j, \mathcal{C}_3^j) \cap P^{\perp} = \begin{cases} \bigoplus_{\epsilon^{k(j)}=1} W^{\epsilon}(\mathcal{C}_1^j, \mathcal{C}_2^j, \mathcal{C}_3^j) & \text{if } k(j) \neq n-1, \\ \bigoplus_{\substack{\epsilon^{k(j)}=1 \\ \epsilon \neq 1}} W^{\epsilon}(\mathcal{C}_1^j, \mathcal{C}_2^j, \mathcal{C}_3^j) & \text{if } k(j) = n-1. \end{cases}$$

Hence,

$$\dim W(\mathcal{C}_1^j, \mathcal{C}_2^j, \mathcal{C}_3^j) \cap \mathcal{P}^{\perp} = \begin{cases} 6k(j) & \text{if } k(j) \neq n-1, \\ 6k_j - 6 & \text{if } k(j) = n-1. \end{cases}$$

With the convention that $W^{\epsilon}(\mathcal{C}_1^j, \mathcal{C}_2^j, \mathcal{C}_3^j) = 0$ if ϵ is not a $k(j)^{\text{th}}$ root of unity,

$$\sum_{j=1}^{m} W(\mathcal{C}_1^j, \mathcal{C}_2^j, \mathcal{C}_3^j) \cap \mathcal{P}^{\perp} = \begin{cases} \bigoplus_{\epsilon} \sum_{j=1}^{m} W^{\epsilon}(\mathcal{C}_1^j, \mathcal{C}_2^j, \mathcal{C}_3^j) & \text{if } k(j) \neq n-1, \\ \bigoplus_{\substack{\epsilon^{k(j)}=1 \\ \epsilon \neq 1}} W^{\epsilon}(\mathcal{C}_1^j, \mathcal{C}_2^j, \mathcal{C}_3^j) & \text{if } k(j) = n-1. \end{cases}$$

Now by Lemma 3.6.4, $\dim \sum_{j=1}^{m} W(\mathcal{C}_1^j, \mathcal{C}_2^j, \mathcal{C}_3^j) \cap \mathcal{P}^{\perp} \leq \sum_{j=1}^{m} 6k(j) - 6$, and by Lemmas 3.6.5 and 3.6.6, $\dim \sum_{j=1}^{m} W(\mathcal{C}_1^j, \mathcal{C}_2^j, \mathcal{C}_3^j) \cap \mathcal{P}^{\perp} \geq \sum_{j=1}^{m} 6k(j) - 6$. Thus,

$$\dim \sum_{j=1}^{m} W(\mathcal{C}_1^j, \mathcal{C}_2^j, \mathcal{C}_3^j) \cap \mathcal{P}^{\perp} = \begin{cases} \sum_{j=1}^{m} 6k(j) - 6 & \text{if } k(j) \neq n-1, \\ 6(n-1) - 6 & \text{if } k(j) = n-1. \end{cases}$$

Since $\sum_{j=1}^{m} k(j) = n-1$, we find in both cases that

$$\dim \sum_{j=1}^{m} W(\mathcal{C}_1^j, \mathcal{C}_2^j, \mathcal{C}_3^j) \cap \mathcal{P}^{\perp} = 6n - 12.$$

Adding 5, the dimension of the primary module $\mathcal{P}$, gives the result. $\square$

Corollary 3.6.8 *With Notation 3.6.1,*

$$\dim E_i^*(\sum_{j=1}^{m} W(\mathcal{C}_1^j, \mathcal{C}_2^j, \mathcal{C}_3^j)) = \begin{cases} 1 & \text{if } i = 0, \\ n-1 & \text{if } i = 1, 2, 3, \\ 3n-5 & \text{if } i = 4. \end{cases}$$

Proof. Note that $[\![p]\!]$ is a basis for $E_0^*(\sum_{j=1}^m W(\mathcal{C}_1^j, \mathcal{C}_2^j, \mathcal{C}_3^j))$. For $i = 1$, the set of vectors, $\{[\![(r_p, \dot{c}, \dot{e})]\!] | \ (r_p, \dot{c}, \dot{e}) \in X\}$ forms a basis for $E_1^*(\sum_{j=1}^m W(\mathcal{C}_1^j, \mathcal{C}_2^j, \mathcal{C}_3^j))$ of dimension $n-1$. Similarly for $i = 2, 3$. For $i = 4$, note that $\sum_{j=1}^m W(\mathcal{C}_1^j, \mathcal{C}_2^j, \mathcal{C}_3^j)$ is the orthogonal sum

$$\sum_{j=1}^{m} W(\mathcal{C}_1^j, \mathcal{C}_2^j, \mathcal{C}_3^j) = \bigoplus_{i=0}^{4} E_i^*(\sum_{j=1}^{m} W(\mathcal{C}_1^j, \mathcal{C}_2^j, \mathcal{C}_3^j)).$$

Hence the dimension of $E_4^*(\sum_{j=1}^m W(\mathcal{C}_1^j, \mathcal{C}_2^j, \mathcal{C}_3^j)) = 6n - 7 - 3(n-1) - 1 = 3n - 5$. $\square$

Corollary 3.6.9 *With Notation 3.6.1, for* $i = 0$, 1, 2, 3,

$$E_i^* V \subset \sum_{j=1}^{m} W(\mathcal{C}_1^j, \mathcal{C}_2^j, \mathcal{C}_3^j).$$

Proof. By Corollary 3.6.8, $E_0^* V \subseteq [\![p]\!]$. Each $v \in \Gamma_i$ $(i = 1, 2, 3)$ is in some cycle, so $[\![v]\!]$ is in the corresponding cycle module. Hence $E_i^* V \subseteq \sum_{j=1}^{m} W(\mathcal{C}_1^j, \mathcal{C}_2^j, \mathcal{C}_3^j)$. $\square$

3.7 The Fourth Subconstituent

The cycle modules do not account for all irreducible $\mathcal{T}$-modules, as $E_4^* V$ is not contained in their sum. In this section we show that the part of $E_4^* V$ not contained in the cycle modules decomposes into mutually isomorphic one-dimensional $\mathcal{T}$-modules.

Lemma 3.7.1 *With Notation 3.6.1, pick any nonzero vector* $v \in E_4^* V$ *which is orthogonal to all* $W(\mathcal{C}_1^j, \mathcal{C}_2^j, \mathcal{C}_3^j)$ $(1 \leq j \leq m)$. *Write* $v = \sum_{x \in \Gamma_4(p)} \alpha_x [\![x]\!]$. *Then for* $1 \leq i \leq 3$ *and* *for* $1 \leq k \leq n$, $k \neq p(i)$, $\sum_{\substack{x \in \Gamma_4(p) \\ x(i)=k}} \alpha_x = 0$.

Proof. For $0 \leq i \leq 3$, $E_i^* V \subseteq \sum_{\ell=1}^m W(\mathcal{C}_1^\ell, \mathcal{C}_2^\ell, \mathcal{C}_3^\ell)$, and $E_i^* A_j E_4^* v \in E_i^* V$. Thus $E_i^* A_j E_4^* v = 0$ $(0 \leq i, j \leq 3)$. Hence,

$$\begin{aligned} 0 &= E_1^* A_2 E_4^* v = \sum_{x \in \Gamma_4(p)} \alpha_x [\![r_p, x(2), L(r_p, x(2))]\!] \\ &= \sum_{k \neq c_p} \sum_{x(2)=k} \alpha_x [\![r_p, x(2), L(r_p, x(2))]\!]. \end{aligned}$$

For $k_1 \neq k_2$, the vectors $\sum_{x(2)=k_1} [\![r_p, x(2), L(r_p, x(2))]\!]$ and $\sum_{x(2)=k_2} [\![r_p, x(2), L(r_p, x(2))]\!]$ are linearly independent by Lemma 2.3.5. Thus $\sum_{x(2)=k} \alpha_x [\![r_p, x(2), L(r_p, x(2))]\!] = 0$, hence $\sum_{x(j)=k} \alpha_x = 0$ for $j = 2$. Similarly $\sum_{x(j)=k} \alpha_x = 0$ for $j = 1, 3$. $\square$

Theorem 3.7.2 *With Notation 3.6.1, pick any nonzero vector* $v \in E_4^* V$ *which is orthogonal to all* $W(\mathcal{C}_1^j, \mathcal{C}_2^j, \mathcal{C}_3^j)$ $(1 \leq j \leq m)$. *Then* v *spans a one-dimensional irreducible* $\mathcal{T}$*-module. We denote this* $\mathcal{T}$*-module by* $F(v)$. *The action of the generators* $E_i^* A_j E_k^*$ *on this element is*

$E_4^* A_0 E_4^* v = v$,

$E_4^* A_i E_4^* v = -v$ *for* $i = 1, 2$, *and 3*,

$E_4^* A_4 E_4^* v = 2v$,

$E_i^* A_j E_k^* v = 0$ *for all other* i, j, *and* k.

Proof. The generators $E_i^* A_j E_k^*$ of $\mathcal{T}$ that act on v in a nonzero manner have $ijk \in \{044, 124, 134, 234, 214, 314, 324, 404, 414, 424, 434, 444\}$. By the choice of v, $E_i^* A_j E_4^* v = 0$

for $1 \leq i, j \leq 3$, $i \neq j$. By (2.1.1), $E_4^* A_0 E_4^* v = E_4^* v = v$. Now

$$
\begin{aligned}
E_4^* A_1 E_4^* v &= \sum_{x \in \Gamma_4(p)} \alpha_x E_4^* A_1 E_4^* [\![x]\!] \\
&= \sum_{x \in \Gamma_4(p)} \alpha_x \big(\sum_{y \in \Gamma_4(p)} \alpha_x [\![y]\!] - [\![x]\!] \big) \\
&= \sum_{\substack{x \in \Gamma_4(p) \\ y_1 = x_1}} \alpha_x [\![y]\!] - \sum_{x \in \Gamma_4(p)} \alpha_x [\![x]\!] \\
&= - \sum_{x \in \Gamma_4(p)} \alpha_x [\![x]\!] = -v.
\end{aligned}
$$

Since $\sum_{\substack{x \in \Gamma_4(p) \\ y_1 = x_1}} \alpha_x [\![y]\!] = 0$ by Lemma 3.7.1. Similarly,

$$E_4^* A_2 E_4^* v = E_4^* A_3 E_4^* v = -v.$$

Also,

$$E_4^* A_4 E_4^* v = E_4^* (J - A_0 - A_1 - A_2 - A_3) E_4^* v = E_4^* J E_4^* - v + 3v = 2v.$$

Note that $E_4^* J E_4^* v = 0$ since $v \perp [\![\Gamma_4(p)]\!]$. $\square$

Corollary 3.7.3 *With reference to Theorem 3.7.2, the $\mathcal{T}$-module isomorphism class of $F(v)$ is independent of the choice of v. The multiplicity of $F(v)$ is $n^2 - 6n + 7$.*

Proof. That all $F(v)$ are isomorphic follows from Theorem 3.7.2 by an argument similar to that in the proof of Lemma 3.6.2. We now consider the multiplicity of $F(v)$. Note that

$$E_4^* V = E_4^* \sum_{j=1}^{m} W(\mathcal{C}_1^j, \mathcal{C}_2^j, \mathcal{C}_3^j) \bigoplus E_4^* \big(\sum_{j=1}^{m} W(\mathcal{C}_1^j, \mathcal{C}_2^j, \mathcal{C}_3^j) \big)^{\perp}.$$

The dimension of $E_4^* V$ is $n^2 - 3n + 2$, and the dimension of $E_4^* (\sum_{j=1}^{m} W(\mathcal{C}_1^j, \mathcal{C}_2^j, \mathcal{C}_3^j))$ is $3n - 5$. Hence the the dimension of the complement is $n^2 - 6n + 7$. By Lemma 3.7.1, the orthogonal complement is the sum of isomorphic copies of one-dimensional modules isomorphic to $F(v)$. Thus $F(v)$ has multiplicity $n^2 - 6n + 7$. $\square$

3.8 Other Results

We present a few additional results. We first note that we now have a complete decomposition of V into irreducible $\mathcal{T}$-modules.

Lemma 3.8.1 *With Notation 3.6.1, let $v_1, v_2, \ldots, v_{n^2-6n+7}$ be an orthogonal basis for $(\sum_{j=1}^{m} W(\mathcal{C}_1^j, \mathcal{C}_2^j, \mathcal{C}_3^j))^{\perp}$. Then*

$$V = \mathcal{P} \bigoplus \sum_{j=1}^{m} \sum_{\substack{\epsilon \\ \epsilon^{k(j)}=1}} W^{\epsilon}(\mathcal{C}_1^j, \mathcal{C}_2^j, \mathcal{C}_3^j) \bigoplus_{i=1}^{n^2-6n+7} F(v_i).$$

Proof. Straightforward from Theorem 3.4.6, Lemma 3.6.6, and Corollaries 3.6.7, 3.6.8, 3.6.9, 3.7.3. $\square$

Lemma 3.8.2 *The isomorphism class of each non-primary irreducible $\mathcal{T}$-module W is uniquely determined by the eigenvalue of $E_1^* A_2 E_3^* A_1 E_2^* A_2 E_1^*$ associated with $E_1^* W$.*

Proof. Clear from Lemma 3.6.2 and Corollary 3.7.3. $\square$

3.9 Cayley Tables of Finite Groups

The Cayley table of a finite group is a Latin square. We describe the cycle structure of these examples.

Theorem 3.9.1 *Let G be a finite group, and let L denote the Cayley table of G. Consider the subconstituent algebra with respect to (g, h, gh) of the Bose-Mesner algebra of L. Then the cycle structure of L with respect to (g, h, gh) is $1^{\iota} 2^{(|G|-\iota-1)/2}$, where ι is the number of elements in G of order 2.*

Proof. Compute

$$\begin{aligned}
E_1^* A_3 E_2^* A_1 E_3^* A_2 E_1^*(g, b, gb) &= E_1^* A_3 E_2^* A_1 E_3^*(ghb^{-1}, b, gh) \\
&= E_1^* A_3 E_2^*(ghb^{-1}, h, ghb^{-1}h) \\
&= (g, hb^{-1}h, ghb^{-1}h).
\end{aligned}$$

Now repeat this computation:

$$\begin{aligned}
E_1^* A_3 E_2^* A_1 E_3^* A_2 E_1^*(g, hb^{-1}h, ghb^{-1}h) &= E_1^* A_3 E_2^* A_1 E_3^*(gbh^{-1}, hb^{-1}h, gh) \\
&= E_1^* A_3 E_2^*(gbh^{-1}, h, gb) \\
&= (g, b, gb).
\end{aligned}$$

Thus the order of $E_1^* A_3 E_2^* A_1 E_3^* A_2 E_1^*$ is one or two. Observe that (g, b, gb) forms a one-cycle if and only if $b = hb^{-1}h$ if and only if $(hb)^2 = e$. Since b can be any element of G other than h, there is exactly one one-cycle for each element of G of order 2 (since we may exclude the identity, which has order 1). $\square$

Corollary 3.9.2 *With reference to Theorem 3.9.1, the cycle structure is* $1^{|G|-1}$ *if and only if* $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$.

Proof. The cycle structure is $1^{|G|-1}$ if and only if $b = hb^{-1}h$ for all b if and only if $(hb^{-1})^2 = e$ for all $b \in G \backslash h$ if and only if $x^2 = e$ for all $x \in G \backslash e$ if and only if $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$. $\square$

3.10 Small Latin Squares

We report results for Latin squares of order $n \leq 4$ which were excluded from much of our discussion up to this point. We shall see that the small size of these Latin squares forces the irreducible $\mathcal{T}$-modules to have smaller dimension that predicted for larger Latin squares.

Example 3.10.1 There is a unique Latin square of order 1, namely (1). Note that

$A_1 = A_2 = A_3 = A_4 = 0$. The standard module has basis $[(1,1,1)] = [\Gamma_0(1,1,1)]$, so it is spanned by the primary module.

Example 3.10.2 The unique Latin square of order 2 up to main class equivalence is the Cayley table of $\mathbb{Z}_2$:

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

Note that $A_1 = A_2$ and $A_4 = 0$. With respect to any base point p, there is a single interleaved 1-cycle $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$. The standard module is the summof the primary module with basis $\{[\![\Gamma_0(p)]\!], [\![\Gamma_1(p)]\!], [\![\Gamma_3(p)]\!]\}$, and $W^1(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$ which has basis $\{u_1^1\}$.

Example 3.10.3 The unique Latin square of order 3 up to main class equivalence is the Cayley table of $\mathbb{Z}_3$:

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}.$$

With respect to any base point p, there is a single interleaved 2-cycle $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ and two elements $\Gamma_4(p)$. The standard module is the sum of the primary module $\mathcal{P}$ with basis $\{[\![\Gamma_0(p)]\!], [\![\Gamma_1(p)]\!], [\![\Gamma_2(p)]\!], [\![\Gamma_3(p)]\!], [\![\Gamma_4(p)]\!]\}$, $W^{-1}(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$ with basis $\{u_1^{-1}, u_2^{-1}, u_3^{-1}, v_{-1}^3\}$. For $i = 1, 2, 3$, $[\![\Gamma_i(p)]\!]$ is the sum of two vectors and u_i^{-1} is the difference of the same two vectors. Similarly for v_{-1}^3 and $[\![\Gamma_4(p)]\!]$.

Example 3.10.4 The Cayley table of $\mathbb{Z}_4$ represents one of the two main class of Latin squares of order 4:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{pmatrix}.$$

With respect to any base point p, there is a an interleaved 1-cycle $\mathcal{C}_1^1, \mathcal{C}_2^1, \mathcal{C}_3^1$ and and interleaved 2-cycle $\mathcal{C}_1^2, \mathcal{C}_2^2, \mathcal{C}_3^2$. The standard module is the sum of the primary module $\mathcal{P}$ with basis $\{[\![\Gamma_0(p)]\!], [\![\Gamma_1(p)]\!], [\![\Gamma_2(p)]\!], [\![\Gamma_3(p)]\!], [\![\Gamma_4(p)]\!]\}$, $W^1(\mathcal{C}_1^2, \mathcal{C}_2^2, \mathcal{C}_3^2)$ with basis $\{u_1^1, u_2^1, u_3^1, v_1^1, v_2^1\}$, and $W^{-1}(\mathcal{C}_1^2, \mathcal{C}_2^2, \mathcal{C}_3^2)$ with basis $\{u_1^{-1}, u_2^{-1}, u_3^{-1}, v_1^{-1}, v_2^{-1}, v_3^{-1}\}$. Note here $\dim W^1(\mathcal{C}_1^2, \mathcal{C}_2^2, \mathcal{C}_3^2) = 5 < 6$.

Example 3.10.5 The Cayley table of $\mathbb{Z}_2 \times \mathbb{Z}_2$ represents the second of the two main class of Latin squares of order 4:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{pmatrix}.$$

With respect to any base point p, there are three interleaved 1-cycle $\mathcal{C}_1^i, \mathcal{C}_2^i, \mathcal{C}_3^i$ for $i =$ 1, 2, 3. The standard module is the sum of the primary module $\mathcal{P}$ which has basis $\{[\![\Gamma_0(p)]\!], [\![\Gamma_1(p)]\!], [\![\Gamma_2(p)]\!], [\![\Gamma_3(p)]\!], [\![\Gamma_4(p)]\!]\}$, $W^1(\mathcal{C}_1^1, \mathcal{C}_2^1, \mathcal{C}_3^1)$ with basis $\{u_1^1, u_2^1, u_3^1, v_1^1, v_2^1\}$, $W^1(\mathcal{C}_1^2, \mathcal{C}_2^2, \mathcal{C}_3^2)$ with basis $\{u_1^1, u_2^1, u_3^1, v_1^1, v_2^1\}$, and $F(v)$ with basis v. The 6 elements of the fourth subconstituent are arranged so that two appear in each of 3 rows and in each of 3 columns and two have each of 3 values. Form v as a ± 1-linear combination of the corresponding characteristic vectors so that the entries with a common row, column, or entry have have opposite sign. Note here $\dim W^1(\mathcal{C}_1^1, \mathcal{C}_2^1, \mathcal{C}_3^1) = \dim W^1(\mathcal{C}_1^2, \mathcal{C}_2^2, \mathcal{C}_3^2) = 5 <$ 6.

4 Strongly Regular Graphs from a Latin Square

One may define several strongly regular graphs using a Latin square. In this chapter we describe the local spectrum and subconstituent algebra of these strongly regular graphs. We use the results of the previous chapter to do so after recalling some background material.

4.1 Strongly Regular Graphs

We recall some facts about strongly regular graphs. See [17, 18] for more information.

Definition 4.1.1 A finite, undirected graph Γ without loops or multiple edges is said to be *strongly regular (SRG) with parameters* (ν, k, λ, μ) if it has ν many vertices, each vertex has exactly k many neighbors, any two adjacent vertices have exactly λ many common neighbors, and any two non-adjacent vertices have exactly μ many common neighbors. A strongly regular graph is said to be *trivial* when $\mu = 0$, in which case it is a disjoint union of cliques of the same size.

Let $\Gamma = (X, R)$ be a nontrivial strongly regular graph with parameters (ν, k, λ, μ), and let $A \in \mathbb{M}_X$ denote the adjacency matrix of Γ. Then $A_0 = I$, $A_1 = A$, and $A_2 = J - A - I$ are the Hadamard idempotents of a Bose-Mesner algebra $\mathcal{M}$. The primitive idempotents of $\mathcal{M}$ are the maximal projections onto the eigenspaces of A. k is an eigenvalue of A; the other two eigenvalues r and s are the roots of the quadratic equation $\theta^2 + (\mu - \lambda)\theta + (\mu - k) = 0$ and $k > r > 0$, $s \leq -1$. Let E_0, E_1, and E_2 denote the primitive idempotents associated with k, r, and s, respectively. Then E_0, E_1, and E_2 have respective multiplicities 1, $m_r = (s+1)k(k-s)/(\mu(s-r))$, and $m_s = \nu - 1 - m_r$. The disjoint union of p many copies of K_q is a trivial SRG with parameters $(pq, q-1, q-2, 0)$. The eigenvalues of this graph are k and -1 with respective multiplicities p and $pq-p$.

Recall that the complement of Γ is a strongly regular graph with parameters $(\nu, \nu-k-1, \nu-2k+\mu-2, \nu-2k+\lambda)$ and eigenvalues $\nu-k-1$, $-s-1$, and $-r-1$. However, Γ and its complement have the same Bose-Mesner algebra and subconstituent algebra with respect to each point.

We recall the subconstituent algebra of a strongly regular graph [85] (cf. [18]).

Lemma 4.1.2 *[85] Let $\mathcal{M}$ denote the Bose-Mesner algebra of a nontrivial strongly regular graph Γ with parameters (ν, k, λ, μ) and eigenvalues k, r, s. Fix a base point p, and let $\mathcal{T}$ denote the subconstituent algebra of $\mathcal{M}$ with respect to p. Let $A^* = \rho(\nu E_1)$.*

(i) *There is a unique irreducible $\mathcal{T}$-module of dimension three. It has an ordered basis Ω^0 such that*

$$[A]_{\Omega^0} = \begin{pmatrix} 0 & k & 0 \\ 1 & \lambda & k-\mu-1 \\ 0 & \mu & k-\mu \end{pmatrix}, \qquad [A^*]_{\Omega^0} = \begin{pmatrix} k & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & s \end{pmatrix}.$$

This is the primary module $\mathcal{P}$ of $\mathcal{T}$.

(ii) *For possibly many distinct numbers $\theta \notin \{r, s\}$, there may be an irreducible $\mathcal{T}$-module which has an ordered basis Ω^1_θ such that*

$$[A]_{\Omega^1_\theta} = \begin{pmatrix} \theta & r-\theta \\ \theta-s & r+s-\theta \end{pmatrix}, \qquad [A^*]_{\Omega^1_\theta} = \begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix}.$$

We say that such a module is of type $U(\theta)$.

(iii) *There may be a one-dimensional irreducible $\mathcal{T}$-module of such that*

$$[A]_{\Omega^1_r} = \begin{pmatrix} r \end{pmatrix}, \qquad [A^*]_{\Omega^1_r} = \begin{pmatrix} r \end{pmatrix}.$$

We say that such a module is of type $U_{1,1}(r)$.

(iv) *There may be a one-dimensional irreducible $\mathcal{T}$-module such that*

$$[A]_{\Omega_s^2} = \begin{pmatrix} s \end{pmatrix}, \qquad [A^*]_{\Omega_s^2} = \begin{pmatrix} s \end{pmatrix}.$$

We say that such a module is of type $U_{2,2}(s)$.

(v) *There may be a one-dimensional irreducible $\mathcal{T}$-module such that*

$$[A]_{\Omega_{s,r}^1} = \begin{pmatrix} s \end{pmatrix}, \qquad [A^*]_{\Omega_{s,r}^1} = \begin{pmatrix} r \end{pmatrix}.$$

We say that such a module is of type $U_{2,1}(s,r)$.

(vi) *There may be a one-dimensional irreducible $\mathcal{T}$-module such that*

$$[A]_{\Omega_{r,s}^2} = \begin{pmatrix} r \end{pmatrix}, \qquad [A^*]_{\Omega_{r,s}^2} = \begin{pmatrix} s \end{pmatrix}.$$

We say that such a module is of type $U_{1,2}(r,s)$.

(vii) *$\mathcal{T}$ has no other possible irreducible modules.*

4.2 Strongly Regular Graphs from a Latin Square

Lemma 4.2.1 *[5] With Notation 3.2.1,*

(i) *Each of A_1, A_2, A_3 is the adjacency matrix of strongly regular graphs with parameters $(n^2, n-1, n-2, 0)$ and eigenvalues $n-1$, -1 with respective multiplicities n, $n(n-1)$. These strongly regular graphs are isomorphic and said to be of type $L(n,1)$. The complements of these strongly regular graphs have respective adjacency matrices $A_4+A_2+A_3$, $A_4+A_1+A_3$, $A_4+A_1+A_2$ and parameters $(n^2, n^2-n, n^2-2n+2, n^2-n)$.*

(ii) *Each of A_1+A_2, A_1+A_3, A_2+A_3 is the adjacency matrix of strongly regular graphs with parameters $(n^2, 2(n-1), n-1, 2)$ and eigenvalues $2(n-1)$, $n-2$, -2 with respective multiplicities 1, $2(n-1)$, and $(n-1)^2$. These strongly regular graphs are isomorphic and said to be of type $L(n,2)$. The complements of these strongly regular graphs have respective adjacency matrices A_4+A_3, A_4+A_2, A_4+A_1 and parameters $(n^2, n^2-2n+1, n^2-4n+6, n^2-3n+3)$.*

(iii) *$A_1 + A_2 + A_3$ is the adjacency matrix of strongly regular graphs with parameters $(n^2, 3n-3, n, 6)$ and eigenvalues $3n-3$, $n-3$, and -3 with respective multiplicities 1, $3(n-1)$, and $(n-2)(n-1)$. These strongly regular graphs are isomorphic and said to be of type $L(n,3)$. The complement of this strongly regular graphs has adjacency matrix A_4 and parameters $(n^2, n^2-3n+2, n^2-6n+10, n^2-5n+6)$.*

Definition 4.2.2 With Notation 3.2.1, define G_1, G_2, and G_3 to be the graphs with vertex set X and respective adjacency matrices A_1, $A_1 + A_2$, and $A_1 + A_2 + A_3$.

The adjacency matrices of G_i $(i = 1, 2, 3)$ are formed by fusing some of the Hadamard idempotents of the Bose-Mesner algebra of L (cf. [9, 76]). Latin squares arise in this context in a special way [56, 57]. In general the relationships between the subconstituent algebras of a Bose-Mesner algebra and its fusions is rather subtle. Thus we shall consider all the SRG's of Lemma 4.2.1 as fusions of $\mathcal{M}$.

4.3 Fusions

The adjacency relation of a strongly regular graph associated with a Latin square is formed by fusing some of the relations R_1, R_2, and R_3 of the association scheme defined by a Latin square. We comment on this perspective.

Notation 4.3.1 Let X denote a finite nonempty set. Let $\mathcal{N}$ and $\mathcal{M}$ denote $(e+1)$- and $(d+1)$-dimensional Bose-Mesner algebras on X, respectively. Let $\{B_i\}_{i=0}^{e}$ and $\{A_i\}_{i=0}^{d}$ denote the respective Hadamard idempotents of $\mathcal{N}$ and $\mathcal{M}$, and let $\{F_i\}_{i=0}^{e}$ and $\{E_i\}_{i=0}^{d}$ denote the respective primitive idempotents of $\mathcal{N}$ and $\mathcal{M}$. Fix $p \in X$. Let $\{B_i^*\}_{i=0}^{e}$ and $\{A_i^*\}_{i=0}^{d}$ denote the respective dual Hadamard idempotents of $\mathcal{N}$ and $\mathcal{M}$ with respect to p, and let $\{F_i^*\}_{i=0}^{e}$ and $\{E_i^*\}_{i=0}^{d}$ denote the respective dual idempotents of $\mathcal{N}$ and $\mathcal{M}$ with respect to p. Let $\mathcal{S}$ and $\mathcal{T}$ denote the respective subconstituent algebras of $\mathcal{N}$ and $\mathcal{M}$ with respect to p.

Theorem 4.3.2 *[9, 76] With Notation 4.3.1, let P and Q denote the eigenmatrix and dual eigenmatrix of $\mathcal{M}$, respectively. Then the following are equivalent.*

(i) *$\mathcal{N}$ is a Bose-Mesner subalgebra of $\mathcal{M}$.*

(ii) *There exists a pair of partitions* $\Lambda = \{\Lambda_0 = \{0\}, \Lambda_1, \ldots, \Lambda_d\}$ *and* $\Lambda' = \{\Lambda'_0 = \{0\}, \Lambda'_1, \ldots, \Lambda'_d\}$ *of* $\{0, 1, \ldots, e\}$ *such that for all* h, i $(0 \leq h, i \leq d)$

$$B_h = \sum_{h' \in \Lambda_h} A_{h'}, \qquad {}^{\mathrm{t}}B_h = \sum_{\ell' \in \Lambda_\ell} A_{\ell'} \qquad \textit{for some } \ell \ (0 \leq \ell \leq d),$$

$$\sum_{h' \in \Lambda_h} P(h', j) = \sum_{h' \in \Lambda_h} P(h', k) \qquad \textit{for all } j, k \in \Lambda'_i.$$

(iii) *There exists a pair of partitions* $\Lambda = \{\Lambda_0 = \{0\}, \Lambda_1, \ldots, \Lambda_d\}$ *and* $\Lambda' = \{\Lambda'_0 = \{0\}, \Lambda'_1, \ldots, \Lambda'_d\}$ *of* $\{0, 1, \ldots, e\}$ *such that for all* h, i $(0 \leq h, i \leq d)$

$$F_h = \sum_{h' \in \Lambda'_h} E_{h'}, \qquad {}^{\mathrm{t}}F_h = \sum_{\ell' \in \Lambda'_\ell} E_{\ell'} \qquad \textit{for some } \ell \ (0 \leq \ell \leq d),$$

$$\sum_{h' \in \Lambda'_h} Q(h', j) = \sum_{h' \in \Lambda'_h} Q(h', k) \qquad \textit{for all } j, k \in \Lambda_i.$$

Suppose (i)–(iii) hold. Then the partitions Λ *in (ii) and (iii) coincide, and the partitions* Λ' *in (ii) and (iii) coincide. Moreover* Λ *uniquely determines* Λ' *and vice versa. We refer to the Bose-Mesner subalgebra* $\mathcal{M}$ *as the* (Λ, Λ')-fusion of $\mathcal{N}$.

Corollary 4.3.3 *With Notation 4.3.1, assume that* $\mathcal{N}$ *is the* (Λ, Λ')*-fusion of* $\mathcal{M}$*. Then the following hold.*

(i) $F^*_i = \sum_{h' \in \Lambda_h} E^*_i$ $(0 \leq i \leq d)$.

(ii) $B^*_i = \sum_{h' \in \Lambda'_h} A^*_i$ $(0 \leq i \leq d)$.

Proof. Clear from Theorem 4.3.2 since ρ is linear. □

Lemma 4.3.4 *With Notation 4.3.1, assume that* $\mathcal{N}$ *is the* (Λ, Λ')*-fusion of* $\mathcal{M}$*. Then every* $\mathcal{T}$*-module is an* $\mathcal{S}$*-modules.*

Proof. Clear from Theorem 4.3.2 and Corollary 4.3.3. □

Despite Lemma 4.3.4, the isomorphism classes of irreducible $\mathcal{S}$-modules are not obviously determined by those of irreducible $\mathcal{T}$-modules. For example, an irreducible $\mathcal{T}$-module may decompose further into the sum of irreducible $\mathcal{S}$-modules (this is necessarily the case for the primary module). Another complication is the possibility that non-isomorphic $\mathcal{T}$-modules are isomorphic as $\mathcal{S}$-modules. We shall see that this is the case for W^{ϵ} and $W^{\bar{\epsilon}}$.

Latin squares are closely related to a rather interesting property concerning fusions.

Definition 4.3.5 An e-dimensional Bose-Mesner algebra is said to be *amorphous* whenever for any partition $\Lambda = \{\Lambda_0 = \{0\}, \Lambda_1, \ldots, \Lambda_d\}$ of $\{0, 1, \ldots, e\}$, there is a partition $\Lambda' = \{\Lambda'_0 = \{0\}, \Lambda'_1, \ldots, \Lambda'_d\}$ of $\{0, 1, \ldots, e\}$ such that there is a Bose-Mesner subalgebra of $\mathcal{M}$ which is the (Λ, Λ')-fusion of $\mathcal{M}$.

Definition 4.3.6 A Bose-Mesner algebra is said to be of *(negative) Latin square type* whenever every $(0,1)$-matrix with zero diagonal is the adjacency matrix of strongly regular graph with $\mu = s(s+1)$ (respectively $\mu = r(r+1)$), where the strongly regular graph has eigenvalues $k > s > r$.

Theorem 4.3.7 *[56, 57] A Bose-Mesner algebra is amorphous if and only if it is of Latin square type or negative Latin square type.*

4.4 G_3

To use the theory of Section 3.4, we will need to distinguish the objects which arise from the Bose-Mesner algebra of a Latin square and those which arise from these SRG's.

Notation 4.4.1 Let L denote the Latin square of order $n \geq 5$. Let X be the set $\{(i, j, L(i, j)) \,|\, 1 \leq i, j \leq n\}$. Define Γ_i $(0 \leq i \leq 4)$ relative to the relations in equations (3.1.1)-(3.1.5). Let $\mathcal{M}$ denote the Bose-Mesner algebra of L, and let $\{A_i\}_{i=0}^{4}$ denote the Hadamard idempotents of $\mathcal{M}$. Fix a base point $p \in X$. Let $\{E_i^*\}_{i=0}^{4}$ denote the dual idempotents and let $\mathcal{T}$ denote the subconstituent algebra of $\mathcal{M}$ with respect to p.

We now describe the irreducible modules and local spectrum of G_3.

Notation 4.4.2 With reference to Notation 4.4.1, let G_3 be as in Definition 4.2.2. Let $\{B_i\}_{i=0}^{2}$ be the Hadamard idempotents of the Bose-Mesner algebra $\mathcal{N}$ of G_3, where $B = B_1$ is the adjacency matrix of G_3. Let $F_i^* = \rho(B_i)$ be the dual idempotents of $\mathcal{N}$ with respect to p, and let S denote the suconstituent algebra of $\mathcal{N}$ with respect to p. Let N_3 be the neighbors of p in G_3.

Lemma 4.4.3 *With Notation 4.4.1 and 4.4.2, the following hold.*

(i) $N_3 = \Gamma_1(p) \cup \Gamma_2(p) \cup \Gamma_3(p)$.

(ii) $\Gamma_1(p)$, $\Gamma_2(p)$, *and* $\Gamma_3(p)$ *are mutually disjoint and the induced subgraphs of* G *on* $\Gamma_1(p)$, $\Gamma_2(p)$, *and* $\Gamma_3(p)$ *are* $(n-1)$*-cliques.*

(iii) *Let* Δ *be the subgraph of the induced subgraph of* G_3 *on* N_3 *formed by removing the edges with both endpoints in* R_1*, both endpoint in* R_2*, or both endpoint in* R_3*. Then* Δ *is a disjoint union of cycles: For each interleaved triple of* k*-cycles* $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ *as in Lemma 3.2.5 the following points form a* $3k$*-cycle in* Δ*:* (r_p, c_1, e_1), (r_1, c_p, e_1), (r_1, c_2, e_p), (r_p, c_2, e_2), (r_2, c_p, e_2), (r_2, c_3, e_p), $\ldots$, (r_p, c_k, e_k), (r_k, c_p, e_k), (r_k, c_1, e_p), (r_p, c_1, e_1).

Proof. The neighbors of $p = (p_r, p_c, p_e)$ share a common entry with p, so $N_3 = \Gamma_1(p) \cup \Gamma_2(p) \cup \Gamma_3(p)$. Each $\Gamma_i(p)$ is a clique of size $n-1$ by construction. This union is disjoint since any two entries uniquely determine the third. Part (iii) follows from Lemma 3.2.5 (see also [35]). $\square$

Lemma 4.4.4 *With Notation 4.4.1 and 4.4.2, the primary* $\mathcal{T}$*-module is the direct sum of 3 irreducible* $\mathcal{S}$*-modules:*

(i) *The primary* $\mathcal{S}$*-module with basis* $\{[\![\Gamma_0(p)]\!], [\![\Gamma_1(p)]\!] + [\![\Gamma_2(p)]\!] + [\![\Gamma_3(p)]\!], [\![\Gamma_4(p)]\!]\}$.

(ii) *A module of type* $U_{1,1}(n-3)$ *with basis* $\{[\![\Gamma_1(p)]\!] - [\![\Gamma_3(p)]\!]\}$.

(iii) *A module of type* $U_{1,1}(n-3)$ *with basis* $\{[\![\Gamma_2(p)]\!] - [\![\Gamma_3(p)]\!]\}$.

Proof. (i): Clear.

(ii): Let $\tilde{B}$ be the principal minor of $F_1^* B F_1^*$ induced by elements of N_3. Observe that $\tilde{B}$ is the adjacency matrix of the induced subgraph Γ on the neighbors of p. First we claim that $[\![\Gamma_1(p)]\!] - [\![\Gamma_3(p)]\!]$ and $[\![\Gamma_2(p)]\!] - [\![\Gamma_3(p)]\!]$ are linearly independent vectors in the eigenspace of $\tilde{B}$ associated with $n-3$. Each vertex of $\Gamma_1(p)$ is adjacent to $n-2$ vertices of $\Gamma_1(p)$, each vertex of $\Gamma_2(p)$ is adjacent to exactly one vertex of $\Gamma_1(p)$, and each vertex of $\Gamma_3(p)$ is adjacent to exactly one vertex of $\Gamma_1(p)$. Thus $\tilde{B}[\![\Gamma_1(p)]\!] = (n-2)[\![\Gamma_1(p)]\!] + [\![\Gamma_2(p)]\!] + [\![\Gamma_3(p)]\!]$. Similarly, $\tilde{B}[\![\Gamma_2(p)]\!] = (n-2)[\![\Gamma_2(p)]\!] + [\![\Gamma_1(p)]\!] + [\![\Gamma_3(p)]\!]$ and $\tilde{B}[\![\Gamma_3(p)]\!] = (n-2)[\![\Gamma_3(p)]\!] + [\![\Gamma_1(p)]\!] + [\![\Gamma_2(p)]\!]$. Thus $[\![\Gamma_1(p)]\!] - [\![\Gamma_3(p)]\!]$ and $[\![\Gamma_2(p)]\!] - [\![\Gamma_3(p)]\!]$ are eigenvectors for $\tilde{B}$ associated with $n-3$. They are clearly linearly independent since they have some nonzero entries in mutually distinct positions. $\square$

Theorem 4.4.5 *With Notation 4.4.1 and 4.4.2, assume $n \geq 5$. Let $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ be an interleaved triple of k-cycles.*

(i) *$W^1(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$ is the direct sum of 3 irreducible $\mathcal{S}$-modules:*

- **(a)** *A module of type $U(1)$ with basis $\{u_1^1 + u_2^1 + u_3^1,\ v_1^1 + v_2^1 + v_3^1\}$.*
- **(b)** *A module of type $U(-2)$ with basis $\{u_1^1 - 2u_2^1 + u_3^1,\ v_1^1 - 2v_2^1 + v_3^1\}$.*
- **(c)** *A module of type $U(-2)$ with basis $\{u_1^1 + 2u_2^1 - 3u_3^1,\ v_1^1 + 2v_2^1 - 3v_3^1\}$.*

(ii) *If k is even, then $W^{-1}(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$ is the direct sum of 4 irreducible $\mathcal{S}$-modules:*

- **(a)** *A module of type $U(0)$ with basis $\{u_1^{-1} + 2u_2^{-1} + u_3^{-1},\ v_1^{-1} + v_3^{-1}\}$.*
- **(b)** *A module of type $U(0)$ with basis $\{u_1^{-1} - u_2^{-1} - 2u_3^{-1},\ v_1^{-1} - v_2^{-1}\}$.*
- **(c)** *A module of type $U_{1,2}(-3, n-3)$ with basis $\{u_1^{-1} - u_2^{-1} + u_3^{-1}\}$.*
- **(d)** *A module of type $U_{2,1}(n-3, -3)$ with basis $\{v_1^{-1} + v_2^{-1} - v_3^{-1}\}$.*

(iii) *For each root of unity $\epsilon \notin \{1, -1\}$ with order which divides k, let θ_i^ϵ $(i = 1, 2, 3)$ be the roots of the polynomial $x^3 + 3x^2 - 2(1 + Re\epsilon)$. Then $W^\epsilon(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$ is the orthogonal direct sum of 3 irreducible $\mathcal{S}$-modules: For $i = 1, 2, 3$ there is a module of type $U(\theta_i^\epsilon)$ with basis $\{u_1^\epsilon + b_i u_2^\epsilon + c_i u_3^\epsilon,\ (b_i + c_i)v_1^\epsilon + (1 + \epsilon^{-1}c_i)v_2^\epsilon + (1 + b_i)v_3^\epsilon\}$, where $b_i = \frac{1+\epsilon+\theta_i^\epsilon}{(\theta_i^\epsilon)^2 + 2\theta_i^\epsilon}$ and $c_i = \frac{1+\epsilon+\epsilon\theta_i^\epsilon}{(\theta_i^\epsilon)^2 + 2\theta_i^\epsilon}$.*

Proof. (iii): Let $u = u_1^\epsilon + bu_2^\epsilon + cu_3^\epsilon$ be an eigenvector of $F_1^* B_1 F_1^*$ with eigenvalue λ, ie, $F_1^* B_1 F_1^* u = \lambda(u_1^\epsilon + bu_2^\epsilon + cu_3^\epsilon)$. Expanding $F_1^* = E_1^* + E_2^* + E_3^*$ and $B_1 = A_1 + A_2 + A_3$

and decomposing into subconstituents gives

$$\begin{aligned}
(E_1^*A_1E_1^* + E_2^*A_3E_1^* + E_3^*A_2E_1^*)u_1^\epsilon - \lambda u_1^\epsilon &= 0,\\
(E_2^*A_2E_2^* + E_1^*A_3E_2^* + E_3^*A_1E_2^*)bu_2^\epsilon - \lambda bu_2^\epsilon &= 0,\\
(E_3^*A_3E_3^* + E_1^*A_2E_3^* + E_2^*A_1E_3^*)cu_3^\epsilon - \lambda cu_3^\epsilon &= 0.
\end{aligned}$$

By Figures 3.6–3.9,

$$-1 - \lambda + b + \epsilon^{-1}c = 1 + (-1-\lambda)b + c = \epsilon + b + (-1-\lambda)c = 0. \tag{4.4.1}$$

By elementary linear algebra these equations are linearly independent unless $\epsilon = \pm 1$, which doesn't occur in (iii). Eliminating b and c gives that λ is a root of the cubic equation $x^3 + 3x^2 - 2(Re(\epsilon) + 1) = 0$. Note that $-1 \leq Re(\epsilon) \leq 1$, so this equation has 3 real roots between 1 and -3. They are distinct since $\epsilon \neq \pm 1$. Write θ_i^ϵ, $i = 1, 2, 3$ to denote these roots. Each is associated with an eigenvector say, u_i for $F_1^*B_1F_1^*$ as described in the statement. Applying $F_2^*B_1F_1^*$ to u_i for some $i = 1, 2, 3$ gives, $v_i = (b_i + c_i)v_1^\epsilon + (1 + \epsilon c_i)v_2^\epsilon + (1 + b_i)v_3^\epsilon$. For each $i = 1, 2, 3$, the set of vectors $\{u_i, v_i\}$ is linearly independent. It is closed under the action of all other generators of $\mathcal{S}$, and it is a subset of the $\mathcal{T}$-module, $W^\epsilon(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$. So u_i and v_i span a two-dimensional $\mathcal{S}$-submodule. Observe that this module is of type $U(\theta_i^\epsilon)$. Note that for $i \neq j$, $U(\theta_i^\epsilon)$ and $U(\theta_j^\epsilon)$ are not isomorphic since the $\theta_i^\epsilon \neq \theta_j^\epsilon$.
(i): For $\epsilon = 1$, the system of equations (4.4.1) becomes:

$$-1 - \lambda + b + c = 1 - (1+\lambda)b + c = 1 + b - (1+\lambda)c = 0.$$

This system of equations has no solution if $\lambda = 0$. Also if $\lambda = -2$, then $b = -1 - c$ and there are two linearly independent eigenvectors associated with the eigenvalue -2 as described in (i)(b,c). If $\lambda \notin \{-2, 0\}$, then λ has to satisfy the quadratic equation: $x^2 + x - 2$, ie $\lambda = 1$. In this case, $c = 1$ and $b = 1$. Hence $u_1^1 + u_2^1 + u_3^1$ is an eigenvector of $F_1^*B_1F_1^*$ with an eigenvalue $\lambda = 1$. Now $F_2^*B_1F_1^*(u_1^1 + u_2^1 + u_3^1) = v_1^1 + v_2^1 + v_3^1$, and all other actions on this vector are linear combinations of these two vectors. Moreover, these two vectors are linearly independent, so they form a basis for an $\mathcal{S}$-submodule of

$W^1(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$. Similarly $u_1^1 - 2u_2^1 + u_3^1$ and $v_1^1 - 2v_2^1 + v_3^1$ form a basis for an $\mathcal{S}$-submodule of $W^1(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$. Also $u_1^1 + 2u_2^1 - 3u_3^1$ and $v_1^1 + 2v_2^1 - 3v_3^1$ form a basis for an $\mathcal{S}$-submodule of $W^1(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$.

(ii): For $\epsilon = -1$, the system of equations (4.4.1) becomes:

$$-1 - \lambda + b - c = 1 - (1 + \lambda)b + c = -1 + b - (1 + \lambda)c = 0.$$

If $\lambda \neq 0$, then $b = -1$, $c = 1$ and $\lambda = -3$. Since λ is an eigenvalue of B, $u_1^{-1} - u_2^{-1} + u_3^{-1}$ is a basis for a one-dimensional $\mathcal{S}$-submodule of $W^{-1}(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$. If $\lambda = 0$, then $b = c + 1$ which gives two linearly independent eigenvectors $u_1^{-1} + 2u_2^{-1} + u_3^{-1}$ and $u_1^{-1} - u_2^{-1} - 2u_3^{-1}$ associated with $\lambda = 0$. As in (i), $u_1^{-1} + 2u_2^{-1} + u_3^{-1}$ and $v_1^{-1} + v_3^{-1}$ form a basis for a two-dimensional $\mathcal{S}$-submodule of $W^{-1}(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$, and $u_1^{-1} - u_2^{-1} - 2u_3^{-1}$ and $v_1^{-1} - v_2^{-1}$ form basis for a two-dimensional irreducible $\mathcal{S}$-submodule of $W^{-1}(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$. Note that $F_2^* B_1 F_2^*(v_1^{-1} + v_2^{-1} - v_3^{-1}) = 4(v_1^{-1} + v_2^{-1} - v_3^{-1})$, and all other actions on this vector is 0. Hence $v_1^{-1} + v_2^{-1} - v_3^{-1}$ forms a basis for a one-dimensional irreducible $\mathcal{S}$-submodule of $W^{-1}(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$. □

Lemma 4.4.6 *With Notation 4.4.1 and 4.4.2, for each $v \in E_4^* V$ which is orthogonal to all irreducible $\mathcal{T}$-modules constructed via Lemma 3.4.2 and Lemma 3.4.4, $F(v)$ is a one-dimensional irreducible $\mathcal{S}$-module of type $U_{2,2}(-3)$.*

Corollary 4.4.7 *With Notation 4.4.1 and 4.4.2, the spectrum of the induced subgraph of G_3 on N_3 is determined as follows.*

- **(i)** *The modules of lemma 4.4.4 contribute 1 to the multiplicity of $3(n-1)$ and 2 to the multiplicity of $(n-3)$.*
- **(ii)** *Let $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ be an interleaved k-cycle. Then*
  - **(a)** *$\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ contributes one to the multiplicity of 1 and two to the multiplicity of -2.*
  - **(b)** *If k is even, then $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ contributes two to the multiplicity of 0 and one to the multiplicity of -3.*

(c) *For each k^{th} root of unity $\epsilon \neq \pm 1$, $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ contributes one to the multiplicity of each root of $x^3 + 3x^2 - 2(1 + Re(\epsilon))$.*

Proof. These are the eigenvalues of $F_1^* B F_1^*$ on each irreducible $\mathcal{S}$-module. □

since ϵ is on the If $n \geq 7$, then no interleaved cycle contributes to the multiplicity of $n-3$. If $n = 6$, then a 4-cycle contributes two to the multiplicity of the $n - 3 = 3$. If $n = 5$, then $\epsilon = 1$ contributes one to the multiplicity of $n - 3 = 2$. If $n \leq 4$, then no interleaved cycle contributes to the multiplicity of $n - 3$. Note that $W^{\epsilon}(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$ and $W^{\bar{\epsilon}}(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$ are isomorphic as $\mathcal{S}$-modules since the irreducibles that they decompose into are isomorphic. In particular, they have the same contribution to the local spectrum.

4.5 G_2

We now report results for G_2 without proofs.

Notation 4.5.1 With reference to Notation 4.4.1, let G_2 be as in Definition 4.2.2. Let $\mathcal{N}$ be the Bose-Mesner algebra of G_2, and let S denote the suconstituent algebra of $\mathcal{N}$ with respect to p. Let N_2 be the neighbors of p in G_2.

Lemma 4.5.2 *With Notation 4.4.1 and 4.5.1, $N_2 = \Gamma_1(p) \cup \Gamma_2(p)$, $\Gamma_1(p)$ and $\Gamma_2(p)$ are disjoint, and the induced subgraphs of G_2 on $\Gamma_1(p)$ and $\Gamma_2(p)$ are $(n-1)$-cliques.*

Lemma 4.5.3 *With Notation 4.4.1 and 4.5.1, the primary $\mathcal{T}$-module is the orthogonal direct sum of 3 irreducible $\mathcal{S}$-modules:*

(i) *The primary $\mathcal{S}$-module with basis $\{[\![\Gamma_0(p)]\!], [\![\Gamma_1(p)]\!] + [\![\Gamma_2(p)]\!], [\![\Gamma_3(p)]\!] + [\![\Gamma_4(p)]\!]\}$.*

(ii) *A module of type $U_{1,1}(n-2)$ with basis $\{[\![\Gamma_1(p)]\!] - [\![\Gamma_2(p)]\!]\}$.*

(iii) *A module of type $U_{2,2}(-2)$ with basis $\{(n-2)[\![\Gamma_3(p)]\!] - [\![\Gamma_4(p)]\!]\}$.*

Theorem 4.5.4 *Assume $n \geq 5$. With Notation 4.4.1 and 4.5.1, let $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ be an interleaved triple of k-cycles.*

(i) *$W^1(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$ is the direct sum of 4 irreducible $\mathcal{S}$-modules:*

(a) *A module of type* $U(-1)$ *with basis* $\{u_1^1 + u_2^1,\ v_1^1 + v_2^1 + 2u_3^1\}$.

(b) *A module of type* $U(-1)$ *with basis* $\{u_1^1 - u_2^1,\ v_2^1 - v_1^1\}$.

(c) *A module of type* $U_{2,2}(-2)$ *with basis* $\{v_3^1 + u_3^1\}$.

(d) *A module of type* $U_{2,2}(-2)$ *with basis* $\{v_1^1 + v_2^1 + v_3^1 - (n-4)u_3^1\}$.

(ii) *If* k *is even, then* $W^{-1}(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$ *is the direct sum of 4 irreducible* $\mathcal{S}$*-modules:*

(a) *A module of type* $U(-1)$ *with basis* $\{u_1^{-1} + u_2^{-1},\ v_1^{-1} + v_2^{-1}\}$.

(b) *A module of type* $U(-1)$ *with basis* $\{u_1^{-1} - u_2^{-1},\ v_2^{-1} - v_1^{-1} - 2u_3^{-1}\}$.

(c) *A module of type* $U_{2,2}(-2)$ *with basis* $\{v_2^{-1} - v_1^{-1} + (n-3)u_3^{-1}\}$.

(d) *A module of type* $U_{2,2}(-2)$ *with basis* $\{v_1^{-1} + v_2^{-1} + (n-1)v_3^{-1}\}$.

(iii) *For each root of unity* $\epsilon \notin \{1, -1\}$ *with order which divides* k, $W^{\epsilon}(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$ *is the direct sum of 4 irreducible* $\mathcal{S}$*-modules:*

(a) *A module of type* $U(-1)$ *with basis* $\{u_1^\epsilon + u_2^\epsilon,\ v_1^\epsilon + v_2^\epsilon + (1+\epsilon)u_3^\epsilon\}$.

(b) *A module of type* $U(-1)$ *with basis* $\{u_1^\epsilon - u_2^\epsilon,\ v_2^\epsilon - v_1^\epsilon + (\epsilon - 1)u_3^\epsilon\}$.

(c) *A module of type* $U_{2,2}(-2)$ *with basis* $\{(\epsilon+1)u_3^\epsilon + v_1^\epsilon + v_2^\epsilon + (n-1)v_3^\epsilon\}$.

(d) *A module of type* $U_{2,2}(-2)$ *with basis* $\{(1+\epsilon^2+2\epsilon(n-2))u_3^\epsilon + (1-\epsilon)v_1^\epsilon + (\epsilon - 1)v_2^\epsilon + (1+\epsilon)(n-1)v_3^\epsilon\}$.

Lemma 4.5.5 *With Notation 4.4.1 and 4.5.1, for each* $v \in E_4^*V$ *which is orthogonal to all irreducible* $\mathcal{T}$*-modules constructed via Lemma 3.4.2 and Lemma 3.4.4,* $F(v)$ *is a one-dimensional irreducible* $\mathcal{S}$*-module of type* $U_{2,2}(-2)$.

Corollary 4.5.6 *With Notation 4.4.1 and 4.5.1, the spectrum of the induced subgraph of* G_2 *on* N_2 *is* $(n-2)^2$, $(-1)^{2(n-2)}$.

4.6 G_1

We report results for G_1 without proofs.

Notation 4.6.1 With reference to Notation 4.4.1, let G_1 be as in Definition 4.2.2. Let $\mathcal{N}$ be the Bose-Mesner algebra of G_1, and let S denote the suconstituent algebra of $\mathcal{N}$ with respect to p. Let N_1 be the neighbors of p in G_1.

Lemma 4.6.2 *With Notation 4.4.1 and 4.6.1, $N_1 = \Gamma_1(p)$, and the induced subgraph of G_1 on N_1 is an $(n-1)$-clique.*

Lemma 4.6.3 *With Notation 4.4.1 and 4.6.1, the primary $\mathcal{T}$-module is the orthogonal direct sum of 3 irreducible $\mathcal{S}$-modules:*

(i) *The primary $\mathcal{S}$-module with basis $\{[\![\Gamma_0(p)]\!], [\![\Gamma_1(p)]\!], [\![\Gamma_2(p)]\!] + [\![\Gamma_3(p)]\!] + [\![\Gamma_4(p)]\!]\}$.*

(ii) *A module of type $U_{2,2}(-1)$ with basis $\{[\![\Gamma_2(p)]\!] - [\![\Gamma_3(p)]\!]\}$.*

(iii) *A module of type $U_{2,2}(-1)$ with basis $\{(n-2)[\![\Gamma_3(p)]\!] - [\![\Gamma_4(p)]\!]\}$.*

Theorem 4.6.4 *Assume $n \geq 5$. With Notation 4.4.1 and 4.6.1, let $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ be an interleaved triple of k-cycles. For each root of unity ϵ with order which divides k, $W^\epsilon(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$ is the orthogonal direct sum of 6 irreducible $\mathcal{S}$-modules:*

(i) *A module of type $U_{1,1}(-1)$ with basis $\{u_1^\epsilon\}$.*

(ii) *A module of type $U_{2,2}(n-1)$ with basis $\{u_2^\epsilon + u_3^\epsilon + v_1^\epsilon\}$.*

(iii) *A module of type $U_{2,2}(-1)$ with basis $\{u_2^\epsilon + v_3^\epsilon\}$.*

(iv) *A module of type $U_{2,2}(-1)$ with basis $\{\epsilon u_2^\epsilon + v_2^\epsilon\}$.*

(v) *A module of type $U_{2,2}(-1)$ with basis $\{(2-n)u_2^\epsilon + v_1^\epsilon\}$.*

(vi) *A module of type $U_{2,2}(-1)$ with basis $\{-u_2^\epsilon + u_3^\epsilon\}$.*

Lemma 4.6.5 *With Notation 4.4.1 and 4.6.1, for each $v \in E_4^* V$ which is orthogonal to all irreducible $\mathcal{T}$-modules constructed via Theorems 2.3.8,* **??**, *and* **??**, *$F(v)$ is a one-dimensional irreducible $\mathcal{S}$-module of type $U_{2,2}(-1)$.*

Corollary 4.6.6 *With Notation 4.4.1 and 4.6.1, the spectrum of the induced subgraph of G_1 on N_1 is $(n-2)$, $(-1)^{n-2}$.*

5 Isomorphisms

In this chapter we compare several notions of isomorphism for subconstituent algebras. Latin squares provide a rich source of examples and counter-examples.

5.1 Isomorphisms of Bose-Mesner Algebra

We recall some notions of isomorphism for Bose-Mesner algebras. See [5, 41] for more details. For completeness we provide proofs of some basic results.

Definition 5.1.1 Let $\mathcal{M}$ and $\mathcal{M}'$ denote Bose-Mesner algebras on X and X', respectively.

(i) $\mathcal{M}$ and $\mathcal{M}'$ are said to be *algebraically isomorphic* if there exists a linear bijection $\phi : \mathcal{M} \to \mathcal{M}'$ such that $\phi(AB) = \phi(A)\phi(B)$ and $\phi(A \circ B) = \phi(A) \circ \phi(B)$ for all A, $B \in \mathcal{M}$.

(ii) $\mathcal{M}$ and $\mathcal{M}'$ are said to be *combinatorially isomorphic* if there exists a bijection $\alpha : X \to X'$ such that $\mathcal{M}' = \{A' \in \mathbb{M}_{X'} \mid \text{ for some } A \in \mathcal{M}, A'(\alpha(x), \alpha(y)) = A(x, y) \forall x, y \in X\}$.

Lemma 5.1.2 *With reference to Definition 5.1.1, if $\mathcal{M}$ and $\mathcal{M}'$ are combinatorially isomorphic, then they are algebraically isomorphic.*

Proof. We may represent the combinatorial isomophism induced by α by conjugation by a permutation matrix $P \in \mathbb{M}_{X,X'}$. Then clearly for all A, $B \in \mathcal{M}$, $P^{-1}ABP = P^{-1}APP^{-1}BP$ and $P^{-1}A \circ BP = P^{-1}AP \circ P^{-1}BP$. Thus conjugation by P defines an algebraic isomorphism. $\square$

Lemma 5.1.3 *With reference to Definition 5.1.1, if $\mathcal{M}$ and $\mathcal{M}'$ are combinatorially isomorphic, then there exists a bijection $\beta : \{0, 1, \ldots, d\} \to \{0, 1, \ldots, d'\}$ such that $A'_{\beta(i)}(\alpha(x), \alpha(y)) = A_i(x, y)$, where $\{A_i\}_{i=0}^{d}$ and $\{A'_i\}_{i=0}^{d'}$ denote the respective Hadamard idempotents of $\mathcal{M}$ and $\mathcal{M}'$.*

Proof. Observe that the set $\{B'_i\}_{i=0}^{d}$ defined by $B'_i(\alpha(x), \alpha(y)) = A_i(x, y)$ for all $x, y \in X$ satisfies (2.1.1) – (2.1.3), so it is the basis of Hadamard idempotents of $\mathcal{M}'$. The result follows from the uniqueness of the basis of Hadamard idempotents of a Bose-Mesner algebra. $\square$

Theorem 5.1.4 *[10] Two Bose-Mesner algebras are algebraically isomorphic if and only if they have the same intersection numbers relative to some orderings of the Hadamard idempotents.*

Proof. Define a linear map $f : \mathcal{M} \to \mathcal{M}'$ taking $A_i \in \mathcal{M}$ to $A'_i \in \mathcal{M}'$. Now $f(A_i A_j) = f(\sum_k p_{ij}^k A_k) = \sum_k p_{ij}^k A'_k$, and $f(A_i)f(A_j) = A'_i A'_j = \sum_k p_{ij}^{k'} A'_k$, and so $f(A_i A_j) = f(A_i)f(A_j)$ if and only if $p_{ij}^k = p_{ij}^{k'}$. Note that $f(A_i \circ A_j) = \delta_{ij} f(A_i) = \delta_{ij} A'_i = A'_i \circ A'_i = f(A'_i) \circ f(A'_i)$. $\square$

To discuss combinatorial isomorphism of Bose-Mesner algebras we recall commutative association scheme and their isomorphisms.

Definition 5.1.5 A d-class *commutative association scheme* is a pair $(X, \{R_i\}_{i=0}^{d})$, where X is a finite non-empty set and the R_i are relations on $X \times X$ such that

(i) $R_0 = \{(x, x) \mid x \in X\}$;

(ii) for all $x, y \in X$, there exists an i $(0 \le i \le d)$ such that $(x, y) \in R_i$;

(iii) for all i $(0 \le i \le d)$ there exists i' $(0 \le i' \le d)$ such that $(x, y) \in R_i$ if and only if $(y, x) \in R_{i'}$;

(iv) for all h, i, j $(0 \le h, i, j \le d)$ there exists a scalar p_{ij}^h such that for all $(x, y) \in R_h$, $|\{z \in X \mid (x, z) \in R_i, (z, y) \in R_j\}| = p_{ij}^h$.

(v) $p_{ij}^h = p_{ji}^h$ for all i, j, k $(0 \le i, j, k \le d)$.

Definition 5.1.6 Two association schemes $(X, \{R_i\}_{i=0}^d)$ and $(X', \{R'_i\}_{i=0}^{d'})$ are *isomorphic* whenever there exist bijections $\alpha : X \to X'$ and $\beta : \{0, 1, 2, \ldots, d\} \to \{0, 1, 2, \ldots, d'\}$ such that xR_iy if and only if $\alpha(x)R'_{\beta(i)}\alpha(y)$ for all $x, y \in X$.

Lemma 5.1.7 *[5, 10, 17, 41]*

(i) *Let $(X, \{R_i\}_{i=0}^d)$ denote a d-class commutative association scheme. Then the matrices $A_i \in \mathbb{M}_X$ $(0 \leq i \leq d)$ defined by $A_i(x, y) = 1$ if $(x, y) \in R_i$ and 0 otherwise are the Hadamard idempotents of a Bose-Mesner algebra.*

(ii) *Let $\{A_i\}_{i=0}^d$ be the Hadamard idempotents of a Bose-Mesner algebra on X. Then the relations R_i $(0 \leq i \leq d)$ on X defined by $(x, y) \in R_i$ if $A_i(x, y) = 1$ are the relations of a d-class commutative association scheme.*

Theorem 5.1.8 *[5] Two Bose-Mesner algebras are combinatorially isomorphic if and only if the related association schemes are isomorphic.*

We note that the converse of Lemma 5.1.2 fails. There are many non-isomorphic association schemes which have isomorphic Bose-Mesner algebras. We shall see that this is the case for many Latin squares.

5.2 Isomorphisms of Subconstituent Algebras

We discuss some notions of isomorphism for subconstituent algebras. Depending upon how much information about the related association scheme/Bose-Mesner algebra one retains upon passing to the subconstituent algebra, several notions of isomorphism may be distinguished. We use the following notation.

Notation 5.2.1 Let $\mathcal{M}$ and $\mathcal{M}'$ denote Bose-Mesner algebras on X and X', respectively. Let $\{A_i\}_{i=0}^d$ and $\{A'_i\}_{i=0}^{d'}$ denote the respective Hadamard idempotents of $\mathcal{M}$ and $\mathcal{M}'$. Fix $p \in X$ and $p' \in X'$, and let $\{E_i^*\}_{i=0}^d$, $\{E_i^{*\prime}\}_{i=0}^d$ denote the respective dual idempotents of $\mathcal{M}$ and $\mathcal{M}'$ with respect to p and p'. Let $\mathcal{T}$ and $\mathcal{T}'$ denote the respective subconstituent algebras of $\mathcal{M}$ and $\mathcal{M}'$.

Definition 5.2.2 Adopt Notation 5.2.1.

(i) We say that $\mathcal{T}$ and $\mathcal{T}'$ are *combinatorially isomorphic* whenever there is a permutation matrix $P \in \mathbb{M}_{X,X'}$ and a bijection $\beta : \{0, 1, \ldots, d\} \to \{0, 1, \ldots, d'\}$ such that $A_i P = P A'_{\beta(i)}$ and $E^*_i P = P E^{*\prime}_{\beta(i)}$ $(0 \leq i \leq d)$.

(ii) We say that $\mathcal{T}$ and $\mathcal{T}'$ are *structurally isomorphic* whenever there is an invertible matrix H and a bijection $\beta : \{0, 1, \ldots, d\} \to \{0, 1, \ldots, d'\}$ such that $A_i H = H A'_{\beta(i)}$ and $E^*_i H = H E^{*\prime}_{\beta(i)}$ $(0 \leq i \leq d)$.

(iii) We say that $\mathcal{T}$ and $\mathcal{T}'$ are *Bose-Mesner isomorphic* whenever there is an algebra isomorphism $\sigma : \mathcal{T} \to \mathcal{T}'$ and a bijection $\beta : \{0, 1, \ldots, d\} \to \{0, 1, \ldots, d'\}$ such that $\sigma(A_i) = A'_{\beta(i)}$ and $\sigma(E^*_i) = E^{*\prime}_{\beta(i)}$ $(0 \leq i \leq d)$.

(iv) We say that $\mathcal{T}$ and $\mathcal{T}'$ are *abstractly isomorphic* whenever there is an algebra isomorphism from $\mathcal{T}$ to $\mathcal{T}'$.

Combinatorial isomorphism for Bose-Mesner algebras necessarily map Hadamard idempotent to Hadamard idempotents (Lemma 5.1.3). Since subconstituent algebras ignore entrywise multiplication, we make this assumption in Definition 5.2.2(i).

Clearly combinatorially isomorphic implies structurally isomorphic implies Bose-Mesner isomorphic implies abstractly isomorphic. In Section 5.4 we shall use Latin squares to show that the reverse implications fail. Before we do this we discuss these various notions of isomorphism in further detail.

Lemma 5.2.3 *With Notation 5.2.1, let α be a bijection from X to X', let β be a bijection from $\{0, 1, 2, \ldots, d\}$ to $\{0, 1, 2, \ldots, d'\}$, and let P be the permutation matrix of α. Then the following are equivalent.*

(i) *α is a combinatorial isomorphism from $\mathcal{M}$ to $\mathcal{M}'$ which satisfies $\alpha(p) = p'$ and β is related to α as in Lemma 5.1.3.*

(ii) *(P, β) is a combinatorial isomorphism from $\mathcal{T}$ to $\mathcal{T}'$.*

Proof. (i)⇒(ii): Compute for all x', $y' \in X'$:

$$\begin{aligned}(PA_iP^{-1})(x',y') &= A_i(\alpha^{-1}(x'),\alpha^{-1}(y')\\ &= A_i(x,y) = A'_{\beta(i)}(\alpha(x),\alpha(y))\\ &= A'_{\beta(i)}(x',y'),\end{aligned}$$

and

$$\begin{aligned}PE_i^*P^{-1}(x',y') &= E_i^*(\alpha^{-1}(x'),\alpha^{-1}(y'))\\ &= \delta_{x,y}E_i^*(x,x) = \delta_{x,y}A_i(p,x)\\ &= \delta_{\alpha(x),\alpha(y)}A'_{\beta(i)}(\alpha(p),\alpha(x))\\ &= \delta_{\alpha(x),\alpha(y)}E^{*\prime}_{\beta(i)}(\alpha(x),\alpha(x)) = E^{*\prime}_{\beta(i)}(\alpha(x),\alpha(y))\\ &= E^{*\prime}_{\beta(i)}(x',y').\end{aligned}$$

(ii)⇒(i): Compute for all x, $y \in X$:

$$A_i(x,y) = (P^{-1}A_iP)(x,y) = A'_{\beta(i)}(\alpha(x),\alpha(y))$$

and

$$\begin{aligned}E_i^*(x,y) &= \delta_{x,y}E_i^*(x,x) = \delta_{x,y}A_i(p,x)\\ &= \delta_{x,y}(P^{-1}A'_{\beta(i)}P)(p,x)\\ &= \delta_{\alpha(x),\alpha(y)}A'_{\beta(i)}(\alpha(p),\alpha(x))\\ &= \delta_{\alpha(x),\alpha(y)}A'_{\beta(i)}(p',x')\\ &= = E^{*\prime}_{\beta(i)}(x',y') = E^{*\prime}_{\beta(i)}(\alpha(x),\alpha(y)).\end{aligned}$$

□

We give module theoretic descriptions of structural and Bose-Mesner isomorphism. We need the following notion for modules.

Definition 5.2.4 With Notation 5.2.1, fix ordering of the Hadamard and dual idempotents. Let W and W' denote $\mathcal{T}$- and $\mathcal{T}'$-modules, respectively. We say that W and W' are *similar* (relative to the ordering of the idempotents) whenever W and W' have respective ordered basis Ω and Ω' such that $[A_i]_\Omega = [A'_i]_{\Omega'}$, $[E^*_i]_\Omega = [E^{*\prime}_i]_{\Omega'}$ $(0 \leq i \leq d)$, where $[B]_O$ denotes the matrix representing B with respect to the basis O.

Lemma 5.2.5 *With Notation 5.2.1, fix ordering of the Hadamard and dual idempotents. Let W and W' denote $\mathcal{T}$- and $\mathcal{T}'$-modules, respectively. Then W and W' are similar with respect to this ordering if and only if for any basis Ω of W and any basis Ω' of W', the pairs $[A_i]_\Omega$, $[A'_i]_{\Omega'}$ and $[E^*_i]_\Omega$, $[E^{*\prime}_i]_{\Omega'}$ $(0 \leq i \leq d)$ are simultaneously similar.*

Proof. If W and W' are similar, then $\Omega' = I\Omega$, where $I \in \mathbb{M}_{\Omega',\Omega}$ is the identity matrix, so $[A_i]_\Omega$ is similar to $[A'_i]_{\Omega'}$ and $[E^*_i]_\Omega$ is similar to $[E^{*\prime}_i]_{\Omega'}$. If $[A_i]_\Omega$ is similar to $[A'_i]_{\Omega'}$, then $Q[A_i]_\Omega = [A'_i]_{\Omega'}Q$ for some invertible matrix Q. Making a change of basis with matrix Q^{-1} gives the equality between $[A_i]_\Omega$, and $[A'_i]_{\Omega'}$. Likewise for $[E^*_i]_\Omega$ and $[E^{*\prime}_i]_{\Omega'}$. $\square$

To show that modules are not similar, it suffices to show that the A_i and A'_i have distinct spectra on the module.

Theorem 5.2.6 *With Notation 5.2.1, $\mathcal{T}$ and $\mathcal{T}'$ are structurally isomorphic if and only if for some orderings of the Hadamard and dual idempotents the following hold.*

- **(i)** *Every irreducible $\mathcal{T}$-module is similar to some irreducible $\mathcal{T}'$-module relative to the given orderings of the idempotents.*
- **(ii)** *If W and W' are respectively similar irreducible modules of $\mathcal{T}$ and $\mathcal{T}'$, then* $\mathrm{mult}(W) = \mathrm{mult}(W')$.

Proof. Let $\Lambda = \{\lambda_1, \ldots \lambda_s\}$ and $\Lambda' = \{\lambda'_1, \ldots \lambda'_s\}$ be index sets for the isomorphism classes of irreducible $\mathcal{T}$- and $\mathcal{T}'$-modules, respectively. Let $\varphi_{\lambda_1}, \ldots, \varphi_{\lambda_n}$, and $\varphi'_{\lambda'_1}, \ldots, \varphi'_{\lambda'_n}$ be the corresponding primitive central idempotent of $\mathcal{T}$ and $\mathcal{T}'$. Recall that $\varphi_{\lambda_i}V$ is the sum of all irreducible $\mathcal{T}$-modules in the isomorphism class indexed by λ_i, and that $V = \sum_{i=1}^n \varphi_{\lambda_i}V$. Similarly, $V' = \sum_{i=1}^m \varphi'_{\lambda'_i}V'$.

Suppose $\mathcal{T}$ and $\mathcal{T}'$ are structurally isomorphic. Then there exists an invertible matrix H and a bijection $\beta : \{0, 1, \ldots, d\} \to \{0, 1, \ldots, d'\}$ such that $A_iH = HA'_{\beta(i)}$ and $E^*_iH =$

$HE^{*\prime}_{\beta(i)}$ $(0 \leq i \leq d)$. Define a map $\mu : V' \to V$ by $\mu(v') = Hv'$. Then μ maps irreducible $\mathcal{T}'$-modules to similar irreducible $\mathcal{T}$-modules. Indeed if W is an irreducible $\mathcal{T}$-module with basis $\{w_1, w_2, \ldots, w_k\}$, then $E_i^{*\prime}\mu(v') = E_i^* Hv' = HE_j^{*\prime}v' \in HW' = W'$ and $A_i\mu(v') = A_i Hv' = HA_j' v' \in HW'$. Thus HW' is a $\mathcal{T}$-module. Also note that H witnesses that W and W' are similar. Since H is invertible, it respects linear independence, so $\text{mult}(W) = \text{mult}(\mu(W))$.

Now suppose (i) and (ii) hold. For $1 \leq i \leq s$, take a direct decomposition of $\varphi_{\lambda_i} V$ into irreducible $\mathcal{T}$-modules $W_1^i \oplus W_2^i \oplus \cdots W_m^i$, where $m = \text{mult}(\lambda_i)$. For $1 \leq j \leq m$, fix a basis $\Omega_j^i = \{w_k^{ij}\}_{k=0}^d$ of W_j^i, where $d = \dim W_j^i$, such that $[A_\ell]_{\Omega_{j_1}^i} = [A_\ell]_{\Omega_{j_2}^i}$ for $1 \leq j_1, j_2 \leq m$. Say the irreducible $\mathcal{T}'$ modules in $\varphi'_{\lambda_i} V'$ are similar to those in $\varphi_{\lambda_i} V$. For $1 \leq i \leq s$, take a direct decomposition $\varphi'_{\lambda_i} V'$ into irreducible $\mathcal{T}'$-modules $W_1^{i\prime} \oplus W_2^{i\prime} \oplus \cdots W_m^{i\prime}$. The number of direct summands for each i agree by (ii). For $1 \leq j \leq m$, fix a basis $\Omega_j^{i\prime} = \{w_k^{ij\prime}\}_{k=0}^d$ of $W_j^{i\prime}$ such that $[A_\ell]_{\Omega_j^i} = [A'_\ell]_{\Omega_j^{i\prime}}$ and $[E_\ell^*]_{\Omega_j^i} = [E_\ell^{*\prime}]_{\Omega_j^{i\prime}}$ for all ℓ. Define a matrix H by $Hw_k^{ij} = w_k^{ij\prime}$. Observe that multiplication by H respects linear independence, so it maps the n^2-dimensional vector space V to a n n^2-dimensional vector subspace of V', so it maps V onto V'. Thus H is invertible. One now checks using similarity that $A_\ell H = HA'_\ell$ and $E_\ell^* H = HE_\ell^{*\prime}$.

To regain the initial ordering of the Hadamard idempotents, some permutation β may be required. Then (H, β), defines a structural isomorphism. □

Theorem 5.2.7 *With Notation 5.2.1, $\mathcal{T}$ and $\mathcal{T}'$ are Bose-Mesner isomorphic if and only if for some orderings of the Hadamard and dual idempotents every irreducible $\mathcal{T}$-module is similar to some irreducible $\mathcal{T}'$-module and every irreducible $\mathcal{T}'$-module is similar to some irreducible $\mathcal{T}$-module.*

Proof. Assume that σ and β define a Bose-Mesner isomorphism from $\mathcal{T}$ to $\mathcal{T}'$, that is, $\sigma(A_i) = A'_{\beta(i)}$ and $\sigma(E_i^*) = E^{*\prime}_{\beta(i)}$ for all i. Let W' be an irreducible $\mathcal{T}'$-modules. Define a $\mathcal{T}$-module structure on W' by $A_i v' = \sigma(A_i)v' = A'_{\beta(i)}v'$ and $E_i^* v' = \sigma(E_i^*)v' = E^{*\prime}_{\beta(i)}v'$ for all $v' \in W'$. Observe that $[A_i]_\Omega = [A'_{\beta(i)}]_\Omega$ and $[E_i^*]_\Omega = [E^{*\prime}_{\beta(i)}]_\Omega$ for any basis Ω of W'. Thus for each irreducible $\mathcal{T}'$-module there is a similar $\mathcal{T}$-module. A similar argument using σ^{-1} shows that for each irreducible $\mathcal{T}$-module there is a similar $\mathcal{T}'$-module.

By Wedderburn theory each $\varphi_\lambda \mathcal{T}$ is isomorphic to a full complex matrix algebra. One such an isomorphism is defined as follows. Fix an irreducible $\mathcal{T}$-module $W \subseteq \varphi_\lambda V$, and fix an ordered basis Ω of W. Let $s = \dim W$. Define $\mu_\lambda : \varphi_\lambda \mathcal{T} \to \mathbb{M}_s$ by $\mu_\lambda(A) = [A]_\Omega$, where $A \in \varphi_\lambda \mathcal{T}$. Then μ_λ is clearly an isomorphism. since $\mathcal{T} = \bigoplus_{\lambda \in \Lambda} \varphi_\lambda \mathcal{T}$, we may define an isomoprhism $\mu = \sum_{\lambda \in \Lambda} \mu_\lambda$, which maps $\mathcal{T}$ to the direct sum of full complex matrix algebras. We have similar maps for $\mathcal{T}'$, denoted with $\quad '$.

Assume that for each irreducible $\mathcal{T}'$-module there is a similar $\mathcal{T}$-module and that for each irreducible $\mathcal{T}$-module there is a similar $\mathcal{T}'$-module. Then there is a bijection $\gamma : \Lambda \to \Lambda'$ between the respective index sets for the isomorphism classes of irreducible modules for $\mathcal{T}$ and $\mathcal{T}'$. By similarlity, taking an appropriate choice of basis in the above construction we have $\mu_\lambda(A_i) = \mu'_{\lambda'}(A'_{\beta(i)})$ and $\mu_\lambda(E^*_i) = \mu'_{\lambda'}(E^{*\prime}_{\beta(i)})$. Now ${\mu'}^{-1}\mu$ defines an isomorphism with the desired properties. $\square$

Theorem 5.2.8 *With Notation 5.2.1, $\mathcal{T}$ and $\mathcal{T}'$ are abstractly isomorphic if and only if they have the same number of isomorphism classes of irreducible modules of each dimension.*

Proof. Subconstituent algebras are semisimple. By Wedderburn theory, there is one direct summand in the full matrix decomposition for each isomorphism class of irreducible $\mathcal{T}$-modules, it is $d \times d$ where the module has dimension d. $\square$

5.3 Equivalences of Latin Squares

We discuss some notions of equivalence for Latin squares. It turns out that combinatorial isomorphism of their Bose-Mesner algebras coincides with one of these equivalences. See [42, 43, 69] for further discussion of these equivalence relations.

Definition 5.3.1 Let L be a Latin square of order n with symbol set $\{1, 2, \ldots, n\}$, and let $X(L) = \{(i, j, L(i, j)) \,|\, 1 \leq i, j \leq n\}$. Note that L is uniquely determined by $X(L)$. The Latin square property gives that uniformly permuting the coordinates of the triples in $X(L)$ gives a set $X(L')$ for some Latin square L'. In this case L and L' are said to be *conjugates* of one another.

Lemma 5.3.2 *Any permutation of the relations R_1, R_2, R_3 associated with a Latin square induces a combinatorial automorphism of the associated Bose-Mesner algebra.*

Proof. Clear from the definition of a combinatorial isomorphism. □

Lemma 5.3.3 *The Bose-Mesner algebras of conjugate Latin squares are combinatorially isomorphic.*

Proof. Immediate from Lemma 5.3.2. □

Definition 5.3.4 Latin squares L and L' are said to be *isotopic* whenever $L'(f(i), g(j)) = h(L(i, j))$ for some row permutation f, column permutation g, and bijection h from the symbols of L to the symbols of L'.

Lemma 5.3.5 *Isotopic Latin squares have combinatorially isomorphic Bose-Mesner algebras.*

Proof. Suppose L and L' are isotopic via isotopy (f, g, h). Let $X = \{(i, j, L(i, j))\}_{i,j=1}^{n}$ and $X' = \{(f(i), g(j), h(L(i, j)))\}_{i,j=1}^{n}$. Denote the association schemes defined by L and L' by $(X, \{R_i\}_{i=0}^{4})$ and $(X', \{R'_i\}_{i=0}^{4})$, respectively. Define $\alpha : X \to X'$ such that $\alpha((i, j, L(i, j))) = (f(i), g(j), h(L(i, j)))$. Clearly α is a bijection. Let β denote the identity map on $\{0, 1, 2, 3, 4\}$. Now α and β define an association scheme isomorphism, so the Bose-Mesner algebras of L and L' are combinatorially isomorphic. □

Definition 5.3.6 The *main class* of a Latin square is the union of the isotopy classes of its conjugates. Two Latin squares L and L' are *main class equivalent* if they belong to the same main class, that is, if L is isotopic to a conjugate of L'.

In Tables 5.1 and 5.2 we recall some counts of Latin squares from the On-line Encyclopedia of Integer Sequences (OEIS) [82].

Theorem 5.3.7 *Main class equivalent Latin squares have combinatorially isomorphic Bose-Mesner algebras.*

n	Reduced Latin squares of size n Sequence A000315 in OEIS	All Latin squares of size n Sequence A002860 in OEIS
1	1	1
2	1	2
3	1	12
4	4	576
5	56	161280
6	9408	812851200
7	16942080	61479419904000
8	535281401856	108776032459082956800
9	377597570964258816	5524751496156892842531225600
10	7580721483160132811489280	9982437658213039871725064756920320000

Table 5.1: The numbers of Latin squares of various sizes

n	Main classes Sequence A003090 in OEIS	Isotopy classes Sequence A040082 in OEIS
1	1	1
2	1	1
3	1	1
4	2	2
5	2	2
6	12	22
7	147	564
8	283657	1676267
9	19270853541	115618721533
10	34817397894749939	208904371354363006

Table 5.2: Equivalence classes of Latin squares

Proof. Clear from Lemmas 5.3.3 and 5.3.5. □

Lemma 5.3.8 *The algebraic isomorphism class of the Bose-Mesner algebra of a Latin square depends only upon its order.*

Proof. Immediate from Theorems 3.1.1 and 5.1.4. □

5.4 Isomorphisms and Latin Squares

Notation 5.4.1 Let L denote a Latin square of order $n \geq 5$ and with symbol set $\{1, 2, \ldots, n\}$. Let X denote the set $\{(i, j, L(i, j)) \,|\, 1 \leq i, j \leq n\}$. Let $\mathcal{M}$ denote the Bose-Mesner algebra of L. Fix $p = (r_p, c_p, e_p) \in X$, and let $\mathcal{T}$ denote the subconstitutent algebra of $\mathcal{M}$ with respect to (r_p, c_p, e_p). We use the same notation for a second Latin square L', with a $\;'$ attached.

Theorem 5.4.2 *With Notation 5.4.1,*

(i) *Suppose L and L' are main class equivalent via conjugation β and isotopy $\iota = (f, g, h)$. Also suppose $\beta(\iota(p)) = p'$. Then $\mathcal{T}$ and $\mathcal{T}'$ are combinatorially isomorphic.*

(ii) *Suppose $\mathcal{T}$ and $\mathcal{T}'$ are combinatorially isomorphic. Then L and L' are main class equivalent.*

Proof. (i) Extend β by $\beta(0) = 0$, $\beta(4) = 4$. Let $\alpha : X \to X'$ be the composition of β and ι. Observe that α is a bijection. Let $P \in \mathbb{M}_{X,X'}$ be the matrix of α. Then (P, β) is a combinatorial isomorphism from $\mathcal{T}$ to $\mathcal{T}'$.

(ii) Let (P, β) be a combinatorial isomorphism from $\mathcal{T}$ to $\mathcal{T}'$. Observe that β fixes 0 and 4, so let γ be the restriction of β to $\{1, 2, 3\}$, and view γ as permuting the coordinates of X', that is, a conjugation map. We first introduce a Latin square $\hat{L}$ conjugate to L by γ. Clearly L and $\hat{L}$ are main class equivalent. Denote the objects associated with $\hat{L}$ with $\hat{}$. Thus γ induces a combinatorial isomorphism from $\mathcal{T}$ to $\hat{\mathcal{T}}$ sending $A \in \mathcal{T}$ to $\hat{A} \in \hat{\mathcal{T}}$ with $\hat{A}(\gamma(x), \gamma(y)) = A(x, y)$. In particular, A_i maps to $\hat{A}_{\beta(i)}$ and E^*_i maps to $\hat{E}^*_{\beta(i)}$.

Define a permutation $\alpha : \hat{X} \to X'$ by $\alpha(\gamma^{-1}(\hat{x})) = Px$. We claim that this induces an isotopy of $\hat{L}$ to L. Say $\hat{x} = (\hat{r}_1, \hat{c}_1, \hat{e}_1)$, $\hat{w} = (\hat{r}_1, \hat{c}_2, \hat{e}_2) \in \hat{X}$. Then $\alpha(\hat{x}) = (r'_1, c'_1, e'_1)$

and $\alpha(\hat{w}) = (r_1'', c_2', e_2')$. But since P is part of a combinatorial isomorphism, $r_1' = r_1''$. Thus there is a well-defined bijection f from the rows of $\hat{L}$ to the rows of L' induced by α. Similarly there are bijections g and h mapping columns and entries of L to columns and entries of L'. Now (f, g, h) is an isotopy. Thus $\hat{L}$ and L' are main class equivalent. □

Theorem 5.4.3 *With Notation 5.4.1, $\mathcal{T}$ and $\mathcal{T}'$ are structurally isomorphic if and only if L and L' have the same cycle structures with respect to p and p'.*

Proof. We note that the primary module and one-dimensional modules in the fourth subconstituent have the same multiplicities and dimensions for $\mathcal{T}$ and $\mathcal{T}'$ and are the corresponding modules are similar by Theorems 2.3.8 and 3.7.2.

Suppose $\mathcal{T}$ and $\mathcal{T}'$ are structurally isomorphic. Then the multi-set of roots of unity arising in connection with the irreducible modules are the same for $\mathcal{T}$ and $\mathcal{T}'$. Take a primitive root of unity with the highest order. Necessarily there is an interleaved cycle for both L and L' of length equal to that order. Take away the other roots of unity that must arise in connection with that cycle module. Arguing by induction shows that L and L' have the cycle structures with respect to p and p'.

Suppose L and L' have the same cycle structures with respect to p and p'. Form a correspondence between interleaved cycles of L and L' of the same length. Observe the corresponding cycle modules are similar by Theorem 3.3.1, so their irreducible submodules are as well. Thus $\mathcal{T}$ and $\mathcal{T}'$ are structurally isomorphic by Theorem 5.2.6. □

In Table 5.3 we recall the number of partitions of the integers from 1 to 10. The partitions correspond to possible cycle structures of the interleaved cycles. Hence the maximum possible number of structural isomorphism classes of subconstituent algebras for Latin squares of order n is the number of partitions of $n - 1$. Compare how small these values are versus those of Table 5.2.

Theorem 5.4.4 *With Notation 5.4.1, $\mathcal{T}$ and $\mathcal{T}'$ are Bose-Mesner isomorphic if and only if L and L' have cycle structures with respect to p and p' with the same divisors of the lengths of the cycles (ignoring multiplicity).*

n	Partitions of $n-1$ sequence A000041 in OEIS
2	1
3	2
4	3
5	5
6	7
7	11
8	15
9	22
10	30

Table 5.3: Number of possible structural isomorphism classes

Proof. The isomorphism class of an irreducible submodule of a cycle module is determined by the associated root of unity. The roots of unity arising in a cycle module of length k have order a divisor of k, and a root arises for every such divisor. The result follows from Theorem 5.2.7. $\square$

In light of Theorem 5.4.4, subconstituent algebras of Latin squares with cycle structures 4^2, $2^2 4$, $1^2 24$, and $1^4 4$ are Bose-Mesner isomorphic, as the distinct divisors in each case are 1, 2, and 4. The associated roots of unity are 1, -1, $\mathbf{i}$, $-\mathbf{i}$, where $\mathbf{i}^2 = -1$. Note that 4 is not a divisor of $1^6 2$ so a related subconstituent algebra will not be Bose-Mesner isomorphic to those on this list.

n	Possible Bose-Mesner isomorphism classes
5	4
6	6
7	7
8	10
9	12
10	19

Table 5.4: Number of possible Bose-Mesner isomorphism classes

Theorem 5.4.5 *With Notation 5.4.1, $\mathcal{T}$ and $\mathcal{T}'$ are abstractly isomorphic if and only if the number of distinct roots of unity with order a divisor of any cycle lengths (less one if there is just one cycle) is the same for each Latin square with respect to the base points.*

Proof. Referring to the proof of Theorem 5.2.8, in the Latin square case there is a 5-dimensional primary module, there are $n^2 - 6n + 7$ mutually isomorphism 1-dimensional module and for each cycle of length k there are k roots of unity, but when two modules are constructed with the same root they are isomorphic. □

In light of Theorem 5.4.5, subconstituent algebras of Latin squares with cycle structures 123 and 15 are abstractly isomorphic since each uses 5 distinct roots of unity to define the irreducible modules.

Corollary 5.4.6 *With Notation 5.4.1,the subconstituent algebra $\mathcal{T}$, of a Latin square L with respect to p is isomorphic (as a complex algebra) to $\mathbb{M}_5 \oplus \mathbb{M}_6^{\ell} \oplus \mathbb{M}_1$, where ℓ is the number of mutually nonisomorphic irreducible $\mathcal{T}$-modules of dimension 6.*

Proof. Clear form Theorem 5.4.5. □

n	Possible abstract isomorphism classes
5	3
6	4
7	5
8	6
9	7
10	8

Table 5.5: Number of possible abstract isomorphism classes

Example 5.4.7 Consider the Latin square

$$L = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 5 & 3 \\ 3 & 5 & 2 & 1 & 4 \\ 4 & 3 & 5 & 2 & 1 \\ 5 & 1 & 4 & 3 & 2 \end{pmatrix}$$

with respect to base points $p = (1, 1, 1)$ and $p' = (1, 3, 3)$. The respective cycle structures are 4^1 and $1^1\,3^1$. By Lemmas 3.4.2 and 3.4.4, the irreducible modules with respect to p are associated with roots of unity -1, $\mathbf{i}$, $-\mathbf{i}$, where $\mathbf{i}^2 = -1$ and the irreducible modules with respect to p' are associated with roots of unity 1, ϵ, ϵ^2, where ϵ is a primitive cubed root of unity. In each case there are three isomorphism classes of 6-dimensional irreducible modules. In particular, the subconstituent algebras $\mathcal{T}(p)$ and $\mathcal{T}'(p')$ of L are abstractly isomorphic by Theorem 5.4.5. However they are not Bose-Mesner isomorphic by Theorem 5.4.4.

Example 5.4.8 Consider the Latin square

$$L = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 5 & 1 & 6 & 3 \\ 3 & 5 & 6 & 2 & 4 & 1 \\ 4 & 6 & 1 & 3 & 2 & 5 \\ 5 & 1 & 4 & 6 & 3 & 1 \\ 6 & 3 & 2 & 5 & 1 & 4 \end{pmatrix}$$

with respect to base points $p = (5, 1, 2)$ and $p' = (6, 6, 4)$. The respective cycle structures are $1^1\ 2^2$ and $1^3\ 2^1$. By Lemmas 3.4.2 and 3.4.4, the irreducible modules with respect to p are associated with roots of unity 1 and -1, and the irreducible modules with respect to p' are also associated with roots of unity 1, -1. In the former case, the modules associated with 1 and -1 have respective multiplicities 3 and 2, and in the latter case the modules associated with 1 and -1 have respective multiplicities 4 and 1. Thus the subconstituent algebras are Bose-Mesner isomorphic by Theorem 5.4.4, but not structurally isomorphic by Theorem 5.4.3.

Example 5.4.9 Consider the Latin squares

$$L = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 5 & 6 & 4 \\ 3 & 1 & 2 & 6 & 4 & 5 \\ 4 & 6 & 5 & 1 & 3 & 2 \\ 5 & 4 & 6 & 2 & 1 & 3 \\ 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix}, \quad L' = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 3 & 6 & 5 \\ 3 & 5 & 6 & 1 & 4 & 2 \\ 4 & 6 & 5 & 2 & 3 & 1 \\ 5 & 3 & 1 & 6 & 2 & 4 \\ 6 & 4 & 2 & 5 & 1 & 3 \end{pmatrix}$$

with respect to base points $p = (1,1,1)$ and $p' = (3,3,6)$. Their respective cycle structures are $1^3\ 2^1$, and $1^3\ 2^1$. Thus the subconstituent algebras $\mathcal{T}(p)$ and $\mathcal{T}'(p')$ of L and L' are structurally isomorphic by Theorem 5.4.3, but not combinatorially isomorphic by Theorem 5.4.2.

Example 5.4.10 Consider the Latin squares

$$L = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 4 & 5 & 6 & 7 & 1 \\ 3 & 4 & 5 & 6 & 7 & 1 & 2 \\ 4 & 5 & 6 & 7 & 1 & 2 & 3 \\ 5 & 6 & 7 & 1 & 2 & 3 & 4 \\ 7 & 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}, \quad L' = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 4 & 1 & 5 & 6 & 7 & 3 \\ 3 & 1 & 7 & 2 & 4 & 5 & 6 \\ 4 & 5 & 2 & 6 & 7 & 3 & 1 \\ 5 & 6 & 4 & 7 & 3 & 1 & 2 \\ 7 & 3 & 6 & 1 & 2 & 5 & 5 \end{pmatrix}.$$

It turns out that with respect to every base point, the cycle structure of both is 2^3. In particular, even considering all cycle structures can not distinguish between all Latin squares. We note that these are the smallest examples.

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Appendices

Appendix A: Permitted Roots of Unity

cycle structure	$(\epsilon_i, \mathrm{mult}(\epsilon_i))$
4^1	$\begin{array}{ccc} -1 & -i & i \\ 1 & 1 & 1 \end{array}$
$3^1 1^1$	$\begin{array}{ccc} 1 & -\sqrt[3]{-1} & (-1)^{2/3} \\ 2 & 1 & 1 \end{array}$
2^2	$\begin{array}{cc} -1 & 1 \\ 2 & 2 \end{array}$
$2^1 1^2$	$\begin{array}{cc} -1 & 1 \\ 1 & 3 \end{array}$
1^4	$\begin{array}{c} 1 \\ 4 \end{array}$

Table 6: Irreducible $\mathcal{T}$-modules for $n = 5$

cycle structure	$(\epsilon_i, \mathrm{mult}(\epsilon_i))$
5^1	$\begin{array}{cccc} -\sqrt[5]{-1} & (-1)^{2/5} & -(-1)^{3/5} & (-1)^{4/5} \\ 1 & 1 & 1 & 1 \end{array}$
$4^1 1^1$	$\begin{array}{cccc} -1 & -i & i & 1 \\ 1 & 1 & 1 & 2 \end{array}$
$3^1 2^1$	$\begin{array}{cccc} -1 & 1 & -\sqrt[3]{-1} & (-1)^{2/3} \\ 1 & 2 & 1 & 1 \end{array}$
$3^1 1^2$	$\begin{array}{ccc} 1 & -\sqrt[3]{-1} & (-1)^{2/3} \\ 3 & 1 & 1 \end{array}$
$2^2 1^1$	$\begin{array}{cc} -1 & 1 \\ 2 & 3 \end{array}$
$2^1 1^3$	$\begin{array}{cc} -1 & 1 \\ 1 & 4 \end{array}$
1^5	$\begin{array}{c} 1 \\ 5 \end{array}$

Table 7: Irreducible $\mathcal{T}$-modules for $n = 6$

cycle structure	$(\epsilon_i, \mathrm{mult}(\epsilon_i))$				
6^1	-1	$-\sqrt[3]{-1}$	$\sqrt[3]{-1}$	$-(-1)^{2/3}$	$(-1)^{2/3}$
	1	1	1	1	1
$5^1 1^1$	1	$-\sqrt[5]{-1}$	$(-1)^{2/5}$	$-(-1)^{3/5}$	$(-1)^{4/5}$
	2	1	1	1	1
$4^1 2^1$	-1	$-i$	i	1	
	2	1	1	2	
$4^1 1^2$	-1	$-i$	i	1	
	1	1	1	3	
3^2	1	$-\sqrt[3]{-1}$	$(-1)^{2/3}$		
	2	2	2		
$3^1 2^1 1^1$	-1	1	$-\sqrt[3]{-1}$	$(-1)^{2/3}$	
	1	3	1	1	
$3^1 1^3$	1	$-\sqrt[3]{-1}$	$(-1)^{2/3}$		
	4	1	1		
2^3	-1	1			
	3	3			
$2^2 1^2$	-1	1			
	2	4			
$2^1 1^4$	-1	1			
	1	5			
1^6	1				
	6				

Table 8: Irreducible $\mathcal{T}$-modules for $n = 7$

cycle structure	$(\epsilon_i, \text{mult}(\epsilon_i))$
7^1	$-\sqrt[7]{-1}$ $(-1)^{2/7}$ $-(-1)^{3/7}$ $(-1)^{4/7}$ $-(-1)^{5/7}$ $(-1)^{6/7}$ 1 1 1 1 1 1
$6^1 1^1$	-1 1 $-\sqrt[3]{-1}$ $\sqrt[3]{-1}$ $-(-1)^{2/3}$ $(-1)^{2/3}$ 1 2 1 1 1 1
$5^1 2^1$	-1 1 $-\sqrt[5]{-1}$ $(-1)^{2/5}$ $-(-1)^{3/5}$ $(-1)^{4/5}$ 1 2 1 1 1 1
$5^1 1^2$	1 $-\sqrt[5]{-1}$ $(-1)^{2/5}$ $-(-1)^{3/5}$ $(-1)^{4/5}$ 3 1 1 1 1
$4^1 3^1$	-1 $-i$ i 1 $-\sqrt[3]{-1}$ $(-1)^{2/3}$ 1 1 1 2 1 1
$4^1 2^1 1^1$	-1 $-i$ i 1 2 1 1 3
$4^1 1^3$	-1 $-i$ i 1 1 1 1 4
$3^2 1^1$	1 $-\sqrt[3]{-1}$ $(-1)^{2/3}$ 3 2 2
$3^1 2^2$	-1 1 $-\sqrt[3]{-1}$ $(-1)^{2/3}$ 2 3 1 1
$3^1 2^1 1^2$	-1 1 $-\sqrt[3]{-1}$ $(-1)^{2/3}$ 1 4 1 1
$3^1 1^4$	1 $-\sqrt[3]{-1}$ $(-1)^{2/3}$ 5 1 1
$2^3 1^1$	-1 1 3 4
$2^2 1^3$	-1 1 2 5
$2^1 1^5$	-1 1 1 6
1^7	1 7

Table 9: Irreducible $\mathcal{T}$-modules for $n = 8$

cycle structure	$(\epsilon_i, \mathrm{mult}(\epsilon_i))$						
8^1	-1	$-i$	i	$-\sqrt[4]{-1}$	$\sqrt[4]{-1}$	$-(-1)^{3/4}$	$(-1)^{3/4}$
	1	1	1	1	1	1	1
$7^1 1^1$	1	$-\sqrt[7]{-1}$	$(-1)^{2/7}$	$-(-1)^{3/7}$	$(-1)^{4/7}$	$-(-1)^{5/7}$	$(-1)^{6/7}$
	2	1	1	1	1	1	1
$6^1 2^1$	-1	1	$-\sqrt[3]{-1}$	$\sqrt[3]{-1}$	$-(-1)^{2/3}$	$(-1)^{2/3}$	
	2	2	1	1	1	1	
$6^1 1^2$	-1	1	$-\sqrt[3]{-1}$	$\sqrt[3]{-1}$	$-(-1)^{2/3}$	$(-1)^{2/3}$	
	1	3	1	1	1	1	
$5^1 3^1$	1	$-\sqrt[5]{-1}$	$-\sqrt[3]{-1}$	$(-1)^{2/5}$	$-(-1)^{3/5}$	$(-1)^{2/3}$	$(-1)^{4/5}$
	2	1	1	1	1	1	1
$5^1 2^1 1^1$	-1	1	$-\sqrt[5]{-1}$	$(-1)^{2/5}$	$-(-1)^{3/5}$	$(-1)^{4/5}$	
	1	3	1	1	1	1	
$5^1 1^3$	1	$-\sqrt[5]{-1}$	$(-1)^{2/5}$	$-(-1)^{3/5}$	$(-1)^{4/5}$		
	4	1	1	1	1		
4^2	-1	$-i$	i	1			
	2	2	2	2			
$4^1 3^1 1^1$	-1	$-i$	i	1	$-\sqrt[3]{-1}$	$(-1)^{2/3}$	
	1	1	1	3	1	1	
$4^1 2^2$	-1	$-i$	i	1			
	3	1	1	3			
$4^1 2^1 1^2$	-1	$-i$	i	1			
	2	1	1	4			
$4^1 1^4$	-1	$-i$	i	1			
	1	1	1	5			
$3^2 2^1$	-1	1	$-\sqrt[3]{-1}$	$(-1)^{2/3}$			
	1	3	2	2			
$3^2 1^2$	1	$-\sqrt[3]{-1}$	$(-1)^{2/3}$				
	4	2	2				

Table 10: Irreducible $\mathcal{T}$-modules for $n = 9$, pt. 1

cycle structure	$(\epsilon_i, \mathrm{mult}(\epsilon_i))$
$3^1 2^2 1^1$	$\begin{array}{cccc} -1 & 1 & -\sqrt[3]{-1} & (-1)^{2/3} \\ 2 & 4 & 1 & 1 \end{array}$
$3^1 2^1 1^3$	$\begin{array}{cccc} -1 & 1 & -\sqrt[3]{-1} & (-1)^{2/3} \\ 1 & 5 & 1 & 1 \end{array}$
$3^1 1^5$	$\begin{array}{ccc} 1 & -\sqrt[3]{-1} & (-1)^{2/3} \\ 6 & 1 & 1 \end{array}$
2^4	$\begin{array}{cc} -1 & 1 \\ 4 & 4 \end{array}$
$2^3 1^2$	$\begin{array}{cc} -1 & 1 \\ 3 & 5 \end{array}$
$2^2 1^4$	$\begin{array}{cc} -1 & 1 \\ 2 & 6 \end{array}$
$2^1 1^6$	$\begin{array}{cc} -1 & 1 \\ 1 & 7 \end{array}$
1^8	$\begin{array}{c} 1 \\ 8 \end{array}$

Table 11: Irreducible $\mathcal{T}$-modules for $n = 9$, pt. 2

cycle	$(\epsilon_i, \mathrm{mult}(\epsilon_i))$								
10^1	-1	$-\sqrt[5]{-1}$	$\sqrt[5]{-1}$	$-(-1)^{2/5}$	$(-1)^{2/5}$	$-(-1)^{3/5}$	$(-1)^{3/5}$	$-(-1)^{4/5}$	$(-1)^{4/5}$
	1	1	1	1	1	1	1	1	1
$9^1 1^1$	1	$-\sqrt[9]{-1}$	$(-1)^{2/9}$	$-\sqrt[3]{-1}$	$(-1)^{4/9}$	$-(-1)^{5/9}$	$(-1)^{2/3}$	$-(-1)^{7/9}$	$(-1)^{8/9}$
	2	1	1	1	1	1	1	1	1
$8^1 2^1$	-1	$-i$	i	1	$-\sqrt[4]{-1}$	$\sqrt[4]{-1}$	$-(-1)^{3/4}$	$(-1)^{3/4}$	
	2	1	1	2	1	1	1	1	
$8^1 1^2$	-1	$-i$	i	1	$-\sqrt[4]{-1}$	$\sqrt[4]{-1}$	$-(-1)^{3/4}$	$(-1)^{3/4}$	
	1	1	1	3	1	1	1	1	
$7^1 3^1$	1	$-\sqrt[7]{-1}$	$(-1)^{2/7}$	$-\sqrt[3]{-1}$	$-(-1)^{3/7}$	$(-1)^{4/7}$	$(-1)^{2/3}$	$-(-1)^{5/7}$	$(-1)^{6/7}$
	2	1	1	1	1	1	1	1	1
$7^1 2^1 1^1$	-1	1	$-\sqrt[7]{-1}$	$(-1)^{2/7}$	$-(-1)^{3/7}$	$(-1)^{4/7}$	$-(-1)^{5/7}$	$(-1)^{6/7}$	
	1	3	1	1	1	1	1	1	
$7^1 1^3$	1	$-\sqrt[7]{-1}$	$(-1)^{2/7}$	$-(-1)^{3/7}$	$(-1)^{4/7}$	$-(-1)^{5/7}$	$(-1)^{6/7}$		
	4	1	1	1	1	1	1		
$6^1 4^1$	-1	$-i$	i	1	$-\sqrt[3]{-1}$	$\sqrt[3]{-1}$	$-(-1)^{2/3}$	$(-1)^{2/3}$	
	2	1	1	2	1	1	1	1	
$6^1 3^1 1^1$	-1	1	$-\sqrt[3]{-1}$	$\sqrt[3]{-1}$	$-(-1)^{2/3}$	$(-1)^{2/3}$			
	1	3	2	1	1	2			
$6^1 2^2$	-1	1	$-\sqrt[3]{-1}$	$\sqrt[3]{-1}$	$-(-1)^{2/3}$	$(-1)^{2/3}$			
	3	3	1	1	1	1			
$6^1 2^1 1^2$	-1	1	$-\sqrt[3]{-1}$	$\sqrt[3]{-1}$	$-(-1)^{2/3}$	$(-1)^{2/3}$			
	2	4	1	1	1	1			
$6^1 1^4$	-1	1	$-\sqrt[3]{-1}$	$\sqrt[3]{-1}$	$-(-1)^{2/3}$	$(-1)^{2/3}$			
	1	5	1	1	1	1			
$5^1 4^1 1^1$	-1	$-i$	i	1	$-\sqrt[5]{-1}$	$(-1)^{2/5}$	$-(-1)^{3/5}$	$(-1)^{4/5}$	
	1	1	1	3	1	1	1	1	
$5^1 3^1 2^1$	-1	1	$-\sqrt[5]{-1}$	$-\sqrt[3]{-1}$	$(-1)^{2/5}$	$-(-1)^{3/5}$	$(-1)^{2/3}$	$(-1)^{4/5}$	
	1	3	1	1	1	1	1	1	
$5^1 3^1 1^2$	1	$-\sqrt[5]{-1}$	$-\sqrt[3]{-1}$	$(-1)^{2/5}$	$-(-1)^{3/5}$	$(-1)^{2/3}$	$(-1)^{4/5}$		
	4	1	1	1	1	1	1		
$5^1 2^2 1^1$	-1	1	$-\sqrt[5]{-1}$	$(-1)^{2/5}$	$-(-1)^{3/5}$	$(-1)^{4/5}$			
	2	4	1	1	1	1			

Table 12: Irreducible $\mathcal{T}$-modules for $n = 10$, pt. 1

cycle	$(\epsilon_i, \mathrm{mult}(\epsilon_i))$					
5^2	1	$-\sqrt[5]{-1}$	$(-1)^{2/5}$	$-(-1)^{3/5}$	$(-1)^{4/5}$	
	2	2	2	2	2	
$5^1 2^1 1^3$	-1	1	$-\sqrt[5]{-1}$	$(-1)^{2/5}$	$-(-1)^{3/5}$	$(-1)^{4/5}$
	1	5	1	1	1	1
$5^1 1^5$	1	$-\sqrt[5]{-1}$	$(-1)^{2/5}$	$-(-1)^{3/5}$	$(-1)^{4/5}$	
	6	1	1	1	1	
$4^2 2^1$	-1	$-i$	i	1		
	3	2	2	3		
$4^2 1^2$	-1	$-i$	i	1		
	2	2	2	4		
$4^1 3^2$	-1	$-i$	i	1	$-\sqrt[3]{-1}$	$(-1)^{2/3}$
	1	1	1	3	2	2
$4^1 3^1 2^1 1^1$	-1	$-i$	i	1	$-\sqrt[3]{-1}$	$(-1)^{2/3}$
	2	1	1	4	1	1
$4^1 3^1 1^3$	-1	$-i$	i	1	$-\sqrt[3]{-1}$	$(-1)^{2/3}$
	1	1	1	5	1	1
$4^1 2^3$	-1	$-i$	i	1		
	4	1	1	4		
$4^1 2^2 1^2$	-1	$-i$	i	1		
	3	1	1	5		
$4^1 2^1 1^4$	-1	$-i$	i	1		
	2	1	1	6		
$4^1 1^6$	-1	$-i$	i	1		
	1	1	1	7		
$3^3 1^1$	1	$-\sqrt[3]{-1}$	$(-1)^{2/3}$			
	4	3	3			

Table 13: Irreducible $\mathcal{T}$-modules for $n = 10$, pt. 2

cycle	$(\epsilon_i, \mathrm{mult}(\epsilon_i))$			
$3^2 2^2$	-1	1	$-\sqrt[3]{-1}$	$(-1)^{2/3}$
	2	4	2	2
$3^2 2^1 1^2$	-1	1	$-\sqrt[3]{-1}$	$(-1)^{2/3}$
	1	5	2	2
$3^2 1^4$	1	$-\sqrt[3]{-1}$	$(-1)^{2/3}$	
	6	2	2	
$3^1 2^3 1^1$	-1	1	$-\sqrt[3]{-1}$	$(-1)^{2/3}$
	3	5	1	1
$3^1 2^2 1^3$	-1	1	$-\sqrt[3]{-1}$	$(-1)^{2/3}$
	2	6	1	1
$3^1 2^1 1^5$	-1	1	$-\sqrt[3]{-1}$	$(-1)^{2/3}$
	1	7	1	1
$3^1 1^7$	1	$-\sqrt[3]{-1}$	$(-1)^{2/3}$	
	8	1	1	
2^5	-1	1		
	5	5		
$2^4 1^2$	-1	1		
	4	6		
$2^3 1^4$	-1	1		
	3	7		
$2^2 1^6$	-1	1		
	2	8		
$2^1 1^8$	-1	1		
	1	9		
1^{10}	1			
	10			

Table 14: Irreducible $\mathcal{T}$-modules for $n = 10$, pt. 3

Appendix B: Computer Code

Routines for Latin Squares Bose-Mesner Algebras

```
Off[General::"spell1", N::"meprec"]
```

Latin Squares from source stream

A complete list of main class representatives up to order 8 is available from Gordon Royale's webpage **http://cs.anu.edu.au/~bdm/data/latin.html**. This data is presented in files consisting of the representatives of a given size n. Each line consists of n^2 ASCII numerals 0 through n-1, the first n numerals give the first row of the Latin square, the second n numerals give the second row, etc. We use this data by first setting 3 global variables to describe the input:

```
LSSource=OpenRead["latin_mc7.txt"];  (*Change file name as required *)
LSOrder=Sqrt[StringLength[Read[LSSource, Word]]];
LSSourceLength=SetStreamPosition[LSSource, Infinity]/(LSOrder^2+1);
```

LSSource gives the input file, **LSOrder** gives the order of the Latin Squares in the input file, and **LSSourceLength** gives the number of Latin Squares in the file. We then use **LSFromStream[k]** to retrieve the k-th Latin Square in the file.

```
LSFromStream[k_]:=
Module[{RawLS},
SetStreamPosition[LSSource,
(k - 1) * (LSOrder^2 + 1) - If[k===1, 0, 1]];
```

```
RawLS = Read[LSSource, Word];
Table[
ToExpression[StringTake[RawLS, {LSOrder i + j}]]+
1,
 {i, 0, LSOrder − 1}, {j, 1, LSOrder}]]
```

Unlike the data of Royale, we name our entires 1 through n. This allows us to use the entry to index a position.

Sample set up

Set source of Latin Squares

```
LSSource = OpenRead["./latin_mc7.txt"];
LSOrder = Sqrt[StringLength[Read[LSSource, Word]]];
LSSourceLength =
SetStreamPosition[LSSource, Infinity]/
(LSOrder^2 + 1);
```

Set Instance to study

Set the Latin square to be studied, and the base point to be considered. We set some new global variables, that need not come from a file as described above.

```
LSToStudy = LSFromStream[10];
LSToStudyOrder = Length[LSToStudy];
BaseToStudy = 1;
```

This data can be changed during the run to study other Latin Squares or other baes points. The value of the parameter BaseToStudy is a number from1 to **LSToStudyOrder**^2. This is explained immediately below.

The set X

We need to turn the two-dimensional data of positions of a Latin Square into one-dimensional indexes for our Bose-Mesner algebra and vice-versa. That is to say we must go between viewing X as a set of numbers between 1 and n^2 and a triple (i,j, L(i,j)). We order the triples (i,j,L(i,j)) lexographicaly: (i,j,L(i,j)) < (i', j', L(i',j')) if i<i' or if i=i' and j<j'. We take the convention
that the rows and columns are indexed by 1, 2,, ... n. Thus we may index the elements of X with a single number between1 and n^2.

The functions **rowof**, **colof**, and **entryof** take an number from 1 to n^2 and return the row/column/entry of element of X of that index. The function **Xof** returns all three as a triple.

```
rowof[nn_]:=Quotient[nn − 1, LSToStudyOrder] + 1
colof[nn_]:=If[Mod[nn, LSToStudyOrder] > 0,
 Mod[nn, LSToStudyOrder],
LSToStudyOrder]
entryof[pos_]:= LSToStudy[[rowof[pos], colof[pos]]];
posof[i_, j_]:=(j) + LSToStudyOrder ∗ (i − 1)
Xof[pos_Integer]:=
{rowof[pos], colof[pos], entryof[pos]}
Xof[pos_List]:=Map[Xof, pos]
```

Dual Bose-Mesner algebra

```
ρ[A_, b_]:=DiagonalMatrix[A[[b]]]
```

BM algebra

Now that a Latin square is fixed we may define the Hadamard idempotents of the associated Bose-Mesner algebra. These matrices need to be redefined wheenver the Latin

Square to be studied is changed.

```
A[0] = IdentityMatrix[LSToStudyOrder^2];
J = Table[1, {i, 1, LSToStudyOrder^2},
{j, 1, LSToStudyOrder^2}];
A[1] =
Table[If[And[rowof[i] == rowof[j], i ≠ j], 1, 0],
 {i, 1, LSToStudyOrder^2},
{j, 1, LSToStudyOrder^2}];
A[2] =
Table[If[And[colof[i] == colof[j], i ≠ j], 1, 0],
 {i, 1, LSToStudyOrder^2},
{j, 1, LSToStudyOrder^2}];
A[3] =
Table[
If[And[LSToStudy[[rowof[i], colof[i]]]
  ==
LSToStudy[[rowof[j], colof[j]]],
 i ≠ j],
 1, 0],
 {i, 1, LSToStudyOrder^2},
{j, 1, LSToStudyOrder^2}];
A[4] = J − (A[0] + A[1] + A[2] + A[3]);
```

dual BM algebra

Now that a Latin square and base point are fixed we may define the dual idempotents of the associated subconstituent algebra.

```
Es[0]:=ρ[A[0], BaseToStudy];
Es[1]:=ρ[A[1], BaseToStudy];
```

```
Es[2]:=ρ[A[2], BaseToStudy];
Es[3]:=ρ[A[3], BaseToStudy];
Es[4]:=ρ[A[4], BaseToStudy];
```

SRGs

```
B1[0] = A[0]; B1[1] = A[1];
B1[2] = A[2] + A[3] + A[4];
Fs1[0] = Es[0]; Fs1[1] = Es[1];
Fs1[2] = Es[2] + Es[3] + Es[4];
B2[0] = A[0]; B2[1] = A[1] + A[2];
B3[2] = A[3] + A[4];
Fs2[0] = Es[0]; Fs2[1] = Es[1] + Es[2];
Fs2[2] = Es[3] + Es[4];
B3[0] = A[0]; B3[1] = A[1] + A[2] + A[3];
B3[2] = A[4];
Fs3[0] = Es[0]; Fs3[1] = Es[1] + Es[2] + Es[3];
Fs3[2] = Es[4];
```

Latin square structure and the subconstituent algebra

Interleaved cycles, cycle structure of Latin Squares

The routine **InterleavedCycles[L, p]** takes as input a Latin Square L and a base point p. It returns the full list of interleaved cycles The each interleaved cycle consists of three list–C1, C2, C3–each with elements ordered as usual.

```
InterleavedCycles[L_, p_]:=Module[
{Lprow = DeleteCases[
Table[{p[[1]], i, L[[p[[1]], i]]},
```

$\{i, 1, \text{Length}[L[[1]]]\}]$,

$\{p[[1]], p[[2]], L[[p[[1]], p[[2]]]]\}]$,

(*Alltriples$(i, j, L(i, j))$inthesame

rowas$p$, excpet$p$*)

Lpcol = DeleteCases[

Table[$\{i, p[[2]], L[[i, p[[2]]]]\}$,

$\{i, 1, \text{Length}[L[[1]]]\}]$,

$\{p[[1]], p[[2]], L[[p[[1]], p[[2]]]]\}]$,

(*Alltriples$(i, j, L(i, j))$inthesame

columnas$p$, excpet$p$*)

Lpent = DeleteCases[

Map[Function[Append[#, L[[#[[1]], #[[2]]]]]],

Position[L,

$L[[p[[1]], p[[2]]]]]]$,

$\{p[[1]], p[[2]], L[[p[[1]], p[[2]]]]\}]$,

(*Alltriples$(i, j, L(i, j))$withthe

sameentryas$p$, excpet$p$*)

atr, atc, ate, currentrcycle, currentccycle,

currentecycle, Interleavedcyclelist = {}},

While[Length[Lprow] > 0,

currentrcycle = {};

currentccycle = {};

currentecycle = {};

atr = Lprow[[1]];

(*begin with first row element not yet used*)

While[Not[MemberQ[currentrcycle, atr]],

currentrcycle = Append[currentrcycle, atr];

atc =

```
Select[Lpcol, Function[#[[3]]===atr[[3]] ]][[
1]];
currentccycle = Append[currentccycle, atc];
ate =
Select[Lpent, Function[#[[1]]===atc[[1]]]][[
1]];
currentecycle = Append[currentecycle, ate];
atr =
Select[Lprow, Function[#[[2]]===ate[[2]]]][[
1]]; ];
Interleavedcyclelist =
Prepend[Interleavedcyclelist,
 {currentrcycle,
currentccycle, currentecycle}];
Lprow = Select[Lprow,
Function[Not[MemberQ[currentrcycle, #]]]];
 (*removetheelementsofthisrowcycle
fromavailableelements*)
];
Interleavedcyclelist
(* Return Interleavedcyclelist *)
]
```

The routine **LSCycleStructure[L, p]** takes as input a Latin Square L and a base point p.It returns a sorted list consisting of the lengths of the interleaved cycles,

```
LSCycleStructure[L_, p_]:=Module[
{Lprow = DeleteCases[
Table[{p[[1]], i, L[[p[[1]], i]]},
{i, 1, Length[L[[1]]]}],
```

```
{p[[1]], p[[2]], L[[p[[1]], p[[2]]]]}],
Lpcol = DeleteCases[
Table[{i, p[[2]], L[[i, p[[2]]]]},
{i, 1, Length[L[[1]]]}],
{p[[1]], p[[2]], L[[p[[1]], p[[2]]]]}],
Lpent = DeleteCases[
Map[Function[Append[#, L[[#[[1]], #[[2]]]]]],
 Position[L, L[[p[[1]], p[[2]]]]]],
{p[[1]], p[[2]], L[[p[[1]], p[[2]]]]}],
atr, atc, ate, currentcycle, cyclelen = {}},
(*Print[Lprow]; Print[Lpcol]; Print[Lpent]; *)
While[Length[Lprow] > 0,
currentcycle = {};
atr = Lprow[[1]];
While[Not[MemberQ[currentcycle, atr]],
currentcycle = Append[currentcycle, atr];
atc =
Select[Lpcol, Function[#[[3]]===atr[[3]] ]][[
1]];
ate =
Select[Lpent, Function[#[[1]]===atc[[1]]]][[
1]];
atr =
Select[Lprow, Function[#[[2]]===ate[[2]]]][[
1]]; ];
cyclelen = Prepend[cyclelen,
Length[currentcycle]];
(*Print[currentcycle]; *)
```

```
Lprow = Select[Lprow,
Function[Not[MemberQ[currentcycle, #]]]];
];
Sort[cyclelen]
]
```

The routine **AllCycleStructureInPlace[L]** takes as input a Latin Square and returns an n-by-n array consting of the cycle strucute of L with respect to the position of the cycle strucute.

```
AllCycleStructureInPlace[L_]:=
Table[LSCycleStructure[L, {i, j}],
{i, 1, Length[L[[1]]]}, {j, 1, Length[L[[1]]]}]
```

Modules

Return a vector of length l with a 1 in position i.

```
charvec[i_, l_]:=Table[If[k == i, 1, 0], {k, 1, l}]
```

Primary module

Return the primary module in a global variable

```
PrimaryModule:=
Map[Function[#.Table[1,  {i, LSToStudyOrder^2}]],
Table[Es[i], {i, 0, 4}]]
```

Intermediate modules

The function **Tmod[IC, div, ep]** needs an interleaved cycle, say of order k, a divisor m of k, and an mth root of unity.

We can produce the modules between the irreducibles and the cycle modules, inclusive.

We produce a cycle module by using div=1 and ep=1

```
Tmod[IC_, div_, ep_]:=
If[And[Mod[Length[IC[[1]]], div] == 0,
N[ep^div] == 1],
Module[
{u =
Map[Function[{y},
Table[Apply[Plus,
MapIndexed[Function[{x, z},
ep^z[[1]] * charvec[posof[x[[1]], x[[2]]],
LSToStudyOrder^2]],
Take[y, {offset, -1,
Length[IC[[1]]]/div}]]],
{offset, 1, Length[IC[[1]]]/div}]], IC]},
{u[[1]], u[[2]], u[[3]],
Map[Function[Es[4].A[1].Es[3].#], u[[3]]],
Map[Function[Es[4].A[2].Es[1].#], u[[1]]],
Map[Function[Es[4].A[3].Es[2].#], u[[2]]]}
], {}]

kthroots[k_]:=x/.Solve[x^k == 1]
```

Produce cycle modules

```
AllCycleMod:=
Module[
{ICs = InterleavedCycles[LSToStudy,
{rowof[BaseToStudy], colof[BaseToStudy]}]},
Map[Function[Tmod[#, 1, 1]], ICs]]
```

Produce irreducible modules

```
PossibleEpsilon[IC_]:=
(x/.Solve[x^Length[IC[[1]]] == 1])
```

haven't built in case that the interleaved cycle has length n-1 and epsilon =1 (this is part of the trivial module).

IrredTmod[IC_, ep_] returns the irreducible T-module of a cycle module associated with IC and the root of unity ep.

```
IrredTmod[IC_, ep_]:=
If[ep^Length[IC[[1]]] == 1,
Module[
{u =
Map[Function[{y},
Apply[Plus, MapIndexed[
Function[{x, z},
ep^z[[1]] * charvec[posof[x[[1]], x[[2]]],
LSToStudyOrder^2]], y]]], IC]},
{u[[1]], u[[2]], u[[3]], Es[4].A[1].Es[3].u[[3]],
Es[4].A[2].Es[1].u[[1]],
Es[4].A[3].Es[2].u[[2]]}-
KroneckerDelta[ep, 1]
Length[IC[[1]]]/(LSToStudyOrder - 1)
{Es[1].J[[1]], Es[2].J[[1]], Es[3].J[[1]],
Es[4].J[[1]], Es[4].J[[1]], Es[4].J[[1]]}
], {}]
```

return a global variable with all 6-dimensional irreducible T-modules.

```
All6dimIrredTmod:=
Module[
```

```
{ICs = InterleavedCycles[LSToStudy,
{rowof[BaseToStudy], colof[BaseToStudy]}]},
Flatten[
MapThread[Function[{IC, epvalues},
Map[Function[IrredTmod[IC, #]], epvalues]],
 {ICs,
Map[PossibleEpsilon, ICs]}], 1]]
TmodParam[irred_]:=
ep/.
Solve[
(irred[[1]].Es[1].A[2].Es[3].A[1].Es[2].
A[3].Es[1]) == ep * irred[[1]], ep][[1]]
```

Fourth subconstituent's one-dimensional modules

We produce a set of vectors orthogonal to all of the cycle modules and the primary module. We use the GramSchmidt procedure from the Mathematica package LinearAlgebra`Orthogonalization`. Recall that GramSchmidt preserves the span of the first i vector for all i, and dependent vectors become zero.

```
<< LinearAlgebra`Orthogonalization`
NonZeroVecQ[v_]:=
Select[v, Function[Not[# == 0]], 1]!={}
fourth:=
Module[{PM = PrimaryModule, SD = All6dimIrredTmod,
b4, m4},
b4 = Map[Function[charvec[#,  LSToStudyOrder^2]],
Flatten[Position[PrimaryModule[[5]], 1]]];
m4 = Prepend[Flatten[Map[Function[Take[#, -3]], SD],
1], PM[[5]]];
```

```
Select[
Take[GramSchmidt[Join[m4, b4],
InnerProduct → (Conjugate[#1].#2&)],
{Length[m4] + 1, −1}], NonZeroVecQ]]
```

if we already computed all 6 dim irred modules, there is no need to do so again.

```
fourth2[SD_]:=Module[{PM = PrimaryModule, b4, m4},
b4 = Map[Function[charvec[#, LSToStudyOrder^2]],
Flatten[Position[PrimaryModule[[5]], 1]]];
m4 = Prepend[Flatten[Map[Function[Take[#, −3]], SD],
1], PM[[5]]];
Select[
Take[GramSchmidt[Join[m4, b4],
InnerProduct → (Conjugate[#1].#2&)],
{Length[m4] + 1, −1}], NonZeroVecQ]]
```

Irred T-module into irreducible modules for SRGs

Take irreducible T-modules as input and return the irreducible S-modules into which it decomposes.

```
SRG3mod[im_]:=Module[{ep = TmodParam[im]},
Which[
ep == 1,
{{im[[1]] + im[[2]] + im[[3]],
im[[4]] + im[[5]] + im[[6]]},
{im[[1]] − 2im[[2]] + im[[3]],
im[[4]] − 2im[[5]] + im[[6]]},
{im[[1]] + 2im[[2]] − 3im[[3]],
im[[4]] + 2im[[5]] − 3im[[6]]}},
ep == −1,
```

```
{{im[[1]] + 2im[[2]] + im[[3]], im[[4]] + im[[6]]},
{im[[1]] − im[[2]] − 2im[[3]], im[[4]] − im[[5]]},
{im[[1]] − im[[2]] + im[[3]]},
{im[[4]] + im[[5]] − im[[6]]}},
True,
Module[
{thlist =
θ/.
Solve[θ^3 + 3θ^2 − 2(1 + Re[ep]) == 0],
bc},
bc =
Map[{(1 + ep + #)/(#^2 + 2#),
(1 + ep + ep ∗ #)/(#^2 + 2#)}&, thlist];
Map[{im[[1]] + #[[1]]im[[2]] + #[[2]]im[[3]],
(#[[1]] + #[[2]])im[[4]]+
(1 + #[[2]]/ep)im[[5]] + (1 + #[[1]])im[[6]]}&,
bc]]
]]
SRG2mod[im_]:=Module[{ep = TmodParam[im]},
Which[
ep == 1,
{{im[[1]] + im[[2]], im[[4]] + im[[5]] + 2im[[3]]},
{im[[1]] − im[[2]], im[[5]] − im[[4]]},
{im[[3]] + im[[6]]},
{im[[4]] + im[[5]] + im[[6]]−
(LSToStudyOrder − 4)im[[3]]}},
ep == −1,
{{im[[1]] + im[[2]], im[[4]] + im[[5]]},
```

```
{im[[1]] − im[[2]], im[[5]] − im[[4]] − 2im[[3]]},
{im[[4]] − im[[3]] + (LSToStudyOrder − 3)im[[3]]},
{im[[4]] + im[[5]] + im[[6]]+
(LSToStudyOrder − 1)im[[6]]}},
True,
{{im[[1]] + im[[2]],
im[[4]] + im[[5]] + (1 + ep)im[[3]]},
{im[[1]] − im[[2]],
im[[5]] − im[[4]] + (ep − 1)im[[3]]},
{(1 + ep)im[[3]] + im[[4]] + im[[5]]+
(LSToStudyOrder − 1)im[[6]]},
{(1 + ep^2 + 2ep(LSToStudyOrder − 2))im[[3]]+
(1 − ep)im[[4]] + (ep − 1)im[[5]]+
(1 + ep)(LSToStudyOrder − 1)im[[6]]}}
]]
SRG1mod[im_]:=Module[{ep = TmodParam[im]},
{{im[[1]]}, {im[[2]] + im[[3]] + im[[4]]},
{im[[2]] + im[[6]]},
{ ep ∗ im[[2]] +  im[[5]]},
{(2 − LSToStudyOrder)im[[2]] + im[[4]]},
{−im[[2]] + im[[3]]}}]
```

The rountine **matrep[AM, md]** returns the matrix representing AM with respect to a basis md.

```
matrep[AM_, mbas_]:=
Module[{vars = Table[x[i], {i, 1, Length[mbas]}]},
MatrixForm[
Flatten[
```

vars/.Map[Solve[# == vars.mbas]&, mbas. AM], 1]]]

the routine urep[th] returns a matrix representing a module of type U(th) (given eigenvalues r, s of the adjacenty matrix)

urep[th_, r_, s_]:=

{{th, (*r*) − th}, {th − (*s*), (*r* + *s*) − th}}

The bases the SRG modules differ from those in the "type U( )" descrption by scalar multiples. Note that the diagonal entries are invariant and the product of the off-diagonals are invariant.

Sample run

Set Instance to study

LSToStudy = LSFromStream[64];

LSToStudyOrder = Length[LSToStudy];

BaseToStudy = 1;

MatrixForm[LSToStudy]

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 5 & 1 & 3 & 6 & 7 & 4 \\ 3 & 1 & 2 & 5 & 7 & 4 & 6 \\ 4 & 3 & 6 & 7 & 1 & 2 & 5 \\ 5 & 7 & 4 & 6 & 2 & 1 & 3 \\ 6 & 4 & 7 & 1 & 3 & 5 & 2 \\ 7 & 6 & 5 & 2 & 4 & 3 & 1 \end{pmatrix}$$

LSCycleStructure[LSToStudy, {1, 1}]

$\{1, 2, 3\}$

LSCycleStructure[LSToStudy, {1, 2}]

$\{6\}$

LSCycleStructure[LSToStudy, {7, 7}]

$\{1, 1, 1, 1, 2\}$

AllCycleStructureInPlace[LSToStudy]

$\{\{\{1,2,3\},\{6\},\{6\},\{6\},\{6\},\{2,4\},\{1,2,3\}\},$

$\{\{6\},\{6\},\{1,5\},\{3,3\},\{1,2,3\},\{2,2,2\},\{2,4\}\},$

$\{\{6\},\{1,5\},\{1,2,3\},\{1,5\},\{6\},\{6\},\{1,2,3\}\},\{\{2,4\},$

$\{2,2,2\},\{1,2,3\},\{6\},\{1,1,2,2\},\{1,1,4\},\{1,2,3\}\},$

$\{\{6\},\{1,5\},\{1,5\},\{6\},\{1,1,4\},\{1,1,2,2\},\{1,1,4\}\},$

$\{\{2,2,2\},\{1,2,3\},\{1,2,3\},\{1,2,3\},\{1,5\},$

$\{1,5\},\{1,1,1,3\}\},\{\{1,2,3\},\{1,5\},\{1,2,3\},$

$\{1,1,2,2\},\{2,4\},\{1,5\},\{1,1,1,1,2\}\}\}$

Note that in this example, every possible cycle structure except for one (that with all one cycles) occures for some base point.

LSToStudy = LSFromStream[39];

LSToStudyOrder = Length[LSToStudy];

BaseToStudy = 1;

MatrixForm[LSToStudy]

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 4 & 1 & 6 & 7 & 5 & 3 \\ 3 & 7 & 6 & 5 & 1 & 2 & 4 \\ 4 & 3 & 5 & 1 & 6 & 7 & 2 \\ 5 & 6 & 7 & 2 & 3 & 4 & 1 \\ 6 & 1 & 2 & 7 & 4 & 3 & 5 \\ 7 & 5 & 4 & 3 & 2 & 1 & 6 \end{pmatrix}$$

LSCycleStructure[LSToStudy, {7, 5}]

$\{1,1,1,1,1,1\}$

The cycle structure missing from the previous example does occur in the next example.

LSToStudy = LSFromStream[38];

LSToStudyOrder = Length[LSToStudy];

BaseToStudy = 1;

MatrixForm[LSToStudy]

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 4 & 1 & 5 & 6 & 7 & 3 \\ 3 & 1 & 7 & 2 & 4 & 5 & 6 \\ 4 & 5 & 2 & 6 & 7 & 3 & 1 \\ 5 & 6 & 4 & 7 & 3 & 1 & 2 \\ 6 & 7 & 5 & 3 & 1 & 2 & 4 \\ 7 & 3 & 6 & 1 & 2 & 4 & 5 \end{pmatrix}$$

AllCycleStructureInPlace[LSToStudy]

{{{2, 2, 2}, {2, 2, 2}, {2, 2, 2},
{2, 2, 2}, {2, 2, 2}, {2, 2, 2}, {2, 2, 2}},
{{2, 2, 2}, {2, 2, 2}, {2, 2, 2}, {2, 2, 2}, {2, 2, 2},
{2, 2, 2}, {2, 2, 2}}, {{2, 2, 2}, {2, 2, 2},
{2, 2, 2}, {2, 2, 2}, {2, 2, 2}, {2, 2, 2}, {2, 2, 2}},
{{2, 2, 2}, {2, 2, 2}, {2, 2, 2}, {2, 2, 2}, {2, 2, 2},
{2, 2, 2}, {2, 2, 2}}, {{2, 2, 2}, {2, 2, 2},
{2, 2, 2}, {2, 2, 2}, {2, 2, 2}, {2, 2, 2}, {2, 2, 2}},
{{2, 2, 2}, {2, 2, 2}, {2, 2, 2}, {2, 2, 2}, {2, 2, 2},
{2, 2, 2}, {2, 2, 2}}, {{2, 2, 2}, {2, 2, 2},
{2, 2, 2}, {2, 2, 2}, {2, 2, 2}, {2, 2, 2}, {2, 2, 2}}}

In this example the cycle structure is the same with respect to every base point.(It is main class equivalent to the integers mod 7).

We have detected no connection between cycle structure with respect to different base points of the same Latin square in general.

Let us pick an interesting example and consider its subconstituent algebra.

LSToStudy = LSFromStream[64];

LSToStudyOrder = Length[LSToStudy];

BaseToStudy = 1;

BM algebra

$A[0] = \text{IdentityMatrix}[\text{LSToStudyOrder\^{}2}];$

$J = \text{Table}[1, \{i, 1, \text{LSToStudyOrder\^{}2}\},$

$\{j, 1, \text{LSToStudyOrder\^{}2}\}];$

$A[1] =$

$\text{Table}[\text{If}[\text{And}[\text{rowof}[i] == \text{rowof}[j], i \neq j], 1, 0],$

$\{i, 1, \text{LSToStudyOrder\^{}2}\},$

$\{j, 1, \text{LSToStudyOrder\^{}2}\}];$

$A[2] =$

$\text{Table}[\text{If}[\text{And}[\text{colof}[i] == \text{colof}[j], i \neq j], 1, 0],$

$\{i, 1, \text{LSToStudyOrder\^{}2}\},$

$\{j, 1, \text{LSToStudyOrder\^{}2}\}];$

$A[3] =$

$\text{Table}[$

$\text{If}[\text{And}[\text{LSToStudy}[[\text{rowof}[i], \text{colof}[i]]]$

$==$

$\text{LSToStudy}[[\text{rowof}[j], \text{colof}[j]]],$

$i \neq j],$

$1, 0],$

$\{i, 1, \text{LSToStudyOrder\^{}2}\},$

$\{j, 1, \text{LSToStudyOrder\^{}2}\}];$

$A[4] = J - (A[0] + A[1] + A[2] + A[3]);$

dual BM algebra

$\text{Es}[0] := \rho[A[0], \text{BaseToStudy}];$

$\text{Es}[1] := \rho[A[1], \text{BaseToStudy}];$

$\text{Es}[2] := \rho[A[2], \text{BaseToStudy}];$

Es[3]:=ρ[A[3], BaseToStudy];

Es[4]:=ρ[A[4], BaseToStudy];

Since vectors are listed at rows, these matrices act on the right. This contrasts with the paper where the vectors are columns and the matrices act on the left. Thus the difference is just a transposition.

Study examples

ICs = InterleavedCycles[LSToStudy,
{rowof[BaseToStudy], colof[BaseToStudy]}]

$\{\{\{\{1,7,7\}\},\{\{7,1,7\}\},\{\{7,7,1\}\}\},$
$\{\{\{1,4,4\},\{1,5,5\},\{1,6,6\}\},$
$\{\{4,1,4\},\{5,1,5\},\{6,1,6\}\},$
$\{\{4,5,1\},\{5,6,1\},\{6,4,1\}\}\},\{\{\{1,2,2\},\{1,3,3\}\},$
$\{\{2,1,2\},\{3,1,3\}\},\{\{2,3,1\},\{3,2,1\}\}\}\}$

The first element is a 1-cycle

ICs[[1]]

$\{\{\{1,7,7\}\},\{\{7,1,7\}\},\{\{7,7,1\}\}\}$

The second is a 3-cycle

ICs[[2]]

$\{\{\{1,4,4\},\{1,5,5\},\{1,6,6\}\},$
$\{\{4,1,4\},\{5,1,5\},\{6,1,6\}\},$
$\{\{4,5,1\},\{5,6,1\},\{6,4,1\}\}\}$

the third is a 2-cycle.

ICs[[3]]

$\{\{\{1,2,2\},\{1,3,3\}\},$
$\{\{2,1,2\},\{3,1,3\}\},\{\{2,3,1\},\{3,2,1\}\}\}$

Some of the cycle modules:

Tmod[ICs[[1]], 1, 1]

{{{0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}},
{{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0}},
{{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1}},
{{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 0}},
{{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0,
0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0,
0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0}},
{{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0,
0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0,
0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}}}

Tmod[ICs[[3]], 1, 1]

{{{0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
{0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}},
{{0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},

{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}},
{{0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}},
{{0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 1, 1, 1, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1,
1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}},
{{0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0,
0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0},
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1,
0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0,
0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0}},
{{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1,
0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1,
0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0},
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0}}}

Data given as list of 6 lists: the first three lists are the characterists vectors of the elements of the cycles in the corresponding subconstituents. The last three are the characteristic vectors of those in the fourth subconstiutents.

We turn to irreducible modules

c2 = ICs[[2]]; (*a3 − cycle*)

PossibleEpsilon[c2]

$\{1, -(-1)^{1/3}, (-1)^{2/3}\}$

2IrredTmod[c2, 1]

$\{\{0, -1, -1, 1, 1, 1, -1, 0, 0, 0, 0, 0, 0, 0,$
$0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,$
$0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\},$
$\{0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, -1, 0,$
$0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0,$
$0, 0, 1, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0\},$
$\{0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, -1,$
$0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0,$
$1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1\},$
$\{0, 0, 0, 0, 0, 0, 0, 0, -1, 0, -1, -1, -1, -1, 0, 0,$
$-1, -1, -1, -1, -1, 0, 1, 1, 1, 0, 1, 1, 0, 1, 1, 1, 1,$
$0, 1, 0, 1, 1, 0, 1, 1, 1, 0, -1, -1, -1, -1, -1, 0\},$
$\{0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 1, 1, 1, -1, 0, 0, -1,$
$1, 1, 1, -1, 0, -1, -1, 1, 0, 1, -1, 0, -1, -1, 1, 1,$
$0, -1, 0, -1, -1, 0, 1, 1, -1, 0, -1, -1, 1, 1, 1, 0\},$
$\{0, 0, 0, 0, 0, 0, 0, 0, 1, 0, -1, 1, -1, 1, 0, 0, -1,$
$1, -1, 1, 1, 0, -1, 1, -1, 0, -1, 1, 0, -1, 1, 1, -1,$
$0, -1, 0, 1, -1, 0, -1, 1, -1, 0, 1, 1, -1, 1, -1, 0\}\}$

TmodParam[%]

1

IrredTmod $[$c2, $-(-1)^{1/3}]$

$\{\{0,0,0,-(-1)^{1/3},(-1)^{2/3},1,0,0,0,0,0,0,0,$
$0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,$
$0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0\},$
$\{0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,$
$0,0,0,-(-1)^{1/3},0,0,0,0,0,0,(-1)^{2/3},0,0,$
$0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0\},$
$\{0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,$
$0,0,0,0,0,0,0,-(-1)^{1/3},0,0,0,0,0,0,0,$
$(-1)^{2/3},0,0,0,0,1,0,0,0,0,0,0,0,0,0\},$
$\{0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,$
$0,0,0,-(-1)^{1/3},-(-1)^{1/3},-(-1)^{1/3},0,-(-1)^{1/3},$
$-(-1)^{1/3},0,(-1)^{2/3},(-1)^{2/3},(-1)^{2/3},(-1)^{2/3},0,$
$(-1)^{2/3},0,1,1,0,1,1,1,0,0,0,0,0,0,0\},$
$\{0,0,0,0,0,0,0,0,0,0,-(-1)^{1/3},(-1)^{2/3},1,0,$
$0,0,0,-(-1)^{1/3},(-1)^{2/3},1,0,0,0,0,-(-1)^{1/3},$
$0,1,0,0,0,0,-(-1)^{1/3},(-1)^{2/3},0,0,0,0,0,$
$0,(-1)^{2/3},1,0,0,0,0,-(-1)^{1/3},(-1)^{2/3},1,0\},$
$\{0,0,0,0,0,0,0,0,(-1)^{2/3},0,0,1,0,-(-1)^{1/3},$
$0,0,0,(-1)^{2/3},0,-(-1)^{1/3},1,0,0,1,0,0,0,$
$(-1)^{2/3},0,0,-(-1)^{1/3},1,0,0,0,0,-(-1)^{1/3},0,$
$0,0,(-1)^{2/3},0,0,1,(-1)^{2/3},0,-(-1)^{1/3},0,0\}\}$

TmodParam[%]

$-(-1)^{1/3}$

IrredTmod $[\mathbf{c2},(-1)^{2/3}]$

$\{\{0,0,0,(-1)^{2/3},-(-1)^{1/3},1,0,0,0,0,0,0,0,$
$0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,$
$0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0\},$
$\{0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,$

$0, 0, 0, (-1)^{2/3}, 0, 0, 0, 0, 0, 0, -(-1)^{1/3}, 0, 0,$
$0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\},$
$\{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,$
$0, 0, 0, 0, 0, 0, 0, (-1)^{2/3}, 0, 0, 0, 0, 0, 0, 0,$
$-(-1)^{1/3}, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0\},$
$\{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,$
$0, 0, 0, 0, (-1)^{2/3}, (-1)^{2/3}, (-1)^{2/3}, 0, (-1)^{2/3},$
$(-1)^{2/3}, 0, -(-1)^{1/3}, -(-1)^{1/3}, -(-1)^{1/3}, -(-1)^{1/3},$
$0, -(-1)^{1/3}, 0, 1, 1, 0, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0\},$
$\{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, (-1)^{2/3}, -(-1)^{1/3}, 1, 0,$
$0, 0, 0, (-1)^{2/3}, -(-1)^{1/3}, 1, 0, 0, 0, 0, (-1)^{2/3},$
$0, 1, 0, 0, 0, 0, (-1)^{2/3}, -(-1)^{1/3}, 0, 0, 0, 0, 0,$
$0, -(-1)^{1/3}, 1, 0, 0, 0, 0, (-1)^{2/3}, -(-1)^{1/3}, 1, 0\},$
$\{0, 0, 0, 0, 0, 0, 0, 0, -(-1)^{1/3}, 0, 0, 1, 0, (-1)^{2/3},$
$0, 0, 0, -(-1)^{1/3}, 0, (-1)^{2/3}, 1, 0, 0, 1, 0, 0, 0,$
$-(-1)^{1/3}, 0, 0, (-1)^{2/3}, 1, 0, 0, 0, 0, (-1)^{2/3}, 0,$
$0, 0, -(-1)^{1/3}, 0, 0, 1, -(-1)^{1/3}, 0, (-1)^{2/3}, 0, 0\}\}$

TmodParam[%]

$(-1)^{2/3}$

SRGs

B1[0] = A[0]; B1[1] = A[1];
B1[2] = A[2] + A[3] + A[4];
Fs1[0] = Es[0]; Fs1[1] = Es[1];
Fs1[2] = Es[2] + Es[3] + Es[4];
B2[0] = A[0]; B2[1] = A[1] + A[2];
B2[2] = A[3] + A[4];
Fs2[0] = Es[0]; Fs2[1] = Es[1] + Es[2];

Fs2[2] = Es[3] + Es[4];

B3[0] = A[0]; B3[1] = A[1] + A[2] + A[3];

B3[2] = A[4];

Fs3[0] = Es[0]; Fs3[1] = Es[1] + Es[2] + Es[3];

Fs3[2] = Es[4];

Study SRG's

S3M1 = Simplify[SRG3mod[IrredTmod[c2, 1]]];

The bases the SRG modules differ from those in the "type U( )" descrption by scalar multiples. Note that the diagonal entries are invariant and the product of the off-diagonals are invariant.

matrep[B3[1], S3M1[[1]]]

$$\begin{pmatrix} 1 & 2 \\ 6 & 0 \end{pmatrix}$$

MatrixForm[urep[1, LSToStudyOrder − 3, −3]]

$$\begin{pmatrix} 1 & 3 \\ 4 & 0 \end{pmatrix}$$

matrep[B3[1], S3M1[[2]]]

$$\begin{pmatrix} -2 & -1 \\ -6 & 3 \end{pmatrix}$$

MatrixForm[urep[−2, LSToStudyOrder − 3, −3]]

$$\begin{pmatrix} -2 & 6 \\ 1 & 3 \end{pmatrix}$$

matrep[B3[1], S3M1[[3]]]

$$\begin{pmatrix} -2 & -1 \\ -6 & 3 \end{pmatrix}$$

S3M2 = SRG3mod [IrredTmod [c2, $(-1)^{2/3}$]] ;

Chop[N[matrep[B3[1], S3M2[[1]]]]]

$$\begin{pmatrix} 0.532089 & 1. \\ 12.249 & 0.467911 \end{pmatrix}$$

MatrixForm[urep[0.5320888862379565`,

LSToStudyOrder − 3, −3]]

$$\begin{pmatrix} 0.532089 & 3.46791 \\ 3.53209 & 0.467911 \end{pmatrix}$$

3.4679111137620433` ∗ 3.5320888862379567`

12.249

Chop[*N*[matrep[B3[1], S3M2[[2]]]]]

$$\begin{pmatrix} -0.652704 & 1. \\ 10.9213 & 1.6527 \end{pmatrix}$$

MatrixForm[urep[−0.6527036446661388`,

LSToStudyOrder − 3, −3]]

$$\begin{pmatrix} -0.652704 & 4.6527 \\ 2.3473 & 1.6527 \end{pmatrix}$$

4.652703644666139` ∗ 2.347296355333861`

10.9213

S2M1 = Simplify[SRG2mod[IrredTmod[c2, 1]]];

matrep[B2[1], S2M1[[1]]]

$$\begin{pmatrix} -1 & 1 \\ 6 & 4 \end{pmatrix}$$

MatrixForm[urep[−1, LSToStudyOrder − 2, −2]]

$$\begin{pmatrix} -1 & 6 \\ 1 & 4 \end{pmatrix}$$

matrep[B2[1], S2M1[[3]]]

$$\begin{pmatrix} -2 \end{pmatrix}$$

S2M2 = SRG2mod $\left[\textbf{IrredTmod}\left[\textbf{c2}, (-1)^{2/3}\right]\right]$ **;**

Chop[*N*[matrep[B2[1], S2M2[[1]]]]]

$$\begin{pmatrix} -1. & 1. \\ 6. & 4. \end{pmatrix}$$

Chop[N[matrep[B2[1], S2M2[[2]]]]]

$$\begin{pmatrix} -1. & 1. \\ 6. & 4. \end{pmatrix}$$

Chop[N[matrep[B2[1], S2M2[[3]]]]]

$$\begin{pmatrix} -2. \end{pmatrix}$$

Chop[N[matrep[B2[1], S2M2[[4]]]]]

$$\begin{pmatrix} -2. \end{pmatrix}$$

S1M2 = SRG1mod [IrredTmod [c2, $(-1)^{2/3}$]] ;

Simplify[matrep[B1[1], S1M2[[1]]]]

$$\begin{pmatrix} -1 \end{pmatrix}$$

Simplify[matrep[B1[1], S1M2[[2]]]]

$$\begin{pmatrix} 6 \end{pmatrix}$$

Simplify[matrep[B1[1], S1M2[[3]]]]

$$\begin{pmatrix} -1 \end{pmatrix}$$

Simplify[matrep[B1[1], S1M2[[4]]]]

$$\begin{pmatrix} -1 \end{pmatrix}$$

Simplify[matrep[B1[1], S1M2[[5]]]]

$$\begin{pmatrix} -1 \end{pmatrix}$$

Simplify[matrep[B1[1], S1M2[[6]]]]

$$\begin{pmatrix} -1 \end{pmatrix}$$

Verify cycle module actions

a = AllCycleMod; (* too long to show output*)

prmd = PrimaryModule;

md = 2; (* specify which cycle module *)

ZeroVectorsQ[v_]:=Union[Flatten[v]]==={0}

act on u_1,i

```
Map[ZeroVectorsQ,
{Union[a[[md, 1]].(Es[1].A[1].Es[1] + A[0])]−
{prmd[[2]]},
a[[md, 1]].Es[1].A[2].Es[3] − RotateRight[a[[md, 3]]],
a[[md, 1]].Es[1].A[2].Es[4] − a[[md, 5]],
a[[md, 1]].Es[1].A[3].Es[4] − a[[md, 6]],
a[[md, 1]].Es[1].A[3].Es[2] − a[[md, 2]],
Union[a[[md, 1]].Es[1].A[1].Es[0]] − {prmd[[1]]}}]
```

{True, True, True, True, True, True}

act on u2,i

```
Map[ZeroVectorsQ,
{Union[a[[md, 2]].(Es[2].A[2].Es[2] + A[0])]−
{PrimaryModule[[3]]},
a[[md, 2]].Es[2].A[3].Es[1] − a[[md, 1]],
a[[md, 2]].Es[2].A[3].Es[4] − a[[md, 6]],
a[[md, 2]].Es[2].A[1].Es[4] − a[[md, 4]],
a[[md, 2]].Es[2].A[1].Es[3] − a[[md, 3]]}]
```

{True, True, True, True, True}

act on u3,i

```
Map[ZeroVectorsQ,
{Union[a[[md, 3]].(Es[3].A[3].Es[3] + A[0])]−
{PrimaryModule[[4]]},
a[[md, 3]].Es[3].A[1].Es[2] − a[[md, 2]],
a[[md, 3]].Es[3].A[1].Es[4] − a[[md, 4]],
a[[md, 3]].Es[3].A[2].Es[4]−
RotateLeft[a[[md, 5]]],
a[[md, 3]].Es[3].A[2].Es[1] − RotateLeft[a[[md, 1]]]}]
```

{True, True, True, True, True}

act on v1,i

```
Map[ZeroVectorsQ,
{a[[md, 4]].Es[4].A[1].Es[4]/(LSToStudyOrder − 3)−
a[[md, 4]],
Union[a[[md, 4]].(−Es[4].A[3].Es[4] − A[0])−
a[[md, 6]]] + {PrimaryModule[[5]]},
Union[a[[md, 4]].(−Es[4].A[3].Es[1]) − a[[md, 1]]]+
{PrimaryModule[[2]]},
Union[a[[md, 4]].(−Es[4].A[3].Es[2]) − a[[md, 2]]]+
{PrimaryModule[[3]]},
a[[md, 4]].(Es[4].A[1].Es[2])/(LSToStudyOrder − 2)−
a[[md, 2]],
a[[md, 4]].(Es[4].A[1].Es[3])/(LSToStudyOrder − 2)−
a[[md, 3]],
Union[a[[md, 4]].(−Es[4].A[2].Es[3]) − a[[md, 3]]]+
{PrimaryModule[[4]]},
Union[a[[md, 4]].(−Es[4].A[2].Es[1])−
RotateLeft[a[[md, 1]]]] + {PrimaryModule[[2]]},
Union[a[[md, 4]].(−Es[4].A[2].Es[4] − A[0])−
RotateLeft[a[[md, 5]]]] + {PrimaryModule[[5]]}}]
```

{True, True, True, True, True, True, True, True, True}

act on v2,i

```
Map[ZeroVectorsQ,
{a[[md, 5]].Es[4].A[2].Es[4]/(LSToStudyOrder − 3)−
a[[md, 5]],
Union[a[[md, 5]].(−Es[4].A[1].Es[4] − A[0])−
RotateRight[a[[md, 4]]]] + {PrimaryModule[[5]]},
```

Union[a[[md, 5]].(−Es[4].A[1].Es[2])−
RotateRight[a[[md, 2]]]] + {PrimaryModule[[3]]},
Union[a[[md, 5]].(−Es[4].A[1].Es[3])−
RotateRight[a[[md, 3]]]] + {PrimaryModule[[4]]},
a[[md, 5]].(Es[4].A[2].Es[3])/(LSToStudyOrder − 2)−
RotateRight[a[[md, 3]]],
a[[md, 5]].(Es[4].A[2].Es[2])/(LSToStudyOrder − 2)−
a[[md, 1]],
Union[a[[md, 5]].(−Es[4].A[3].Es[1]) − a[[md, 1]]]+
{PrimaryModule[[2]]},
Union[a[[md, 5]].(−Es[4].A[3].Es[2]) − a[[md, 2]]]+
{PrimaryModule[[3]]},
Union[a[[md, 5]].(−Es[4].A[3].Es[4] − A[0])−
a[[md, 6]]] + {PrimaryModule[[5]]}}]

{True, True, True, True, True, False, True, True, True}

act on v3,i

Map[ZeroVectorsQ,
{a[[md, 6]].Es[4].A[3].Es[4]/(LSToStudyOrder − 3)−
a[[md, 6]],
Union[a[[md, 6]].(−Es[4].A[2].Es[4] − A[0])−
a[[md, 5]]] + {PrimaryModule[[5]]},
Union[a[[md, 6]].(−Es[4].A[2].Es[3])−
RotateRight[a[[md, 3]]]] + {PrimaryModule[[4]]},
Union[a[[md, 6]].(−Es[4].A[2].Es[1]) − a[[md, 1]]]+
{PrimaryModule[[2]]},
a[[md, 6]].(Es[4].A[3].Es[1])/(LSToStudyOrder − 2)−
a[[md, 1]],
a[[md, 6]].(Es[4].A[3].Es[2])/(LSToStudyOrder − 2)−

$\boldsymbol{a}$**[[md, 2]],**
Union[$\boldsymbol{a}$**[[md, 6]].(−Es[4].**$\boldsymbol{A}$**[1].Es[2]) −** $\boldsymbol{a}$**[[md, 2]]]+**
{PrimaryModule[[3]]},
Union[$\boldsymbol{a}$**[[md, 6]].(−Es[4].**$\boldsymbol{A}$**[1].Es[3]) −** $\boldsymbol{a}$**[[md, 3]]]+**
{PrimaryModule[[4]]},
Union[$\boldsymbol{a}$**[[md, 6]].(−Es[4].**$\boldsymbol{A}$**[1].Es[4] −** $\boldsymbol{A}$**[0])−**
$\boldsymbol{a}$**[[md, 4]]] + {PrimaryModule[[5]]}}]**
{True, True, True, True, True, True, True, True, True}

About the Author

Ibtisam Daqqa received a Bachelors Degree in Mathematics from Applied Science University in 1995 and a Master's Degree in Mathematics from University of Jordan in 1998. In 1999 she started teaching as an instructor of Mathematics at the Hashemite University in Jordan. She entered the Ph.D. program as a teaching assistant at the University of South Florida in 2002. While in the Ph.D. program at the University of South Florida, Mrs. Daqqa taught several undergraduate classes and submitted two papers for publication. She also made several presentations at AMS sectional and national meetings.