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Application of the Riemann-Hilbert method to soliton solutions of a nonlocal reverse-spacetime Sasa-Satsuma equation and a higher-order reverse-time NLS-type equation

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by

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Dedication

I dedicate this dissertation to my beloved mother, Youssria Abdou, who devoted all her life to my family, my siblings and myself. I also dedicate it to my beautiful beloved little daughters, Cera and Celine.

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Abstract

For many years, the study of integrable systems has been one of the most fascinating branches of mathematics and has been thought to be an interesting area for both mathematicians and physicists alike. Many natural phenomena can be predicted by using integrable systems, particularly by studying their different solutions, as well as analyzing and exploring their properties and structures. They are commonly found in nonlinear optics, plasmas, ocean and water waves, gravitational fields, and fluid dynamics. Typical examples of integrable systems include the Korteweg-de Vries (KdV) equation, the nonlinear Schrödinger (NLS) equation, and the Kadomtsev-Petviashvili (KP) equation. Solitons are intrinsic solutions for these equations, and various types of solitons can be obtained, such as bright and dark solitons, lump and rogue waves, and breathers.

In the dissertation, we present and investigate a novel nonlocal nonlinear reverse-spacetime Sasa-Satsuma equation, which is a KdV-type equation. Furthermore, we analyze it and determine its Hamiltonian structure. This equation is derived from an AKNS spectral problem involving a nonlocal 5×5 matrix. Also, as part of our investigation, we also develop a higher-order nonlocal reverse-time NLS-type equation that originates from the analysis of a local 4×4 matrix spectral problem. By using vectors lying in the kernel of the Jost solutions, we can generate soliton solutions for the nonlocal Sasa-Satsuma and the nonlocal NLS-type equations using the Riemann-Hilbert problem with the real line being the contour.

When the reflection coefficients vanish, the jump matrix is taken to be the identity matrix, which provides explicit soliton solutions through the corresponding Riemann-Hilbert problem. This allows us to explore the dynamical behaviors for both equations; the nonlocal reverse-spacetime Sasa-Satsuma equation and the higher-order nonlocal reverse-time NLS-type equation. These dynamical behaviors depend on the configuration of the eigenvalues in the spectral plane and how they are chosen, sometimes under specific conditions.

Chapter 1

Introduction

1.1 History and background

1.1.1 The KdV equation

In 1834, Scott Russell observed an unusual wave on the Edinburgh-Glasgow canal, which he called ”the great wave” [1]. He described it as ”...a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed.”. This was the first reported description of a solitary wave. A solitary wave is a stable localized wave that keeps its form and speed as it travels through space. Further, S. Russell’s interest continued and later he showed that this wave can be created in lab by dropping some weight on the water at the edge of a tunnel. He eventually obtained an explicit formula for its speed. Hence, he showed that it is a gravity wave, in the sense that, its amplitude a and speed c depend on the gravity force acting on the weight dropped, as well as on the water depth h. He showed that the speed of the wave is

$$c^2 = g(h + a) \tag{1.1}$$

Later in 1871 and 1876, using equations of motion, both J. Boussineq and L. Rayleigh obtained Russell’s formula for the wave speed and also a stabilized profile of the wave [2]-[3]

$$u(x, t) = asech^2(\beta(x - ct)) \tag{1.2}$$

where β is a parameter depending on h and a. But they were unsuccessful to present any mathematical model that have the profile equation as a solution. Later on, the work was completed by Diederik Korteweg and his student Gustav de Vries in 1895, where they obtained a simple mathematical model for shallow water waves, namely the Korteweg-de Vries (KdV) equation [4]:

$$u_t - 6uu_x + u_{xxx} = 0. \tag{1.3}$$

Thus, the KdV equation arose as a mathematical equation that has the solitary wave as a one-class solution for it. The terms uu_x and u_{xxx} correspond to the dissipative and dispersive phenomena in nonlinear interactions. The KdV describes waves of finite small amplitudes in long periods of time. It turns out that this partial differential equation is exactly solvable, i.e., it has solutions that can be precisely obtained. In addition, it has numerous applications in real life. For instance, it arises in the flow of a rotating fluid, low temperature plasmas, pressure waves, longitudinal waves, etc. [5]-[7]. Another interesting equation with even more applications than the KdV equation is the well-known nonlinear Schrödinger equation (NLS):

$$\mathrm{i}u_t + u_{xx} + |u|^2 u = 0. \tag{1.4}$$

This equation first appeared in a linear form in 1925 by Erwin Schrödinger who used it to explain the wave function in quantum mechanics. Interestingly, the NLS equation can be derived from the KdV equation [8]. Thus, the NLS equation also describes small amplitude waves in deep inviscid fluid. Moreover, the NLS is very important and has enormous applications in many of our real life problems. It is significantly useful in nonlinear fiber optics, low and high temperature plasma, Bose-Einstein condensate, propagation of waves beams and even in biological systems on a molecular level, etc. Since these two equations describe the time evolution of waves in a medium or a space, we call them ”evolution” equations.

Another equation that is derived from the KdV equation, is the modified-KdV equation (mKdV), which can be derived as follows:

Using the **Miura transformation**

$$u = v^2 \pm v_x \tag{1.5}$$

the KdV equation (1.3) is rewritten as:

$$\Big(2v \pm \frac{\partial}{\partial x}\Big)(v_t - 6v^2 v_x + v_{xxx}) = 0. \tag{1.6}$$

Thus, if v is a solution of the mKdV equation

$$v_t - 6v^2 v_x + v_{xxx} = 0, \tag{1.7}$$

then u is a solution of the KdV equation [9]-[10].

Remark 1.1.1. Note that the inverse of the latter result is not true. Since the ker$(2v \pm \partial_x) \neq 0$. Hence, the Bäcklund transformation is a mapping from v to u.

This huge research effort and tremendous study of the KdV and NLS equations led to the discovery of many interesting evolution equations. However, before we look into these equations, their properties and solutions, Hamiltonian structures, etc..., we take a brief look at some of the previous literature and then introduce some techniques for solving these equations, along with some basic notions and definitions, as we proceed.

1.1.2 A literature review

In 1974, M. J. Ablowitz, D. J. Kaup, A. C. Newell and H. Segur obtained some of the fundemental integrable evolution equations, like the NLS and the KdV equations from simple reductions of the general AKNS scheme [11]. For the NLS equation, they started with the so-called AKNS scattering problem:

$$\psi_x = U\psi. \tag{1.8}$$

Here $\psi = \psi(x,t)$ is a two-component vector $\psi(x,t) = (\psi_1(x,t), \psi_2(x,t))^T$ and $\mathbf{u}$ is the vector; $\mathbf{u} = (q(x,t),\, r(x,t))^T$, q, r are complex valued functions (potentials), with $q, r \to 0$ sufficiently rapidly as $x \to \pm\infty$ and λ is a spectral parameter. The matrix $U = U(\mathbf{u}, \lambda)$ is defined as:

$$U = \begin{pmatrix} -\mathrm{i}\lambda & q(x,t) \\ r(x,t) & \mathrm{i}\lambda \end{pmatrix}. \tag{1.9}$$

The time evolution equation associated with (1.8) is:

$$\psi_t = V\psi \tag{1.10}$$

where,

$$V = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}. \tag{1.11}$$

with

$$A = \mathrm{i}2\lambda^2 + \mathrm{i}q(x,t)r(x,t), \tag{1.12}$$

$$B = -2\lambda q(x,t) - \mathrm{i}q_x(x,t), \tag{1.13}$$

$$C = -2\lambda r(x,t) + \mathrm{i}r_x(x,t). \tag{1.14}$$

The compatibility condition:

$$\psi_{xt} = \psi_{tx} \tag{1.15}$$

gives the system:

$$\begin{cases} \mathrm{i}q_t(x,t) = q_{xx}(x,t) - 2r(x,t)q^2(x,t), \\ -\mathrm{i}r_t(x,t) = r_{xx}(x,t) - 2q(x,t)r^2(x,t). \end{cases} \tag{1.16}$$

For five decades the standard AKNS symmetry reduction:

$$r(x,t) = \sigma \overset{*}{q}(x,t), \quad \sigma = \pm 1 \tag{1.17}$$

was used to reduce the system (1.16) to the scalar local NLS equation:

$$\mathrm{i}q_t(x,t) = q_{xx}(x,t) - 2\sigma|q(x,t)|^2 q(x,t). \tag{1.18}$$

In 2016, Ablowitz and Musslimani found new symmetry reductions that reduce the system (1.16) to new nonlocal nonlinear NLS-type equations and mKdV-type evolution equations [12]-[13]. Moreover, they showed that the obtained equations are integrable and PT symmetric, i.e., they are invariant under the joint transformation $x \to -x$, $t \to -t$ and $\mathrm{i} \to -\mathrm{i}$. The three examples of the NLS-type reductions they found are: the reverse-time symmetry, the PT preserving symmetry and the reverse space-time symmetry:

$$r(x,t) = \sigma q(x,-t), \tag{1.19}$$

$$r(x,t) = \sigma \overset{*}{q}(-x,t), \tag{1.20}$$

$$r(x,t) = \sigma q(-x,-t), \quad q \in \mathbb{C}. \tag{1.21}$$

They reduce the system (1.16) to:

$$\mathrm{i}q_t(x,t) = q_{xx}(x,t) - 2\sigma q^2(x,t) q(x,-t), \tag{1.22}$$

$$\mathrm{i}q_t(x,t) = q_{xx}(x,t) - 2\sigma \overset{*}{q}(-x,t) q^2(x,t), \tag{1.23}$$

$$\mathrm{i}q_t(x,t) = q_{xx}(x,t) - 2\sigma q(-x,-t) q^2(x,t), \tag{1.24}$$

the nonlocal reverse-time, reverse-space and reverse-spacetime NLS equations, respectively. These equations are integrable and PT symmetric. Additionally, they also found nonlocal mKdV equations, by taking

$$A = -\mathrm{i}4\lambda^3 - \mathrm{i}2\lambda qrq_xr - qr_x \tag{1.25}$$

$$B = 4\lambda^2 q + \mathrm{i}2\lambda q_x + 2q^2 r - q_{xx} \tag{1.26}$$

$$C = 4\lambda^2 r - \mathrm{i}2\lambda r_x + 2qr^2 - r_{xx} \tag{1.27}$$

the compatibility condition gives the coupled system:

$$\begin{cases} q_t(x,t) = -q_{xxx}(x,t) + 6q(x,t)r(x,t)q_x(x,t), \\ r_t(x,t) = -r_{xxx}(x,t) + 6r(x,t)q(x,t)r_x(x,t). \end{cases} \tag{1.28}$$

Under two new nonlocal reductions found by Ablowitz and Musslimani, namely the PT symmetric complex reverse-spacetime symmetry and the PT symmetric real reverse-spacetime symmetry:

$$r(x,t) = \sigma \overset{*}{q}(-x,-t), \tag{1.29}$$

$$r(x,t) = \sigma q(-x,-t), \quad q \in \mathbb{R}. \tag{1.30}$$

the system (1.28) reduces to:

$$q_t(x,t) = -q_{xxx}(x,t) + 6\sigma q(x,t)\overset{*}{q}(-x,-t)q_x(x,t), \tag{1.31}$$

$$q_t(x,t) = -q_{xxx}(x,t) + 6\sigma q(x,t)q(-x,-t)q_x(x,t), \tag{1.32}$$

the complex and real nonlocal reverse-spacetime mKdV-type equations, respectively.

Later in 2019, in the framework of Riemann-Hilbert and inverse scattering, Jianke Yang obtained the fundamental and the general multi-soliton solutions to the previously mentioned nonlocal NLS equations. As a consequence, this resulted in different kinds of soliton structures and dynamical behaviors depending on the choice and configuration of the eigenvalues in the spectral plane [14]-[15].

1.1.3 Motivation

Based on the previous literature, most known integrable systems have been derived from ”local AKNS hierarchies”. One can also produce nonlocal integrable systems by using nonlocal reductions, such as those proposed by M. Ablowitz and Z. Musslimani. In some cases, these reductions, however, are not always sufficient to guarantee the integrability of the resulting systems. Considering this, we construct a nonlocal integrable hierarchy, where the nonlocal nature is embodied within the hierarchy’s structure. The latter construction allows nonlocal systems to be constructed without using reductions and guarantees integrability. A second novel result is the derivation of the bi-Hamiltonian structures for a new nonlocal two-component Sasa-Satsuma equation.

1.2 Preliminaries

1.2.1 Hirota bilinear form

The **Hirota bilinear form** or commonly known as, the **bilinear form** was introduced by Ryogo Hirota in 1971. He applied it to a wide variety of nonlinear evolution equations [16]-[17]. His method states that under suitable transformations, the original nonlinear equation can be mapped to a new bilinear equation. The resulting equation can be written in bilinear derivatives form, that is, with a new bilinear differential operator acting on it. In most simple cases, the transformations are taken to be

$$u(x,t) = a\frac{\partial^2}{\partial x^2}\log f \tag{1.33}$$

or

$$u(x,t) = b\frac{\partial}{\partial x}\log f \tag{1.34}$$

where a and b are constants and $f_x, f_{xx}, \ldots \to 0$ as $x \to \pm\infty$. Hirota beautifully defined the bilinear operator by:

$$D_t^m D_x^n (f \cdot g) = \Big(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\Big)^m \Big(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\Big)^n f(x,t)g(x',t')\Big|_{\substack{x'=x\\ t'=t}} \tag{1.35}$$

For example, D_x^4 and D_tD_x give:

$$D_x^4(f \cdot g) = f_{xxxx}g - 4f_{xxx}g_x + 6f_{xx}g_{xx} - 4f_xg_{xxx} + fg_{xxxx}, \tag{1.36}$$

$$D_tD_x(f \cdot g) = f_{xt}g - f_tg_x - f_xg_t + fg_{xt}. \tag{1.37}$$

Example 1. Consider the KdV equation:

$$u_t - 6uu_x + u_{xxx} = 0, \tag{1.38}$$

where $u = u(x,t)$. The N-soliton solutions of this equation can, in general, be written in the form (1.33). Substituting that in the KdV equation (1.38) and integrating, we get:

$$ff_{xt} - f_xf_t + ff_{xxxx} - 4f_xf_{xxx} + 3f_{xx}^2 = 0 \tag{1.39}$$

where f can be taken as the determinant of an appropriate matrix. Using the Hirota method, we can obtain the soliton solutions by transforming u to f and solve the corresponding bilinear equation. To do that, write f as an integral power series in ε, that is

$$f = \sum_{n=0}^{\infty} \varepsilon^n f_n(x,t), \tag{1.40}$$

where $f_0 = 1$. Now since,

$$D_x^4(f \cdot f) = 2(ff_{xxxx} - 4f_xf_{xxx} + 3f_{xx}^2), \tag{1.41}$$

$$D_tD_x(f \cdot f) = 2(ff_{xt} - f_xf_t), \tag{1.42}$$

from (1.39), equations (1.41) and (1.42) suggest that $D_tD_x + D_x^4$ is a bilinear operator of the KdV equation. For simplicity of calculations, denote the bilinear operator $\mathbf{BL} = D_tD_x + D_x^4$. Using the operator $\mathbf{BL}$ on the expansion of f, we see that

$$\begin{aligned}\mathbf{BL}(f \cdot f) = \mathbf{BL}(1 \cdot 1) + \varepsilon\mathbf{BL}(f_1 \cdot 1 + 1 \cdot f_1) \\ + \varepsilon^2\mathbf{BL}(f_2 \cdot 1 + f_1 \cdot f_1 + 1 \cdot f_2) + \varepsilon^3\mathbf{BL}(f_3 \cdot 1 + f_2 \cdot f_1 + f_1 \cdot f_2 + 1 \cdot f_3) \\ + \ldots + \epsilon^N\mathbf{BL}(\sum_{i=0}^{N} f_{N-i} \cdot f_i) + \ldots = 0.\end{aligned} \tag{1.43}$$

Since the coefficients of ε^N for $N \geq 1$ must all be zero, $\mathbf{BL}(f_1 \cdot 1 + 1 \cdot f_1) = 0$ implies that

$$\Big(\frac{\partial}{\partial t} + \frac{\partial^3}{\partial x^3}\Big)f_1 = 0. \tag{1.44}$$

A direct observable solution for (1.44) is of exponential form,

$$f_1 = e^{k_1x+l_1t+\theta}. \tag{1.45}$$

By substituting this, we can easily obtain the dispersion relation $l_1 = -k_1^3$. Thus $f_1 = e^{kx-k^3t+\theta}$ generates the one-soliton solution for the KdV equation

$$u(x,t) = -2k^2\frac{e^{kx-k^3t+\theta}}{(1+e^{kx-k^3t+\theta})^2}. \tag{1.46}$$

For the two-soliton solution, we can apply the same idea. First we notice that since the equation (1.44) is linear, we can have an arbitrary exponential terms. Let

$$f_1 = e^{k_1x+l_1t+\theta_1} + e^{k_2x+l_2t+\theta_2} \tag{1.47}$$

Using the next relations in the coefficients of powers of ε:

$$2\mathbf{BL}(f_2 \cdot 1) = \mathbf{BL}(f_1 \cdot f_1), \tag{1.48}$$

$$2\mathbf{BL}(f_3 \cdot 1) = -\mathbf{BL}(f_1 \cdot f_2 + f_2 \cdot f_1), \tag{1.49}$$

we can solve for f_2 and deduce at $\varepsilon = 1$ after a similar procedure that

$$u(x,t) = -2\frac{\mathsf{N}(x,t)}{\mathsf{D}(x,t)}, \tag{1.50}$$

where

$$\begin{aligned} \mathsf{N}(x,t) &= (k_1+k_2)^2(k_1-k_2)^2\Bigg[k_1^2 e^{(k_1+2k_2)x-(k_1^3+2k_2^3)t+(\theta_1+2\theta_2)} + k_2^2 e^{(2k_1+k_2)x-(2k_1^3+k_2^3)t+(2\theta_1+\theta_2)} \\ &\quad + 2(k_1+k_2)^2 e^{(k_1+k_2)x-(k_1^3+k_2^3)t+(\theta_1+\theta_2)}\Bigg], \end{aligned} \tag{1.51}$$

$$\mathsf{D}(x,t) = \Bigg[(k_1-k_2)^2 e^{(k_1+k_2)x-(k_1^3+k_2^3)t+(\theta_1+\theta_2)} + (k_1+k_2)^2(e^{k_1x-k_1^3t+\theta_1} + e^{k_2x-k_2^3t+\theta_2}+1)\Bigg]^2, \tag{1.52}$$

is the two-soliton solution for the KdV equation [18]-[20].

1.2.2 Bäcklund transformation

Bäcklund Transformations originally arose in differential geometry and differential equations as a solving technique in the 1880s. It was primarily used in geometrical transformations of surfaces that share the associated properties in different spaces [20]-[21]. To see how the Bäcklund transformation works, let's begin with the definition.

Definition 1.2.1. Given two uncoupled partial differential equations:

$$P(u) = 0, \tag{1.53}$$

$$Q(v) = 0, \tag{1.54}$$

where $u = u(x,t)$, $v = v(x,t)$ and P, Q are, in general, nonlinear operators. Let

$$F_1 = F_1(x, t, u, v, u_t, v_t, u_x, v_x, \ldots) = 0, \tag{1.55}$$
$$\vdots$$
$$F_n = F_n(x, t, u, v, u_t, v_t, u_x, v_x, \ldots) = 0, \tag{1.56}$$

be a set of relations between u and v. Then $\{F_1, \ldots, F_n\}$ is called a **Bäcklund transformation** if $P(u) = 0$ implies $Q(v) = 0$, and vice versa. If $P = Q$, then $\{F_1, \ldots, F_n\}$ is called an **auto-Bäcklund transformation**.

Example 2. *Solitary wave solutions of the KdV equation*

Let u and v be solutions of the KdV equation and the mKdV equation, respectively. Since the KdV equation is Galilean invariant, that is invariant under the transformations

$$x \to x - \lambda t, \quad t \to t, \quad u \to u + \frac{1}{6}\lambda, \tag{1.57}$$

then from the Miura transformation, we have

$$u - \lambda = v^2 \pm v_x, \tag{1.58}$$

where λ is a constant. Introduce

$$u_1 = w_{1x} = \lambda + v^2 + v_x, \tag{1.59}$$
$$u_2 = w_{2x} = \lambda + v^2 - v_x. \tag{1.60}$$

We can eliminate v from equations (1.59) and (1.60), to get the auxiliary equations relating w_1 and

w_2:

$$(w_1 + w_2)_x = 2\lambda + \frac{1}{2}(w_1 - w_2)^2, \tag{1.61}$$

$$(w_1 - w_2)_t = 3(w_{1x}^2 - w_{2x}^2) - (w_1 - w_2)_{xxx}. \tag{1.62}$$

These equations constitute an auto-Bäcklund transformation for the KdV equation. Adding equations (1.61) and (1.62), we see that:

$$w_{1t} - 3w_{1x}^2 + w_{1xxx} = w_{2t} - 3w_{2x}^2 + w_{2xxx}, \tag{1.63}$$

thus w_1 satisfies the potential KdV equation if and only if w_2 does. Equations (1.61) and (1.62) allow us to obtain solutions for w_1 and w_2 from an initial zero solution. For example, start by taking $w_2(x,t) = 0$ for all x and t. Thus equation (1.61) becomes

$$w_{1x} = 2\lambda + \frac{1}{2}w_1^2. \tag{1.64}$$

Integrating and using equation (1.62), the final solution can be written as:

$$w_1(x,t) = -2c\tanh(c(x - x_0 - 4c^2t)), \tag{1.65}$$

where $c^2 = -\lambda$ and $\lambda > 0$. Thus, the set of solitary wave solutions for the KdV is given by

$$u_1(x,t) = -2c^2 sech^2(c(x - x_0 - 4c^2t)), \quad c > \frac{1}{2}|w_1|^2. \tag{1.66}$$

As a result, by applying this procedure again, we can construct new solutions, i.e., w_{12}, w_{21} from w_1 and w_2 by this iteration process and successively obtain solutions $u_2, u_3, \ldots$ to the KdV equation, starting with a given solution $u_1 = w_{1x}$.

1.2.3 Inverse scattering transform

The inverse scattering transform (IST) is a method of solving Cauchy problems of nonlinear integrable partial differential equations (PDEs). It is analogous to the Fourier transform, by which we transform a linear PDE with initial conditions, $u(x,0)$, $u_t(x,0)$ to an ordinary differential equation (ODE) equation with a Fourier coefficient $A(k)$. Then solving this ODE by finding the time evolution of the Fourier transform; $A(k)e^{-\mathrm{i}\omega(k)t}$ and finally using the Fourier inverse transform to get the solution $u(x,t)$ of the original PDE problem.

In a similar pattern, the inverse scattering transform reduces some nonlinear PDE to a class of linear PDEs or a system of solvable linear ODEs. It consists of three main steps [22]-[27]:

1. **The direct scattering:** it maps a decaying potential $u(x,0) \in L^2$ to scattering data $S(\lambda_i)$ at an initial time (i.e. $t = 0$), where the λ_i are time-independent eigenvalues of the associated spectral problems. The initial condition $u(x,0)$ determines normalized eigenfunctions which are characterized by the scattering data.

2. **The time evolution:** it tells us how the scattering data $S(\lambda_i, t = 0)$ evolute in time to $S(\lambda_i, t = t_1)$, t_1 is arbitrary, in other words, it gives the scattering data at any time t. The scattering data needed are the transmission and reflection coefficients, respectively (they are the coefficients of the reflected and transmitted waves of eigenfunctions), the normalization constants (a constant depending on λ_i to normalize the eigenfunctions) and the associated eigenvalues.

3. **The inverse scattering:** it reconstructs the solution $u(x,t)$ of the nonlinear PDE from the evolved scattering data $S(\lambda_i, t)$ by solving certain integral equations, i.e., the Gel'fand-Levitan-Marchenko (GLM) equations or a Riemann-Hilbert problem.

The following diagram summaries the process of the inverse scattering transform:

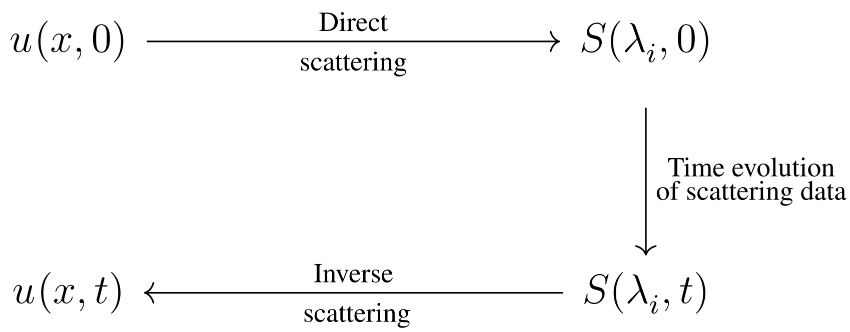


Figure 1.: Inverse scattering procedure

1.2.4 Lax pair

In 1968, Peter Lax introduced what is known as the ”Lax pair” [28]. He used some transformations like the Miura transformation and the Riccati transformation along with a compatability condition to relate nonlinear PDEs to two linear operators L and M.

Let’s consider the KdV equation to see how to obtain a Lax pair. Consider $u(x,t)$ and $v(x,t)$, the solutions of the KdV and mKdV equations:

$$u_t - 6uu_x + u_{xxx} = 0, \tag{1.67}$$

$$v_t - 6v^2v_x + v_{xxx} = 0. \tag{1.68}$$

Introduce a time-dependent eigenfunction ψ and recall the Miura transformation $u = v^2 + v_x - \lambda$. Let

$$v = \frac{\psi_x}{\psi} \tag{1.69}$$

be a Riccati transformation for v. Under this transformation the Miura relation can be written as a second-order linear ODE

$$\psi_{xx} - u(x,t)\psi = \lambda\psi, \tag{1.70}$$

or equivalently

$$L\psi = \lambda\psi, \tag{1.71}$$

where $L = \partial_x^2 - u(x,t)$ and λ the eigenvalue, called the *spectral parameter*, is independent of time or isospectral, in other words, $\lambda_t = 0$. Now substituting (1.69) into the mKdV equation (1.68) and eliminating v, one can obtain after some manipulations, the time evolution equation for ψ, that is:

$$\psi_t = M\psi, \tag{1.72}$$

where $M = 3u_x + 6u\partial_x - 4\partial_x^3 + C$, with some constant of integration C. Thus, we have the Lax equations:

$$L\psi = \lambda\psi, \tag{1.73}$$

$$\psi_t = M\psi, \tag{1.74}$$

and L, M are said to be the Lax pair for the KdV equation. The linear differential operators L and M satisfy an isospectral property:

$$L_t + [L, M] = 0. \tag{1.75}$$

To show this, we start by taking the t-derivative of equation (1.73) to get:

$$L_t\psi + L\psi_t = \lambda_t\psi + \lambda\psi_t \tag{1.76}$$

$$\Rightarrow L_t\psi + LM\psi = \lambda M\psi \tag{1.77}$$

$$\Rightarrow L_t\psi + LM\psi - ML\psi = 0 \tag{1.78}$$

$$\Rightarrow (L_t + [L, M])\psi = 0. \tag{1.79}$$

Since the Lax pair has nontrivial solutions $(\psi \neq 0)$, thus when $u(x,t)$ satisfies the KdV equation then the linear differential operators L, M satisfy the corresponding isospectral property (1.75).

1.2.5 Hamiltonian structure

Definition 1.2.2. An evolution differential equation possess a *Hamiltonian structure* (with respect to J), if it can be written in the form [29]-[32]:

$$u_t = J\frac{\delta H}{\delta u}, \tag{1.80}$$

where J is called a *Hamiltonian operator* with coefficients depending on $x, u, u_x, \ldots$ etc., and H is the functional $H = H(x, u, u_x, \ldots)$, called a *Hamiltonian functional*, in addition, the *variational derivative* is defined by:

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - \frac{d}{dx}\frac{\partial}{\partial u_x} + \frac{d^2}{dx^2}\frac{\partial^2}{\partial u_{xx}} - \ldots + (-1)^n\frac{d^n}{dx^n}\frac{\partial^n}{\partial u_{nx}} + \ldots. \tag{1.81}$$

Along with a Hamiltonian structure, we define a Poisson bracket:

$$\{f, g\}_J = \int_{-\infty}^{\infty} \Big[\Big(\frac{\delta f}{\delta u}\Big)^T J\frac{\delta g}{\delta u}\Big]dx \tag{1.82}$$

which satisfies the following elementary properties of the Poisson bracket:

1. Skew-symmetric: $\{f, g\} = -\{g, f\}$.
2. The Jacobi identity: $\{f, \{g, h\}\} + cyclic(f, g, h) = 0$.

Moreover, if an evolution equation can be written as:

$$u_t = K = J_1\frac{\delta H_1}{\delta u} = J_2\frac{\delta H_2}{\delta u}, \tag{1.83}$$

then, it is said to have a *bi-Hamiltonian* structure, where $\{J_1, J_2\}$ is a Hamiltonian pair, and H_1, H_2 are two Hamiltonian functionals. Here a Hamiltonian pair means that

$$C_1J_1 + C_2J_2 \tag{1.84}$$

where C_1, C_2 are arbitrary constants, is again a Hamiltonian operator.

Let $K_0 = J_1 \frac{\delta H_1}{\delta u}$, where H_1 is the first Hamiltonian functional and let Φ be the recursion operator $\Phi = J_2 J_1^{-1}$. Define

$$K_m = \Phi K_{m-1}, \quad m \geq 1, \tag{1.85}$$

then, there exists a sequence of Hamiltonian functionals, H_m for $m \geq 1$, such that they satisfy the following properties:

1. The skew-symmetry, that is for any Hamiltonian functional H_i, we have:

$$\{H_i, H_i\}_{J_1} = \{H_i, H_i\}_{J_2} = 0, \quad i \geq 1. \tag{1.86}$$

2. The Hamiltonian functionals H_i are in involution with respect to either Poisson brackets:

$$\{H_i, H_j\}_{J_1} = \{H_i, H_j\}_{J_2} = 0, \quad i, j \geq 1. \tag{1.87}$$

Example 3. *The bi-Hamiltonian structure of the KdV equation*

The KdV equation has two different Hamiltonian structures [33]. The first one arises from the conserved energy:

$$H_1 = \int_{-\infty}^{\infty} (u^3 + \frac{1}{2} u_x^2) dx, \tag{1.88}$$

associated with the Hamiltonian operator $J_1 = \partial_x$, while the second structure arises from the conserved momentum:

$$H_2 = \int_{-\infty}^{\infty} \frac{1}{2} u^2 dx, \tag{1.89}$$

with the associated Hamiltonian operator $J_2 = -\partial_x^3 + 4u\partial_x + 2u_x$. To show that ∂_x is a Hamiltonian operator, we can check if it satisfies the antisymmetry and the Jacobi identity.

For the anti-symmetry, we see that:

$$\{f,g\}+\{g,f\} = \int_{-\infty}^{\infty}\Big(\frac{\delta f}{\delta u}\partial_x\frac{\delta g}{\delta u}+\frac{\delta g}{\delta u}\partial_x\frac{\delta f}{\delta u}\Big)dx \tag{1.90}$$
$$= \int_{-\infty}^{\infty}\partial_x\Big(\frac{\delta f}{\delta u}\frac{\delta g}{\delta u}\Big)dx$$
$$= 0.$$

Also, without loss of generality, let $f = f(x,u)$, $g = g(x,u)$ and $h = h(x,u)$, ($f = f(x,u,u_x,\ldots)$ can be generalized). Then since

$$\{f,\{g,h\}\} = \int_{-\infty}^{\infty}\Big(\frac{\delta f}{\delta u}\partial_x\frac{\delta}{\delta u}\{g,h\}\Big)dx \tag{1.91}$$
$$= -\int_{-\infty}^{\infty}\Big(\partial_x\Big(\frac{\delta f}{\delta u}\Big)\frac{\delta}{\delta u}\{g,h\}\Big)dx$$

and using

$$\partial_x\frac{\delta f}{\delta u} = f_{ux}+f_{uu}u_x \tag{1.92}$$

$$\frac{\delta}{\delta u}\{g,h\} = \frac{\delta}{\delta u}\int_{-\infty}^{\infty}\Big(\frac{\delta g}{\delta u}\partial_x\frac{\delta h}{\delta u}\Big)dx \tag{1.93}$$
$$= \frac{\delta}{\delta u}\int_{-\infty}^{\infty}g_u(h_{ux}+h_{uu}u_x)dx$$
$$= g_{uu}h_{ux}-h_{uu}g_{ux}.$$

Thus,

$$\{f,\{g,h\}\}+cyclic(f,g,h) = \int_{-\infty}^{\infty}(f_{ux}+f_{uu}u_x)(g_{uu}h_{ux}-h_{uu}g_{ux})dx+cyclic(f,g,h) = 0. \tag{1.94}$$

In addition, it can also be easily checked that

$$
\begin{aligned}
u_t = J_1 \frac{\delta H_1}{\delta u} &= \partial_x \frac{\delta}{\delta u} \int\limits_{-\infty}^{\infty} (u^3 + \frac{1}{2} u_x^2) dx \\
&= \partial_x \Big[\frac{\partial}{\partial u}(u^3 + \frac{1}{2}u_x^2) - \frac{d}{dx}\frac{\partial}{\partial u_x}(u^3 + \frac{1}{2}u_x^2) \Big] \\
&= \partial_x (3u^2 - u_{xx}) \\
&= 6uu_x - u_{xxx}.
\end{aligned}
\tag{1.95}
$$

Also,

$$
\begin{aligned}
u_t = J_2 \frac{\delta H_2}{\delta u} &= (-\partial_x^3 + 4u\partial_x + 2u_x) \frac{\delta}{\delta u} \int\limits_{-\infty}^{\infty} \frac{1}{2} u^2 dx \\
&= (-\partial_x^3 + 4u\partial_x + 2u_x) \frac{\partial}{\partial u} \frac{1}{2} u^2 \\
&= 6uu_x - u_{xxx}.
\end{aligned}
\tag{1.96}
$$

1.2.6 Trace identity

Hamiltonian structures of evolution equations can be formulated from what is known as the *trace identity*. In 1989, G. Tu showed that using the stationary zero-curvature equation:

$$
W_x = [U, W], \tag{1.97}
$$

we can obtain a hierarchy of evolution equations along with its Hamiltonians (conserved densities) by applying a trace identity [34].

Starting from a matrix semi-simple Lie algebra $\mathfrak{g}$; i.e. a direct sum of simple Lie algebras (non-abelian Lie algebras that have no trivial ideals), Tu showed that if W is a solution of (1.97), then we can prove the following trace identity:

$$\frac{\delta}{\delta \mathbf{u}} \int \Big\langle W, \frac{\partial U}{\partial \lambda} \Big\rangle dx = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \Big[\lambda^{\gamma} \Big\langle W, \frac{\partial U}{\partial \mathbf{u}} \Big\rangle \Big], \tag{1.98}$$

where $\langle \cdot, \cdot \rangle$ is the Cartan-Killing form, $\gamma = -\frac{\lambda}{2} \frac{d}{d\lambda} ln|\langle W, W \rangle|$ and $\mathbf{u}$ is a column vector of functions.

Recall that the Killing form of a Lie algebra over some field $\mathbb{F}$ is a bilinear map $\langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{F}$:

$$\langle a, b \rangle = tr(ad(a)ad(b)), \quad a, b \in \mathfrak{g}, \tag{1.99}$$

where $ad(x)$ is the adjoint linear transformatiom from $\mathfrak{g}$ to itself, defined by:

$$ad(x)y := [x, y], \quad y \in \mathfrak{g}, \tag{1.100}$$

where $[\cdot, \cdot]$ is the Lie product of $\mathfrak{g}$. In the case of semi-simple Lie algebra, $\langle \cdot, \cdot \rangle$ is non-degenerate, symmetric and the above Cartan-Killing form is the trace (see p.10789 of [31]), and so we can have

$$\langle a, b \rangle = tr(ab), \quad a, b \in \mathfrak{g}, \tag{1.101}$$

where $\mathfrak{g}$ is a matrix Lie algebra. As a result, the formula (1.98) is reduced to the following trace identity:

$$\frac{\delta}{\delta \mathbf{u}} \int tr\Big(W \frac{\partial U}{\partial \lambda} \Big) dx = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \Big[\lambda^{\gamma} tr\Big(W \frac{\partial U}{\partial \mathbf{u}} \Big) \Big], \tag{1.102}$$

with $\gamma = -\frac{\lambda}{2} \frac{d}{d\lambda} ln|tr(W^2)|$. We will use this method to derive Hamiltonian conserved quantities.

Chapter 2

Examples of hierarchies of integrable equations

In this chapter, we will take a look on different hierarchies and how integrable equations arise by using the stationary zero-curvature equation and the trace identity previously derived. In addition, we apply the trace identity to obtain Hamiltonian structures of the resulting integrable equations.

2.1 AKNS hierarchy

We begin with the most well-known AKNS hierarchy. Let's start with the spectral problem:

$$-i\varphi_x = U\varphi, \tag{2.1}$$

where U is the matrix defined by

$$U = U(\mathbf{u}, \lambda) = \begin{pmatrix} \alpha_1\lambda & q \\ r & \alpha_2\lambda \end{pmatrix}, \tag{2.2}$$

where λ is the spectral parameter and $\mathbf{u}$ is the 2-dimensional vector $\mathbf{u} = (q, r)^T$.

To derive the associated integrable hierarchy, we start by solving the stationary zero curvature equation

$$W_x = \mathrm{i}[U, W] \tag{2.3}$$

with the solution W to be in the following form:

$$W = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{2.4}$$

So, the stationary zero curvature equation is equivalent to the set of equations:

$$a_x = \mathrm{i}(-rb + qc) = -d_x, \tag{2.5}$$

$$b_x = \mathrm{i}(\alpha\lambda b - qa + qd), \tag{2.6}$$

$$c_x = \mathrm{i}(-\alpha\lambda c + ra - rd), \tag{2.7}$$

where $\alpha = \alpha_1 - \alpha_2$. Expanding the matrix W in Laurent series, that is,

$$W = \sum_{m=0}^{\infty} W_m \lambda^{-m} \tag{2.8}$$

with

$$a = \sum_{m=0}^{\infty} a^{[m]} \lambda^{-m}, \quad b = \sum_{m=0}^{\infty} b^{[m]} \lambda^{-m}, \tag{2.9}$$

$$c = \sum_{m=0}^{\infty} c^{[m]} \lambda^{-m}, \quad d = \sum_{m=0}^{\infty} d^{[m]} \lambda^{-m}, \tag{2.10}$$

we see that the system (2.5)-(2.7) gives the recursion relations:

$$a_x^{[0]} = 0,\ b^{[0]} = 0,\ c^{[0]} = 0,\ d_x^{[0]} = 0, \tag{2.11}$$

$$b^{[m+1]} = \frac{1}{\alpha}(-\mathrm{i}b_x^{[m]} + qa^{[m]} - qd^{[m]}), \tag{2.12}$$

$$c^{[m+1]} = \frac{1}{\alpha}(\mathrm{i}c_x^{[m]} + ra^{[m]} - rd^{[m]}), \qquad m \geq 0. \tag{2.13}$$

$$a_x^{[m+1]} = \mathrm{i}(-rb^{[m+1]} + qc^{[m+1]}) = -d_x^{[m+1]}, \tag{2.14}$$

By choosing the initial values:

$$a^{[0]} = \beta_1, \quad \text{and} \quad d^{[0]} = \beta_2, \tag{2.15}$$

where the β_1, β_2 are arbitrary real constants, and taking zero constants of integration, we uniquely generate:

$$a^{[1]} = 0 = d^{[1]},\ b^{[1]} = \frac{\beta}{\alpha}q,\ c^{[1]} = \frac{\beta}{\alpha}r, \tag{2.16}$$

$$a^{[2]} = -\frac{\beta}{\alpha^2}qr = -d^{[2]},\ b^{[2]} = -\mathrm{i}\frac{\beta}{\alpha^2}q_x,\ c^{[2]} = \mathrm{i}\frac{\beta}{\alpha^2}r_x, \tag{2.17}$$

$$a^{[3]} = -\mathrm{i}\frac{\beta}{\alpha^3}(qr_x - q_xr) = -d^{[3]}, \tag{2.18}$$

$$b^{[3]} = -\frac{\beta}{\alpha^3}(q_{xx} + 2q^2r),\ c^{[3]} = -\frac{\beta}{\alpha^3}(r_{xx} + 2qr^2), \tag{2.19}$$

$$a^{[4]} = -\frac{\beta}{\alpha^4}(q_xr_x - 3q^2r^2 - q_{xx}r - qr_{xx}) = -d^{[4]}, \tag{2.20}$$

$$b^{[4]} = \mathrm{i}\frac{\beta}{\alpha^4}(q_{xxx} + 6qq_xr),\ c^{[4]} = -\mathrm{i}\frac{\beta}{\alpha^4}(r_{xxx} + 6qrr_x), \tag{2.21}$$

$$a^{[5]} = \mathrm{i}\frac{\beta}{\alpha^5}(qr_{xxx} - q_{xxx}r + 6q^2rr_x - 6qq_xr^2 + q_{xx}r_x - q_xr_{xx}) = -d^{[5]}, \tag{2.22}$$

$$b^{[5]} = \frac{\beta}{\alpha^5}(q_{xxxx} + 6q^3r^2 + 8qq_{xx}r + 2q^2r_{xx} + 4qq_xr_x + 6q_x^2r), \tag{2.23}$$

$$c^{[5]} = \frac{\beta}{\alpha^5}(r_{xxxx} + 6q^2r^3 + 8qrr_{xx} + 2q_{xx}r^2 + 4q_xrr_x + 6qr_x^2), \tag{2.24}$$

$$\begin{aligned} a^{[6]} = &-\frac{\beta}{\alpha^6}\Big(q_{xxxx}r + qr_{xxxx} + q_{xx}r_{xx} - q_xr_{xxx} - q_{xxx}r_x \\ &+ 10(q^3r^3 + qq_{xx}r^2 + q^2rr_{xx}) + 5(q^2r_x^2 + q_x^2r^2)\Big) = -d^{[6]}, \end{aligned} \tag{2.25}$$

$$\begin{aligned} b^{[6]} = &-\mathrm{i}\frac{\beta}{\alpha^6}\Big(q_{xxxxx} + 10(qq_{xxx}r + q_x^2r_x + qq_xr_{xx} + qq_{xx}r_x) \\ &+ 20q_xq_{xx}r + 30q^2q_xr^2\Big), \end{aligned} \tag{2.26}$$

$$\begin{aligned} c^{[6]} = &\,\mathrm{i}\frac{\beta}{\alpha^6}\Big(r_{xxxxx} + 10(qrr_{xxx} + q_xr_x^2 + q_{xx}rr_x + q_xrr_{xx}) \\ &+ 20qr_xr_{xx} + 30q^2r^2r_x\Big), \end{aligned} \tag{2.27}$$

where $\beta = \beta_1 - \beta_2$. To find a soliton hierarchy, we begin by introducing a series of Lax matrices

$$V^{[m]} = (\lambda^m W)_+ + \mathit{\Delta}_m, \quad \text{for} \quad m \geq 0, \tag{2.28}$$

where $+$ means to take the polynomial part. Taking the modification terms $\mathit{\Delta}_m = 0$, for $m \geq 0$ we can generate the soliton hierarchy

$$\mathbf{u}_{t_m} = K_m(x, t, q, r, q_x, r_x, \ldots), \quad m \geq 0, \tag{2.29}$$

associated with the zero curvature equations

$$U_{t_m} - V_x^{[m]} + \mathrm{i}[U, V^{[m]}] = 0, \quad m \geq 0. \tag{2.30}$$

Upon using the zero curvature equations along with the zero modification terms and the recursion relations (2.12)-(2.14), the hierarchy can be written explicitly in the following form:

$$\mathbf{u}_{t_m} = \begin{pmatrix} q \\ r \end{pmatrix}_{t_m} = \mathrm{i}\alpha \begin{pmatrix} b^{[m+1]} \\ -c^{[m+1]} \end{pmatrix}. \tag{2.31}$$

For $m = 2$, $m = 3$ and $m = 5$, we obtain the corresponding systems:

$$\begin{cases} q_{t_2} &= -\mathrm{i}\frac{\beta}{\alpha^2}(q_{xx} + 2q^2 r), \\ r_{t_2} &= \mathrm{i}\frac{\beta}{\alpha^2}(r_{xx} + 2qr^2), \end{cases} \tag{2.32}$$

$$\begin{cases} q_{t_3} &= -\frac{\beta}{\alpha^3}(q_{xxx} + 6qq_x r), \\ r_{t_3} &= \frac{\beta}{\alpha^3}(r_{xxx} + 6qrr_x), \end{cases} \tag{2.33}$$

$$\begin{cases} q_{t_5} &= \frac{\beta}{\alpha^5}\Big(q_{xxxxx} + 10(qq_{xxx}r + q_x^2 r_x + qq_x r_{xx} + qq_{xx}r_x) + 20q_x q_{xx} r + 30q^2 q_x r^2\Big), \\ r_{t_5} &= -\frac{\beta}{\alpha^5}\Big(r_{xxxxx} + 10(qrr_{xxx} + q_x r_x^2 + q_{xx} r r_x + q_x r r_{xx}) + 20qr_x r_{xx} + 30q^2 r^2 r_x\Big), \end{cases} \tag{2.34}$$

respectively. Taking $r = \overset{*}{q}$ for the first system and $r = q$ for the second and third, these systems reduce respectively to the NLS equation, the third-order and the fifth-order mKdV equations:

$$q_{t_2} + \mathrm{i}\frac{\beta}{\alpha^2}(q_{xx} + 2|q|^2 q) = 0, \tag{2.35}$$

$$q_{t_3} + \frac{\beta}{\alpha^3}(q_{xxx} + 6q^2 q_x) = 0, \tag{2.36}$$

$$q_{t_5} - \frac{\beta}{\alpha^5}(q_{xxxxx} + 10q_x^3 + 10q^2 q_{xxx} + 30q^4 q_x + 40qq_x q_{xx}) = 0. \tag{2.37}$$

2.1.1 Bi-Hamiltonian structure of the AKNS hierarchy

In this section, we derive a bi-Hamiltonian structure of the hierarchy (2.31). To proceed, we use the trace identity

$$\frac{\delta}{\delta \mathbf{u}} \int tr\Big(W\frac{\partial U}{\partial \lambda}\Big)dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\Big[\lambda^{\gamma} tr\Big(W\frac{\partial U}{\partial \mathbf{u}}\Big)\Big], \tag{2.38}$$

where $\gamma = -\frac{\lambda}{2}\frac{d}{d\lambda} ln|tr(W^2)|$. We have

$$tr\Big(W\frac{\partial U}{\partial \lambda}\Big) = \alpha_1 \sum_{m=0}^{\infty} a^{[m]}\lambda^{-m} + \alpha_2 \sum_{m=0}^{\infty} d^{[m]}\lambda^{-m}, \tag{2.39}$$

$$tr\Big(W\frac{\partial U}{\partial q}\Big) = \sum_{m=0}^{\infty} c^{[m]}\lambda^{-m}, \quad tr\Big(W\frac{\partial U}{\partial r}\Big) = \sum_{m=0}^{\infty} b^{[m]}\lambda^{-m}. \tag{2.40}$$

Plugging these into the trace identity and matching the powers of λ^{-m-1}, we see that

$$\frac{\delta}{\delta \mathbf{u}} \int \Big(\alpha_1 a^{[m+1]} + \alpha_2 d^{[m+1]}\Big)dx = (\gamma - m)\begin{pmatrix} c^{[m]} \\ b^{[m]} \end{pmatrix}, \quad m \geq 1. \tag{2.41}$$

When $m = 0$, we deduce that $\gamma = 0$. Thus, the Hamiltonians can be taken as

$$\mathcal{H}_m = -\frac{\mathrm{i}}{m} \int \Big(\alpha_1 a^{[m+1]} + \alpha_2 d^{[m+1]} \Big) dx, \quad m \geq 1. \tag{2.42}$$

Explicitly, the first three Hamiltonians read:

$$\mathcal{H}_1 = \frac{\beta}{\alpha} \int (qr) dx, \tag{2.43}$$

$$\mathcal{H}_2 = \mathrm{i} \frac{\beta}{2\alpha^2} \int (q r_x - q_x r) dx, \tag{2.44}$$

$$\mathcal{H}_3 = \frac{\beta}{3\alpha^3} \int (q_x r_x - 3q^2 r^2 - q_{xx} r - q r_{xx}) dx. \tag{2.45}$$

From (2.41), we deduce

$$\frac{\delta \mathcal{H}_m}{\delta \mathbf{u}} = \mathrm{i} \begin{pmatrix} c^{[m]} \\ b^{[m]} \end{pmatrix}, \quad m \geq 1. \tag{2.46}$$

Hence from the above equalities, the bi-Hamiltonian structure reads:

$$\mathbf{u}_{t_m} = K_m = J_1 \frac{\delta \mathcal{H}_{m+1}}{\delta \mathbf{u}} = J_2 \frac{\delta \mathcal{H}_m}{\delta \mathbf{u}}, \quad m \geq 1, \tag{2.47}$$

where the Hamiltonian pair are $\{J_1, J_2\}$, with $J_2 = J_1 \Phi$. Explicitly we have

$$J_1 = \begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix}, \quad \Phi = \frac{\mathrm{i}}{\alpha} \begin{pmatrix} \partial + 2r\partial^{-1} q & -2r\partial^{-1} r \\ 2q\partial^{-1} q & -\partial - 2q\partial^{-1} r \end{pmatrix}, \tag{2.48}$$

and the recursive formula is defined by:

$$\begin{pmatrix} c^{[m+1]} \\ b^{[m+1]} \end{pmatrix} = \Phi \begin{pmatrix} c^{[m]} \\ b^{[m]} \end{pmatrix}, \quad m \geq 1. \tag{2.49}$$

2.2 Kaup-Newell (KN) hierarchy

To begin with the KN hierarchy, we similarly consider the spectral problem:

$$\varphi_x = U\varphi, \tag{2.50}$$

In this case, U is the matrix defined by

$$U = U(\mathbf{u}, \lambda) = \begin{pmatrix} \lambda^2 & \lambda p \\ \lambda q & -\lambda^2 \end{pmatrix}, \tag{2.51}$$

where λ is the spectral potential and $\mathbf{u}$ is the 2-dimensional vector $\mathbf{u} = (p, q)^T$.

First, we solve the following stationary zero curvature equation to obtain an associated integrable hierarchy:

$$W_x = [U, W] \tag{2.52}$$

assuming the solution W is as follows:

$$W = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}. \tag{2.53}$$

This implies a set of equations that corresponds to the stationary zero curvature equation:

$$a_x = -\lambda q b + \lambda p c, \tag{2.54}$$

$$b_x = -2\lambda p a + 2\lambda^2 b, \tag{2.55}$$

$$c_x = 2\lambda q a - 2\lambda^2 c. \tag{2.56}$$

In the Laurent series, the matrix W is expanded as follows:

$$W = \sum_{m=0}^{\infty} W_m \lambda^{-m} \tag{2.57}$$

with its components being given by

$$a=\sum_{m=0}^{\infty}a^{[m]}\lambda^{-2m},\quad b=\sum_{m=0}^{\infty}b^{[m]}\lambda^{-2m-1},\quad c=\sum_{m=0}^{\infty}c^{[m]}\lambda^{-2m-1}. \tag{2.58}$$

then (2.54)-(2.56) give the following recursion relations:

$$a_x^{[m+1]}=-\frac{1}{2}(qb_x^{[m]}+pc_x^{[m]}), \tag{2.59}$$

$$b^{[m+1]}=\frac{1}{2}b_x^{[m]}+pa^{[m+1]},\qquad m\geq 0. \tag{2.60}$$

$$c^{[m+1]}=-\frac{1}{2}c_x^{[m]}+qa^{[m+1]}, \tag{2.61}$$

With the initial values chosen as follows:

$$a^{[0]}=1,\quad b^{[0]}=p,\quad c^{[0]}=q, \tag{2.62}$$

we can uniquely generate:

$$a^{[1]}=-\frac{1}{2}pq,\ b^{[1]}=\frac{1}{2}(p_x-p^2q),\ c^{[1]}=-\frac{1}{2}(q_x+pq^2), \tag{2.63}$$

$$a^{[2]}=\frac{3}{8}pq+\frac{1}{4}pq_x-\frac{1}{4}p_xq, \tag{2.64}$$

$$b^{[2]}=\frac{1}{4}p_{xx}+\frac{3}{8}p^3q^2-\frac{3}{4}pp_xq,\ c^{[2]}=\frac{1}{4}q_{xx}+\frac{3}{8}p^2q^3+\frac{3}{4}pqq_x, \tag{2.65}$$

$$a^{[3]}=\frac{1}{8}p_xq_x-\frac{5}{16}p^3q^3+\frac{3}{8}pp_xq^2-\frac{3}{8}p^2qq_x-\frac{1}{8}p_{xx}q-\frac{1}{8}pq_{xx}, \tag{2.66}$$

$$b^{[3]}=\frac{1}{8}p_{xxx}-\frac{1}{4}pp_xq_x-\frac{1}{8}p^2q_{xx}-\frac{3}{8}p_x^2q-\frac{1}{2}pp_{xx}q+\frac{15}{16}p^2p_xq^2-\frac{5}{16}p^4q^3, \tag{2.67}$$

$$c^{[3]}=-\frac{1}{8}q_{xxx}-\frac{1}{4}p_xqq_x-\frac{1}{8}p_{xx}q^2-\frac{3}{8}pq_x^2-\frac{1}{2}pqq_{xx}-\frac{15}{16}p^2q^2q_x-\frac{5}{16}p^3q^4. \tag{2.68}$$

The soliton hierarchy is determined by a series of Lax matrices:

$$V^{[m]}=(\lambda^{2m+2}W)_{+}+\mathit{\Delta}_m,\quad \text{for}\quad m\geq 0. \tag{2.69}$$

Based on the following modification term

$$\Delta_m = \begin{pmatrix} -a^{[m+1]} & 0 \\ 0 & a^{[m+1]} \end{pmatrix}, \quad m \geq 0. \tag{2.70}$$

A soliton hierarchy can be generated as follows:

$$\mathbf{u}_{t_m} = K_m(x, t, p, q, p_x, q_x, \ldots), \quad m \geq 0, \tag{2.71}$$

from the zero curvature equations

$$U_{t_m} - V_x^{[m]} + [U, V^{[m]}] = 0, \quad m \geq 0. \tag{2.72}$$

The hierarchy can be written explicitly in the following form using the zero curvature equations, the modification terms, and recursion relations (2.59)-(2.61):

$$\mathbf{u}_{t_m} = \begin{pmatrix} p \\ q \end{pmatrix}_{t_m} = \begin{pmatrix} b^{[m]} \\ c^{[m]} \end{pmatrix}_x . \tag{2.73}$$

For $m = 1$ and $m = 2$, the following systems are derived:

$$\begin{cases} p_{t_1} &= \frac{1}{2}p_{xx} - pp_xq - \frac{1}{2}p^2q_x, \\ q_{t_1} &= -\frac{1}{2}q_{xx} - pqq_x - \frac{1}{2}p_xq^2, \end{cases} \tag{2.74}$$

$$\begin{cases} p_{t_2} &= \frac{1}{4}p_{xxx} - \frac{3}{4}pp_{xx}q - \frac{3}{4}pp_xq_x - \frac{3}{4}p_x^2q + \frac{9}{8}p^2p_xq^2 + \frac{3}{4}p^3qq_x, \\ q_{t_2} &= \frac{1}{4}q_{xxx} + \frac{3}{4}pqq_{xx} + \frac{3}{4}p_xqq_x + \frac{3}{4}pq_x^2 + \frac{9}{8}p^2q^2q_x + \frac{3}{4}pp_xq^3. \end{cases} \tag{2.75}$$

Taking $q = p$ these two systems reduce respectively to the following second-order derivative NLS equation and the mKdV-type equation:

$$p_{t_1} = \frac{1}{2}p_{xx} - \frac{3}{2}p^2 p_x, \tag{2.76}$$

$$p_{t_2} = \frac{1}{4}p_{xxx} - \frac{3}{2}pp_x^2 - \frac{3}{4}p^2 p_{xx} + \frac{15}{8}p^4 p_x. \tag{2.77}$$

2.2.1 Bi-Hamiltonian structure of the KN hierarchy

In order to derive the bi-Hamiltonian structure of the hierarchy (2.73), we use the trace identity (2.38). We have the components:

$$tr\Big(W\frac{\partial U}{\partial \lambda}\Big) = 4\sum_{m=0}^{\infty} a^{[m]}\lambda^{-2m+1} + q\sum_{m=0}^{\infty} b^{[m]}\lambda^{-2m-1} + p\sum_{m=0}^{\infty} c^{[m]}\lambda^{-2m-1}, \tag{2.78}$$

$$tr\Big(W\frac{\partial U}{\partial p}\Big) = \sum_{m=0}^{\infty} c^{[m]}\lambda^{-2m}, \quad tr\Big(W\frac{\partial U}{\partial q}\Big) = \sum_{m=0}^{\infty} b^{[m]}\lambda^{-2m}. \tag{2.79}$$

As a result of plugging these into the trace identity and matching the powers of λ^{-2m-1}, we observe the following:

$$\frac{\delta}{\delta \mathbf{u}}\int \Big(4a^{[m+1]} + qb^{[m]} + pc^{[m]}\Big)dx = (\gamma - 2m)\begin{pmatrix} c^{[m]} \\ b^{[m]} \end{pmatrix}, \quad m \geq 1. \tag{2.80}$$

When $m = 1$, we deduce that $\gamma = 0$. Consequently, the Hamiltonians can be taken for this hierarchy as follows:

$$\mathcal{H}_m = -\frac{1}{2m}\int \Big(4a^{[m+1]} + qb^{[m]} + pc^{[m]}\Big)dx, \quad m \geq 1. \tag{2.81}$$

Here, the first three Hamiltonians read explicitly as follows:

$$\mathcal{H}_1 = -\frac{1}{4}\int (pq_x - p_x q + p^2q^2)dx, \tag{2.82}$$
$$\mathcal{H}_2 = \frac{1}{16}\int (pq_{xx} + p_{xx}q - 2p_xq_x + 3p^2qq_x - 3pp_xq^2 + 2p^3q^3)dx, \tag{2.83}$$
$$\begin{aligned}\mathcal{H}_3 = -\frac{1}{48}\int (&pq_{xxx} - p_{xxx}q + 2p_{xx}q_x - 2p_xq_{xx} + 2p^2q_x^2 + 2p_x^2q^2 \\ &- 4pp_xqq_x + 5pp_{xx}q^2 + 5p^2qq_{xx} + \frac{15}{2}p^3q^2q_x - \frac{15}{2}p^2p_xq^3 + \frac{15}{4}p^4q^4)dx.\end{aligned} \tag{2.84}$$

We can conclude from (2.80) that

$$\frac{\delta\mathcal{H}_m}{\delta\mathbf{u}} = \begin{pmatrix} c^{[m]} \\ b^{[m]} \end{pmatrix}, \quad m \geq 1. \tag{2.85}$$

So, from the above equalities, we derive the bi-Hamiltonian structure as follows:

$$\mathbf{u}_{t_m} = J_1\frac{\delta\mathcal{H}_{m+1}}{\delta\mathbf{u}} = J_2\frac{\delta\mathcal{H}_m}{\delta\mathbf{u}}, \quad m \geq 1, \tag{2.86}$$

the Hamiltonian pair $\{J_1, J_2\}$ are as follows:

$$J_1 = \begin{pmatrix} -2p\partial^{-1}p & 2 + 2p\partial^{-1}q \\ -2 + 2q\partial^{-1}p & -2q\partial^{-1}q \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix}. \tag{2.87}$$

2.3 TC hierarchy

Let us begin with the following isospectral problem for the TC hierarchy:

$$\varphi_x = U\varphi, \tag{2.88}$$

where we set the matrix U to be:

$$U = U(\mathbf{u}, \lambda) = \begin{pmatrix} 0 & 1 + \frac{q+r}{2\lambda} \\ \lambda + \frac{q-r}{2} & 0 \end{pmatrix}, \tag{2.89}$$

where λ is the spectral potential and $\mathbf{u}$ is the 2-dimensional vector $\mathbf{u} = (q, r)^T$.

As before, in order to determine an associated integrable hierarchy, we first solve the stationary zero curvature equation (2.52), with the matrix W being taken the form:

$$W = \begin{pmatrix} \frac{1}{2}a & \frac{1}{2\lambda}(b+c) \\ \frac{1}{2}(b-c) & -\frac{1}{2}a \end{pmatrix}. \tag{2.90}$$

Thus, from the stationary zero curvature equation, the following equations are obtained:

$$a_x = -\lambda^{-1}qc - 2c + \lambda^{-1}rb, \tag{2.91}$$

$$b_x = -ra, \tag{2.92}$$

$$c_x = -2\lambda a - qa. \tag{2.93}$$

Upon expanding the matrix W as in the Laurent series (2.57), that is, with $a = \sum_{m=0}^{\infty} a^{[m]}\lambda^{-m}$, $b = \sum_{m=0}^{\infty} b^{[m]}\lambda^{-m}$ and $c = \sum_{m=0}^{\infty} c^{[m]}\lambda^{-m}$, the system (2.91)-(2.93) gives the recursion relations:

$$a_x^{[0]} = -2c^{[0]},\ b_x^{[0]} = -ra^{[0]}, \tag{2.94}$$

$$a^{[m+1]} = \frac{1}{2}(-qa^{[m]} - c_x^{[m]}), \tag{2.95}$$

$$b^{[m+1]} = -\partial^{-1}ra^{[m+1]}, \qquad m \geq 0. \tag{2.96}$$

$$c^{[m+1]} = \frac{1}{2}(rb^{[m]} - qc^{[m]} - a_x^{[m+1]}), \tag{2.97}$$

Starting from the initial values:

$$c^{[0]} = 0,\, a^{[0]} = 0, \quad \text{and} \quad b^{[0]} = \beta, \tag{2.98}$$

where β is an arbitrary real constant, and taking zero constants of integration, we can work out that:

$$a^{[1]} = 0,\, b^{[1]} = 0,\, c^{[1]} = \frac{1}{2}\beta r, \tag{2.99}$$

$$a^{[2]} = -\frac{1}{4}\beta r_x,\, b^{[2]} = \frac{1}{8}\beta r^2,\, c^{[2]} = \frac{1}{8}\beta(r_{xx} - 2qr), \tag{2.100}$$

$$a^{[3]} = -\frac{1}{16}\beta(r_{xxx} - 4qr_x - 2q_x r), \tag{2.101}$$

$$b^{[3]} = -\frac{1}{32}\beta(r_x^2 + 4qr^2 - 2rr_{xx}), \tag{2.102}$$

$$c^{[3]} = \frac{1}{32}\beta(r_{xxxx} + 4q^2 r + 2r^3 - 6qr_{xx} - 6q_x r_x - 2q_{xx} r). \tag{2.103}$$

We begin by computing a series of Lax matrices in order to determine a soliton hierarchy

$$V^{[m]} = (\lambda^m W)_+ + \Delta_m, \quad \text{for} \quad m \geq 0. \tag{2.104}$$

While taking into account the modification terms:

$$\Delta_m = \begin{pmatrix} 0 & \frac{1}{2\lambda}(\frac{q}{r}c^{[m]} - b^{[m]}) \\ \frac{1}{2}(\frac{q}{r}c^{[m]} - b^{[m]}) & 0 \end{pmatrix}, \tag{2.105}$$

we can generate the soliton hierarchy

$$\mathbf{u}_{t_m} = K_m(x, t, q, r, q_x, r_x, \ldots), \quad m \geq 0, \tag{2.106}$$

associated with the zero curvature equations

$$U_{t_m} - V_x^{[m]} + [U, V^{[m]}] = 0, \quad m \geq 0. \tag{2.107}$$

Thus, one can easily show that the hierarchy can be expressed explicitly as

$$\mathbf{u}_{t_m} = \begin{pmatrix} q \\ r \end{pmatrix}_{t_m} = \begin{pmatrix} \partial(\frac{q}{r}c^{[m]} - b^{[m]}) \\ \partial c^{[m]} - \frac{q}{r}\partial b^{[m]} \end{pmatrix}, \quad m \geq 0. \tag{2.108}$$

For $m = 2$ and $m = 3$, we obtain the systems:

$$\begin{cases} q_{t_2} &= -\frac{\beta}{8r^2}(4qq_x r^2 + 2r^3 r_x + qr_x r_{xx} - qrr_{xxx} - q_x r r_{xx}), \\ r_{t_2} &= -\frac{\beta}{8}(4qr_x + 2q_x r - r_{xxx}), \end{cases} \tag{2.109}$$

$$\begin{cases} q_{t_3} &= \frac{\beta}{32r^2}(12q^2 q_x r^2 + 12qr^3 r_x + 6q_x r^4 + 6q^2 r_x r_{xx} - 6q^2 r r_{xxx} + 6qq_x r_x^2 - 6qq_{xx} r r_x - 2r^3 r_{xxx} \\ & -2qq_{xxx}r^2 - 18qq_x r r_{xx} - 6q_x^2 r r_x - 2q_x q_{xx} r^2 - qr_x r_{xxxx} + qrr_{xxxxx} + q_x r r_{xxxx}), \\ r_{t_3} &= \frac{\beta}{32}(12q^2 r_x + 12qq_x r + 6r^2 r_x - 8qr_{xxx} - 2q_{xxx} r - 12q_x r_{xx} - 8q_{xx} r_x + r_{xxxxx}), \end{cases} \tag{2.110}$$

respectively. With $\beta = 8$ and taking $r = q$, the first system reduces to the KdV equation:

$$q_{t_2} + 6qq_x - q_{xxx} = 0, \tag{2.111}$$

while the second one reduces to the fifth-order mKdV-type equation:

$$q_{t_3} - \frac{15}{2}q^2 q_x + \frac{5}{2}qq_{xxx} + 5q_x q_{xx} - \frac{1}{4}q_{xxxxx} = 0. \tag{2.112}$$

2.3.1 Bi-Hamiltonian structure of the TC hierarchy

To obtain a bi-Hamiltonian structure of the hierarchy (2.108), we continue to use the trace identity (2.38), to have

$$tr\Big(W\frac{\partial U}{\partial \lambda}\Big) = \frac{1}{2\lambda}\sum_{m=0}^{\infty}(b^{[m]}+c^{[m]})\lambda^{-m} - \frac{1}{4\lambda^2}(q+r)\sum_{m=0}^{\infty}(b^{[m]}-c^{[m]})\lambda^{-m}, \tag{2.113}$$

$$tr\Big(W\frac{\partial U}{\partial q}\Big) = \frac{1}{2\lambda}\sum_{m=0}^{\infty}b^{[m]}\lambda^{-m}, \quad tr\Big(W\frac{\partial U}{\partial r}\Big) = -\frac{1}{2\lambda}\sum_{m=0}^{\infty}c^{[m]}\lambda^{-m}. \tag{2.114}$$

After substituting these components into the trace identity and matching the powers of λ^{-m-2}, we get

$$\frac{\delta}{\delta \mathbf{u}}\int\Big(\frac{1}{2}(b^{[m+1]}+c^{[m+1]}) - \frac{1}{4}(q+r)(b^{[m]}-c^{[m]})\Big)dx = \frac{1}{2}(\gamma-m-1)\begin{pmatrix} b^{[m]} \\ -c^{[m]} \end{pmatrix}, \quad m\geq 0. \tag{2.115}$$

We deduce that $\gamma = \frac{1}{2}$ when $m=0$. Therefore, the Hamiltonians can be taken as

$$\mathcal{H}_m = -\frac{1}{2m+1}\int\Big(2(b^{[m+1]}+c^{[m+1]}) - (q+r)(b^{[m]}-c^{[m]})\Big)dx, \quad m\geq 0. \tag{2.116}$$

The first three Hamiltonians are:

$$\mathcal{H}_0 = \beta\int q\,dx, \tag{2.117}$$

$$\mathcal{H}_1 = -\frac{\beta}{12}\int(3r^2 + r_{xx})dx, \tag{2.118}$$

$$\mathcal{H}_2 = -\frac{\beta}{160}\int(-20qr^2 - 2r_x^2 + 8rr_{xx} - 8qr_{xx} - 12q_xr_x - 4q_{xx}r + 2r_{xxxx})dx. \tag{2.119}$$

We derive from (2.115) that

$$\frac{\delta \mathcal{H}_m}{\delta \mathbf{u}} = \begin{pmatrix} b^{[m]} \\ -c^{[m]} \end{pmatrix}, \quad m\geq 0, \tag{2.120}$$

and the bi-Hamiltonian structure is:

$$\mathbf{u}_{t_m} = K_m = J_1 \frac{\delta \mathcal{H}_m}{\delta \mathbf{u}} = J_2 \frac{\delta \mathcal{H}_{m-1}}{\delta \mathbf{u}}, \quad m \geq 1, \tag{2.121}$$

where $\{J_1, J_2\}$ are the Hamiltonian pair, with $J_2 = J_1 \Phi$, and

$$J_1 = \begin{pmatrix} -\partial & -\partial(\frac{q}{r}\cdot) \\ -\frac{q}{r}\partial & -\partial \end{pmatrix}, \quad \Phi = \begin{pmatrix} -\frac{1}{2}\partial^{-1}q\partial & -\frac{1}{2}\partial^{-1}r\partial \\ -\frac{1}{2}r + \frac{1}{4}\partial(\frac{q}{r})\partial & -\frac{1}{2}q + \frac{1}{4}\partial^2 \end{pmatrix}. \tag{2.122}$$

The recursive formula is defined by:

$$\begin{pmatrix} b^{[m+1]} \\ -c^{[m+1]} \end{pmatrix} = \Phi \begin{pmatrix} b^{[m]} \\ -c^{[m]} \end{pmatrix}, \quad m \geq 0. \tag{2.123}$$

2.4 TA hierarchy

Starting with the spectral problem:

$$\varphi_x = U\varphi, \tag{2.124}$$

where U is the matrix defined by

$$U = U(\mathbf{u}, \lambda) = \begin{pmatrix} 0 & 1 \\ \lambda + q + \frac{r}{\lambda} & 0 \end{pmatrix}, \tag{2.125}$$

where λ is the spectral parameter and $\mathbf{u}$ is the 2-dimensional vector $\mathbf{u} = (q, r)^T$. We use the stationary zero curvature equation (2.52) to derive an integrable hierarchy, with W being the matrix:

$$W = \begin{pmatrix} \frac{1}{2}a & \frac{1}{2\lambda}(b+c) \\ \frac{1}{2}(b-c) & -\frac{1}{2}a \end{pmatrix}. \tag{2.126}$$

The stationary zero curvature equation gives the following equations:

$$a_x = -\lambda^{-1}(b+c)q - \lambda^{-2}(b+c)r - 2c, \tag{2.127}$$

$$b_x = (q+\lambda^{-1}r)a, \tag{2.128}$$

$$c_x = -(q+\lambda^{-1}r+2\lambda)a. \tag{2.129}$$

Expanding the matrix W in Laurent series (2.57), with $a = \sum_{m=0}^{\infty} a^{[m]}\lambda^{-m}$, $b = \sum_{m=0}^{\infty} b^{[m]}\lambda^{-m}$ and $c = \sum_{m=0}^{\infty} c^{[m]}\lambda^{-m}$, using the equations (2.127)-(2.129), we get the recursion relations:

$$a^{[0]} = 0,\ c^{[0]} = 0, b_x^{[0]} = 0, \tag{2.130}$$

$$a_x^{[1]} = -2c^{[1]} - qb^{[0]},\ b_x^{[1]} = qa^{[1]}, c_x^{[1]} = -qa^{[1]} - 2a^{[2]}, \tag{2.131}$$

$$a^{[m+1]} = \frac{1}{2}(-qa^{[m]} - c_x^{[m]}), \tag{2.132}$$

$$b^{[m+1]} = -\partial^{-1}ra^{[m+1]}, \qquad m \geq 1. \tag{2.133}$$

$$c^{[m+1]} = \frac{1}{2}(rb^{[m]} - qc^{[m]} - a_x^{[m+1]}), \tag{2.134}$$

Taking the initial values to be:

$$b^{[0]} = 2\beta,\ a^{[1]} = 0, \tag{2.135}$$

and rewriting, we get

$$b^{[1]} = 0,\ c^{[1]} = -\beta q, \tag{2.136}$$

$$a_x^{[m]} = -q(b^{[m-1]} + c^{[m-1]}) - r(b^{[m-2]} + c^{[m-2]}) - 2c^{[m]}, \tag{2.137}$$

$$b_x^{[m]} = qa^{[m]} + ra^{[m-1]}, \qquad m \geq 2. \tag{2.138}$$

$$c_x^{[m]} = -qa^{[m]} - ra^{[m-1]} - 2a^{[m+1]}, \tag{2.139}$$

Let $d^{[m]} = \frac{1}{2}(b^{[m]} + c^{[m]})$. Thus, through some manipulations the above system can be rewritten in the simplified form:

$$d^{[0]} = \beta, \quad d^{[1]} = -\frac{1}{2}\beta q \tag{2.140}$$

$$d^{[m+1]} = \Big(\frac{1}{4}\partial^2 - q + \frac{1}{2}\partial^{-1}q_x\Big)d^{[m]} + \Big(-r + \frac{1}{2}\partial^{-1}r_x\Big)d^{[m-1]}, \quad m \geq 1. \tag{2.141}$$

As before, in this case we need the non-zero modification terms:

$$\varDelta_m = \begin{pmatrix} 0 & -d^{[m]}\lambda^{-1} \\ -d^{[m]} + r d^{[m-1]}\lambda^{-1} & 0 \end{pmatrix}, \quad m \geq 1. \tag{2.142}$$

Hence, we can construct the soliton hierarchy by using the zero curvature equations:

$$\mathbf{u}_{t_m} = \begin{pmatrix} q \\ r \end{pmatrix}_{t_m} = \begin{pmatrix} -2d_x^{[m]} \\ 2r d_x^{[m-1]} + r_x d^{[m-1]} \end{pmatrix}, \quad m \geq 1. \tag{2.143}$$

For $m = 2$ and $m = 3$, we obtain the systems:

$$\begin{cases} q_{t_2} &= \frac{1}{4}\beta(q_{xxx} - 6qq_x + 4r_x), \\ r_{t_2} &= -\beta(rq_x + \frac{1}{2}r_x q), \end{cases} \tag{2.144}$$

$$\begin{cases} q_{t_3} &= \frac{\beta}{16}(q_{xxxxx} + 4r_{xxx} - 20q_x q_{xx} - 24qr_x - 10qq_{xxx} - 24q_x r + 30q^2 q_x), \\ r_{t_3} &= -\frac{\beta}{8}(2q_{xxx}r + q_{xx}r_x + 12rr_x - 3q^2 r_x - 12qq_x r), \end{cases} \tag{2.145}$$

respectively. Taking $\beta = 4$ and $r = 0$, the first system reduces to the KdV equation (2.111) as well, while the second one reduces to the fifth-order mKdV-type equation (2.112).

2.4.1 Bi-Hamiltonian structure of the TA hierarchy

Let's derive a bi-Hamiltonian structure of the soliton hierarchy (2.143). As usual, we use the trace identity (2.38), to obtain

$$tr\Big(W\frac{\partial U}{\partial \lambda}\Big) = \frac{1}{2}(\lambda^{-1} - r\lambda^{-3})\sum_{m=0}^{\infty}(b^{[m]} + c^{[m]})\lambda^{-m}, \tag{2.146}$$

$$tr\Big(W\frac{\partial U}{\partial q}\Big) = \frac{1}{2\lambda}\sum_{m=0}^{\infty}(b^{[m]} + c^{[m]})\lambda^{-m}, \tag{2.147}$$

$$tr\Big(W\frac{\partial U}{\partial r}\Big) = \frac{1}{2\lambda^2}\sum_{m=0}^{\infty}(b^{[m]} + c^{[m]})\lambda^{-m}. \tag{2.148}$$

As a result of matching the powers of λ^{-m} and using the trace identity, we observe

$$\frac{\delta}{\delta \mathbf{u}}\int\Big(d^{[m-1]} - rd^{[m-3]}\Big)dx = (\gamma - m + 1)\begin{pmatrix} d^{[m-2]} \\ d^{[m-3]} \end{pmatrix}, \quad m \geq 3. \tag{2.149}$$

The case where $m = 0$ implies that $\gamma = \frac{1}{2}$. Therefore, the Hamiltonians can be taken as

$$\mathcal{H}_m = -\frac{2}{2m+1}\int\Big(d^{[m+1]} - rd^{[m-1]}\Big)dx, \quad m \geq 1. \tag{2.150}$$

The first three Hamiltonians are:

$$\mathcal{H}_1 = \frac{\beta}{12}\int(q_{xx} - 3q^2 + 12r)dx, \tag{2.151}$$

$$\mathcal{H}_2 = \frac{\beta}{80}\int(q_{xxxx} + 10q^3 - 40qr - 10qq_{xx} - 5q_x^2 + 4r_{xx})dx, \tag{2.152}$$

$$\begin{aligned}\mathcal{H}_3 = \frac{\beta}{448}\int(&q_{xxxxxx} + 4r_{xxxx} - 30q_xq_{xxx} - 48q_xr_x - 128r^2 - 56q_{xx}r \\ &- 20q_{xx}^2 - 14qq_{xxxx} - 40qr_{xx} + 80qq_x^2 + 192q^2r + 70q^2q_{xx} - 40q^4)dx,\end{aligned} \tag{2.153}$$

and from (2.149), we can write

$$\frac{\delta \mathcal{H}_m}{\delta \mathbf{u}} = \begin{pmatrix} d^{[m]} \\ d^{[m-1]} \end{pmatrix}, \quad m \geq 1. \tag{2.154}$$

The bi-Hamiltonian structure thus reads:

$$\mathbf{u}_{t_m} = K_m = J_1 \frac{\delta \mathcal{H}_m}{\delta \mathbf{u}} = J_2 \frac{\delta \mathcal{H}_{m-1}}{\delta \mathbf{u}}, \quad m \geq 2, \tag{2.155}$$

with the Hamiltonian pair are $\{J_1, J_2\}$, with $J_2 = J_1 \Phi$ and

$$J_1 = \begin{pmatrix} -2\partial & 0 \\ 0 & 2r\partial + r_x \end{pmatrix}, \quad \Phi = \begin{pmatrix} \frac{1}{4}\partial^2 - q + \frac{1}{2}\partial^{-1} q_x & -r + \frac{1}{2}\partial^{-1} r_x \\ 1 & 0 \end{pmatrix}. \tag{2.156}$$

with the recursive formula:

$$\begin{pmatrix} d^{[m+1]} \\ d^{[m]} \end{pmatrix} = \Phi \begin{pmatrix} d^{[m]} \\ d^{[m-1]} \end{pmatrix}, \quad m \geq 1. \tag{2.157}$$

2.5 Boiti-Pempinelli-Tu (BPT) hierarchy

We begin with the spatial spectral problem:

$$\varphi_x = U\varphi, \tag{2.158}$$

where U is the matrix defined by

$$U = U(\mathbf{u}, \lambda) = \begin{pmatrix} \lambda + \frac{1}{2}\lambda^{-1} s & \frac{1}{2}(q + \lambda^{-1} r) \\ \frac{1}{2}(q - \lambda^{-1} r) & -\lambda - \frac{1}{2}\lambda^{-1} s \end{pmatrix}, \tag{2.159}$$

where λ is the spectral potential and $\mathbf{u}$ is the 3-dimensional vector $\mathbf{u} = (q, r, s)^T$. After solving the stationary zero curvature equation (2.52) with the solution W being taken as:

$$W = \begin{pmatrix} \frac{1}{2}a & \frac{1}{2}(b+c) \\ \frac{1}{2}(b-c) & -\frac{1}{2}a \end{pmatrix}. \tag{2.160}$$

We obtain the system of equations

$$a_x = -qc + \lambda^{-1} rb, \tag{2.161}$$

$$b_x = 2\lambda c - \lambda^{-1} ra + \lambda^{-1} sc, \tag{2.162}$$

$$c_x = 2\lambda b - qa + \lambda^{-1} sb. \tag{2.163}$$

Expanding the matrix W in Laurent series (2.57), with $a = \sum_{m=0}^{\infty} a^{[m]}\lambda^{-m}$, $b = \sum_{m=0}^{\infty} b^{[m]}\lambda^{-m}$ and $c = \sum_{m=0}^{\infty} c^{[m]}\lambda^{-m}$, we get the corresponding recursion relations:

$$a_x^{[0]} = 0,\ b^{[0]} = 0,\ c^{[0]} = 0,\ c^{[1]} = 0,\ b^{[1]} = \frac{1}{2}qa^{[0]}, \tag{2.164}$$

$$a^{[m+1]} = \partial^{-1}(-qc^{[m+1]} + rb^{[m]}), \tag{2.165}$$

$$b^{[m+1]} = \frac{1}{2}(c_x^{[m]} + qa^{[m]} - sb^{[m-1]}), \qquad m \geq 1. \tag{2.166}$$

$$c^{[m+1]} = \frac{1}{2}(b_x^{[m]} + ra^{[m-1]} - sc^{[m-1]}), \tag{2.167}$$

By choosing the initial value:

$$a^{[0]} = 2\beta, \tag{2.168}$$

with β is an arbitrary real constant, and taking zero constants of integration we uniquely generate the concrete expressions:

$$\begin{aligned}
&a^{[1]} = 0,\ b^{[1]} = \beta q,\ c^{[1]} = 0,\\
&a^{[2]} = -\frac{1}{4}\beta q^2,\ b^{[2]} = 0,\ c^{[2]} = \beta(\frac{1}{2}q_x + r),\\
&a^{[3]} = 0,\ b^{[3]} = \frac{1}{4}\beta(q_{xx} + 2r_x - \frac{1}{2}q^3 - 2qs),\ c^{[3]} = 0,\\
&a^{[4]} = \frac{\beta}{64}(3q^4 + 16q^2 s + 4q_x^2 + 16q_x r + 16r^2 - 8qq_{xx} - 16qr_x),\ b^{[4]} = 0,\\
&c^{[4]} = \frac{\beta}{8}(q_{xxx} - \frac{3}{2}q^2 q_x - q^2 r - 4q_x s - 4rs - 2qs_x + 2r_{xx}),
\end{aligned}$$

$$\begin{aligned}
&a^{[5]} = 0,\\
&b^{[5]} = \frac{\beta}{128}(8q_{xxxx} + 16r_{xxx} + 3q^5 + 24q^3 s - 20qq_x^2 + 16qr^2 - 20q^2 q_{xx}\\
&\quad - 24q^2 r_x + 32qs^2 - 48q_x s_x - 32rs_x - 16qs_{xx} - 48q_{xx}s - 64r_x s),\\
&c^{[5]} = 0,
\end{aligned}$$

$$\begin{aligned}
a^{[6]} = &-\frac{\beta}{512}(16qq_{xxxx} + 5q^6 + 48q^4 s - 40q^3 q_{xx} - 48q^3 r_x + 96q^2 s^2 + 48q^2 r^2\\
&+ 48q^2 q_x r - 20q^2 q_x^2 - 32qs - 128qqs - 192qr_x s - 64qq_x s_x + 128rs\\
&+ 192q_x rs + 64q_x^2 s + 32qr_{xxx} - 32q_{xxx}r - 16q_x q_{xxx} - 64rr_{xx} - 32q_x r_{xx}\\
&+ 8q_{xx}^2 + 32q_{xx}r_x + 32r_x^2),\ b^{[6]} = 0,\\
c^{[6]} = &\frac{\beta}{256}(8q_{xxxxx} + 48q^2 rs - 16qq_{xx}r - 20q_x^3 - 48qq_x r_x + 96qss_x + 8q_x^2 r - 16qs_{xxx} + 32r^3\\
&+ 96q^2 q_x s - 20q^2 q_{xxx} + 6q^4 r + 64rs^2 - 80qq_x q_{xx} + 16r_{xxxx} + 15q^4 q_x - 64q_{xxx}s - 96r_{xx}s\\
&- 96q_{xx}s_x - 32rs_{xx} - 96r_x s_x - 64q_x s_{xx} + 24q^3 s_x + 48q_x r^2 - 24q^2 r_{xx} + 96q_x s^2).
\end{aligned} \tag{2.169}$$

By taking the modification terms $\varDelta_m = \mathbf{0}_3$, $m \geq 0$, with $\mathbf{0}_3$ being the 3×3 zero matrix, we can generate the soliton hierarchy:

$$\mathbf{u}_{t_m} = \begin{pmatrix} q \\ r \\ s \end{pmatrix}_{t_m} = \begin{pmatrix} 2c^{[2m]} \\ -sb^{[2m-1]} \\ -rb^{[2m-1]} \end{pmatrix}, \quad m \geq 1. \tag{2.170}$$

For $m = 2$ and $m = 3$, we obtain the systems:

$$\begin{cases} q_{t_2} &= \frac{\beta}{4}(q_{xxx} + 2r_{xx} - \frac{3}{2}q^2 q_x - q^2 r - 2qs_x - 4q_x s - 4rs), \\ r_{t_2} &= -\frac{\beta}{4}s(q_{xx} + 2r_x - \frac{1}{2}q^3 - 2qs), \\ s_{t_2} &= -\frac{\beta}{4}r(q_{xx} + 2r_x - \frac{1}{2}q^3 - 2qs), \end{cases} \tag{2.171}$$

$$\begin{cases} q_{t_3} &= \frac{\beta}{128}(8q_{xxxxx} + 48q^2 rs - 16qq_{xx}r - 20q_x^3 - 48qq_x r_x + 96qss_x + 8q_x^2 r - 16qs_{xxx} \\ & +32r^3 + 96q^2 q_x s - 20q^2 q_{xxx} + 6q^4 r + 64rs^2 - 80qq_x q_{xx} + 16r_{xxxx} + 15q^4 q_x \\ & -64q_{xxx}s - 96r_{xx}s - 96q_{xx}s_x - 32rs_{xx} - 96r_x s_x - 64q_x s_{xx} + 24q^3 s_x + 48q_x r^2 \\ & -24q^2 r_{xx} + 96q_x s^2) \\ r_{t_3} &= -\frac{\beta}{128}s(8q_{xxxx} + 16r_{xxx} + 3q^5 + 24q^3 s - 20qq_x^2 + 16qr^2 - 20q^2 q_{xx} - 24q^2 r_x + 32qs^2 \\ & -48q_x s_x - 32rs_x - 16qs_{xx} - 48q_{xx}s - 64r_x s), \\ s_{t_3} &= -\frac{\beta}{128}r(8q_{xxxx} + 16r_{xxx} + 3q^5 + 24q^3 s - 20qq_x^2 + 16qr^2 - 20q^2 q_{xx} - 24q^2 r_x + 32qs^2 \\ & -48q_x s_x - 32rs_x - 16qs_{xx} - 48q_{xx}s - 64r_x s), \end{cases} \tag{2.172}$$

respectively. Taking $\beta = 4$ and $r = s = 0$, the first system reduces to the mKdV equation too:

$$q_{t_2} = q_{xxx} - \frac{3}{2}q^2 q_x, \tag{2.173}$$

while the second one reduces to the fifth-order mKdV-type equation:

$$q_{t_3} = \frac{1}{4}(q_{xxxxx} - 10qq_xq_{xx} + \frac{15}{8}q^4q_x - \frac{5}{2}q_x^3 - \frac{5}{2}q^2q_{xxx}). \tag{2.174}$$

2.5.1 Bi-Hamiltonian structure of the BPT hierarchy

For the BPT hierarchy, the components of the trace identity (2.38) are

$$tr\Big(W\frac{\partial U}{\partial \lambda}\Big) = (1 - \frac{1}{2}\lambda^{-2}s)\sum_{m=0}^{\infty} a^{[m]}\lambda^{-m} + \frac{1}{2}\lambda^{-2}r\sum_{m=0}^{\infty} c^{[m]}\lambda^{-m} \tag{2.175}$$

$$tr\Big(W\frac{\partial U}{\partial q}\Big) = \frac{1}{2}\sum_{m=0}^{\infty} b^{[m]}\lambda^{-m}, \quad tr\Big(W\frac{\partial U}{\partial r}\Big) = -\frac{1}{2\lambda}\sum_{m=0}^{\infty} c^{[m]}\lambda^{-m}, \tag{2.176}$$

$$tr\Big(W\frac{\partial U}{\partial s}\Big) = \frac{1}{2\lambda}\sum_{m=0}^{\infty} a^{[m]}\lambda^{-m}. \tag{2.177}$$

Substituting them in the trace identity and matching the powers of λ^{-m}, we obtain

$$\frac{\delta}{\delta \mathbf{u}}\int \Big(a^{[2m+2]} - \frac{1}{2}sa^{[2m]} + \frac{1}{2}rc^{[2m]}\Big)dx = \frac{1}{2}(\gamma - 2m + 1)\begin{pmatrix} b^{[2m+1]} \\ -c^{[2m]} \\ a^{[2m]} \end{pmatrix}, \quad m \geq 0. \tag{2.178}$$

The case where $m = 0$ requires that $\gamma = 0$. Therefore, the Hamiltonians for the soliton hierarchy (2.170) can be taken as

$$\mathcal{H}_m = -\frac{2}{2m+1}\int \Big(a^{[2m+2]} - \frac{1}{2}sa^{[2m]} + \frac{1}{2}rc^{[2m]}\Big)dx, \quad m \geq 0. \tag{2.179}$$

The first three Hamiltonians are the following:

$$\mathcal{H}_0 = \frac{\beta}{2}\int (q^2 + 4s)dx, \tag{2.180}$$

$$\mathcal{H}_1 = -\frac{\beta}{96}\int (3q^4 + 24q^2s - 16qr_x - 8qq_{xx} + 48r^2 + 32q_xr + 4q_x^2)dx, \tag{2.181}$$

$$
\mathcal{H}_2 = \frac{\beta}{1280}\int (5q^6 + 16qq_{xxxx} - 256qr_xs + 96q^2q_xr - 64qq_xs_x + 384q_xrs \tag{2.182}
$$

$$
- 160qq_{xx}s + 64qrs_x + 60q^4s + 160q^2s^2 + 80q^2r^2 + 320r^2s - 128rr_{xx}
$$

$$
- 16q_xq_{xxx} + 80q_x^2s - 20q^2q_x^2 - 32q_xr_{xx} + 32q_{xx}r_x - 40q^3q_{xx} - 64q_{xxx}r
$$

$$
+ 32qr_{xxx} - 32q^2s_{xx} + 32r_x^2 + 8q_{xx}^2 - 48q^3r_x)dx. \tag{2.183}
$$

From (2.178), we deduce

$$
\frac{\delta \mathcal{H}_m}{\delta \mathbf{u}} = \begin{pmatrix} b^{[2m+1]} \\ -c^{[2m]} \\ a^{[2m]} \end{pmatrix}, \quad m \geq 0, \tag{2.184}
$$

and the bi-Hamiltonian structure:

$$
\mathbf{u}_{t_m} = K_m = J_1 \frac{\delta \mathcal{H}_m}{\delta \mathbf{u}} = J_2 \frac{\delta \mathcal{H}_{m-1}}{\delta \mathbf{u}}, \quad m \geq 1, \tag{2.185}
$$

in which the Hamiltonian pair are $\{J_1, J_2\}$ and $J_2 = J_1 \Phi$, explicitly we have

$$
J_1 = \begin{pmatrix} 0 & -2 & 0 \\ 2 & \partial & -q \\ 0 & q & -\partial \end{pmatrix}, \tag{2.186}
$$

$$
\Phi = \frac{1}{4}\begin{pmatrix} \partial^2 - 2s - q^2 + q\partial^{-1}(2r + q_x) & \partial s - q\partial^{-1}(qs) & \partial r - q\partial^{-1}(qr) \\ -2\partial & -2s & -2r \\ -2q + \partial^{-1}(2q_x + 4r) & -2\partial^{-1}(qs) & -2\partial^{-1}(qr) \end{pmatrix}. \tag{2.187}
$$

Here, the recursive formula is given by:

$$
\begin{pmatrix} b^{[2m+1]} \\ -c^{[2m]} \\ a^{[2m]} \end{pmatrix} = \Phi \begin{pmatrix} b^{[2m-1]} \\ -c^{[2m-2]} \\ a^{[2m-2]} \end{pmatrix}, \quad m \geq 1. \tag{2.188}
$$

Chapter 3

Riemann-Hilbert problems of a non-local reverse-time AKNS system of sixth-order and its exact soliton solutions

3.1 Introduction

In this chapter, we investigate the solvability of a nonlinear nonlocal reverse-time six-component sixth-order AKNS system. We use reverse-time reduction to reduce the coupled system to an integrable sixth-order NLS-type equation. Starting from the spectral problem of the AKNS system, a Riemann-Hilbert problem will be formulated. Soliton solutions are generated by using vectors in the kernel of the matrix Jost solutions. In the case of zero reflection coefficients, the jump matrix is identity, and the corresponding Riemann-Hilbert problem gives solitons, which allows to explore soliton dynamics [35]. We formulate the AKNS hierarchy for the six-component AKNS system of sixth-order and solve the resulting Riemann-Hilbert problem, with the contour being the real line [37]-[44].

3.2 Six-component AKNS hierarchy system of sixth-order

3.2.1 Six-component AKNS soliton hierarchy of coupled sixth-order integrable systems

Consider the 4×4 matrix spatial spectral problem [39]

$$\varphi_x = \mathrm{i}U\varphi, \tag{3.1}$$

where φ is the eigenfunction and $U(u,\lambda)$ the spectral matrix is given by

$$U(u,\lambda)=\begin{pmatrix}\alpha_1\lambda & p_1 & p_2 & p_3\\ r_1 & \alpha_2\lambda & 0 & 0\\ r_2 & 0 & \alpha_2\lambda & 0\\ r_3 & 0 & 0 & \alpha_2\lambda\end{pmatrix}=\lambda\Lambda+P(u), \tag{3.2}$$

where $\Lambda=diag(\alpha_1,\alpha_2,\alpha_2,\alpha_2)$, λ is the spectral parameter, α_1,α_2 are two distinct real constants, $u=(p,r^T)^T$ is a vector of six potentials, where $p=(p_1,p_2,p_3)$ and $r=(r_1,r_2,r_3)^T$ are vector functions of (x,t) and $\{p_i,r_i\}_{i=1,2,3}\in\mathcal{S}(\mathbb{R})$, the Schwartz space, and

$$P=\begin{pmatrix}0 & p_1 & p_2 & p_3\\ r_1 & 0 & 0 & 0\\ r_2 & 0 & 0 & 0\\ r_3 & 0 & 0 & 0\end{pmatrix}. \tag{3.3}$$

Let's construct the AKNS system of sixth-order. To do so, we need to solve the stationary zero curvature equation

$$W_x=\mathrm{i}[U,W], \tag{3.4}$$

to which

$$W=\begin{pmatrix}a & b_1 & b_2 & b_3\\ c_1 & d_{11} & d_{12} & d_{13}\\ c_2 & d_{21} & d_{22} & d_{23}\\ c_3 & d_{31} & d_{32} & d_{33}\end{pmatrix} \tag{3.5}$$

is a solution, where a, b_i, c_i, d_{ij} are scalar components for $i, j \in \{1, 2, 3\}$. From the stationary zero curvature equation, we get:

$$\begin{cases} a_x & = \mathrm{i}\big(- \sum\limits_{i=1}^{3} b_i r_i + \sum\limits_{i=1}^{3} c_i p_i \big), \\ b_{i,x} & = \mathrm{i}(\alpha\lambda b_i - a p_i + d_{1i} p_1 + d_{2i} p_2 + d_{3i} p_3), \quad i \in \{1, 2, 3\}, \\ c_{i,x} & = \mathrm{i}(-\alpha\lambda c_i + a r_i - d_{i1} r_1 - d_{i2} r_2 - d_{i3} r_3), \quad i \in \{1, 2, 3\}, \\ d_{ij,x} & = \mathrm{i}(b_j r_i - c_i p_j), \quad i, j \in \{1, 2, 3\}, \end{cases} \tag{3.6}$$

where $\alpha = \alpha_1 - \alpha_2$. We expand W in Laurent series:

$$W = \sum_{m=0}^{\infty} W_m \lambda^{-m} \quad \text{with} \quad W_m = \begin{pmatrix} a^{[m]} & b_1^{[m]} & b_2^{[m]} & b_3^{[m]} \\ c_1^{[m]} & d_{11}^{[m]} & d_{12}^{[m]} & d_{13}^{[m]} \\ c_2^{[m]} & d_{21}^{[m]} & d_{22}^{[m]} & d_{23}^{[m]} \\ c_3^{[m]} & d_{31}^{[m]} & d_{32}^{[m]} & d_{33}^{[m]} \end{pmatrix}, \tag{3.7}$$

explicitly, i.e., set

$$a = \sum_{m=0}^{\infty} a^{[m]} \lambda^{-m}, \quad b_j = \sum_{m=0}^{\infty} b_j^{[m]} \lambda^{-m}, \tag{3.8}$$

$$c_j = \sum_{m=0}^{\infty} c_j^{[m]} \lambda^{-m}, \quad d_{jk} = \sum_{m=0}^{\infty} d_{jk}^{[m]} \lambda^{-m}, \tag{3.9}$$

for $j,k \in \{1,2,3\}$. The system (3.6) generates the recursion relations:

$$b_j^{[0]} = 0,\ c_j^{[0]} = 0, \quad \text{for} \quad j \in \{1,2,3\}, \tag{3.10}$$

$$a_x^{[0]} = 0, \tag{3.11}$$

$$d_{jk,x}^{[0]} = 0, \quad \text{for} \quad j,k \in \{1,2,3\}, \tag{3.12}$$

$$b_j^{[m+1]} = \frac{1}{\alpha}(-\mathrm{i}b_{j,x}^{[m]} + a^{[m]}p_j - d_{1j}^{[m]}p_1 - d_{2j}^{[m]}p_2 - d_{3j}^{[m]}p_3), \quad j \in \{1,2,3\}, \tag{3.13}$$

$$c_j^{[m+1]} = \frac{1}{\alpha}(\mathrm{i}c_{j,x}^{[m]} + a^{[m]}r_j - d_{j1}^{[m]}r_1 - d_{j2}^{[m]}r_2 - d_{j3}^{[m]}r_3), \quad j \in \{1,2,3\}, \tag{3.14}$$

$$a_x^{[m]} = \mathrm{i}(-\sum_{j=1}^{3} b_j^{[m]}r_j + \sum_{j=1}^{3} c_j^{[m]}p_j), \tag{3.15}$$

$$d_{jk,x}^{[m]} = \mathrm{i}(b_k^{[m]}r_j - c_j^{[m]}p_k), \quad j,k \in \{1,2,3\}, \tag{3.16}$$

where $m \geq 0$. Particularly, we can work out

$$\begin{cases} a^{[0]} &= \beta_1, \quad a^{[1]} = 0, \quad a^{[2]} = -\frac{\beta}{\alpha^2}\mathbf{T}_{0,0}, \quad a^{[3]} = -\mathrm{i}\frac{\beta}{\alpha^3}(\mathbf{T}_{0,1} - \mathbf{T}_{1,0}), \\ a^{[4]} &= \frac{\beta}{\alpha^4}\Big[3\mathbf{T}_{0,0}^2 + \mathbf{T}_{0,2} - \mathbf{T}_{1,1} + \mathbf{T}_{2,0}\Big], \\ a^{[5]} &= \mathrm{i}\frac{\beta}{\alpha^5}\Big[6\mathbf{T}_{0,0}(\mathbf{T}_{0,1} - \mathbf{T}_{1,0}) + \mathbf{T}_{0,3} - \mathbf{T}_{3,0} + \mathbf{T}_{2,1} - \mathbf{T}_{1,2}\Big], \\ a^{[6]} &= -\frac{\beta}{\alpha^6}\Big[10\mathbf{T}_{0,0}^3 + 10\mathbf{T}_{0,0}(\mathbf{T}_{0,2} + \mathbf{T}_{2,0}) + 5\Big(\mathbf{T}_{1,0}^2 + \mathbf{T}_{0,1}^2\Big) \\ &\qquad + (\mathbf{T}_{0,4} + \mathbf{T}_{4,0} - \mathbf{T}_{1,3} - \mathbf{T}_{3,1} + \mathbf{T}_{2,2})\Big], \\ a^{[7]} &= -\mathrm{i}\frac{\beta}{\alpha^7}\Big[30\mathbf{T}_{0,0}^2(\mathbf{T}_{0,1} - \mathbf{T}_{1,0}) + 5\mathbf{T}_{0,0}(\mathbf{T}_{2,1} - \mathbf{T}_{1,2}) + 10\mathbf{T}_{0,0}(\mathbf{T}_{0,3} - \mathbf{T}_{3,0}) \\ &\qquad + 10\mathbf{T}_{1,1}(\mathbf{T}_{0,1} - \mathbf{T}_{1,0}) + 5(\mathbf{T}_{0,1}\mathbf{T}_{2,0} - \mathbf{T}_{1,0}\mathbf{T}_{0,2}) + 20(\mathbf{T}_{0,1}\mathbf{T}_{0,2} - \mathbf{T}_{1,0}\mathbf{T}_{2,0}) \\ &\qquad + 5(\mathbf{T}_{5,0} - \mathbf{T}_{4,1} + \mathbf{T}_{3,2} - \mathbf{T}_{2,3} + \mathbf{T}_{1,4} - \mathbf{T}_{0,5})\Big], \end{cases} \tag{3.17}$$

$$
\begin{cases}
b_k^{[0]} &= 0, \quad b_k^{[1]} = \frac{\beta}{\alpha} p_k, \quad b_k^{[2]} = -\mathrm{i}\frac{\beta}{\alpha^2} p_{k,x}, \quad b_k^{[3]} = -\frac{\beta}{\alpha^3}\Big[p_{k,xx} + 2\mathbf{T}_{0,0} p_k\Big], \\
b_k^{[4]} &= \mathrm{i}\frac{\beta}{\alpha^4}\Big[p_{k,xxx} + 3\mathbf{T}_{0,0} p_{k,x} + 3\mathbf{T}_{1,0} p_k\Big], \\
b_k^{[5]} &= \frac{\beta}{\alpha^5}\Big[p_{k,xxxx} + 4\mathbf{T}_{0,0} p_{k,xx} + (6\mathbf{T}_{1,0} + 2\mathbf{T}_{0,1}) p_{k,x} + (4\mathbf{T}_{2,0} + 2\mathbf{T}_{1,1} + 2\mathbf{T}_{0,2} + 6\mathbf{T}_{0,0}^2) p_k\Big], \\
b_k^{[6]} &= -\mathrm{i}\frac{\beta}{\alpha^6}\Bigg[p_{k,xxxxx} + 5\mathbf{T}_{0,0} p_{k,xxx} + (10\mathbf{T}_{1,0} + 5\mathbf{T}_{0,1}) p_{k,xx} \\
&\quad + \Bigg(10\mathbf{T}_{2,0} + 5\mathbf{T}_{0,2} + 10\mathbf{T}_{1,1} + 10\mathbf{T}_{0,0}^2\Bigg) p_{k,x} + \Bigg(5\mathbf{T}_{3,0} + 5\mathbf{T}_{2,1} + 5\mathbf{T}_{1,2} + 20\mathbf{T}_{0,0}\mathbf{T}_{1,0}\Bigg) p_k\Bigg], \\
b_k^{[7]} &= -\frac{\beta}{\alpha^7}\Bigg[p_{k,xxxxxx} + 6\mathbf{T}_{0,0} p_{k,xxxx} + (9\mathbf{T}_{0,1} + 15\mathbf{T}_{1,0}) p_{k,xxx} \\
&\quad + (15\mathbf{T}_{0,0}^2 + 11\mathbf{T}_{0,2} + 20\mathbf{T}_{2,0} + 25\mathbf{T}_{1,1}) p_{k,xx} \\
&\quad + \Bigg(\mathbf{T}_{0,0}(15\mathbf{T}_{0,1} + 45\mathbf{T}_{1,0}) + 15\mathbf{T}_{3,0} + 4\mathbf{T}_{0,3} + 20\mathbf{T}_{1,2} + 25\mathbf{T}_{2,1}\Bigg) p_{k,x} \\
&\quad + \Bigg((20\mathbf{T}_{0,0}^3 + \mathbf{T}_{0,0}(20\mathbf{T}_{0,2} + 35\mathbf{T}_{2,0} + 25\mathbf{T}_{1,1})) + 10\mathbf{T}_{0,1}^2 \\
&\qquad + 25\mathbf{T}_{1,0}^2 + 20\mathbf{T}_{1,0}\mathbf{T}_{0,1} + 2\mathbf{T}_{0,4} + 6\mathbf{T}_{4,0} + 4\mathbf{T}_{1,3} + 9\mathbf{T}_{3,1} + 11\mathbf{T}_{2,2}\Bigg) p_k\Bigg],
\end{cases}
\tag{3.18}
$$

$$
\begin{cases}
c_k^{[0]} &= 0, \quad c_k^{[1]} = \frac{\beta}{\alpha} r_k, \quad c_k^{[2]} = \mathrm{i}\frac{\beta}{\alpha^2} r_{k,x}, \quad c_k^{[3]} = -\frac{\beta}{\alpha^3}\Big[r_{k,xx} + 2\mathbf{T}_{0,0} r_k\Big], \\
c_k^{[4]} &= -\mathrm{i}\frac{\beta}{\alpha^4}\Big[r_{k,xxx} + 3\mathbf{T}_{0,0} r_{k,x} + 3\mathbf{T}_{0,1} r_k\Big], \\
c_k^{[5]} &= \frac{\beta}{\alpha^5}\Big[r_{k,xxxx} + 4\mathbf{T}_{0,0} r_{k,xx} + (6\mathbf{T}_{0,1} + 2\mathbf{T}_{1,0}) r_{k,x} + (4\mathbf{T}_{0,2} + 2\mathbf{T}_{1,1} + 2\mathbf{T}_{2,0} + 6\mathbf{T}_{0,0}^2) r_k\Big], \\
c_k^{[6]} &= \mathrm{i}\frac{\beta}{\alpha^6}\Bigg[r_{k,xxxxx} + 5\mathbf{T}_{0,0} r_{k,xxx} + (10\mathbf{T}_{0,1} + 5\mathbf{T}_{1,0}) r_{k,xx} \\
&\quad + \Bigg(10\mathbf{T}_{0,2} + 5\mathbf{T}_{2,0} + 10\mathbf{T}_{1,1} + 10\mathbf{T}_{0,0}^2\Bigg) r_{k,x} + \Bigg(5\mathbf{T}_{0,3} + 5\mathbf{T}_{1,2} + 5\mathbf{T}_{2,1} + 20\mathbf{T}_{0,0}\mathbf{T}_{0,1}\Bigg) r_k\Bigg] \\
c_k^{[7]} &= -\frac{\beta}{\alpha^7}\Bigg[r_{k,xxxxxx} + 6\mathbf{T}_{0,0} r_{k,xxxx} + (15\mathbf{T}_{0,1} + 9\mathbf{T}_{1,0}) r_{k,xxx} \\
&\quad + (15\mathbf{T}_{0,0}^2 + 20\mathbf{T}_{0,2} + 11\mathbf{T}_{2,0} + 25\mathbf{T}_{1,1}) r_{k,xx} \\
&\quad + \Bigg(\mathbf{T}_{0,0}(45\mathbf{T}_{0,1} + 15\mathbf{T}_{1,0}) + 4\mathbf{T}_{3,0} + 15\mathbf{T}_{0,3} + 25\mathbf{T}_{1,2} + 20\mathbf{T}_{2,1}\Bigg) r_{k,x} \\
&\quad + \Bigg((20\mathbf{T}_{0,0}^3 + \mathbf{T}_{0,0}(35\mathbf{T}_{0,2} + 20\mathbf{T}_{2,0} + 25\mathbf{T}_{1,1})) + 25\mathbf{T}_{0,1}^2 \\
&\qquad + 10\mathbf{T}_{1,0}^2 + 20\mathbf{T}_{1,0}\mathbf{T}_{0,1} + 6\mathbf{T}_{0,4} + 2\mathbf{T}_{4,0} + 9\mathbf{T}_{1,3} + 4\mathbf{T}_{3,1} + 11\mathbf{T}_{2,2}\Bigg) r_k\Bigg],
\end{cases}
\tag{3.19}
$$

where $k \in \{1,2,3\}$.

$$
\begin{cases}
d_{kj}^{[0]} &= \beta_2, \text{ for } k=j, \quad \text{and} \quad d_{kj}^{[0]}=0, \text{ for } k\neq j, \quad \text{where} \quad k,j\in\{1,2,3\} \\
d_{kj}^{[1]} &= 0, \quad d_{kj}^{[2]} = \frac{\beta}{\alpha^2} p_j r_k, \quad d_{kj}^{[3]} = -\mathrm{i}\frac{\beta}{\alpha^3}(p_{j,x}r_k - p_j r_{k,x}), \\
d_{kj}^{[4]} &= -\frac{\beta}{\alpha^4}\Big[3\mathbf{T}_{0,0}p_j r_k + p_{j,xx}r_k - p_{j,x}r_{k,x} + p_j r_{k,xx}\Big], \\
d_{kj}^{[5]} &= \mathrm{i}\frac{\beta}{\alpha^5}\Big[2(\mathbf{T}_{1,0}-\mathbf{T}_{0,1})p_j r_k + 4\mathbf{T}_{0,0}(p_{j,x}r_k - p_j r_{k,x}) \\
&\quad + p_{j,xxx}r_k - p_j r_{k,xxx} + p_{j,x}r_{k,xx} - p_{j,xx}r_{k,x}\Big], \\
d_{kj}^{[6]} &= \frac{\beta}{\alpha^6}\Big[\Big(\mathbf{T}_{0,0}^2 + 5(\mathbf{T}_{0,2}+\mathbf{T}_{1,1}+\mathbf{T}_{2,0})\Big)p_j r_k + 5\mathbf{T}_{0,1}p_j r_{k,x} + 5\mathbf{T}_{1,0}p_{j,x}r_k \\
&\quad + 5\mathbf{T}_{0,0}(p_j r_{k,xx} - p_{j,x}r_{k,x} + p_{j,xx}r_k) \\
&\quad + p_{j,xxxx}r_k - p_{j,xxx}r_{k,x} + p_{j,xx}r_{k,xx} - p_{j,x}r_{k,xxx} + p_j r_{k,xxxx}\Big] \\
d_{kj}^{[7]} &= -\mathrm{i}\frac{\beta}{\alpha^7}\Big[\Big(\mathbf{T}_{0,0}(\mathbf{T}_{1,0}-\mathbf{T}_{0,1}) - 4(\mathbf{T}_{3,0}-\mathbf{T}_{0,3}) + (\mathbf{T}_{2,1}-\mathbf{T}_{1,2})\Big)p_j r_k \\
&\quad + \Big(15\mathbf{T}_{0,0}^2 + 8\mathbf{T}_{0,2} + 11\mathbf{T}_{2,0} + 13\mathbf{T}_{1,1}\Big)p_{j,x}r_k - \Big(15\mathbf{T}_{0,0}^2 + 11\mathbf{T}_{0,2} + 8\mathbf{T}_{2,0} + 13\mathbf{T}_{1,1}\Big)p_j r_{k,x} \\
&\quad + \Big(3\mathbf{T}_{0,1} + 9\mathbf{T}_{1,0}\Big)p_{j,xx}r_k + \Big(3\mathbf{T}_{0,1} - 3\mathbf{T}_{1,0}\Big)p_{j,x}r_{k,x} - \Big(9\mathbf{T}_{0,1} + 3\mathbf{T}_{1,0}\Big)p_j r_{k,xx} \\
&\quad + 6\mathbf{T}_{0,0}(p_{j,xxx}r_k - p_{j,xx}r_{k,x} + p_{j,x}r_{k,xx} - p_j r_{k,xxx}) \\
&\quad + p_{j,xxxxx}r_k - p_{j,xxxx}r_{k,x} + p_{j,xxx}r_{k,xx} - p_{j,xx}r_{k,xxx} + p_{j,x}r_{k,xxxx} - p_j r_{k,xxxxx}\Big],
\end{cases}
\tag{3.20}
$$

where $\beta = \beta_1 - \beta_2$ and

$$
\begin{cases}
\mathbf{T}_{0,0} = \sum_{j=1}^{3} p_j r_j,\ \mathbf{T}_{0,1} = \sum_{j=1}^{3} p_j r_{j,x},\ \mathbf{T}_{1,0} = \sum_{j=1}^{3} p_{j,x} r_j, \\
\mathbf{T}_{0,2} = \sum_{j=1}^{3} p_j r_{j,xx},\ \mathbf{T}_{2,0} = \sum_{j=1}^{3} p_{j,xx} r_j,\ \mathbf{T}_{1,1} = \sum_{j=1}^{3} p_{j,x} r_{j,x}, \\
\mathbf{T}_{0,3} = \sum_{j=1}^{3} p_j r_{j,xxx},\ \mathbf{T}_{1,2} = \sum_{j=1}^{3} p_{j,x} r_{j,xx},\ \mathbf{T}_{2,1} = \sum_{j=1}^{3} p_{j,xx} r_{j,x},\ \mathbf{T}_{3,0} = \sum_{j=1}^{3} p_{j,xxx} r_j, \\
\mathbf{T}_{0,4} = \sum_{j=1}^{3} p_j r_{j,xxxx},\ \mathbf{T}_{1,3} = \sum_{j=1}^{3} p_{j,x} r_{j,xxx},\ \mathbf{T}_{2,2} = \sum_{j=1}^{3} p_{j,xx} r_{j,xx}, \\
\mathbf{T}_{3,1} = \sum_{j=1}^{3} p_{j,xxx} r_{j,x}\ \mathbf{T}_{4,0} = \sum_{j=1}^{3} p_{j,xxxx} r_j.
\end{cases}
$$

We always assume that $b^{[m]} = (b_1^{[m]}, b_2^{[m]}, b_3^{[m]})$ and $c^{[m]} = (c_1^{[m]}, c_2^{[m]}, c_3^{[m]})^T$, for $m \in \{1,2,3,4,5,6,7\}$. To derive the sixth-order six-component AKNS integrable system, we take the Lax matrices

$$V^{[6]} = V^{[6]}(u,\lambda) = (\lambda^6 W)_+ = \sum_{m=0}^{6} W_m \lambda^{6-m}, \tag{3.21}$$

by setting the modification terms to be zero.

We begin with the spatial and temporal equations of the spectral problems, with the associated Lax pair $\{U, V\}$:

$$\varphi_x = \mathrm{i}U\varphi, \tag{3.22}$$

$$\varphi_t = \mathrm{i}V\varphi, \tag{3.23}$$

where $V = V^{[6]}$ and φ is the eigenfunction [39].

The Lax matrix operator V is determined by the compatibility condition $\varphi_{xt} = \varphi_{tx}$ which leads to the zero curvature equation:

$$U_t - V_x + \mathrm{i}[U,V] = 0, \tag{3.24}$$

which gives the six-component system of soliton equations

$$u_t = \begin{pmatrix} p^T \\ r \end{pmatrix}_t = \mathrm{i}\begin{pmatrix} \alpha b^{[7]T} \\ -\alpha c^{[7]}, \end{pmatrix} \tag{3.25}$$

where $b^{[7]}$ and $c^{[7]}$ are defined earlier, and

$$V = \begin{pmatrix} \mathbf{V}_{11} & \mathbf{V}_{12} & \mathbf{V}_{13} & \mathbf{V}_{14} \\ \mathbf{V}_{21} & \mathbf{V}_{22} & \mathbf{V}_{23} & \mathbf{V}_{24} \\ \mathbf{V}_{31} & \mathbf{V}_{32} & \mathbf{V}_{33} & \mathbf{V}_{34} \\ \mathbf{V}_{41} & \mathbf{V}_{42} & \mathbf{V}_{43} & \mathbf{V}_{44} \end{pmatrix}, \tag{3.26}$$

where

$$\mathbf{V}_{11}=a^{[0]}\lambda^6+\sum_{l=2}^{6}a^{[l]}\lambda^{6-l},\ \mathbf{V}_{12}=\sum_{l=1}^{6}b_1^{[l]}\lambda^{6-l},\ \mathbf{V}_{13}=\sum_{l=1}^{6}b_2^{[l]}\lambda^{6-l},\ \mathbf{V}_{14}=\sum_{l=1}^{6}b_3^{[l]}\lambda^{6-l}, \tag{3.27}$$

$$\mathbf{V}_{21}=\sum_{l=1}^{6}c_1^{[l]}\lambda^{6-l},\ \mathbf{V}_{22}=d_{11}^{[0]}\lambda^6+\sum_{l=2}^{6}d_{11}^{[l]}\lambda^{6-l},\ \mathbf{V}_{23}=\sum_{l=2}^{6}d_{12}^{[l]}\lambda^{6-l},\ \mathbf{V}_{24}=\sum_{l=2}^{6}d_{13}^{[l]}\lambda^{6-l}, \tag{3.28}$$

$$\mathbf{V}_{31}=\sum_{l=1}^{6}c_2^{[l]}\lambda^{6-l},\ \mathbf{V}_{32}=\sum_{l=2}^{6}d_{21}^{[l]}\lambda^{6-l},\ \mathbf{V}_{33}=d_{22}^{[0]}\lambda^6+\sum_{l=2}^{6}d_{22}^{[l]}\lambda^{6-l},\ \mathbf{V}_{34}=\sum_{l=2}^{6}d_{23}^{[l]}\lambda^{6-l}, \tag{3.29}$$

$$\mathbf{V}_{41}=\sum_{l=1}^{6}c_3^{[l]}\lambda^{6-l},\ \mathbf{V}_{42}=\sum_{l=2}^{6}d_{31}^{[l]}\lambda^{6-l},\ \mathbf{V}_{43}=\sum_{l=2}^{6}d_{32}^{[l]}\lambda^{6-l},\ \mathbf{V}_{44}=d_{33}^{[0]}\lambda^6+\sum_{l=2}^{6}d_{33}^{[l]}\lambda^{6-l}. \tag{3.30}$$

Thus, we deduce the coupled AKNS system of sixth-order equations [39]:

$$\begin{aligned}
p_{k,t}=-\mathrm{i}\frac{\beta}{\alpha^6}\Bigg[&p_{k,xxxxxx}+6(\sum_{i=1}^{3}p_ir_i)p_{k,xxxx}+(9\sum_{i=1}^{3}p_ir_{i,x}+15\sum_{i=1}^{3}p_{i,x}r_i)p_{k,xxx}\\
&+\Bigg(15(\sum_{i=1}^{3}p_ir_i)^2+11\sum_{i=1}^{3}p_ir_{i,xx}+20\sum_{i=1}^{3}p_{i,xx}r_i+25\sum_{i=1}^{3}p_{i,x}r_{i,x}\Bigg)p_{k,xx}\\
&+\Bigg((\sum_{i=1}^{3}p_ir_i)(15\sum_{i=1}^{3}p_ir_{i,x}+45\sum_{i=1}^{3}p_{i,x}r_i)+15\sum_{i=1}^{3}p_{i,xxx}r_i+4\sum_{i=1}^{3}p_ir_{i,xxx}\\
&+20\sum_{i=1}^{3}p_{i,x}r_{i,xx}+25\sum_{i=1}^{3}p_{i,xx}r_{i,x}\Bigg)p_{k,x}\\
&+\Bigg(20(\sum_{i=1}^{3}p_ir_i)^3+(\sum_{i=1}^{3}p_ir_i)(20\sum_{i=1}^{3}p_ir_{i,xx}+35\sum_{i=1}^{3}p_{i,xx}r_i+25\sum_{i=1}^{3}p_{i,x}r_{i,x})\\
&+10(\sum_{i=1}^{3}p_ir_{i,x})^2+20(\sum_{i=1}^{3}p_{i,x}r_i)(\sum_{i=1}^{3}p_ir_{i,x})+25(\sum_{i=1}^{3}p_{i,x}r_i)^2\\
&+2\sum_{i=1}^{3}p_ir_{i,xxxx}+4\sum_{i=1}^{3}p_{i,x}r_{i,xxx}+11\sum_{i=1}^{3}p_{i,xx}r_{i,xx}+9\sum_{i=1}^{3}p_{i,xxx}r_{i,x}+6\sum_{i=1}^{3}p_{i,xxxx}r_i\Bigg)p_k\Bigg],
\end{aligned}$$

$$
\begin{aligned}
r_{k,t} = \mathrm{i}\frac{\beta}{\alpha^6}\Bigg[& r_{k,xxxxxx} + 6(\sum_{i=1}^{3} p_i r_i) r_{k,xxxx} + (15\sum_{i=1}^{3} p_i r_{i,x} + 9\sum_{i=1}^{3} p_{i,x} r_i) r_{k,xxx} \\
& + \Big(15(\sum_{i=1}^{3} p_i r_i)^2 + 20\sum_{i=1}^{3} p_i r_{i,xx} + 11\sum_{i=1}^{3} p_{i,xx} r_i + 25\sum_{i=1}^{3} p_{i,x} r_{i,x}\Big) r_{k,xx} \\
& + \Big((\sum_{i=1}^{3} p_i r_i)(45\sum_{i=1}^{3} p_i r_{i,x} + 15\sum_{i=1}^{3} p_{i,x} r_i) + 4\sum_{i=1}^{3} p_{i,xxx} r_i + 15\sum_{i=1}^{3} p_i r_{i,xxx} \\
& + 25\sum_{i=1}^{3} p_{i,x} r_{i,xx} + 20\sum_{i=1}^{3} p_{i,xx} r_{i,x}\Big) r_{k,x} \\
& + \Big(20(\sum_{i=1}^{3} p_i r_i)^3 + (\sum_{i=1}^{3} p_i r_i)(35\sum_{i=1}^{3} p_i r_{i,xx} + 20\sum_{i=1}^{3} p_{i,xx} r_i + 25\sum_{i=1}^{3} p_{i,x} r_{i,x}) \\
& + 25(\sum_{i=1}^{3} p_i r_{i,x})^2 + 20(\sum_{i=1}^{3} p_{i,x} r_i)(\sum_{i=1}^{3} p_i r_{i,x}) + 10(\sum_{i=1}^{3} p_{i,x} r_i)^2 \\
& + 6\sum_{i=1}^{3} p_i r_{i,xxxx} + 9\sum_{i=1}^{3} p_{i,x} r_{i,xxx} + 11\sum_{i=1}^{3} p_{i,xx} r_{i,xx} + 4\sum_{i=1}^{3} p_{i,xxx} r_{i,x} + 2\sum_{i=1}^{3} p_{i,xxxx} r_i\Big) r_k\Bigg],
\end{aligned}
\tag{3.31}
$$

where $k \in \{1, 2, 3\}$.

3.2.2 Nonlocal reverse-time six-component AKNS system of sixth-order

We study the nonlocal reverse-time by considering specific reductions for the spectral matrix

$$U^T(x, -t, -\lambda) = -CU(x, t, \lambda)C^{-1}, \tag{3.32}$$

where $C = \begin{pmatrix} 1 & 0 \\ 0 & \Sigma \end{pmatrix}$ and Σ is a constant invertible symmetric 3×3 matrix, in other words $\det \Sigma \neq 0$ and $\Sigma^T = \Sigma$, see [38].

Because $U(x, t, \lambda) = \lambda \Lambda + P(x, t)$, for $P = \begin{pmatrix} 0 & p \\ r & 0 \end{pmatrix}$, using the reduction (3.32) we can easily prove that

$$P^T(x, -t) = -CP(x, t)C^{-1}. \tag{3.33}$$

It follows from (3.33) that

$$p^T(x,-t) = -\Sigma r(x,t) \quad \text{i.e.} \quad r(x,t) = -\Sigma^{-1} p^T(x,-t). \tag{3.34}$$

Similarly from $V(x,t,\lambda) = \lambda^6 \Omega + Q(x,t,\lambda)$ along with (3.34), one can prove with a tedious calculation that

$$Q^T(x,-t,-\lambda) = CQ(x,t,\lambda)C^{-1}, \tag{3.35}$$

and

$$V^T(x,-t,-\lambda) = CV(x,t,\lambda)C^{-1}, \tag{3.36}$$

where $\Omega = diag(\beta_1, \beta_2, \beta_2, \beta_2)$.

It is interesting that the two nonlocal Lax matrices $U^T(x,-t,-\lambda)$ and $V^T(x,-t,-\lambda)$ satisfy the equivalent zero curvature equation:

$$U_t^T(x,-t,-\lambda) + V_x^T(x,-t,-\lambda) + \mathrm{i}\big[U^T(x,-t,-\lambda), V^T(x,-t,-\lambda)\big] = 0. \tag{3.37}$$

By taking $\Sigma = diag(\rho_1^{-1}, \rho_2^{-1}, \rho_3^{-1})$, where ρ_1, ρ_2, ρ_3 are non-zero real, we deduce from (3.34) the nonlocal relation between the components of the vectors p and r, that is

$$r_i(x,t) = -\rho_i p_i(x,-t) \quad \text{for} \quad i \in \{1,2,3\}. \tag{3.38}$$

Hence, we can reduce the coupled equations (3.31) to the nonlocal reverse-time sixth-order equation:

$$
\begin{aligned}
p_{k,t}(x,t) = -\mathrm{i}\frac{\beta}{\alpha^6}\Bigg[& p_{k,xxxxxx}(x,t) \\
& - 6\Big(\sum_{i=1}^{3}\rho_i p_i(x,t)p_i(x,-t)\Big)p_{k,xxxx} \\
& - \Big(9\sum_{i=1}^{3}\rho_i p_i(x,t)p_{i,x}(x,-t) + 15\sum_{i=1}^{3}\rho_i p_{i,x}(x,t)p_i(x,-t)\Big)p_{k,xxx} \\
& + \Big(15\big(\sum_{i=1}^{3}\rho_i p_i(x,t)p_i(x,-t)\big)^2 - 11\sum_{i=1}^{3}\rho_i p_i(x,t)p_{i,xx}(x,-t) \\
& \qquad - 20\sum_{i=1}^{3}\rho_i p_{i,xx}(x,t)p_i(x,-t) - 25\sum_{i=1}^{3}\rho_i p_{i,x}(x,t)p_{i,x}(x,-t)\Big)p_{k,xx} \\
& + \Bigg(\Big(\sum_{i=1}^{3}\rho_i p_i(x,t)p_i(x,-t)\Big)\Big(15\sum_{i=1}^{3}\rho_i p_i(x,t)p_{i,x}(x,-t) + 45\sum_{i=1}^{3}\rho_i p_{i,x}(x,t)p_i(x,-t)\Big) \\
& \qquad - 15\sum_{i=1}^{3}\rho_i p_{i,xxx}(x,t)p_i(x,-t) - 4\sum_{i=1}^{3}\rho_i p_i(x,t)p_{i,xxx}(x,-t) \\
& \qquad - 20\sum_{i=1}^{3}\rho_i p_{i,x}(x,t)p_{i,xx}(x,-t) - 25\sum_{i=1}^{3}\rho_i p_{i,xx}(x,t)p_{i,x}(x,-t)\Bigg)p_{k,x} \\
& + \Bigg(-20\Big(\sum_{i=1}^{3}\rho_i p_i(x,t)p_i(x,-t)\Big)^3 + \Big(\sum_{i=1}^{3}\rho_i p_i(x,t)p_i(x,-t)\Big)\Big(20\sum_{i=1}^{3}\rho_i p_i(x,t)p_{i,xx}(x,-t) \\
& + 35\sum_{i=1}^{3}\rho_i p_{i,xx}(x,t)p_i(x,-t) + 25\sum_{i=1}^{3}\rho_i p_{i,x}(x,t)p_{i,x}(x,-t)\Big) \\
& + 10\big(\sum_{i=1}^{3}\rho_i p_i(x,t)p_{i,x}(x,-t)\big)^2 + 20\Big(\sum_{i=1}^{3}\rho_i p_{i,x}(x,t)p_i(x,-t)\Big)\Big(\sum_{i=1}^{3}\rho_i p_i(x,t)p_{i,x}(x,-t)\Big) \\
& + 25\big(\sum_{i=1}^{3}\rho_i p_{i,x}(x,t)p_i(x,-t)\big)^2 - 2\sum_{i=1}^{3}\rho_i p_i(x,t)p_{i,xxxx}(x,-t) \\
& - 4\sum_{i=1}^{3}\rho_i p_{i,x}(x,t)p_{i,xxx}(x,-t) - 11\sum_{i=1}^{3}\rho_i p_{i,xx}(x,t)p_{i,xx}(x,-t) \\
& - 9\sum_{i=1}^{3}\rho_i p_{i,xxx}(x,t)p_{i,x}(x,-t) - 6\sum_{i=1}^{3}\rho_i p_{i,xxxx}(x,t)p_i(x,-t)\Bigg)p_k\Bigg]
\end{aligned}
$$

for $k \in \{1,2,3\}$.

We can see that when all $\rho_i < 0$ for $i \in \{1,2,3\}$, the dispersive term and the nonlinear terms attract. Hence, we obtain the focusing nonlocal reverse-time six-component sixth-order equation. Otherwise, if ρ_i's are not all negative for $i \in \{1,2,3\}$, then we have combined focussing and defocussing cases.

3.3 Riemann-Hilbert problems

The spatial and temporal spectral problem of the six-component sixth-order AKNS equations can be written:

$$\varphi_x = iU\varphi = i(\lambda\Lambda + P)\varphi, \tag{3.39}$$

$$\varphi_t = iV\varphi = i(\lambda^6\Omega + Q)\varphi, \tag{3.40}$$

where $\Omega = diag(\beta_1, \beta_2, \beta_2, \beta_2)$, and

$$Q = \begin{pmatrix} \mathbf{Q}_{11} & \mathbf{Q}_{12} & \mathbf{Q}_{13} & \mathbf{Q}_{14} \\ \mathbf{Q}_{21} & \mathbf{Q}_{22} & \mathbf{Q}_{23} & \mathbf{Q}_{24} \\ \mathbf{Q}_{31} & \mathbf{Q}_{32} & \mathbf{Q}_{33} & \mathbf{Q}_{34} \\ \mathbf{Q}_{41} & \mathbf{Q}_{42} & \mathbf{Q}_{43} & \mathbf{Q}_{44} \end{pmatrix}, \tag{3.41}$$

with components

$$\mathbf{Q}_{11} = \sum_{l=2}^{6} a^{[l]}\lambda^{6-l},\ \mathbf{Q}_{12} = \sum_{l=1}^{6} b_1^{[l]}\lambda^{6-l},\ \mathbf{Q}_{13} = \sum_{l=1}^{6} b_2^{[l]}\lambda^{6-l},\ \mathbf{Q}_{14} = \sum_{l=1}^{6} b_3^{[l]}\lambda^{6-l}, \tag{3.42}$$

$$\mathbf{Q}_{21} = \sum_{l=1}^{6} c_1^{[l]}\lambda^{6-l},\ \mathbf{Q}_{22} = \sum_{l=2}^{6} d_{11}^{[l]}\lambda^{6-l},\ \mathbf{Q}_{23} = \sum_{l=2}^{6} d_{12}^{[l]}\lambda^{6-l},\ \mathbf{Q}_{24} = \sum_{l=2}^{6} d_{13}^{[l]}\lambda^{6-l}, \tag{3.43}$$

$$\mathbf{Q}_{31} = \sum_{l=1}^{6} c_2^{[l]}\lambda^{6-l},\ \mathbf{Q}_{32} = \sum_{l=2}^{6} d_{21}^{[l]}\lambda^{6-l},\ \mathbf{Q}_{33} = \sum_{l=2}^{6} d_{22}^{[l]}\lambda^{6-l},\ \mathbf{Q}_{34} = \sum_{l=2}^{6} d_{23}^{[l]}\lambda^{6-l}, \tag{3.44}$$

$$\mathbf{Q}_{41} = \sum_{l=1}^{6} c_3^{[l]}\lambda^{6-l},\ \mathbf{Q}_{42} = \sum_{l=2}^{6} d_{31}^{[l]}\lambda^{6-l},\ \mathbf{Q}_{43} = \sum_{l=2}^{6} d_{32}^{[l]}\lambda^{6-l},\ \mathbf{Q}_{44} = \sum_{l=2}^{6} d_{33}^{[l]}\lambda^{6-l}. \tag{3.45}$$

Throughout the dissertation, we assume that $\alpha = \alpha_1 - \alpha_2 < 0$ and $\beta = \beta_1 - \beta_2 < 0$.

To find soliton solutions we begin with an initial condition $(p(x,0), r^T(x,0))^T$ and evolute in time to reach $(p(x,t), r^T(x,t))^T$. Taking p_j and r_j in Schwartz space, they will decay exponentially, and so, $p_j \to 0$ and $r_j \to 0$ as $x, t \to \pm\infty$ for $j \in \{1,2,3\}$. Therefore from the spectral problems (3.39) and (3.40), the asymptotic behaviour of the fundamental matrix φ can be written as

$$\varphi(x,t) \rightsquigarrow e^{i\lambda\Lambda x+i\lambda^6\Omega t}. \tag{3.46}$$

Hence, the solution of the spectral problems can be written in the form:

$$\varphi(x,t) = \psi(x,t)e^{i\lambda\Lambda x+i\lambda^6\Omega t}. \tag{3.47}$$

The Jost solution of the eigenfunction (3.47) requires that [20, 45]

$$\psi(x,t) \to I_4, \quad \text{as} \quad x,t \to \pm\infty, \tag{3.48}$$

where I_4 is the 4×4 identity matrix. The Lax pair (3.39) and (3.40) can be rewritten in terms of ψ using equation (3.47), which gives the equivalent expression of the spectral problems

$$\psi_x = i\lambda[\Lambda, \psi] + iP\psi, \tag{3.49}$$

$$\psi_t = i\lambda^6[\Omega, \psi] + iQ\psi. \tag{3.50}$$

To construct the Riemann-Hilbert problems and their solutions in the reflectionless case, we are going to use the adjoint spectral problems of $\varphi_x = iU\varphi$ and $\varphi_t = iV\varphi$. Their adjoints are

$$\tilde{\varphi}_x = -i\tilde{\varphi}U, \tag{3.51}$$

$$\tilde{\varphi}_t = -i\tilde{\varphi}V, \tag{3.52}$$

and the equivalent adjoint spectral problems read

$$\tilde{\psi}_x = -i\lambda[\tilde{\psi}, \Lambda] - i\tilde{\psi}P, \tag{3.53}$$

$$\tilde{\psi}_t = -i\lambda^6[\tilde{\psi}, \Omega] - i\tilde{\psi}Q. \tag{3.54}$$

Because $tr(iP) = 0$ and $tr(iQ) = 0$, using Liouvilles's formula [45], it is easy to see that the $(det(\psi))_x = 0$, that is, $det(\psi)$ is a constant, and utilizing the boundary condition (3.48), we conclude

$$det(\psi) = 1, \tag{3.55}$$

and hence the Jost matrix ψ is invertible.

Furthermore, as $\psi_x^{-1} = -\psi^{-1}\psi_x\psi^{-1}$, we can derive from (3.49),

$$\psi_x^{-1} = -i\lambda[\psi^{-1}, \Lambda] - i\psi^{-1}P. \tag{3.56}$$

We can also show that both satisfies the temporal adjoint equation (3.54) as well.

Notice that if the eigenfunction $\psi(x, t, \lambda)$ is a solution of the spectral problem (3.49), then $\psi^{-1}(x, t, \lambda)$ is a solution of the adjoint spectral problem (3.53), implying that $C\psi^{-1}(x, t, \lambda)$ is also a solution of (3.53) with the same eigenvalue because $\psi_x^{-1} = -\psi^{-1}\psi_x\psi^{-1}$. In a similar way, the nonlocal $\psi^T(x, -t, -\lambda)C$ is also a solution of the spectral adjoint problem (3.53). Since the boundary condition is the same for both solutions as $x \to \pm\infty$, this guarantees the uniqueness of the solution, and so

$$\psi^T(x, -t, -\lambda) = C\psi^{-1}(x, t, \lambda)C^{-1}. \tag{3.57}$$

As a result, if λ is an eigenvalue of the spectral problems, then $-\lambda$ is also an eigenvalue and the relation (3.57) holds.

Now, we are going to work with the spatial spectral problem (3.49), assuming that the time is $t = 0$.

For the notation simplicity, we denote Y^+ and Y^- to indicate the boundary conditions are set as $x \to \infty$ and $x \to -\infty$, respectively.

We know that

$$\psi^\pm \to I_4 \quad \text{when} \quad x \to \pm\infty. \tag{3.58}$$

From (3.47), this allows us to write

$$\varphi^\pm = \psi^\pm e^{i\lambda\Lambda x}. \tag{3.59}$$

Both φ^+ and φ^- satisfy the spectral spatial differential equation (3.39), i.e., both are two solutions of that equation. Also, note that both $(\psi^{+})^{-1}$ and $(\psi^{-})^{-1}$ satisfies the spatial adjoint equation (3.53). Thus, they are linearly dependent, hence there exists a scattering matrix $S(\lambda)$, such that

$$\varphi^{-} = \varphi^{+} S(\lambda). \tag{3.60}$$

Substituting (3.59) into (3.60) leads to

$$\psi^{-} = \psi^{+} e^{i\lambda\Lambda x} S(\lambda) e^{-i\lambda\Lambda x}, \quad \text{for} \quad \lambda \in \mathbb{R}, \tag{3.61}$$

where

$$S(\lambda) = (s_{jk})_{4\times 4} = \begin{pmatrix} s_{11} & s_{12} & s_{13} & s_{14} \\ s_{21} & s_{22} & s_{23} & s_{24} \\ s_{31} & s_{32} & s_{33} & s_{34} \\ s_{41} & s_{42} & s_{43} & s_{44} \end{pmatrix}. \tag{3.62}$$

Given that $det(\psi^{\pm}) = 1$, we obtain

$$det(S(\lambda)) = 1. \tag{3.63}$$

In addition, we can show from (3.61) and (3.57) that $S(\lambda)$ possess the involution relation

$$S^{T}(-\lambda) = CS^{-1}(\lambda)C^{-1}. \tag{3.64}$$

We deduce from (3.64) that

$$\hat{s}_{11}(\lambda) = s_{11}(-\lambda), \tag{3.65}$$

where the inverse scattering data matrix $S^{-1} = (\hat{s}_{jk})_{4\times 4}$ for $j, k \in \{1, 2, 3, 4\}$.

From $\psi^{-} = \psi^{+} e^{i\lambda\Lambda x} S(\lambda) e^{-i\lambda\Lambda x}$, $\psi^{\pm} \to I_4$ when $x \to \pm\infty$. In order to formulate Riemann-Hilbert problems we need to analyse the analyticity of the Jost matrix $\psi^{\pm}$.

To do so, we can use the Volterra integral equations to write the solutions $\psi^{\pm}$ in a uniquely manner by using the spatial spectral problem (3.39):

$$\psi^-(x,\lambda) = I_4 + i \int\limits_{-\infty}^{x} e^{i\lambda\Lambda(x-y)} P(y)\psi^-(y,\lambda) e^{i\lambda\Lambda(y-x)} dy, \tag{3.66}$$

$$\psi^+(x,\lambda) = I_4 - i \int\limits_{x}^{+\infty} e^{i\lambda\Lambda(x-y)} P(y)\psi^+(y,\lambda) e^{i\lambda\Lambda(y-x)} dy. \tag{3.67}$$

We denote the matrix ψ^- to be

$$\psi^- = \begin{pmatrix} \psi^-_{11} & \psi^-_{12} & \psi^-_{13} & \psi^-_{14} \\ \psi^-_{21} & \psi^-_{22} & \psi^-_{23} & \psi^-_{24} \\ \psi^-_{31} & \psi^-_{32} & \psi^-_{33} & \psi^-_{34} \\ \psi^-_{41} & \psi^-_{42} & \psi^-_{43} & \psi^-_{44} \end{pmatrix}. \tag{3.68}$$

and ψ^+ is denoted similarly. So from (3.66) the components of the first column of ψ^- are

$$\psi^-_{11} = 1 + i \int_{-\infty}^{x} (p_1(y)\psi^-_{21}(y,\lambda) + p_2(y)\psi^-_{31}(y,\lambda) + p_3(y)\psi^-_{41}(y,\lambda)) dy, \tag{3.69}$$

$$\psi^-_{21} = i \int_{-\infty}^{x} r_1(y)\psi^-_{11}(y,\lambda) e^{-i\lambda\alpha(x-y)} dy, \tag{3.70}$$

$$\psi^-_{31} = i \int_{-\infty}^{x} r_2(y)\psi^-_{11}(y,\lambda) e^{-i\lambda\alpha(x-y)} dy, \tag{3.71}$$

$$\psi^-_{41} = i \int_{-\infty}^{x} r_3(y)\psi^-_{11}(y,\lambda) e^{-i\lambda\alpha(x-y)} dy. \tag{3.72}$$

Similarly, the components of the second column of ψ^- are

$$\psi^-_{12} = i \int_{-\infty}^{x} \Big(p_1(y)\psi^-_{22}(y,\lambda) + p_2(y)\psi^-_{32}(y,\lambda) + p_3(y)\psi^-_{42}(y,\lambda) \Big) e^{i\lambda\alpha(x-y)} dy, \tag{3.73}$$

$$\psi^-_{22} = 1 + i \int_{-\infty}^{x} r_1(y)\psi^-_{12}(y,\lambda) dy, \tag{3.74}$$

$$\psi^-_{32} = i \int_{-\infty}^{x} r_2(y)\psi^-_{12}(y,\lambda) dy, \tag{3.75}$$

$$\psi^-_{42} = i \int_{-\infty}^{x} r_3(y)\psi^-_{12}(y,\lambda) dy, \tag{3.76}$$

and the components of the third column of ψ^- are

$$\psi^-_{13} = i\int_{-\infty}^{x}\Big(p_1(y)\psi^-_{23}(y,\lambda)+p_2(y)\psi^-_{33}(y,\lambda)+p_3(y)\psi^-_{43}(y,\lambda)\Big)e^{i\lambda\alpha(x-y)}dy, \tag{3.77}$$

$$\psi^-_{23} = i\int_{-\infty}^{x} r_1(y)\psi^-_{13}(y,\lambda)dy, \tag{3.78}$$

$$\psi^-_{33} = 1+i\int_{-\infty}^{x} r_2(y)\psi^-_{13}(y,\lambda)dy, \tag{3.79}$$

$$\psi^-_{43} = i\int_{-\infty}^{x} r_3(y)\psi^-_{13}(y,\lambda)dy, \tag{3.80}$$

and finally the components of the fourth column of ψ^- are

$$\psi^-_{14} = i\int_{-\infty}^{x}\Big(p_1(y)\psi^-_{24}(y,\lambda)+p_2(y)\psi^-_{34}(y,\lambda)+p_3(y)\psi^-_{44}(y,\lambda)\Big)e^{i\lambda\alpha(x-y)}dy, \tag{3.81}$$

$$\psi^-_{24} = i\int_{-\infty}^{x} r_1(y)\psi^-_{14}(y,\lambda)dy, \tag{3.82}$$

$$\psi^-_{34} = i\int_{-\infty}^{x} r_2(y)\psi^-_{14}(y,\lambda)dy, \tag{3.83}$$

$$\psi^-_{44} = 1+i\int_{-\infty}^{x} r_3(y)\psi^-_{14}(y,\lambda)dy. \tag{3.84}$$

Recall that $\alpha < 0$. If $Im(\lambda) > 0$ and $y < x$ then, $Re(e^{-i\lambda\alpha(x-y)})$ decays exponentially and so each integral of the first column of ψ^- converges. As a result, the components of the first column of ψ^-, are analytic in the upper half complex plane for $\lambda \in \mathbb{C}_+$, and continuous for $\lambda \in \mathbb{C}_+ \cup \mathbb{R}$.

In the same way, for $y > x$, the components of the last three columns of ψ^+ are analytic in the upper half plane for $\lambda \in \mathbb{C}_+$ and continuous for $\lambda \in \mathbb{C}_+ \cup \mathbb{R}$.

It is worth mentioning the case when $Im(\lambda) < 0$, then the first column of ψ^+ is analytic in the lower half plane for $\lambda \in \mathbb{C}_-$ and continuous for $\lambda \in \mathbb{C}_- \cup \mathbb{R}$, and the components of the last three columns of ψ^- are analytic in the lower half plane for $\lambda \in \mathbb{C}_-$ and continuous for $\lambda \in \mathbb{C}_- \cup \mathbb{R}$.

Now, let us construct the associated Riemann-Hilbert problems. To construct the analytic matrix P^+ in the upper-half plane, we note that

$$\psi^{\pm} = \varphi^{\pm}e^{-i\lambda\Lambda x}. \tag{3.85}$$

Let $\psi_j^\pm$ be the jth column of $\psi^\pm$ for $j \in \{1,2,3,4\}$, hence the first Jost matrix solution can be taken as

$$P^+(x,\lambda) = (\psi_1^-, \psi_2^+, \psi_3^+, \psi_4^+) = \psi^- H_1 + \psi^+ H_2, \tag{3.86}$$

where $H_1 = diag(1,0,0,0)$ and $H_2 = diag(0,1,1,1)$.

Therefore, P^+ is then analytic for $\lambda \in \mathbb{C}_+$ and continuous for $\lambda \in \mathbb{C}_+ \cup \mathbb{R}$.

For the lower-half plane, we can construct $P^- \in \mathbb{C}_-$ which is the analytic counterpart of $P^+ \in \mathbb{C}_+$. To do so, we utilize the equivalent spectral adjoint equation (3.56). Because $\tilde{\psi}^\pm = (\psi^\pm)^{-1}$ and $\varphi^\pm = \psi^\pm e^{i\lambda\Lambda x}$, we have

$$(\psi^\pm)^{-1} = e^{i\lambda\Lambda x}(\varphi^\pm)^{-1}. \tag{3.87}$$

Let $\tilde{\psi}_j^\pm$ be the jth row of $\tilde{\psi}^\pm$ for $j \in \{1,2,3,4\}$. As above, we can get

$$P^-(x,\lambda) = \left(\tilde{\psi}_1^-, \tilde{\psi}_2^+, \tilde{\psi}_3^+, \tilde{\psi}_4^+\right)^T = H_1(\psi^-)^{-1} + H_2(\psi^+)^{-1}. \tag{3.88}$$

Hence, P^- is analytic for $\lambda \in \mathbb{C}_-$ and continuous for $\lambda \in \mathbb{C}_- \cup \mathbb{R}$.

Since both ψ^- and ψ^+ satisfy

$$\psi^T(x,-t,-\lambda) = C\psi^{-1}(x,t,\lambda)C^{-1}, \tag{3.89}$$

using (3.86), we have

$$P^+(x,-t,-\lambda) = \psi^-(x,-t,-\lambda)H_1 + \psi^+(x,-t,-\lambda)H_2, \tag{3.90}$$

or equivalently

$$(P^+)^T(x,-t,-\lambda) = H_1^T(\psi^-)^T(x,-t,-\lambda) + H_2^T(\psi^+)^T(x,-t,-\lambda). \tag{3.91}$$

Substituting (3.89) in (3.91), we have the nonlocal involution property

$$(P^+)^T(x,-t,-\lambda) = CP^-(x,t,\lambda)C^{-1}. \tag{3.92}$$

Employing the analyticity of both P^+ and P^-, we can construct the Riemann-Hilbert problems

$$P^-P^+ = J, \tag{3.93}$$

where $J = e^{i\lambda\Lambda x}(H_1 + H_2 S)(H_1 + S^{-1}H_2)e^{-i\lambda\Lambda x}$ for $\lambda \in \mathbb{R}$.

Replacing (3.61) in (3.86), we have

$$P^+(x,\lambda) = \psi^+(e^{i\lambda\Lambda x}Se^{-i\lambda\Lambda x}H_1 + H_2). \tag{3.94}$$

Because $\psi^+(x,\lambda) \to I_4$ when $x \to +\infty$, we get

$$\lim_{x\to+\infty} P^+ = \begin{pmatrix} s_{11}(\lambda) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{for} \quad \lambda \in \mathbb{C}_+ \cup \mathbb{R}. \tag{3.95}$$

In the same way,

$$\lim_{x\to-\infty} P^- = \begin{pmatrix} \hat{s}_{11}(\lambda) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{for} \quad \lambda \in \mathbb{C}_- \cup \mathbb{R}. \tag{3.96}$$

Thus if we choose

$$G^+(x,\lambda) = P^+(x,\lambda)\begin{pmatrix} s_{11}^{-1}(\lambda) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad (G^-)^{-1}(x,\lambda) = \begin{pmatrix} \hat{s}_{11}^{-1}(\lambda) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} P^-(x,\lambda)\ , \tag{3.97}$$

the two generalized matrices $G^+(x,\lambda)$ and $G^-(x,\lambda)$ generate the matrix Riemann-Hilbert problems on the real line for the six-component AKNS system of sixth-order given by

$$G^+(x,\lambda) = G^-(x,\lambda)G_0(x,\lambda), \quad \text{for} \quad \lambda \in \mathbb{R}, \tag{3.98}$$

where the jump matrix $G_0(x,\lambda)$ can be cast as

$$G_0(x,\lambda) = \begin{pmatrix} \hat{s}_{11}^{-1}(\lambda) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} J \begin{pmatrix} s_{11}^{-1}(\lambda) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.99}$$

This reads

$$G_0(x,\lambda) = \begin{pmatrix} s_{11}^{-1}\hat{s}_{11}^{-1} & \hat{s}_{12}\hat{s}_{11}^{-1}e^{i\lambda\alpha x} & \hat{s}_{13}\hat{s}_{11}^{-1}e^{i\lambda\alpha x} & \hat{s}_{14}\hat{s}_{11}^{-1}e^{i\lambda\alpha x} \\ s_{21}s_{11}^{-1}e^{-i\lambda\alpha x} & 1 & 0 & 0 \\ s_{31}s_{11}^{-1}e^{-i\lambda\alpha x} & 0 & 1 & 0 \\ s_{41}s_{11}^{-1}e^{-i\lambda\alpha x} & 0 & 0 & 1 \end{pmatrix}, \tag{3.100}$$

and its canonical normalization conditions are:

$$G^+(x,\lambda) \to I_4 \quad \text{as} \quad \lambda \in \mathbb{C}_+ \cup \mathbb{R} \to \infty, \tag{3.101}$$

$$G^-(x,\lambda) \to I_4 \quad \text{as} \quad \lambda \in \mathbb{C}_- \cup \mathbb{R} \to \infty. \tag{3.102}$$

From (3.92) along with (3.97) and using (3.65), we deduce the nonlocal involution property

$$(G^+)^T(x,-t,-\lambda) = C(G^-)^{-1}(x,t,\lambda)C^{-1}. \tag{3.103}$$

Furthermore, we derive from (3.99) and (3.65), the following nonlocal involution property for G_0:

$$G_0^T(x,-t,-\lambda) = CG_0(x,t,\lambda)C^{-1}. \tag{3.104}$$

3.3.1 Time evolution of the scattering data

Reaching this point, we need to determine the scattering data as they evolute in time. In order to do that , we differentiate equation (3.61) with respect to time t and applying (3.50) gives

$$S_t = i\lambda^6[\Omega, S], \tag{3.105}$$

and thus

$$S_t = \begin{pmatrix} 0 & i\beta\lambda^6 s_{12} & i\beta\lambda^6 s_{13} & i\beta\lambda^6 s_{14} \\ -i\beta\lambda^6 s_{21} & 0 & 0 & 0 \\ -i\beta\lambda^6 s_{31} & 0 & 0 & 0 \\ -i\beta\lambda^6 s_{41} & 0 & 0 & 0 \end{pmatrix}. \tag{3.106}$$

As a result, we have

$$\begin{cases} s_{12}(t,\lambda) = s_{12}(0,\lambda)e^{i\beta\lambda^6 t}, \\ s_{13}(t,\lambda) = s_{13}(0,\lambda)e^{i\beta\lambda^6 t}, \\ s_{14}(t,\lambda) = s_{14}(0,\lambda)e^{i\beta\lambda^6 t}, \\ s_{21}(t,\lambda) = s_{21}(0,\lambda)e^{-i\beta\lambda^6 t}, \\ s_{31}(t,\lambda) = s_{31}(0,\lambda)e^{-i\beta\lambda^6 t}, \\ s_{41}(t,\lambda) = s_{41}(0,\lambda)e^{-i\beta\lambda^6 t}, \end{cases} \tag{3.107}$$

and $s_{11}, s_{22}, s_{23}, s_{24}, s_{32}, s_{33}, s_{34}, s_{42}, s_{43}, s_{44}$ are constants.

3.4 Soliton solutions

3.4.1 General case

The determinant of the matrix $G^{\pm}$ determines the type of soliton solutions generated using the Riemann-Hilbert problems. In the regular case, when $det(G^{\pm}) \neq 0$, we obtain the unique solution. In the non-regular case, that is to say when $det(G^{\pm}) = 0$, it could generate discrete eigenvalues in the spectral plane. This non-regular case can be transformed into the regular case to solve for soliton solutions [45].

From (3.94) and $det(\psi^{\pm}) = 1$, we can show that

$$det(P^{+}(x,\lambda)) = s_{11}(\lambda), \tag{3.108}$$

and in the same way,

$$det(P^{-}(x,\lambda)) = \hat{s}_{11}(\lambda). \tag{3.109}$$

Because $det(S(\lambda)) = 1$, this implies that $S^{-1}(\lambda) = \Big(cof(S(\lambda))\Big)^{T}$ and

$$\hat{s}_{11} = \begin{vmatrix} s_{22} & s_{23} & s_{24} \\ s_{32} & s_{33} & s_{34} \\ s_{42} & s_{43} & s_{44} \end{vmatrix}, \tag{3.110}$$

which should be zero for the non-regular case.

To give rise to soliton solutions, we need the solutions of $det(P^{+}(x,\lambda)) = det(P^{-}(x,\lambda)) = 0$ to be simple. When $det(P^{+}(x,\lambda)) = s_{11}(\lambda) = 0$, we assume $s_{11}(\lambda)$ has simple zeros producing discrete eigenvalues $\lambda_k \in \mathbb{C}_+$ for $k \in \{1, 2, ..., N\}$, while for $det(P^{-}(x,\lambda)) = \hat{s}_{11}(\lambda) = 0$, we assume $\hat{s}_{11}(\lambda)$ has simple zeros producing discrete eigenvalues $\hat{\lambda}_k \in \mathbb{C}_-$ for $k \in \{1, 2, ..., N\}$, which are the poles of the transmission coefficients [20].

From $\hat{s}_{11}(\lambda) = s_{11}(-\lambda)$ and $det(P^{\pm}(x,\lambda)) = 0$, we have the nonlocal involution relation

$$\hat{\lambda} = -\lambda. \tag{3.111}$$

Each $Ker(P^{+}(x,\lambda_k))$ contains only a single column vector v_k, and similarly each $Ker(P^{-}(x,\hat{\lambda}_k))$ contains only a single row vector $\hat{v}_k$ such that:

$$P^{+}(x,\lambda_k)v_k = 0 \quad \text{for} \quad k \in \{1, 2, ..., N\}, \tag{3.112}$$

and

$$\hat{v}_k P^{-}(x,\hat{\lambda}_k) = 0 \quad \text{for} \quad k \in \{1, 2, ..., N\}. \tag{3.113}$$

To obtain explicit soliton solutions, we take $G_0 = I_4$ in the Riemann-Hilbert problems. This will force the reflection coefficients $s_{21} = s_{31} = s_{41} = 0$ and $\hat{s}_{12} = \hat{s}_{13} = \hat{s}_{14} = 0$.

In that case, the Riemann-Hilbert problems can be presented as follows [26]:

$$G^+(x, \lambda) = I_4 - \sum_{k,j=1}^{N} \frac{v_k (M^{-1})_{kj} \hat{v}_j}{\lambda - \hat{\lambda}_j}, \tag{3.114}$$

and

$$(G^-)^{-1}(x, \lambda) = I_4 + \sum_{k,j=1}^{N} \frac{v_k (M^{-1})_{kj} \hat{v}_j}{\lambda - \lambda_k}, \tag{3.115}$$

where $M = (m_{kj})_{N\times N}$ is a matrix defined by

$$m_{kj} = \begin{cases} \frac{\hat{v}_k v_j}{\lambda_j - \hat{\lambda}_k}, & \text{if} \quad \lambda_j \neq \hat{\lambda}_k, \\ 0, & \text{if} \quad \lambda_j = \hat{\lambda}_k, \end{cases} \qquad k, j \in \{1, 2, ..., N\}. \tag{3.116}$$

Since the zeros λ_k and $\hat{\lambda}_k$ are constants, they are independent of space and time. We can explore the spatial and temporal evolution of the scattering vectors $v_k(x, t)$ and $\hat{v}_k(x, t)$ for $1 \leq k \leq N$.

Taking the x-derivative of both sides of the equation

$$P^+(x, \lambda_k) v_k = 0, \quad 1 \leq k \leq N, \tag{3.117}$$

and knowing that P^+ satisfies the spectral spatial equivalent equation (3.49), along with (3.112), we obtain

$$P^+(x, \lambda_k) \left(\frac{dv_k}{dx} - i\lambda_k \Lambda v_k \right) = 0 \quad \text{for} \quad k, j \in \{1, 2, ..., N\}. \tag{3.118}$$

In a similar manner, taking the t-derivative and using the temporal equation (3.50) and (3.112), we acquire

$$P^+(x, \lambda_k) \left(\frac{dv_k}{dt} - i\lambda_k^6 \Omega v_k \right) = 0 \quad \text{for} \quad k, j \in \{1, 2, ..., N\}. \tag{3.119}$$

For the adjoint spectral equations (3.53) and (3.54) , we can obtain the following similar results

$$\left(\frac{d\hat{v}_k}{dx} + i\hat{\lambda}_k \hat{v}_k \mathit{\Lambda}\right) P^{-}(x, \hat{\lambda}_k) = 0, \tag{3.120}$$

and

$$\left(\frac{d\hat{v}_k}{dt} + i\hat{\lambda}_k^6 \hat{v}_k \mathit{\Omega}\right) P^{-}(x, \hat{\lambda}_k) = 0. \tag{3.121}$$

Because v_k is a single vector in the kernel of P^{+}, so $\frac{dv_k}{dx} - i\lambda_k \mathit{\Lambda} v_k$ and $\frac{dv_k}{dt} - i\lambda_k^6 \mathit{\Omega} v_k$ are scalar multiples of v_k.

Hence without loss of generality, we can take the space dependence of v_k to be:

$$\frac{dv_k}{dx} = i\lambda_k \mathit{\Lambda} v_k, \quad 1 \le k \le N \tag{3.122}$$

and the time dependence of v_k as:

$$\frac{dv_k}{dt} = i\lambda_k^6 \mathit{\Omega} v_k, \quad 1 \le k \le N. \tag{3.123}$$

In this way, we can conclude that

$$v_k(x,t) = e^{i\lambda_k \mathit{\Lambda} x + i\lambda_k^6 \mathit{\Omega} t} w_k \quad \text{for} \quad k \in \{1, 2, ..., N\}, \tag{3.124}$$

by solving equations (3.122) and (3.123), where w_k is a constant column vector. Likewise, we get

$$\hat{v}_k(x,t) = \hat{w}_k e^{-i\hat{\lambda}_k \mathit{\Lambda} x - i\hat{\lambda}_k^6 \mathit{\Omega} t} \quad \text{for} \quad k \in \{1, 2, ..., N\}, \tag{3.125}$$

where $\hat{w}_k$ is a constant row vector.

From (3.112) and using the formula (3.92), it is easy to see

$$v_k^T(x, -t, \lambda_k)(P^{+})^T(x, -t, \lambda_k) = v_k^T(x, -t, \lambda_k) C P^{-}(x, t, -\lambda_k) C^{-1} = 0. \tag{3.126}$$

Because $v_k^T(x,-t,\lambda_k)CP^-(x,t,-\lambda_k)$ can be zero and using (3.113), this leads to

$$v_k^T(x,-t,\lambda_k)CP^-(x,t,\hat{\lambda}_k) = \hat{v}_k(x,t,\hat{\lambda}_k)P^-(x,t,\hat{\lambda}_k) \tag{3.127}$$

$$= \hat{v}_k(x,t,-\lambda_k)P^-(x,t,-\lambda_k) = 0. \tag{3.128}$$

From (3.111), we have $\hat{\lambda}_k = -\lambda_k$ for $k \in \{1,2,...,N\}$, and we can take

$$\hat{v}_k(x,t,\hat{\lambda}_k) = v_k^T(x,-t,\lambda_k)C. \tag{3.129}$$

Thus, the involution relations (3.124) and (3.125) give

$$v_k(x,t) = e^{i\lambda_k \Lambda x + i\lambda_k^6 \Omega t} w_k, \tag{3.130}$$

$$\hat{v}_k(x,t) = w_k^T e^{-i\hat{\lambda}_k \Lambda x - i\hat{\lambda}_k^6 \Omega t} C. \tag{3.131}$$

Because the jump matrix $G_0 = I_4$, we can solve the Riemann-Hilbert problem precisely. As a result, we can determine the potentials by computing the matrix P^+. Because P^+ is analytic, we can expand G^+ as follows:

$$G^+(x,\lambda) = I_4 + \frac{1}{\lambda}G_1^+(x) + O(\frac{1}{\lambda^2}), \quad \text{when} \quad \lambda \to \infty. \tag{3.132}$$

Because G^+ satisfies the spectral problem, substituting it in (3.49) and matching the coefficients of the same power of $\frac{1}{\lambda}$, at order $O(1)$, we get

$$P = -[\Lambda, G_1^+]. \tag{3.133}$$

If we denote

$$G_1^+ = \begin{pmatrix} (G_1^+)_{11} & (G_1^+)_{12} & (G_1^+)_{13} & (G_1^+)_{14} \\ (G_1^+)_{21} & (G_1^+)_{22} & (G_1^+)_{23} & (G_1^+)_{24} \\ (G_1^+)_{31} & (G_1^+)_{32} & (G_1^+)_{33} & (G_1^+)_{34} \\ (G_1^+)_{41} & (G_1^+)_{42} & (G_1^+)_{43} & (G_1^+)_{44} \end{pmatrix}, \tag{3.134}$$

then

$$P=-[\Lambda,G_1^+]=\begin{pmatrix} 0 & -\alpha(G_1^+)_{12} & -\alpha(G_1^+)_{13} & -\alpha(G_1^+)_{14} \\ \alpha(G_1^+)_{21} & 0 & 0 & 0 \\ \alpha(G_1^+)_{31} & 0 & 0 & 0 \\ \alpha(G_1^+)_{41} & 0 & 0 & 0 \end{pmatrix}. \tag{3.135}$$

Consequently, we can recover the potentials p_j and r_j for $j \in \{1,2,3\}$:

$$\begin{aligned} p_1 &= -\alpha(G_1^+)_{12}, \quad r_1 = \alpha(G_1^+)_{21}, \\ p_2 &= -\alpha(G_1^+)_{13}, \quad r_2 = \alpha(G_1^+)_{31}, \\ p_3 &= -\alpha(G_1^+)_{14}, \quad r_3 = \alpha(G_1^+)_{41}. \end{aligned} \tag{3.136}$$

It can be seen from (3.132) that

$$G_1^+ = \lambda \lim_{\lambda\to\infty}(G^+(x,\lambda)-I_4), \tag{3.137}$$

and then using equation (3.114), we deduce

$$G_1^+ = -\sum_{k,j=1}^{N} v_k(M^{-1})_{k,j}\hat{v}_j. \tag{3.138}$$

In addition, by the use of equations (3.33) and (3.133), we can easily prove the following nonlocal involution property

$$(G_1^+)^T(x,-t) = CG_1^+(x,t)C^{-1}. \tag{3.139}$$

By substituting (3.138) into (3.136) and using (3.130) and (3.131), we generate the N-soliton solution to the nonlocal reverse-time six-component AKNS system of sixth-order

$$p_i = \alpha\sum_{k,j=1}^{N} v_{k,1}(M^{-1})_{kj}\hat{v}_{j,i+1} \quad \text{for} \quad i\in\{1,2,3\}, \tag{3.140}$$

where w_k is an arbitrary constant column vector in $\mathbb{C}^4$, and

$$v_k = (v_{k,1}, v_{k,2}, v_{k,3}, ..., v_{k,n+1})^T,\ \hat{v}_k = (\hat{v}_{k,1}, \hat{v}_{k,2}, \hat{v}_{k,3}, ..., \hat{v}_{k,n+1}).$$

3.5 Exact soliton solutions and their dynamics

3.5.1 Explicit one-soliton solution and its dynamics

A general explicit solution for a single soliton in the reverse-time case when $N = 1$, and $w_1 = (w_{11}, w_{12}, w_{13}, w_{14})^T$, with $\lambda_1 \in \mathbb{C}$ an arbitrary, with $\hat{\lambda}_1 = -\lambda_1$, the solution is given by

$$p_1(x,t) = \frac{2\rho_2\rho_3\lambda_1(\alpha_1-\alpha_2)w_{11}w_{12}e^{i\lambda_1(\alpha_1+\alpha_2)x+i\lambda_1^6(\beta_1-\beta_2)t}}{\rho_1\rho_2\rho_3 w_{11}^2 e^{2i\lambda_1\alpha_1 x} + (\rho_2\rho_3 w_{12}^2 + \rho_1\rho_3 w_{13}^2 + \rho_1\rho_2 w_{14}^2)e^{2i\lambda_1\alpha_2 x}}, \tag{3.141}$$

$$p_2(x,t) = \frac{2\rho_1\rho_3\lambda_1(\alpha_1-\alpha_2)w_{11}w_{13}e^{i\lambda_1(\alpha_1+\alpha_2)x+i\lambda_1^6(\beta_1-\beta_2)t}}{\rho_1\rho_2\rho_3 w_{11}^2 e^{2i\lambda_1\alpha_1 x} + (\rho_2\rho_3 w_{12}^2 + \rho_1\rho_3 w_{13}^2 + \rho_1\rho_2 w_{14}^2)e^{2i\lambda_1\alpha_2 x}}, \tag{3.142}$$

$$p_3(x,t) = \frac{2\rho_1\rho_2\lambda_1(\alpha_1-\alpha_2)w_{11}w_{14}e^{i\lambda_1(\alpha_1+\alpha_2)x+i\lambda_1^6(\beta_1-\beta_2)t}}{\rho_1\rho_2\rho_3 w_{11}^2 e^{2i\lambda_1\alpha_1 x} + (\rho_2\rho_3 w_{12}^2 + \rho_1\rho_3 w_{13}^2 + \rho_1\rho_2 w_{14}^2)e^{2i\lambda_1\alpha_2 x}}. \tag{3.143}$$

We can get the amplitude of p_1:

$$|p_1(x,t)| = 2Ae^{-\beta t Im(\lambda_1^6)} \tag{3.144}$$

where

$$A = \left| \frac{2\lambda_1\rho_2\rho_3(\alpha_1-\alpha_2)w_{11}w_{12}e^{-Im(\lambda_1(\alpha_1+\alpha_2)x)}}{\rho_1\rho_2\rho_3 w_{11}^2 e^{2i\lambda_1\alpha_1 x} + (\rho_2\rho_3 w_{12}^2 + \rho_1\rho_3 w_{13}^2 + \rho_1\rho_2 w_{14}^2)e^{2i\lambda_1\alpha_2 x}} \right|. \tag{3.145}$$

We can see from p_1 that the frequency is zero, thus the velocity is zero. Hence the one-soliton is not a travelling wave, and it is stationary in space.

Fixing $x = x_0$, the amplitude is $|p_1(x,t)| = 2A|_{x=x_0} e^{-\beta t Im(\lambda_1^6)}$. If $Im(\lambda_1^6) < 0$ the amplitude decays exponentially, while it grows exponentially for $Im(\lambda_1^6) > 0$ and when $Im(\lambda_1^6) = 0$, the amplitude remains constant over the time.

In this reverse-time case, the resulting one-soliton does not collapse, either it strictly increases, decreases or stays constant.

From the spectral plane, let $\lambda_1 = \xi + i\eta = |\lambda_1|e^{i\theta}$, where $|\lambda_1| > 0$, and $0 < \theta < 2\pi$ then:

$$\text{if} \begin{cases} \theta \in \left\{(\frac{n}{6}\pi, \frac{n+1}{6}\pi)\right\}, \text{then the amplitude of the soliton is increasing for } n = \{0,2,4,\ldots\}, \\ \theta \in \left\{(\frac{n}{6}\pi, \frac{n+1}{6}\pi)\right\}, \text{the amplitude of the soliton is decreasing for } n = \{1,3,5,\ldots\}, \\ \theta \in \left(\frac{n}{6}\bmod\{n\}\right)\pi, \text{the amplitude of the soliton is constant for } n = \{0,1,2,3,4,5,\ldots\}, \\ \theta \in \{n\pi\}, \text{ we obtain one breather with constant amplitude for } n = \{0,1,2,3,4,5,\ldots\}. \end{cases} \tag{3.146}$$

This illustration is shown by the figure below.

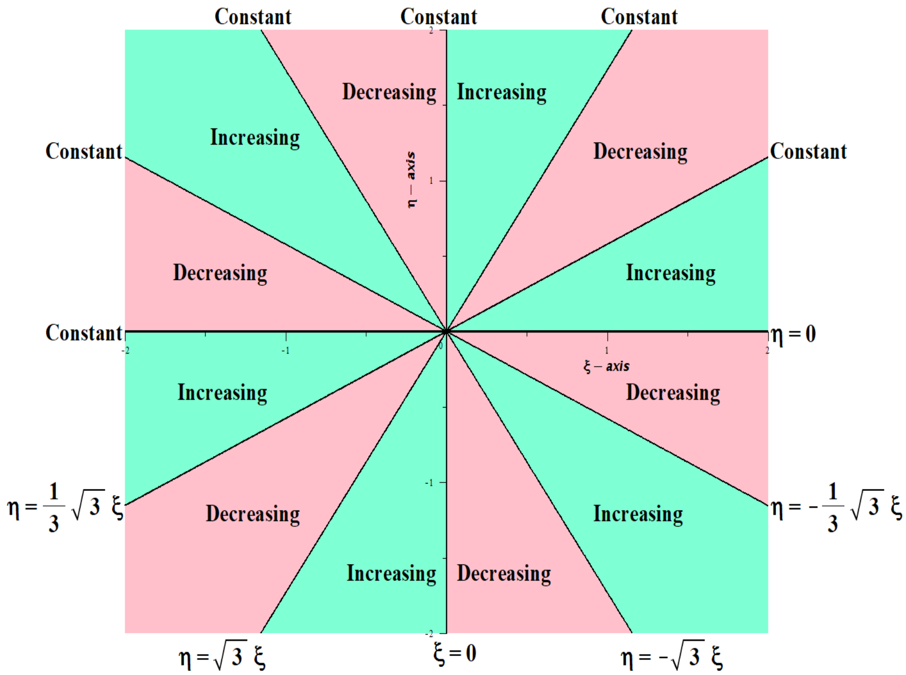


Figure 2.: Spectral plane of eigenvalues.

For the one-soliton solution, when λ_1 does not lie on the real axis, the imaginary axis or the trisectors, i.e. $\lambda \notin \{\xi, \mathrm{i}\eta, (1 \pm \mathrm{i}\sqrt{3})\xi, (1 \pm \mathrm{i}\frac{1}{\sqrt{3}})\xi\}$, the amplitude of the potential grows and decays exponentially, if $Im(\lambda_1^6) > 0$ and $Im(\lambda_1^6) < 0$, respectively. In Figure 3 and Figure 4, we have two examples where the amplitude grows and decays exponentially.

The amplitude does not change when $Im(\lambda_1^6) = 0$, i.e. that means when λ_1 belongs to the real axis, the imaginary axis or the trisectors. If λ_1 lies on the imaginary axis or the trisectors, then we have a fundamental soliton as seen in figure 5. If λ_1 is purely imaginary, then the Lax matrix $U(u, \lambda)$ is a skew-Hermitian matrix. On the other hand, if λ_1 lies on the real axis, we have a breather which is a periodic one-soliton with period $\frac{\pi}{|\alpha\lambda_1|}$ as seen in figure 6. This is a consequence of the Lax matrix $U(u, \lambda)$ being a Hermitian matrix.

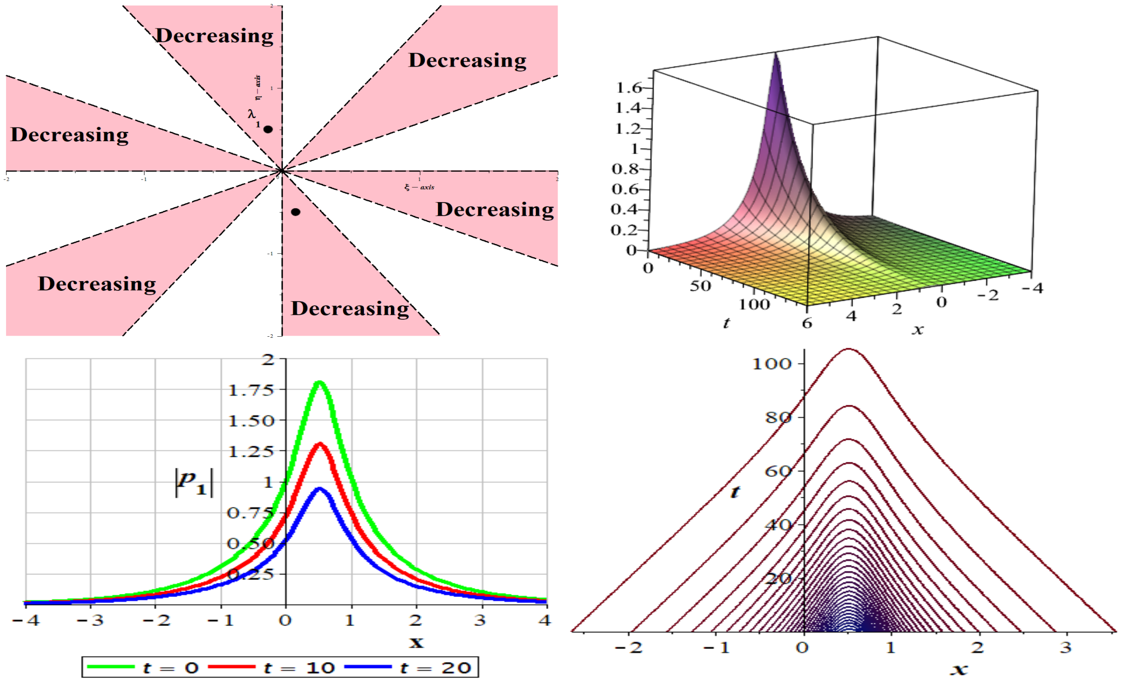


Figure 3.: Spectral plane along with 3D, 2D and contour plots of $|p_1|$ in the focussing case of the one-soliton with parameters $(\rho_1, \rho_2, \rho_3, \alpha_1, \alpha_2, \beta_1, \beta_2) = (-1, -2, -1, -1, 1, -1, 1)$, $\lambda_1 = -0.1 + 0.5i$, $w_1 = (1, i, 2 + i, 1)$.

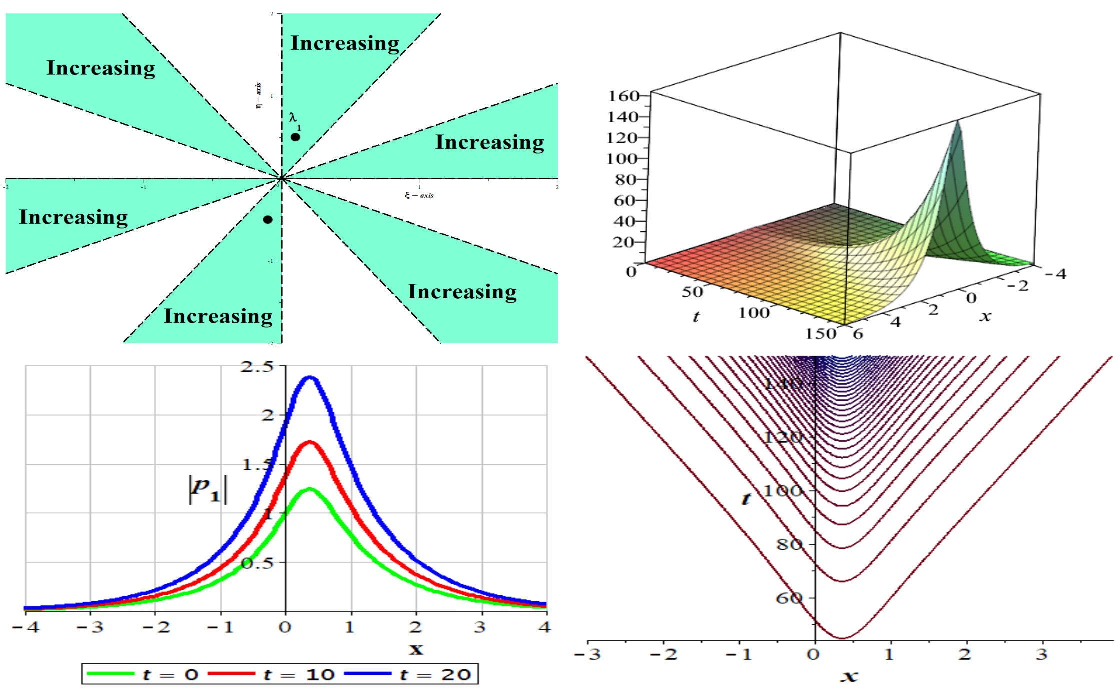


Figure 4.: Spectral plane along with 3D, 2D and contour plots of $|p_1|$ in the focussing case of the one-soliton with parameters $(\rho_1, \rho_2, \rho_3, \alpha_1, \alpha_2, \beta_1, \beta_2) = (-1, -2, -1, -1, 1, -1, 1)$, $\lambda_1 = 0.1 + 0.5i$, $w_1 = (1, i, 2 + i, 1)$.

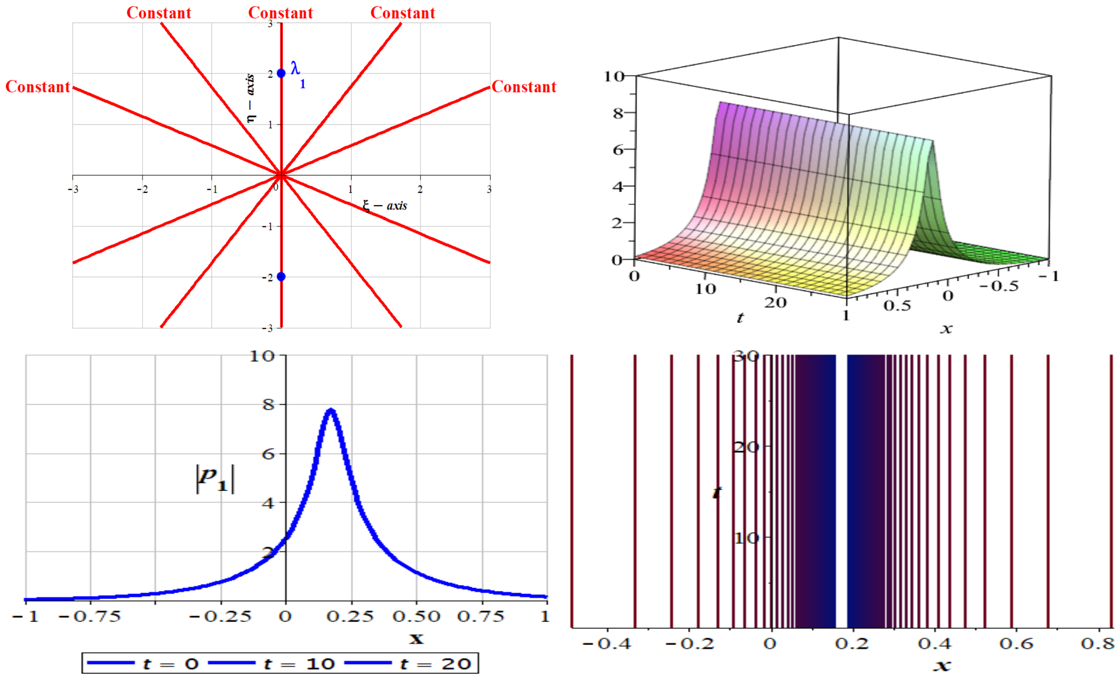


Figure 5.: Spectral plane along with 3D, 2D and contour plots of $|p_1|$ of the one-soliton with parameters $(\rho_1, \rho_2, \rho_3, \alpha_1, \alpha_2, \beta_1, \beta_2) = (1, -2, -1, -1, 1, -1, 1)$, $\lambda_1 = 2i$, $w_1 = (1, i, 2 + i, 1)$.

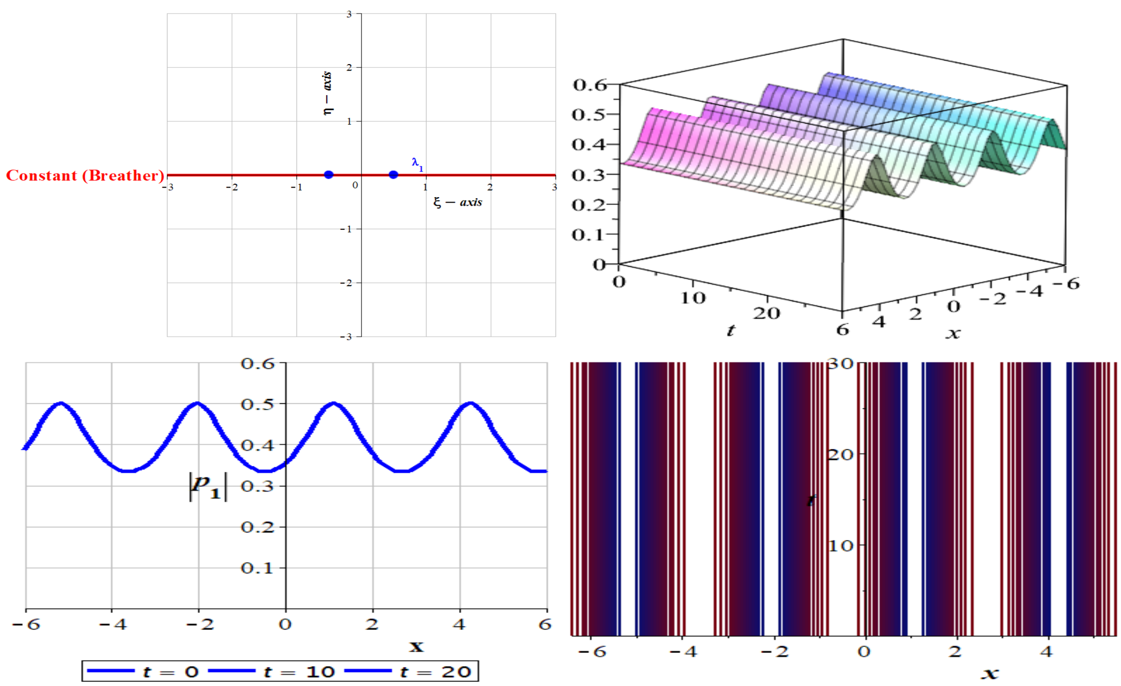


Figure 6.: Spectral plane along with 3D, 2D and contour plots of $|p_1|$ of the one-soliton with parameters $(\rho_1, \rho_2, \rho_3, \alpha_1, \alpha_2, \beta_1, \beta_2) = (1, 1, 1, -1, 1, -1, 1)$, $\lambda_1 = 0.5$, $w_1 = (1, i, 2 + i, 1)$.

Remark 3.5.1. In the case of one-soliton solutions, when the AKNS spectral problem (3.22) has higher even-orders (NLS-type), i.e., $\lambda_1^{2m} = |\lambda_1|^{2m} e^{i2m\theta}$, $m \in \mathbb{N}$, the amplitude of p_1 can be written in the form:

$$|p_1(x,t)| = Ae^{-Im(\lambda_1^{2m}(\beta_1-\beta_2)t+\lambda_1(\alpha_1+\alpha_2)x)} \tag{3.147}$$

where A is a constant. From (3.147) if $Im(\lambda_1^{2m}) = |\lambda_1|^{2m} sin(2m\theta) = 0$, that gives the partition of the complex plane by $2m$-sectors.

If $0 < |\lambda_1| < 1$, then $\lim_{m\to\infty} Im(\lambda_1^{2m}) = 0$, and this means for any λ_1 lying inside the disk of radius 1, the soliton has a constant amplitude.

If $|\lambda_1| = 1$, i.e., it lies on the circle of radius 1, then the amplitude $|p_1(x,t)|$ will be bounded by $Ae^{\beta t} \leq |p_1(x,t)| \leq Ae^{-\beta t}$, when $\beta = \beta_1 - \beta_2 < 0$.

If $|\lambda_1| > 1$, then $\lim_{m\to\infty} Im(\lambda_1^{2m}) \to \pm\infty$, and so the amplitude will grow exponentially or it will decay to zero exponentially.

3.5.2 Explicit two-soliton solution and its dynamics

A general explicit two-soliton solution in the reverse-time case when $N = 2$, $w_1 = (w_{11}, w_{12}, w_{13}, w_{14})^T$, $w_2 = (w_{21}, w_{22}, w_{23}, w_{24})^T$, $(\lambda_1, \lambda_2) \in \mathbb{C}^2$ are arbitrary, and $\hat{\lambda}_1 = -\lambda_1$, $\hat{\lambda}_2 = -\lambda_2$, is given if $\lambda_1 \neq -\lambda_2$ by

$$p_1(x,t) = 2\rho_2\rho_3(\lambda_1+\lambda_2)(\alpha_1-\alpha_2)\frac{A(x,t)}{B(x,t)}, \tag{3.148}$$

$$p_2(x,t) = 2\rho_1\rho_3(\lambda_1+\lambda_2)(\alpha_1-\alpha_2)\frac{C(x,t)}{B(x,t)}, \tag{3.149}$$

$$p_3(x,t) = 2\rho_1\rho_2(\lambda_1+\lambda_2)(\alpha_1-\alpha_2)\frac{D(x,t)}{B(x,t)}, \tag{3.150}$$

where

$$\begin{aligned} A(x,t) = e^{i[\lambda_2^6(\beta_1-\beta_2)t+\lambda_2(\alpha_1+\alpha_2)x]}.&\bigg[\Big(w_{22}M(\lambda_1+\lambda_2) - 2w_{12}K\lambda_1\Big)w_{21}\lambda_2 e^{i2\alpha_2\lambda_1 x} \\ &- \rho_1\rho_2\rho_3(\lambda_1-\lambda_2)w_{11}^2 w_{21}w_{22}\lambda_2 e^{i2\alpha_1\lambda_1 x}\bigg] \\ +e^{i[\lambda_1^6(\beta_1-\beta_2)t+\lambda_1(\alpha_1+\alpha_2)x]}.&\bigg[\Big(w_{12}N(\lambda_1+\lambda_2) - 2w_{22}K\lambda_2\Big)w_{11}\lambda_1 e^{i2\alpha_2\lambda_2 x} \\ &+ \rho_1\rho_2\rho_3(\lambda_1-\lambda_2)w_{11}w_{12}w_{21}^2\lambda_1 e^{i2\alpha_1\lambda_2 x}\bigg], \end{aligned} \tag{3.151}$$

$$B(x,t) = -4\rho_1\rho_2\rho_3\lambda_1\lambda_2 w_{11}w_{21}Ke^{i(\lambda_1+\lambda_2)(\alpha_1+\alpha_2)x} \cdot \left[e^{i(\lambda_1^6-\lambda_2^6)(\beta_1-\beta_2)t} + e^{-i(\lambda_1^6-\lambda_2^6)(\beta_1-\beta_2)t}\right]$$
$$+ \rho_1\rho_2\rho_3 w_{21}^2 M(\lambda_1+\lambda_2)^2 e^{i2(\alpha_1\lambda_2+\alpha_2\lambda_1)x} + \rho_1\rho_2\rho_3 w_{11}^2 N(\lambda_1+\lambda_2)^2 e^{i2(\alpha_1\lambda_1+\alpha_2\lambda_2)x}$$
$$+ \rho_1^2\rho_2^2\rho_3^2 w_{11}^2 w_{21}^2(\lambda_1-\lambda_2)^2 e^{i2\alpha_1(\lambda_1+\lambda_2)x} + \left[(\lambda_1^2+\lambda_2^2)MN + (2MN-4K^2)\lambda_1\lambda_2\right]e^{i2\alpha_2(\lambda_1+\lambda_2)x}, \tag{3.152}$$

$$C(x,t) = e^{i[\lambda_2^6(\beta_1-\beta_2)t+\lambda_2(\alpha_1+\alpha_2)x]} \cdot \Big[\Big(w_{23}M(\lambda_1+\lambda_2) - 2w_{13}K\lambda_1\Big)w_{21}\lambda_2 e^{i2\alpha_2\lambda_1 x}$$
$$- \rho_1\rho_2\rho_3(\lambda_1-\lambda_2)w_{11}^2 w_{21}w_{23}\lambda_2 e^{i2\alpha_1\lambda_1 x}\Big]$$
$$+e^{i[\lambda_1^6(\beta_1-\beta_2)t+\lambda_1(\alpha_1+\alpha_2)x]} \cdot \Big[\Big(w_{13}N(\lambda_1+\lambda_2) - 2w_{23}K\lambda_2\Big)w_{11}\lambda_1 e^{i2\alpha_2\lambda_2 x}$$
$$+ \rho_1\rho_2\rho_3(\lambda_1-\lambda_2)w_{11}w_{13}w_{21}^2\lambda_1 e^{i2\alpha_1\lambda_2 x}\Big], \tag{3.153}$$

$$D(x,t) = e^{i[\lambda_2^6(\beta_1-\beta_2)t+\lambda_2(\alpha_1+\alpha_2)x]} \cdot \Big[\Big(w_{24}M(\lambda_1+\lambda_2) - 2w_{14}K\lambda_1\Big)w_{21}\lambda_2 e^{i2\alpha_2\lambda_1 x}$$
$$- \rho_1\rho_2\rho_3(\lambda_1-\lambda_2)w_{11}^2 w_{21}w_{24}\lambda_2 e^{i2\alpha_1\lambda_1 x}\Big]$$
$$+e^{i[\lambda_1^6(\beta_1-\beta_2)t+\lambda_1(\alpha_1+\alpha_2)x]} \cdot \Big[\Big(w_{14}N(\lambda_1+\lambda_2) - 2w_{24}K\lambda_2\Big)w_{11}\lambda_1 e^{i2\alpha_2\lambda_2 x}$$
$$+ \rho_1\rho_2\rho_3(\lambda_1-\lambda_2)w_{11}w_{14}w_{21}^2\lambda_1 e^{i2\alpha_1\lambda_2 x}\Big], \tag{3.154}$$

and $M = \rho_2\rho_3 w_{12}^2 + \rho_1\rho_3 w_{13}^2 + \rho_1\rho_2 w_{14}^2$, $N = \rho_2\rho_3 w_{22}^2 + \rho_1\rho_3 w_{23}^2 + \rho_1\rho_2 w_{24}^2$ and $K = \rho_2\rho_3 w_{12}w_{22} + \rho_1\rho_3 w_{13}w_{23} + \rho_1\rho_2 w_{14}w_{24}$.

For the two-soliton dynamics, we notice that either both the two solitons are moving (repeatedly or not) in opposite directions or both are stationary, i.e., they don't move with respect to space. In figure 7, we have two travelling waves that move in opposite directions, keeping the same amplitude before and after interaction in an elastic collision, that is, no energy radiation emitted [45]. Whereas in figure 8, the amplitudes of the two waves change after interaction to new constant amplitudes resembling Manakov waves [46].

In figure 9, we have two solitons with exponentially decaying amplitude and stationary over the time, i.e., they do not move in space. On the other hand, we can have as in figure 10, two solitons with exponentially decaying amplitude but moving apart over the time.

Aside, if both λ_1 and λ_2 lie on the real axis, then we will obtain breather solitons with the time period $\frac{2\pi}{\beta(\lambda_1^6-\lambda_2^6)}$. An example is shown in figure 11, where the breather waves coincide for $t = 5$ and $t = 6.015873016$.

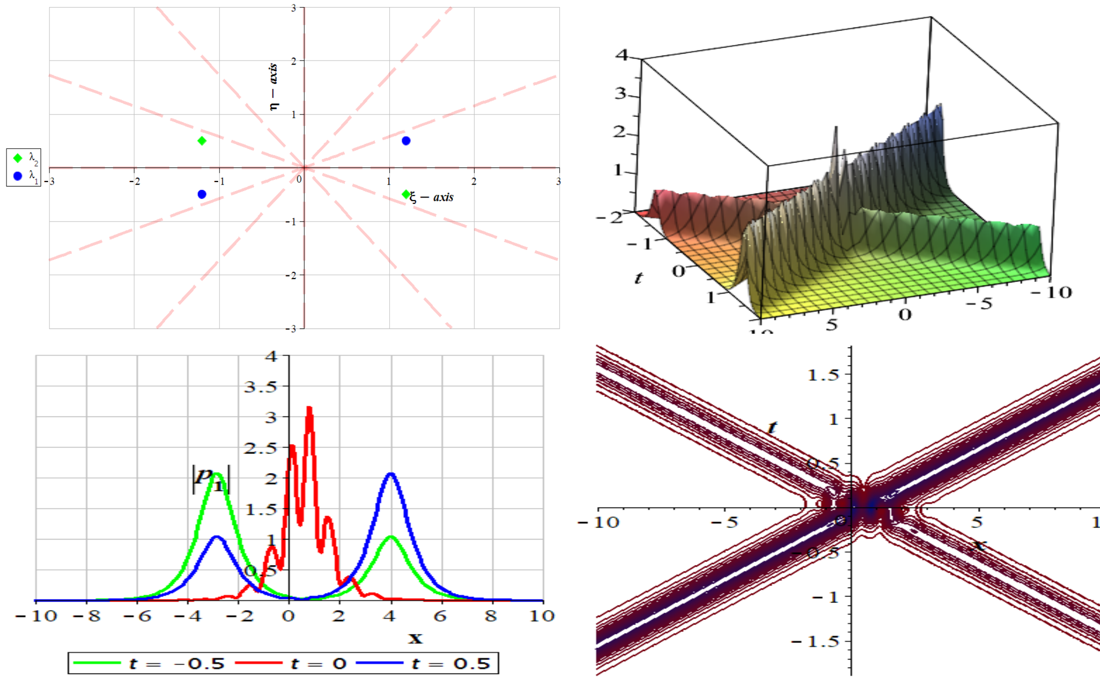


Figure 7.: Spectral plane along with 3D, 2D and contour plots of $|p_1|$ of the two-soliton with parameters $(\rho_1, \rho_2, \rho_3, \alpha_1, \alpha_2, \beta_1, \beta_2) = (-1, -1, 1, -2, 1, -1, 2)$, $(\lambda_1, \lambda_2) = (1.2 + 0.5i, -1.2 + 0.5i)$, $w_1 = (1, 1 - 3i, -i, 1 + i)$, $w_2 = (2, 1 - 3i, -i, 1 + i)$.

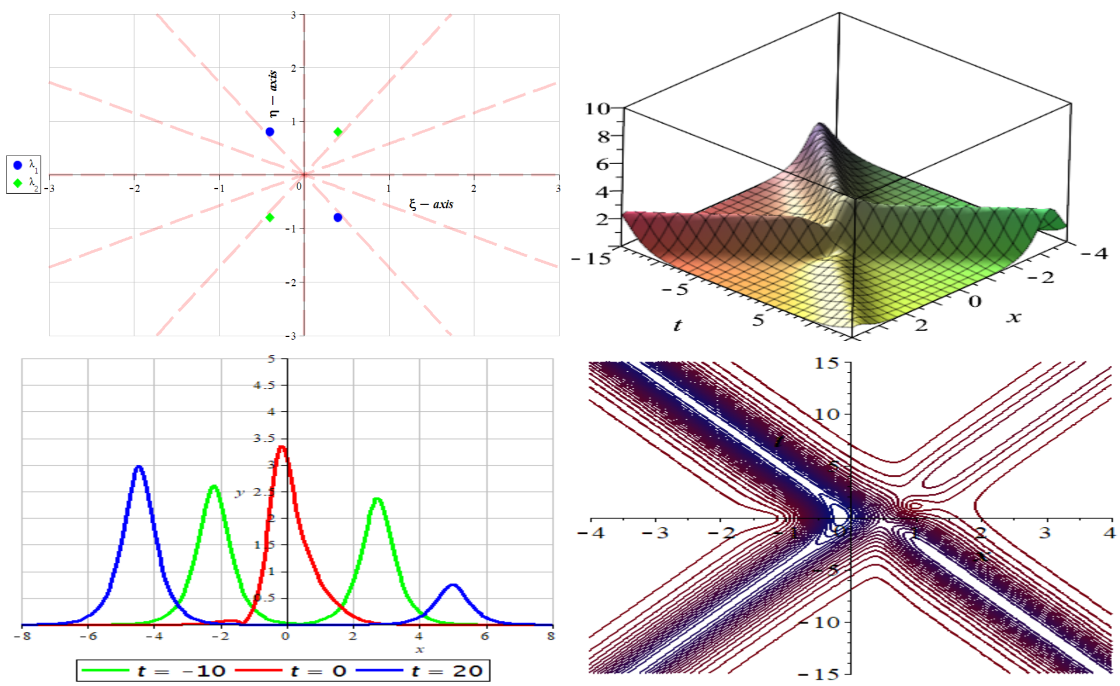


Figure 8.: Spectral plane along with 3D, 2D and contour plots of $|p_1|$ of the two-soliton with parameters $(\rho_1, \rho_2, \rho_3, \alpha_1, \alpha_2, \beta_1, \beta_2) = (-1, 1, -1, -2, 1, -2, 1)$, $(\lambda_1, \lambda_2) = (-0.4 + 0.8i, 0.4 + 0.8i)$, $w_1 = (1, 1 - i, -0.1 + i, 1 + i)$, $w_2 = (-1 + 2i, 1 - 0.1i, 3 + i, 0)$.

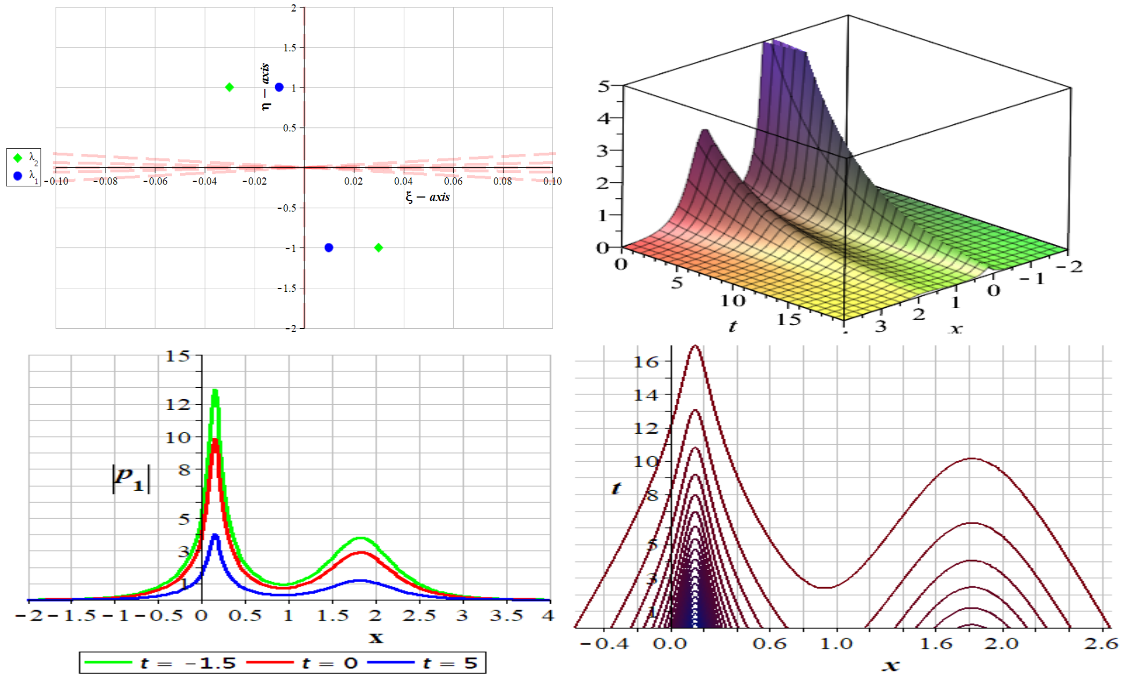


Figure 9.: Spectral plane along with 3D, 2D and contour plots of $|p_1|$ in the focussing case of the two-soliton with parameters $(\rho_1, \rho_2, \rho_3, \alpha_1, \alpha_2, \beta_1, \beta_2) = (-1, -2, -3, -2, 1, -1, 2)$, $(\lambda_1, \lambda_2) = (-0.01 + i, -0.03 + i)$, $w_1 = (1, 0, 2 + i, 0)$, $w_2 = (1, 2 - i, 0, 1)$.

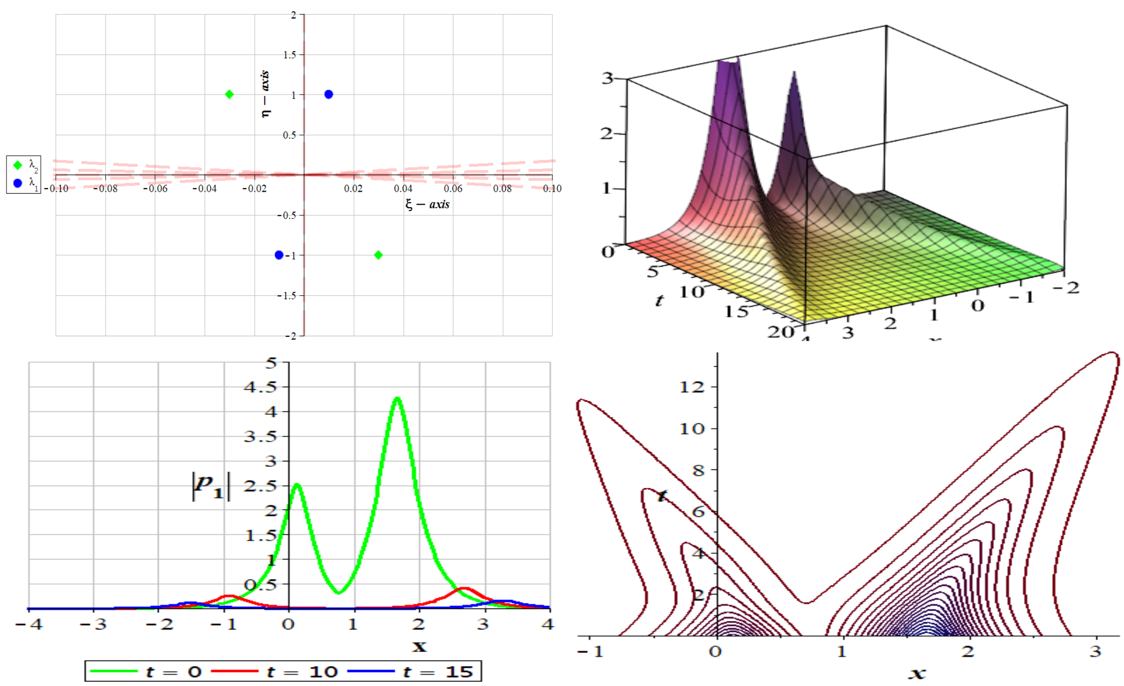


Figure 10.: Spectral plane along with 3D, 2D and contour plots of $|p_1|$ in the focussing case of the two-soliton with parameters $(\rho_1, \rho_2, \rho_3, \alpha_1, \alpha_2, \beta_1, \beta_2) = (-1, -2, -3, -2, 1, -1, 2)$, $(\lambda_1, \lambda_2) = (0.01 + i, -0.03 + i)$, $w_1 = (1, i, 2 + i, 1)$, $w_2 = (1, 2 - i, i, 1)$.

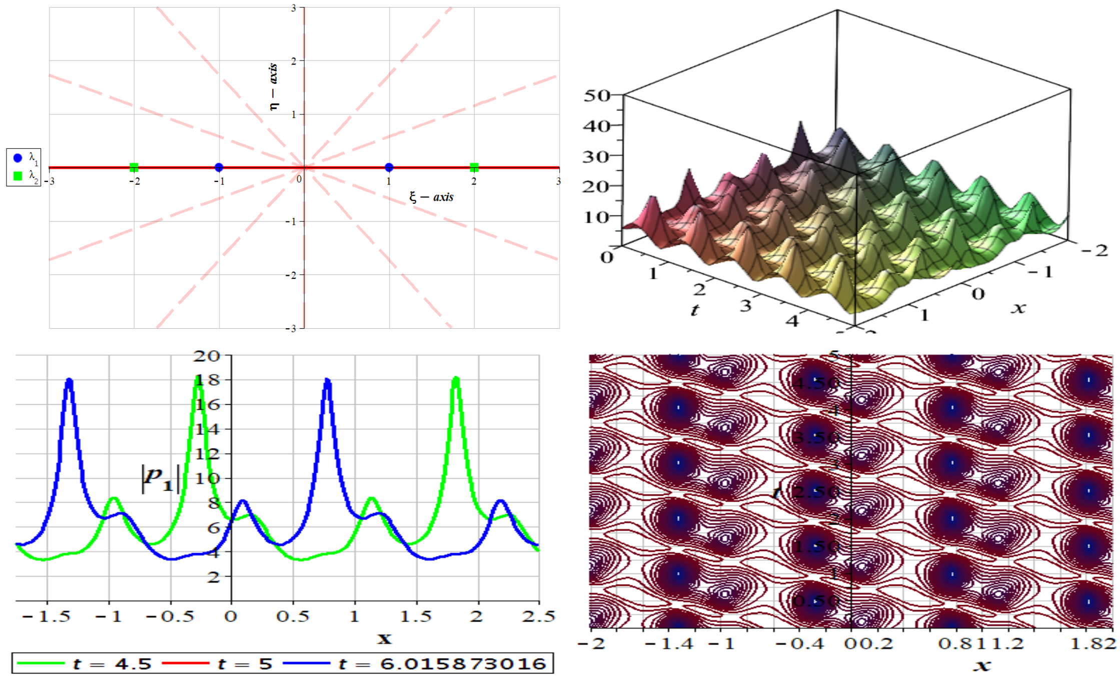


Figure 11.: Spectral plane along with 3D, 2D and contour plots of $|p_1|$ in the focussing case of the two-soliton with parameters $(\rho_1, \rho_2, \rho_3, \alpha_1, \alpha_2, \beta_1, \beta_2) = (-1, -1, -1, -2, 1, -1/64\pi, 1/64\pi)$, $(\lambda_1, \lambda_2) = (1, 2)$, $w_1 = (1, 0, 2+i, 1-i)$, $w_2 = (-1, 1-2i, -i, 0)$.

3.5.3 Dynamics of a three-soliton solution

The three-soliton solution is given, for which we take $N = 3$, $w_1 = (w_{11}, w_{12}, w_{13}, w_{14})^T$, $w_2 = (w_{21}, w_{22}, w_{23}, w_{24})^T$, $w_3 = (w_{31}, w_{32}, w_{33}, w_{34})^T$, $(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{C}^3$, and $\hat{\lambda}_1 = -\lambda_1$, $\hat{\lambda}_2 = -\lambda_2$, $\hat{\lambda}_3 = -\lambda_3$, by

$$p_1 = \alpha \sum_{k,j=1}^{3} v_{k,1} (M^{-1})_{kj} \hat{v}_{j,2}, \tag{3.155}$$

$$p_2 = \alpha \sum_{k,j=1}^{3} v_{k,1} (M^{-1})_{kj} \hat{v}_{j,3}, \tag{3.156}$$

$$p_3 = \alpha \sum_{k,j=1}^{3} v_{k,1} (M^{-1})_{kj} \hat{v}_{j,4}. \tag{3.157}$$

Without loss of generality, for the three-soliton solution, we take all three eigenvalues in the upper-half plane in such a way that $\lambda_j \neq \lambda_k$ for $j, k \in \{1, 2, 3\}$.

Here, we can look at some examples of the three-soliton solution dynamics. We have two solitons moving in opposite directions interacting with one stationary soliton. After the interaction either the three solitons keep their amplitudes or the amplitudes change to new constant amplitudes. An example is shown in figure 12.

Another behaviour could be the interaction of three solitons that are embedded into two solitons after the interaction, where the stationary soliton keeps or changes its amplitude after collision as seen in figure 13. We may have the opposite case where the two solitons unfold to three solitons.

A different class of behaviour shows that three solitons can interact and get embedded into a single soliton after interaction.

In figure 14, we have three solitons, of which two are moving in opposite directions and one is stationary. All of them have constant amplitudes before interaction. After the interaction, they are embedded into a one stationary soliton with constant amplitude. We may have the opposite case where one stationary soliton unfolds to three different solitons, each keeping its amplitude.

In figure 15, we also have three solitons, two solitons moving in opposite directions interacting with an exponentially decreasing stationary soliton. In that case, after the interaction, they are embedded into one stationary decreasing soliton over the time due to an effect of energy radiation.

In contrast, we can have one stationary increasing soliton that unfolds to three soltions, of which two of are moving in opposite directions keeping their amplitude while the other is stationary and increasing exponentially over the time.

In figure 16, we have three solitons, two solitons moving in opposite directions interacting with an exponentially decreasing stationary soliton. They are embedded into a one moving soliton that keeps its constant amplitude after the interaction where the stationary soliton vanish.

In contrary, one moving soliton can also unfold to three different solitons, where two are moving in opposite direct keeping the amplitude and the other is increasing exponentially over the time [47].

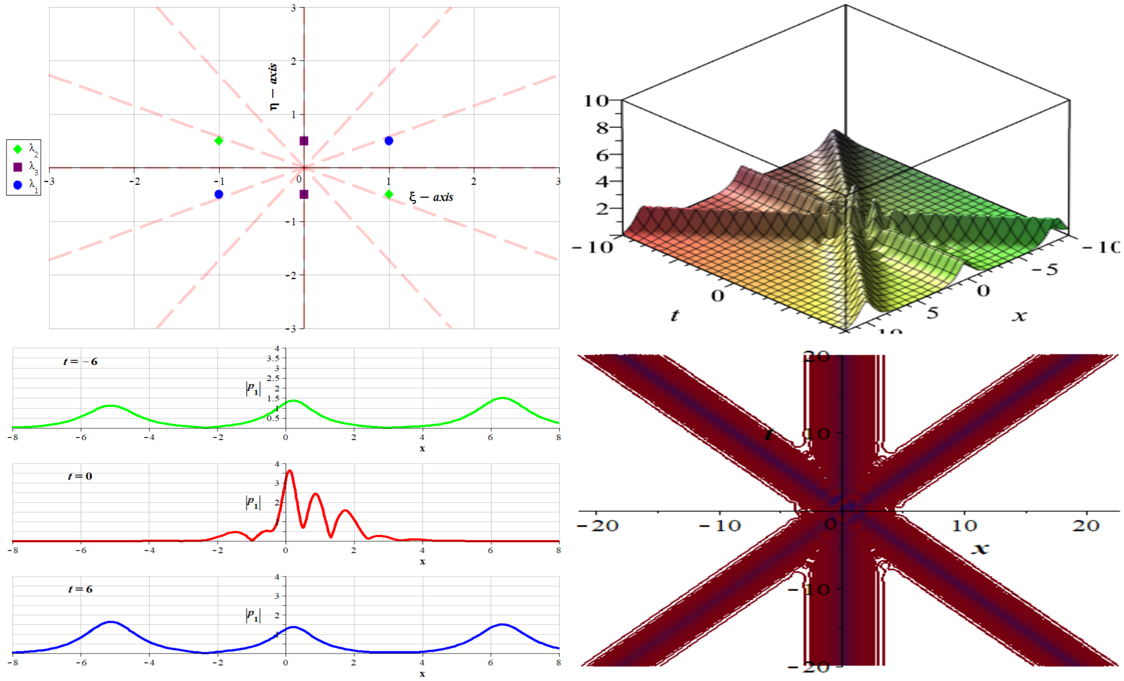


Figure 12.: Spectral plane along with 3D, 2D and contour plots of $|p_1|$ in the focussing case of the three-soliton with parameters $(\rho_1, \rho_2, \rho_3, \alpha_1, \alpha_2, \beta_1, \beta_2) = (-1, -1, -1, -2, 1, -1, 1)$, $(\lambda_1, \lambda_2, \lambda_3) = (1 + 0.5i, -1 + 0.5i, 0.5i)$, $w_1 = (1, 1 + 2i, 0, 0)$, $w_2 = (-1, 1 - 2i, 0, 0)$, $w_3 = (2 + i, 1 + 2i, 1, 2i)$.

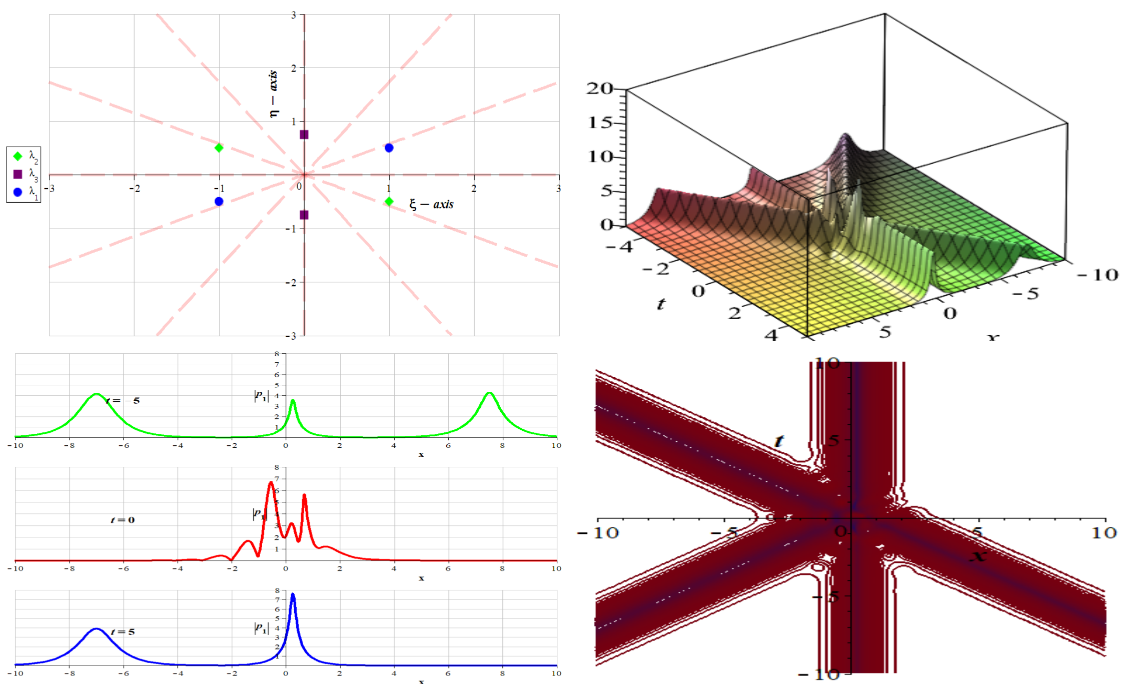


Figure 13.: Spectral plane along with 3D, 2D and contour plots of $|p_1|$ of the three-soliton with parameters $(\rho_1, \rho_2, \rho_3, \alpha_1, \alpha_2, \beta_1, \beta_2) = (1, 1, 1, -2, 1, -2, 1)$, $(\lambda_1, \lambda_2, \lambda_3) = (1 + 0.5i, -1 + 0.5i, 0.75i)$, $w_1 = (1, 0, 2 + i, 1 - i)$, $w_2 = (-1, 1 - 2i, -i, 0)$, $w_3 = (2 + i, 1 + 2i, 1, 2i)$.

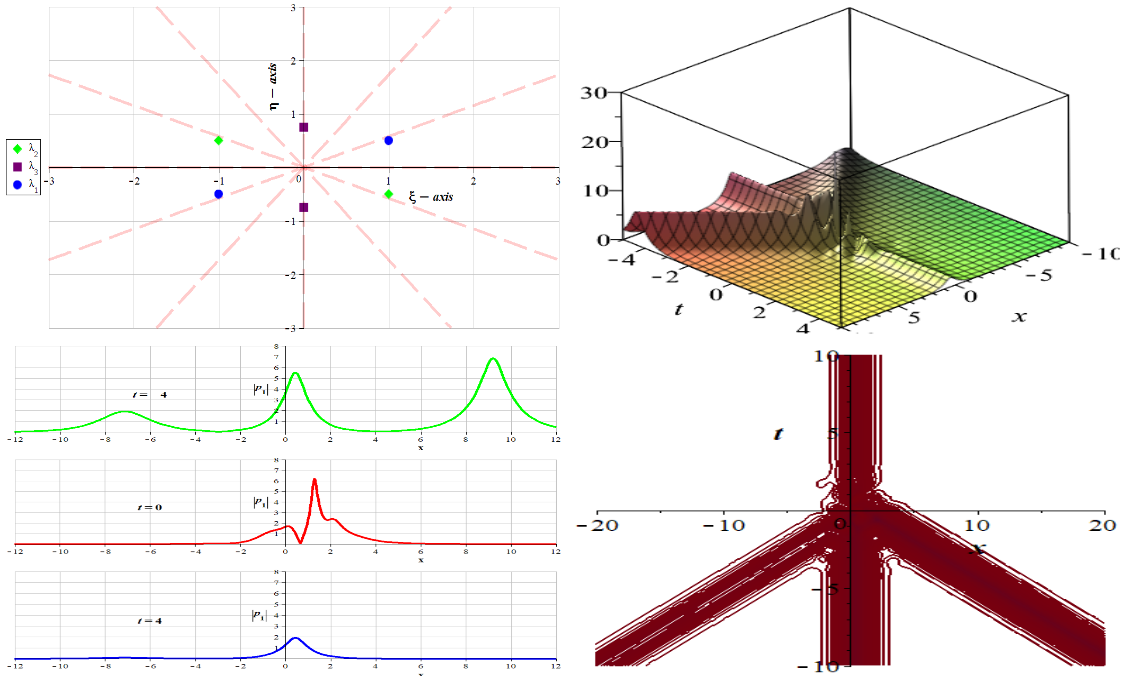


Figure 14.: Spectral plane along with 3D, 2D and contour plots of $|p_1|$ of the three-soliton with parameters $(\rho_1, \rho_2, \rho_3, \alpha_1, \alpha_2, \beta_1, \beta_2) = (1, 1, 1, -1, 1, -2, 1)$, $(\lambda_1, \lambda_2, \lambda_3) = (1 + 0.5i, -1 + 0.5i, 0.75i)$, $w_1 = (1, 0, 2 + i, 1 - i)$, $w_2 = (-1, 5 - 2i, -i, 0)$, $w_3 = (2 + i, 1 + 2i, 1, 2i)$.

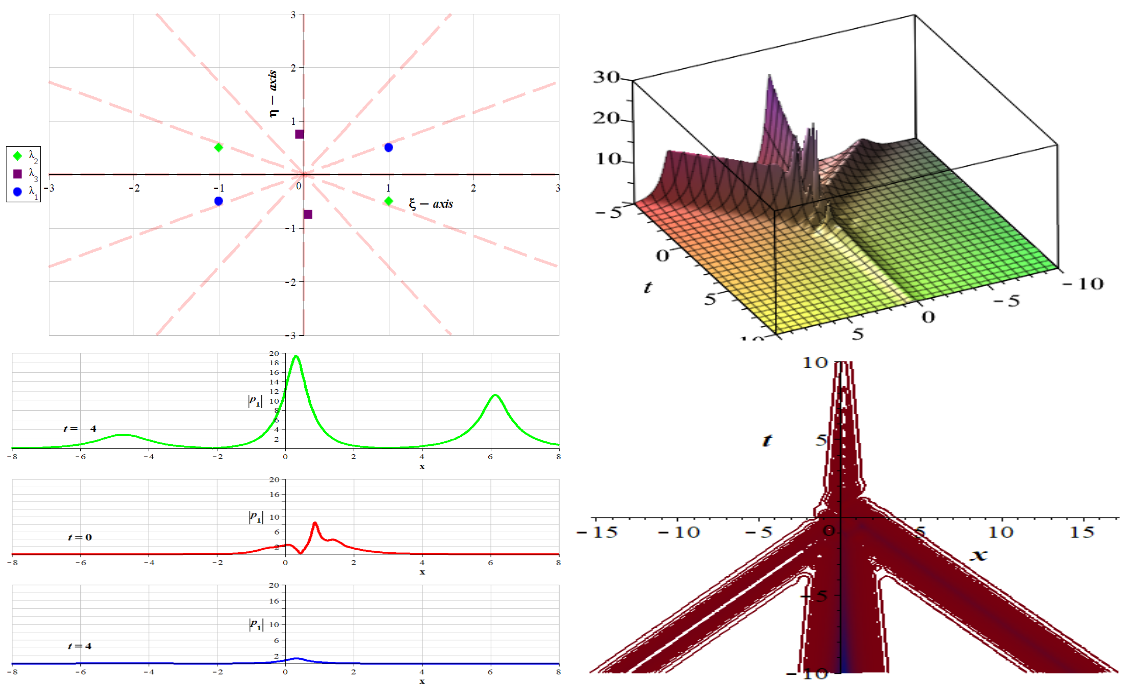


Figure 15.: Spectral plane along with 3D, 2D and contour plots of $|p_1|$ of the three-soliton with parameters $(\rho_1, \rho_2, \rho_3, \alpha_1, \alpha_2, \beta_1, \beta_2) = (1, 1, 1, -2, 1, -2, 1)$, $(\lambda_1, \lambda_2, \lambda_3) = (1 + 0.5i, -1 + 0.5i, -0.05 + 0.75i)$, $w_1 = (1, 0, 2 + i, 1 - i)$, $w_2 = (-1, 5 - 2i, -i, 0)$, $w_3 = (2 + i, 1 + 2i, 1, 2i)$.

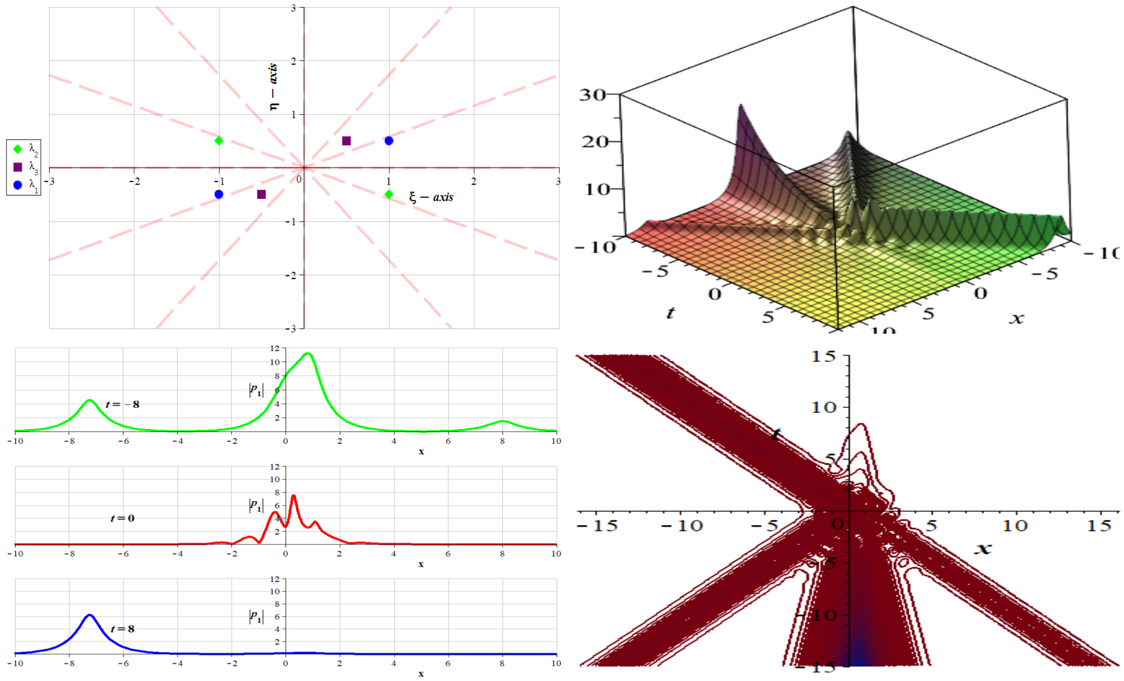


Figure 16.: Spectral plane along with 3D, 2D and contour plots of $|p_1|$ in the focussing case of the three-soliton with parameters $(\rho_1, \rho_2, \rho_3, \alpha_1, \alpha_2, \beta_1, \beta_2) = (-1, -1, -1, -2, 1, -1, 1)$, $(\lambda_1, \lambda_2, \lambda_3) = (1+0.5i, -1+0.5i, 0.5+0.5i)$, $w_1 = (1, 0, 2+i, 1-i)$, $w_2 = (-1, 1-2i, -i, 0)$, $w_3 = (2+i, 1+2i, 1, 2i)$.

3.5.4 Dynamics of a four-soliton solution

The four-soliton solution is given, for which we take $N = 4$, $w_1 = (w_{11}, w_{12}, w_{13}, w_{14})^T$, $w_2 = (w_{21}, w_{22}, w_{23}, w_{24})^T$, $w_3 = (w_{31}, w_{32}, w_{33}, w_{34})^T$, $w_4 = (w_{41}, w_{42}, w_{43}, w_{44})^T$, $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in \mathbb{C}^4$, and $\hat{\lambda}_1 = -\lambda_1$, $\hat{\lambda}_2 = -\lambda_2$, $\hat{\lambda}_3 = -\lambda_3$, $\hat{\lambda}_4 = -\lambda_4$, by

$$p_1 = \alpha \sum_{k,j=1}^{4} v_{k,1}(M^{-1})_{kj}\hat{v}_{j,2}, \tag{3.158}$$

$$p_2 = \alpha \sum_{k,j=1}^{4} v_{k,1}(M^{-1})_{kj}\hat{v}_{j,3}, \tag{3.159}$$

$$p_3 = \alpha \sum_{k,j=1}^{4} v_{k,1}(M^{-1})_{kj}\hat{v}_{j,4}. \tag{3.160}$$

Without loss of generality, for the four-soliton solution, we take all four eigenvalues in the upper-half plane in such a way that $\lambda_j \neq \lambda_k$ for $j, k \in \{1, 2, 3, 4\}$.

For the four-soliton dynamics, we have the interactions of four solitons. Two of them can be stationary or all the four solitons are moving.

Figure 17 exhibits the interaction of two exponentially increasing solitons moving in opposite directions and interacting with two moving solitons with constant amplitudes. After interaction the middle two solitons keep moving with increasing amplitudes, while the two other solitons keep moving with constant amplitudes.

We can notice that the middle two solitons can decrease exponentially while moving and interacting with the other two solitons.

Remark 3.5.2. The speed of the far right and left solitons are larger than the speed of the middle two solitons such that all four solitons collide together.

Another behaviour is shown in figure 18, where two solitons moving in opposite directions interact with two stationary solitons with constant amplitudes. After the interaction, the two stationary solitons remain stationary and the two moving solitons continue to move in opposite directions, but their amplitudes can change to new constant amplitudes or it can stay unchanged.

As for figure 19, we have the interaction of four moving solitons. Two waves move in the same direction and interact with the other two waves coming from the opposite direction. After the interaction, each of the four solitons can keep its amplitude unchanged or its amplitude can change to a new constant amplitude. In the case that each soliton keeps its amplitude before and after the interaction, we have four travelling waves.

Finally in figure 20, four moving solitons are embedded into three moving solitons. After the interaction each soliton keeps its amplitude unchanged or it can be changed to a new constant amplitude over the time.

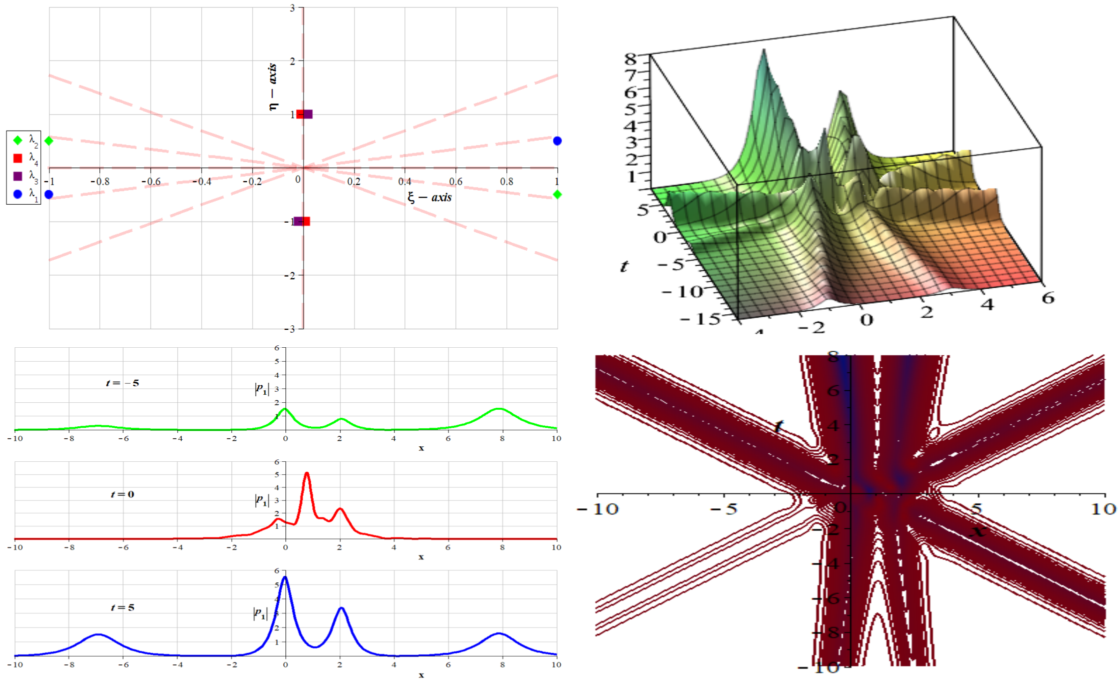


Figure 17.: Spectral plane along with 3D, 2D and contour plots of $|p_1|$ of the four-soliton with parameters $(\rho_1, \rho_2, \rho_3, \alpha_1, \alpha_2, \beta_1, \beta_2) = (1, -2, 1, -2, 1, -2, 1)$, $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (1 + 0.5i, -1 + 0.5i, 0.02 + i, -0.01 + i)$, $w_1 = (1 - 2i, 1 + 3i, -i, 1 + i)$, $w_2 = (-1 + 2i, 1 - 3i, i, 1 - i)$, $w_3 = (1 + i, 1 + 2i, 0, 2i)$, $w_4 = (1, i, 2 + i, 1)$.

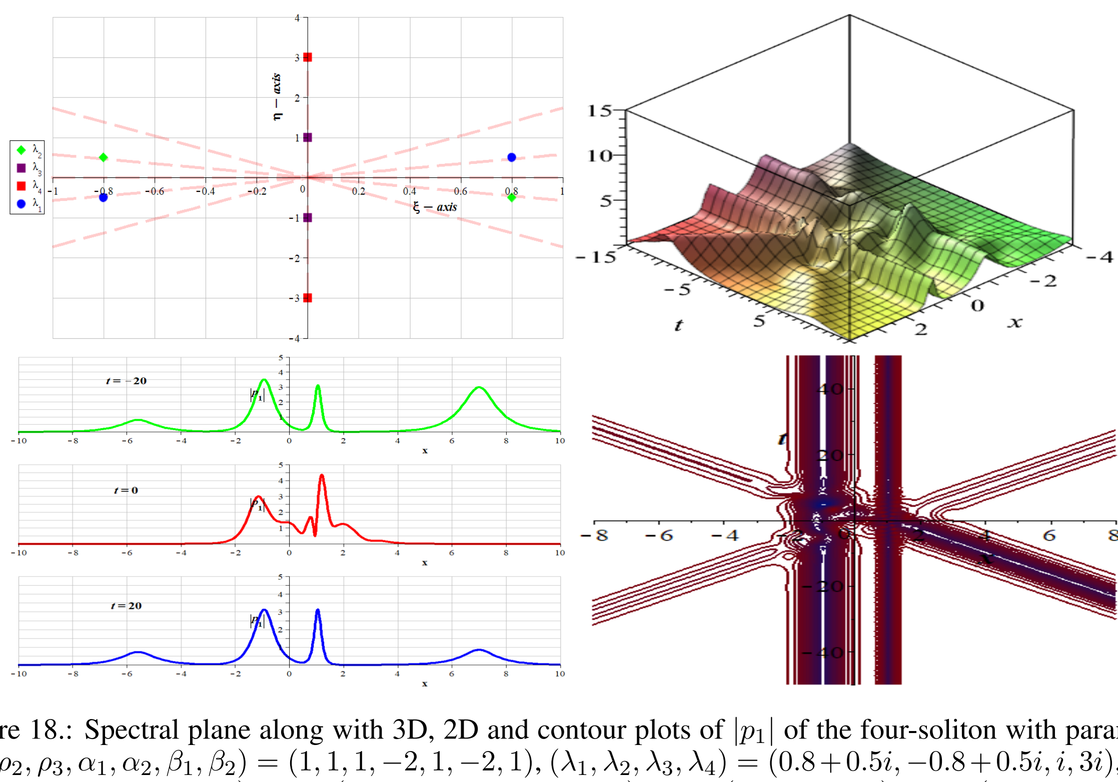


Figure 18.: Spectral plane along with 3D, 2D and contour plots of $|p_1|$ of the four-soliton with parameters $(\rho_1, \rho_2, \rho_3, \alpha_1, \alpha_2, \beta_1, \beta_2) = (1, 1, 1, -2, 1, -2, 1)$, $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (0.8 + 0.5i, -0.8 + 0.5i, i, 3i)$, $w_1 = (1 - 0.5i, 1 + 3i, -i, 1 + i)$, $w_2 = (-1 + 2i, 1 - 1.5i, i, 1 - i)$, $w_3 = (30, i, 2 + i, 1)$, $w_4 = (-0.0005, 1, 2, 1)$.

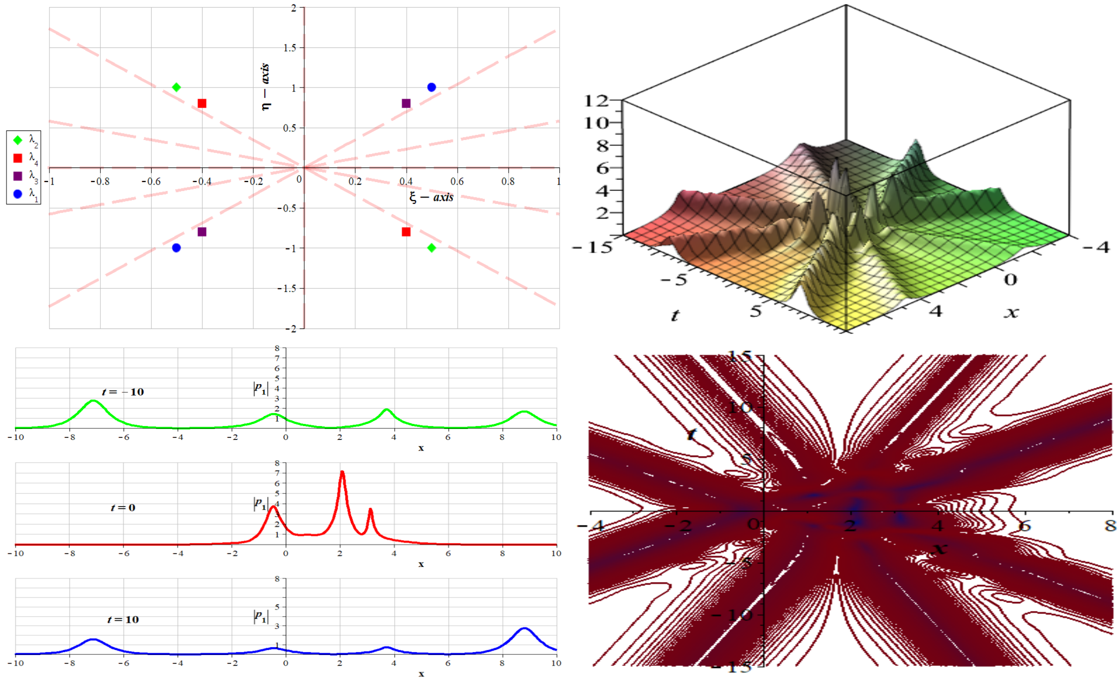


Figure 19.: Spectral plane along with 3D, 2D and contour plots of $|p_1|$ in the focussing case of the four-soliton with parameters $(\rho_1, \rho_2, \rho_3, \alpha_1, \alpha_2, \beta_1, \beta_2) = (-1, -1, -1, -1, 1, -1, 1)$, $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (0.5 + i, -0.5 + i, 0.4 + 0.8i, -0.4 + 0.8i,)$, $w_1 = (-1.5 + 2i, 2 - 3i, i, 1 - i)$, $w_2 = (3 + 2i, -1 + 3i, -i, 1 + i)$, $w_3 = (i, 1, 1 - 2i, 1 - i)$, $w_4 = (-i, 1 - 2i, 1, 1 + i)$.

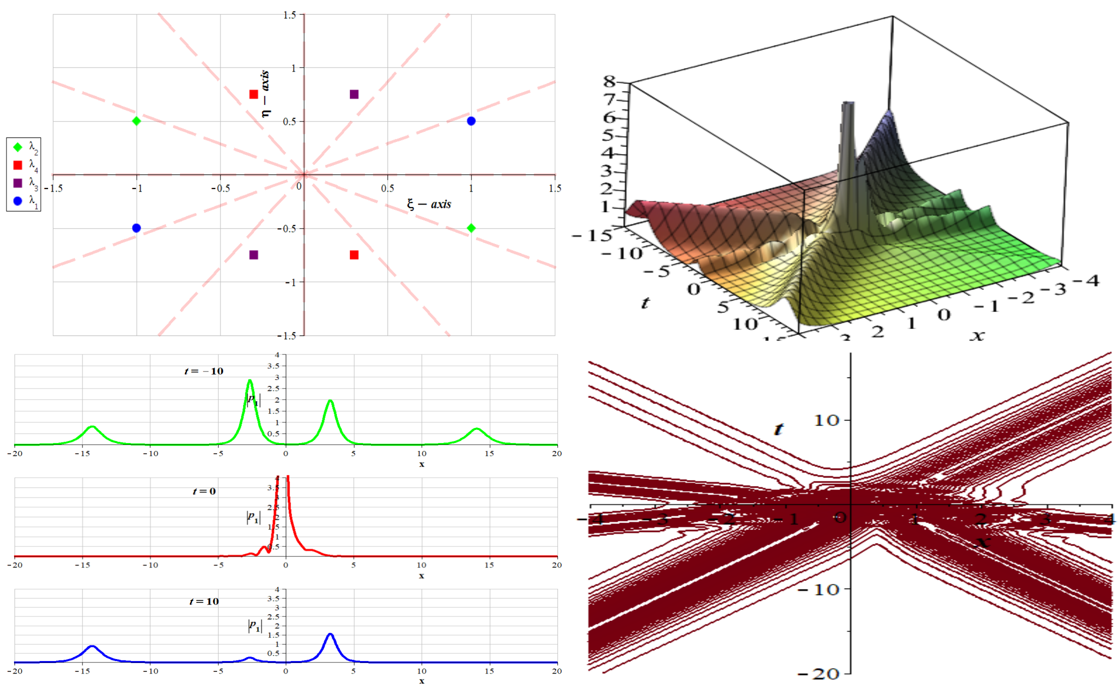


Figure 20.: Spectral plane along with 3D, 2D and contour plots of $|p_1|$ in the focussing case of the four-soliton with parameters $(\rho_1, \rho_2, \rho_3, \alpha_1, \alpha_2, \beta_1, \beta_2) = (-1, -1, -1, -2, 1, -2, 1)$, $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (1 + 0.5i, -1 + 0.5i, 0.3 + 0.75i, -0.3 + 0.75i)$, $w_1 = (1 + 4i, 0, 2 + i, i)$, $w_2 = (-1 + 4i, i, 1 - 2i, 0)$, $w_3 = (-2 + i, 0.5 + i, -2 + i, 1)$, $w_4 = (2 + i, 1 - 0.5i, 1, 1)$.

3.5.5 Remarks

An interesting observation is that we can explore the one and two soliton solutions for any nth-order (n is even) six-component AKNS integrable system for our spectral matrix $U(u,\lambda)$.

That is the general explicit one-soliton solution for a reverse-time nth-order six-component system when $\hat{\lambda}_1 = -\lambda_1$ is given by

$$p_1(x,t) = \frac{2\rho_2\rho_3\lambda_1(\alpha_1-\alpha_2)w_{11}w_{12}e^{i\lambda_1(\alpha_1+\alpha_2)x+i\lambda_1^n(\beta_1-\beta_2)t}}{\rho_1\rho_2\rho_3 w_{11}^2 e^{2i\lambda_1\alpha_1 x} + (\rho_2\rho_3 w_{12}^2 + \rho_1\rho_3 w_{13}^2 + \rho_1\rho_2 w_{14}^2)e^{2i\lambda_1\alpha_2 x}}, \tag{3.161}$$

similarly for $p_2(x,t)$ and $p_3(x,t)$.

As for the two-soliton solution, the general explicit solution when $\hat{\lambda}_1 = -\lambda_1$, $\hat{\lambda}_2 = -\lambda_2$, if $\lambda_1 \neq -\lambda_2$, is given by

$$p_1(x,t) = 2\rho_2\rho_3(\lambda_1+\lambda_2)(\alpha_1-\alpha_2)\frac{A(x,t)}{B(x,t)}, \tag{3.162}$$

where

$$\begin{aligned} A(x,t) = e^{i[\lambda_2^n(\beta_1-\beta_2)t+\lambda_2(\alpha_1+\alpha_2)x]}.&\Bigg[\Big(w_{22}M(\lambda_1+\lambda_2) - 2w_{12}K\lambda_1\Big)w_{21}\lambda_2 e^{i2\alpha_2\lambda_1 x} \\ &- \rho_1\rho_2\rho_3(\lambda_1-\lambda_2)w_{11}^2 w_{21}w_{22}\lambda_2 e^{i2\alpha_1\lambda_1 x}\Bigg] \\ +e^{i[\lambda_1^n(\beta_1-\beta_2)t+\lambda_1(\alpha_1+\alpha_2)x]}.&\Bigg[\Big(w_{12}N(\lambda_1+\lambda_2) - 2w_{22}K\lambda_2\Big)w_{11}\lambda_1 e^{i2\alpha_2\lambda_2 x} \\ &+ \rho_1\rho_2\rho_3(\lambda_1-\lambda_2)w_{11}w_{12}w_{21}^2\lambda_1 e^{i2\alpha_1\lambda_2 x}\Bigg], \end{aligned} \tag{3.163}$$

$$\begin{aligned} B(x,t) = &-4\rho_1\rho_2\rho_3\lambda_1\lambda_2 w_{11}w_{21}K e^{i(\lambda_1+\lambda_2)(\alpha_1+\alpha_2)x}\cdot\Big[e^{i(\lambda_1^n-\lambda_2^n)(\beta_1-\beta_2)t} + e^{-i(\lambda_1^n-\lambda_2^n)(\beta_1-\beta_2)t}\Big] \\ &+ \rho_1\rho_2\rho_3 w_{21}^2 M(\lambda_1+\lambda_2)^2 e^{i2(\alpha_1\lambda_2+\alpha_2\lambda_1)x} + \rho_1\rho_2\rho_3 w_{11}^2 N(\lambda_1+\lambda_2)^2 e^{i2(\alpha_1\lambda_1+\alpha_2\lambda_2)x} \\ &+ \rho_1^2\rho_2^2\rho_3^2 w_{11}^2 w_{21}^2(\lambda_1-\lambda_2)^2 e^{i2\alpha_1(\lambda_1+\lambda_2)x} \\ &+ \Big[(\lambda_1^2+\lambda_2^2)MN + (2MN-4K^2)\lambda_1\lambda_2\Big]e^{i2\alpha_2(\lambda_1+\lambda_2)x}, \end{aligned} \tag{3.164}$$

and $M = \rho_2\rho_3 w_{12}^2 + \rho_1\rho_3 w_{13}^2 + \rho_1\rho_2 w_{14}^2$, $N = \rho_2\rho_3 w_{22}^2 + \rho_1\rho_3 w_{23}^2 + \rho_1\rho_2 w_{24}^2$ and $K = \rho_2\rho_3 w_{12} w_{22} + \rho_1\rho_3 w_{13} w_{23} + \rho_1\rho_2 w_{14} w_{24}$. Similarly we can present the formulas for $p_2(x,t)$ and $p_3(x,t)$.

For higher-order non-local reverse-time AKNS systems, the N-soliton exhibits a similar dynamics (or combinations of dynamics) to the ones discussed and in the previous work [35].

Chapter 4

Nonlocal reverse-spacetime Sasa-Satsuma equation

4.1 Introduction

We will discuss in this chapter the new nonlocal reverse-spacetime two-component Sasa-Satsuma equation given by [48]-[50]:

$$
\begin{aligned}
u_t(x,t) = -\frac{\beta}{\alpha^3}\Big[& u_{xxx}(x,t) \\
& - 3\big(|u(x,t)|^2 + |u(-x,-t)|^2 + |v(x,t)|^2 + |v(-x,-t)|^2\big) u_x(x,t) \\
& - 3\big(|u(x,t)|^2 + |u(-x,-t)|^2 + |v(x,t)|^2 + |v(-x,-t)|^2\big)_x u(x,t) \\
& + 3\Big(u(x,t)\overset{*}{u}_x(x,t) - u(-x,-t)\overset{*}{u}_x(-x,-t) \\
& \qquad + v(x,t)\overset{*}{v}_x(x,t) - v(-x,-t)\overset{*}{v}_x(-x,-t)\Big)u(x,t)\Big],
\end{aligned} \tag{4.1}
$$

$$
\begin{aligned}
v_t(x,t) = -\frac{\beta}{\alpha^3}\Big[& v_{xxx}(x,t) \\
& - 3\big(|u(x,t)|^2 + |u(-x,-t)|^2 + |v(x,t)|^2 + |v(-x,-t)|^2\big) v_x(x,t) \\
& - 3\big(|u(x,t)|^2 + |u(-x,-t)|^2 + |v(x,t)|^2 + |v(-x,-t)|^2\big)_x v(x,t) \\
& + 3\Big(u(x,t)\overset{*}{u}_x(x,t) - u(-x,-t)\overset{*}{u}_x(-x,-t) \\
& \qquad + v(x,t)\overset{*}{v}_x(x,t) - v(-x,-t)\overset{*}{v}_x(-x,-t)\Big)v(x,t)\Big].
\end{aligned} \tag{4.2}
$$

Unlike the previous reverse-time sixth-order NLS-type equation which was obtained from a nonlocal reduction through a local AKNS hierarchy, we derive this Sasa-Satsuma equation from a nonlocal hierarchy. Starting with a nonlocal 5×5 spectral matrix problem, we formulate a kind of Riemann-Hilbert problems for the above nonlocal two-component Sasa-Satsuma equation with still the real line being its contour, and solve the resulting Riemann-Hilbert problems with identity jump matrix to present its soliton solutions [36]-[42]. We explore the one- and two-soliton solution and classify the different cases for the explicit two-soliton solution according to the configuration of the eigenvalues in the spectral plane. As a final step, we explore their dynamical behaviors.

4.2 Two-component nonlocal hierarchy

Consider the nonlocal 5×5 matrix AKNS spatial spectral problem [39]

$$\varphi_x = \mathrm{i}U\varphi, \tag{4.3}$$

where φ is the eigenfunction and the spectral matrix $U(\mathbf{u}, \lambda)$ is given by

$$U(\mathbf{u},\lambda) = \begin{pmatrix} \alpha_1\lambda & u(x,t) & u(-x,-t) & v(x,t) & v(-x,-t) \\ -\overset{*}{u}(x,t) & \alpha_2\lambda & 0 & 0 & 0 \\ -\overset{*}{u}(-x,-t) & 0 & \alpha_2\lambda & 0 & 0 \\ -\overset{*}{v}(x,t) & 0 & 0 & \alpha_2\lambda & 0 \\ -\overset{*}{v}(-x,-t) & 0 & 0 & 0 & \alpha_2\lambda \end{pmatrix} = \lambda\Lambda + P(\mathbf{u}), \tag{4.4}$$

where $\Lambda = diag(\alpha_1, \alpha_2 I_4)$, λ is the spectral parameter, α_1, α_2 are two distinct real constants and $\mathbf{u}$ is the column vector of two potentials, where we assume that u and xu belong to the L^2 space and

$$P = \begin{pmatrix} 0 & u(x,t) & u(-x,-t) & v(x,t) & v(-x,-t) \\ -\overset{*}{u}(x,t) & 0 & 0 & 0 & 0 \\ -\overset{*}{u}(-x,-t) & 0 & 0 & 0 & 0 \\ -\overset{*}{v}(x,t) & 0 & 0 & 0 & 0 \\ -\overset{*}{v}(-x,-t) & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.5}$$

Remark 4.2.1. One can see that the matrix U has the symmetry properties:

$$\begin{cases} U^\dagger(-x,-t,-\overset{*}{\lambda}) = -C_1 U(x,t,\lambda) C_1^{-1}, \\ U(-x,-t,-\lambda) = -C_2 U(x,t,\lambda) C_2^{-1}, \end{cases} \tag{4.6}$$

where

$$C_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}. \tag{4.7}$$

Note that C_j are symmetric and orthogonal matrices, i.e., $C_j = C_j^T$ and $C_j = C_j^{-1}$, for $j \in \{1, 2\}$.

In addition, since $U(x, t, \lambda) = \lambda \Lambda + P(x, t)$, we can easily prove that

$$\begin{cases} P^\dagger(-x, -t) = -C_1 P(x, t) C_1^{-1}, \\ P(-x, -t) = -C_2 P(x, t) C_2^{-1}. \end{cases} \tag{4.8}$$

Let's construct the associated two-component Sasa-Satsuma soliton hierarchy. To do so, we need to solve the stationary zero curvature equation

$$W_x = \mathrm{i}[U, W], \tag{4.9}$$

for

$$W = \begin{pmatrix} a & b_1 & b_2 & b_3 & b_4 \\ c_1 & d_{11} & d_{12} & d_{13} & d_{14} \\ c_2 & d_{21} & d_{22} & d_{23} & d_{24} \\ c_3 & d_{31} & d_{32} & d_{33} & d_{34} \\ c_4 & d_{41} & d_{42} & d_{43} & d_{44} \end{pmatrix}, \tag{4.10}$$

where a, b_j, c_j, d_{jk} are scalar components for $j, k \in \{1, 2, 3, 4\}$. From the stationary zero curvature equation, we get:

$$a_x = \mathrm{i}\Big[\overset{*}{u}(x) b_1 + \overset{*}{u}(-x) b_2 + \overset{*}{v}(x) b_3 + \overset{*}{v}(-x) b_4 + u(x) c_1 + u(-x) c_2 + v(x) c_3 + v(-x) c_4\Big], \tag{4.11}$$

$$
\begin{aligned}
&d_{11,x} = -\mathrm{i}\big[\overset{*}{u}(x)b_1 + u(x)c_1\big], \quad && d_{12,x} = -\mathrm{i}\big[\overset{*}{u}(x)b_2 + u(-x)c_1\big],\\
&d_{21,x} = -\mathrm{i}\big[\overset{*}{u}(-x)b_1 + u(x)c_2\big], && d_{22,x} = -\mathrm{i}\big[\overset{*}{u}(-x)b_2 + u(-x)c_2\big],\\
&d_{31,x} = -\mathrm{i}\big[\overset{*}{v}(x)b_1 + u(x)c_3\big], && d_{32,x} = -\mathrm{i}\big[\overset{*}{v}(x)b_2 + u(-x)c_3\big],\\
&d_{41,x} = -\mathrm{i}\big[\overset{*}{v}(-x)b_1 + u(x)c_4\big], && d_{42,x} = -\mathrm{i}\big[\overset{*}{v}(-x)b_2 + u(-x)c_4\big],
\end{aligned} \tag{4.12}
$$

$$
\begin{aligned}
&d_{13,x} = -\mathrm{i}\big[\overset{*}{u}(x)b_3 + v(x)c_1\big], \quad && d_{14,x} = -\mathrm{i}\big[\overset{*}{u}(x)b_4 + v(-x)c_1\big],\\
&d_{23,x} = -\mathrm{i}\big[\overset{*}{u}(-x)b_3 + v(x)c_2\big], && d_{24,x} = -\mathrm{i}\big[\overset{*}{u}(-x)b_4 + v(-x)c_2\big],\\
&d_{33,x} = -\mathrm{i}\big[\overset{*}{v}(x)b_3 + v(x)c_3\big], && d_{34,x} = -\mathrm{i}\big[\overset{*}{v}(x)b_4 + v(-x)c_3\big],\\
&d_{43,x} = -\mathrm{i}\big[\overset{*}{v}(-x)b_3 + v(x)c_4\big], && d_{44,x} = -\mathrm{i}\big[\overset{*}{v}(-x)b_4 + v(-x)c_4\big],
\end{aligned} \tag{4.13}
$$

$$b_{1,x} = \mathrm{i}\big[\alpha\lambda b_1 - u(x)a + u(x)d_{11} + u(-x)d_{21} + v(x)d_{31} + v(-x)d_{41}\big], \tag{4.14}$$

$$b_{2,x} = \mathrm{i}\big[\alpha\lambda b_2 - u(-x)a + u(x)d_{12} + u(-x)d_{22} + v(x)d_{32} + v(-x)d_{42}\big], \tag{4.15}$$

$$b_{3,x} = \mathrm{i}\big[\alpha\lambda b_3 - v(x)a + u(x)d_{13} + u(-x)d_{23} + v(x)d_{33} + v(-x)d_{43}\big], \tag{4.16}$$

$$b_{4,x} = \mathrm{i}\big[\alpha\lambda b_4 - v(-x)a + u(x)d_{14} + u(-x)d_{24} + v(x)d_{34} + v(-x)d_{44}\big], \tag{4.17}$$

$$c_{1,x} = \mathrm{i}\big[\alpha\lambda c_1 - \overset{*}{u}(x)a + \overset{*}{u}(x)d_{11} + \overset{*}{u}(-x)d_{12} + \overset{*}{v}(x)d_{13} + \overset{*}{v}(-x)d_{14}\big], \tag{4.18}$$

$$c_{2,x} = \mathrm{i}\big[\alpha\lambda c_2 - \overset{*}{u}(-x)a + \overset{*}{u}(x)d_{21} + \overset{*}{u}(-x)d_{22} + \overset{*}{v}(x)d_{23} + \overset{*}{v}(-x)d_{24}\big], \tag{4.19}$$

$$c_{3,x} = \mathrm{i}\big[\alpha\lambda c_3 - \overset{*}{v}(x)a + \overset{*}{u}(x)d_{31} + \overset{*}{u}(-x)d_{32} + \overset{*}{v}(x)d_{33} + \overset{*}{v}(-x)d_{34}\big], \tag{4.20}$$

$$c_{4,x} = \mathrm{i}\big[\alpha\lambda c_4 - \overset{*}{v}(-x)a + \overset{*}{u}(x)d_{41} + \overset{*}{u}(-x)d_{42} + \overset{*}{v}(x)d_{43} + \overset{*}{v}(-x)d_{44}\big], \tag{4.21}$$

where $\alpha = \alpha_1 - \alpha_2$.

We expand W in Laurent series:

$$W = \sum_{m=0}^{\infty} W_m \lambda^{-m} \quad \text{with} \quad W_m = \begin{pmatrix} a^{[m]} & b_1^{[m]} & b_2^{[m]} & b_3^{[m]} & b_4^{[m]} \\ c_1^{[m]} & d_{11}^{[m]} & d_{12}^{[m]} & d_{13}^{[m]} & d_{14}^{[m]} \\ c_2^{[m]} & d_{21}^{[m]} & d_{22}^{[m]} & d_{23}^{[m]} & d_{24}^{[m]} \\ c_3^{[m]} & d_{31}^{[m]} & d_{32}^{[m]} & d_{33}^{[m]} & d_{34}^{[m]} \\ c_4^{[m]} & d_{41}^{[m]} & d_{42}^{[m]} & d_{43}^{[m]} & d_{44}^{[m]} \end{pmatrix}, \tag{4.22}$$

explicitly,

$$a=\sum_{m=0}^{\infty} a^{[m]}\lambda^{-m},\quad d_{jk}=\sum_{m=0}^{\infty} d_{jk}^{[m]}\lambda^{-m},\ i\in\{1,2,3,4\},$$
$$b_j=\sum_{m=0}^{\infty} b_j^{[m]}\lambda^{-m},\ i\in\{1,\ldots,4\},\quad c_j=\sum_{m=0}^{\infty} c_j^{[m]}\lambda^{-m},\ j,k\in\{1,\ldots,4\},$$

where $m\geq 0$. The system (4.12) generates the recursive relations:

$$b_j^{[0]}=c_j^{[0]}=0,\quad j\in\{1,\ldots,4\}, \tag{4.23}$$

$$\begin{aligned} a_x^{[m]}=\mathrm{i}\Big[&\overset{*}{u}(x)b_1^{[m]}+\overset{*}{u}(-x)b_2^{[m]}+\overset{*}{v}(x)b_3^{[m]}+\overset{*}{v}(-x)b_4^{[m]}\\ &+u(x)c_1^{[m]}+u(-x)c_2^{[m]}+v(x)c_3^{[m]}+v(-x)c_4^{[m]}\Big], \end{aligned} \tag{4.24}$$

$$\begin{aligned} &d_{11,x}^{[m]}=-\mathrm{i}\big[\overset{*}{u}(x)b_1^{[m]}+u(x)c_1^{[m]}\big], && d_{12,x}^{[m]}=-\mathrm{i}\big[\overset{*}{u}(x)b_2^{[m]}+u(-x)c_1^{[m]}\big],\\ &d_{21,x}^{[m]}=-\mathrm{i}\big[\overset{*}{u}(-x)b_1^{[m]}+u(x)c_2^{[m]}\big], && d_{22,x}^{[m]}=-\mathrm{i}\big[\overset{*}{u}(-x)b_2^{[m]}+u(-x)c_2^{[m]}\big],\\ &d_{31,x}^{[m]}=-\mathrm{i}\big[\overset{*}{v}(x)b_1^{[m]}+u(x)c_3^{[m]}\big], && d_{32,x}^{[m]}=-\mathrm{i}\big[\overset{*}{v}(x)b_2^{[m]}+u(-x)c_3^{[m]}\big],\\ &d_{41,x}^{[m]}=-\mathrm{i}\big[\overset{*}{v}(-x)b_1^{[m]}+u(x)c_4^{[m]}\big], && d_{42,x}^{[m]}=-\mathrm{i}\big[\overset{*}{v}(-x)b_2^{[m]}+u(-x)c_4^{[m]}\big], \end{aligned} \tag{4.25}$$

$$\begin{aligned} &d_{13,x}^{[m]}=-\mathrm{i}\big[\overset{*}{u}(x)b_3^{[m]}+v(x)c_1^{[m]}\big], && d_{14,x}^{[m]}=-\mathrm{i}\big[\overset{*}{u}(x)b_4^{[m]}+v(-x)c_1^{[m]}\big],\\ &d_{23,x}^{[m]}=-\mathrm{i}\big[\overset{*}{u}(-x)b_3^{[m]}+v(x)c_2^{[m]}\big], && d_{24,x}^{[m]}=-\mathrm{i}\big[\overset{*}{u}(-x)b_4^{[m]}+v(-x)c_2^{[m]}\big],\\ &d_{33,x}^{[m]}=-\mathrm{i}\big[\overset{*}{v}(x)b_3^{[m]}+v(x)c_3^{[m]}\big], && d_{34,x}^{[m]}=-\mathrm{i}\big[\overset{*}{v}(x)b_4^{[m]}+v(-x)c_3^{[m]}\big],\\ &d_{43,x}^{[m]}=-\mathrm{i}\big[\overset{*}{v}(-x)b_3^{[m]}+v(x)c_4^{[m]}\big], && d_{44,x}^{[m]}=-\mathrm{i}\big[\overset{*}{v}(-x)b_4^{[m]}+v(-x)c_4^{[m]}\big], \end{aligned} \tag{4.26}$$

$$b_1^{[m+1]}=\frac{1}{\alpha}\big[-\mathrm{i}b_{1,x}^{[m]}+u(x)a^{[m]}-u(x)d_{11}^{[m]}-u(-x)d_{21}^{[m]}-v(x)d_{31}^{[m]}-v(-x)d_{41}^{[m]}\big], \tag{4.27}$$
$$b_2^{[m+1]}=\frac{1}{\alpha}\big[-\mathrm{i}b_{2,x}^{[m]}+u(-x)a^{[m]}-u(x)d_{12}^{[m]}-u(-x)d_{22}^{[m]}-v(x)d_{32}^{[m]}-v(-x)d_{42}^{[m]}\big], \tag{4.28}$$
$$b_3^{[m+1]}=\frac{1}{\alpha}\big[-\mathrm{i}b_{3,x}^{[m]}+v(x)a^{[m]}-u(x)d_{13}^{[m]}-u(-x)d_{23}^{[m]}-v(x)d_{33}^{[m]}-v(-x)d_{43}^{[m]}\big], \tag{4.29}$$
$$b_4^{[m+1]}=\frac{1}{\alpha}\big[-\mathrm{i}b_{4,x}^{[m]}+v(-x)a^{[m]}-u(x)d_{14}^{[m]}-u(-x)d_{24}^{[m]}-v(x)d_{34}^{[m]}-v(-x)d_{44}^{[m]}\big], \tag{4.30}$$

$$c_1^{[m+1]} = \frac{1}{\alpha}\big[\mathrm{i}c_{1,x}^{[m]} - \overset{*}{u}(x)a^{[m]} + \overset{*}{u}(x)d_{11}^{[m]} + \overset{*}{u}(-x)d_{12}^{[m]} + \overset{*}{v}(x)d_{13}^{[m]} + \overset{*}{v}(-x)d_{14}^{[m]}\big], \tag{4.31}$$

$$c_2^{[m+1]} = \frac{1}{\alpha}\big[\mathrm{i}c_{2,x}^{[m]} - \overset{*}{u}(-x)a^{[m]} + \overset{*}{u}(x)d_{21}^{[m]} + \overset{*}{u}(-x)d_{22}^{[m]} + \overset{*}{v}(x)d_{23}^{[m]} + \overset{*}{v}(-x)d_{24}^{[m]}\big], \tag{4.32}$$

$$c_3^{[m+1]} = \frac{1}{\alpha}\big[\mathrm{i}c_{3,x}^{[m]} - \overset{*}{v}(x)a^{[m]} + \overset{*}{u}(x)d_{31}^{[m]} + \overset{*}{u}(-x)d_{32}^{[m]} + \overset{*}{v}(x)d_{33}^{[m]} + \overset{*}{v}(-x)d_{34}^{[m]}\big], \tag{4.33}$$

$$c_4^{[m+1]} = \frac{1}{\alpha}\big[\mathrm{i}c_{4,x}^{[m]} - \overset{*}{v}(-x)a^{[m]} + \overset{*}{u}(x)d_{41}^{[m]} + \overset{*}{u}(-x)d_{42}^{[m]} + \overset{*}{v}(x)d_{43}^{[m]} + \overset{*}{v}(-x)d_{44}^{[m]}\big]. \tag{4.34}$$

The first few involved functions can be worked out as follows:

$$\begin{cases} a^{[0]} &= \beta_1, \\ a^{[1]} &= 0, \\ a^{[2]} &= \frac{\beta}{\alpha^2}\big(|u(x,t)|^2 + |u(-x,-t)|^2 + |v(x,t)|^2 + |v(-x,-t)|^2\big), \\ a^{[3]} &= 2\frac{\beta}{\alpha^3}\mathrm{Im}\big(u_x(x,t)\overset{*}{u}(x,t) + u(-x,-t)\overset{*}{u}_x(-x,-t) \\ & \qquad +v_x(x,t)\overset{*}{v}(x,t) + v(-x,-t)\overset{*}{v}_x(-x,-t)\big), \\ a^{[4]} &= \frac{\beta}{\alpha^4}\Big[3\Big(|u(x,t)|^2 + |u(-x,-t)|^2 + |v(x,t)|^2 + |v(-x,-t)|^2\Big)^2 \\ & -2\mathrm{Re}\big(u(x,t)\overset{*}{u}_{xx}(x,t) + u(-x,-t)\overset{*}{u}_{xx}(-x,-t) \\ & \qquad +v(x,t)\overset{*}{v}_{xx}(x,t) + v(-x,-t)\overset{*}{v}_{xx}(-x,-t)\big) \\ & +\Big(|u_x(x,t)|^2 + |u_x(-x,-t)|^2 + |v_x(x,t)|^2 + |v_x(-x,-t)|^2\Big)\Big], \end{cases} \tag{4.35}$$

$$\begin{cases} b_1^{[0]} &= 0, \\ b_1^{[1]} &= \frac{\beta}{\alpha}u(x,t), \\ b_1^{[2]} &= -\mathrm{i}\frac{\beta}{\alpha^2}u_x(x,t), \\ b_1^{[3]} &= -\frac{\beta}{\alpha^3}\Big[u_{xx}(x,t) - 2\Big(|u(x,t)|^2 + |u(-x,-t)|^2 + |v(x,t)|^2 + |v(-x,-t)|^2\Big)u(x,t)\Big], \\ b_1^{[4]} &= \mathrm{i}\frac{\beta}{\alpha^4}\Big[u_{xxx}(x,t) + 3\mathbf{T}_1 u_x(x,t) + 3\mathbf{T}_2 u(x,t)\Big], \end{cases} \tag{4.36}$$

$$
\begin{cases}
b_3^{[0]} &= 0,\\
b_3^{[1]} &= \frac{\beta}{\alpha} v(x,t),\\
b_3^{[2]} &= -\mathrm{i}\frac{\beta}{\alpha^2} v_x(x,t),\\
b_3^{[3]} &= -\frac{\beta}{\alpha^3}\Big[v_{xx}(x,t) - 2\Big(|u(x,t)|^2 + |u(-x,-t)|^2 + |v(x,t)|^2 + |v(-x,-t)|^2\Big)v(x,t)\Big],\\
b_3^{[4]} &= \mathrm{i}\frac{\beta}{\alpha^4}\Big[v_{xxx}(x,t) + 3\mathbf{T}_1 v_x(x,t) + 3\mathbf{T}_2 v(x,t)\Big],
\end{cases}
\tag{4.37}
$$

$$
\begin{cases}
d_{11}^{[0]} &= \beta_2,\\
d_{11}^{[1]} &= 0,\\
d_{11}^{[2]} &= -\frac{\beta}{\alpha^2}|u(x,t)|^2,\\
d_{11}^{[3]} &= 2\frac{\beta}{\alpha^3}\mathrm{Im}\Big(\overset{*}{u}_x(x,t)u(x,t)\Big),\\
d_{11}^{[4]} &= \frac{\beta}{\alpha^4}\Big[3\mathbf{T}_1|u(x,t)|^2 + \overset{*}{u}(x,t)u_{xx}(x,t) - |u_x(x,t)|^2 + \overset{*}{u}_{xx}(x,t)u(x,t)\Big],
\end{cases}
\tag{4.38}
$$

$$
\begin{cases}
d_{21}^{[0]} &= 0,\\
d_{21}^{[1]} &= 0,\\
d_{21}^{[2]} &= -\frac{\beta}{\alpha^2}\overset{*}{u}(-x,-t)u(x,t),\\
d_{21}^{[3]} &= \mathrm{i}\frac{\beta}{\alpha^3}\Big(\overset{*}{u}(-x,-t)u_x(x,t) + \overset{*}{u}_x(-x,-t)u(x,t)\Big),\\
d_{21}^{[4]} &= \frac{\beta}{\alpha^4}\Big[3\mathbf{T}_1\overset{*}{u}(-x,-t)u(x,t) + \overset{*}{u}(-x,-t)u_{xx}(x,t) + \overset{*}{u}_x(-x,-t)u_x(x,t) + \overset{*}{u}_{xx}(-x,-t)u(x,t)\Big],
\end{cases}
\tag{4.39}
$$

$$
\begin{cases}
d_{31}^{[0]} &= 0,\\
d_{31}^{[1]} &= 0,\\
d_{31}^{[2]} &= -\frac{\beta}{\alpha^2}u(x,t)\overset{*}{v}(x,t),\\
d_{31}^{[3]} &= -\mathrm{i}\frac{\beta}{\alpha^3}\Big(-\overset{*}{v}(x,t)u_x(x,t) + \overset{*}{v}_x(x,t)u(x,t)\Big),\\
d_{31}^{[4]} &= \frac{\beta}{\alpha^4}\Big[3\mathbf{T}_1\overset{*}{v}(x,t)u(x,t) + \overset{*}{v}(x,t)u_{xx}(x,t) - \overset{*}{v}_x(x,t)u_x(x,t) + \overset{*}{v}_{xx}(x,t)u(x,t)\Big],
\end{cases}
\tag{4.40}
$$

$$\begin{cases} d_{41}^{[0]} &= 0, \\ d_{41}^{[1]} &= 0, \\ d_{41}^{[2]} &= -\frac{\beta}{\alpha^2} u(x,t)\overset{*}{v}(-x,-t), \\ d_{41}^{[3]} &= \mathrm{i}\frac{\beta}{\alpha^3}\Big(\overset{*}{v}(-x,-t)u_x(x,t)+\overset{*}{v}_x(-x,-t)u(x,t)\Big), \\ d_{41}^{[4]} &= \frac{\beta}{\alpha^4}\Big[3\mathbf{T}_1\overset{*}{v}(-x,-t)u(x,t)+\overset{*}{v}(-x,-t)u_{xx}(x,t)+\overset{*}{v}_x(-x,-t)u_x(x,t)+\overset{*}{v}_{xx}(-x,-t)u(x,t)\Big], \end{cases} \tag{4.41}$$

$$\begin{cases} d_{13}^{[0]} &= 0, \\ d_{13}^{[1]} &= 0, \\ d_{13}^{[2]} &= -\frac{\beta}{\alpha^2}\overset{*}{u}(x,t)v(x,t), \\ d_{13}^{[3]} &= -\mathrm{i}\frac{\beta}{\alpha^3}\Big(-\overset{*}{u}(x,t)v_x(x,t)+\overset{*}{u}_x(x,t)v(x,t)\Big), \\ d_{13}^{[4]} &= \frac{\beta}{\alpha^4}\Big[3\mathbf{T}_1\overset{*}{u}(x,t)v(x,t)+\overset{*}{u}(x,t)v_{xx}(x,t)-\overset{*}{u}_x(x,t)v_x(x,t)+\overset{*}{u}_{xx}(x,t)v(x,t)\Big], \end{cases} \tag{4.42}$$

$$\begin{cases} d_{23}^{[0]} &= 0, \\ d_{23}^{[1]} &= 0, \\ d_{23}^{[2]} &= -\frac{\beta}{\alpha^2}\overset{*}{u}(-x,-t)v(x,t), \\ d_{23}^{[3]} &= \mathrm{i}\frac{\beta}{\alpha^3}\Big(\overset{*}{u}(-x,-t)v_x(x,t)+\overset{*}{u}_x(-x,-t)v(x,t)\Big), \\ d_{23}^{[4]} &= \frac{\beta}{\alpha^4}\Big[3\mathbf{T}_1\overset{*}{u}(-x,-t)v(x,t)+\overset{*}{u}(-x,-t)v_{xx}(x,t)+\overset{*}{u}_x(-x,-t)v_x(x,t)+\overset{*}{u}_{xx}(-x,-t)v(x,t)\Big], \end{cases} \tag{4.43}$$

$$
\begin{cases}
d_{33}^{[0]} &= \beta_2, \\
d_{33}^{[1]} &= 0, \\
d_{33}^{[2]} &= -\frac{\beta}{\alpha^2}|v(x,t)|^2, \\
d_{33}^{[3]} &= -2\frac{\beta}{\alpha^3}\mathrm{Im}\Big(\overset{*}{v}(x,t)v_x(x,t)\Big), \\
d_{33}^{[4]} &= \frac{\beta}{\alpha^4}\Big[3\mathbf{T}_1|v(x,t)|^2 + \overset{*}{v}(x,t)v_{xx}(x,t) - |v_x(x,t)|^2 + \overset{*}{v}_{xx}(x,t)v(x,t)\Big],
\end{cases}
\tag{4.44}
$$

$$
\begin{cases}
d_{43}^{[0]} &= 0, \\
d_{43}^{[1]} &= 0, \\
d_{43}^{[2]} &= -\frac{\beta}{\alpha^2}v(x,t)\overset{*}{v}(-x,-t), \\
d_{43}^{[3]} &= \mathrm{i}\frac{\beta}{\alpha^3}\Big(\overset{*}{v}(-x,-t)v_x(x,t) + \overset{*}{v}_x(-x,-t)v(x,t)\Big), \\
d_{43}^{[4]} &= \frac{\beta}{\alpha^4}\Big[3\mathbf{T}_1\overset{*}{v}(-x,-t)v(x,t) + \overset{*}{v}(-x,-t)v_{xx}(x,t) + \overset{*}{v}_x(-x,-t)v_x(x,t) + \overset{*}{v}_{xx}(-x,-t)v(x,t)\Big],
\end{cases}
\tag{4.45}
$$

where $\beta = \beta_1 - \beta_2$, and

$$
\begin{cases}
\mathbf{T}_1 = -\Big(|u(x,t)|^2 + |u(-x,-t)|^2 + |v(x,t)|^2 + |v(-x,-t)|^2\Big), \\
\mathbf{T}_2 = -u_x(x,t)\overset{*}{u}(x,t) + u_x(-x,-t)\overset{*}{u}(-x,-t) - v_x(x,t)\overset{*}{v}(x,t) + v_x(-x,-t)\overset{*}{v}(-x,-t), \\
\mathbf{T}_3 = -u(x,t)\overset{*}{u}_x(x,t) + u(-x,-t)\overset{*}{u}_x(-x,-t) - v(x,t)\overset{*}{v}_x(x,t) + v(-x,-t)\overset{*}{v}_x(-x,-t).
\end{cases}
$$

Remark 4.2.2. Under the symmetry relations (4.8), one can show that W satisfies the equations:

$$
\begin{cases}
W^\dagger(-x,-t,-\overset{*}{\lambda}) = C_1 W(x,t,\lambda) C_1^{-1}, \\
W(-x,-t,-\lambda) = C_2 W(x,t,\lambda) C_2^{-1},
\end{cases}
\tag{4.46}
$$

for a solution W to the stationary zero curvature equation. Using the Laurent expansion (4.22) of W, we get the relations:

$$b_1^{[m]}(-x,-t) = (-1)^m \overset{*}{c}_2^{[m]}(x,t), \quad b_2^{[m]}(-x,-t) = (-1)^m \overset{*}{c}_1^{[m]}(x,t), \tag{4.47}$$

$$b_3^{[m]}(-x,-t) = (-1)^m \overset{*}{c}_4^{[m]}(x,t), \quad b_4^{[m]}(-x,-t) = (-1)^m \overset{*}{c}_3^{[m]}(x,t), \tag{4.48}$$

$$\overset{*}{a}^{[m]}(-x,-t) = (-1)^m a^{[m]}(x,t), \quad \overset{*}{d}_{11}^{[m]}(-x,-t) = (-1)^m d_{22}^{[m]}(x,t), \tag{4.49}$$

$$\overset{*}{d}_{21}^{[m]}(-x,-t) = (-1)^m d_{21}^{[m]}(x,t), \quad \overset{*}{d}_{31}^{[m]}(-x,-t) = (-1)^m d_{24}^{[m]}(x,t), \tag{4.50}$$

$$\overset{*}{d}_{41}^{[m]}(-x,-t) = (-1)^m d_{23}^{[m]}(x,t), \quad \overset{*}{d}_{12}^{[m]}(-x,-t) = (-1)^m d_{12}^{[m]}(x,t), \tag{4.51}$$

$$\overset{*}{d}_{32}^{[m]}(-x,-t) = (-1)^m d_{14}^{[m]}(x,t), \quad \overset{*}{d}_{42}^{[m]}(-x,-t) = (-1)^m d_{13}^{[m]}(x,t), \tag{4.52}$$

$$\overset{*}{d}_{33}^{[m]}(-x,-t) = (-1)^m d_{44}^{[m]}(x,t), \quad \overset{*}{d}_{43}^{[m]}(-x,-t) = (-1)^m d_{43}^{[m]}(x,t), \tag{4.53}$$

$$\overset{*}{d}_{34}^{[m]}(-x,-t) = (-1)^m d_{34}^{[m]}(x,t), \tag{4.54}$$

and

$$b_1^{[m]}(-x,-t) = (-1)^{m+1} b_2^{[m]}(x,t), \quad b_3^{[m]}(-x,-t) = (-1)^{m+1} b_4^{[m]}(x,t), \tag{4.55}$$

$$a^{[m]}(-x,-t) = (-1)^m a^{[m]}(x,t), \tag{4.56}$$

$$d_{11}^{[m]}(-x,-t) = (-1)^m d_{22}^{[m]}(x,t), \quad d_{12}^{[m]}(-x,-t) = (-1)^m d_{21}^{[m]}(x,t), \tag{4.57}$$

$$d_{13}^{[m]}(-x,-t) = (-1)^m d_{24}^{[m]}(x,t), \quad d_{14}^{[m]}(-x,-t) = (-1)^m d_{23}^{[m]}(x,t), \tag{4.58}$$

$$d_{31}^{[m]}(-x,-t) = (-1)^m d_{42}^{[m]}(x,t), \quad d_{32}^{[m]}(-x,-t) = (-1)^m d_{41}^{[m]}(x,t), \tag{4.59}$$

$$d_{33}^{[m]}(-x,-t) = (-1)^m d_{44}^{[m]}(x,t), \quad d_{34}^{[m]}(-x,-t) = (-1)^m d_{43}^{[m]}(x,t). \tag{4.60}$$

We introduce the Lax matrix

$$V^{[m]} = V^{[m]}(\mathbf{u},\lambda) = (\lambda^m W)_+ = \sum_{i=0}^{m} W_i \lambda^{m-i}$$
$$= \sum_{i=0}^{m} \begin{pmatrix} a^{[i]}\lambda^{m-i} & b_1^{[i]}\lambda^{m-i} & b_2^{[i]}\lambda^{m-i} & b_3^{[i]}\lambda^{m-i} & b_4^{[i]}\lambda^{m-i} \\ c_1^{[i]}\lambda^{m-i} & d_{11}^{[i]}\lambda^{m-i} & d_{12}^{[i]}\lambda^{m-i} & d_{13}^{[i]}\lambda^{m-i} & d_{14}^{[i]}\lambda^{m-i} \\ c_2^{[i]}\lambda^{m-i} & d_{21}^{[i]}\lambda^{m-i} & d_{22}^{[i]}\lambda^{m-i} & d_{23}^{[i]}\lambda^{m-i} & d_{24}^{[i]}\lambda^{m-i} \\ c_3^{[i]}\lambda^{m-i} & d_{31}^{[i]}\lambda^{m-i} & d_{32}^{[i]}\lambda^{m-i} & d_{33}^{[i]}\lambda^{m-i} & d_{34}^{[i]}\lambda^{m-i} \\ c_4^{[i]}\lambda^{m-i} & d_{41}^{[i]}\lambda^{m-i} & d_{42}^{[i]}\lambda^{m-i} & d_{43}^{[i]}\lambda^{m-i} & d_{44}^{[i]}\lambda^{m-i} \end{pmatrix},$$

where the modification terms are zero, and we get the spatial and temporal equations of the spectral problems [39], with the associated Lax pair $\{U, V^{[m]}\}$:

$$\varphi_x = \mathrm{i}U\varphi, \tag{4.61}$$

$$\varphi_{t_m} = \mathrm{i}V^{[m]}\varphi. \tag{4.62}$$

The compatibility conditions of equations (4.61)-(4.62) give rise to the zero curvature equations

$$U_{t_m} - V_x^{[m]} + \mathrm{i}[U, V^{[m]}] = 0. \tag{4.63}$$

This gives the two-component Sasa-Satsuma integrable hierarchy

$$\mathbf{u}_{t_m} = K_m(\mathbf{u}) = \begin{pmatrix} u(x,t) \\ v(x,t) \end{pmatrix}_{t_m} = \mathrm{i}\alpha \begin{pmatrix} b_1^{[m+1]} \\ b_3^{[m+1]} \end{pmatrix}, \quad m \geq 0. \tag{4.64}$$

This soliton hierarchy possesses a bi-Hamiltonian structure

$$\mathbf{u}_{t_m} = \Phi^m K_0 = J_1 \frac{\delta \mathcal{H}_{m+1}}{\delta \mathbf{u}} = J_2 \frac{\delta \mathcal{H}_m}{\delta \mathbf{u}}, \tag{4.65}$$

where $\mathcal{H}_m$ are the Hamiltonian functionals and Φ is the recursion operator.

We derive from the recursive relations (4.27)-(4.34) and the relations (4.47)-(4.60), the following recursive formula between $\{b_1^{[m+1]}, b_3^{[m+1]}\}$ and $\{b_1^{[m]}, b_3^{[m]}\}$:

$$\begin{pmatrix} b_1^{[m+1]} \\ b_3^{[m+1]} \end{pmatrix} = \Phi \begin{pmatrix} b_1^{[m]} \\ b_3^{[m]} \end{pmatrix}, \tag{4.66}$$

where the recursion operator Φ reads:

$$\Phi = \frac{\mathrm{i}}{\alpha} \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix}, \tag{4.67}$$

with components

$$\begin{aligned}
\Phi_{11} &= -\partial + u(-x)\partial^{-1}\overset{*}{u}(-x) + v(x)\partial^{-1}\overset{*}{v}(x) + v(-x)\partial^{-1}\overset{*}{v}(-x) + \mathrm{i}4u(x)\partial^{-1}\mathrm{Im}(\overset{*}{u}(x)) \\
&\quad + \mathrm{i}2(-1)^{-m-1}u(x)\partial^{-1}\mathrm{Im}(\overset{*}{u}(-x)\Gamma_-) + (-1)^{-m}u(-x)\partial^{-1}u(x)\overset{*}{\Gamma}_-, && (4.68) \\
\Phi_{12} &= \mathrm{i}2u(x)\partial^{-1}\mathrm{Im}(\overset{*}{v}(x)) + \mathrm{i}2(-1)^{-m-1}u(x)\partial^{-1}\mathrm{Im}(\overset{*}{v}(-x)\Gamma_-) - v(x)\partial^{-1}u(x)\overset{*}{\Gamma}_+ \\
&\quad + (-1)^{-m}v(-x)\partial^{-1}u(x)\overset{*}{\Gamma}_-, && (4.69) \\
\Phi_{21} &= \mathrm{i}2v(x)\partial^{-1}\mathrm{Im}(\overset{*}{u}(x)) + \mathrm{i}2(-1)^{-m-1}v(x)\partial^{-1}\mathrm{Im}(\overset{*}{u}(-x)\Gamma_-) - u(x)\partial^{-1}v(x)\overset{*}{\Gamma}_+ \\
&\quad + (-1)^{-m}u(-x)\partial^{-1}v(x)\overset{*}{\Gamma}_-, && (4.70) \\
\Phi_{22} &= -\partial + v(-x)\partial^{-1}\overset{*}{v}(-x) + u(x)\partial^{-1}\overset{*}{u}(x) + u(-x)\partial^{-1}\overset{*}{u}(-x) + \mathrm{i}4v(x)\partial^{-1}\mathrm{Im}(\overset{*}{v}(x)) \\
&\quad + \mathrm{i}2(-1)^{-m-1}v(x)\partial^{-1}\mathrm{Im}(\overset{*}{v}(-x)\Gamma_-) + (-1)^{-m}v(-x)\partial^{-1}v(x)\overset{*}{\Gamma}_-, && (4.71)
\end{aligned}$$

and where the operators $\Gamma_-, \overset{*}{\Gamma}_\pm$ are defined by:

$$\Gamma_- f(x,t) = f(-x,-t), \tag{4.72}$$

$$\overset{*}{\Gamma}_+ f(x,t) = \overset{*}{f}(x,t), \tag{4.73}$$

$$\overset{*}{\Gamma}_- f(x,t) = \overset{*}{f}(-x,-t), \tag{4.74}$$

with Γ_+ being the identity operator, i.e., $\Gamma_+ f(x,t) = \mathrm{Id} f(x,t) = f(x,t)$.

4.2.1 Nonlocal two-component Sasa-Satsuma equation

To derive a two-component nonlocal Sasa-Satsuma equation, we take the Lax matrix

$$V^{[3]} = V^{[3]}(\mathbf{u}, \lambda) = (\lambda^3 W)_+. \tag{4.75}$$

The spatial and temporal equations of the spectral problems (4.61) and (4.62) with the associated Lax pair $\{U, V^{[3]}\}$ read

$$\varphi_x = \mathrm{i}U\varphi, \tag{4.76}$$

$$\varphi_{t_3} = \mathrm{i}V^{[3]}\varphi, \tag{4.77}$$

while the zero curvature equation

$$U_{t_3} - V^{[3]}_x + \mathrm{i}[U, V^{[3]}] = 0 \tag{4.78}$$

gives the two-component Sasa-Satsuma equation

$$\mathbf{u}_{t_3} = \begin{pmatrix} u(x,t) \\ v(x,t) \end{pmatrix}_{t_3} = \mathrm{i}\alpha \begin{pmatrix} b_1^{[4]} \\ b_3^{[4]} \end{pmatrix}. \tag{4.79}$$

The $V^{[3]}$ reads

$$V^{[3]} = \begin{pmatrix} V_{11} & V_{12} & V_{13} & V_{14} & V_{15} \\ V_{21} & V_{22} & V_{23} & V_{24} & V_{25} \\ V_{31} & V_{32} & V_{33} & V_{34} & V_{35} \\ V_{41} & V_{42} & V_{43} & V_{44} & V_{45} \\ V_{51} & V_{52} & V_{53} & V_{54} & V_{55} \end{pmatrix}, \tag{4.80}$$

where

$$V_{11}=a^{[0]}\lambda^3+a^{[1]}\lambda^2+a^{[2]}\lambda+a^{[3]}, \qquad V_{21}=c_1^{[0]}\lambda^3+c_1^{[1]}\lambda^2+c_1^{[2]}\lambda+c_1^{[3]},$$

$$V_{12}=b_1^{[0]}\lambda^3+b_1^{[1]}\lambda^2+b_1^{[2]}\lambda+b_1^{[3]}, \qquad V_{22}=d_{11}^{[0]}\lambda^3+d_{11}^{[1]}\lambda^2+d_{11}^{[2]}\lambda,+d_{11}^{[3]},$$

$$V_{13}=b_2^{[0]}\lambda^3+b_2^{[1]}\lambda^2+b_2^{[2]}\lambda+b_2^{[3]}, \qquad V_{23}=d_{12}^{[0]}\lambda^3+d_{12}^{[1]}\lambda^2+d_{12}^{[2]}\lambda+d_{12}^{[3]},$$

$$V_{14}=b_3^{[0]}\lambda^3+b_3^{[1]}\lambda^2+b_3^{[2]}\lambda+b_3^{[3]}, \qquad V_{24}=d_{13}^{[0]}\lambda^3+d_{13}^{[1]}\lambda^2+d_{13}^{[2]}\lambda+d_{13}^{[3]},$$

$$V_{15}=b_4^{[0]}\lambda^3+b_4^{[1]}\lambda^2+b_4^{[2]}\lambda+b_4^{[3]}, \qquad V_{25}=d_{14}^{[0]}\lambda^3+d_{14}^{[1]}\lambda^2+d_{14}^{[2]}\lambda+d_{14}^{[3]},$$

$$V_{31}=c_2^{[0]}\lambda^3+c_2^{[1]}\lambda^2+c_2^{[2]}\lambda+c_2^{[3]}, \qquad V_{41}=c_3^{[0]}\lambda^3+c_3^{[1]}\lambda^2+c_3^{[2]}\lambda+c_3^{[3]},$$

$$V_{32}=d_{21}^{[0]}\lambda^3+d_{21}^{[1]}\lambda^2+d_{21}^{[2]}\lambda,+d_{21}^{[3]}, \qquad V_{42}=d_{31}^{[0]}\lambda^3+d_{31}^{[1]}\lambda^2+d_{31}^{[2]}\lambda+d_{31}^{[3]},$$

$$V_{33}=d_{22}^{[0]}\lambda^3+d_{22}^{[1]}\lambda^2+d_{22}^{[2]}\lambda,+d_{22}^{[3]}, \qquad V_{43}=d_{32}^{[0]}\lambda^3+d_{32}^{[1]}\lambda^2+d_{32}^{[2]}\lambda+d_{32}^{[3]},$$

$$V_{34}=d_{23}^{[0]}\lambda^3+d_{23}^{[1]}\lambda^2+d_{23}^{[2]}\lambda,+d_{23}^{[3]}, \qquad V_{44}=d_{33}^{[0]}\lambda^3+d_{33}^{[1]}\lambda^2+d_{33}^{[2]}\lambda+d_{33}^{[3]},$$

$$V_{35}=d_{24}^{[0]}\lambda^3+d_{24}^{[1]}\lambda^2+d_{24}^{[2]}\lambda+d_{24}^{[3]}, \qquad V_{45}=d_{34}^{[0]}\lambda^3+d_{34}^{[1]}\lambda^2+d_{34}^{[2]}\lambda+d_{34}^{[3]},$$

$$V_{51}=c_4^{[0]}\lambda^3+c_4^{[1]}\lambda^2+c_4^{[2]}\lambda+c_4^{[3]},$$

$$V_{52}=d_{41}^{[0]}\lambda^3+d_{41}^{[1]}\lambda^2+d_{41}^{[2]}\lambda+d_{41}^{[3]},$$

$$V_{53}=d_{42}^{[0]}\lambda^3+d_{42}^{[1]}\lambda^2+d_{42}^{[2]}\lambda+d_{42}^{[3]},$$

$$V_{54}=d_{43}^{[0]}\lambda^3+d_{43}^{[1]}\lambda^2+d_{43}^{[2]}\lambda+d_{43}^{[3]},$$

$$V_{55}=d_{44}^{[0]}\lambda^3+d_{44}^{[1]}\lambda^2+d_{44}^{[2]}\lambda+d_{44}^{[3]}.$$

The matrix V exhibits the properties of symmetry:

$$\begin{cases} V^{[3]\dagger}(-x,-t,-\overset{*}{\lambda}) = -C_1 V^{[3]}(x,t,\lambda)C_1^{-1}, \\ V^{[3]}(-x,-t,-\lambda) = -C_2 V^{[3]}(x,t,\lambda)C_2^{-1}. \end{cases} \tag{4.81}$$

The equation (4.79) leads explicitly to the nonlocal reverse-spacetime Sasa-Satsuma equation:

$$u_{t_3}(x,t) = -\frac{\beta}{\alpha^3}\big[u_{xxx}(x,t) + 3\mathbf{T}_1 u_x(x,t) + 3\mathbf{T}_2 u(x,t)\big], \tag{4.82}$$

$$v_{t_3}(x,t) = -\frac{\beta}{\alpha^3}\big[v_{xxx}(x,t) + 3\mathbf{T}_1 v_x(x,t) + 3\mathbf{T}_2 v(x,t)\big], \tag{4.83}$$

where $\mathbf{T}_1$ and $\mathbf{T}_2$ are defined by

$$\mathbf{T}_1 = -\Big(|u(x,t)|^2 + |u(-x,-t)|^2 + |v(x,t)|^2 + |v(-x,-t)|^2\Big),$$

$$\mathbf{T}_2 = -u_x(x,t)\overset{*}{u}(x,t) + u_x(-x,-t)\overset{*}{u}(-x,-t) - v_x(x,t)\overset{*}{v}(x,t) + v_x(-x,-t)\overset{*}{v}(-x,-t).$$

4.2.2 Bi-Hamiltonian structure

We start to find a bi-Hamiltonian structure of the soliton hierarchy (4.64). To do so, we are going to use the trace identity

$$\frac{\delta}{\delta \mathbf{u}}\int tr\big(W\frac{\partial U}{\partial \lambda}\big)dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\Big[\lambda^{\gamma} tr\big(W\frac{\partial U}{\partial \mathbf{u}}\big)\Big], \tag{4.84}$$

where

$$\gamma = -\frac{\lambda}{2}\frac{d}{d\lambda}\ln|tr\big(W^2\big)|. \tag{4.85}$$

Let $\mathbf{u} = (u, v, p_1, p_2, p_3, q_1, q_2, q_3)^T$ be a representation of the components of the P matrix, such that:

$$p_1 = u(-x,-t),\ p_2 = \overset{*}{u},\ p_3 = \overset{*}{u}(-x,-t), \tag{4.86}$$

$$q_1 = v(-x,-t),\ q_2 = \overset{*}{v},\ q_3 = \overset{*}{v}(-x,-t), \tag{4.87}$$

and $\mathbf{u}_t = K_m(u, v, p_i, q_i, u_x, v_x, p_{i,x}, q_{i,x}, \ldots)$, this can be written equivalently as $\mathbf{u}_t = K_m(u, v, u_x, v_x, \ldots)$. Thus, from the matrix U, one can easily compute $\frac{\partial U}{\partial \mathbf{u}}$ to obtain,

$$tr\Big[W\frac{\partial U}{\partial \lambda}\Big] = \alpha_1 a(x,t) + \alpha_2\sum_{i=1}^{4} d_{ii}(x,t),$$

$$tr\Big[W\frac{\partial U}{\partial u(x,t)}\Big] = c_1,\ tr\Big[W\frac{\partial U}{\partial v(x,t)}\Big] = c_3.$$

By substituting these in the trace identity formula (4.84), and matching the powers of λ^{-m-1}, we get

$$\frac{\delta}{\delta \mathbf{u}} \int \Big(\alpha_1 a^{[m+1]} + \alpha_2 (1 + (-1)^{-m-1} \overset{*}{\Gamma}_-)(d_{11}^{[m+1]} + d_{33}^{[m+1]}) \Big) dx = (\gamma - m) \begin{pmatrix} -\overset{*}{\Gamma}_+ b_1^{[m]} \\ -\overset{*}{\Gamma}_+ b_3^{[m]} \end{pmatrix}, \quad m \geq 1. \tag{4.88}$$

where $\frac{\delta}{\delta \mathbf{u}}$ is defined as:

$$\frac{\delta}{\delta \mathbf{u}} = \begin{pmatrix} \frac{\delta}{\delta u(x,t)} \\ \frac{\delta}{\delta v(x,t)} \end{pmatrix}. \tag{4.89}$$

When $m = 2$, this yields $\gamma = 0$. Hence, the Hamiltonians can be taken as

$$\mathcal{H}_m = -\frac{\mathrm{i}}{m} \int \alpha_1 a^{[m+1]} + \alpha_2 (1 + (-1)^{-m-1} \overset{*}{\Gamma}_-)(d_{11}^{[m+1]} + d_{33}^{[m+1]}) dx, \quad m \geq 1, \tag{4.90}$$

and we have

$$\frac{\delta \mathcal{H}_m}{\delta \mathbf{u}} = \mathrm{i} \begin{pmatrix} -\overset{*}{\Gamma}_+ b_1^{[m]} \\ -\overset{*}{\Gamma}_+ b_3^{[m]} \end{pmatrix}, \quad m \geq 0. \tag{4.91}$$

Since

$$\mathbf{u}_{t_m} = \mathrm{i}\alpha \begin{pmatrix} b_1^{[m+1]} \\ b_3^{[m+1]} \end{pmatrix} = J_1 \frac{\delta \mathcal{H}_{m+1}}{\delta \mathbf{u}} = J_1 \begin{pmatrix} -\mathrm{i}\overset{*}{\Gamma}_+ b_1^{[m+1]} \\ -\mathrm{i}\overset{*}{\Gamma}_+ b_3^{[m+1]} \end{pmatrix} \tag{4.92}$$

$$= J_1 \Phi \frac{\delta \mathcal{H}_m}{\delta \mathbf{u}} = J_2 \frac{\delta \mathcal{H}_m}{\delta \mathbf{u}}, \tag{4.93}$$

where $\overset{*}{\Gamma}_+(c\overset{*}{\Gamma}_+ f) = \overset{*}{c} f$, and c is any complex number.

Moreover, we can deduce the Hamiltonian pair J_1 and J_2 as follows:

$$J_1 = \begin{pmatrix} \alpha \overset{*}{\Gamma}_+ & 0 \\ 0 & \alpha \overset{*}{\Gamma}_+ \end{pmatrix} \tag{4.94}$$

and

$$J_2 = \Phi J_1 = \mathrm{i} \begin{pmatrix} \Phi_{11} \overset{*}{\Gamma}_+ & \Phi_{12} \overset{*}{\Gamma}_+ \\ \Phi_{21} \overset{*}{\Gamma}_+ & \Phi_{22} \overset{*}{\Gamma}_+ \end{pmatrix}. \tag{4.95}$$

Using the matrix U, we can find the first four Hamiltonian functionals:

$$\mathcal{H}_1 = -\mathrm{i}\frac{\beta}{\alpha}\int \Big(|u(x,t)|^2 + |u(-x,-t)|^2 + |v(x,t)|^2 + |v(-x,-t)|^2\Big)dx, \tag{4.96}$$
$$\mathcal{H}_2 = -\mathrm{i}\frac{\beta}{\alpha^2}\int \mathrm{Im}\Big(u_x(x,t)\overset{*}{u}(x,t) + u(-x,-t)\overset{*}{u}_x(-x,-t)$$
$$+ v_x(x,t)\overset{*}{v}(x,t) + v(-x,-t)\overset{*}{v}_x(-x,-t)\Big)dx,$$
$$\mathcal{H}_3 = -\mathrm{i}\frac{\beta}{3\alpha^3}\int \Big[3\Big(|u(x,t)|^2 + |u(-x,-t)|^2 + |v(x,t)|^2 + |v(-x,-t)|^2\Big)^2$$
$$- 2\mathrm{Re}\Big(u(x,t)\overset{*}{u}_{xx}(x,t) + u(-x,-t)\overset{*}{u}_{xx}(-x,-t) + v(x,t)\overset{*}{v}_{xx}(x,t) + v(-x,-t)\overset{*}{v}_{xx}(-x,-t)\Big)$$
$$+ \Big(|u_x(x,t)|^2 + |u_x(-x,-t)|^2 + |v_x(x,t)|^2 + |v_x(-x,-t)|^2\Big)\Big]dx,$$
$$\mathcal{H}_4 = -\mathrm{i}\frac{\beta}{2\alpha^4}\int \Big[6\Big(|u(x,t)|^2 + |u(-x,-t)|^2 + |v(x,t)|^2 + |v(-x,-t)|^2\Big)\cdot$$
$$\Big(\mathrm{Im}\Big(u_x(x,t)\overset{*}{u}(x,t) + u(-x,-t)\overset{*}{u}_x(-x,-t) + v_x(x,t)\overset{*}{v}(x,t) + v(-x,-t)\overset{*}{v}_x(-x,-t)\Big)\Big)$$
$$-\mathrm{Im}\Big(u_{xxx}(x,t)\overset{*}{u}(x,t) + \overset{*}{u}_{xxx}(-x,-t)u(-x,-t) + v_{xxx}(x,t)\overset{*}{v}(x,t) + \overset{*}{v}_{xxx}(-x,-t)v(-x,-t)\Big)$$
$$-\mathrm{Im}\Big(u_x(x,t)\overset{*}{u}_{xx}(x,t) + u_{xx}(-x,-t)\overset{*}{u}_x(-x,-t) + v_x(x,t)\overset{*}{v}_{xx}(x,t) + v_{xx}(-x,-t)\overset{*}{v}_x(-x,-t)\Big)\Big]dx.$$

4.3 Riemann-Hilbert problems

The spatial and temporal spectral problems of the two-component nonlocal Sasa-Satsuma equation can be written as:

$$\varphi_x = iU\varphi = i(\lambda\Lambda + P)\varphi, \tag{4.97}$$

$$\varphi_t = iV^{[3]}\varphi = i(\lambda^3\Omega + Q)\varphi, \tag{4.98}$$

where $\Lambda = diag(\alpha_1, \alpha_2 I_4)$, $\Omega = diag(\beta_1, \beta_2 I_4)$, and

$$P = \begin{pmatrix} 0 & u(x,t) & u(-x,-t) & v(x,t) & v(-x,-t) \\ -\overset{*}{u}(x,t) & 0 & 0 & 0 & 0 \\ -\overset{*}{u}(-x,-t) & 0 & 0 & 0 & 0 \\ -\overset{*}{v}(x,t) & 0 & 0 & 0 & 0 \\ -\overset{*}{v}(-x,-t) & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{4.99}$$

$$Q = \begin{pmatrix} Q_{11} & Q_{12} & Q_{13} & Q_{14} & Q_{15} \\ Q_{21} & Q_{22} & Q_{23} & Q_{24} & Q_{25} \\ Q_{31} & Q_{32} & Q_{33} & Q_{34} & Q_{35} \\ Q_{41} & Q_{42} & Q_{43} & Q_{44} & Q_{45} \\ Q_{51} & Q_{52} & Q_{53} & Q_{54} & Q_{55} \end{pmatrix}, \tag{4.100}$$

with components

$$Q_{11} = a^{[1]}\lambda^2 + a^{[2]}\lambda + a^{[3]}, \qquad Q_{21} = c_1^{[0]}\lambda^3 + c_1^{[1]}\lambda^2 + c_1^{[2]}\lambda + c_1^{[3]},$$
$$Q_{12} = b_1^{[0]}\lambda^3 + b_1^{[1]}\lambda^2 + b_1^{[2]}\lambda + b_1^{[3]}, \qquad Q_{22} = d_{11}^{[1]}\lambda^2 + d_{11}^{[2]}\lambda, + d_{11}^{[3]},$$
$$Q_{13} = b_2^{[0]}\lambda^3 + b_2^{[1]}\lambda^2 + b_2^{[2]}\lambda + b_2^{[3]}, \qquad Q_{23} = d_{12}^{[0]}\lambda^3 + d_{12}^{[1]}\lambda^2 + d_{12}^{[2]}\lambda + d_{12}^{[3]},$$
$$Q_{14} = b_3^{[0]}\lambda^3 + b_3^{[1]}\lambda^2 + b_3^{[2]}\lambda + b_3^{[3]}, \qquad Q_{24} = d_{13}^{[0]}\lambda^3 + d_{13}^{[1]}\lambda^2 + d_{13}^{[2]}\lambda + d_{13}^{[3]},$$
$$Q_{15} = b_4^{[0]}\lambda^3 + b_4^{[1]}\lambda^2 + b_4^{[2]}\lambda + b_4^{[3]}, \qquad Q_{25} = d_{14}^{[0]}\lambda^3 + d_{14}^{[1]}\lambda^2 + d_{14}^{[2]}\lambda + d_{14}^{[3]},$$

$$Q_{31} = c_2^{[0]}\lambda^3 + c_2^{[1]}\lambda^2 + c_2^{[2]}\lambda + c_2^{[3]}, \qquad Q_{41} = c_3^{[0]}\lambda^3 + c_3^{[1]}\lambda^2 + c_3^{[2]}\lambda + c_3^{[3]},$$
$$Q_{32} = d_{21}^{[0]}\lambda^3 + d_{21}^{[1]}\lambda^2 + d_{21}^{[2]}\lambda, + d_{21}^{[3]}, \qquad Q_{42} = d_{31}^{[0]}\lambda^3 + d_{31}^{[1]}\lambda^2 + d_{31}^{[2]}\lambda + d_{31}^{[3]},$$
$$Q_{33} = d_{22}^{[1]}\lambda^2 + d_{22}^{[2]}\lambda, + d_{22}^{[3]}, \qquad Q_{43} = d_{32}^{[0]}\lambda^3 + d_{32}^{[1]}\lambda^2 + d_{32}^{[2]}\lambda + d_{32}^{[3]},$$
$$Q_{34} = d_{23}^{[0]}\lambda^3 + d_{23}^{[1]}\lambda^2 + d_{23}^{[2]}\lambda, + d_{23}^{[3]}, \qquad Q_{44} = d_{33}^{[1]}\lambda^2 + d_{33}^{[2]}\lambda + d_{33}^{[3]},$$
$$Q_{35} = d_{24}^{[0]}\lambda^3 + d_{24}^{[1]}\lambda^2 + d_{24}^{[2]}\lambda + d_{24}^{[3]}, \qquad Q_{45} = d_{34}^{[0]}\lambda^3 + d_{34}^{[1]}\lambda^2 + d_{34}^{[2]}\lambda + d_{34}^{[3]},$$

$$Q_{51} = c_4^{[0]}\lambda^3 + c_4^{[1]}\lambda^2 + c_4^{[2]}\lambda + c_4^{[3]},$$
$$Q_{52} = d_{41}^{[0]}\lambda^3 + d_{41}^{[1]}\lambda^2 + d_{41}^{[2]}\lambda + d_{41}^{[3]},$$
$$Q_{53} = d_{42}^{[0]}\lambda^3 + d_{42}^{[1]}\lambda^2 + d_{42}^{[2]}\lambda + d_{42}^{[3]},$$
$$Q_{54} = d_{43}^{[0]}\lambda^3 + d_{43}^{[1]}\lambda^2 + d_{43}^{[2]}\lambda + d_{43}^{[3]},$$
$$Q_{55} = d_{44}^{[1]}\lambda^2 + d_{44}^{[2]}\lambda + d_{44}^{[3]}.$$

Here we assume that $\alpha = \alpha_1 - \alpha_2 < 0$ and $\beta = \beta_1 - \beta_2 < 0$, where $\beta_1 + 4\beta_2 = 0$.

To find soliton solutions, we begin with an initial condition $\{u(x, 0), v(x, 0)\}$ and evolute in time to reach $\{u(x, t), v(x, t)\}$. Assume that u and v decay exponentially, i.e., $u \to 0$ and $v \to 0$ as $x, t \to \pm\infty$. Therefore from the spectral problems (4.97) and (4.98), the asymptotic behaviour of the fundamental matrix φ can be written as

$$\varphi(x, t) \rightsquigarrow e^{i\lambda \Lambda x + i\lambda^3 \Omega t}. \tag{4.101}$$

Hence, the solution of the spectral problems can be written in the form:

$$\varphi(x, t) = \psi(x, t) e^{i\lambda \Lambda x + i\lambda^3 \Omega t}. \tag{4.102}$$

The Jost solution of the eigenfunction (4.102) requires that [20, 45]

$$\psi(x, t) \to I_5, \quad \text{as} \quad x, t \to \pm\infty, \tag{4.103}$$

where I_5 is the 5×5 identity matrix. We denote

$$\psi^{\pm} \to I_5, \quad \text{when} \quad x \to \pm\infty. \tag{4.104}$$

Using equation (4.102), the spectral problems (4.97) and (4.98) can be written equivalently as:

$$\psi_x = i\lambda[\Lambda, \psi] + iP\psi, \tag{4.105}$$

$$\psi_{t_3} = i\lambda^3[\Omega, \psi] + iQ\psi. \tag{4.106}$$

To construct the Riemann-Hilbert problems and their solutions in the reflectionless case, we are going to use the adjoint scattering equations of the spectral problems $\varphi_x = iU\varphi$ and $\varphi_{t_3} = iV^{[3]}\varphi$. Their adjoints are

$$\tilde{\varphi}_x = -i\tilde{\varphi}U, \tag{4.107}$$

$$\tilde{\varphi}_{t_3} = -i\tilde{\varphi}V^{[3]}, \tag{4.108}$$

and the equivalent spectral adjoint equations read

$$\tilde{\psi}_x = -i\lambda[\tilde{\psi}, \mathit{\Lambda}] - i\tilde{\psi}P, \tag{4.109}$$

$$\tilde{\psi}_{t_3} = -i\lambda^3[\tilde{\psi}, \mathit{\Omega}] - i\tilde{\psi}Q. \tag{4.110}$$

Because $tr(iP) = 0$ and $tr(iQ) = 0$, using Liouville's formula [45], it is easy to see that the $(det(\psi))_x = 0$, that is, $det(\psi)$ is a constant, and utilizing the boundary condition (4.103), we conclude

$$det(\psi) = 1, \tag{4.111}$$

and hence the Jost matrix ψ is invertible.

Furthermore, as $\psi_x^{-1} = -\psi^{-1}\psi_x\psi^{-1}$, we can derive from (4.105),

$$\psi_x^{-1} = -i\lambda[\psi^{-1}, \mathit{\Lambda}] - i\psi^{-1}P. \tag{4.112}$$

Thus, we can see that both $(\psi^+)^{-1}$ and $(\psi^-)^{-1}$ satisfies the spatial adjoint equation (4.109). We can also show that both satisfies the temporal adjoint equation (4.110) as well.

It can be shown that if eigenfunction $\psi(x, t, \lambda)$ is a solution to the spectral problem (4.105), then $\psi^{-1}(x, t, \lambda)$ is a solution to the adjoint spectral problem (4.109).

This implies that $C_1\psi^{-1}(x, t, \lambda)$ is also a solution of (4.109) with the same eigenvalue, because $\psi_x^{-1} = -\psi^{-1}\psi_x\psi^{-1}$. In a similar way, the nonlocal $\psi^\dagger(-x, -t, -\overset{*}{\lambda})C_1$ is also a solution of the spectral adjoint problem (4.109). Since the boundary condition is the same for both solutions as $x \to \pm\infty$, this guarantees the uniqueness of the solution, so

$$\psi^\dagger(-x, -t, -\overset{*}{\lambda}) = C_1\psi^{-1}(x, t, \lambda)C_1^{-1}. \tag{4.113}$$

As a result, if λ is an eigenvalue of equation (4.105) or (4.109), then $-\overset{*}{\lambda}$ is also an eigenvalue and the relation (4.113) is satisfied.

In the same way, one can prove that $\psi(x, t, \lambda)C_2^{-1}$ and $C_2^{-1}\psi(-x, -t, -\lambda)$ satisfy (4.105), using the boundary condition and by uniqueness of the solution, we can also derive

$$\psi(-x,-t,-\lambda)=C_2\psi(x,t,\lambda)C_2^{-1}. \tag{4.114}$$

Now, we are going to work with the spatial spectral problem (4.105), assuming that the time is $t = 0$. For notation simplicity, we denote Y^+ and Y^- to indicate the boundary conditions are set as $x \to \infty$ and $x \to -\infty$, respectively.

We know that

$$\psi^{\pm} \to I_5 \quad \text{when} \quad x \to \pm\infty. \tag{4.115}$$

From (4.102), we can write

$$\varphi^{\pm} = \psi^{\pm} e^{i\lambda\Lambda x}. \tag{4.116}$$

Both φ^+ and φ^- satisfy the spectral spatial differential equation (4.97), i.e. both are two solutions of that equation. Thus, they are linearly dependent. So, there exists a scattering matrix $S(\lambda)$ such that

$$\varphi^- = \varphi^+ S(\lambda), \tag{4.117}$$

and substituting (4.116) into (4.117), leads to

$$\psi^- = \psi^+ e^{i\lambda\Lambda x} S(\lambda) e^{-i\lambda\Lambda x}, \quad \text{for} \quad \lambda \in \mathbb{R}, \tag{4.118}$$

where

$$S(\lambda) = (s_{ij})_{5\times5} = \begin{pmatrix} s_{11} & s_{12} & s_{13} & s_{14} & s_{15} \\ s_{21} & s_{22} & s_{23} & s_{24} & s_{25} \\ s_{31} & s_{32} & s_{33} & s_{34} & s_{35} \\ s_{41} & s_{42} & s_{43} & s_{44} & s_{45} \\ s_{41} & s_{42} & s_{43} & s_{44} & s_{55} \end{pmatrix}. \tag{4.119}$$

Given that $det(\psi^{\pm}) = 1$, we obtain

$$det(S(\lambda)) = 1. \tag{4.120}$$

In addition, we can show from (4.113) and (4.118) that $S(\lambda)$ possesses the involution relation

$$S^{\dagger}(-\overset{*}{\lambda}) = C_1 S^{-1}(\lambda) C_1^{-1}. \tag{4.121}$$

We deduce from (4.121) that

$$\hat{s}_{11}(\lambda) = \overset{*}{s}_{11}(-\overset{*}{\lambda}), \tag{4.122}$$

where the inverse of the scattering data matrix is denoted by $S^{-1} = (\hat{s}_{jk})_{5\times 5}$ for $j, k \in \{1, 2, 3, 4, 5\}$. We can show similarly from (4.114) and (4.118) that $S(\lambda)$ satisfies

$$S(-\lambda) = C_2 S(\lambda) C_2^{-1}. \tag{4.123}$$

This leads us to deduce

$$s_{11}(-\lambda) = s_{11}(\lambda). \tag{4.124}$$

In order to formulate Riemann-Hilbert problems, we need to analyse the analyticity of the Jost matrix $\psi^{\pm}$. Our solutions $\psi^{\pm}$ to this problem can be uniquely written by using the Volterra integral equations in conjunction with the spatial spectral problem (4.97):

$$\psi^{-}(x,\lambda) = I_5 + i \int_{-\infty}^{x} e^{i\lambda(x-y)\Lambda} P(y) \psi^{-}(y,\lambda) e^{i\lambda(y-x)\Lambda} dy, \tag{4.125}$$

$$\psi^{+}(x,\lambda) = I_5 - i \int_{x}^{+\infty} e^{i\lambda(x-y)\Lambda} P(y) \psi^{+}(y,\lambda) e^{i\lambda(y-x)\Lambda} dy. \tag{4.126}$$

We denote the matrix ψ^{-} to be

$$\psi^{-} = \begin{pmatrix} \psi_{11}^{-} & \psi_{12}^{-} & \psi_{13}^{-} & \psi_{14}^{-} & \psi_{15}^{-} \\ \psi_{21}^{-} & \psi_{22}^{-} & \psi_{23}^{-} & \psi_{24}^{-} & \psi_{25}^{-} \\ \psi_{31}^{-} & \psi_{32}^{-} & \psi_{33}^{-} & \psi_{34}^{-} & \psi_{35}^{-} \\ \psi_{41}^{-} & \psi_{42}^{-} & \psi_{43}^{-} & \psi_{44}^{-} & \psi_{45}^{-} \\ \psi_{51}^{-} & \psi_{52}^{-} & \psi_{53}^{-} & \psi_{54}^{-} & \psi_{55}^{-} \end{pmatrix}, \tag{4.127}$$

and ψ^+ is denoted similarly. Thus from (4.125) the components of the first column of ψ^- are

$$\begin{aligned}
\psi_{11}^- &= 1 + i\int_{-\infty}^{x} (u(y)\psi_{21}^-(y,\lambda) + u(-y)\psi_{31}^-(y,\lambda) + v(y)\psi_{41}^-(y,\lambda) + v(-y)\psi_{51}^-(y,\lambda))dy,\\
\psi_{21}^- &= -i\int_{-\infty}^{x} \overset{*}{u}(y)\psi_{11}^-(y,\lambda)e^{-i\lambda\alpha(x-y)}dy, \quad \psi_{31}^- = -i\int_{-\infty}^{x} \overset{*}{u}(-y)\psi_{11}^-(y,\lambda)e^{-i\lambda\alpha(x-y)}dy,\\
\psi_{41}^- &= -i\int_{-\infty}^{x} \overset{*}{v}(y)\psi_{11}^-(y,\lambda)e^{-i\lambda\alpha(x-y)}dy, \quad \psi_{51}^- = -i\int_{-\infty}^{x} \overset{*}{v}(-y)\psi_{11}^-(y,\lambda)e^{-i\lambda\alpha(x-y)}dy.
\end{aligned}$$

Similarly, the components of the second column of ψ^- are

$$\begin{aligned}
\psi_{12}^- &= i\int_{-\infty}^{x} \Big(u(y)\psi_{22}^-(y,\lambda) + u(-y)\psi_{32}^-(y,\lambda) + v(y)\psi_{42}^-(y,\lambda) + v(-y)\psi_{52}^-(y,\lambda)\Big)e^{i\lambda\alpha(x-y)}dy,\\
\psi_{22}^- &= 1 - i\int_{-\infty}^{x} \overset{*}{u}(y)\psi_{12}^-(y,\lambda)dy, \quad \psi_{32}^- = -i\int_{-\infty}^{x} \overset{*}{u}(-y)\psi_{12}^-(y,\lambda)dy,\\
\psi_{42}^- &= -i\int_{-\infty}^{x} \overset{*}{v}(y)\psi_{12}^-(y,\lambda)dy, \quad \psi_{52}^- = -i\int_{-\infty}^{x} \overset{*}{v}(-y)\psi_{12}^-(y,\lambda)dy,
\end{aligned}$$

and the components of the third column of ψ^- are

$$\begin{aligned}
\psi_{13}^- &= i\int_{-\infty}^{x} \Big(u(y)\psi_{23}^-(y,\lambda) + u(-y)\psi_{33}^-(y,\lambda) + v(y)\psi_{43}^-(y,\lambda) + v(-y)\psi_{53}^-(y,\lambda)\Big)e^{i\lambda\alpha(x-y)}dy,\\
\psi_{23}^- &= -i\int_{-\infty}^{x} \overset{*}{u}(y)\psi_{13}^-(y,\lambda)dy, \quad \psi_{33}^- = 1 - i\int_{-\infty}^{x} \overset{*}{u}(-y)\psi_{13}^-(y,\lambda)dy,\\
\psi_{43}^- &= -i\int_{-\infty}^{x} \overset{*}{v}(y)\psi_{13}^-(y,\lambda)dy, \quad \psi_{53}^- = -i\int_{-\infty}^{x} \overset{*}{v}(-y)\psi_{13}^-(y,\lambda)dy,
\end{aligned}$$

the components of the fourth column of ψ^- are

$$\begin{aligned}
\psi_{14}^- &= i\int_{-\infty}^{x} \Big(u(y)\psi_{24}^-(y,\lambda) + u(-y)\psi_{34}^-(y,\lambda) + v(y)\psi_{44}^-(y,\lambda) + v(-y)\psi_{54}^-(y,\lambda)\Big)e^{i\lambda\alpha(x-y)}dy,\\
\psi_{24}^- &= -i\int_{-\infty}^{x} \overset{*}{u}(y)\psi_{14}^-(y,\lambda)dy, \quad \psi_{34}^- = -i\int_{-\infty}^{x} \overset{*}{u}(-y)\psi_{14}^-(y,\lambda)dy,\\
\psi_{44}^- &= 1 - i\int_{-\infty}^{x} \overset{*}{v}(y)\psi_{14}^-(y,\lambda)dy, \quad \psi_{54}^- = -i\int_{-\infty}^{x} \overset{*}{v}(-y)\psi_{14}^-(y,\lambda)dy,
\end{aligned}$$

and finally the components of the fifth column of ψ^- are

$$\psi^-_{15} = i\int_{-\infty}^{x}\Big(u(y)\psi^-_{25}(y,\lambda)+u(-y)\psi^-_{35}(y,\lambda)+v(y)\psi^-_{45}(y,\lambda)+v(-y)\psi^-_{55}(y,\lambda)\Big)e^{i\lambda\alpha(x-y)}dy,$$
$$\psi^-_{25} = -i\int_{-\infty}^{x}\overset{*}{u}(y)\psi^-_{15}(y,\lambda)dy, \quad \psi^-_{35} = -i\int_{-\infty}^{x}\overset{*}{u}(-y)\psi^-_{15}(y,\lambda)dy,$$
$$\psi^-_{45} = -i\int_{-\infty}^{x}\overset{*}{v}(y)\psi^-_{15}(y,\lambda)dy, \quad \psi^-_{55} = 1-i\int_{-\infty}^{x}\overset{*}{v}(-y)\psi^-_{15}(y,\lambda)dy.$$

Recall that $\alpha < 0$. If $Im(\lambda) > 0$ and $y < x$ then, $Re(e^{-i\lambda\alpha(x-y)})$ decays exponentially and so each integral of the first column of ψ^- converges. As a result, the components of the first column of ψ^-, are analytic in the upper half complex plane for $\lambda \in \mathbb{C}_+$, and continuous for $\lambda \in \mathbb{C}_+ \cup \mathbb{R}$.
In the same way for $y > x$, the components of the last four columns of ψ^+ are analytic in the upper half plane for $\lambda \in \mathbb{C}_+$ and continuous for $\lambda \in \mathbb{C}_+ \cup \mathbb{R}$.

It is worth mentioning the case when $Im(\lambda) < 0$, then the first column ψ^+ is analytic in the lower half plane for $\lambda \in \mathbb{C}_-$ and continuous for $\lambda \in \mathbb{C}_- \cup \mathbb{R}$, and the components of the last four columns of ψ^- are analytic in the lower half plane for $\lambda \in \mathbb{C}_-$ and continuous for $\lambda \in \mathbb{C}_- \cup \mathbb{R}$.

Now, let us construct the Riemann-Hilbert problems. To construct the Jost matrix in the upper-half plane we note that

$$\psi^{\pm} = \varphi^{\pm}e^{-i\lambda\Lambda x}. \tag{4.128}$$

Let $\psi^{\pm}_j$ be the jth column of $\psi^{\pm}$ for $j \in \{1,2,3,4,5\}$. Hence the first Jost matrix solution can be taken as

$$P^+(x,\lambda) = (\psi^-_1, \psi^+_2, \psi^+_3, \psi^+_4, \psi^+_5) = \psi^- H_1 + \psi^+ H_2, \tag{4.129}$$

where $H_1 = diag(1,0,0,0,0)$ and $H_2 = diag(0,1,1,1,1)$.

Therefore P^+ is analytic for $\lambda \in \mathbb{C}_+$ and continuous for $\lambda \in \mathbb{C}_+ \cup \mathbb{R}$.

For the lower-half plane, we can construct $P^- \in \mathbb{C}_-$ which is the analytic counterpart of $P^+ \in \mathbb{C}_+$. We do this by utilizing the equivalent spectral adjoint equation (4.112). Because $\tilde{\psi}^{\pm} = (\psi^{\pm})^{-1}$ and
$\varphi^{\pm} = \psi^{\pm}e^{i\lambda\Lambda x}$, we have

$$(\psi^{\pm})^{-1} = e^{i\lambda\Lambda x}(\varphi^{\pm})^{-1}. \tag{4.130}$$

Let $\tilde{\psi}_j^{\pm}$ be the jth row of $\tilde{\phi}^{\pm}$ for $j \in \{1,2,3,4,5\}$. As above, we can get

$$P^-(x,\lambda) = \left(\tilde{\psi}_1^-, \tilde{\psi}_2^+, \tilde{\psi}_3^+, \tilde{\psi}_4^+, \tilde{\psi}_5^+\right)^T = H_1(\psi^-)^{-1} + H_2(\psi^+)^{-1}. \tag{4.131}$$

Hence, P^- is analytic for $\lambda \in \mathbb{C}_-$ and continuous for $\lambda \in \mathbb{C}_- \cup \mathbb{R}$.

Since both ψ^- and ψ^+ satisfy

$$\psi^\dagger(-x,-t,-\overset{*}{\lambda}) = C_1\psi^{-1}(x,t,\lambda)C_1^{-1}, \tag{4.132}$$

using (4.129), we have

$$P^+(-x,-t,-\overset{*}{\lambda}) = \psi^-(-x,-t,-\overset{*}{\lambda})H_1 + \psi^+(-x,-t,-\overset{*}{\lambda})H_2 \tag{4.133}$$

or equivalently

$$(P^+)^\dagger(-x,-t,-\overset{*}{\lambda}) = H_1(\psi^-)^\dagger(-x,-t,-\overset{*}{\lambda}) + H_2(\psi^+)^\dagger(-x,-t,-\overset{*}{\lambda}). \tag{4.134}$$

Substituting (4.132) in (4.134), we have the nonlocal symmetry property

$$(P^+)^\dagger(-x,-t,-\overset{*}{\lambda}) = C_1P^-(x,t,\lambda)C_1^{-1}. \tag{4.135}$$

One can prove as well that

$$P^+(-x,-t,-\lambda) = C_2P^+(x,t,\lambda)C_2^{-1}. \tag{4.136}$$

Employing analyticity of both P^+ and P^-, one can construct the Riemann-Hilbert problems

$$P^-P^+ = J, \tag{4.137}$$

where $J = e^{i\lambda\Lambda x}(H_1 + H_2S)(H_1 + S^{-1}H_2)e^{-i\lambda\Lambda x}$ for $\quad \lambda \in \mathbb{R}$.

Replacing (4.118) in (4.129), we have

$$P^{+}(x,\lambda)=\psi^{+}(e^{i\lambda\Lambda x}Se^{-i\lambda\Lambda x}H_1+H_2). \tag{4.138}$$

Because $\psi^{+}(x,\lambda)\to I_5$ when $x\to+\infty$, we get

$$\lim_{x\to+\infty}P^{+}=\begin{pmatrix}s_{11}(\lambda) & 0\\ 0 & I_4\end{pmatrix},\quad \text{for}\quad \lambda\in\mathbb{C}_{+}\cup\mathbb{R}. \tag{4.139}$$

In the same way,

$$\lim_{x\to-\infty}P^{-}=\begin{pmatrix}\hat{s}_{11}(\lambda) & 0\\ 0 & I_4\end{pmatrix},\quad \text{for}\quad \lambda\in\mathbb{C}_{-}\cup\mathbb{R}. \tag{4.140}$$

Thus, if we choose

$$G^{+}(x,\lambda)=P^{+}(x,\lambda)\begin{pmatrix}s_{11}^{-1}(\lambda) & 0\\ 0 & I_4\end{pmatrix}\quad \text{and}\quad (G^{-})^{-1}(x,\lambda)=\begin{pmatrix}\hat{s}_{11}^{-1}(\lambda) & 0\\ 0 & I_4\end{pmatrix}P^{-}(x,\lambda)\ , \tag{4.141}$$

the two generalized matrices $G^{+}(x,\lambda)$ and $G^{-}(x,\lambda)$ generate the matrix Riemann-Hilbert problems on the real line for the resulting two-component nonlocal Sasa-Satsuma equation, given by

$$G^{+}(x,\lambda)=G^{-}(x,\lambda)G_0(x,\lambda),\quad \text{for}\quad \lambda\in\mathbb{R}, \tag{4.142}$$

where the jump matrix $G_0(x,\lambda)$ can be cast as

$$G_0(x,\lambda)=\begin{pmatrix}\hat{s}_{11}^{-1}(\lambda) & 0\\ 0 & I_4\end{pmatrix}J\begin{pmatrix}s_{11}^{-1}(\lambda) & 0\\ 0 & I_4\end{pmatrix}, \tag{4.143}$$

which reads

$$G_0(x,\lambda) = \begin{pmatrix} s_{11}^{-1}\hat{s}_{11}^{-1} & \hat{s}_{12}\hat{s}_{11}^{-1}e^{i\lambda\alpha x} & \hat{s}_{13}\hat{s}_{11}^{-1}e^{i\lambda\alpha x} & \hat{s}_{14}\hat{s}_{11}^{-1}e^{i\lambda\alpha x} & \hat{s}_{15}\hat{s}_{11}^{-1}e^{i\lambda\alpha x} \\ s_{21}s_{11}^{-1}e^{-i\lambda\alpha x} & 1 & 0 & 0 & 0 \\ s_{31}s_{11}^{-1}e^{-i\lambda\alpha x} & 0 & 1 & 0 & 0 \\ s_{41}s_{11}^{-1}e^{-i\lambda\alpha x} & 0 & 0 & 1 & 0 \\ s_{51}s_{11}^{-1}e^{-i\lambda\alpha x} & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{4.144}$$

and whose canonical normalization conditions are:

$$G^{+}(x,\lambda) \to I_5 \quad \text{as} \quad \lambda \in \mathbb{C}_{+} \cup \mathbb{R} \to \infty, \tag{4.145}$$

$$G^{-}(x,\lambda) \to I_5 \quad \text{as} \quad \lambda \in \mathbb{C}_{-} \cup \mathbb{R} \to \infty. \tag{4.146}$$

From (4.135) along with (4.141) and using (4.122), we deduce the nonlocal involution properties

$$\begin{cases} (G^{+})^{\dagger}(-x,-t,-\overset{*}{\lambda}) = C_1 (G^{-})^{-1}(x,t,\lambda) C_1^{-1}, \\ G^{+}(-x,-t,-\lambda) = C_2 G^{+}(x,t,\lambda) C_2^{-1}. \end{cases} \tag{4.147}$$

Furthermore, from (4.143),(4.122) and (4.142),(4.147), we derive the following nonlocal involution properties for G_0

$$\begin{cases} G_0^{\dagger}(-x,-t,-\overset{*}{\lambda}) = C_1 G_0(x,t,\lambda) C_1^{-1}, \\ G_0(-x,-t,-\lambda) = C_2 G_0(x,t,\lambda) C_2^{-1}, \end{cases} \quad \lambda \in \mathbb{R}. \tag{4.148}$$

4.3.1 Time evolution of the scattering data

At this point, we have to determine how the scattering data evolves over time. In order to do that, we differentiate equation (4.118) with respect to time t and applying (4.106) gives

$$S_t = i\lambda^3 [\Omega, S], \tag{4.149}$$

and thus

$$S_t = \begin{pmatrix} 0 & i\beta\lambda^3 s_{12} & i\beta\lambda^3 s_{13} & i\beta\lambda^3 s_{14} & \beta\lambda^3 s_{15} \\ -i\beta\lambda^3 s_{21} & 0 & 0 & 0 & 0 \\ -i\beta\lambda^3 s_{31} & 0 & 0 & 0 & 0 \\ -i\beta\lambda^3 s_{41} & 0 & 0 & 0 & 0 \\ -i\beta\lambda^3 s_{51} & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.150}$$

As a result, we have

$$\begin{cases} s_{12}(t,\lambda) = s_{12}(0,\lambda)e^{i\beta\lambda^3 t}, \quad s_{21}(t,\lambda) = s_{21}(0,\lambda)e^{-i\beta\lambda^3 t}, \\ s_{13}(t,\lambda) = s_{13}(0,\lambda)e^{i\beta\lambda^3 t}, \quad s_{31}(t,\lambda) = s_{31}(0,\lambda)e^{-i\beta\lambda^3 t}, \\ s_{14}(t,\lambda) = s_{14}(0,\lambda)e^{i\beta\lambda^3 t}, \quad s_{41}(t,\lambda) = s_{41}(0,\lambda)e^{-i\beta\lambda^3 t}, \\ s_{15}(t,\lambda) = s_{15}(0,\lambda)e^{i\beta\lambda^3 t}, \quad s_{51}(t,\lambda) = s_{51}(0,\lambda)e^{-i\beta\lambda^3 t}, \end{cases} \tag{4.151}$$

and $s_{11}, s_{2j}, s_{3j}, s_{4j}, s_{5j}$ are constants for $j \in \{2, ..., 5\}$.

4.4 Soliton solutions

4.4.1 General case

Based on the Riemann-Hilbert problems, the type of soliton solutions generated is determined by the determinant of the matrix $G^{\pm}$. When $det(G^{\pm}) \neq 0$, the regular case leads to a unique solution. On the other hand, the non-regular case $det(G^{\pm}) = 0$, generates discrete eigenvalues in the spectral plane. To solve for soliton solutions, we can transform the non-regular case into the regular case.

The following can be shown from (4.138) and $det(\psi^{\pm}) = 1$

$$det(P^+(x,\lambda)) = s_{11}(\lambda), \tag{4.152}$$

$$det(P^-(x,\lambda)) = \hat{s}_{11}(\lambda). \tag{4.153}$$

Since $det(S(\lambda)) = 1$, thus it follows that $S^{-1}(\lambda) = \Big(cof(S(\lambda))\Big)^T$. So

$$\hat{s}_{11} = \begin{vmatrix} s_{22} & s_{23} & s_{24} & s_{25} \\ s_{32} & s_{33} & s_{34} & s_{35} \\ s_{42} & s_{43} & s_{44} & s_{45} \\ s_{52} & s_{53} & s_{54} & s_{55} \end{vmatrix}, \tag{4.154}$$

which should be zero for the non-regular case.

The solutions to $det(P^+(x,\lambda)) = det(P^-(x,\lambda)) = 0$ have to be simple in order to obtain soliton solutions. In the case of $det(P^+(x,\lambda)) = s_{11}(\lambda) = 0$, we assume $s_{11}(\lambda)$ has simple zeros generating discrete eigenvalues $\lambda_k \in \mathbb{C}_+$ for $k \in \{1, 2, ..., 2N_1 = N\}$, whereas in the case of $det(P^-(x,\lambda)) = \hat{s}_{11}(\lambda) = 0$, we assume $\hat{s}_{11}(\lambda)$ has simple zeros generating discrete eigenvalues $\hat{\lambda}_k \in \mathbb{C}_-$ for $k \in \{1, 2, ..., 2N_1 = N\}$.

From $\hat{s}_{11}(\lambda) = \overset{*}{s}_{11}(-\overset{*}{\lambda})$ and $det(P^{\pm}(x,\lambda)) = 0$, one can see that if $\lambda \in \mathbb{C}_+$, then $-\overset{*}{\lambda} \in \mathbb{C}_+$. Also, from $s_{11}(-\lambda) = s_{11}(\lambda)$ and $det(P^{\pm}(x,\lambda)) = 0$, we deduce that if $\lambda \in \mathbb{C}_+$, then $-\lambda \in \mathbb{C}_-$. In other words,

$$\text{if} \quad \lambda \in \mathbb{C}_+, \quad \text{then} \quad \begin{cases} -\overset{*}{\lambda} \in \mathbb{C}_+, \\ -\lambda \in \mathbb{C}_-, \end{cases} \quad \lambda \notin \mathrm{i}\mathbb{R}. \tag{4.155}$$

If $\lambda = \mathrm{i}m \in \mathrm{i}\mathbb{R}$, for $m > 0$, the couple $(\lambda, -\overset{*}{\lambda}) \in \mathbb{C}_+^2$ coincide, forcing $\hat{\lambda} = -\lambda = -im \in \mathbb{C}_-$.

To make this clearer, we can view the choices of the eigenvalues in a more systematic way. Recall that the Riemann-Hilbert problem requires the same number of eigenvalues in the upper-half plane and in the lower-half plane. Assume $\lambda_k \in \mathbb{C}_+$ for all $k = 1, 2, \ldots, 2N_1$. Fix n for $1 \leq n \leq N_1$ and λ_n lies off the imaginary axis. The eigenvalues are given by the N_1-couples $(\lambda_n, \lambda_{N_1+n}) = (\lambda, -\overset{*}{\lambda}) \in \mathbb{C}_+^2$, which are assumed to be the zeros of $det(P^+(x,\lambda)) = 0$. For any λ_n, the choice of λ_{N_1+n} depends on λ_n, that is, $\lambda_n = -\overset{*}{\lambda}_{N_1+n}$, where λ_n is freely chosen. If λ_n lies on the imaginary axis, then the coupled pair coincide.

In the lower-half plane, we have $\hat{\lambda}_k \in \mathbb{C}_-$ for all $k = 1, 2, \ldots, 2N_1$ and similarly the eigenvalues are given by the N_1-couples $(\hat{\lambda}_n, \hat{\lambda}_{N_1+n}) = (-\lambda, \overset{*}{\lambda}) \in \mathbb{C}_-^2$, which are assumed to be the zeros of $det(P^-(x,\lambda)) = 0$, and $\hat{\lambda}_n = -\overset{*}{\hat{\lambda}}_{N_1+n}$.

In other words, if λ_n is not pure imaginary, then the scheme of the eigenvalues take the form

$$(\lambda_n, \lambda_{N_1+n}, \hat{\lambda}_n, \hat{\lambda}_{N_1+n}) = (\lambda, -\overset{*}{\lambda}, -\lambda, \overset{*}{\lambda}). \tag{4.156}$$

Each $Ker(P^+(x,\lambda_k))$ contains only a single column vector v_k, similarly each $Ker(P^-(x,\hat{\lambda}_k))$ contains only a single row vector $\hat{\mathrm{v}}_k$ such that:

$$P^+(x,\lambda_k)\mathrm{v}_k = 0 \quad \text{for} \quad k \in \{1,2,...,2N_1\}, \tag{4.157}$$

and

$$\hat{\mathrm{v}}_k P^-(x,\hat{\lambda}_k) = 0 \quad \text{for} \quad k \in \{1,2,...,2N_1\}. \tag{4.158}$$

To obtain explicit soliton solutions, we take $G_0 = I_5$ in the Riemann-Hilbert problems. This will force the reflection coefficients $s_{21} = s_{31} = s_{41} = s_{51} = 0$ and $\hat{s}_{12} = \hat{s}_{13} = \hat{s}_{14} = \hat{s}_{15} = 0$.
In that case, the Riemann-Hilbert problems can be presented as follows [26]:

$$G^+(x,\lambda) = I_5 - \sum_{k,j=1}^{N} \frac{\mathrm{v}_k (M^{-1})_{kj} \hat{\mathrm{v}}_j}{\lambda - \hat{\lambda}_j}, \tag{4.159}$$

and

$$(G^-)^{-1}(x,\lambda) = I_5 + \sum_{k,j=1}^{N} \frac{\mathrm{v}_k (M^{-1})_{kj} \hat{\mathrm{v}}_j}{\lambda - \lambda_k}, \tag{4.160}$$

where $M = (m_{kj})_{N\times N}$ is a matrix defined by [26]

$$m_{kj} = \begin{cases} \frac{\hat{\mathrm{v}}_k \mathrm{v}_j}{\lambda_j - \hat{\lambda}_k}, & \text{if} \quad \lambda_j \neq \hat{\lambda}_k, \\ 0, & \text{if} \quad \lambda_j = \hat{\lambda}_k, \end{cases} \quad k,j \in \{1,2,...,N\}. \tag{4.161}$$

Since the zeros λ_k and $\hat{\lambda}_k$ are constants, because they are independent of space and time, we can explore the spatial and temporal evolution of the scattering vectors $\mathrm{v}_k(x,t)$ and $\hat{\mathrm{v}}_k(x,t)$, $1 \leq k \leq N$.

Taking the x-derivative of both sides of the equation

$$P^+(x,\lambda_k)\mathrm{v}_k = 0, \quad 1 \leq k \leq N, \tag{4.162}$$

and knowing that P^{+} satisfies the spectral spatial equivalent equation (4.105) together with (4.157), we obtain

$$P^{+}(x,\lambda_k)\left(\frac{d\mathrm{v}_k}{dx}-i\lambda_k\varLambda\mathrm{v}_k\right)=0\quad\text{for}\quad k,j\in\{1,2,...,N\}.\tag{4.163}$$

In a similar manner, taking the t-derivative and using the temporal equation (4.106) with (4.157), we acquire

$$P^{+}(x,\lambda_k)\left(\frac{d\mathrm{v}_k}{dt}-i\lambda_k^3\varOmega\mathrm{v}_k\right)=0\quad\text{for}\quad k,j\in\{1,2,...,N\}.\tag{4.164}$$

For the adjoint spectral equations (4.109) and (4.110), we can obtain the following similar results

$$\left(\frac{d\hat{\mathrm{v}}_k}{dx}+i\hat{\lambda}_k\hat{\mathrm{v}}_k\varLambda\right)P^{-}(x,\hat{\lambda}_k)=0,\tag{4.165}$$

and

$$\left(\frac{d\hat{\mathrm{v}}_k}{dt}+i\hat{\lambda}_k^3\hat{\mathrm{v}}_k\varOmega\right)P^{-}(x,\hat{\lambda}_k)=0.\tag{4.166}$$

Because v_k is a single vector in the kernel of P^{+}, so $\frac{d\mathrm{v}_k}{dx}-\mathrm{i}\lambda_k\varLambda\mathrm{v}_k$ and $\frac{d\mathrm{v}_k}{dt}-\mathrm{i}\lambda_k^3\varOmega\mathrm{v}_k$ are scalar multiples of v_k.

Hence without loss of generality, we can take the space dependence of v_k to be:

$$\frac{d\mathrm{v}_k}{dx}=i\lambda_k\varLambda\mathrm{v}_k,\quad 1\leq k\leq N\tag{4.167}$$

and the time dependence of v_k as:

$$\frac{d\mathrm{v}_k}{dt}=i\lambda_k^3\varOmega\mathrm{v}_k,\quad 1\leq k\leq N.\tag{4.168}$$

so, we can conclude that

$$\mathrm{v}_k(x,t)=\mathrm{v}_k(x,t,\lambda_k)=e^{i\lambda_k\varLambda x+i\lambda_k^3\varOmega t}\mathrm{w}_k\quad\text{for}\quad k\in\{1,2,...,N\},\tag{4.169}$$

by solving equations (4.167) and (4.168). Likewise, we get

$$\hat{\mathrm{v}}_k(x,t)=\hat{\mathrm{v}}_k(x,t,\hat{\lambda}_k)=\hat{\mathrm{w}}_k e^{-i\hat{\lambda}_k\varLambda x-i\hat{\lambda}_k^3\varOmega t}\quad\text{for}\quad k\in\{1,2,...,N\},\tag{4.170}$$

where w_k and $\hat{\mathrm{w}}_k$ are constant column and row vectors in $\mathbb{C}^5$, respectively. In addition, they need to satisfy the orthogonality condition:

$$\hat{\mathrm{w}}_k \mathrm{w}_l = 0, \quad \text{when} \quad \lambda_l = \hat{\lambda}_k, \quad 1 \leq k, l \leq N. \tag{4.171}$$

From (4.157) and using the formula (4.135), it is easy to see

$$\mathrm{v}_k^\dagger(-x, -t, \overset{*}{\lambda}_k)(P^+)^\dagger(-x, -t, \overset{*}{\lambda}_k) = \mathrm{v}_k^\dagger(-x, -t, \overset{*}{\lambda}_k) C_1 P^-(x, t, -\lambda_k) C_1 = 0. \tag{4.172}$$

Because $\mathrm{v}_k^\dagger(-x, -t, -\overset{*}{\lambda}_k) C_1 P^-(x, t, \lambda_k)$ can be zero and using (4.158) this leads to

$$\mathrm{v}_k^\dagger(-x, -t, \overset{*}{\lambda}_k) C_1 P^-(x, t, -\lambda_k) = \mathrm{v}_k^\dagger(-x, -t, \overset{*}{\lambda}_k) C_1 P^-(x, t, -\lambda_k) \tag{4.173}$$

$$= \hat{\mathrm{v}}_k(x, t, \hat{\lambda}_k) P^-(x, t, \hat{\lambda}_k) = 0, \tag{4.174}$$

thus, we can take

$$\hat{\mathrm{v}}_k(x, t, \hat{\lambda}_k) = \mathrm{v}_k^\dagger(-x, -t, \overset{*}{\lambda}_k) C_1. \tag{4.175}$$

Therefore, the involution relations (4.169) and (4.170) give

$$\mathrm{v}_k(x, t) = e^{i\lambda_k \Lambda x + i\lambda_k^3 \Omega t} \mathrm{w}_k, \tag{4.176}$$

$$\hat{\mathrm{v}}_k(x, t) = \mathrm{w}_k^\dagger e^{-i\hat{\lambda}_k \Lambda x - i\hat{\lambda}_k^3 \Omega t} C_1. \tag{4.177}$$

Now, in order to satisfy the orthogonality condition (4.171), one can notice that we require:

$$\mathrm{w}_k^\dagger C_1 \mathrm{w}_l = 0, \quad \text{as} \quad \lambda_l = \hat{\lambda}_k, \quad 1 \leq k, l \leq N. \tag{4.178}$$

As a consequence, $\hat{\lambda}_k = \lambda_k$ still occurs for $\lambda_k \in \mathrm{i}\mathbb{R}$ and $\hat{\lambda}_k = -\overset{*}{\lambda}_k$ holds, when $\lambda_k \neq \hat{\lambda}_k$.

Because the jump matrix $G_0 = I_5$, we can solve the Riemann-Hilbert problem precisely. As a result, we can determine the potentials by computing the matrix P^+. Because P^+ is analytic, we can expand G^+ as follows:

$$G^+(x, \lambda) = I_5 + \frac{1}{\lambda} G_1^+(x) + O\Big(\frac{1}{\lambda^2}\Big), \quad \text{when} \quad \lambda \to \infty. \tag{4.179}$$

Because G^+ satisfies the spectral problem, substituting it in (4.105) and matching the coefficients of the same power of $\frac{1}{\lambda}$, at order $O(1)$, we get

$$P = -[\Lambda, G_1^+]. \tag{4.180}$$

If we denote

$$G_1^+ = \begin{pmatrix} (G_1^+)_{11} & (G_1^+)_{12} & (G_1^+)_{13} & (G_1^+)_{14} & (G_1^+)_{15} \\ (G_1^+)_{21} & (G_1^+)_{22} & (G_1^+)_{23} & (G_1^+)_{24} & (G_1^+)_{25} \\ (G_1^+)_{31} & (G_1^+)_{32} & (G_1^+)_{33} & (G_1^+)_{34} & (G_1^+)_{35} \\ (G_1^+)_{41} & (G_1^+)_{42} & (G_1^+)_{43} & (G_1^+)_{44} & (G_1^+)_{45} \\ (G_1^+)_{51} & (G_1^+)_{52} & (G_1^+)_{53} & (G_1^+)_{54} & (G_1^+)_{55} \end{pmatrix} \tag{4.181}$$

then

$$P = -[\Lambda, G_1^+] = \begin{pmatrix} 0 & -\alpha(G_1^+)_{12} & -\alpha(G_1^+)_{13} & -\alpha(G_1^+)_{14} & -\alpha(G_1^+)_{15} \\ \alpha(G_1^+)_{21} & 0 & 0 & 0 & 0 \\ \alpha(G_1^+)_{31} & 0 & 0 & 0 & 0 \\ \alpha(G_1^+)_{41} & 0 & 0 & 0 & 0 \\ \alpha(G_1^+)_{51} & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.182}$$

Matching the components of (4.182) to the components of the P matrix, P can be rewritten in the form:

$$P = \begin{pmatrix} 0 & -\alpha(G_1^+(x))_{12} & -\alpha(G_1^+(-x))_{12} & -\alpha(G_1^+(x))_{14} & -\alpha(G_1^+(-x))_{14} \\ \alpha(\overset{*}{G}{}_1^+(x))_{12} & 0 & 0 & 0 & 0 \\ \alpha(\overset{*}{G}{}_1^+(-x))_{12} & 0 & 0 & 0 & 0 \\ \alpha(\overset{*}{G}{}_1^+(x))_{14} & 0 & 0 & 0 & 0 \\ \alpha(\overset{*}{G}{}_1^+(-x))_{14} & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.183}$$

As a result, we can recover the potentials u and v as

$$u(x,t) = -\alpha(G_1^+(x,t))_{12}, \tag{4.184}$$

$$v(x,t) = -\alpha(G_1^+(x,t))_{14}. \tag{4.185}$$

It can be seen from (4.179) that

$$G_1^+ = \lambda \lim_{\lambda\to\infty} (G^+(x,\lambda) - I_5), \tag{4.186}$$

and then using equation (4.159), we deduce

$$G_1^+ = -\sum_{k,j=1}^{N} \mathrm{v}_k (M^{-1})_{k,j} \hat{\mathrm{v}}_j, \tag{4.187}$$

where

$$\mathrm{v}_k = (\mathrm{v}_{k,1}, \mathrm{v}_{k,2}, \mathrm{v}_{k,3}, \mathrm{v}_{k,4}, \mathrm{v}_{k,5})^T, \hat{\mathrm{v}}_k = (\hat{\mathrm{v}}_{k,1}, \hat{\mathrm{v}}_{k,2}, \hat{\mathrm{v}}_{k,3}, \hat{\mathrm{v}}_{k,4}, \hat{\mathrm{v}}_{k,5}).$$

In addition, by the use of equations (4.8) and (4.180), we can easily prove the symmetry relation

$$\begin{cases} (G_1^+)^\dagger(-x,-t) = C_1 G_1^+(x,t) C_1^{-1} \\ G_1^+(-x,-t) = C_2 G_1^+(x,t) C_2^{-1} \end{cases} \tag{4.188}$$

We deduce that the specific Riemann-Hilbert problem solutions determined by (4.159)-(4.161), satisfy (4.147). Hence the matrix G_1^+ posses the symmetry relation (4.188), which is genereted from the nonlocal symmetry (4.6).

Now, by substituting (4.187) into (4.184) and using (4.176) and (4.177), we generate the N-soliton solution to the nonlocal reverse-spacetime two-component AKNS system of third-order

$$u(x,t) = \alpha \sum_{k,j=1}^{N} \mathrm{v}_{k,1} (M^{-1})_{kj} \hat{\mathrm{v}}_{j,2}, \tag{4.189}$$

$$v(x,t) = \alpha \sum_{k,j=1}^{N} \mathrm{v}_{k,1} (M^{-1})_{kj} \hat{\mathrm{v}}_{j,4}. \tag{4.190}$$

4.5 Exact soliton solutions and their dynamics

4.5.1 Explicit one-soliton solution

For a general explicit formula for the one-soliton solution of the Sasa-Satsuma equation (4.1) and (4.2), i.e., when $N = 1$, we choose $\lambda_1 = \mathrm{i}m$ and $\hat{\lambda}_1 = -\mathrm{i}m$, where $m > 0$, in order to fulfill condition (4.155). This

Sasa-Satsuma equation requires a further orthogonality condition

$$\mathrm{w}_k^\dagger C_2 \mathrm{w}_l = 0, \quad \text{for} \quad \lambda_l = \hat{\lambda}_k, \quad 1 \leq k, l \leq N. \tag{4.191}$$

which will impose the choice of w_1 to be:

$$\mathrm{w}_1 = (\mathrm{w}_{11}, \mathrm{w}_{12}, \mathrm{w}_{12}, \mathrm{w}_{14}, \mathrm{w}_{14})^T. \tag{4.192}$$

in order to satisfy the Sasa-Satsuma equation (4.1). As a consequence, the solution to the two-component nonlocal reverse-spacetime Sasa-Satsuma equation (4.82)-(4.83) reads

$$u(x,t) = \frac{\mathrm{i}2\alpha m \mathrm{w}_{11} \overset{*}{\mathrm{w}}_{12}}{\mathbf{A} e^{\alpha m x - \beta m^3 t} + \mathbf{B} e^{-\alpha m x + \beta m^3 t}}, \tag{4.193}$$

$$v(x,t) = \frac{\mathrm{i}2\alpha m \mathrm{w}_{11} \overset{*}{\mathrm{w}}_{14}}{\mathbf{A} e^{\alpha m x - \beta m^3 t} + \mathbf{B} e^{-\alpha m x + \beta m^3 t}}, \tag{4.194}$$

where

$$\mathbf{A} = 2(|\mathrm{w}_{12}|^2 + |\mathrm{w}_{14}|^2), \quad \mathbf{B} = |\mathrm{w}_{11}|^2. \tag{4.195}$$

4.5.1.1 The dynamics of the one-soliton

For the one-soliton, the soliton moves with speed $V = \frac{\beta}{\alpha} m^2$ along the line $x = \frac{\beta}{\alpha} m^2 t$. In that case, the amplitude is given by

$$|u(x,t)| = \frac{\alpha 2m |\mathrm{w}_{11}||\mathrm{w}_{12}|}{\mathbf{A} + \mathbf{B}}. \tag{4.196}$$

The amplitude of the moving soliton stays constant as seen in figure 21. In the case when $\lambda_1 = m$ is real, we get a breather with period $\frac{\pi}{|\beta m^3|}$ as in figure 22.

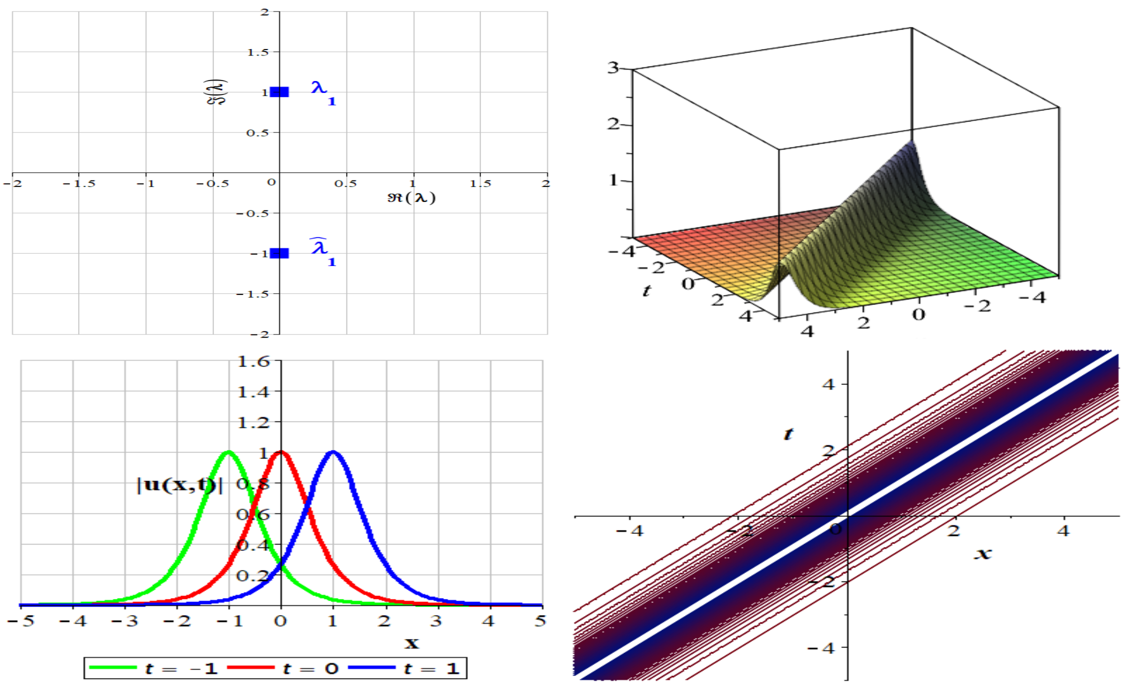


Figure 21.: Spectral plane along with 3D, 2D and contours plots of $|u(x,t)|$ of the one-soliton with parameters $(\alpha, \beta) = (-2, -2)$, $(\lambda_1, \hat{\lambda}_1) = (i, -i)$, $\mathrm{w}_1 = (1, 0.5, 0.5, 0.5, 0.5)$.

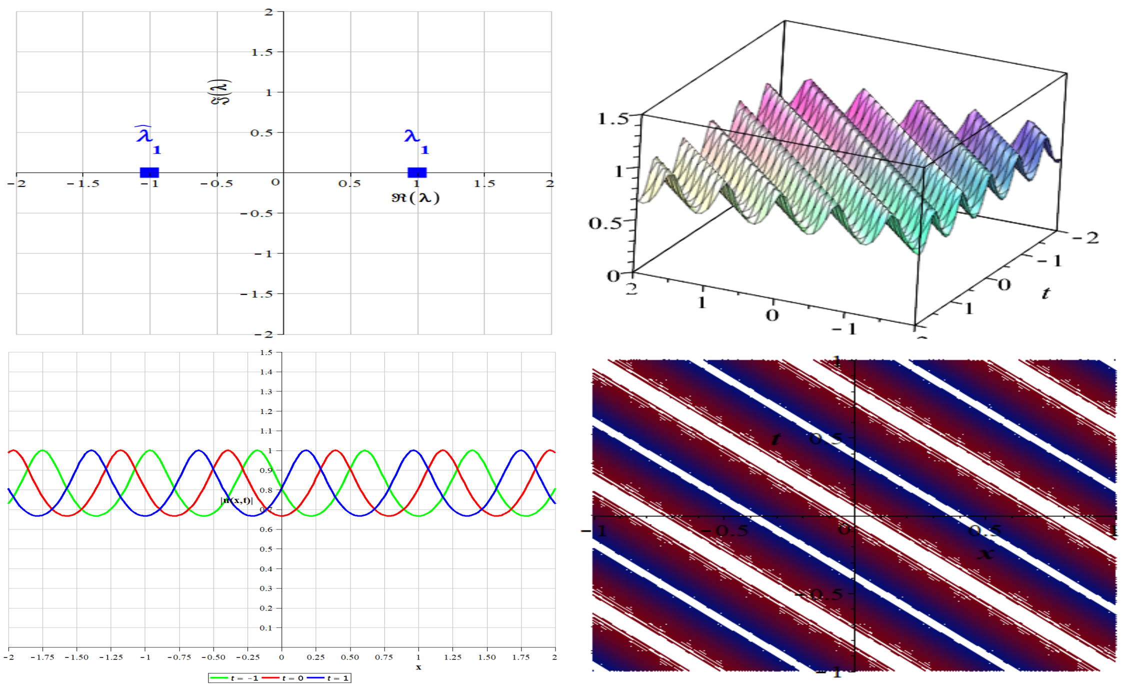
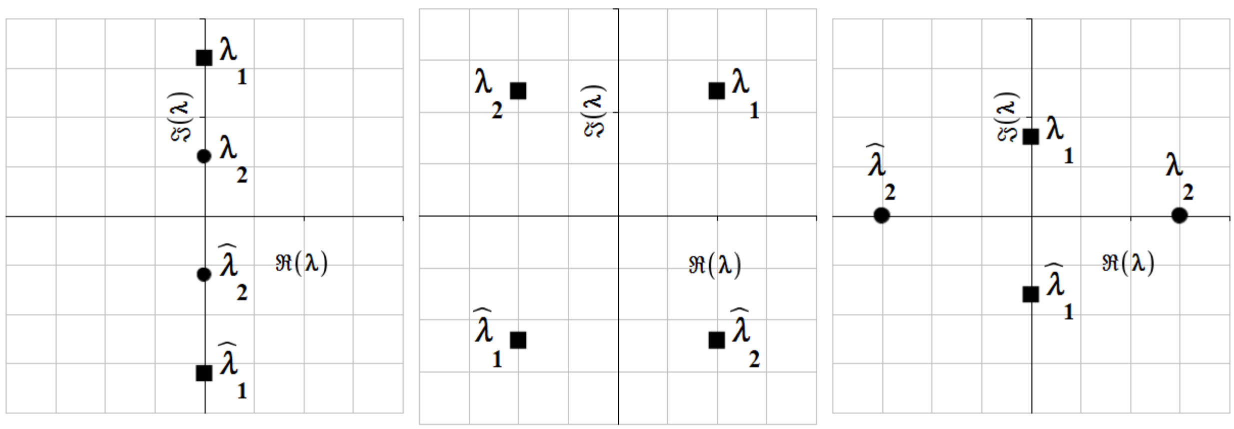


Figure 22.: Spectral plane along with 3D, 2D and contours plots of $|u(x,t)|$ of the one-soliton breather with parameters $(\alpha, \beta) = (-4, -4)$, $(\lambda_1, \hat{\lambda}_1) = (1, -1)$, $\mathrm{w}_1 = (1, -0.5, -0.5, 1.5, 1.5)$.

4.5.2 Two-soliton solutions

For a general explicit formula for two-soliton solutions of the Sasa-Satsuma equation (4.1) and (4.2), i.e., when $N = 2$, the configuration of the eigenvalues for this equation is given by $(\lambda_1, \lambda_2, \hat{\lambda}_1, \hat{\lambda}_2) = (\lambda, -\overset{*}{\lambda}, -\lambda, \overset{*}{\lambda})$. As a result, we have three distinct cases as shown in figure 23. In all cases, the eigenvalues $\lambda_1, \lambda_2 \in \mathbb{C}_+ \cup \mathbb{R}$ and $\hat{\lambda}_1, \hat{\lambda}_2 \in \mathbb{C}_- \cup \mathbb{R}$ are all taken to be distinct, i.e., $\lambda_1 \neq \lambda_2$ and $\hat{\lambda}_1 \neq \hat{\lambda}_2$.



(a) Case I (b) Case II (c) Case III

Figure 23.: Spectral planes of two-soliton eigenvalues cases

4.5.2.1 Explicit two-soliton solutions: Case I

If all eigenvalues in the complex plane are pure imaginary, that is $\lambda_1 = \mathrm{i}m_1$, $\lambda_2 = \mathrm{i}m_2$, $\hat{\lambda}_1 = -\mathrm{i}m_1$, $\hat{\lambda}_2 = -\mathrm{i}m_2$, for $m_1, m_2 > 0$ and $\mathrm{w}_1 = (\mathrm{w}_{11}, \mathrm{w}_{12}, \mathrm{w}_{12}, \mathrm{w}_{14}, \mathrm{w}_{14})^T$, then for simplicity of the solution, we take $\mathrm{w}_2 = \mathrm{w}_1$. Hence, the solution in this nonlocal reverse-spacetime case is given by:

$$u(x,t) = -\mathrm{i}\alpha \mathrm{w}_{11} \overset{*}{\mathrm{w}}_{12} \frac{\mathsf{N}_1(x,t)}{\mathsf{D}_1(x,t)}, \tag{4.197}$$

$$v(x,t) = \mathrm{i}\alpha \mathrm{w}_{11} \overset{*}{\mathrm{w}}_{14} \frac{\mathsf{N}_1(x,t)}{\mathsf{D}_1(x,t)}, \tag{4.198}$$

where

$$
\begin{aligned}
\mathsf{N}_1(x,t) = {} & m_1\mathcal{A}_1 e^{-\left(\alpha_1 m_1+\alpha_2(m_1+2m_2)\right)x+\left(\beta_1 m_1^3+\beta_2(m_1^3+2m_2^3)\right)t} \\
& - m_2\mathcal{A}_1 e^{-\left(\alpha_1 m_2+\alpha_2(2m_1+m_2)\right)x+\left(\beta_1 m_2^3+\beta_2(2m_1^3+m_2^3)\right)t} \\
& + m_1\mathcal{A}_2 e^{-\left(\alpha_2 m_1+\alpha_1(m_1+2m_2)\right)x+\left(\beta_2 m_1^3+\beta_1(m_1^3+2m_2^3)\right)t} \\
& - m_2\mathcal{A}_2 e^{-\left(\alpha_2 m_2+\alpha_1(2m_1+m_2)\right)x+\left(\beta_2 m_2^3+\beta_1(2m_1^3+m_2^3)\right)t}
\end{aligned} \tag{4.199}
$$

and

$$
\begin{aligned}
\mathsf{D}_1(x,t) = {} & \mathcal{A}_3 e^{-2\alpha_2(m_1+m_2)x+2\beta_2(m_1^3+m_2^3)t} + \mathcal{A}_4 e^{-2(\alpha_1 m_2+\alpha_2 m_1)x+2(\beta_1 m_2^3+\beta_2 m_1^3)t} \\
& + 2\mathcal{A}_5 e^{-(\alpha_1+\alpha_2)(m_1+m_2)x+(\beta_1+\beta_2)(m_1^3+m_2^3)t} + \mathcal{A}_4 e^{-2(\alpha_1 m_1+\alpha_2 m_2)x+2(\beta_1 m_1^3+\beta_2 m_2^3)t} \\
& + \mathcal{A}_6 e^{-2\alpha_1(m_1+m_2)x+2\beta_1(m_1^3+m_2^3)t},
\end{aligned} \tag{4.200}
$$

where the coefficients are

$$
\begin{aligned}
& \mathcal{A}_1 = 4(m_1^2-m_2^2)(|\mathrm{w}_{12}|^2+|\mathrm{w}_{14}|^2),\ \mathcal{A}_2 = 2(m_1^2-m_2^2)|\mathrm{w}_{11}|^2, \\
& \mathcal{A}_3 = 4(m_1-m_2)^2(|\mathrm{w}_{12}|^2+|\mathrm{w}_{14}|^2)^2,\ \mathcal{A}_4 = 2(m_1+m_2)^2|\mathrm{w}_{11}|^2(|\mathrm{w}_{12}|^2+|\mathrm{w}_{14}|^2), \\
& \mathcal{A}_5 = -8m_1m_2|\mathrm{w}_{11}|^2(|\mathrm{w}_{12}|^2+|\mathrm{w}_{14}|^2),\ \mathcal{A}_6 = (m_1-m_2)^2|\mathrm{w}_{11}|^4.
\end{aligned}
$$

4.5.2.2 The dynamics of the two-soliton solution: Case I

If the eigenvalues $\lambda_1 = -\hat{\lambda}_1$ and $\lambda_2 = -\hat{\lambda}_2$, then the two solitons move in the same direction before and after the elastic collision, where the faster soliton overtakes the slower one. An overlay of two traveling waves is shown in figure 24, in which the amplitude pre and post collision remains unchanged, and the speed of the soliton S_2 is larger than the speed of the soliton S_1.

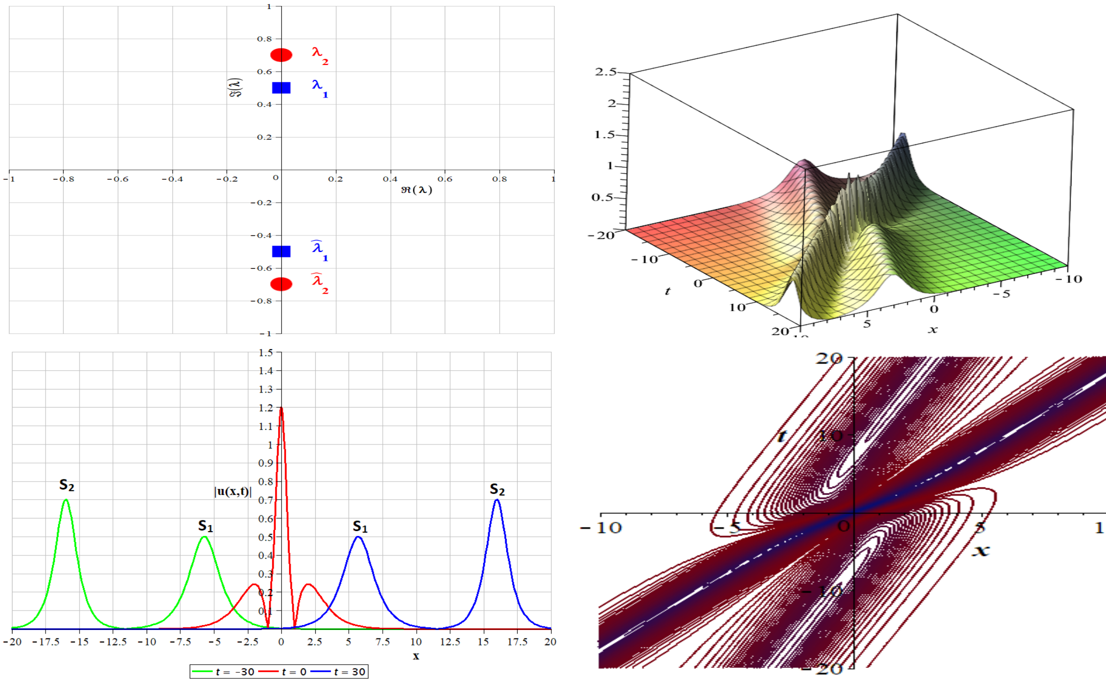


Figure 24.: Spectral plane along with 3D, 2D and contours plots of $|u(x,t)|$ of the two solitons interaction with parameters $(\alpha, \beta) = (-2, -2)$, $(\lambda_1, \lambda_2, \hat{\lambda}_1, \hat{\lambda}_2) = (0.5\mathrm{i}, 0.7\mathrm{i}, -0.5\mathrm{i}, -0.7\mathrm{i})$, $\mathrm{w}_1 = \mathrm{w}_2 = (1, 0.5, 0.5, 0.5, 0.5)$.

4.5.2.3 Explicit two-soliton solutions: Case II

In that case, if $\lambda_1, \lambda_2 \in \mathbb{C}_+$ are not pure imaginary, then the involution property (4.155) requires that $\lambda_2 = -\overset{*}{\lambda}_1$, while in the lower half-plane $\hat{\lambda}_1 = -\lambda_1$ and $\hat{\lambda}_2 = \overset{*}{\lambda}_1$.

Let $\mathrm{w}_1 = (\mathrm{w}_{11}, \mathrm{w}_{12}, \mathrm{w}_{12}, \mathrm{w}_{14}, \mathrm{w}_{14})^T$ and $\mathrm{w}_2 = \mathrm{w}_1$. the solution in this nonlocal reverse-spacetime case is given by:

$$u(x,t) = -\alpha \mathrm{w}_{11} \overset{*}{\mathrm{w}}_{12} \frac{\mathsf{N}_2(x,t)}{\mathsf{D}_2(x,t)}, \tag{4.201}$$

$$v(x,t) = -\alpha \mathrm{w}_{11} \overset{*}{\mathrm{w}}_{14} \frac{\mathsf{N}_2(x,t)}{\mathsf{D}_2(x,t)}, \tag{4.202}$$

where

$$\begin{aligned}\mathsf{N}_2(x,t) =& \overset{*}{\lambda}_1\mathcal{B}_1 e^{\mathrm{i}\left(-\alpha_1\overset{*}{\lambda}_1+\alpha_2(2\lambda_1-\overset{*}{\lambda}_1)\right)x+\mathrm{i}\left(-\beta_1\overset{*}{\lambda}{}_1^3+\beta_2(2\lambda_1^3-\overset{*}{\lambda}{}_1^3)\right)t} \\ &+ \overset{*}{\lambda}_1\mathcal{B}_2 e^{\mathrm{i}\left(\alpha_1(2\lambda_1-\overset{*}{\lambda}_1)-\alpha_2\overset{*}{\lambda}_1\right)x+\mathrm{i}\left(\beta_1(2\lambda_1^3-\overset{*}{\lambda}{}_1^3)-\beta_2\overset{*}{\lambda}{}_1^3\right)t} \\ &+ \lambda_1\mathcal{B}_2 e^{\mathrm{i}\left(\alpha_1(\lambda_1-2\overset{*}{\lambda}_1)+\alpha_2\lambda_1\right)x+\mathrm{i}\left(\beta_1(\lambda_1^3-2\overset{*}{\lambda}{}_1^3)+\beta_2\lambda_1^3\right)t} \\ &+ \lambda_1\mathcal{B}_1 e^{\mathrm{i}\left(\alpha_1\lambda_1+\alpha_2(\lambda_1-2\overset{*}{\lambda}_1)\right)x+\mathrm{i}\left(\beta_1\lambda_1^3+\beta_2(\lambda_1^3-2\overset{*}{\lambda}{}_1^3)\right)t},\end{aligned} \tag{4.203}$$

$$\begin{aligned}\mathsf{D}_2(x,t) =& \mathcal{B}_3 e^{-4\alpha_2\mathrm{Im}(\lambda_1)x-4\beta_2\mathrm{Im}(\lambda_1^3)t} + \mathcal{B}_4 e^{\mathrm{i}2(\alpha_1\lambda_1-\alpha_2\overset{*}{\lambda}_1)x+\mathrm{i}2(\beta_1\lambda_1^3-\beta 2\overset{*}{\lambda}{}_1^3)t} \\ &+ 2\mathcal{B}_5 e^{-2(\alpha_1+\alpha_2)\mathrm{Im}(\lambda_1)x-2(\beta_1+\beta_2)\mathrm{Im}(\lambda_1^3)t} + \mathcal{B}_4 e^{-\mathrm{i}2(\alpha_1\overset{*}{\lambda}_1-\alpha_2\lambda_1)x-\mathrm{i}2(\beta_1\overset{*}{\lambda}{}_1^3-\beta 2\lambda_1^3)t} \\ &+ \mathcal{B}_6 e^{-4\alpha_1\mathrm{Im}(\lambda_1)x-4\beta_1\mathrm{Im}(\lambda_1^3)t},\end{aligned} \tag{4.204}$$

with coefficients

$$\begin{aligned}&\mathcal{B}_1 = \mathrm{i}16\mathrm{Im}(\lambda_1)\mathrm{Re}(\lambda_1)(|\mathrm{w}_{12}|^2+|\mathrm{w}_{14}|^2),\ \mathcal{B}_2 = \mathrm{i}8\mathrm{Im}(\lambda_1)\mathrm{Re}(\lambda_1)|\mathrm{w}_{11}|^2, \\ &\mathcal{B}_3 = -\left(4\mathrm{Re}(\lambda_1)(|\mathrm{w}_{12}|^2+|\mathrm{w}_{14}|^2)\right)^2,\ \mathcal{B}_4 = 8(\mathrm{Im}(\lambda_1))^2|\mathrm{w}_{11}|^2(|\mathrm{w}_{12}|^2+|\mathrm{w}_{14}|^2), \\ &\mathcal{B}_5 = -8|\lambda_1|^2|\mathrm{w}_{11}|^2(|\mathrm{w}_{12}|^2+|\mathrm{w}_{14}|^2),\ \mathcal{B}_6 = -\left(2|\mathrm{w}_{11}|^2\mathrm{Re}(\lambda_1)\right)^2.\end{aligned}$$

4.5.2.4 The dynamics of the two-soliton solution: Case II

In this configuration of the eigenvalues, the two solitons S_1 and S_2 move in the same direction as shown in figure 25. The soliton wave S_2 with the higher speed overtakes the wave S_1 and after the collision, the wave S_1 gains speed and overtakes S_2. Therefore, we have a continuously occurring phenomenon of periodic elastic collisions.

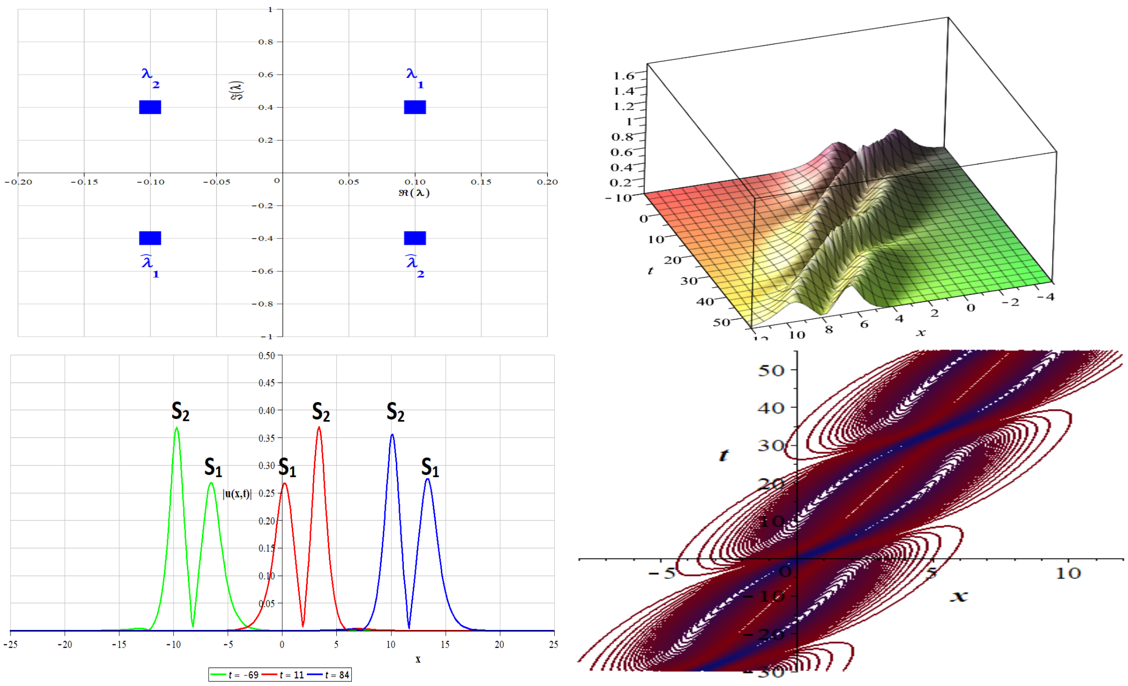


Figure 25.: Spectral plane along with 3D, 2D and contours plots of $|u(x,t)|$ of the two solitons interaction with parameters $(\alpha, \beta) = (-3, -3)$, $(\lambda_1, \lambda_2, \hat{\lambda}_1, \hat{\lambda}_2) = (0.1 + 0.4\mathrm{i}, -0.1 + 0.4\mathrm{i}, -0.1 - 0.4\mathrm{i}, 0.1 - 0.4\mathrm{i})$, $\mathrm{w}_1 = \mathrm{w}_2 = (1, 0.5, 0.5, 1 + \mathrm{i}, 1 + \mathrm{i})$.

4.5.2.5 Explicit two-soliton solutions: Case III

In that case, if $\lambda_1 = \mathrm{i}m \in \mathrm{i}\mathbb{R}_+$ is pure imaginary and $\lambda_2 = n \in \mathbb{R}_+$, then the involution property (4.155) requires that $\hat{\lambda}_1 = -\mathrm{i}m$ and $\hat{\lambda}_2 = -n$.

Let $\mathrm{w}_1 = \mathrm{w}_2 = (\mathrm{w}_{11}, \mathrm{w}_{12}, \mathrm{w}_{12}, \mathrm{w}_{14}, \mathrm{w}_{14})^T$. The solution for this nonlocal reverse-spacetime case reads:

$$u(x,t) = -2\alpha(m^2 + n^2)\mathrm{w}_{11}\overset{*}{\mathrm{w}}_{12}\frac{\mathsf{N}_3(x,t)}{\mathsf{D}_3(x,t)}, \tag{4.205}$$

$$v(x,t) = -2\alpha(m^2 + n^2)\mathrm{w}_{11}\overset{*}{\mathrm{w}}_{14}\frac{\mathsf{N}_3(x,t)}{\mathsf{D}_3(x,t)}, \tag{4.206}$$

where

$$\begin{aligned}\mathsf{N}_3(x,t) = {} & \mathcal{C}_1 e^{\left(-(\alpha_1+\alpha_2)m + \mathrm{i}2\alpha_2 n\right)x + \left((\beta_1+\beta_2)m^3 + \mathrm{i}2\beta_2 n^3\right)t} + \mathcal{C}_2 e^{\left(\mathrm{i}(\alpha_1+\alpha_2)n - 2\alpha_2 m\right)x + \left(\mathrm{i}(\beta_1+\beta_2)n^3 + 2\beta_2 m^3\right)t} \\ & + \mathcal{C}_3 e^{\left(\mathrm{i}(\alpha_1+\alpha_2)n - 2\alpha_1 m\right)x + \left(\mathrm{i}(\beta_1+\beta_2)n^3 + 2\beta_1 m^3\right)t} + \mathcal{C}_4 e^{\left(-(\alpha_1+\alpha_2)m + \mathrm{i}2\alpha_1 n\right)x + \left((\beta_1+\beta_2)m^3 + \mathrm{i}2\beta_1 n^3\right)t},\end{aligned} \tag{4.207}$$

$$
\begin{aligned}
\mathsf{D}_3(x,t) = {} & \mathcal{C}_5 e^{\left(-2\alpha_2(m-\mathrm{i}n)\right)x+\left(2\beta_2(m^3+\mathrm{i}n^3)\right)t} + \mathcal{C}_6 e^{\left(-2\alpha_1 m+\mathrm{i}2\alpha_2 n\right)x+\left(2\beta_1 m^3+\mathrm{i}2\beta_2 n^3\right)t} \\
& + \mathcal{C}_7 e^{\left(-(\alpha_1+\alpha_2)(m-\mathrm{i}n)\right)x+\left((\beta_1+\beta_2)(m^3+\mathrm{i}n^3)\right)t} + \mathcal{C}_8 e^{\left(-2\alpha_2 m+\mathrm{i}2\alpha_1 n\right)x+\left(2\beta_2 m^3+\mathrm{i}2\beta_1 n^3\right)t} \\
& + \mathcal{C}_9 e^{\left(-2\alpha_1(m-\mathrm{i}n)\right)x+\left(2\beta_1(m^3+\mathrm{i}n^3)\right)t},
\end{aligned} \tag{4.208}
$$

where the coefficients are

$\mathcal{C}_1 = \mathrm{i}2m(|\mathrm{w}_{12}|^2+|\mathrm{w}_{14}|^2)$, $\mathcal{C}_2 = -2n(|\mathrm{w}_{12}|^2+|\mathrm{w}_{14}|^2)$, $\mathcal{C}_3 = -n|\mathrm{w}_{11}|^2$, $\mathcal{C}_4 = \mathrm{i}m|\mathrm{w}_{11}|^2$,

$\mathcal{C}_5 = -4(\mathrm{i}2mn+m^2-n^2)(|\mathrm{w}_{12}|^2+|\mathrm{w}_{14}|^2)^2$, $\mathcal{C}_6 = 2(\mathrm{i}2mn-m^2+n^2)|\mathrm{w}_{11}|^2(|\mathrm{w}_{12}|^2+|\mathrm{w}_{14}|^2)$,

$\mathcal{C}_7 = -\mathrm{i}16mn|\mathrm{w}_{11}|^2(|\mathrm{w}_{12}|^2+|\mathrm{w}_{14}|^2)$, $\mathcal{C}_8 = 2(\mathrm{i}2mn-m^2+n^2)|\mathrm{w}_{11}|^2(|\mathrm{w}_{12}|^2+|\mathrm{w}_{14}|^2)$,

$\mathcal{C}_9 = -(\mathrm{i}2mn+m^2-n^2)|\mathrm{w}_{11}|^4$.

4.5.2.6 The dynamics of the two-soliton solution: Case III

Taking a look at this dynamics, we can observe a soliton moving in one direction, and a breather moving in the opposite direction. They interact continuously while the soliton travels through the breather. This is shown in figure 26.

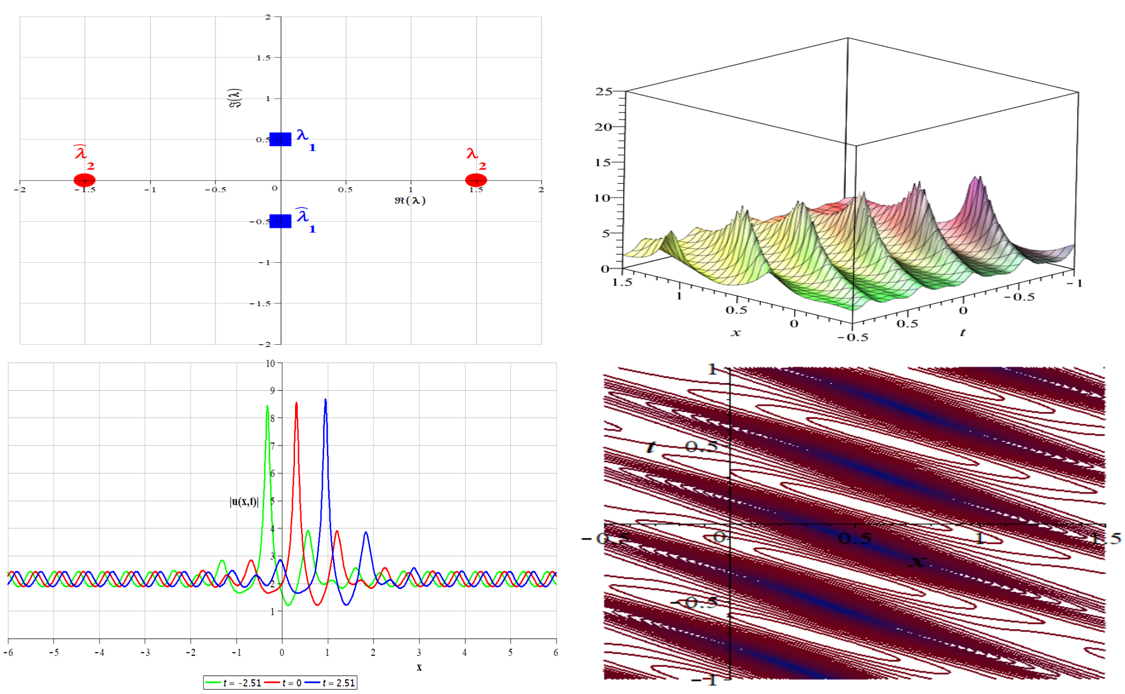


Figure 26.: Spectral plane along with 3D, 2D and contours plots of $|u(x,t)|$ of the continuous interaction between the soliton wave and the breather. The parameters are $(\alpha,\beta) = (-4,-4)$, $(\lambda_1,\lambda_2,\hat{\lambda}_1,\hat{\lambda}_2) = (0.5\mathrm{i}, 1.5, -0.5\mathrm{i}, -1.5)$, $\mathrm{w}_1 = \mathrm{w}_2 = (1, -1+\mathrm{i}, -1+\mathrm{i}, 1+\mathrm{i}, 1+\mathrm{i})$.

4.5.3 Breathers

In this particular case, the configuration (4.156) compels a two-soliton breather to behave as a one-soliton breather, if all eigenvalues are real. That is, since $\lambda_1 \neq \lambda_2$ and $\hat{\lambda}_1 \neq \hat{\lambda}_2$, then λ_2 and $\hat{\lambda}_2$ are redundant and we take $\lambda_2 = \hat{\lambda}_2 = 0$, which reduces to the one-soliton breather solution, previously mentioned (figure 22).

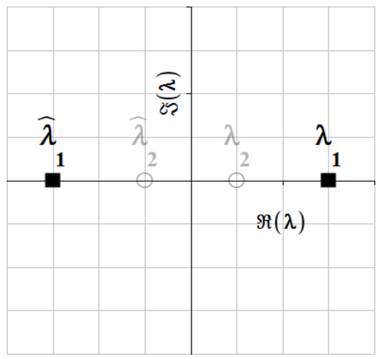


Figure 27.: Spectral plane of the two-soliton breather

4.6 Remarks

In this chapter, we investigated a nonlocal reverse-spacetime two-component Sasa-Satsuma equation. The nonlocality is embedded within the framework of a nonlocal integrable hierarchy, thus resulting in this equation. This technique allows the construction of nonlocal systems without reductions and ensures their integrability.

Chapter 5

Conclusion

To summarize, we investigated a nonlocal reverse-spacetime two-component Sasa-Satsuma equation, which was derived from a nonlocal hierarchy. This equation presents a special configuration of the eigenvalues in the spectral plane, that is $(\lambda, -\overset{*}{\lambda}, -\lambda, \overset{*}{\lambda})$ must hold whenever the eigenvalue λ is not pure imaginary. In the case where λ is pure imaginary, then the latter configuration reduces to $(\lambda,-\lambda)$. In contrary, the reverse-time sixth-order NLS-type equation exhibits the simple eigenvalue configuration $(\lambda,-\lambda)$ in the spectral plane. This configuration $(\lambda,-\lambda)$ of eigenvalues makes the Riemann-Hilbert problem easier to be solved in the reverse-time than in the reverse-spacetime [14]. As can be seen from the eigenvalues configuration of the nonlocal Sasa-Satsuma equation, which involves two symmetry relations. The first is associated with time, and the second with space. Further, a kind of soliton solutions was generated, and the Hamiltonian structure was derived for the resulting nonlocal Sasa-Satsuma equation.

Furthermore, looking at the dynamics, the reverse-spacetime equations exhibit very different dynamical behaviors than reverse-time and reverse-space equations [14]. For instance, in the reverse-spacetime Sasa-Satsuma equation, the one-soliton is a moving soliton, while in the reverse-time and reverse-space NLS-type equation, it is stationary [35].

It is also noteworthy that the two solitons collide elastically and move in the same direction in the reverse-spacetime Sasa-Satsuma equation resembling fundamental solitons, whereas in the reverse-time NLS-type equation, two solitons coming from opposite directions can collide in an elastic or inelastic manner.

At last one can ask: in general, can we construct nonlocal hierarchies starting with a nonlocal spectral matrix to obtain nonlocal integrable systems?

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