April 2024

Quandle Rings, Idempotents and Cocycle Invariants of Knots

Dipali Swain
University of South Florida

Quandle Rings, Idempotents and Cocycle Invariants of Knots

by

Dipali Swain

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
Department of Mathematics and Statistics
College of Arts and Sciences
University of South Florida

Major Professor: Mohamed Elhamdadi, Ph.D.
Dmytro Savchuk, Ph.D.
Boris Shekhtman, Ph.D.
Nasir Ghani, Ph.D.

Date of Approval:
April 04, 2024

Keywords: Quandles, Quandle rings, Idempotents, Quandle 2 cocycle, Knots and links

Dedication

To BABA who pushed me to take up Mathematics in high school, and BOU for her dream to become a Mathematician.

Acknowledgments

I want to express my sincere gratitude to Professor Mohamed Elhamdadi for his amazing guidance and mentorship during my academic journey. His insightful publications and thought-provoking questions about knot theory have really shaped my research path. The regular meetings and interesting conversations with Professor Elhamdadi have been crucial in helping me think critically and creatively, significantly advancing my research.

I extend my gratitude to Dr. Mahender Singh for his mentorship while writing the first paper with him. His impeccable knowledge of algebra helped me understand quandles much better and improved me as a mathematician. I would also thank Dr. Manpreet Singh for his support specifically in Python programming and improving my mathematical rigour.

I also want to thank the committee members, Dr. Dmytro Savchuk and Dr. Boris Shekhtman for reading my drafts and providing valuable feedback throughout the dissertation process. Special thanks to Dr. Masahico Saito for reading both my thesis and my second paper and providing valuable feedback for the same.

Lastly, a big shout-out to my husband Sudheendra for his unwavering support. His generosity and understanding have allowed me to pursue my academic goals, who always believed in my dreams and helped me realize them.

Table of Contents

List of Tables

List of Figures

Abstract

Quandles are sets with self-distributive binary operations that axiomatize the three Reidemeister moves in classical knot theory. In an attempt to bring ring theoretic techniques to the study of quandles, a theory of quandle rings analogous to the classical theory of group rings where several interconnections between quandles and their associated quandle rings have been explored. Functoriality of the construction implies that morphisms of quandle rings give a natural enhancement of the well-known quandle coloring and quandle 2 cocycle invariant of knots and links.

The dissertation is structured into two main parts. In the first part, we delve into quandle rings obtained from non-trivial quandles over rings. We demonstrate that integral quandle rings emerging from non-trivial involutory coverings possess infinitely many non-trivial idempotents which, themselves form quandles, contributing to a comprehensive understanding of their structure. Applying these findings to knot theory, we deduce that the quandle ring associated with the knot quandle of a non-trivial long knot exhibits non-trivial idempotents. Furthermore, we explore free products of quandles and establish that integral quandle rings of free quandles exclusively feature trivial idempotents, yielding an infinite family of such quandles.

In the second part, we focus on leveraging idempotents in quandle rings to enhance the quandle 2-cocycle invariant of knots and links. By combining idempotents with state sum invariants of knots, we successfully distinguish all 12965 prime oriented knots with up to 13 crossings, utilizing only 21 connected quandles and three quandles made of idempotents in quandle rings. Additionally, we distinguish from knots their mirror images using the same set of 24 quandles.

Chapter 1: Preliminaries

1.1 Introduction

The field of knot theory revolves around the examination and the study of embeddings of the unit circle $\mathbb{S}^1$ in the three-dimensional Euclidean space $\mathbb{R}^3$ or its compactification $\mathbb{S}^3$. Such embeddings are known as knots and they become the focal point of investigation. Equivalence between two knots is established under the criterion of the existence of an ambient isotopy that transforms one knot into the other which is further found in the topological and geometric properties inherent in these embeddings, providing a foundation for the systematic understanding and classification of knots. In other words, two knots K and K' are said to be equivalent (denoted by $K \cong K'$) if there exists a continuous map $f : [0,1] \times \mathbb{R}^3 \to \mathbb{R}^3$ such that $f(0, K) = K$ and $f(1, K) = K'$ and $f(t, \cdot)$ is a homeomorphism for all t. One of the problems in Knot theory is on classifying knots. Knots are considered equivalent if we can smoothly transform one into the other. This is an equivalence relation on the set of knots and thus we obtain equivalence classes of knots. An invariant of knots is a function that is constant within each equivalence class. Invariants of knots are used to distinguish knots. For instance, if a knot invariant gives different values for two knots, thus the knots are not equivalent. These invariants can take well defined algebraic forms, such as numbers, polynomials, or groups, and serve as algebraic tools in the classification of knots.

Reidemeister [29] showed that the study of knots in the 3-space is the same as the study of knot diagrams in the plane modulo the planar isotopy and the so called Reidemeister moves I, II and III as shown in Figure 1.2. More precisely, he proved that two links are equivalent if and only if any link diagram of one can be transformed to any link diagram of the other by a finite sequence of Reidemeister moves and planar isotopies.

Axiomatisation of the three Reidemeister moves of planar diagrams of knots and links in the 3-space led to algebraic structures known as quandles [23]. Besides being fundamental to knot theory, these structures arise in a variety of contexts such as set-theoretic solutions to the Yang-Baxter equation [5], Yetter-Drinfeld Modules [13], Riemannian symmetric spaces [26], Hopf algebras [1] and mapping class groups [35, 36, 37], to name a few.

In an attempt to bring ring theoretic techniques to the study of quandles, a theory of quandle rings analogous to the classical theory of group rings has been proposed in [3], where several interconnections between quandles and their associated quandle rings have been explored. Functoriality of the construction implies that morphisms of quandle rings give a natural enhancement of the well-known quandle coloring invariant of knots and links. Quandle rings of non-trivial quandles are non-associative, and it has been proved in [14] that these rings are not even power-associative, which is the other end of the spectrum of associativity. Furthermore, quandle rings of non-trivial quandles over rings of characteristic more than three cannot be alternative or Jordan algebras since alternative and Jordan algeras are power associative [4].

The objective of this thesis is to explore idempotents in quandle rings and their relation with quandle coverings. It is shown that integral quandle rings of finite quandles with non-trivial coverings over nice base quandles admit infinitely many non-trivial idempotents which form quandle. The quandles that admit a set of non trivial idempotents which form a quandle are used to construct stronger invariant of knots by combining cocycle invariant of knots and idempotents.

This thesis is organised as follows: In Chapter 1 we review some basics of knot theory and algebraic structures from knots. We discuss some knot invariants using these algebraic structures such as quandle coloring and quandle 2 cocycle invariant. In Chapter 2 we define quandle rings analogous to group rings and explore various properties of these quandle rings in particular. In Chapter 3, we consider the notion of idempotents in quandle rings and give the following sufficient condition on a quandle X for its quandle ring $\mathbf{k}[X]$ over the ring $\mathbf{k}$ to admit non-trivial idempotents (Proposition 3.2.8):

Let X be a quandle containing a trivial subquandle Y of order more than one. Then $\mathbf{k}[X]$ has non-trivial idempotents.

We also relate idempotents with quandle coverings in the following (see statement of Proposition 3.3.6):

If L is a non-trivial long knot (see Definition 3.3.5), then the quandle ring $\mathbf{k}[Q(L)]$ of its knot quandle $Q(L)$ has non-trivial idempotents.

As one of the main results of Chapter 3, we prove that if $p : X \to Y$ is a non-trivial quandle covering such that X is involutory and $\mathbf{k}[Y]$ has only trivial idempotents, then $\mathbf{k}[X]$ has many non-trivial idempotents and we give their precise description in the following theorem (Theorem 3.3.2).

Let $p : X \to Y$ be a non-trivial quandle covering where X is an involutory quandle. If $\mathbf{k}[Y]$ has only trivial idempotents, then the set of idempotents of $\mathbf{k}[X]$ is

$$\begin{aligned}\mathcal{I}(\mathbf{k}[X]) \;=\; \Big\{ &\sum_{y\in J}\Big(\sum_{x\in I_y,\ \sum \alpha_x=0} \alpha_x\big(e_x + e_{x**x_0}\big)\Big) + \Big(\sum_{x'\in I_{y_0},\ \sum \alpha_{x'}=1} \alpha_{x'} e_{x'}\Big) \;\Big|\\ &J \in \mathcal{F}(Y),\ \ I_y \in \mathcal{F}(p^{-1}(y)),\ \ I_{y_0} \in \mathcal{F}(p^{-1}(y_0)),\ \ x_0 \in I_{y_0},\ \ y_0 \in Y,\ \ \alpha_x, \alpha_{x'} \in \mathbf{k}\Big\}.\end{aligned}$$

We also consider free products of quandles and overcome the lack of associativity in quandles through an appropriate length function for elements in free products. As the second main result, we prove the following result (Theorem 3.4.3):

Let FQ_n be the free quandle of rank $n \geq 1$. Then $\mathbb{Z}[FQ_n]$ has only trivial idempotents.

This gives an infinite family of quandles whose integral quandle rings have only trivial idempotents. Lastly, as an application in Chapter 4 we use *idempotents* in quandle rings in combination with the state sum invariants of knots to distinguish all of the 12965 prime oriented knots up to 13 crossings using *only* 21 *connected* quandles and three quandles made of idempotents in quandle rings. We also distinguish all knots up to 13 crossings from their mirror images using the same 24 quandles. Furthermore, we distinguish all of the 2977 prime oriented knots up to 12 crossings using *only* 10 connected quandles and three quandles made of idempotents in quandle rings improving a result in [7] .

1.2 Review of Knot Theory

A *knot* is the image of an embedding of a circle $\mathbb{S}^1$ into the 3-sphere $\mathbb{S}^3$. A knot is said to be *oriented* if there is a preferred direction to travel around the knot. Two knots are considered to be *equivalent* if one can be transformed into the other by a continuous deformation. Precisely,

Definition 1.2.1. Two knots K and K' are said to be equivalent (denoted by$K \cong K'$) if there exists a continuous map $f : [0,1] \times \mathbb{R}^3 \to \mathbb{R}^3$ such that $f(0,K) = K$ and $f(1,K) = K'$ and $f(t,\cdot)$ is a homeomorphism for all t.

A *link* is a collection of disjoint union of finitely many knots. Each knot in a link is termed as a component. Thus, a knot is a link with one component. Equivalence of links can be defined in the same manner as that of knots. It is easy to note that each link can be projected on the plane $\mathbb{R}^2$ or on the 2-sphere $\mathbb{S}^2$. A projection is said to be generic if there are only finitely many multiple points, and that the multiple points are only transversal double points.

1.2.1 Knot Diagrams

Of fundamental importance to the classification of knots and links, is the concept of a diagram of a knot or a link. This is a generically immersed closed plane curve together with over/under crossing information corresponding to each double point.

Definition 1.2.2. A link diagram is a generic projection of a link with the information of over- and under-crossing arcs at the double points.

It is easy to see that such a diagram always exists. The following figure gives some diagrams of the knot, trefoil and figure eight knot.

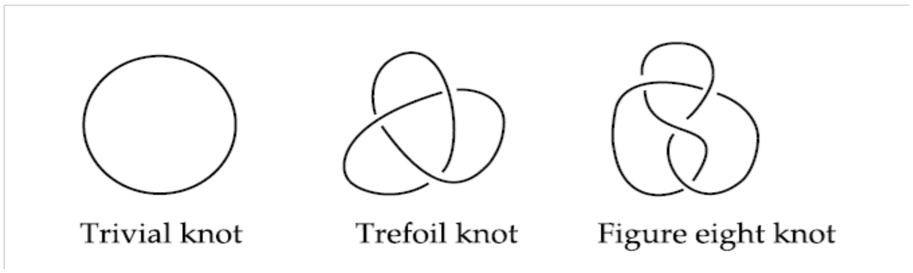


Figure 1.1 Examples of knot diagrams

In 1920s, Reidemeister [29] showed that the study of equivalence classes of links in $\mathbb{S}^3$ is equivalent to the study of link diagrams on the plane modulo three local moves known as the *Reidemeister moves* (See Figure 1.2).

Theorem 1.2.1. *[29] Two links are equivalent if and only if their link diagrams are related by a finite sequence of Reidemeister moves and planar isotopies (orientation preserving homeomorphisms of plane onto itself).*

The above interpretation of links in terms of their diagrams is one of the most important results in knot theory which has lead to the study of links from a combinatorial perspective. As a result, various invariants have been constructed for the classifications of knots.

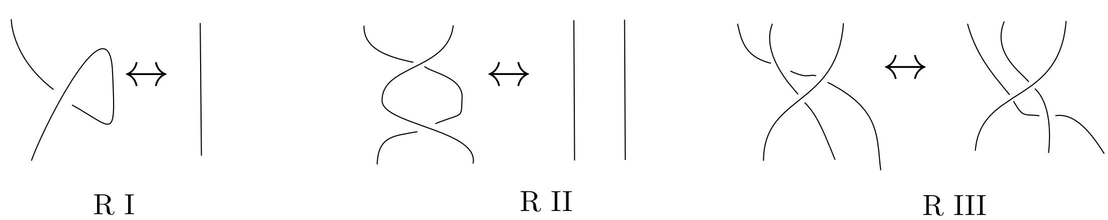


Figure 1.2 Reidemeister moves for link diagrams

Let K_1 and K_2 be two oriented knots. Then the connected sum [31] of K_1 and K_2, denoted by $K_1 \# K_2$, is shown below in the Figure 1.3

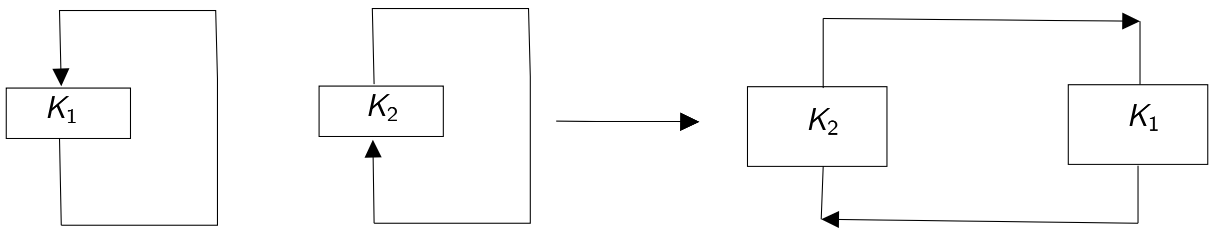


Figure 1.3 Connected sum of two oriented knots

Definition 1.2.3. A knot is said to be prime if it cannot be written as a connected sum of two non-trivial knots. For example, torus knots (knots which lie on the surface of a torus) are prime.

1.2.2 Classical Braids

A *geometric braid* [31] on n strands is a subset β of $\mathbb{R}^2 \times I$ consisting of n disjoint closed intervals such that following conditions are satisfied :

- $\beta \cap (\mathbb{R}^2 \times 0) = \{(1,0,0),(2,0,0),\ldots(n,0,0)\}$,
- $\beta \cap (\mathbb{R}^2 \times 1) = \{(1,0,1),(2,0,1),\ldots(n,0,1)\}$,
- each strand of β intersects with $\mathbb{R}^2 \times \{t\}$ on a point for all $t \in [0,1]$.

Two geometric braids β_1 and β_2 are said to be isotopic if there exists an ambient isotopy

$$f : (\mathbb{R}^2 \times I) \times I \to \mathbb{R}^2 \times I$$

such that $f(\beta_1, 0) = \beta_1$, $f(\beta_1, 1) = \beta_2$ and $f(\beta_1, t)$ is geometric braid at each time t.

Clearly, isotopy induces an equivalence relation on the set of geometric braids on n strands. These equivalence classes are called *braids*. As in case of links, geometric braids can be studied via diagrams on the plane.

Two braid diagrams are said to be equivalent if they are related by a finite sequence of planar isotopies and local moves shown in Figure 1.2.

Definition 1.2.4. The braid group B_n is the group with a presentation having n - 1 generators $\sigma_1, \dots, \sigma_{n-1}$ and following set of relations:

- $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i-j| \geq 2$ and $i, j \in \{1, 2, \dots, n-1\}$,
- $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for $i \in \{1, \dots n-2\}$.

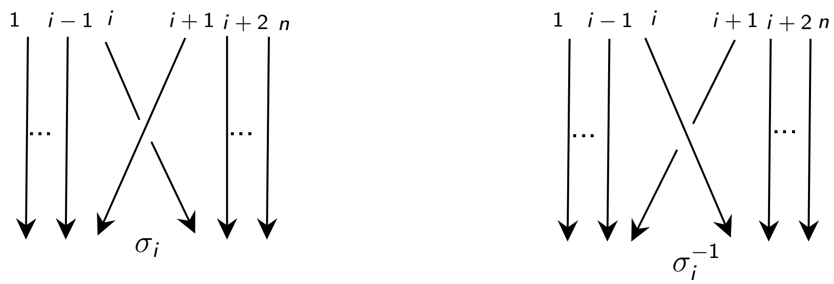


Figure 1.4 Generators of the braid group B_n

The closure of a braid is the link $Cl(b)$ obtained from b by connecting the lower ends of the braid with the upper ends; (see Figure 1.5). Obviously, isotopic braids generate isotopic links. Closures of braids are usually taken to be oriented: all strands of the braid are oriented from the top to the bottom (See Figure 1.4).

Theorem 1.2.5. Each link can be represented as the closure of a braid.

One can obtain a link diagram from a braid diagram. By closure of a braid diagram D, we mean a diagram obtained by connecting the boundary points of D having the same second coordinate with smooth non-intersecting arcs. Obviously, closure of a braid is a well-defined operation as closures of any two equivalent braid diagrams give equivalent link diagrams. From now onwards, we will denote the closure of a braid β by $Cl(\beta)$. Knots are, in particular, closed braids. Figure 1.5 illustrates the braid diagram of the trefoil knot.

Definition 1.2.6. For a knot K, the *braid index* [25], denoted $b(K)$, is the fewest number of strings needed to express K as a closed braid. The *braid length* [25] of the knot K is the fewest number of crossings needed to express K as a closed braid. For example, in Figure 1.5, the braid index and the braid length for the trefoil respectively are 2 and 3.

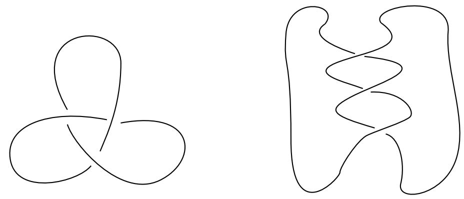


Figure 1.5 Braid diagram for the trefoil knot

1.2.3 Symmetries in Knots

If a knot is viewed as an oriented homeomorphism class of an oriented pair, $K = (\mathbb{S}^3, \mathbb{S}^1)$, with $\mathbb{S}^i$ homeomorphic to $\mathbb{S}^i$, there are four oriented knots associated to any particular knot K. In addition to K itself, there is the reverse, $r(K) = (\mathbb{S}^3, -\mathbb{S}^1)$, the concordance inverse, $-K = (-\mathbb{S}^3, -\mathbb{S}^1)$, and the mirror image, $m(K) = (-\mathbb{S}^3, \mathbb{S}^1)$[25].

Definition 1.2.7. [7] By a symmetry we mean that a knot K remains unchanged under one of r, m, rm. As in the definition of symmetry type in [25] we say that a knot K is

- *reversible* if the only symmetry it has is $K = r(K)$,
- *negative amphicheiral* if the only symmetry it has is $K = rm(K)$,
- *positive amphicheiral* if the only symmetry it has is $K = m(K)$,
- *fully amphicheiral* if it has all three symmetries, that is, $K = r(K) = m(K) = rm(K)$,
- *chiral* if $K \neq r(K) \neq m(K) \neq rm(K)$.

Note that there exists non-equivalent knots which are not mirror images with isomorphic knot groups.

Figure 1.6 gives diagrams of the trefoil knot and its mirror image.

Definition 1.2.8. [30] The Wirtinger presentation is a finite group presentation of the fundamental group of the complement of a knot in 3-space.

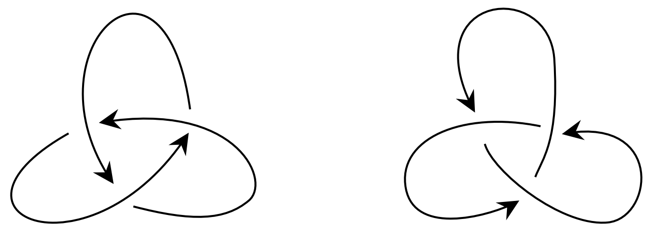


Figure 1.6 Left hand trefoil and Right hand trefoil

For an oriented knot K, let $\mathcal{D}(K)$ be the knot diagram of K. We label each arc in D with x_i and define the relation at each crossing as shown in the Figure 1.7 below Let r_i denote each ith relation obtained at each crossing. Then, the group $\pi(\mathbb{R}^3 \setminus K) = \{x_1, x_2, \dots, x_n; r_1, \dots r_n\}$

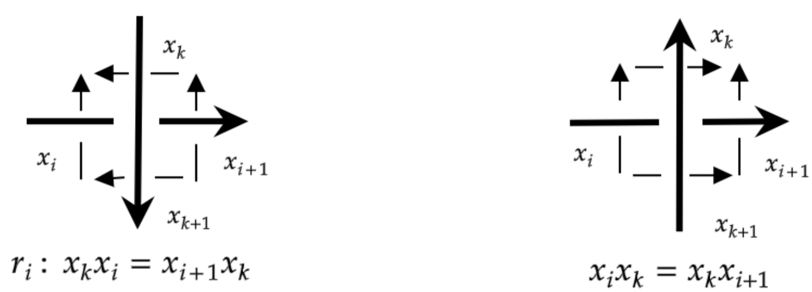


Figure 1.7 Wirtinger relations on crossings

Theorem 1.2.2. *[10, 31] An n-component link L is trivial if and only if $\pi_1(C(L))$ is isomorphic to the free group of rank n.*

Chapter 2: Algebraic Structures from Knots

2.1 Quandles

In this section, we introduce the main objects of our study.

Definition 2.1.1. A *quandle* is a non-empty set X with a binary operation $(x, y) \mapsto x * y$ satisfying the following axioms:

- $x * x = x$ for all $x \in X$,
- For any $x, y \in Q$ there exists a unique $z \in X$ such that $x = z * y$.
- $(x * y) * z = (x * z) * (y * z)$ for all $x, y, z \in X$.

Using oritented links, we can give quandle crossing relation as follows

Some basic examples of quandles are as follows:

Example 2.1.2. The set $X = \{1, 2, 3, \dots, n\}$ with binary operation $x * y = x$ for all $x, y \in X$. In other words, the map $S_x : X \times X \to X$ such that $S_y(x) = x * y$ is an identity map. This set X, equipped with the binary operation $*$ is known as *trivial quandle* denoted by T_n. A cayley table for trivial quandle T_n of order n looks like the following:

$*$	1	2	3	$\cdots$	n
1	1	1	1	$\dots$	1
2	2	2	2	$\cdots$	2
3	3	3	3	$\dots$	3
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$
n	n	n	n	$\cdots$	n

Example 2.1.3. Let $X = \{0, 1, 2\}$ with operation $x * y = 2y - x \ (mod\ 3)$ is a quandle. The cayley table is as follows:

$*$	0	1	2
0	0	2	1
1	2	1	0
2	1	0	2

Example 2.1.4. Let G be a multiplicative group. Then the binary operation $x * y = yxy^{-1}$ is a quandle on G known as conjugation quandle denoted by $Conj(G)$. Conjugacy classes in groups are a rich source of quandles. Another way of defining a quandle operation on such groups G can be obtained by the following example.

Example 2.1.5. Given a multiplicative group G, define a binary operation by $x*y = yx^{-1}y$. Then $(G, *)$ is a quandle known as *core quandle* denoted by $Core(G)$. For abelian groups (written additively) the operation becomes $x * y = 2y - x$. In particular, the cyclic group of order $n \geq 2$ gives the *dihedral quandle* $R_n = \{0, 1, 2, \ldots, n - 1\}$ of order n. For $n = 3$, (Example 2.1.3) represents R_3.

Example 2.1.6. Let G be a group and let $\phi \in Aut(G)$ be a group automorphism. Define a binary operation $*$ such that $x * y = \phi(xy^{-1})y$. Then, the set G equipped with $*$ forms a quandle known as the *generalised Alexander quandle* of G with respect to ϕ.

Definition 2.1.7. Let $(X, *)$ and $(Y, \star)$ be two quandles and $f : (X, *) \to (Y, \star)$ be a map. Then,

- f is a *quandle homomorphism* if $f(x * y) = f(x) \star f(y)$.
- f is a *quandle isomorphsim* if f is a bijective quandle homomorphism.
- f is a *quandle automorphsim* if f is a quandle isomorphism of X with itself.

Let X be a quandle. We denote set of all automorphisms of X as $Aut(X)$. For a given element $x \in X$, the inner automorphism induced by x is a map $S_x : X \to X$ such that

$S_x(y) = y * x$. The subgroup of $Aut(X)$ generated by the set $\{S_x | x \in X\}$ is known as the inner automorphism group of X, and is denoted by $Inn(X)$. Henceforth, the word *orbit* would correspond to an orbit in X under the action of $Inn(X)$.

Remark 2.1.8. It turns out that quandle axioms are simply algebraic formulations of the three Reidemeister moves of planar diagrams of knots and links in the 3-space which can be seen below.

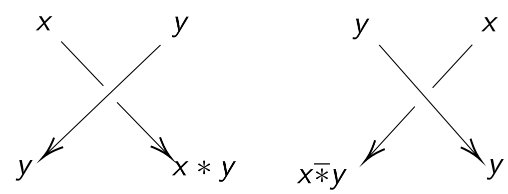


Figure 2.1 Quandle crossing relation

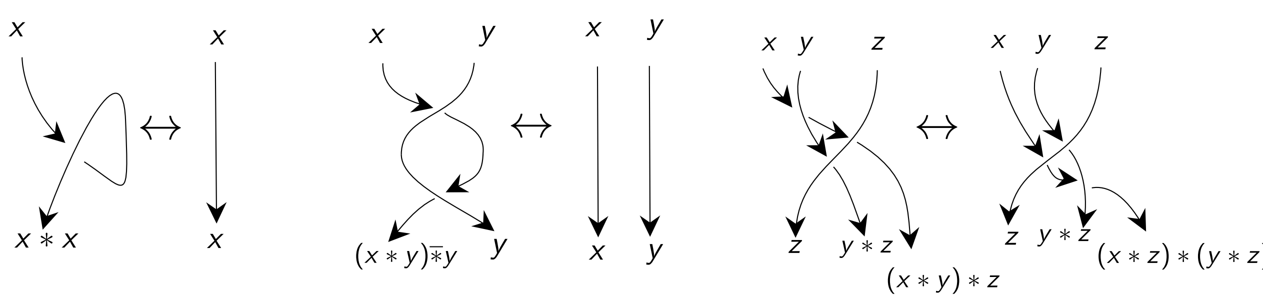


Figure 2.2 Quandle axioms from Reidemeister I, II and III

Besides being fundamental to knot theory, quandles arise in a variety of contexts such as set-theoretic solutions to the Yang-Baxter equation [5], Yetter-Drinfeld Modules [13], Riemannian symmetric spaces [26], Hopf algebras [1] and mapping class groups [35, 36, 37], to name a few.

Definition 2.1.9. A quandle X is called *connected* [31] if the inner automorphism group $Inn(X)$ acts transitively on X. In other words, X cannot be written as disjoint union of its orbits. For example, the dihedral quandle R_{2n+1} is connected, whereas R_{2n} is not.

Definition 2.1.10. A quandle X is said to be *involutory* [31] if for each $x \in X$, the inner automorphism S_x is an involution, that is, $(y * x) * x = y$ for all $x, y \in X$. For example, for any group G, the core quandle $Core(G)$ is involutory, whereas $Conj(F_n)$ is not involutory for a free group F_n of rank $n \geq 2$.

Definition 2.1.11. A quandle X is called *latin* if each left multiplication $L_x : X \to X$ given by $L_x(y) = x * y$ for $y \in X$, is bijective, and called *semi-latin* if each L_x is injective.

Example 2.1.12. For example, dihedral quandles R_{2n+1} are latin. A simple way of checking latin quandles is to observe that elements in each row are non-repeating. (Refer to Example 2.1.3)

Definition 2.1.13. A quandle X is said to be *simple* [31] if for any quandle Y, every quandle homomorphism $X \to Y$ is either injective or constant. For example, if G is a simple group, then $Core(G)$ is a simple quandle. On the other hand, the dihedral quandle R_{2n} is neither latin nor simple.

Definition 2.1.14. A quandle X is said to be *commutative* [31] if for any $x, y \in X$, $x * y = y * x$. For example, R_3 (see Example 2.1.3) is commutative but R_4 is not. Furthermore, X is said to be *quasicommutative* if at least one of the following holds:

1. $x * y = y * x$,
2. $x * y = x \bar{*} y$,
3. $x \bar{*} y = y * x$,
4. $x \bar{*} y = y \bar{*} x$.

Every commutative quandle is quasicommutative but not vice versa. For example, Consider group $(\mathbb{R}, +)$ and its automorphism $\phi : \mathbb{R} \to \mathbb{R}$ defined as $\phi(x) = 2x$. Then, the Alexander quandle $\mathbb{R}$ with respect to ϕ is quasi-commutative but not commutative.

2.2 Knot Quandles

Analogous to knot group also known as fundamental group of a knot K, we define the fundamental quandle or Knot quandle for a knot K

Definition 2.2.1. Let K be an oriented knot. We label each arc of K with a unique arbitrary symbol and at each crossing note the relation given in Figure 2.1.The resulting quandle associated to this knot K is a free quandle generated by these symbols modulo the relations denoted by $Q(K)$. That is, the quandle ssociated to K is the set of formal strings of arc labels separated by $*$ and $\bar{*}$ using parentheses to indicate association. Furthermore, to each homeomorphism of knots we assign the unique isomorphism of quandles induced by the Reidemeister moves (refer to Figure 2.2). We call $Q(K)$ as *the fundamental quandle or the knot quandle* of the knot K.

There exists a covariant functor between the category of knot diagrams and the category of its respective knot quandles [24].

Theorem 2.2.1. *[24, 28] Let K and K' be two oriented knots in the 3-sphere, $\mathbb{S}^3$, and let : $\mathcal{D}(K)$ and $\mathcal{D}(K')$ be their associated knot diagrams. Then K' is ambient isotopic to either K or the mirror image of K with direction reversed if and only if the fundamental quandles of K and K' are isomorphic.*

Below is an example of the figure-8 knot and its associated knot quandle

Example 2.2.2. Consider the following knot K. We label each of its arcs with symbols a, b, c and d. At each crossing, we use the relation given in Figure 2.1 to note down the relation.

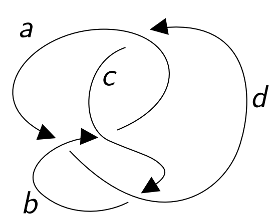


Figure 2.3 Figure 8 knot

Then, the knot quandle $Q(K)$ is a free quandle with the symbols and relations given as

$$Q(K) = < a,\ b,\ c,\ d\ :\ d * a = c,\ b * c = a,\ a \bar{*} b = d,\ d \bar{*} c = b >$$

which can further be reduced to three generators

$$Q(K) = < a,\ b,\ c\ :\ (a \bar{*} b) * a = c,\ b * c = a,\ (a \bar{*} b) * c = b >$$

2.3 Quandle Cocycle Invariants of Knots and Links

Let X be a quandle and K be a classical knot or a link diagram. Let $\mathcal{R}$ be a set of arcs. Then a coloring of K by the quandle X is a map from $\mathcal{R}$ to X such that at every crossing, the relations in Figure 2.4 hold. Alternately, it is a homomorphism from the knot quandle (see for example [17, 24, 28]) of K to the quandle X.

Theorem 2.3.1. *Let $Q(K)$ be the knot quandle for a knot K and X be some finite quandle. Then $|Hom(Q(K), X|$ is invariant under the Reidemeister moves, thus forming an invariant for knots.*

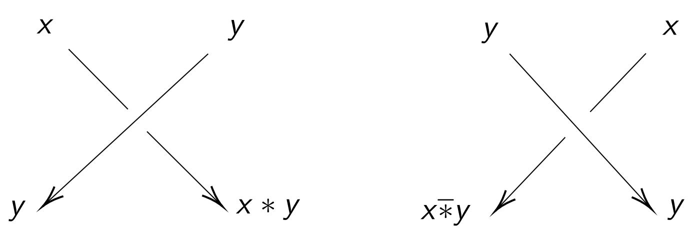


Figure 2.4 Rules of colorings at crossing

Let X be a finite quandle and let A be an abelian group. Let $C_n^{\mathrm{R}}(X)$ be the free abelian group generated by n-tuples $(x_1, \dots, x_n)$ of elements X. We define a homomorphism ∂_n : $C_n^{\mathrm{R}}(X) \to C_{n-1}^{\mathrm{R}}(X)$ by

$$\begin{aligned} &\partial_n(x_1, x_2, \dots, x_n) \\ &\quad = \sum_{i=2}^{n} (-1)^i \left[(x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \right. \\ &\quad \left. - (x_1 * x_i, x_2 * x_i, \dots, x_{i-1} * x_i, x_{i+1}, \dots, x_n) \right] \end{aligned}$$

for $n \geq 2$ and $\partial_n = 0$ for $n \leq 1$. Then $C_*^{\mathrm{R}}(X) = \{C_n^{\mathrm{R}}(X), \partial_n\}$ is a chain complex.

Let $C_n^{\mathsf{D}}(X)$ be the subset of $C_n^{\mathsf{R}}(X)$ generated by n-tuples $(x_1, \dots, x_n)$ with $x_i = x_{i+1}$ for some $i \in \{1, \dots, n-1\}$ if $n \geq 2$; otherwise let $C_n^{\mathsf{D}}(X) = 0$. If X is a quandle, then $\partial_n(C_n^{\mathsf{D}}(X)) \subset C_{n-1}^{\mathsf{D}}(X)$ and $C_*^{\mathsf{D}}(X) = \{C_n^{\mathsf{D}}(X), \partial_n\}$ is a sub-complex of $C_*^{\mathsf{R}}(X)$. Consider the quotient complex $\{C_*^{\mathsf{Q}}(X)\}$ with $C_n^{\mathsf{Q}}(X) = C_n^{\mathsf{R}}(X)/C_n^{\mathsf{D}}(X)$. For quandles, the chain and cochain complexes with coefficient in an abelian group A are given by

$$C_*^{\mathsf{Q}}(X; A) = C_*^{\mathsf{Q}}(X) \otimes A, \qquad \partial = \partial \otimes \mathsf{id};$$
$$C^*_{\mathsf{Q}}(X; A) = \mathsf{Hom}(C_*^{\mathsf{Q}}(X), A), \qquad \delta = \mathsf{Hom}(\partial, \mathsf{id}).$$

The nth *quandle homology group* and the nth *quandle cohomology group* [6] of a quandle X with coefficient group A are given by

$$H_n^{\mathsf{Q}}(X; A) = H_n(C_*^{\mathsf{Q}}(X; A)), \quad H^n_{\mathsf{Q}}(X; A) = H^n(C^*_{\mathsf{Q}}(X; A)).$$

For more details on quandle cohomology see [6]. In this thesis we will focus on low dimensional cohomology and precisely 2-cocycles as they are needed to define the quandle cocycle invariant of knots. A function $\phi : X \times X \to A$ is called a quandle 2-cocycle if it satisfies the 2-cocycle condition:

$$\phi(x, y) - \phi(x, z) + \phi(x * y, z) - \phi(x * z, y * z) = 0; \quad \forall x, y, z \in X \tag{2.3.1}$$

and

$$\phi(x, x) = 0, \quad \forall x \in X \tag{2.3.2}$$

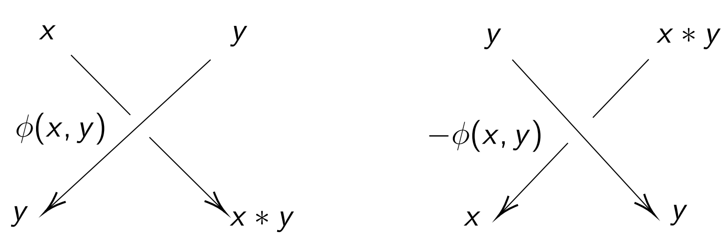


Figure 2.5 Boltzmann weights at crossing

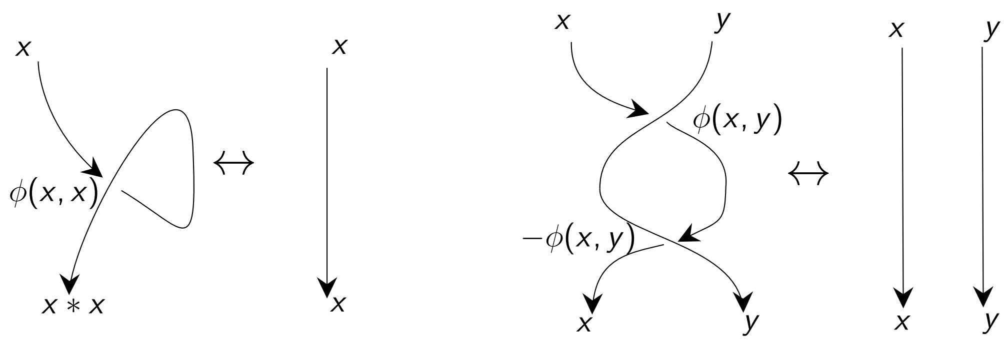


Figure 2.6 Boltzmann weights from Reidemeister I and II

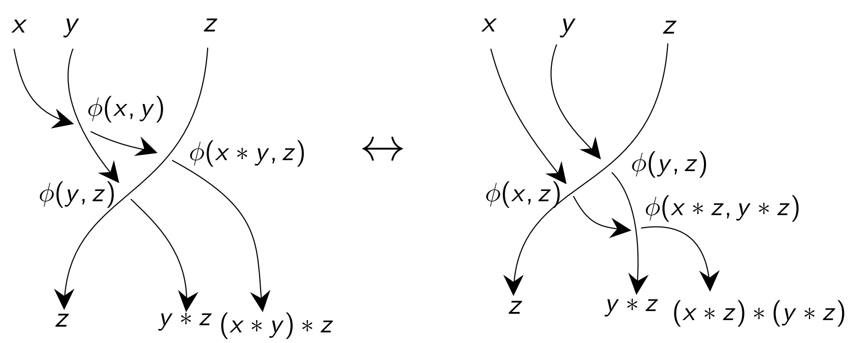


Figure 2.7 The quandle 2-cocycle condition (2.3.1) from Reidemeister III

Let X be a quandle and $\phi : X \times X \to A$ be a 2-cocycle. Consider a knot K and let $\mathcal{C}_X(K)$ be a coloring of K. The Boltzmann weight at a crossing τ is defined by $\phi(x, y)^\epsilon$, where ϵ is the sign of the crossings (see Figure 2.5). Thus one sees that equation (2.3.1) can be obtained from Figure 2.7.

Definition 2.3.1. [6] Let X be a quandle and ϕ be a 2-cocycle with coefficient in an abelian group A. Let $\mathcal{D}(K)$ be a diagram of a knot K. The state sum of the knot diagram $\mathcal{D}(K)$ is given by

$$\Phi(\mathcal{D}) = \sum_{\mathcal{C}} \prod_{\tau} \phi(x, y)^\epsilon \tag{2.3.3}$$

where the product is taken over all crossings of $\mathcal{D}$ and the sum is taken over all the possible colorings of $\mathcal{D}$.

Observe that in Definition 2.3.1, the group A is assumed to be multiplicative group. The Boltzmann state sum is an element of the group ring of A i.e. $\Phi(\mathcal{D}) \in \mathbb{Z}[A]$.

Theorem 2.3.2. [6, Theorem 4.4, page 3954] Let ϕ be a 2-cocycle with coefficient in an abelian group A. Let $\mathcal{D}(K)$ be a diagram of a knot K. The state sum $\Phi(\mathcal{D})$ is invariant under the three Reidemeister moves, thus it is denoted by $\Phi_\phi(K)$.

Chapter 3: Idempotents, Free Products and Quandle Coverings

In this chapter we explore idempotents in quandle rings, specifically their connection to quandle coverings. We establish that integral quandle rings arising from non-trivial involutory coverings over well-behaved base quandles possess infinitely many non-trivial idempotents, offering a complete characterization of these idempotents. Notably, the collected idempotents constitute a quandle in their own right. Applying these results to knot theory, we infer that the quandle ring of the knot quandle for a non-trivial long knot exhibits non-trivial idempotents. Additionally, we investigate free products of quandles, proving that integral quandle rings of free quandles exclusively feature trivial idempotents, yielding an infinite family of such quandles. We extend our analysis to describe idempotents in quandle rings associated with unions and specific twisted unions of quandles. This work contributes to the mathematical understanding of quandle structures and their relationships with idempotents in diverse algebraic settings. The present chapter is based on [19].

3.1 Group Rings

Let $\mathbf{k}$ be a field and let G be a multiplicative group. A *group ring* $\mathbf{k}[G]$ is an associative $\mathbf{k}$-algebra with the elements of G as a basis and with multiplication defined distributively using the group multiplication in G. To be more precise, $\mathbf{k}[G]$ consists of all formal finite sums of the form

$$\alpha = \sum_{x \in G} a_x \cdot x$$

with $a_x \in \mathbf{k}$. Here, finiteness means all of the coefficients a_x are zero except finitely many.

If $(\mathbf{k}, +, \cdot)$ is an integral domain implies an associative and commutative ring with unity and without zero-divisors with unity.

Definition 3.1.1. Let $\mathbf{k}[G]$ be a group ring where $\mathbf{k}$ is an associative ring with unity. Then, the surjective ring homomorphism defined by $\varphi : \mathbf{k}[G] \to \mathbf{k}$ is called the *augmentation map* given by

$$\varphi\Big(\sum_{g\in G} \alpha_g g\Big) = \sum_{g\in G} \alpha_g$$

3.2 Quandle Rings

Definition 3.2.1. Let $(X, *)$ be a quandle and $\mathbf{k}$ an integral domain with unity $\mathbf{1}$. Let $\mathbf{k}[X]$ be the set of all formal expressions of the form $\sum_{x\in X} \alpha_x e_x$, where each e_x is a unique symbol corresponding to $x \in X$ and $\alpha_x \in \mathbf{k}$ such that all $\alpha_x = 0$ except finitely many. The addition in $\mathbf{k}[X]$ is defined as usual and the multiplication is given by

$$\Big(\sum_{x\in X} \alpha_x e_x\Big)\Big(\sum_{y\in X} \beta_y e_y\Big) = \sum_{x,y\in X} \alpha_x \beta_y e_{x*y},$$

where $x, y \in X$ and $\alpha_x, \beta_y \in \mathbf{k}$. This turns $\mathbf{k}[X]$ into a ring (rather a $\mathbf{k}$-algebra) called the *quandle ring* of X with coefficients in $\mathbf{k}$.

Clearly, the multiplication is distributive with respect to addition from both left and right, and $\mathbf{k}[X]$ forms a ring, which we call the *quandle ring* of X with coefficients in the ring $\mathbf{k}$. Since X is non-associative, unless it is a trivial quandle, it follows that $\mathbf{k}[X]$ is a non-associative ring, in general. It follows that $\{e_x \mid x \in X\}$ forms a basis for the $\mathbf{k}$-algebra $\mathbf{k}[X]$.

Remark 3.2.2. Observe that a quandle with a left multiplicative identity has only one element. For, let $e \in X$ be the left identity of X. Then $e * x = x$ for all $x \in X$. But, we have $x * x = x$. Now, by axiom invertibility of right multiplication, we must have $e = x$ for all $x \in X$, and hence $X = \{e\}$. Thus, $\mathbf{k}[X]$ is a non-associative ring without unity, unless X is a singleton.

Definition 3.2.3. Let X be a quandle. Consider $\mathbf{k}[X]$ as the quandle ring where $\mathbf{k}$ is a ring with unity. Then, the surjective ring homomorphism $\varepsilon : \mathbf{k}[X] \to \mathbf{k}$ given by

$$\varepsilon\Big(\sum_{x\in X} \alpha_x e_x\Big) = \sum_{x\in X} \alpha_x$$

is called the *augmentation map*. The kernel of ε is a two-sided ideal of $\mathbf{k}[X]$, called the *augmentation ideal* of $\mathbf{k}[X]$.We make a distinction between the product in a quandle and the product in its associated quandle ring.

Since $\mathbf{k}[X]$ is a ring without unity, it is desirable to embed it into a ring with unity. The ring

$$\mathbf{k}^\circ[X] := \mathbf{k}[X] \oplus \mathbf{k}e,$$

where e is a symbol (not in X) satisfying $e\big(\sum_i \alpha_i x_i\big) = \sum_i \alpha_i x_i = \big(\sum_i \alpha_i x_i\big)e$, is called the *extended quandle ring* of X. For convenience, we denote the unity $1e$ of $\mathbf{k}^\circ[X]$ by e. We can extend the augmentation map to $\varepsilon : \mathbf{k}^\circ[Q] \to \mathbf{k}$ and define the *extended augmentation ideal* as

$$\Delta_{\mathbf{k}^\circ}(Q) := \ker(\varepsilon : \mathbf{k}^\circ[Q] \to \mathbf{k}).$$

As before, it is easy to see that the set $\{x - e \mid x \in X\}$ is a basis for $\Delta_{\mathbf{k}^\circ}(X)$ as an $\mathbf{k}$-module.

Definition 3.2.4. Let X be a quandle and $\mathbf{k}$ an associative ring. Then $\{e_x - e_y \mid x, y \in X\}$ is a generating set for $\Delta_{\mathbf{k}}(X)$ as an $\mathbf{k}$-module. Further, if $x_0 \in X$ is a fixed element, then the set $\big\{e_x - e_{x_0} \mid x \in X \setminus \{x_0\}\big\}$ is a basis for $\Delta_{\mathbf{k}}(X)$ as an $\mathbf{k}$-module.

Using Definition 3.2.4, the article [3] makes the following obseervation

1. Let X be a quandle and $\mathbf{k}$ an associative ring. Then $x * y + y * x \equiv x + y \mod \Delta_{\mathbf{k}}^2(X)$ for all $x, y \in X$.

2. Let X be a trivial quandle, Y a subquandle of X and $\mathbf{k}$ an associative ring. Then $\Delta_{\mathbf{k}}(Y)$ is a two-sided ideal of $\mathbf{k}[X]$.

3. Let X be a quandle and $\mathbf{k}$ an associative ring. Then the quandle X is trivial if and only if $\Delta^2_{\mathbf{k}}(X) = \{0\}$.

A group algebra can be studied using methods of associative algebras. Recall that quandle algebras are not associative for non-trivial quandles. On the other hand, some classes of non associative algebras, for instance, alternative algebras, Jordan algebras and Lie algebras, are well studied. Thus, it is interesting to know whether quandle algebras belong to these classes of algebras.

3.2.1 Idempotents in Quandle Rings

Definition 3.2.5. Let X be a quandle and $\mathbf{k}$ an integral domain with unity. A non-zero element $v \in \mathbf{k}[X]$ is called an *idempotent* if $v^2 = v$. The set of all idempotents of $\mathbf{k}[X]$ is denoted by

$$\mathcal{I}(\mathbf{k}[X]) = \left\{ v \in \mathbf{k}[X] \mid v^2 = v \right\}$$

Unlike in group rings, where the units play a fundamental role in the structure theory of the group ring, quandle rings have idempotents as the natural object since each quandle element is, by definition, an idempotent of the quandle ring i.e $\{e_x \mid x \in X\}$ are idempotents of $\mathbf{k}[X]$, and we refer to them as *trivial idempotents.* A non-trivial idempotent is an element of $\mathbf{k}[X]$ that is not of the form e_x for any $x \in X$.

In ring theory, figuring out idempotents is a key challenge. Likewise, exploring idempotents in a quandle ring is based on the search for new quandles within the quandle ring. To find the set $\mathcal{I}(\mathbf{k}[X])$ of non-zero idempotents in a quandle ring $\mathbf{k}[X]$, we start with a basic idea: every quandle element is an idempotent in its own ring, called "trivial idempotents." Unlike integral group rings, which lack non-trivial idempotents, extended quandle rings have a unique twist. The identity element and elements of the form $e - x$, where $x \in X$, are non-trivial idempotents, offering a distinct perspective on idempotent behavior in quandle rings.

It is a well-known result of Swan [32, p.571] that if G is a finite group, then the group ring $\mathbf{k}[G]$ has a non-trivial idempotent if and only if some prime divisor of $|G|$ is invertible in $\mathbf{k}$. Although, we do not have Lagrange's theorem for finite quandles, a partial one way analogue of this result does hold for finite quandles.

Proposition 3.2.6. Let X be a finite quandle having a subquandle Y with more than one element such that $|Y|$ is invertible in $\mathbf{k}$. Then $\mathbf{k}[X]$ has a non-trivial idempotent.

Proof. Since the subquandle Y has more than one element, a direct check shows that the element

$u = \frac{1}{|Y|}\sum_{y\in Y} e_y$ is a non-trivial idempotent of $\mathbf{k}[X]$.

Remark 3.2.7. The converse of Proposition 3.2.6 does not hold. For example, consider the quandle

$X = \{1, 2, 3\}$ given in terms of its multiplication table as follows:

*	1	2	3
1	1	1	2
2	2	2	1
3	3	3	3

Here, (i, j)-th entry of the matrix represents the element $i * j$. The quandle ring $\mathbb{Z}[X]$ has non-trivial idempotents of the form $\alpha e_1 + (1-\alpha)e_2$ for $\alpha \in \mathbb{Z}$, but X has no subquandle Y with more than one element such that $|Y|$ is invertible in $\mathbb{Z}$.

The following proposition gives a sufficient condition that guarantees the existence of non-trivial idempotents.

Proposition 3.2.8. Let X be a quandle containing a trivial subquandle Y of order more than one. Then $\mathbf{k}[X]$ has non-trivial idempotents.

Proof. Consider the element $u = \sum_{i=1}^{n} \alpha_i e_{y_i}$, where $n \geq 2$, $y_i \in Y$ and $\alpha_i \in \mathbf{k}$ such that $\sum_{i=1}^{n} \alpha_i = 1$.

A direct check shows that $u^2 = u$, and hence u is a non-trivial idempotent of $\mathbf{k}[X]$.

Lemma 3.2.1. *Let X be a faithful quandle. If $x, y \in X$ be two distinct elements such that $x * y = x$, then $y * x = y$.*

Proof. Since $S_y S_x = S_{x*y} S_y$ and $x * y = x$, it follows that S_x and S_y commute.

Thus, the identity $S_x S_y = S_{y*x} S_x$ implies that $S_{y*x} = S_y$. Since X is faithful, we get $y * x = y$, which is desired.

Proposition 3.2.9. Let X be a faithful quandle such that S_x has more than one fixed-point for some $x \in X$. Then $\mathbf{k}[X]$ has non-trivial idempotents.

Proof. Since S_x has a non-trivial fixed-point, we have $y * x = y$ for some $y \in X$ with $y \neq x$. But, X is faithful, and hence by Lemma 3.2.1, we have $x * y = x$. Thus, the set $\{x, y\}$ forms a trivial subquandle in X. From Proposition 3.2.8, we have $(\alpha e_x + (1 - \alpha) e_y)^2 = \alpha e_x + (1 - \alpha) e_y$. Thus $\mathbf{k}[X]$ has non-trivial idempotents.

Proposition 3.2.10. If G is a non-trivial group, then $\mathbf{k}[(G)]$ has non-trivial idempotents.

Proof. Note that, for each non-identity element $x \in G$ and distinct integers i, j, the set $\{x^i, x^j\}$ forms a trivial subquandle. The result now follows from Proposition 3.2.8.

As an application of quandle rings for link quandles, we have the following proposition.

Proposition 3.2.11. Let L be a link containing the Hopf link and $Q(L)$ the corresponding link quandle of L. Then $\mathbf{k}[Q(L)]$ has non-trivial idempotents.

Proof. Let H be the Hopf link. The knot quandle $Q(H)$ is given by

$$Q(H) = < x,\ y :\ x * y = x,\ y * x = y > = \{x\ , y\}$$

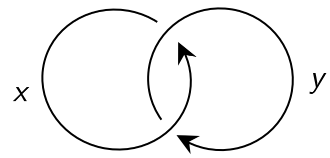


Figure 3.1 Hopf link

It follows from the construction of the link quandle [23, 28] that $Q(L)$ contains $Q(H)$ as a subquandle with two elements. Using Proposition 3.2.8, we have the desired result.

It has been speculated in [4] that connected quandles have only trivial idempotents. We give two examples showing that this is not true in general.

Example 3.2.12. Consider $X = C[8, 1] = \{1, 2, 3, ... 8\}$ (The notation $C[8, 1]$ means that we are considering the first connected quandle of order 8 given in [33]). The right multiplications of X as products of disjoint cycles are as follows:

$S_1 = S_2 = (3\ 6\ 7)(4\ 5\ 8)$, $S_3 = S_4 = (1\ 8\ 6)(2\ 5\ 7)$, $S_5 = S_6 = (1\ 4\ 7)(2\ 3\ 8)$,
$S_7 = S_8 = (1\ 5\ 3)(2\ 6\ 4)$

We see that X has trivial subquandles $\{1, 2\}, \{3, 4\}, \{5, 6\}$ and $\{7, 8\}$. From proposition 3.2.8, we have $e_1 - e_2, e_3 - e_4, e_5 - e_6$ and $e_7 - e_8$ are non-trivial idempotents in $\mathbb{Z}[X]$.

Example 3.2.13. Consider the connected quandle $X = C[12, 9] = \{1, 2, \ldots, 12\}$ of order 12 (As in the previous example $C[12, 9]$ is the ninth connected quandle of order 12 given in [33]). The right multiplications of X as products of disjoint cycles are as follows:

$$S_1 = (5\ 11\ 7\ 9)(6\ 12\ 8\ 10),\quad S_2 = (5\ 12\ 7\ 10)(6\ 11\ 8\ 9),\quad S_3 = (5\ 9\ 7\ 11)(6\ 10\ 8\ 12),$$
$$S_4 = (5\ 10\ 7\ 12)(6\ 9\ 8\ 11),\quad S_5 = (1\ 9\ 3\ 11)(2\ 10\ 4\ 12),\quad S_6 = (1\ 10\ 3\ 12)(2\ 9\ 4\ 11),$$
$$S_7 = (1\ 11\ 3\ 9)(2\ 12\ 4\ 10),\quad S_8 = (1\ 12\ 3\ 10)(2\ 11\ 4\ 9),\quad S_9 = (1\ 7\ 3\ 5)(2\ 8\ 4\ 6),$$
$$S_{10} = (1\ 8\ 3\ 6)(2\ 7\ 4\ 5),\quad S_{11} = (1\ 5\ 3\ 7)(2\ 6\ 4\ 8),\quad S_{12} = (1\ 6\ 3\ 8)(2\ 5\ 4\ 7).$$

We see that X has trivial subquandles $\{1, 2, 3, 4\}$, $\{5, 6, 7, 8\}$ and $\{9, 10, 11, 12\}$. By Proposition 3.2.8, the elements $\alpha e_1 + \beta e_2 + \gamma e_3 + (1 - \alpha - \beta - \gamma)e_4$, $\alpha e_5 + \beta e_6 + \gamma e_7 + (1 -$

$\alpha-\beta-\gamma)e_8$ and $\alpha e_9+\beta e_{10}+\gamma e_{11}+(1-\alpha-\beta-\gamma)e_{12}$ are non-trivial idempotents of $\mathbb{Z}[X]$ for any $\alpha, \beta, \gamma \in \mathbb{Z}$.

A computer assisted check [27] with quandles of order less than seven suggests the following.

Conjecture 3.2.14. If X is a finite latin quandle then the quandle ring $\mathbb{Z}[X]$ has only trivial idempotents.

Every group G can be turned into a quandle (G) by setting $x*y = yx^{-1}y$, and called the *core quandle* of G. For abelian groups G (written additively), the quandle operation becomes $x*y = 2y-x$. In particular, the cyclic group of order $n \geq 2$ gives the dihedral quandle of order n, denoted by $R_n = \{0, 1, 2, \ldots, n-1\}$. As a supporting evidence to Conjecture 3.2.14, we prove the following.

Proposition 3.2.15. Let G be an abelian group without 2 and 3-torsion. Then $\mathbb{Z}[(G)]$ has no non-trivial idempotent built up with at most three distinct basis elements.

Proof. Let $u = \alpha e_x + \beta e_y + \gamma e_z$ be an idempotent of $\mathbb{Z}[(G)]$, where $x, y, z \in G$ are distinct and $\alpha, \beta, \gamma \in \mathbb{Z}$. We have the following two cases:

Case 1: Suppose that precisely two of α, β and γ are non-zero. Without loss of generality, we can take $u = \alpha e_x + \beta e_y$ with $\alpha \neq 0$ and $\beta \neq 0$. Then $u = u^2$ gives

$$\alpha e_x + \beta e_y = \alpha^2 e_x + \beta^2 e_y + \alpha\beta e_{2x-y} + \alpha\beta e_{2y-x}. \tag{3.2.1}$$

Clearly, $e_{2x-y} \neq e_x$ and $e_{2x-y} \neq e_y$ since G has no 2-torsion. Similarly, $e_{2y-x} \neq e_x$ and $e_{2y-x} \neq e_y$. Hence, we must have $\alpha\beta = 0$, a contradiction. Thus, this case does not arise.

Case 2: Suppose that all of α, β and γ are non-zero. Then $u = u^2$ gives

$$\alpha e_x+\beta e_y+\gamma e_z = \alpha^2 e_x+\beta^2 e_y+\gamma^2 e_z+\alpha\beta e_{2x-y}+\alpha\beta e_{2y-x}+\beta\gamma e_{2y-z}+\beta\gamma e_{2z-y}+\alpha\gamma e_{2z-x}+\alpha\gamma e_{2x-z}. \tag{3.2.2}$$

Note that $e_{2y-z} \neq e_{2z-y}$, $e_{2y-x} \neq e_{2x-y}$ and $e_{2z-x} \neq e_{2x-z}$ since G has no 3-torsion. We compare coefficients of e_x on both the sides of (3.2.2). Clearly, $e_x \neq e_y, e_z, e_{2x-y}, e_{2x-z}$. Further, $e_x \neq e_{2y-x}, e_{2z-x}$ since G has no 2-torsion.

Case 2(a): If $e_x = e_{2y-z}$, then

$$\alpha = \alpha^2 + \beta\gamma. \tag{3.2.3}$$

But, $e_x = e_{2y-z}$ also implies that $e_z = e_{2y-x}$. Comparing coefficients of e_z on both the sides of (3.2.2) gives

$$\gamma = \gamma^2 + \alpha\beta. \tag{3.2.4}$$

Adding (3.2.3) and (3.2.4) gives $\alpha + \gamma = \alpha^2 + \gamma^2 + \beta(\alpha + \gamma)$. Now we compare coefficients of e_y on both the sides of (3.2.2). If e_y appears only once on the right hand side of (3.2.2), then $\beta = \beta^2$, and hence $\beta = 1$. This gives $\alpha^2 + \gamma^2 = 0$, which further implies that $\alpha = \gamma = 0$, a contradiction. If $e_y = e_{2x-z}$, then $e_z = e_{2x-y}$, a contradiction. Similarly, if $e_y = e_{2z-x}$, then $e_x = e_{2z-y}$, which is again a contradiction. Hence Case 2(a) does not arise.

Case 2(b): If $e_x = e_{2z-y}$, we proceed as above, and see that this subcase does not arise.

It follows that e_x appears on the right hand side of (3.2.2) precisely once. Hence $\alpha = \alpha^2$, and consequently $\alpha = 1$. Repeating the process for e_y and e_z, we obtain $\beta = 1$ and $\gamma = 1$. But, this gives $\varepsilon(u) = 3$, a contradiction. Hence $\mathbb{Z}[(G)]$ has no non-trivial idempotent built up with at most three distinct basis elements.

Remark 3.2.16. Given a non-empty set X and a ring $\mathbf{k}$, let $\mathbf{k}[X]$ be the free $\mathbf{k}$-module on the set X. Then a binary operation on X can be used to define a ring structure on $\mathbf{k}[X]$ by imitating the construction of a quandle or a group ring. An *idempotent quasigroup* is a set X with a binary operation such that both left and right multiplications by elements of X are bijections of X and $x * x = x$ for all $x \in X$. It is worth mentioning that Conjecture 3.2.14 does not hold if we replace latin quandles by idempotent quasigroups. As a counterexample, consider the idempotent quasigroup with multiplication table as follows:

*	1	2	3	4	5	6	7	8
1	1	3	2	5	6	4	8	7
2	5	2	1	7	8	3	4	6
3	4	6	3	8	7	1	5	2
4	6	8	7	4	3	5	2	1
5	8	7	4	6	5	2	1	3
6	7	4	8	2	1	6	3	5
7	3	5	6	1	2	8	7	4
8	2	1	5	3	4	7	6	8

A direct computation shows that $u = e_2 - e_3 - e_6 + e_7$ is an idempotent of the ring $\mathbb{Z}[X]$. This suggests that a proof of Conjecture 3.2.14 should use the right-distributivity of the quandle in an essential way.

Definition 3.2.17. A quandle X is called *medial* if $(x * y) * (z * w) = (x * z) * (y * w)$ for all $x, y, z, w \in X$. These are precisely the quandles for which the natural map $X \times X \to X$ given by $(x, y) \mapsto x * y$ is a quandle homomorphism, where $X \times X$ is equipped with the product quandle structure. The following result is interesting in its own.

Proposition 3.2.18. Let X be a medial quandle. Then the following hold:

1. The right multiplication by an idempotent is a ring endomorphism of $\mathbf{k}[X]$.
2. If X is finite, then right multiplications by distinct idempotents give distinct ring endomorphisms of $\mathbf{k}[X]$.

Proof. Let $u = \sum_{i=1}^{n} \alpha_i e_i$ be an idempotent of $\mathbf{k}[X]$. Let $\hat{S}_u : \mathbf{k}[X] \to \mathbf{k}[X]$ be the map given by $\hat{S}_u(w) = wu$ for all $w \in \mathbf{k}[X]$. Let e_k, e_l be two basis elements of $\mathbf{k}[X]$. Then, we see that

$$\begin{aligned}
\hat{S}_u(e_k e_l) &= e_{k*l} \Big(\sum_{i,j=1}^{n} \alpha_i \alpha_j e_{i*j} \Big), \quad \text{since } u = u^2 \\
&= \sum_{i,j=1}^{n} \alpha_i \alpha_j e_{(k*l)*(i*j)} \\
&= \sum_{i,j=1}^{n} \alpha_i \alpha_j e_{(k*i)(l*j)}, \quad \text{since } X \text{ is medial} \\
&= \sum_{i,j=1}^{n} \alpha_i \alpha_j e_{k*i} e_{l*j} \\
&= \Big(\sum_{i=1}^{n} \alpha_i e_{k*i} \Big) \Big(\sum_{j=1}^{n} \alpha_j e_{l*j} \Big) \\
&= \hat{S}_u(e_k)\ \hat{S}_u(e_l).
\end{aligned}$$

Since $\hat{S}_u$ is $\mathbf{k}$-linear, it is a ring homomorphism, which proves (1).

For assertion (2), suppose that X is finite of order n. Let $u = \sum_{i=1}^{n} \alpha_i e_i$ and $v = \sum_{i=1}^{n} \beta_i e_i$ be two idempotents of $\mathbf{k}[X]$. If $\hat{S}_u = \hat{S}_v$, then $\sum_{i=1}^{n} \alpha_i e_{k*i} = \hat{S}_u(e_k) = \hat{S}_v(e_k) = \sum_{i=1}^{n} \beta_i e_{k*i}$ for any basis element e_k. But, this gives $\alpha_i = \beta_i$ for all i, which implies that $u = v$.

Remark 3.2.19. Consider the quandle $X = C[6,1]$ from [33] with multiplication table as follows:

*	1	2	3	4	5	6
1	1	1	5	6	3	4
2	2	2	6	5	4	3
3	5	6	3	3	1	2
4	6	5	4	4	2	1
5	3	4	1	2	5	5
6	4	3	2	1	6	6

Take $\mathbf{k} = \mathbb{Q}$, $u = \frac{1}{2}(e_1 + e_2)$ and $\hat{S}_u$ the right multiplication by u. Then u is an idempotent of $\mathbb{Q}[X]$ and $e_3 - e_4 \in \ker(\hat{S}_u)$. Thus, the $\mathbf{k}$-linear map $\hat{S}_u$ need not be injective in general.

Remark 3.2.20. The set of idempotents of a quandle ring fails to satisfy the right-distributivity in general. For example, consider the quandle ring $\mathbb{Z}[X]$ of the quandle X of Remark 3.2.19. Take the idempotents $u = e_1$, $v = e_4$ and $w = \alpha e_5 + (1-\alpha)e_6$. Then a direct check shows that $(uv)w = e_6$, whereas $(uw)(vw) = (2\alpha - 2\alpha^2)e_5 + (2\alpha^2 - 2\alpha + 1)e_6$. This implies that the set of idempotents of a quandle may not form a quandle. The article [18] provides a table of quandles up to order 5 with their idempotents computed with coefficients in $\mathbb{Z}$ and $\mathbb{Z}_2$ and states the cases when such a set of idempotents is a quandle.

In associative algebras, the operator induced by multiplication by an idempotent is a projection onto a subspace, and hence has eigenvalues 1 and 0. For non-associative algebras, the eigenvalues of the operator induced by an idempotent can be arbitrary in general. Given an idempotent v of a non-associative algebra $\mathbf{k}$ over $\mathbb{C}$, let $\sigma(v)$ denote the Peirce spectrum of v, which is the set of all eigenvalues of the operator induced by v. The *Peirce spectrum* of an idempotent v induces the Peirce decomposition of the algebra $\mathbf{k}$, which is the decomposition of $\mathbf{k}$ into a direct sum of corresponding eigenspaces.

Remark 3.2.21. Let X be a non-trivial quandle and $\mathbf{k}$ a field. On the contrary to associative algebras, the right multiplication $\hat{S}_u$ by an idempotent u of $\mathbf{k}[X]$ is not a projection of the underlying $\mathbf{k}$-vector space $\mathbf{k}[X]$. Thus, the spectrum of the idempotent u (defined as the spectrum of the $\mathbf{k}$-linear map $\hat{S}_u$) may be arbitrary.

3.3 Idempotents from Quandle Coverings

In this section, we use quandle coverings for computing idempotents in quandle rings of involutory quandles. The notion of a quandle covering was introduced in the work of Eisermann [11, 12].

Definition 3.3.1. A quandle homomorphism $p : X \to Y$ is called a *quandle covering* if p is surjective and $S_x = S_{x'}$ whenever $p(x) = p(x')$ for any $x, x' \in X$. Clearly, an isomorphism of quandles is a quandle covering, called a *trivial covering*.

Example 3.3.2. Some examples of quandle coverings are:

1. A surjective group homomorphism $p : G \to H$ yields a quandle covering $(G) \to (H)$ if and only if $\ker(p)$ is a central subgroup of G.

2. A surjective group homomorphism $p : G \to H$ yields a quandle covering $(G) \to (H)$ if and only if $\ker(p)$ is a central subgroup of G of exponent two.

3. Let X be a quandle and F a non-empty set viewed as a trivial quandle. Consider $X \times F$ with the product quandle structure $(x, s) * (y, t) = (x * y, s)$. Then the projection $p : X \times F \to X$ given by $(x, s) \to x$ is a quandle covering, called trivial covering with fibre F.

4. Let X be a quandle and A an abelian group. A map $\alpha : X \times X \to A$ is called a quandle 2-cocycle if it satisfies

$$\alpha_{x,y}\ \alpha_{x*y,z} = \alpha_{x,z}\ \alpha_{x*z,y*z}$$

and

$$\alpha_{x,x} = 1$$

for $x, y, z \in X$. Given a 2-cocyle α, the set $X \times A$ turns into a quandle with the binary operation

$$(x, s) * (y, t) = \big(x * y,\ s\ \alpha(x, y)\big),$$

for $x, y \in X$ and $s, t \in A$. The quandle so obtained is called an *extension* of X by A through α, and is denoted by $X \times_\alpha A$. We refer the reader to [1] for generalities and related results. A direct check shows that the projection $p : X \times_\alpha A \to X$ given by $p(x, s) = x$ is a quandle covering.

The following lemma summarises some basic properties of quandle coverings.

Lemma 3.3.1. *If $p : X \to Y$ is a quandle covering, then the following hold:*

1. *Each fibre $p^{-1}(y)$ is a trivial subquandle of X.*

2. *Each inner automorphism of X permutes fibres.*

3. *The fibres over any two elements of the same connected component of Y are isomorphic.*

Proof. If $p : X \to Y$ is a quandle homomorphism, then each fibre $p^{-1}(y)$ is a subquandle of X. Since p is a covering, $S_x = S_{x'}$ whenever $x, x' \in p^{-1}(y)$. This gives $x*x' = S_{x'}(x) = S_x(x) = x$ and $x' * x = S_x(x') = S_{x'}(x') = x'$, which proves assertion (1).

For assertion (2), it is enough to check that if $x_1, x_2 \in p^{-1}(y)$, then $S_x(x_1)$ and $S_x(x_2)$ are in the same fibre. Indeed, $p(S_x(x_1)) = p(x_1 * x) = y * p(x) = p(x_2 * x) = p(S_x(x_2))$, and we are done.

Let y, y' be elements of the same connected component of Y. Then there exists elements $y_1, y_2, \dots, y_n \in Y$ and $\mu_1, \mu_2, \dots, \mu_n \in \{1, -1\}$ such that $y' = y *^{\mu_1} y_1 *^{\mu_2} y_2 \cdots *^{\mu_n} y_n$. Here the parentheses are left normalised. For each i, choose one element $x_i \in p^{-1}(y_i)$. If $x \in p^{-1}(y)$, then we see that

$$p(x *^{\mu_1} x_1 *^{\mu_2} x_2 \cdots *^{\mu_n} x_n) = y *^{\mu_1} y_1 *^{\mu_2} y_2 \cdots *^{\mu_n} y_n = y'.$$

Thus, the inner automorphism $S_{x_n}^{\mu_n} S_{x_{n-1}}^{\mu_{n-1}} \cdots S_{x_1}^{\mu_1}$ maps the fibre $p^{-1}(y)$ bijectively onto $p^{-1}(y')$, which proves (3).

Proposition 3.3.3. If $p : X \to Y$ is a non-trivial quandle covering, then $\mathbf{k}[X]$ has non-trivial idempotents.

Proof. Since p is a non-trivial covering, there is at least one connected component of Y such that $|p^{-1}(y)| \geq 2$ for all elements y in that connected component. By Lemma 3.3.1(1), $p^{-1}(y)$ is a trivial subquandle of X. The result now follows from Proposition 3.2.8.

Example 3.3.4. Consider $X = C[12,3]$ the third connected quandle [33] of order 12. As a set $X = \{1, 2, ..., 12\}$, Its quandle operationc $*$ is given in terms of right multiplications as follows:

$S_1 = (2\ 12\ 5\ 10\ 11)(3\ 8\ 6\ 7\ 4)$ $S_2 = (1\ 11\ 7\ 4\ 12)(3\ 5\ 10\ 6\ 9)$ $S_3 = (1\ 2\ 7\ 6\ 10)(4\ 9\ 8\ 5\ 12)$

$S_4 = (1\ 11\ 6\ 8\ 5)(2\ 7\ 9\ 3\ 12)$ $S_5 = (1\ 12\ 3\ 8\ 10)(2\ 4\ 9\ 6\ 11)$ $S_6 = (1\ 5\ 3\ 4\ 2)(7\ 11\ 10\ 8\ 9)$

$S_7 = (1\ 10\ 8\ 3\ 12)(2\ 11\ 6\ 9\ 4)$ $S_8 = (1\ 12\ 5\ 7\ 11)(3\ 9\ 6\ 10\ 5)$ $S_9 = (2\ 11\ 10\ 5\ 12)(3\ 4\ 7\ 6\ 8)$

$S_{10} = (1\ 5\ 8\ 6\ 11)(2\ 12\ 3\ 9\ 7)$ $S_{11} = (1\ 10\ 6\ 7\ 2)(4\ 12\ 5\ 8\ 9)$ $S_{12} = (1\ 2\ 4\ 3\ 5)(7\ 9\ 8\ 10\ 11)$.

Consider $Y = \{1, \dots, 24\}$. Its quandle operation $\star$ is given in terms of right multiplications as follows:

$S_1 = S_{13} = (2\ 12\ 5\ 10\ 11)(3\ 8\ 6\ 7\ 4)(14\ 24\ 17\ 22\ 23)(15\ 20\ 18\ 19\ 16)$

$S_2 = S_{14} = (1\ 11\ 7\ 4\ 12)(3\ 5\ 10\ 6\ 9)(13\ 23\ 19\ 16\ 24)(15\ 17\ 22\ 18\ 21)$

$S_3 = S_{15} = (1\ 2\ 7\ 6\ 10)(4\ 9\ 8\ 5\ 12)(13\ 14\ 19\ 18\ 22)(16\ 21\ 20\ 17\ 24)$

$S_4 = S_{16} = (1\ 11\ 6\ 8\ 5)(2\ 7\ 9\ 3\ 12)(13\ 23\ 18\ 20\ 17)(14\ 19\ 21\ 15\ 16)$

$S_5 = S_{17} = (1\ 12\ 3\ 8\ 10)(2\ 4\ 9\ 6\ 11)(13\ 24\ 15\ 20\ 22)(14\ 16\ 21\ 18\ 23)$

$S_6 = S_{18} = (1\ 5\ 3\ 4\ 2)(7\ 11\ 10\ 8\ 9)(13\ 17\ 15\ 16\ 14)(19\ 23\ 22\ 20\ 18)$

$S_7 = S_{19} = (1\ 10\ 8\ 3\ 12)(2\ 11\ 6\ 9\ 4)(13\ 22\ 20\ 15\ 24)(14\ 23\ 18\ 21\ 16)$

$S_8 = S_{20} = (1\ 12\ 5\ 7\ 11)(3\ 9\ 6\ 10\ 5)(13\ 24\ 16\ 19\ 23)(15\ 21\ 18\ 22\ 17)$

$S_9 = S_{21} = (2\ 11\ 10\ 5\ 12)(3\ 4\ 7\ 6\ 8)(14\ 23\ 22\ 17\ 24)(15\ 16\ 19\ 18\ 20)$

$S_{10} = S_{22} = (1\ 5\ 8\ 6\ 11)(2\ 12\ 3\ 9\ 7)(13\ 17\ 20\ 18\ 23)(14\ 24\ 15\ 21\ 19)$

$S_{11} = S_{23} = (1\ 10\ 6\ 7\ 2)(4\ 12\ 5\ 8\ 9)(13\ 22\ 18\ 19\ 14)(16\ 24\ 17\ 20\ 21)$

$S_{12} = S_{24} = (1\ 2\ 4\ 3\ 5)(7\ 9\ 8\ 10\ 11)(13\ 14\ 16\ 15\ 17)(19\ 21\ 20\ 22\ 23)$

Let $\phi : (Y, \star) \to (X, *)$ be a map such that $\phi(y) = y(mod 12)$ for all $y \in Y$. It is easy to check that ϕ is a quandle homomorphism as for any $a, b \in Y$, we have $\phi(a * b) = (ab)\ mod\ (12) = (a\ mod\ 12)(b\ mod\ 12) = \phi(a) \star \phi(b)$. By the nature of the map, ϕ is surjective and not injective(as $|X| < |Y|$). Thus, ϕ is a non-trivial covering. Then by Proposition 3.3.3, we have $\mathbf{k}[X]$ has non trivial idempotents. In particular, $\mathbb{Z}_2[X]$ has non-trivial idempotents.

Definition 3.3.5. An embedding $\phi : \mathbb{R} \to \mathbb{R}^3$ is called a *long knot* if there exist $\alpha, \beta \in \mathbb{R}$ such that

$\phi(t) = (0, 0, t)$ for any $t < \alpha$ or $t > \beta$.

The following proposition about long knots is also an immediate application of the Proposition 3.3.3.

Proposition 3.3.6. If L is a non-trivial long knot, then the quandle ring $\mathbf{k}[Q(L)]$ of its knot quandle $Q(L)$ has non-trivial idempotents.

Proof. Let L be a long knot and K its corresponding closed knot defined in the obvious way. Let $Q(L)$ and $Q(K)$ be knot quandles of L and K, respectively. Note that $Q(K)$ is obtained from $Q(L)$ by adjoining one extra relation corresponding to the first and the last arc of L. By [11, Theorem 35], the natural projection $p : Q(L) \to Q(K)$ is a non-trivial quandle covering, and the result follows from Proposition 3.3.3.

Remark 3.3.7. Let $p : X \to Y$ be a quandle covering, and $\mathcal{F}(Y)$ the set of all finite subsets of Y. For each $y \in Y$, let $\mathcal{F}(p^{-1}(y))$ be the set of all finite subsets of $p^{-1}(y)$, and denote a typical element of this set by I_y. The main result of this section is the following theorem.

Theorem 3.3.2. *Let $p : X \to Y$ be a non-trivial quandle covering where X is involutory. If $\mathbf{k}[Y]$ has only trivial idempotents, then the set of idempotents of $\mathbf{k}[X]$ is*

$$\mathcal{I}\big(\mathbf{k}[X]\big) = \Big\{ \sum_{y \in J} \Big(\sum_{x \in I_y, \ \sum \alpha_x = 0} \alpha_x \big(e_x + e_{x * x_0}\big)\Big) + \Big(\sum_{x' \in I_{y_0}, \ \sum \alpha_{x'} = 1} \alpha_{x'} e_{x'} \Big) \ \Big| \ J \in \mathcal{F}(Y), \ \ I_y \in \mathcal{F}(p^{-1}(y)), \ \ I_{y_0} \in \mathcal{F}(p^{-1}(y_0)), \ \ x_0 \in I_{y_0}, \ \ y_0 \in Y, \ \ \alpha_x, \alpha_{x'} \in \mathbf{k} \Big\} \tag{3.3.1}$$

Proof. Since p is a quandle covering, we have $S_x = S_{x'}$ for any $x, x' \in p^{-1}(y)$. Hence the induced automorphisms of the quandle ring $\mathbf{k}[X]$ are identical for any $x, x' \in p^{-1}(y)$. This together with direct computations give

$$\begin{aligned}
&\Big(\sum_{x\in J}\beta_x e_x\Big)\Big(\sum_{x'\in I_y,\ \sum\alpha_{x'}=1}\alpha_{x'}e_{x'}\Big) \qquad (3.3.2)\\
= &\sum_{x'\in I_y,\ \sum\alpha_{x'}=1}\alpha_{x'}\Big(\sum_{x\in J}\beta_x e_x\Big)e_{x'}\\
= &\sum_{x'\in I_y,\ \sum\alpha_{x'}=1}\alpha_{x'}\Big(\sum_{x\in J}\beta_x e_x\Big)e_{x_0}, \quad \text{for any fixed } x_0\in I_y\\
= &\sum_{x'\in I_y,\ \sum\alpha_{x'}=1}\alpha_{x'}\Big(\sum_{x\in J}\beta_x e_{x*x_0}\Big)\\
= &\sum_{x\in J}\beta_x e_{x*x_0}, \quad \text{since} \sum_{x'\in I_y,\ \sum\alpha_{x'}=1}\alpha_{x'}=1,
\end{aligned}$$

and

$$\begin{aligned}
&\Big(\sum_{x\in J}\beta_x e_x\Big)\Big(\sum_{x'\in I_y,\ \sum\alpha_{x'}=0}\alpha_{x'}e_{x'}\Big) \qquad (3.3.3)\\
= &\sum_{x'\in I_y,\ \sum\alpha_{x'}=0}\alpha_{x'}\Big(\sum_{x\in J}\beta_x e_x\Big)e_{x'}\\
= &\sum_{x'\in I_y,\ \sum\alpha_{x'}=0}\alpha_{x'}\Big(\sum_{x\in J}\beta_x e_{x*x_0}\Big), \quad \text{for any fixed } x_0\in I_y\\
= &\Big(\sum_{x'\in I_y,\ \sum\alpha_{x'}=0}\alpha_{x'}\Big)\Big(\sum_{x\in K}\beta_x e_{x*x_0}\Big)\\
= &\ 0, \quad \text{since} \sum_{x'\in I_y,\ \sum\alpha_{x'}=0}\alpha_{x'}=0,
\end{aligned}$$

where $J\in\mathcal{F}(X)$, $I_y\in\mathcal{F}(p^{-1}(y))$, $y\in Y$ and $\beta_x,\alpha_{x'}\in\mathbf{k}$. Let $u=v+w$, where

$$\begin{aligned}
v &= \sum_{y\in J}\Big(\sum_{x\in I_y,\ \sum\alpha_x=0}\alpha_x\big(e_x+e_{x*x_0}\big)\Big),\\
w &= \sum_{x'\in I_{y_0},\ \sum\alpha_{x'}=1}\alpha_{x'}e_{x'},
\end{aligned}$$

$J \in \mathcal{F}(Y)$ and $x_0 \in I_{y_0}$ a fixed element. Equations (3.3.2) and (3.3.3) imply that $w^2 = w$, $wv = 0$ and $v^2 = 0$. Since X is involutory, it follows that $(e_x + e_{x*x_0})e_{x_0} = e_x + e_{x*x_0}$. Consequently, $vw = v$, and hence $u^2 = u$.

For the converse, let u be an idempotent of $\mathbf{k}[X]$. Since X is the disjoint union of fibres of p, we can write u uniquely in the form

$$u = \sum_{y \in J} \Big(\sum_{x \in I_y} \alpha_x e_x \Big)$$

for some $J \in \mathcal{F}(Y)$ and $I_y \in \mathcal{F}(p^{-1}(y))$ for each $y \in J$. If $\hat{p} : \mathbf{k}[X] \to \mathbf{k}[Y]$ is the induced homomorphism of rings, then $\hat{p}(u)$ is an idempotent of $\mathbf{k}[Y]$. It follows from the decomposition of u that

$$\hat{p}(u) = \sum_{y \in J} \Big(\sum_{x \in I_y} \alpha_x \Big) e_y.$$

Since $\mathbf{k}[Y]$ has only trivial idempotents, it follows that either $\hat{p}(u) = 0$ or precisely one of the coefficients of $\hat{p}(u)$ is 1 and all other coefficients are 0. If $\hat{p}(u) = 0$, then $\sum_{x \in I_y} \alpha_x = 0$ for each $y \in J$. Writing

$$u = \sum_{y \in J} \Big(\sum_{x \in I_y,\ \sum \alpha_x = 0} \alpha_x e_x \Big),$$

it follows from (3.3.3) that $u = u^2 = 0$, which is a contradiction as $u \neq 0$. Hence, there exists $y_0 \in J$ such that $\sum_{x' \in I_{y_0}} \alpha_{x'} = 1$ and $\sum_{x \in I_y} \alpha_x = 0$ for all $y \neq y_0$. Then can write $u = v + w$, where

$$v = \sum_{y \in J,\ y \neq y_0} \Big(\sum_{x \in I_y,\ \sum \alpha_x = 0} \alpha_x e_x \Big)$$

and

$$w = \sum_{x' \in I_{y_0},\ \sum \alpha_{x'} = 1} \alpha_{x'} e_{x'}.$$

Again, equations (3.3.2) and (3.3.3) imply that $w^2 = w$, $wv = 0$ and $v^2 = 0$. Thus, we have

$$u = u^2 = v^2 + w^2 + vw + wv = w + vw,$$

and consequently $vw = v$. This implies that

$$\sum_{y\in J,\ y\neq y_0}\Big(\sum_{x\in I_y,\ \sum\alpha_x=0}\alpha_x e_x\Big)e_{x_0} = \sum_{y\in J,\ y\neq y_0}\Big(\sum_{x\in I_y,\ \sum\alpha_x=0}\alpha_x e_x\Big)$$

for some fixed $x_0 \in I_{y_0}$. Comparing coefficients of e_x on both the sides give $\alpha_x = \alpha_{x*x_0}$. Thus, v has the form

$$v = \sum_{y\in J,\ y\neq y_0}\Big(\sum_{x\in I_y,\ \sum\alpha_x=0}\alpha_x\big(e_x + e_{x*x_0}\big)\Big),$$

which completes the proof of the theorem.

Corollary 3.3.3. *If X is a trivial quandle, then*

$$\mathcal{I}\big(\mathbf{k}[X]\big) = \Big\{\sum_{x\in J}\alpha_x e_x \ \mid\ J \in \mathcal{F}(X),\ \alpha_x \in \mathbf{k} \text{ such that } \sum_{x\in J}\alpha_x = 1\Big\}.$$

Proof. If $\{z\}$ is a one element quandle, then the constant map $c : X \to \{z\}$ is a quandle covering. The proof now follows from Theorem 3.3.2.

Corollary 3.3.4. *Let $p : X \to Y$ be a non-trivial quandle covering such that $\mathbf{k}[Y]$ has only trivial idempotents. Then every idempotent of $\mathbf{k}[X]$ has augmentation value 1.*

Proof. The assertion follows from the proof of the converse part of Theorem 3.3.2. Note that we do not need our quandles to be involutory.

It has been shown in [4] that the integral quandle ring of R_3 has only trivial idempotents. A computational check shows that the same assertion holds for the integral quandle ring of R_5 as well. As an application of the preceding theorem, we characterise idempotents in quandle rings of certain dihedral quandles of even order under the assumption of Conjecture 3.2.14.

Corollary 3.3.5. *Let $n = 2m+1$ be an odd integer with $m \geq 1$. Assume that $\mathbf{k}[R_n]$ has only trivial idempotents. Then the set of idempotents of $\mathbf{k}[R_{2n}]$ is given by*

$$\mathcal{I}(\mathbf{k}[R_{2n}]) = \Big\{ (\beta e_j + (1-\beta) e_{n+j}) + \sum_{i=0}^{m} \alpha_i (e_i - e_{n+i} + e_{2j-i} - e_{n+2j-i}) \mid 0 \leq j \leq n-1 \text{ and } \alpha_i, \beta \in \mathbf{k} \Big\}.$$

Proof. Note that the natural map $p : R_{2n} \to R_n$ given by reduction modulo n is a non-trivial quandle covering. Further, for each $i \in R_n$, we have $p^{-1}(i) = \{i, n+i\}$. The result now follows from Theorem 3.3.2.

Proposition 3.3.8. Let $p : X \to Y$ be a non-trivial quandle covering. Then $\mathbf{k}[X]$ has right zero-divisors.

Proof. Let $J \in \mathcal{F}(X)$, $y \in Y$ and $I_y \in \mathcal{F}(p^{-1}(y))$ such that $|I_y| \geq 2$. Then for any $\sum_{x \in I_y, \ \sum \alpha_x = 0} \alpha_x e_x$ and $\sum_{x' \in J} \beta_{x'} e_{x'}$, it follows from (3.3.3) that

$$\Big(\sum_{x' \in J} \beta_{x'} e_{x'} \Big) \Big(\sum_{x \in I_y, \ \sum \alpha_x = 0} \alpha_x e_x \Big) = 0$$

and hence $\sum_{x \in I_y, \ \sum \alpha_x = 0} \alpha_x e_x$ is a right zero-divisor of $k[X]$.

Proposition 3.3.9. Let X be an involutory quandle such that $\mathbf{k}[X]$ has only trivial idempotents. Let A be a non-trivial abelian group and $\alpha : X \times X \to A$ a quandle 2-cocycle satisfying $\alpha_{x*y,y} = \alpha_{x,y}^{-1}$ for all $x, y \in X$. Then the extension $X \times_\alpha A$ is involutory and $\mathbf{k}[X \times_\alpha A]$ has non-trivial idempotents.

Proof. A direct check shows that the condition $\alpha_{x*y,y} = \alpha_{x,y}^{-1}$ is equivalent to $X \times_\alpha A$ being involutory. Since the map $p : X \times_\alpha A \to X$ is a non-trivial quandle covering, the result follows from Theorem 3.3.2. In fact, Theorem 3.3.2 gives the precise set of idempotents.

Proposition 3.3.10. Let $p : X \to Y$ be a quandle covering. Then the set

$$\begin{aligned} I \ = \ & \Big\{ \sum_{y \in J} \Big(\sum_{x \in I_y, \ \sum \alpha_x = 0} \alpha_x (e_x + e_{x*x_0}) \Big) + \Big(\sum_{x' \in I_{y_0}, \ \sum \alpha_{x'} = 1} \alpha_{x'} e_{x'} \Big) \mid \\ & J \in \mathcal{F}(Y), \ \ I_y \in \mathcal{F}(p^{-1}(y)), \ \ I_{y_0} \in \mathcal{F}(p^{-1}(y_0)), \ \ x_0 \in I_{y_0}, \ \ y_0 \in Y, \ \ \alpha_x, \alpha_{x'} \in \mathbf{k} \Big\} \end{aligned}$$

of idempotents is a quandle with respect to the ring multiplication.

Proof. Consider the elements

$$
\begin{aligned}
u &= \sum_{y\in J_1}\Big(\sum_{x\in I_y,\ \sum\alpha_x=0}\alpha_x\big(e_x+e_{x*x_1}\big)\Big)+\Big(\sum_{x'\in I_{y_1},\ \sum\alpha_{x'}=1}\alpha_{x'}e_{x'}\Big),\\
v &= \sum_{y\in J_2}\Big(\sum_{x\in I_y,\ \sum\beta_x=0}\beta_x\big(e_x+e_{x*x_2}\big)\Big)+\Big(\sum_{x'\in I_{y_2},\ \sum\beta_{x'}=1}\beta_{x'}e_{x'}\Big)
\end{aligned}
$$

in the set I, where $J_i\in\mathcal{F}(Y)$ and $I_y\in\mathcal{F}(p^{-1}(y))$, $y_i\in Y$ and $x_i\in I_{y_i}$. Then we have

$$
\begin{aligned}
& uv\\
&= \Big(\sum_{y\in J_1}\Big(\sum_{x\in I_y,\ \sum\alpha_x=0}\alpha_x\big(e_x+e_{x*x_1}\big)\Big)+\Big(\sum_{x'\in I_{y_1},\ \sum\alpha_{x'}=1}\alpha_{x'}e_{x'}\Big)\Big)\\
&\quad \Big(\sum_{y\in J_2}\Big(\sum_{x\in I_y,\ \sum\beta_x=0}\beta_x\big(e_x+e_{x*x_2}\big)\Big)+\Big(\sum_{x'\in I_{y_2},\ \sum\beta_{x'}=1}\beta_{x'}e_{x'}\Big)\Big)\\
&= \Big(\sum_{y\in J_1}\Big(\sum_{x\in I_y,\ \sum\alpha_x=0}\alpha_x\big(e_x+e_{x*x_1}\big)\Big)+\Big(\sum_{x'\in I_{y_1},\ \sum\alpha_{x'}=1}\alpha_{x'}e_{x'}\Big)\Big)\Big(\sum_{x'\in I_{y_2},\ \sum\beta_{x'}=1}\beta_{x'}e_{x'}\Big)\Big),\\
&\quad \text{by (3.3.3)}\\
&= \Big(\sum_{y\in J_1}\Big(\sum_{x\in I_y,\ \sum\alpha_x=0}\alpha_x\big(e_x+e_{x*x_1}\big)\Big)+\Big(\sum_{x'\in I_{y_1},\ \sum\alpha_{x'}=1}\alpha_{x'}e_{x'}\Big)\Big)e_{x_2}, \quad \text{by (3.3.2)}\\
&= \sum_{y\in J_1}\Big(\sum_{x\in I_y,\ \sum\alpha_x=0}\alpha_x\big(e_{x*x_2}+e_{(x*x_1)*x_2}\big)\Big)+\Big(\sum_{x'\in I_{y_1},\ \sum\alpha_{x'}=1}\alpha_{x'}e_{x'*x_2}\Big)\\
&= \sum_{y\in J_1}\Big(\sum_{x*x_2\in I_{y*y_2},\ \sum\alpha_x=0}\alpha_x\big(e_{x*x_2}+e_{(x*x_2)*(x_1*x_2)}\big)\Big)+\Big(\sum_{x'*x_2\in I_{y_1*y_2},\ \sum\alpha_{x'}=1}\alpha_{x'}e_{x'*x_2}\Big),\\
&\quad \text{where } x_1*x_2\in I_{y_1*y_2}.
\end{aligned}
$$

Thus, we have proved that $uv\in I$. The preceding computation also shows that the right multiplication by v is precisely the right multiplication by e_{x_2} for any fixed $x_2\in I_{y_2}$. In other words, the right multiplication by v is the ring automorphism $\hat{S}_{x_2}$ of $\mathbf{k}[X]$. This proves that the set I is a quandle.

As an immediate consequence of Proposition 3.3.10, we have the following.

Corollary 3.3.6. *Let $p : X \to Y$ be a quandle covering such that X is involutory and $\mathbf{k}[Y]$ has only trivial idempotents. Then the following hold:*

1. *The set of all idempotents of $\mathbf{k}[X]$ is a quandle with respect to the ring multiplication.*

2. *The right multiplication by each idempotent of $\mathbf{k}[X]$ is a ring automorphism induced by some trivial idempotent of $\mathbf{k}[X]$.*

Note that Proposition 3.2.18 already proves the endomorphism assertion of Corollary 3.3.6(2) for all medial quandles.

3.4 Idempotents in Quandle Rings of Free Products

Definition 3.4.1. Let $X_i = \langle S_i \mid R_i \rangle$ be a collection of $n \geq 2$ quandles given in terms of presentations. Then their free product $X_1 \star X_2 \star \cdots \star X_n$ is the quandle defined by the presentation

$$X_1 \star X_2 \star \cdots \star X_n = \langle S_1 \sqcup S_2 \sqcup \cdots \sqcup S_n \mid R_1 \sqcup R_2 \sqcup \cdots \sqcup R_n \rangle.$$

Example 3.4.2. The free quandle FQ_n of rank n can be seen as

$$FQ_n = \langle x_1 \rangle \star \langle x_2 \rangle \star \cdots \star \langle x_n \rangle,$$

the free product of n copies of trivial one element quandles $\langle x_i \rangle$.

It follows from the right distributivity axiom in a quandle X that

$$x *^{\epsilon} (y *^{\mu} z) = \big((x *^{-\mu} z) *^{\epsilon} y \big) *^{\mu} z \tag{3.4.1}$$

for all $x, y, z \in X$ and $\epsilon, \mu \in \{-1, 1\}$. For ease of notation, we write a left-associated product

$$((\cdots ((x_0 *^{\epsilon_1} x_1) *^{\epsilon_2} x_2) *^{\epsilon_3} \cdots) *^{\epsilon_{n-1}} x_{n-1}) *^{\epsilon_n} x_n$$

simply as

$$x_0 *^{\epsilon_1} x_1 *^{\epsilon_2} \cdots *^{\epsilon_n} x_n.$$

Remark 3.4.3. A repeated use of equation (3.4.1) gives the following result [34, Lemma 4.4.8].

Lemma 3.4.1. *Let X be a quandle. Then the product*

$$(x_0 *^{\epsilon_1} x_1 *^{\epsilon_2} \cdots *^{\epsilon_m} x_m) *^{\mu_0} (y_0 *^{\mu_1} y_1 *^{\mu_2} \cdots *^{\mu_n} y_n)$$

of two left-associated expressions in X is the left-associated expression

$$x_0 *^{\epsilon_1} x_1 *^{\epsilon_2} \cdots *^{\epsilon_m} x_m *^{-\mu_n} y_n *^{-\mu_{n-1}} y_{n-1} *^{-\mu_{n-2}} \cdots *^{-\mu_1} y_1 *^{\mu_0} y_0 *^{\mu_1} y_1 *^{\mu_2} \cdots *^{\mu_n} y_n.$$

The quandle axioms imply that each element of a quandle X has a canonical left-associated expression $x_0 *^{\epsilon_1} x_1 *^{\epsilon_2} \cdots *^{\epsilon_n} x_n$, where $x_0 \neq x_1$, and if $x_i = x_{i+1}$ for any $1 \leq i \leq n-1$, then $\epsilon_i = \epsilon_{i+1}$.

Lack of associativity in quandles makes it hard to have a normal form for elements in free products of quandles. We overcome this difficulty by defining a length for elements in free products. Let $X = X_1 \star X_2 \star \cdots \star X_n$ be the free product of $n \geq 2$ quandles. Given an element $w \in X$, we define the length $\ell(w)$ of w as

$$\begin{aligned} \ell(w) \quad = \quad \min\Big\{ r \mid & \ w \text{ can be written as a canonical left associated product of } r \\ & \text{elements from } X_1 \sqcup X_2 \sqcup \cdots \sqcup X_n \Big\}. \end{aligned}$$

Notice that each $w \in X$ has a reduced left associated expression attaining the length $\ell(w)$. This can be done by gathering together all the leftmost alphabets in a left associated expression of w that lie in the same component quandle X_i, and rename it as a single element of X_i. This shows that $\ell(w) = 1$ if and only if $w \in X_i$ for some i. Equivalently, $\ell(w) \geq 2$

if and only if $w \in X \setminus (\sqcup_{s=1}^{n} X_s)$. For example, if $x_1, x_2 \in X_i$ and $y_1, y_2 \in X_j$ for $i \neq j$, then $\ell(x_1 * x_2) = 1$, $\ell(x_1 * x_2 *^{-1} x_1) = 1$, $\ell(x_1 * y_1) = 2$, $\ell(x_1 * y_1 * y_2) = 3$ and $\ell(x_1 * y_1 * y_2 * x_2) = 4$.

Note that, if $X = X_1 \star X_2 \star \cdots \star X_n$, then every $u \in \mathbf{k}[X]$ can be written uniquely in the form

$$u = u_1 + u_2 + \cdots + u_n + v, \tag{3.4.2}$$

where each $u_i \in \mathbf{k}[X_i]$, $v = \sum_{k=1}^{m} \gamma_k e_{w_k}$ with each $\ell(w_k) \geq 2$ and $\gamma_k \in \mathbf{k}$.

Proposition 3.4.4. Let $X = X_1 \star X_2 \star \cdots \star X_n$ be the free product of n quandles such that each $\mathbf{k}[X_i]$ has only trivial idempotents. Then any idempotent u of $\mathbf{k}[X]$ can be written uniquely as

$$u = \alpha_1 e_{x_1} + \alpha_2 e_{x_2} + \cdots + \alpha_n e_{x_n} + v,$$

where $x_i \in X_i$, $v = \sum_{k=1}^{m} \gamma_k e_{w_k}$ with $\ell(w_k) \geq 2$ and $\alpha_i, \gamma_k \in \mathbf{k}$ for all i and k.

Proof. For each i, fix an element $z_i \in X_i$. Then the maps $p_i : X \to X_i$ defined by setting

$$p_i(x) = \begin{cases} x & \text{if } x \in X_i, \\ z_i & \text{if } x \in X_j \text{ for } j \neq i. \end{cases}$$

The universal property of free products implies that each p_i is a quandle homomorphism. Let $u = u_1 + u_2 + \cdots + u_n + v$ be an idempotent of $\mathbf{k}[X]$, where each $u_i \in \mathbf{k}[X_i]$, $v = \sum_{k=1}^{m} \gamma_k e_{w_k}$ with $\ell(w_k) \geq 2$ and $\gamma_k \in \mathbf{k}$. Then $\hat{p}_i(u)$ is an idempotent in $\mathbf{k}[X_i]$ for each i. Since each $\mathbf{k}[X_i]$ has only trivial idempotents and

$$\hat{p}_i(u) = u_i + \sum_{j \neq i,\ j=1}^{n} \epsilon(u_j) e_{z_i} + \hat{p}_i(v),$$

it follows that $u_i = \alpha_i e_{x_i}$ for some $x_i \in X_i$ and $\alpha_i \in \mathbf{k}$. Note that if $\epsilon(u_j) \neq 0$ for any $j \neq i$, then $x_i = z_i$. Thus, $u = \alpha_1 e_{x_1} + \alpha_2 e_{x_2} + \cdots + \alpha_n e_{x_n} + v$, and we are done.

Lemma 3.4.2. *Let $X = X_1 \star X_2 \star \cdots \star X_n$ be the free product of n quandles with $n \geq 2$. Let $u \in \mathbf{k}[X]$ be an idempotent and $u = u_1 + u_2 + \cdots + u_n + v$ be its unique decomposition as in* (3.4.2)*. Suppose that*

1. *$\ell(w_k * w_l) \geq 2$ for any k and l.*
2. *$\ell(w_k * x) \geq 2$ for any $x \in \sqcup_{s=1}^{n} X_s$ and any k.*

Then u_i is an idempotent of $\mathbf{k}[X_i]$ for each i.

Proof. Since $u = u^2$, we have

$$u_1 + u_2 + \cdots + u_n + v = u_1^2 + u_2^2 + \cdots + u_n^2 + v^2 + \sum_{i \neq j,\ i,j=1}^{n} u_i u_j + \sum_{i=1}^{n} u_i v + \sum_{j=1}^{n} v u_j. \quad (3.4.3)$$

If $v = 0$, then (3.4.3) takes the form

$$u_1 + u_2 + \cdots + u_n = u_1^2 + u_2^2 + \cdots + u_n^2 + \sum_{i \neq j,\ i,j=1}^{n} u_i u_j. \quad (3.4.4)$$

For $1 \leq i \neq j \leq n$, each basis element of $\mathbf{k}[X]$ appearing in a product $u_i u_j$ corresponds to a quandle element from $X \setminus (\sqcup_{s=1}^{n} X_s)$. For each $1 \leq i \leq n$, gathering all the summands on the right hand side of (3.4.4) corresponding to elements from the quandle X_i implies that $u_i = u_i^2$, which is desired.

Now suppose that $v \neq 0$. For each $1 \leq k, l \leq m$, the condition $\ell(w_k * w_l) \geq 2$ implies that the basis element of $\mathbf{k}[X]$ corresponding to the quandle element $w_k * w_l$ does not appear as a summand for any u_j. Further, each basis element appearing in a product $u_i v$ corresponds to a quandle element of the form $x * w_k$ for some $x \in X_i$ and some $1 \leq k \leq m$. But, we have $\ell(x * w_k) \geq 2$ for such elements. Lastly, the condition $\ell(w_k * x) \geq 2$ for any $x \in \sqcup_{s=1}^{n} X_s$ also implies that the basis element of $\mathbf{k}[X]$ corresponding to the quandle element $w_k * x$ does not appear as a summand for any u_j. For each $1 \leq i \leq n$, gathering together all the summands on the right hand side of (3.4.3) corresponding to elements from the quandle X_i imply that $u_i = u_i^2$, which is desired.

Theorem 3.4.3. *Let FQ_n be the free quandle of rank $n \geq 1$. Then $\mathbb{Z}[FQ_n]$ has only trivial idempotents.*

Proof. An analogue of the Nielsen–Schreier theorem stating that every subquandle of a free quandle is free has been proved recently in [22]. Let $FQ_2 = \langle x \rangle \star \langle y \rangle$ be the free quandle of rank two. Then,

$$FQ_n \cong \langle x \rangle \star \langle x * y \rangle \star \langle x * y * y \rangle \star \cdots \star \langle x * \underbrace{y * y * \cdots * y}_{(n-1) \text{ times}} \rangle$$

and embeds as a subquandle of FQ_2 for each $n \geq 3$. Thus, it suffices to prove that $\mathbb{Z}[FQ_2]$ has only trivial idempotents.

Let $u = \alpha e_x + \beta e_y + v$ be an idempotent of $\mathbb{Z}[FQ_2]$, where $v = \sum_{k=1}^{m} \gamma_k e_{w_k}$ with $\ell(w_k) \geq 2$ and $\alpha, \beta, \gamma_k \in \mathbb{Z}$. If $v = 0$, then Lemma 3.4.2 implies that $\alpha e_x = \alpha^2 e_x$ and $\beta e_y = \beta^2 e_y$. Hence, either $u = e_x$ or $u = e_y$, and u is a trivial idempotent.

Now, suppose that $v \neq 0$. Note that the first two leftmost alphabets in the reduced left associated expression of each w_k are distinct. We claim that $\gamma_k = 1$ for each k. This will be achieved by transforming the idempotent u into a new idempotent such that conditions of Lemma 3.4.2 are satisfied. Fix a k such that $1 \leq k \leq m$ and write

$$w_k = x_0 *^{\epsilon_1} x_1 *^{\epsilon_2} x_2 *^{\epsilon_3} \cdots *^{\epsilon_r} x_r,$$

in its reduced left associated expression, where $x_i \in \{x, y\}$ and $\epsilon_i \in \mathbb{Z}$ for each i. Since the expression is reduced, without loss of generality, we can assume that $x_0 = x$ and $x_1 = y$. Consider the inner automorphism

$$\phi = S_{x_0} S_{x_0} S_{x_1}^{-\epsilon_1} S_{x_2}^{-\epsilon_2} \cdots S_{x_{r-1}}^{-\epsilon_{r-1}} S_{x_r}^{-\epsilon_r}$$

of FQ_2. We analyse the effect of ϕ on each summand of u. First note that $\phi(w_k) = x_0 = x$. Consider any fixed w_i for $i \neq k$ and write $w_i = y_0 *^{\mu_1} y_1 *^{\mu_2} y_2 *^{\mu_3} \cdots *^{\mu_s} y_s$ in its reduced left associated expression, where $y_t \in \{x, y\}$ and $\mu_t \in \mathbb{Z}$ for each t. We have

$$\phi(w_i) = y_0 *^{\mu_1} y_1 *^{\mu_2} y_2 *^{\mu_3} \cdots *^{\mu_s} y_s *^{-\epsilon_r} x_r *^{-\epsilon_{r-1}} x_{r-1} *^{-\epsilon_{r-2}} \cdots *^{-\epsilon_1} x_1 * x_0 * x_0.$$

Considering the cases $s = r$, $s > r$ and $s < r$, and using the fact that the set of alphabets is $\{x, y\}$, we obtain $\ell(\phi(w_i)) \geq 3$. This clearly implies that $\phi(w_i) * x, \phi(w_i) * y \notin \{x, y\}$ for any $i \neq k$. Now consider another w_j for $j \neq k$ and $j \neq i$ and write $w_j = z_0 *^{\nu_1} z_1 *^{\nu_2} z_2 *^{\nu_3} \cdots *^{\nu_l} z_l$ in its reduced left associated expression, where $z_t \in \{x, y\}$ and $\nu_t \in \mathbb{Z}$ for each t. Then Lemma 3.4.1 gives

$$\begin{aligned}
& \phi(w_i) * \phi(w_j) \\
= \; & \phi(w_i * w_j) \\
= \; & \big((y_0 *^{\mu_1} y_1 *^{\mu_2} \cdots *^{\mu_s} y_s)(z_0 *^{\nu_1} z_1 *^{\nu_2} \cdots *^{\nu_l} z_l)\big) \\
& *^{-\epsilon_r} x_r *^{-\epsilon_{r-1}} x_{r-1} *^{-\epsilon_{r-2}} \cdots *^{-\epsilon_1} x_1 * x_0 * x_0 \\
= \; & y_0 *^{\mu_1} y_1 *^{\mu_2} \cdots *^{\mu_s} y_s *^{-\nu_l} z_l *^{-\nu_{l-1}} z_{l-1} *^{-\nu_{l-2}} \cdots *^{-\nu_1} z_1 * z_0 *^{\nu_1} z_1 *^{\nu_2} \cdots *^{\nu_l} z_l \\
& *^{-\epsilon_r} x_r *^{-\epsilon_{r-1}} x_{r-1} *^{-\epsilon_{r-2}} \cdots *^{-\epsilon_1} x_1 * x_0 * x_0.
\end{aligned}$$

As before, by comparing $\ell(w_i * w_j)$ and r, we obtain $\ell\big(\phi(w_i) * \phi(w_j)\big) \geq 3$. If α and β are non-zero, then $\ell(\phi(x)), \ell(\phi(y)) \geq 3$ for the same reason. Thus, the only summand of the idempotent $\phi(u) = \alpha e_{\phi(x)} + \beta e_{\phi(y)} + \sum_{k=1}^{m} \gamma_k e_{\phi(w_k)}$ that corresponds to an element from $\{x, y\}$ is $\phi(w_k)$, and all the summands corresponding to $\phi(w_i)$ for $i \neq k$ satisfy the conditions of Lemma 3.4.2. Thus, we obtain $\gamma_k e_{\phi(w_k)} = (\gamma_k e_{\phi(w_k)})^2$, and hence $\gamma_k = 1$, which proves the claim. On plugging this information back to u, we can write $u = \alpha e_x + \beta e_y + \sum_{k=1}^{m} e_{w_k}$.

Since u is an idempotent, we have

$$\alpha e_x + \beta e_y + \sum_{k=1}^{m} e_{w_k} = \alpha^2 e_x + \beta^2 e_y + \sum_{k,\ l=1}^{m} e_{w_k * w_l} + \alpha\beta e_{x*y} + \alpha\beta e_{y*x}$$
$$+\alpha \sum_{k=1}^{m} e_{x*w_k} + \alpha \sum_{k=1}^{m} e_{w_k*x} + \beta \sum_{k=1}^{m} e_{y*w_k} + \beta \sum_{k=1}^{m} e_{w_k*y}.$$

Comparing coefficients of e_x gives

$$\alpha = \alpha^2, \quad \alpha = \alpha^2 + \sum_{w_k*w_l=x} 1, \quad \alpha = \alpha^2 + \beta \quad \text{or} \quad \alpha = \alpha^2 + \sum_{w_k*w_l=x} 1 + \beta.$$

Similarly, comparing coefficients of e_y gives

$$\beta = \beta^2, \quad \beta = \beta^2 + \sum_{w_k*w_l=y} 1, \quad \beta = \beta^2 + \alpha \quad \text{or} \quad \beta = \beta^2 + \sum_{w_k*w_l=y} 1 + \alpha.$$

A direct check shows that the only possible cases are

$$\alpha = \alpha^2 \quad \text{and} \quad \beta = \beta^2,$$
$$\alpha = \alpha^2 + \beta \quad \text{and} \quad \beta = \beta^2,$$
$$\alpha = \alpha^2 \quad \text{and} \quad \beta = \beta^2 + \alpha,$$
$$\alpha = \alpha^2 + \beta \quad \text{and} \quad \beta = \beta^2 + \alpha.$$

This together with the fact that $\varepsilon(u) = \alpha + \beta + m$ shows that $\alpha = \beta = 0$ and $m = 1$. Hence, $u = e_w$ for some $w \in FQ_2$, and the proof is complete.

Since the link quandle of a trivial link with n components is the free quandle of rank n, we have

Corollary 3.4.4. *If L is a trivial link, then $\mathbb{Z}[Q(L)]$ has only trivial idempotents.*

We denote by $_{\text{ring}}(\mathbf{k}[X])$ the group of $\mathbf{k}$-algebra automorphisms of $\mathbf{k}[X]$, that is, ring automorphisms of $\mathbf{k}[X]$ that are $\mathbf{k}$-linear. Let WB_n be the welded braid group on n-strands.

See [9] for a nice survey of these groups. As an application to automorphisms of quandle rings, we have

Corollary 3.4.5. *$_{ring}\big(\mathbb{Z}[FQ_n]\big) \cong_{quandle} (FQ_n) \cong WB_n$ for each $n \geq 1$.*

Proof. Obviously, each automorphism of FQ_n induces an automorphism of $\mathbb{Z}[FQ_n]$. Conversely, if $\phi \in_{\text{ring}} \big(\mathbb{Z}[FQ_n]\big)$, then ϕ is a bijection of the set $\mathcal{I}\big(\mathbb{Z}[FQ_n]\big)$ of all idempotents. Since $\mathbb{Z}[FQ_n]$ has only trivial idempotents, $FQ_n \cong \mathcal{I}\big(\mathbb{Z}[FQ_n]\big)$ via the map $x \mapsto e_x$, and hence ϕ can be viewed as an automorphism of FQ_n, proving the first isomorphism. The second isomorphism is a well-known result from [21].

3.5 Idempotents in Quandle Rings of Unions

Definition 3.5.1. Let $\{(X_i, *_i)\}_i$ be a family of quandles. Then the binary operation

$$x * y = \begin{cases} x *_i y & \text{if } x, y \in X_i, \\ x & \text{if } x \in X_i \text{ and } y \in X_j \text{ for } i \neq j, \end{cases}$$

turns the disjoint union $\sqcup_i X_i$ into a quandle called the *union quandle*.

Proposition 3.5.2. Let $X = X_1 \sqcup X_2 \sqcup \cdots \sqcup X_n$ be the disjoint union of $n \geq 2$ quandles. Then $\mathbf{k}[X]$ contains idempotents of the following form:

1. $\sum_{j=1}^n \alpha_j u_j$, where $u_j \in \mathbf{k}[X_j]$ is an idempotent with $\varepsilon(u_j) = 1$ for each j and $\sum_{i=1}^n \alpha_i = 1$.
2. $\sum_{j=1}^n u_j$, where $u_i \in \mathbf{k}[X_i]$ is an idempotent with $\varepsilon(u_i) = 1$ and $u_j \in \mathbf{k}[X_j]$ satisfy $u_j^2 = 0$ for each $j \neq i$.
3. $\sum_{j=1}^n \alpha_j (\sum_{x \in X_j} e_x)$, where $|X_j| < \infty$ and $\sum_{i=1}^n \alpha_i |X_i| = 1$.

Proof. We begin by noting that if $u \in \mathbf{k}[X_i]$ and $v \in \mathbf{k}[X_k]$ for $i \neq k$, then $uv = \varepsilon(v)u$. For assertion (1), take $w = \sum_{j=1}^n \alpha_j u_j$, where u_j is an idempotent of $\mathbf{k}[X_j]$ and $\sum_{i=1}^n \alpha_i = 1$. Then we have

$$w^2 = \sum_{i,j=1}^{n} \alpha_i\alpha_j u_i u_j = \sum_{i,j=1}^{n} \alpha_i\alpha_j \varepsilon(u_j) u_i = \sum_{i,j=1}^{n} \alpha_i\alpha_j u_i = \sum_{j=1}^{n} \alpha_j \Big(\sum_{i=1}^{n} \alpha_i u_i \Big) = \sum_{j=1}^{n} \alpha_j w = w.$$

For assertion (2), take $w = \sum_{j=1}^{n} u_j$, where u_i is an idempotent in $\mathbf{k}[X_i]$ and $u_j \in \mathbf{k}[X_j]$ satisfy $u_j^2 = 0$ for each $j \neq i$. Since $\varepsilon(u_j) = 0$ for all $j \neq i$ and $\varepsilon(u_i) = 1$, it follows that

$$w^2 = \sum_{k \neq i, \quad k,j=1}^{n} u_j u_k + \sum_{j=1}^{n} u_j u_i = \sum_{k \neq i, \quad k,j=1}^{n} \varepsilon(u_k) u_j + \sum_{j=1}^{n} \varepsilon(u_i) u_j = w.$$

For assertion (3), suppose that $|X_j| < \infty$ for each j and take $w = \sum_{j=1}^{n} \alpha_j v_j$, where $v_j = \sum_{x \in X_j} e_x$ and $\sum_{i=1}^{n} \alpha_i |X_i| = 1$. Then we see that

$$w^2 = \sum_{i,j=1}^{n} \alpha_i\alpha_j v_i v_j = \sum_{i,j=1}^{n} \alpha_i\alpha_j |X_j| v_i = \sum_{i=1}^{n} \Big(\sum_{j=1}^{n} \alpha_j |X_j| \Big)(\alpha_i v_i) = \sum_{i=1}^{n} \alpha_i v_i = w.$$

Remark 3.5.3. Note that Proposition 3.5.2 holds for arbitrary families of quandles. Further, it appears that the proposition gives all idempotents of the quandle ring of a union of quandles.

The union construction for two quandles has a twisted version when the quandles act on each other by automorphisms (see [2, Proposition 11]). We consider a simple case of this construction when both the quandles are trivial. Note that the automorphism group of a trivial quandle is the permutation group of the underlying set. Let X, Y be trivial quandles, $f \in (X)$ and $g \in (Y)$. For $x \in X$ and $y \in Y$, setting $x * y = f(x)$ and $y * x = g(y)$ defines a quandle structure on the disjoint union $X \sqcup Y$, and we denote this quandle by $X \sqcup_{f,g} Y$. We prove a twisted version of Proposition 3.5.2.

Proposition 3.5.4. Let X and Y be trivial quandles of orders n and m, respectively. Let $f \in Aut(X)$ and $g \in Aut(Y)$ be automorphisms acting transitively on X and Y, respectively.

Then

$$\mathcal{I}(\mathbf{k}[X \sqcup_{f,g} Y]) = \mathcal{I}(\mathbf{k}[X]) \sqcup \mathcal{I}(\mathbf{k}[Y]) \sqcup \Big\{ \alpha\big(\sum_{x\in X} e_x\big) + \beta\big(\sum_{y \in Y} e_y\big) \mid \alpha, \beta \in \mathbf{k} \text{ such that } \alpha n + \beta m = 1 \Big\}.$$

Proof. Note that any $u \in \mathbf{k}[X \sqcup_{f,g} Y]$ can be written uniquely as $u = v + w$, where $v = \sum_{x \in X} \alpha_x e_x \in \mathbf{k}[X]$ and $w = \sum_{y \in Y} \beta_y e_y \in \mathbf{k}[Y]$. If $u = u^2$, then

$$v + w = v^2 + w^2 + vw + wv = \varepsilon(v)v + \varepsilon(w)w + \varepsilon(w)\sum_{x \in X} \alpha_x e_{f(x)} + \varepsilon(v)\sum_{y \in Y} \beta_y e_{g(y)},$$

and consequently

$$v = \varepsilon(v)v + \varepsilon(w)\sum_{x \in X} \alpha_x e_{f(x)} \quad \text{and} \quad w = \varepsilon(w)w + \varepsilon(v)\sum_{y \in Y} \beta_y e_{g(y)}.$$

Comparing coefficients give

$$\alpha_x = \varepsilon(v)\alpha_x + \varepsilon(w)\alpha_{f^{-1}(x)} \tag{3.5.1}$$

and

$$\beta_y = \varepsilon(w)\beta_y + \varepsilon(v)\beta_{g^{-1}(y)} \tag{3.5.2}$$

for all $x \in X$ and $y \in Y$. Adding (3.5.1) for all $x \in X$ gives $\varepsilon(v) = \varepsilon(v)\varepsilon(u)$. Similarly, adding (3.5.2) for all $y \in Y$ gives $\varepsilon(w) = \varepsilon(w)\varepsilon(u)$. If $\varepsilon(u) = 0$, then $\varepsilon(v) = \varepsilon(w) = 0$, and hence $u = 0$, a contradiction. So, we can assume that $\varepsilon(u) = 1$, and hence at least one of $\varepsilon(v)$ or $\varepsilon(w)$ is non-zero. If $\varepsilon(v) \neq 0$, then (3.5.2) gives $\beta_y = \beta_{g^{-1}(y)}$ for all $y \in Y$. Since g acts transitively on Y, it follows that $\beta_y = \beta$ (say) for all $y \in Y$. If $\beta = 0$, then $w = 0$. In this case, $u = \sum_{x \in X} \alpha_x e_x$, where $\sum_{x \in X} \alpha_x = 1$, and hence $u \in \mathcal{I}(\mathbf{k}[X])$. If $\beta \neq 0$, then $\varepsilon(w) = m\beta \neq 0$, and (3.5.1) gives $\alpha_x = \alpha_{f^{-1}(x)}$ for all $x \in X$. Since f also acts transitively on X, it follows that $\alpha_x = \alpha$ (say) for all $x \in X$. Thus, we have

$$u = \alpha\big(\sum_{x \in X} e_x\big) + \beta\big(\sum_{y \in Y} e_y\big),$$

where $n\alpha + m\beta = 1$. Similarly, if $\varepsilon(w) \neq 0$ and $\alpha = 0$, then we get $v = 0$. In this case, $u = \sum_{y \in Y} \beta_y e_y$, where $\sum_{y \in Y} \beta_y = 1$, and hence $u \in \mathcal{I}\big(\mathbf{k}[Y]\big)$. This completes the proof.

3.6 Remarks and Some Open Questions

We conclude with some remarks and open problems motivated by the results in the preceding sections.

1. All the idempotents computed in the preceding sections have augmentation value one, and we believe that this is the case in general.

   Conjecture 3.6.1. Every non-zero idempotent of a quandle ring has augmentation value one.

2. Let $Q(L)$ be the link quandle of a link L in $\mathbb{R}^3$ and X any quandle. It is well-known that the set $\big(Q(L), X\big)$ of all quandle homomorphisms extends the classical Fox colouring invariant of links. A link invariant which determines the quandle coloring invariant is called an *enhancement* of the quandle coloring invariant. Further, an enhancement is proper if there are examples in which the enhancement distinguishes links which have the same quandle coloring invariant. For instance, the quandle cocycle invariant is a proper enhancement arising from quandle cohomology. Since each quandle homomorphism $f : Q(L) \to X$ induces a homomorphism $\hat{f} : \mathbf{k}[Q(L)] \to \mathbf{k}[X]$ of quandle rings, it turns out that ${}_{\text{ring}}\big(\mathbf{k}[Q(L)], \mathbf{k}[X]\big)$ is an enhancement of $\big(Q(L), X\big)$. It is worth exploring whether this enhancement has a cohomolgical interpretation.

3. If a quandle has a subquandle of order two, then Proposition 3.2.8 shows that its quandle ring has non-trivial idempotents. A look at the table of quandles of order upto 35 seems to suggest that every faithful and non-latin quandle has a subquandle of order two.

4. Proposition 3.2.11 shows that the quandle ring of the link quandle of the Hopf link admit non-trivial idempotents. Similarly, Corollary 3.3.6 proves that the quandle ring of the knot quandle of the long knot has non-trivial idempotents. It is interesting to determine idempotents of quandle rings associated to other knots and links.

5. Quandle rings that have only trivial idempotents, quandle rings discussed in [4] and quandle rings covered by Corollary 3.3.6 have the property that the right multiplication by each idempotent is an automorphism of the quandle ring. Proposition 3.2.18 proves that the right multiplication by an idempotent is always a ring endomorphism for medial quandles. Remark 3.2.19 shows that the right multiplication by an idempotent need not be injective over the field of rationals. Further, Remark 3.2.20 shows that idempotents fail to satisfy right-distributivity in general. In view of these observations, it would be interesting to classify quandles for which the set of all idempotents of their quandle rings over appropriate coefficients form a quandle with respect to the ring multiplication.

6. Our proof of Theorem 3.4.3 crucially uses the fact that FQ_2 is the free product of one element quandles. We believe that the result holds for arbitrary free products of quandles whose quandle rings have only trivial idempotents.

Chapter 4: Cocycle Invariants and Idempotents in Quandle Rings

In this chapter, we distinguish all of the 12965 prime oriented knots up to 13 crossings using *only* 21 *connected* quandles and three quandles made of idempotents in quandle rings. We also distinguish the 12965 knots from their mirror image using the same 24 quandles. This chapter is based on [20]

Notation: $C[i,j]$ stands for j-th connected quandle of order i. The knot 12_{a125} represents the 125-th alternating knot with 12 crossings, likewise 12_{n125} represents the 125-th non-alternating knot with 12 crossings.(see [33])

4.1 Distinguishing Knots up to 12 crossings

Recall the Definition 2.3.1 of quandle 2-cocycle : For a quandle X and ϕ, a 2-cocycle with coefficient in an abelian group A, if $\mathcal{D}(K)$ is a diagram of a knot K, then, the state sum of the knot diagram $\mathcal{D}(K)$ is given by

$$\Phi(D) = \sum_{\mathcal{C}} \prod_{\tau} \phi(x, y)^{\epsilon}$$

where the product is taken over all crossings of D and the sum is taken over all the possible colorings of D.

It is known [6] that the coloring invariant of link is weaker than the quandle 2-cocycle invariant of the link.

Below is an example of classes of knots which were not distinguished by coloring [7]. However, we were able to distinguish them by using the 2-cocycle of $C[12,3]$, $\mathcal{I}\big(\mathbb{Z}_2[C[12,3]]\big)$ and $C[13,4]$.

Example 4.1.1. Let $X = C[12, 3]$ be the third connected quandle [33] of order 12. As a set $X = \{1, 2, ..., 12\}$. Its quandle operation is given in terms of right multiplications as follows:

$S_1 = (2\ 12\ 5\ 10\ 11)(3\ 8\ 6\ 7\ 4)$ $S_2 = (1\ 11\ 7\ 4\ 12)(3\ 5\ 10\ 6\ 9)$ $S_3 = (1\ 2\ 7\ 6\ 10)(4\ 9\ 8\ 5\ 12)$

$S_4 = (1\ 11\ 6\ 8\ 5)(2\ 7\ 9\ 3\ 12)$ $S_5 = (1\ 12\ 3\ 8\ 10)(2\ 4\ 9\ 6\ 11)$ $S_6 = (1\ 5\ 3\ 4\ 2)(7\ 11\ 10\ 8\ 9)$

$S_7 = (1\ 10\ 8\ 3\ 12)(2\ 11\ 6\ 9\ 4)$ $S_8 = (1\ 12\ 5\ 7\ 11)(3\ 9\ 6\ 10\ 5)$ $S_9 = (2\ 11\ 10\ 5\ 12)(3\ 4\ 7\ 6\ 8)$

$S_{10} = (1\ 5\ 8\ 6\ 11)(2\ 12\ 3\ 9\ 7)$ $S_{11} = (1\ 10\ 6\ 7\ 2)(4\ 12\ 5\ 8\ 9)$ $S_{12} = (1\ 2\ 4\ 3\ 5)(7\ 9\ 8\ 10\ 11)$.

Using Maple software, we obtained the following 2-cocycle with coefficients in $\mathbb{Z}_2$. The map $\phi : X \times X \to \mathbb{Z}_2$ is given explicitly by

$\phi(3, 2) = 1,$ $\phi(3, 4) = 1,$ $\phi(4, 7) = 1,$ $\phi(4, 11) = 1,$ $\phi(6, 5) = 1,$ $\phi(6, 8) = 1,$

$\phi(7, 6) = 1,$ $\phi(8, 3) = 1,$ $\phi(8, 12) = 1,$ $\phi(9, 2) = 1,$ $\phi(9, 3) = 1,$ $\phi(9, 4) = 1,$

$\phi(9, 5) = 1,$ $\phi(9, 6) = 1,$ $\phi(9, 7) = 1,$ $\phi(9, 8) = 1,$ $\phi(9, 10) = 1,$ $\phi(9, 11) = 1,$

$\phi(9, 12) = 1$ and $\phi(x, y) = 0$ for all other $x, y \in Y$.

Note that this 2-cocycle is not a coboundary since the value of the quandle cocycle invariant of the knot $12n_{368}$ is given by $\Phi_{(C[12,3],\phi)}(12n_{368}) = 40 + 32u$.

The 2-cocycle invariant $\Phi_{(X,\phi)}(K)$ of the knots $K \in \{9_{13}, 9_{14}, 9_{16}, 9_{20}, 9_{23}, 9_{24}, 10_{123}, 12n_{0572}, 12n_{0576}, 12n_{0578}, 12n_{0580}\}$ has value 72.

To distinguish these knots further, we use the following quandle $Y = \mathcal{I}(\mathbb{Z}_2[X])$. As a set, we write $Y = \{1, 2, ..., 24\}$ and we give its quandle structure by listing its right multiplication given below:

$S_1 = S_{13} = (2\ 12\ 5\ 10\ 11)(3\ 8\ 6\ 7\ 4)(14\ 24\ 17\ 22\ 23)(15\ 20\ 18\ 19\ 16)$

$S_2 = S_{14} = (1\ 11\ 7\ 4\ 12)(3\ 5\ 10\ 6\ 9)(13\ 23\ 19\ 16\ 24)(15\ 17\ 22\ 18\ 21)$

$S_3 = S_{15} = (1\ 2\ 7\ 6\ 10)(4\ 9\ 8\ 5\ 12)(13\ 14\ 19\ 18\ 22)(16\ 21\ 20\ 17\ 24)$

$S_4 = S_{16} = (1\ 11\ 6\ 8\ 5)(2\ 7\ 9\ 3\ 12)(13\ 23\ 18\ 20\ 17)(14\ 19\ 21\ 15\ 16)$

$S_5 = S_{17} = (1\ 12\ 3\ 8\ 10)(2\ 4\ 9\ 6\ 11)(13\ 24\ 15\ 20\ 22)(14\ 16\ 21\ 18\ 23)$

$S_6 = S_{18} = (1\ 5\ 3\ 4\ 2)(7\ 11\ 10\ 8\ 9)(13\ 17\ 15\ 16\ 14)(19\ 23\ 22\ 20\ 18)$

$S_7 = S_{19} = (1\ 10\ 8\ 3\ 12)(2\ 11\ 6\ 9\ 4)(13\ 22\ 20\ 15\ 24)(14\ 23\ 18\ 21\ 16)$

$S_8 = S_{20} = (1\ 12\ 5\ 7\ 11)(3\ 9\ 6\ 10\ 5)(13\ 24\ 16\ 19\ 23)(15\ 21\ 18\ 22\ 17)$

$S_9 = S_{21} = (2\ 11\ 10\ 5\ 12)(3\ 4\ 7\ 6\ 8)(14\ 23\ 22\ 17\ 24)(15\ 16\ 19\ 18\ 20)$

$S_{10} = S_{22} = (1\ 5\ 8\ 6\ 11)(2\ 12\ 3\ 9\ 7)(13\ 17\ 20\ 18\ 23)(14\ 24\ 15\ 21\ 19)$

$S_{11} = S_{23} = (1\ 10\ 6\ 7\ 2)(4\ 12\ 5\ 8\ 9)(13\ 22\ 18\ 19\ 14)(16\ 24\ 17\ 20\ 21)$

$S_{12} = S_{24} = (1\ 2\ 4\ 3\ 5)(7\ 9\ 8\ 10\ 11)(13\ 14\ 16\ 15\ 17)(19\ 21\ 20\ 22\ 23)$

with 2-cocycle map $\psi : Y \times Y \to \mathbb{Z}_2$ given by

$$\begin{array}{lllll}
\psi(3,2)=1, & \psi(3,4)=1, & \psi(3,16)=1, & \psi(4,7)=1, & \psi(4,11)=1, \\
\psi(4,19)=1, & \psi(4,23)=1, & \psi(7,6)=1, & \psi(7,10)=1, & \psi(7,18)=1, \\
\psi(7,22)=1, & \psi(8,3)=1, & \psi(8,12)=1, & \psi(8,15)=1, & \psi(8,24)=1, \\
\psi(9,2)=1, & \psi(9,3)=1, & \psi(9,4)=1, & \psi(9,5)=1, & \psi(9,6)=1, \\
\psi(9,7)=1, & \psi(9,8)=1, & \psi(9,10)=1, & \psi(9,11)=1, & \psi(9,12)=1, \\
\psi(9,14)=1, & \psi(9,15)=1, & \psi(9,20)=1, & \psi(9,22)=1, & \psi(9,23)=1,
\end{array}$$

$\psi(9,24)=1,$ and $\psi(x,y)=0$ for all other $x, y \in Y$.

This further breaks down the set of knots into the following partition $\{9_{13}, 9_{14}, 9_{16}, 9_{20}, 9_{23}, 9_{24}\} \sqcup \{10_{123}, 12n_{0572}\} \sqcup \{12n_{0576}\} \sqcup \{12n_{0578}\} \sqcup \{12n_{0580}\}$ since the cocycle invariants for each partition are respectively $144, 106+38u, 58+86u, 120+24u$ and $64+80u$.

To completely distinguish all the knots, we use the following quandle $C[13,4]$. As a set we denote it by $W = \{1, 2, 3, ..., 13\}$. Its quandle operation is given in terms of right multiplications by

$S_1 = (2\ 9\ 13\ 6)(3\ 4\ 12\ 11)(5\ 7\ 10\ 8)$ $S_2 = (1\ 7\ 3\ 10)(4\ 5\ 13\ 12)(6\ 8\ 11\ 9)$

$S_3 = (1\ 13\ 5\ 6)(2\ 8\ 4\ 11)(7\ 9\ 12\ 10)$ $S_4 = (1\ 6\ 7\ 2)(3\ 9\ 5\ 12)(8\ 10\ 13\ 11)$

$S_5 = (1\ 12\ 9\ 11)(2\ 7\ 8\ 3)(4\ 10\ 6\ 13)$ $S_6 = (1\ 5\ 11\ 7)(2\ 13\ 10\ 12)(3\ 8\ 9\ 4)$

$S_7 = (1\ 11\ 13\ 3)(2\ 6\ 12\ 8)(4\ 9\ 10\ 5)$ $S_8 = (1\ 4\ 2\ 12)(3\ 7\ 13\ 9)(5\ 10\ 11\ 6)$

$S_9 = (1\ 10\ 4\ 8)(2\ 5\ 3\ 13)(6\ 11\ 12\ 7)$ $S_{10} = (1\ 3\ 6\ 4)(2\ 11\ 5\ 9)(7\ 12\ 13\ 8)$

$S_{11} = (1\ 9\ 8\ 13)(2\ 4\ 7\ 5)(3\ 12\ 6\ 10)$ $S_{12} = (1\ 2\ 10\ 9)(3\ 5\ 8\ 6)(4\ 13\ 7\ 11)$

$S_{13} = (1\ 8\ 12\ 5)(2\ 3\ 11\ 10)(4\ 6\ 9\ 7)$.

Now we consider the following 2-cocycle map $\vartheta : W \times W \to \mathbb{Z}_3$ given by

Table 4.1 Distinguishing knots using two connected quandles and its idempotents

K	$\Phi_{(X,\phi)}(K)$	$\Psi_{(Y,\psi)}(K)$	$\Theta_{(W,\vartheta)}(K)$
9_{13}	72	144	$11u^2+42u+52$
9_{14}	72	144	$20u^2+5u+14$
9_{16}	72	144	$69+14u$
9_{20}	72	144	$7u^2+13u+20$
9_{23}	72	144	13
9_{24}	72	144	$45u^2+37u+13$
10_{123}	72	$106+38u$	$3u^2+20u+17$
12_{n0572}	72	$106+38u$	$100u^2+70u+11$
12_{n0576}	72	$58+86u$	$20u^2+135u+10$
12_{n0578}	72	$120+24u$	$16u^2+104u+13$
12_{n0580}	72	$64+80u$	$78u^2+90u+11$

$\vartheta(1,12)=1,\quad \vartheta(2,10)=1,\quad \vartheta(3,8)=1,\quad \vartheta(4,6)=1,\quad \vartheta(5,4)=1,$

$\vartheta(6,2)=1,\quad \vartheta(7,13)=1,\quad \vartheta(8,11)=1,\quad \vartheta(9,1)=2,\quad \vartheta(9,2)=2,$

$\vartheta(9,3)=2,\quad \vartheta(9,4)=2,\quad \vartheta(9,5)=2,\quad \vartheta(9,6)=2,\quad \vartheta(9,7)=2,$

$\vartheta(9,8)=2,\quad \vartheta(9,10)=2,\quad \vartheta(9,11)=2,\quad \vartheta(9,12)=2,\quad \vartheta(9,13)=2,$

$\vartheta(10,7)=1,\quad \vartheta(11,5)=1,\quad \vartheta(12,3)=1,\quad \vartheta(13,1)=1,$ and

$\vartheta(x,y)=0$ for all other $x,y\in W$.

Finally, we are able to distinguish all the above knots in the following table:

In a similar manner, we present a table that distinguish some non-alternating knots of 12 crossings.

Table 4.2 Distinguishing non-alternating knots of 12 crossings using two connected quandles and its idempotents

K	$\mathcal{C}_X(K)$	$\Phi_{(X,\phi)}(K)$	$\Psi_{(Y,\psi)}(K)$
$12n_{0573}$	132	$68+64u$	$136+128u$
$12n_{0575}$	132	$48+84u$	$172+92u$
$12n_{0577}$	132	$48+84u$	$144+120u$
$12n_{0579}$	132	$76+56u$	$96+168u$
$12n_{0581}$	192	$94+98u$	$196+188u$
$12n_{0594}$	192	$94+98u$	$240+144u$
$12n_{0574}$	312	$72+240u$	$320+304u$
$12n_{0737}$	312	$72+240u$	$324+300u$

In conclusion, the triplet $(\Phi_{(X,\phi)}(K), \Psi_{(Y,\phi)}(K), \Theta_{(W,\vartheta)}(K))$ distinguishes all the above mentioned knots completely. Following this strategy we were able to distinguish all knots up to 12 crossings using 13 quandles listed in the (see AppendixA).

The following example distinguishes all the knots of 9 crossings using only 3 connected quandles and quandles made of its idempotents in quandle rings.

Example 4.1.2. Consider $X = C[12,3]$, $Y = \mathcal{I}(\mathbb{Z}_2[X])$ $Z = C[13,7]$ and $W = C[16,3]$. Let $\phi : X \times X \to \mathbb{Z}_2$, $\psi : Y \times Y \to \mathbb{Z}_2$, $\gamma : Z \times Z \to \mathbb{Z}_2$, $\vartheta : W \times W \to \mathbb{Z}_2$ be respectively the 2-cocycle maps of X, Y, Z and W. Then the quadruple $(\Phi_{(X,\phi)}(K), \Psi_{(Y,\phi)}(K), \Gamma_{(Z,\gamma)}(K), \Theta_{(W,\vartheta)}(K))$ distinguishes all the knots of 9 crossings fully given below.

Table 4.3 Distinguishing all knots of 9 crossings part 1

K	$\Phi_{(X,\phi)}(K)$	$\Psi_{(Y,\psi)}(K)$	$\Gamma_{(Z,\gamma)}(K)$	$\Theta_{(W,\vartheta)}(K)$
9_1	72	144	13	241
9_{13}	72	144	13	$82 + 86u$
9_{14}	72	144	13	$49 + 104u$
9_{16}	72	144	13	16
9_{20}	72	144	13	$90 + 51u$
9_{23}	72	144	13	$87 + 56u$
9_{24}	72	144	13	$70 + 98u$
9_5	72	$48 + 96u$	13	16
9_6	72	$90 + 54u$	13	16

Similarly, the following example distinguishes knots of 11 crossings and 12 crossings (alternating and non-alternating) using 4 connected quandles and quandles makde out of its idempotents in quandle rings.

Example 4.1.3. Consider the quandles $X = C[12,3]$, $Y = \mathcal{I}(\mathbb{Z}_2[X])$, $Z = C[13,7]$, $W = C[16,3]$ and $V = C[16,4]$ (see Appendix A for these quandles in terms of its right multiplication).

Let $\phi : X \times X \to \mathbb{Z}_3$, $\psi : Y \times Y \to \mathbb{Z}_3$, $\gamma : Z \times Z \to \mathbb{Z}_3$, $\vartheta : W \times W \to \mathbb{Z}_3$, $\zeta : V \times V \to \mathbb{Z}_3$ be respectively the 2-cocycle maps of X, Y, Z, W and V. Then the quintuple $(\Phi_{(X,\phi)}(K), \Psi_{(Y,\phi)}(K), \Gamma_{(Z,\gamma)}(K), \Theta_{(W,\vartheta)}(K),\ \xi_{(V,\zeta)}(K))$ fully distinguishes the given set of knots of 11 crossings and 12 crossings (alternating and non-alternating).

Table 4.4 Distinguishing all knots of 9 crossings part 2

K	$\Phi_{(X,\phi)}(K)$	$\Psi_{(Y,\psi)}(K)$	$\Gamma_{(Z,\gamma)}(K)$	$\Theta_{(W,\vartheta)}(K)$
9_2	12	24	$70+99u$	16
9_8	12	24	$54+106u$	16
9_9	12	24	169	16
9_{10}	12	24	13	16
9_{11}	12	24	$84+85u$	16
9_{12}	12	24	$129+40u$	16
9_{36}	12	24	$114+55u$	16
9_3	12	24	13	$87+160u$
9_4	12	24	13	$85+160u$
9_{26}	12	24	$27+38u$	$85+160u$
9_{43}	12	24	169	$85+160u$
9_7	12	24	$23+42u$	$87+156u$
9_{22}	12	24	13	$87+156u$
9_{44}	12	24	$57+8u$	$87+156u$
9_{15}	12	24	13	$32+48u$
9_{17}	12	24	13	$86+158u$
9_{18}	12	24	13	$90+150u$
9_{19}	12	24	13	$48+32u$
9_{21}	12	24	13	$104+54u$
9_{28}	12	24	13	$96+90u$
9_{31}	12	24	13	$80+112u$
9_{33}	12	24	169	$160+32u$
9_{34}	12	24	169	$54+16u$
9_{42}	12	24	169	$84+116u$
9_{45}	12	24	169	$32+48u$
9_{26}	$36+36u$	$60+84u$	$151+18u$	$104+54u$
9_{29}	$36+36u$	$48+96u$	13	$87+56u$
9_{38}	$40+32u$	$92+52u$	$65+20u$	$110+48u$
9_{39}	$40+32u$	$96+48u$	$65+20u$	$124+34u$
9_{40}	$42+30u$	$48+96u$	169	$124+34u$
9_{47}	$40+32u$	144	169	$48+30u$
9_{49}	$40+32u$	$98+46u$	13	$48+30u$
9_{31}	$40+32u$	$92+52u$	13	$64+64u$
9_{28}	$44+28u$	144	$48+96u$	$64+64u$
9_{35}	132	264	13	$87+148u$
9_{37}	132	264	$118+51u$	$85+146u$
9_{46}	132	$120+144u$	13	$85+146u$
9_{48}	132	$140+124u$	$100+69u$	$85+146u$
9_{41}	132	$120+144u$	13	$87+54u$

Table 4.5 Distinguishing 11 crossings and 12 crossings knots

K	$\Phi_{(X,\phi)}(K)$	$\Psi_{(Y,\phi)}(K)$	$\Gamma_{(Z,\gamma)}(K)$	$\Theta_{(W,\vartheta)}(K)$	$\xi_{(V,\zeta)}(K)$
11_{a127}	72	144	$14u^2+13u+38$	16	16
11_{a130}	72	144	$13u^2+42u+10$	16	$56u^2+80+70$
11_{n174}	72	144	$13u^2+42u+10$	16	$48u^2+80u+78$
11_{n175}	$24u^2+6u+42$	$48u^2+12u+84$	13	$24u^2+48u+78$	16
12_{n388}	$24u^2+6u+42$	$24u^2+48u+72$	13	$24u^2+96u+24$	16
12_{n389}	$30u^2+6u+36$	$42u^2+16u+88$	169	16	16
12_{n390}	$30u^2+6u+36$	$24u^2+48u+72$	$14u^2+13u+38$	16	16
11_{a129}	132	$60u^2+168u+36$	13	$24u^2+96u+24$	16
11_{n176}	132	264	13	$24u^2+90u+30$	16
11_{n177}	132	$72u^2+168u+24$	13	$24u^2+90u+30$	16
11_{n178}	$36u^2+84u+12$	$72u^2+168u+24$	13	$24u^2+48u+78$	16
11_{n179}	$36u^2+84u+12$	$60u^2+168u+36$	13	$24u^2+48u+78$	16
12_{a412}	$32u^2+84u+16$	264	13	16	$24u^2+48u+16$
12_{a413}	$32u^2+84u+16$	264	13	16	$24u^2+42u+22$
11_{a126}	252	504	$85u^2+69u+15$	$24u^2+96u+24$	16
11_{a132}	252	504	$13u^2+42u+10$	$24u^2+90u+30$	16
11_{n172}	252	504	169	16	16
11_{n173}	$120u^2+84u+48$	$240u^2+168u+96$	13	16	16
12_{n387}	$120u^2+84u+48$	$240u^2+168u+96$	169	16	16
12_{n391}	$120u^2+84u+48$	$240u^2+168u+96$	$13u^2+42u+10$	$24u^2+48u+78$	16
11_{a128}	12	24	$14u^2+13u+38$	16	$24u^2+24u+30$
11_{a131}	12	24	$14u^2+13u+38$	16	$12u^2+24u+42$
12_{a418}	12	24	13	16	$56u^2+80u+70$
12_{a419}	12	24	13	16	$16u^2+24u+38$
12_{a417}	12	24	13	$72u^2+24u+48$	$24u^2+16u+48$
11_{a133}	12	24	$85u^2+69u+15$	$60u^2+60u+16$	16
12_{a414}	12	24	$85u^2+69u+15$	$48u^2+32u+16$	16
12_{a415}	12	24	169	$72u^2+24u+48$	$20u^2+24u+34$
12_{a416}	12	24	$13u^2+42u+10$	$72u^2+24u+48$	$24u^2+16u+48$

Based on the above examples, we formulate the following conjecture for knots up to 12 crossings.

Conjecture 4.1.4. Let X be a quandle and $\phi : X \times X \to A$ be a 2-cocycle. Let $Y = \mathcal{I}(\mathbb{Z}_2[X])$ be the set of idempotents in the quandle ring $\mathbb{Z}_2[X]$ such that Y is a quandle and $\psi : Y \times Y \to A$ be a 2-cocycle. Then the cocycle invariant $\Psi_{(Y,\psi)}(K)$ is an enhancement of the cocycle invariant $\Phi_{(X,\phi)}(K)$ for all prime oriented knots up to 12 crossings.

This can further be extended to give the following generalized conjecture

Conjecture 4.1.5. There exists a finite sequence of quandles $(X_1, X_2, X_3, ..., X_k)$ such that $\Psi(K) = (\Phi_{(X_1,\phi_1)}(K), ..., \Phi_{(X_k,\phi_k)}(K), \Phi_{(\mathcal{I}(\mathbb{Z}_2[X_1]),\psi_1)}(K), ..., \Phi_{(\mathcal{I}(\mathbb{Z}_2[X_k]),\psi_k)}(K))$ is an invariant. In other words,

$\Phi(K) = \Phi(K')$ if and only if $K = K'$ for all K in the list of knots up to 12 crossings.

Example 4.1.6. For a knot K let $m(K)$ denote the mirror image of K. We say that a knot K is *positive amphicheiral* if $K = m(K)$. In this example, we give two knots which are not distinguished from their mirror image by the *Jones* polynomial or by the quandle coloring, but we are able to distinguish them using the pair of 2-cocycle invariants $(\Phi_{(X,\phi)}(K), \Phi_{(Y,\psi)}(K))$ where $X = C[12,3]$ and $Y = \mathcal{I}(\mathbb{Z}_2[C[12,3]])$.

Table 4.6 Distinguishing knots from its mirror image

K	$\mathcal{C}_X(K)$	Jones Polynomial	$(\Phi_{(X,\phi)}(K), \Psi_{(Y,\psi)}(K))$
9_{42}	24	$t^{-3} - t^{-2} + t^{-1} - 1 +$	$(24, 48)$
$m(9_{42})$	24	$t - t^2 + t^3$	$(24, 32u + 16)$
$12a_{669}$	24	$-t^{-6} + 2t^{-5} - 4t^{-4} + 6t^{-3} - 7t^{-2} + 9t^{-1} - 9 +$	$(24, 48)$
$m(12a_{669})$	24	$9t - 7t^2 + 6t^3 - 4t^4 + 2t^5 - t^6$	$(24, 12u + 36)$

4.2 Distinguishing Knots of 13 crossings

We extended our computation towards 13 crossing knots for both alternating and non-alternating to support our Conjecture 4.1.5. It turns out that we needed 24 quandles to distinguish these knots up to 13 crossings. For example, consider the quandle $X = C[12,3]$ and $Y = \mathcal{I}(\mathbb{Z}_2[X])$. We compute the 2-cocycle invariants $\Phi_{(X,\phi)}(K)$ and $\Psi_{(Y,\psi)}(K)$ using these two quandles. Now, we iterate the process of taking idempotents and consider the set $W = \mathcal{I}(\mathbb{Z}_2[\mathcal{I}(\mathbb{Z}_2[X])])$. We find that this set W also forms a quandle. Additionally, the 2-cocycle invariant $\Theta_{(W,\vartheta)}(K)$ from W is stronger than the previous 2-cocycle invariant $\Phi_{(X,\phi)}(K)$ and $\Psi_{(Y,\psi)}(K)$ from X and Y respectively. The following table further supports the claim:

Below in another example of knots of 13 crossings (alternating and non-alternating) distinguished fully using 4 connected quandles and quandles made out of its idempotents in quandle rings.

Table 4.7 Distinguishing knots of 13 crossings using iteration of idempotents in quandle rings

K	$\Phi_{(X,\phi)}(K)$	$\Psi_{(Y,\psi)}(K)$	$\Theta_{(W,\theta)}(K)$
13_{120}	132	264	528
13_{482}	132	264	$280+248u$
13_{484}	$48+84u$	$144+120u$	$252+276u$
13_{485}	$48+84u$	$144+120u$	$240+288u$
13_{1596}	$48+84u$	$144+120u$	$360+168u$

Example 4.2.1. Consider $X = C[12,3]$, $Y = \mathcal{I}(\mathbb{Z}_2[X])$, $Z = C[13,7]$, $W = C[16,3]$ and $V = C[16,4]$ (see Appendix A for these quandles in terms of its right multiplication). Let$\phi : X \times X \to \mathbb{Z}_3$, $\psi : Y \times Y \to \mathbb{Z}_3$, $\gamma : Z \times Z \to \mathbb{Z}_3$, $\vartheta : W \times W \to \mathbb{Z}_3$, $\zeta : V \times V \to \mathbb{Z}_3$ be respectively the 2-cocycle maps of X, Y, Z, W and V.

Then the quintuple $(\Phi_{(X,\phi)}(K), \Psi_{(Y,\phi)}(K), \Gamma_{(Z,\gamma)}(K), \Theta_{(W,\vartheta)}(K), \ \xi_{(V,\zeta)}(K))$ fully distinguishes the given set of knots of 13 crossings (alternating and non-alternating) as given below.

Table 4.8 Distinguishing 13 crossings knots

K	$\Phi_{(X,\phi)}(K)$	$\Psi_{(Y,\phi)}(K)$	$\Gamma_{(Z,\gamma)}(K)$	$\Theta_{(W,\vartheta)}(K))$	$\xi_{(V,\zeta)}(K)$
13_{a2363}	12	24	13	$60u^2+60u+16$	16
13_{a2366}	12	24	13	$48u^2+32u+16$	16
13_{a2367}	12	24	169	$24u^2+48u+78$	16
13_{n8806}	12	24	169	$24u^2+96u+24$	16
13_{n8809}	12	24	169	$72u^2+24u+48$	16
13_{n8810}	$24u^2+6u+42$	$24u^2+48u+72$	$14u^2+13u+38$	16	16
13_{n8811}	$24u^2+6u+42$	$24u^2+48u+72$	$13u^2+42u+10$	16	$56u^2+80u+70$
13_{n8807}	$24u^2+6u+42$	$24u^2+48u+72$	$85u^2+69u+15$	16	$56u^2+80u+70$
13_{a2364}	24	$24u^2+16u+8$	13	$24u^2+48u+78$	$24u^2+24u+30$
13_{a2365}	24	$16u^2+16u+16$	13	$24u^2+48u+78$	$24u^2+24u+30$
13_{n8808}	24	$12u^2+20u+16$	13	$24u^2+48u+78$	$20u^2+24u+34$
13_{n8818}	24	$8u^2+16u+24$	13	$24u^2+48u+78$	$20u^2+24u+34$
13_{n8817}	$40u^2+50u+42$	$90u^2+96u+78$	169	$24u^2+96u+24$	16
13_{a2371}	$30u^2+60u+42$	$82u^2+90u+92$	169	$24u^2+96u+24$	16
13_{a2372}	$36u^2+56u+42$	$60u^2+92u+112$	169	$24u^2+96u+24$	16
13_{n8816}	$20u^2+72u+40$	264	$14u^2+13u+38$	16	$56u^2+80u+70$
13_{n8815}	$20u^2+72u+40$	264	$14u^2+13u+38$	16	$20u^2+24u+34$
13_{a2368}	$20u^2+72u+40$	264	$85u^2+69u+15$	16	$24u^2+24u+30$
13_{a2369}	$48u^2+72u+24$	$48u^2+80u+40$	$85u^2+69u+15$	16	$12u^2+24u+42$
13_{a2370}	$48u^2+72u+24$	$80u^2+160u+120$	$13u^2+42u+10$	16	$24u^2+16u+48$

The article [7] defines some similarity of quandles. Precisely, for any two quandles X_1, X_2 and a family of knots $\mathcal{K}$, we say $X_1 \approx X_2$ if $\mathcal{C}_{X_1}(K) = \mathcal{C}_{X_2}(K)$ for every $K \in \mathcal{K}$.

In a similar manner we introduce a more general similarity of quandles using 2-cocycle invariant of knots; i.e. for any two quandles and their respective 2-cocycles given by $(X_1, \phi), (X_2, \psi)$ and a family of knots $\mathcal{K}$, we say $X_1 \sim X_2$ if $\Phi_{(X_1,\phi)}(K) = \Psi_{(X_2,\psi)}(K)$ for every $K \in \mathcal{K}$.

Based on this, we have the following observation for all 12965 prime oriented knots up to 13 crossings and the 24 quandles in our computation:

- There are a total of 3520 classes of $\sim$ consisting more than one quandle for knots up to 13 crossings. For example:
  - $C[12,3] \sim C[12,6]$ for $K \in \mathcal{K} = \{9_2, 9_3, 9_4, 9_7, 9_9, 9_{12}, 9_{13}, 11a_{172}, 11a_{190}, 11a_{191}, 12n_{0370}, 12n_{0371}, 12n_{0373}, 12n_{376}, 13_{3108}\}$.
  - $C[13,7] \sim C[13,10]$ for $K \in \mathcal{K} = \{7_4, 7_6, 7_7, 8_7, 8_9, 8_{13}, 8_{17}, 8_{18}, 10_{46}, 10_{47}, 10_{48}, 10_{49}, 10_{50}, 10_{51}, 10_{52}, 11a_{151}, 11a_{152}, 11a_{171}, 13_{528}, 13_{3109}, 13_{9089}, 13_{9090}\}$.
- From the 3520 classes, there are 1460 classes containing more than two quandles for knots up to 13 crossings.
  - $C[12,3] \sim C[12,4] \sim C[12,6]$ for $K \in \mathcal{K} = \{9_3, 9_5, 9_7, 9_8, 9_9, 12n_{0370}, 12n_{0371}, 12n_{0373}, 12n_{376}\}$.
  - $C[16,3] \sim C[16,4] \sim \mathcal{I}(\mathbb{Z}_2[C[8,1]])$ for $K \in \mathcal{K} = \{12n_{370}, 12n_{371}, 12n_{372}, 12n_{373}, 12n_{374}, 12n_{376}, 13_{4039}, 13_{535}, 13_{544}\}$

4.3 Description of the Algorithm

In this section, we give a complete description of the algorithm. The following algorithm was inspired by http://shell.cas.usf.edu/~saito/Maple/. The algorithm has three steps

- **Step 1.** Given a quandle X, we check when the set of idempotents $\mathcal{I}(\mathbb{Z}_2[X])$ is a quandle.

- **Step 2.** Using X and $\mathcal{I}(\mathbb{Z}_2[X])$, we calculate the colorings of a given knot K using its braid representation.

- **Step 3.** After obtaining the coloring, we calculate the State Sum Invariant of the knot K.

Now we describe each of the 3 steps in details.

- **Checking the set of idempotents $\mathcal{I}(\mathbb{Z}_2[X])$ is a quandle:** We will denote by $\mathcal{X}$ the family of quandles used in our computation. For a given $X \in \mathcal{X}$, the following algorithm checks if $Y = \mathcal{I}(\mathbb{Z}_2[X])$ is a quandle.

  Given a finite set Y with binary operation we use Algorithm 4.1, Algorithm 4.2 and Algorithm 4.3 to check if Y is a quandle. Precisely, we check that the set of idempotents, with multiplication as the binary operation, is a quandle. Let n be the number of elements in Y.

  We first check the right-distributivity axiom for Y. If Y does not satisfy this axiom, we stop the algorithm and return the statement: *The Cayley table does not represent a quandle.* The algorithm for checking right-distributivity is described in Algorithm 4.1 i.e. for all $a, b, c \in Y$, $(a * b) * c = (a * c) * (b * c)$.

Algorithm 4.1 Right Distributivity

Input: Y - the Cayley table representing a candidate quandle

Input: n - the cardinality of the quandle described by Y

```
for a from 1 to n do
  for b from 1 to n do
    for c from 1 to n do
      if Y[Y[a, b], c] is not equal to Y[Y[a, c], Y[b, c]] then

        return False
      end if
    end for
  end for
end for
```

After checking the right-distributivity axiom, we check that right multiplications in Y is invertible. In other words, for any $a, b \in Y$, we verify that $a * (a * b) = a * b$. We check the invertibility of right multiplication by running two loops; one on a and one on b. Fix $a = 1$ and vary on b from 1 to n. If $a*(a*b) \neq a*b$ then return value **False**. If $a * (a * b) = b$ then set $a = 2$, and vary b from 1 to n and so on. The algorithm for invertibility of right multiplication is given below.

Algorithm 4.2 Invertibility of Right Multiplication

Input: Y - the Cayley table representing a candidate quandle

Input: n - the cardinality of the quandle described by Y

```
for a from 1 to n do
  for b from 1 to n do
    if Y[a, Y[a,b]] is not equal to Y[a,b] then
      return False
    end if
  end for
end for
```

Lastly, we use Algorithm 4.3 to check idempotency i.e. for any $a \in Y$, we have $a * a = a$. Fix $a = 1$ and check for $a * a$. If $a * a \neq a$ then return value **False** else proceed in setting $a = 2$ and so on.

Algorithm 4.3 Idempotency Property

Input: Y - the Cayley table representing a candidate quandle

Input: n - the cardinality of the quandle described by Y

```
for a from 1 to n do
  if Y[a, a] is not equal to a then
    return False
  end if
end for
```

- **Finding the colorings of a knot K:** Since every knot is the closure of a braid, to obtain a coloring of the knot, we first color the braid as follows.

  1. Let m be the braid index of the knot K.

2. Let $\vec{x} = (x_1, x_2, x_3, ..., x_i, x_{i+1}, ..., x_m) \in X^m$ be the top color of the braid.
3. At any i^{th} crossing, if it is positive, then $\vec{x}$ becomes $(x_1, ..., x_{i+1}, x_i * x_{i+1}, ..., x_m)$. if the i^{th} crossing is negative, then $\vec{x}$ becomes $(x_1, ..., x_{i+1} \bar{*} x_i, x_i, ..., x_m)$. Let $(y_1, y_2, ..., y_m)$ be the bottom vector of the braid.
4. A solution of the system of equations $y_1 = x_1, ..., \ y_m = x_m$ is a coloring of knot K by the quandle X. We abuse the notation and use $\vec{x}$ to denote this coloring.

Python uses a list to store the braid representation and braid index of knots up to 13 crossings (The data of which is collected from [25] for knots up to 12 crossings and from http://shell.cas.usf.edu/~saito/QuandleColor/12965knotsGAP.txt for all 13 crossing knots). Python also uses a list to store the cayley table of all the 24 quandles used in our computation.

- **Calculating State Sum Invariant of K**

1. Let $\vec{x} = (x_1, x_2, x_3, ..., x_m) \in \mathcal{C}$ be a coloring.
2. At a positive with input color x_i, x_{i+1} , we assign the Boltzmann weight $\phi(x_i, x_{i+1})$ (refer to the left picture of Figure 2.5). Similarly at a negative crossings with output colour x_i, x_{i+1} we assign the Boltzmann weight $-\phi(x_i, x_{i+1})$ (refer to the right picture of Figure 2.5).
3. Now, compute the state sum invariant $\Phi(D) = \sum_{\mathcal{C}} \prod_{\tau} \phi(x, y)^{\epsilon}$ to get the state sum invariant for K

References

[1] N. Andruskiewitsch and M. Graña, *From racks to pointed Hopf algebras*, Adv. Math. 178 (2003), no. 2, 177–243.

[2] V. G. Bardakov, T. Nasybullov and M. Singh, *Automorphism groups of quandles and related groups*, Monatsh. Math. 189 (2019), no. 1, 1–21.

[3] V. G. Bardakov, I. B. S. Passi and M. Singh, *Quandle rings*, J. Algebra Appl. 18 (2019), 1950157, 23 pp.

[4] V. G. Bardakov, I. B. S. Passi and M. Singh, *Zero-divisors and idempotents in quandle rings*, Osaka J. Math. (2022) (to appear), arXiv:2001.06843, 2020.

[5] S. J. Carter, M. Elhamdadi and M. Saito, *Homology theory for the set-theoretic Yang-Baxter equation and knot invariants from generalizations of quandles*, Fund. Math. 184 (2004), 31–54.

[6] S. J. Carter, D. Jelsovsky, S. Kamada, L. Langford and M. Saito *Quandle cohomology and state-sum invariants of knotted curves and surfaces*, Trans. Amer. Math. Soc. 355 (2003), no. 10, 3947–3989.

[7] W. E. Clark, M. Elhamdadi, M. Saito, T. Yeatman, *Quandle Colorings of Knots and Applications*, J. Knot Theory Ramifications, 23(2014), no. 6.

[8] R. H. Crowell and R. H. Fox. Introduction to knot theory. Springer-Verlag, New York-Heidelberg, 1977. Reprint of the 1963 original, Graduate Texts in Mathematics, No. 57.

[9] C. Damiani, *A journey through loop braid groups*, Expo. Math. 35 (2017), no. 3, 252–285.

[10] M. Dehn, *Uber unendliche diskontinuierliche Gruppen.* Math. Ann., 71(1): 116–144, 1911.

[11] M. Eisermann, *Homological characterization of the unknot*, J. Pure Appl. Algebra 177 (2003), no. 2, 131–157.

[12] M. Eisermann, *Quandle coverings and their Galois correspondence*, Fund. Math. 225 (2014), no. 1, 103–168.

[13] M. Eisermann, *Yang-Baxter deformations of quandles and racks*, Algebr. Geom. Topol. 5 (2005), 537–562.

[14] M. Elhamdadi, N. Fernando and B. Tsvelikhovskiy, *Ring theoretic aspects of quandles*, J. Algebra. 526 (2019), 166–187.

[15] M. Elhamdadi, J. Macquarrie and R. Restrepo, *Automorphism groups of quandles*, J. Algebra Appl. 11 (2012), 1250008.

[16] M. Elhamdadi, A. Makhlouf, S. Silvestrov and E. Zappala, *The derivation problem for quandle algebras*, Internat. J. Algebra Comput. (to appear), arXiv:2106.08289v2.

[17] M. Elhamdadi and S. Nelson, *Quandles—an introduction to the algebra of knots*, isbn: 978-1-4704-2213-4, American Mathematical Society, Providence, RI, 74(2015)

[18] M. Elhamdadi, B. Nunez and M. Singh, *Enhancements of link colorings via idempotents of quandle rings*, J. Pure Appl. Algebra, 227, (2023), no. 10, 107400, 16, doi=10.1016/j.jpaa.2023.107400

[19] M. Elhamdadi, B. Nunez, M. Singh and D. Swain, *Idempotents, free products and quandle coverings*, Internat. J. Math. 34 (2023), no. 3, 2350011, 27 pp.

[20] M. Elhamdadi and D. Swain, *State sum invariants of knots from idempotents in quandle rings*, arxiv: https://doi.org/10.48550/arXiv.2402.14661, 14pages.

[21] R. Fenn, R. Rimányi and C. Rourke, *The braid-permutation group*, Topology 36 (1997), no. 1, 123–135.

[22] S. O. Ivanov, G. Kadantsev and K. Kuznetsov, *Subquandles of free quandles*, (2019), arXiv:1904.06571.

[23] D. E. Joyce, *An algebraic approach to symmetry with applications to knot theory*, Ph.D. Thesis, University of Pennsylvania, 1979.

[24] D. E. Joyce, *A classifying invariant of knots, the knot quandle*, J. Pure Appl. Algebra, 23 (1982), no. 1

[25] C. Livingston, H. A. Moore, *KnotInfo, Table of Knot Invariants*, http://www.knotinfo.math.indiana.edu,August23,2022.

[26] O. Loos, *Reflexion spaces and homogeneous symmetric spaces*, Bull. Amer. Math. Soc. 73 (1967) 250–253.

[27] Maple 15 - Magma package-copywrite by Maplesoft, a division of Waterloo Maple, Inc, 1981–2011.

[28] S. V. Matveev, *Distributive groupoids in knot theory*, Mat. Sb. (N.S.) 119(161) (1982), no. 1, 78–88, 160.

[29] K. Reidemeister, Elementare Begr¨undung der Knotentheorie. Abh. Math. Sem. Univ. Hamburg, 5(1):24–32, 1927.

[30] D. Rolfsen. *Knots and Links*, isbn: 0-8218-3436-3, American Mathematical Society, Providence, RI, 74(1976)

[31] M. Singh, *Algebraic Structures in Knot Theory*, Indian Institute of Science Education and Research Mohali (2021), Doctoral Thesis

[32] R. G. Swan, *Induced representations and projective modules*, Ann. of Math. (2) 71 (1960), 552–578.

[33] L. Vendramin, *RIG – a GAP package for racks and quandles*, (2011) https://code.google.com/p/rig.

[34] S. K. Winker, *Quandles, knots invariants and the n-fold branched cover*, Ph.D. Thesis, University of Illinois at Chicago, 1984.

[35] J. Zablow, *Loops, waves, and an "algebra" for Heegaard splittings*, Ph.D. Thesis, City University of New York, 1999.

[36] J. Zablow, *Loops and disks in surfaces and handlebodies*, J. Knot Theory Ramifications 12 (2003), no. 2, 203–223.

[37] J. Zablow, *On relations and homology of the Dehn quandle*, Algebr. Geom. Topol. 8 (2008), no. 1, 19–51.

Appendix A: List of Quandles used in Computation

In this Appendix we provide the list of quandles and some of their idempotent quandles

Note: The notation $C[i, j]$ stands for the j-th connected quandle of order i (see [33]). The right multiplications S_k in the quandle are given by $S_k(l) = l * k$. As permutations, right multiplications are written below as product of cycles.

- $C[8, 1]$

  $S_1 = S_2 = (3\ 6\ 7)(4\ 5\ 8)$ $\quad S_3 = S_4 = (1\ 8\ 6)(2\ 5\ 7)$ $\quad S_5 = S_6 = (1\ 4\ 7)(2\ 3\ 8)$

  $S_7 = S_8 = (1\ 5\ 3)(2\ 6\ 4)$

- $\mathcal{I}(\mathbb{Z}_2[C[8, 1]])$. As a set $\mathcal{I}(\mathbb{Z}_2[C[8, 1]]) = \{1, \dots, 16\}$.

  $S_1 = S_2 = S_9 = S_{10} = (3\ 6\ 7)(4\ 5\ 8)(15\ 11\ 14)(12\ 13\ 16)$

  $S_3 = S_4 = S_{11} = S_{12} = (1\ 8\ 6)(2\ 5\ 7)(13\ 10\ 15)(9\ 16\ 14)$

  $S_5 = S_6 = S_{13} = S_{14} = (1\ 4\ 7)(2\ 3\ 8)(9\ 12\ 15)(10\ 11\ 16)$

  $S_7 = S_8 = S_{15} = S_{16} = (1\ 5\ 3)(2\ 6\ 4)(9\ 13\ 11)(10\ 14\ 12)$

- $C[12, 3]$

  $S_1 = (2\ 12\ 5\ 10\ 11)(3\ 8\ 6\ 7\ 4)$ $\quad S_2 = (1\ 11\ 7\ 4\ 12)(3\ 5\ 10\ 6\ 9)$

  $S_3 = (1\ 2\ 7\ 6\ 10)(4\ 9\ 8\ 5\ 12)$ $\quad S_4 = (1\ 11\ 6\ 8\ 5)(2\ 7\ 9\ 3\ 12)$

  $S_5 = (1\ 12\ 3\ 8\ 10)(2\ 4\ 9\ 6\ 11)$ $\quad S_6 = (1\ 5\ 3\ 4\ 2)(7\ 11\ 10\ 8\ 9)$

  $S_7 = (1\ 10\ 8\ 3\ 12)(2\ 11\ 6\ 9\ 4)$ $\quad S_8 = (1\ 12\ 5\ 7\ 11)(3\ 9\ 6\ 10\ 5)$

  $S_9 = (2\ 11\ 10\ 5\ 12)(3\ 4\ 7\ 6\ 8)$ $\quad S_{10} = (1\ 5\ 8\ 6\ 11)(2\ 12\ 3\ 9\ 7)$

  $S_{11} = (1\ 10\ 6\ 7\ 2)(4\ 12\ 5\ 8\ 9)$ $\quad S_{12} = (1\ 2\ 4\ 3\ 5)(7\ 9\ 8\ 10\ 11).$

- $\mathcal{I}(\mathbb{Z}_2[C[12,3]])$. As a set $\mathcal{I}(\mathbb{Z}_2[C[12,3]]) = \{1, \dots, 24\}$.

  $S_1 = S_{13} = (2\ 12\ 5\ 10\ 11)(3\ 8\ 6\ 7\ 4)(14\ 24\ 17\ 22\ 23)(15\ 20\ 18\ 19\ 16)$

  $S_2 = S_{14} = (1\ 11\ 7\ 4\ 12)(3\ 5\ 10\ 6\ 9)(13\ 23\ 19\ 16\ 24)(15\ 17\ 22\ 18\ 21)$

  $S_3 = S_{15} = (1\ 2\ 7\ 6\ 10)(4\ 9\ 8\ 5\ 12)(13\ 14\ 19\ 18\ 22)(16\ 21\ 20\ 17\ 24)$

  $S_4 = S_{16} = (1\ 11\ 6\ 8\ 5)(2\ 7\ 9\ 3\ 12)(13\ 23\ 18\ 20\ 17)(14\ 19\ 21\ 15\ 16)$

  $S_5 = S_{17} = (1\ 12\ 3\ 8\ 10)(2\ 4\ 9\ 6\ 11)(13\ 24\ 15\ 20\ 22)(14\ 16\ 21\ 18\ 23)$

  $S_6 = S_{18} = (1\ 5\ 3\ 4\ 2)(7\ 11\ 10\ 8\ 9)(13\ 17\ 15\ 16\ 14)(19\ 23\ 22\ 20\ 18)$

  $S_7 = S_{19} = (1\ 10\ 8\ 3\ 12)(2\ 11\ 6\ 9\ 4)(13\ 22\ 20\ 15\ 24)(14\ 23\ 18\ 21\ 16)$

  $S_8 = S_{20} = (1\ 12\ 5\ 7\ 11)(3\ 9\ 6\ 10\ 5)(13\ 24\ 16\ 19\ 23)(15\ 21\ 18\ 22\ 17)$

  $S_9 = S_{21} = (2\ 11\ 10\ 5\ 12)(3\ 4\ 7\ 6\ 8)(14\ 23\ 22\ 17\ 24)(15\ 16\ 19\ 18\ 20)$

  $S_{10} = S_{22} = (1\ 5\ 8\ 6\ 11)(2\ 12\ 3\ 9\ 7)(13\ 17\ 20\ 18\ 23)(14\ 24\ 15\ 21\ 19)$

  $S_{11} = S_{23} = (1\ 10\ 6\ 7\ 2)(4\ 12\ 5\ 8\ 9)(13\ 22\ 18\ 19\ 14)(16\ 24\ 17\ 20\ 21)$

  $S_{12} = S_{24} = (1\ 2\ 4\ 3\ 5)(7\ 9\ 8\ 10\ 11)(13\ 14\ 16\ 15\ 17)(19\ 21\ 20\ 22\ 23)$

- $C[12,4]$

  $S_1 = (5\ 9)(2\ 3\ 4)(6\ 11\ 8\ 10\ 7\ 12)$ $S_2 = (6\ 10)(1\ 4\ 3)(5\ 12\ 7\ 9\ 8\ 11)$

  $S_3 = (7\ 11)(1\ 2\ 4)(5\ 10\ 8\ 9\ 6\ 12)$ $S_4 = (8\ 12)(1\ 3\ 2)(5\ 11\ 6\ 9\ 7\ 10)$

  $S_5 = (1\ 9)(6\ 7\ 8)(2\ 11\ 4\ 10\ 3\ 12)$ $S_6 = (2\ 10)(5\ 8\ 7)(1\ 12\ 3\ 9\ 4\ 11)$

  $S_7 = (3\ 11)(5\ 6\ 8)(1\ 10\ 4\ 9\ 2\ 12)$ $S_8 = (4\ 12)(5\ 7\ 6)(1\ 11\ 2\ 9\ 3\ 10)$

  $S_9 = (1\ 5)(10\ 11\ 12)(2\ 7\ 4\ 6\ 3\ 8)$ $S_{10} = (2\ 6)(9\ 12\ 11)(1\ 8\ 3\ 5\ 4$

  $S_{11} = (3\ 7)(9\ 10\ 12)(1\ 6\ 4\ 5\ 2\ 8)$ $S_{12} = (4\ 8)(9\ 11\ 10)(1\ 7\ 2\ 5\ 3\ 6)$.

- $C[12,6]$

  $S_1 = (3\ 4)(5\ 10)(6\ 9)(8\ 12)(7\ 11)$ $S_2 = (3\ 4)(5\ 9)(6\ 10)(8\ 11)(7\ 12)$

  $S_3 = (1\ 2)(5\ 11)(6\ 12)(7\ 9)(8\ 10)$ $S_4 = (1\ 2)(5\ 12)(6\ 11)(7\ 10)(8\ 11)$

  $S_5 = (1\ 10)(2\ 9)(3\ 11)(4\ 12)(7\ 8)$ $S_6 = (1\ 9)(2\ 10)(3\ 12)(4\ 11)(7\ 8)$

  $S_7 = (1\ 11)(2\ 12)(3\ 9)(4\ 10)(5\ 6)$ $S_8 = (1\ 12)(2\ 11)(3\ 10)(4\ 9)(5\ 6)$

  $S_9 = (1\ 6)(2\ 5)(3\ 7)(4\ 8)(11\ 12)$ $S_{10} = (1\ 5)(2\ 6)(3\ 8)(4\ 7)(11\ 12)$

$S_{11} = (1\ 7)(2\ 8)(3\ 5)(4\ 6)(9\ 10) \quad S_{12} = (1\ 8)(2\ 7)(3\ 6)(4\ 5)(9\ 10).$

- $C[13, 4]$

  $S_1 = (2\ 9\ 13\ 6)(3\ 4\ 12\ 11)(5\ 7\ 10\ 8) \quad S_2 = (1\ 7\ 3\ 10)(4\ 5\ 13\ 12)(6\ 8\ 11\ 9)$

  $S_3 = (1\ 13\ 5\ 6)(2\ 8\ 4\ 11)(7\ 9\ 12\ 10) \quad S_4 = (1\ 6\ 7\ 2)(3\ 9\ 5\ 12)(8\ 10\ 13\ 11)$

  $S_5 = (1\ 12\ 9\ 11)(2\ 7\ 8\ 3)(4\ 10\ 6\ 13) \quad S_6 = (1\ 5\ 11\ 7)(2\ 13\ 10\ 12)(3\ 8\ 9\ 4)$

  $S_7 = (1\ 11\ 13\ 3)(2\ 6\ 12\ 8)(4\ 9\ 10\ 5) \quad S_8 = (1\ 4\ 2\ 12)(3\ 7\ 13\ 9)(5\ 10\ 11\ 6)$

  $S_9 = (1\ 10\ 4\ 8)(2\ 5\ 3\ 13)(6\ 11\ 12\ 7) \quad S_{10} = (1\ 3\ 6\ 4)(2\ 11\ 5\ 9)(7\ 12\ 13\ 8)$

  $S_{11} = (1\ 9\ 8\ 13)(2\ 4\ 7\ 5)(3\ 12\ 6\ 10) \quad S_{12} = (1\ 2\ 10\ 9)(3\ 5\ 8\ 6)(4\ 13\ 7\ 11)$

  $S_{13} = (1\ 8\ 12\ 5)(2\ 3\ 11\ 10)(4\ 6\ 9\ 7).$

- $C[13, 7]$

  $S_1 = (2\ 6\ 13\ 9)(3\ 11\ 12\ 4)(5\ 8\ 10\ 7) \quad S_2 = (1\ 10\ 3\ 7)(4\ 12\ 13\ 5)(6\ 9\ 8\ 11)$

  $S_3 = (1\ 6\ 5\ 13)(2\ 11\ 4\ 8)(7\ 10\ 12\ 9) \quad S_4 = (1\ 2\ 7\ 6)(3\ 12\ 5\ 9)(8\ 11\ 13\ 10)$

  $S_5 = (1\ 11\ 9\ 12)(2\ 3\ 8\ 7)(4\ 13\ 6\ 10) \quad S_6 = (1\ 7\ 11\ 5)(2\ !2\ 10\ 13)(3\ 4\ 9\ 8)$

  $S_7 = (1\ 3\ 13\ 11)(2\ 8\ 12\ 6)(4\ 5\ 10\ 9) \quad S_8 = (1\ 12\ 2\ 4)(3\ 9\ 13\ 7)(5\ 6\ 11\ 10)$

  $S_9 = (1\ 8\ 4\ 10)(2\ 13\ 3\ 5)(6\ 7\ 12\ 11) \quad S_{10} = (1\ 4\ 6\ 3)(2\ 9\ 5\ 11)(7\ 8\ 13\ 12)$

  $S_{11} = (1\ 13\ 8\ 9)(2\ 5\ 7\ 4)(3\ 10\ 6\ 12) \quad S_{12} = (1\ 9\ 10\ 2)(3\ 6\ 8\ 5)(4\ 11\ 7\ 13)$

  $S_{13} = (1\ 5\ 12\ 8)(2\ 10\ 11\ 3)(4\ 7\ 9\ 6).$

- $C[13, 10]$

  $S_1 = (2\ 7\ 11\ 9\ 10\ 3\ 13\ 8\ 4\ 6\ 5\ 12) \quad S_2 = (1\ 9\ 5\ 7\ 6\ 13\ 3\ 8\ 12\ 10\ 11\ 4)$

  $S_3 = (1\ 4\ 9\ 13\ 11\ 12\ 5\ 2\ 10\ 6\ 8\ 7) \quad S_4 = (1\ 12\ 13\ 6\ 3\ 11\ 7\ 9\ 8\ 2\ 5\ 10)$

  $S_5 = (1\ 7\ 4\ 12\ 8\ 10\ 9\ 3\ 6\ 11\ 2\ 13) \quad S_6 = (1\ 2\ 8\ 5\ 13\ 9\ 11\ 10\ 4\ 7\ 12\ 3)$

  $S_7 = (1\ 10\ 12\ 11\ 5\ 8\ 13\ 4\ 2\ 3\ 9\ 6) \quad S_8 = (1\ 5\ 3\ 4\ 10\ 7\ 2\ 11\ 13\ 12\ 6\ 9)$

  $S_9 = (1\ 13\ 7\ 10\ 2\ 6\ 4\ 5\ 11\ 8\ 3\ 12) \quad S_{10} = (1\ 8\ 11\ 3\ 7\ 5\ 6\ 12\ 9\ 4\ 13\ 2)$

  $S_{11} = (1\ 3\ 2\ 9\ 12\ 4\ 8\ 6\ 7\ 13\ 10\ 5) \quad S_{12} = (1\ 11\ 6\ 2\ 4\ 3\ 10\ 13\ 5\ 9\ 7\ 8)$

  $S_{13} = (1\ 6\ 10\ 8\ 9\ 2\ 12\ 7\ 3\ 5\ 4\ 11).$

- $C[16, 3]$

$S_1 = (2\ 3\ 5\ 9\ 16)(4\ 7 13\ 8\ 15)(6\ 11\ 12\ 10\ 14)$ $S_2 = (1\ 4\ 6\ 10\ 15)(3\ 8\ 14\ 7\ 16)(5\ 12\ 11\ 9\ 13)$

$S_3 = (1\ 7\ 11\ 14\ 4)(2\ 5\ 15\ 6\ 13)(8\ 9\ 10\ 12\ 16)$ $S_4 = (1\ 6\ 16\ 5\ 14)(2\ 8\ 12\ 13\ 3)(7\ 10\ 9\ 11\ 15)$

$S_5 = (1\ 13\ 12\ 6\ 7)(2\ 15\ 16\ 14\ 10)(3\ 9\ 4\ 11\ 8)$ $S_6 = (1\ 16\ 15\ 13\ 9)(2\ 14\ 11\ 5\ 8)(3\ 12\ 7\ 4\ 10)$

$S_7 = (1\ 11\ 2\ 9\ 6)(3\ 15\ 10\ 8\ 5)(4\ 13\ 14\ 16\ 12)$ $S_8 = (1\ 10\ 5\ 2\ 12)(3\ 14\ 13\ 15\ 11)(4\ 16\ 9\ 7\ 6)$

$S_9 = (1\ 8\ 10\ 11\ 13)(2\ 6\ 13\ 3\ 4)(5\ 16\ 7\ 12\ 15)$ $S_{10} = (1\ 5\ 13\ 4\ 3)(2\ 7\ 9\ 12\ 14)(6\ 15\ 8\ 11\ 16)$

$S_{11} = (1\ 2\ 4\ 8\ 16)(3\ 6\ 12\ 9\ 15)(5\ 10\ 13\ 7\ 14)$ $S_{12} = (1\ 3\ 7\ 15\ 2)(4\ 5\ 11\ 10\ 16)(6\ 9\ 14\ 8\ 13)$

$S_{13} = (1\ 12\ 3\ 16\ 11)(2\ 10\ 7\ 8\ 6)(4\ 14\ 15\ 9\ 5)$ $S_{14} = (1\ 9\ 8\ 7\ 5)(2\ 11\ 4\ 15\ 12)(3\ 13\ 16\ 10\ 6)$

$S_{15} = (1\ 14\ 9\ 3\ 10)(2\ 16\ 13\ 11\ 7)(4\ 12\ 5\ 6\ 8)$ $S_{16} = (1\ 15\ 14\ 12\ 8)(2\ 13\ 10\ 4\ 9)(3\ 11\ 6\ 5\ 7)$

- $C[16, 4]$

$S_1 = (2\ 13\ 4\ 9\ 3\ 5)(6\ 14\ 16\ 12\ 11\ 7)(8\ 10\ 15)$ $S_2 = (1\ 14\ 3\ 10\ 4\ 6)(5\ 13\ 15\ 11\ 12\ 8)(7\ 9\ 16)$

$S_3 = (1\ 7\ 4\ 15\ 2\ 11)(5\ 8\ 16\ 14\ 10\ 9)(6\ 12\ 13)$ $S_4 = (1\ 12\ 2\ 8\ 3\ 16)(6\ 7\ 15\ 13\ 9\ 10)(5\ 11\ 14)$

$S_5 = (1\ 6\ 9\ 8\ 13\ 7)(2\ 10\ 12\ 16\ 15\ 3)(4\ 14\ 11)$ $S_6 = (1\ 9\ 11\ 15\ 16\ 4)(2\ 5\ 10\ 7\ 14\ 8)(3\ 13\ 12)$

$S_7 = (1\ 4\ 12\ 10\ 14\ 13)(3\ 8\ 11\ 6\ 15\ 5)(2\ 16\ 9)$ $S_8 = (2\ 3\ 11\ 9\ 13\ 14)(4\ 7\ 12\ 5\ 16\ 6)(1\ 15\ 10)$

$S_9 = (1\ 11\ 13\ 10\ 5\ 12)(3\ 15\ 14\ 6\ 8\ 4)(2\ 7\ 16)$ $S_{10} = (2\ 12\ 14\ 9\ 6\ 11)(3\ 4\ 16\ 13\ 5\ 7)(1\ 8\ 15)$

$S_{11} = (1\ 13\ 16\ 8\ 6\ 2)(3\ 9\ 15\ 12\ 7\ 10)(4\ 5\ 14)$ $S_{12} = (1\ 2\ 14\ 15\ 7\ 5)(4\ 10\ 16\ 11\ 8\ 9)(3\ 16\ 13)$

$S_{13} = (1\ 16\ 5\ 15\ 9\ 14)(2\ 4\ 8\ 7\ 11\ 10)(3\ 12\ 6)$ $S_{14} = (1\ 3\ 7\ 8\ 12\ 9)(2\ 15\ 6\ 16\ 10\ 13)(4\ 11\ 5)$

$S_{15} = (2\ 6\ 5\ 9\ 12\ 4)(3\ 14\ 7\ 13\ 11\ 16)(1\ 10\ 8)$ $S_{16} = (1\ 5\ 6\ 10\ 11\ 3)(4\ 13\ 8\ 14\ 12\ 15)(2\ 9\ 7)$

- $C[16, 8]$

$S_1 = (2\ 3\ 5\ 9\ 10\ 12\ 16\ 8\ 15\ 6\ 11\ 14\ 4\ 7\ 13)$ $S_2 = (1\ 4\ 6\ 10\ 9\ 11\ 15\ 7\ 16\ 5\ 12\ 13\ 3\ 8\ 14\ 1)$

$S_3 = (1\ 7\ 11\ 12\ 10\ 14\ 6\ 13\ 8\ 9\ 16\ 2\ 5\ 15\ 4)$ $S_4 = (1\ 6\ 16\ 3\ 2\ 8\ 12\ 11\ 9\ 13\ 5\ 14\ 7\ 10\ 15)$

$S_5 = (1\ 13\ 14\ 16\ 12\ 4\ 11\ 2\ 15\ 10\ 8\ 3\ 9\ 6\ 7)$ $S_6 = (1\ 16\ 9\ 7\ 4\ 10\ 5\ 8\ 2\ 14\ 13\ 15\ 11\ 3\ 12)$

$S_7 = (1\ 11\ 8\ 5\ 3\ 15\ 16\ 14\ 10\ 2\ 9\ 4\ 13\ 12\ 6)$ $S_8 = (1\ 10\ 3\ 14\ 11\ 5\ 2\ 12\ 7\ 6\ 4\ 16\ 15\ 13\ 9)$

$S_9 = (1\ 2\ 4\ 8\ 16\ 7\ 14\ 3\ 6\ 12\ 15\ 5\ 10\ 11\ 13)$ $S_{10} = (1\ 3\ 7\ 15\ 8\ 13\ 4\ 5\ 11\ 16\ 6\ 9\ 12\ 14)$

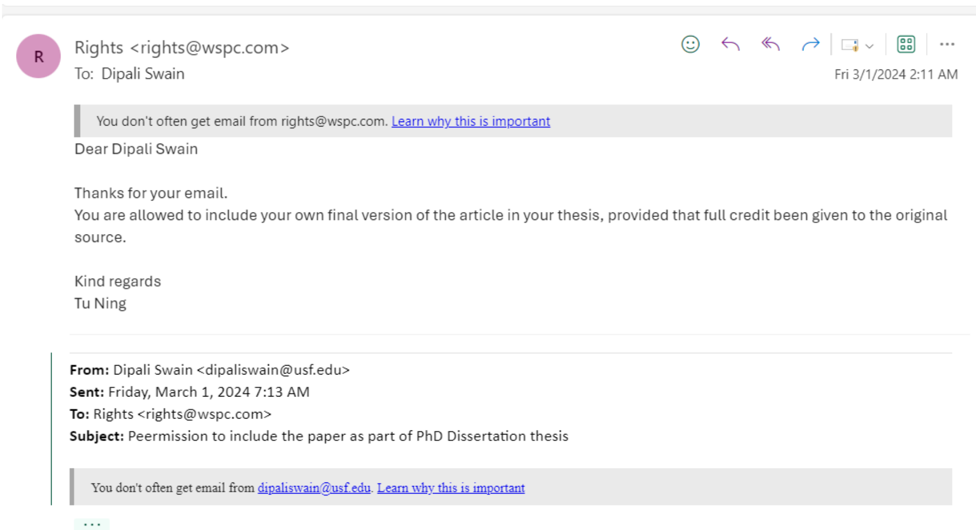
$S_{11} = (1\ 8\ 10\ 13\ 7\ 12\ 9\ 15\ 3\ 4\ 2\ 6\ 14\ 5\ 16)$ $S_{12} = (1\ 5\ 13\ 6\ 15\ 2\ 7\ 9\ 14\ 8\ 11\ 10\ 16\ 4\ 3)$

$S_{13} = (1\ 14\ 15\ 9\ 5\ 6\ 8\ 4\ 12\ 3\ 10\ 7\ 2\ 16\ 11)$ $S_{14} = (1\ 15\ 12\ 2\ 13\ 16\ 10\ 6\ 5\ 7\ 3\ 11\ 4\ 9\ 8)$

$S_{15} = (1\ 12\ 5\ 4\ 14\ 9\ 3\ 16\ 13\ 11\ 7\ 8\ 6\ 2\ 10)$ $S_{16} = (1\ 9\ 2\ 11\ 6\ 3\ 13\ 10\ 4\ 15\ 14\ 12\ 8\ 7\ 5)$

Appendix B: Copyright Clearance Form

Below is the permission for the material included in Chapter 3.



Rights <rights@wspc.com>
To: Dipali Swain
Fri 3/1/2024 2:11 AM

You don't often get email from rights@wspc.com. Learn why this is important

Dear Dipali Swain

Thanks for your email.
You are allowed to include your own final version of the article in your thesis, provided that full credit been given to the original source.

Kind regards
Tu Ning

From: Dipali Swain <dipaliswain@usf.edu>
Sent: Friday, March 1, 2024 7:13 AM
To: Rights <rights@wspc.com>
Subject: Peermission to include the paper as part of PhD Dissertation thesis

You don't often get email from dipaliswain@usf.edu. Learn why this is important

Appendix C: ArXiv License

According to arXiv license information, I have the right to freely utilize material from the papers I have submitted to arXiv, without any specific permission.

arXiv / Help / arXiv License Information

arXiv License Information

arXiv is a repository for scholarly material, and perpetual access is necessary to maintain the scholarly record. As such, arXiv keeps a permanent record of every submission and replacement announced.

arXiv does not ask that copyright be transferred. However, we require sufficient rights to allow us to distribute submitted articles in perpetuity. In order to submit an article to arXiv, the submitter must either:

- grant arXiv.org a non-exclusive and irrevocable license to distribute the article, and certify that the submitter has the right to grant this license;
- certify that the work is available under one of the following Creative Commons licenses and that the submitter has the right to assign this license:
  - Creative Commons Attribution license (CC BY 4.0)
  - Creative Commons Attribution-ShareAlike license (CC BY-SA 4.0)
  - Creative Commons Attribution-Noncommercial-ShareAlike license (CC BY-NC-SA 4.0);
- or dedicate the work to the public domain by associating the Creative Commons Public Domain Dedication (CC0 1.0) with the submission.

In the most common case, authors have the right to grant these licenses because they hold copyright in their own work.

We currently support three of the Creative Commons licenses. If you wish to use a different CC license, then select arXiv's non-exclusive license to distribute in the arXiv submission process and indicate the desired Creative Commons license in the actual article.

Note: if you intend to submit, or have submitted, your article to a journal then you should verify that the license you select during arXiv submission does not conflict with the journal's license or copyright transfer agreement. Many journal agreements permit submission to arXiv using the non-exclusive license to distribute, which arXiv has used since 2004. Yet the CC BY and CC BY-SA licenses permit commercial reuse and may therefore conflict with some journal agreements.

Metadata license

To the extent that the depositor or arxiv has a copyright interest in metadata accompanying the submission, a Creative Commons CC0 1.0 Universal Public Domain Dedication will apply.

https://arxiv.org/help/license 1/3