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On the Subelliptic and Subparabolic Infinity Laplacian in Grushin-Type Spaces

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On The Subelliptic and Subparabolic Infinity Laplacian in Grushin-Type Spaces

by

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A dissertation submitted in partial fulfillment
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DEDICATION

To my parents, brothers, wife, and sons. Thank you for believing in me.

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ABSTRACT

This thesis poses the ∞-*Laplace equation* in *Grushin-type spaces.* Grushin-type spaces $\mathbb{G}$ are defined by the vector fields which serve as a basis for their tangent spaces; by weighting the canonical (Euclidean) directional vectors $\{\partial/\partial x_i\}_{i=1}^n$ by functions ρ_i that obey certain technical assumptions, we produce a class of metric spaces in which certain directions may not be accessible at all points in the space. We prove the existence and uniqueness of *viscosity solutions* to both *Dirichlet problems* and *Cauchy-Dirichlet problems* involving the ∞-Laplacian over bounded Grushin-type domains. The main tool in proving uniqueness of these solutions is a comparison principle for semilinear functions, which we obtain by exploiting the relationship between Euclidean and Grushin-type geometry. We also prove that solutions of certain Cauchy-Dirichlet problems converge to solutions of (time-stationary) Dirichlet problems as we permit the chronological variable t to tend toward ∞.

CHAPTER 1:
INTRODUCTION AND BACKGROUND

Let $\mathbb{R}^n$ have coordinates $(x_1, x_2, \ldots, x_n)$ and let $\{\partial/\partial x_i\}_{i=1}^n$ be the standard vectors (directional derivatives) that are orthonormal under the inner product $(\cdot,\cdot)_{\mathrm{eucl}}$ and related norm $\|\cdot\|_{\mathrm{eucl}}$. We recall that for a smooth function $w : \mathbb{R}^n \to \mathbb{R}^n$, we define the gradient to be the vector

$$Dw := \left(\frac{\partial w}{\partial x_1}, \cdots, \frac{\partial w}{\partial x_n}\right)$$

and the $n \times n$ second derivative matrix $D^2 w$ has entries given by

$$\left[D^2 w\right]_{ij} := \frac{\partial^2 w}{\partial x_i \partial x_j}.$$

Using these derivatives, we can define the ∞-Laplace operator by

$$\Delta_\infty w := -\left(D^2 w \cdot Dw, Dw\right)_{\mathrm{eucl}} = -\sum_{i,j=1}^{n} \frac{\partial w}{\partial x_i} \cdot \frac{\partial w}{\partial x_j} \cdot \frac{\partial^2 w}{\partial x_i \partial x_j}$$

for a sufficiently smooth function $w : \mathbb{R}^n \to \mathbb{R}$ and its parabolic counterpart by

$$f_t + \Delta_\infty f$$

for a sufficiently smooth function $f : \mathbb{R}^n \times [0,T] \to \mathbb{R}$. Note that the parabolic operator contains a time element.

The Euclidean ∞-Laplace operator can be thought of as the formal limit of the Euclidean $\mathtt{p}$-Laplace operator

$$\begin{aligned} \Delta_{\mathtt{p}} w &:= -\operatorname{div}\left(\|Dw\|^{\mathtt{p}-2} \cdot Dw\right) \\ &= -\left(\|Dw\|_{\mathrm{eucl}}^{\mathtt{p}-2} \operatorname{tr}\left(D^2 w\right) + (\mathtt{p}-2)\|Dw\|_{\mathrm{eucl}}^{\mathtt{p}-4} \Delta_\infty w\right) \end{aligned}$$

for $1 < \mathtt{p} < \infty$ as we take $\mathtt{p} \uparrow \infty$. (Here div is the standard Euclidean divergence.) Since weak solutions in the sense of distributions to the homogeneous equation $\Delta_{\mathtt{p}} w = 0$ are minimizers of certain energy integrals, the relationship between $\Delta_{\mathtt{p}}$ and Δ_∞ allows us to treat solutions u of the homogeneous ∞-Laplace Equation

$\Delta_\infty w = 0$ as members of the Sobolev Space $W^{1,\infty}$ such that

$$\|Du\|_\infty \leq \|Dv\|_\infty$$

for all v in $W^{1,\infty}$ such that $u - v$ belong to $W_0^{1,\infty}$. Consequently, solutions to the ∞-Laplacian are useful in applications which seek to minimize the maxima of systems, such as in the construction of load-bearing columns and air conditioning systems (see [24]).

In this dissertation, we will adapt the ∞-Laplace and parabolic ∞-Laplace operators to a sub-Riemannian space where the standard vectors are replaced by a collection $\mathfrak{X} = \{X_i\}_{i=1}^n$ of vector fields satisfying certain technical assumptions. We also replace the Euclidean inner-product $(\cdot,\cdot)_{\text{eucl}}$ by an inner-product $\langle \cdot,\cdot \rangle$ which makes $\mathfrak{X}$ an orthonormal collection (except at certain points) under $\langle \cdot,\cdot \rangle$ and the Euclidean norm $\|\cdot\|_{\text{eucl}}$ by the norm $\|\cdot\|_{\mathfrak{X}}$ induced by $\langle \cdot,\cdot \rangle$. In this setting, the gradient of smooth w *relative to* $\mathfrak{X}$ is now given by

$$\nabla_{\mathbb{G}}\, w := (X_1 w, \cdots, X_n w)$$

and the $n \times n$ symmetrized second derivative matrix $\left(D^2 w\right)^\star$ *relative to* $\mathfrak{X}$ has entries given by

$$\left[\left(D^2 w\right)^\star\right]_{ij} := \frac{1}{2}\left(X_j X_i w + X_i X_j w\right).$$

Note that the actions $X_i w$ and $X_i X_j w$ are first- and second-order directional derivatives, respectively. As a consequence, the ∞-Laplace operator *relative to* $\mathfrak{X}$ is given by

$$\Delta_{\mathfrak{X},\infty} w := -\left\langle \left(D^2 w\right)^\star \cdot \nabla_{\mathbb{G}}\, w, \nabla_{\mathbb{G}}\, w \right\rangle,$$

and our focus now is on solutions to the (homogeneous) equations

$$\Delta_{\mathfrak{X},\infty} w = 0 \tag{1.1}$$

and

$$w_t + \Delta_{\mathfrak{X},\infty} w = 0. \tag{1.2}$$

In this dissertation, as in [24], [27], [25], and others, we will seek solutions to Equations (1.1) and (1.2) in the sense of viscosity solutions. Viscosity theory is a useful tool in the study of partial differential equations: It provides a notion of point-wise estimates for first- and second-order derivatives, and permits the formulation of comparison principles between viscosity sub- and supersolutions. Comparison principles for both Equation (1.1) and Equation (1.2) will be a major focus of this dissertation, as the formulation and proof of the theorems and estimates necessary to establish these theorems depends greatly upon the nature of the sub-Riemannian space in question.

All of our work will take place in the setting of sub-Riemannian manifolds: That is, in n-dimensional manifolds M ($n \geq 2$) for which the tangent space T_pM at each point $p \in M$, called the *horizontal distribution* for M at p, can be a proper subset of $\mathbb{R}^n$ at some points. All those vectors which fail to belong to the horizontal distribution of M are thought of as "missing" directions, and model physical settings in which motion is restricted depending upon the point in space being occupied. Such is the geometry surrounding a bridge: At points off the bridge, motion is unrestricted; however, for points on the bridge, certain directions can not be taken. It is clear that the tangent space for M impacts its geometry. The tangent space also impacts the calculus of M since derivatives in M can be thought of as derivatives in the direction of tangent vectors.

The class of sub-Riemannian manifolds studied herein are called Grushin-type spaces. Initially studied by V. Grushin, for whom they are named, in [22] and [23], these are sub-Riemannian manifolds whose horizontal distribution at p are defined by "weight functions" ρ_k on the canonical Euclidean frame $\{\partial/\partial x_i\}_{i=1}^n$. For example, in the case that $n = 2$, the *Grushin plane* possesses a horizontal distribution at $p = (x_1, x_2)$ which is the span of the vectors

$$X_1(x_1, x_2) := \frac{\partial}{\partial x_1} \text{ and } X_2(x_1, x_2) := x_1 \cdot \frac{\partial}{\partial x_2}.$$

For points off the x_2-axis the space is Riemannian in nature. On the x_2-axis, the vector field X_2 vanishes and hence the only directions of travel are those parallel to the x_1-axis. For this dissertation, the class of Grushin-type spaces under consideration can be thought of as extensions and generalizations of the Grushin plane; they may all be characterized as metric spaces lacking a group law, but for which notions of calculus are preserved.

Analytic and geometric properties of such spaces have been investigated previously by various authors: In [16] the authors detail results concerning geodesics in Grushin-type spaces for $n = 2$; the ∞-Laplace Equation (1.1) and its viscosity solutions were considered in [4], [6], and [10] for Grushin-type spaces whose horizontal distributions were examples of the ones in this dissertation; the fundamental solution to (1.1) was studied in [12] when the weight functions ρ_k were polynomials, and again in [7] for nonpolynomial C^2 weight functions. In this dissertation, our first objective is proving the existence and uniqueness of viscosity solutions to the Dirichlet problem

$$\begin{cases} \Delta_{\mathfrak{X},\infty} w = 0 & \text{in } \Omega \\ w = g & \text{on } \partial\Omega, \end{cases} \tag{DP}$$

for bounded Grushin domains Ω and continuous functions $g : \partial\Omega \to \mathbb{R}$. Subsequently, we shall prove the existence and uniqueness of solutions to the Cauchy-Dirichlet problem

$$\begin{cases} w_t + \Delta_{\mathfrak{X},\infty} w = 0 & \text{in } \Omega \times (0,T) \\ w = g & \text{on } \partial_{\text{par}}\Big(\Omega \times (0,T)\Big), \end{cases} \tag{CDP}$$

where $T > 0$, after which we will address asymptotic behavior of solutions u to

$$\begin{cases} w_t + \Delta_{\mathfrak{X},\infty} w = 0 & \text{in } \Omega \times (0,\infty) \\ w = g & \text{on } \partial_{\text{par}}\Big(\Omega \times (0,\infty)\Big). \end{cases} \tag{1.3}$$

as we allow $t \to \infty$.

The layout of this dissertation will be as follows. In Chapter 2 we will define Grushin-type spaces via their tangent spaces, listing some notable examples of such spaces, and discuss geometric and metric space properties before introducing the calculus of Grushin-type spaces. In Chapter 3 we define the notion of viscosity solutions to the Grushin ∞-Laplace equation via two equivalent methods and then relate these notions to their Euclidean counterparts. In Chapter 4 we address both existence and uniqueness of solutions to Problem (DP); the primary focus of the chapter will be on producing certain useful estimates which permit a comparison principle for solutions. We revisit viscosity theory in Chapter 5 by extending the notion to time-space cylinders over Grushin sets. We conclude in Chapter 6 and address the following: We prove uniqueness of solutions by producing a parabolic variant of our previous comparison principle; establish existence by appealing to techniques similar to the Perron's Method approach discussed in [21]; and finally, under certain restrictions, show that solutions of Equation (1.3) tend to solutions of the (time stationary) ∞-Laplace equation.

CHAPTER 2:
DEFINITION AND PROPERTIES OF GRUSHIN-TYPE SPACES

2.1 Tangent Spaces and Definition For Grushin-Type Spaces

Let $n \geq 2$ and, given some arbitrary point $p = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, write $T_p(\mathbb{R}^n)$ to denote the Euclidean tangent space at p with canonical basis vectors

$$\frac{\partial}{\partial x_k} := e_k. \tag{2.1}$$

We construct a frame $\mathfrak{X}(p) := \{X_i(p)\}_{i=1}^n$ of vector fields by defining vector field X_1 as

$$X_1(p) := \frac{\partial}{\partial x_1}, \tag{2.2}$$

and defining vector fields X_k for $k \geq 2$ by

$$X_k(p) := \rho_k(p) \cdot \frac{\partial}{\partial x_k} = \rho_k(x_1, \ldots, x_{k-1}) \cdot \frac{\partial}{\partial x_k}$$

for functions ρ_k not identically zero on the whole space. We require certain mild technical assumptions on the functions ρ_k when $k \geq 2$:

1. The functions ρ_k depend only upon the first $k-1$ coordinates of p.

2. The functions ρ_k are C^∞ in the Euclidean sense in all of $\mathbb{R}^{k-1}$, which we will henceforth write as $\rho_k \in C^\infty_{\text{eucl}}(\mathbb{R}^{k-1})$.

To retain consistent notation, we may also define X_1 relative to a function ρ_1 and decree $\rho_1 \equiv 1$.

Example 2.1 (The Grushin Plane)**.** Suppose that $n = 2$, define X_1 as before, and define

$$X_2(x_1, x_2) := x_1 \cdot \frac{\partial}{\partial x_2}.$$

These vector fields and the Lie Algebra they induce have been studied in, for example, [22], [23], and [2]. ■

Example 2.2. For any $n \geq 3$, if ρ_k is a polynomial in $x_1, \ldots, x_{k-1}$, then the vector fields $X_1, \ldots X_n$ defined relative to these ρ_k satisfy our assumptions. Solutions to the ∞-Laplace equation (see below and Chapter 3) in such spaces were studied in [4, 6], and in [12] the authors studied the fundamental solution of the p-Laplacian. ■

Example 2.3. Selecting $\rho_k(p) := \sin(x_1 + \cdots + x_{k-1})$ for each k, the frame $\mathfrak{X}$ satisfies our assumptions. ■

Setting $\mathfrak{g}(p) := \operatorname{span}\mathfrak{X}$, we produce a Lie Algebra which may be endowed with a singular (at points where at least one ρ_k is zero) inner-product $\langle\cdot,\cdot\rangle$ that makes $\mathfrak{X}$ an orthonormal basis for $\mathfrak{g}$. We may then define the Grushin-type space $\mathbb{G}$ to be the collection of all points $p = (x_1, \ldots, x_n) \in \mathbb{R}^n$ with tangent space $\mathfrak{g}(p)$ at p.

We may also define the *exponential mapping* for $\mathbb{G}$ by following the procedure outlined in the appendix for [30, pp. 141-146]. Fixing any $p \in \mathbb{G}$ and letting $\xi := (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{R}^n$, the initial value problem

$$\begin{cases} \gamma'(t) &= \displaystyle\sum_{i=1}^{n} \xi_i X_i(\gamma(t)) \\ \gamma(0) &= p \end{cases} \tag{2.3}$$

possesses a solution γ_p so long as we require that

$$\xi \in \operatorname{span}\{X_i(p) \in \mathfrak{X}(p) \,:\, X_i(p) \neq 0\},$$

where it should be noted that such a choice is always possible owing to the assumptions on the vector fields X_i. The exponential mapping is then defined to be $\Theta_p(\xi) := \gamma_p(1)$; near p the mapping Θ_p induces a system of *exponential coordinates* via its differential mapping $D\Theta_p$.

Example 2.4. Let $n = 2$ and

$$\begin{cases} X_1(x_1, x_2) &:= \dfrac{\partial}{\partial x_1} \\ X_2(x_1, x_2) &:= x_1^2 \cdot \dfrac{\partial}{\partial x_2}; \end{cases}$$

let $p = (x_1^p, x_2^p)$ and $\xi = (\xi_1, \xi_2) \in \operatorname{span}\{X_i(p) \in \mathfrak{X}(p) \,:\, X_i(p) \neq 0\}$ be given. (Note that for our case we must have $\xi \in \operatorname{span}\{X_1(p)\}$ if $x_1^p = 0$.) We will calculate $\Theta_p(\xi)$ explicitly.

Under the given assumptions, Equation (2.3) becomes

$$\begin{cases} \gamma'(t) &= \xi_1 \dfrac{\partial}{\partial x_1} + \xi_2 \left(x_1(t)\right)^2 \dfrac{\partial}{\partial x_2} \\ \gamma(0) &= (x_1^p, x_2^p). \end{cases}$$

We obtain the first coordinate $x_1(t)$ for $\gamma(t)$ by a straightforward integration:

$$x_1(t) = \int_0^t \xi_1 \, dt = \xi_1 t + x_1^p,$$

where we have applied the initial condition $\gamma(0) = (x_1^p, x_2^p)$. This implies that $x_2'(t) = \xi_2 \left(\xi_1 t + x_1^p\right)^2$; applying our initial condition once again,

$$x_2(t) = \int_0^t \xi_2 \left(\xi_1 s + x_1^p\right)^2 \, ds = \frac{\xi_1^2 \xi_2}{3} t^3 + \xi_1 \xi_2 x_1^p t + \xi_2 (x_1^p)^2 t + x_2^p.$$

We may now apply the definition of Θ_p to find that

$$\Theta_p(\xi) = \left(x_1^p + \xi_1, x_2^p + \frac{1}{3} \left(\xi_1^2 \xi_2 + 3\xi_1 \xi_2 x_1^p + 3\xi_2 (x_1^p)^2 \right) \right). \quad \blacksquare$$

2.2 Properties of Grushin-Type Spaces

Grushin-type spaces as defined above share certain properties which can be shown to hold *a priori*. Observe that if $\rho(p_0) = 0$ for some $k \geq 2$, then we will have $X_k(p_0) = 0$ and hence $\dim \mathfrak{g}(p) < n$. As a consequence of dimension of the tangent space at a point relying on the point itself, it follows that $\mathbb{G}$ is not a group.

Additionally, given two points $p, q \in \mathbb{G}$, it can be shown that there exists a *horizontal curve* connecting p and q – more precisely, there exists $\gamma : [0,1] \to \mathbb{G}$ such that $\gamma(0) = p, \gamma(1) = q$, and $\gamma'(t) \in \mathfrak{g}\left(\gamma(t)\right)$ for $t \in (0,1)$. In the case of vector fields such as in Examples 2.1 and 2.2, the existence of such curves can be shown as a consequence of *Chow's Theorem*. Recalling that for given vector fields A, B we define their Lie Bracket to be the vector field $[A, B] := AB - BA$, note that if $k < \ell$ then direct calculation yields

$$\begin{aligned} [X_k, X_\ell](p) &= \rho_k(x_1, \ldots, x_{k-1}) \cdot \frac{\partial \rho_\ell}{\partial x_k}(x_1, \ldots, x_{\ell-1}) \cdot \frac{\partial}{\partial x_\ell} \\ &= \left(\rho_k \frac{\partial \rho_\ell}{\partial x_k} \right)(p) \cdot \frac{\partial}{\partial x_\ell}. \end{aligned} \tag{2.4}$$

If there exists some finite iteration of brackets $[X_{k_1}, [X_{k_2}, [\cdots [X_{k_r}, X_\ell] \cdots]]]$ resulting in a nonzero coefficient on $\partial / \partial x_\ell$ for each point $p \in \mathbb{G}$ (e.g. the cases of vector fields defined by polynomials as in Examples 2.1 and

2.2), then $\mathbb{G}$ is said to satisfy *Hörmander's Condition*:

$$\begin{gathered}\text{The vector fields } X_1, \ldots, X_n \text{ together with the iterated Lie Brackets} \\ [X_k, X_\ell], [X_j, [X_k, X_\ell]], \Big[X_i, [X_j, [X_k, X_\ell]]\Big], \ldots \text{ span } \mathbb{R}^n.\end{gathered} \tag{H}$$

We then apply the following:

Theorem 2.5 (Chow's Theorem)**.** *Let M be a connected manifold with tangent space $\mathfrak{m} = \operatorname{span}\{Y_1, \ldots, Y_n\}$ such that $Y_1, \ldots, Y_n$ satisfy Hörmander's Condition. Then each pair of points in M are connected by a horizontal curve.*

In general, however, it may be that $\mathbb{G}$ fails to satisfy Hörmander's Condition. In such cases, Chow's Theorem does not apply; however, since $X_1 \neq 0$ for all p, it is always possible to construct piecewise horizontal curves between points, concatenating as necessary to connect points. (See the example on [2, p. 18] and the discussion appearing on [29, p. 355].)

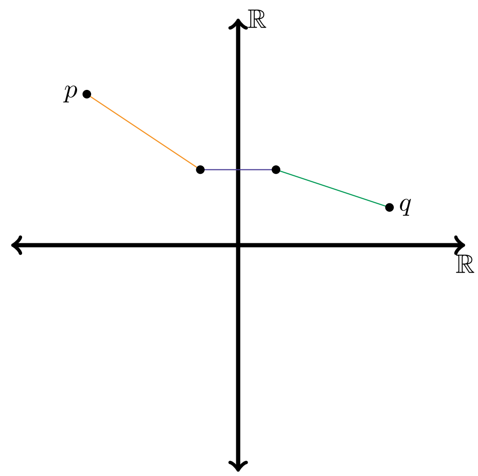

Figure 1: An example of piecewise horizontal curves in the space defined in Example 2.1.

With the existence of horizontal curves between points of $\mathbb{G}$ established, we may now define a notion of distance on $\mathbb{G}$ which respects the geometry of $\mathbb{G}$, called the *Carnot-Caratheodory metric* (more simply, the CC-metric), and which we denote by $d_{CC}(\cdot, \cdot)$. For a given pair of points $p, q \in \mathbb{G}$, we write $\Gamma_{p,q}$ to denote the collection of all horizontal curves γ connecting p and q – as indicated above, it is known that $\Gamma_{p,q} \neq \varnothing$. Recalling the singular inner-product $\langle \cdot, \cdot \rangle$ and defining $\|Y\|_{\mathfrak{X}} := \langle Y, Y \rangle^{1/2}$ for members $Y \in \mathfrak{g}$, we then define

$$d_{CC}(p, q) := \inf_{\Gamma_{p,q}} \int_0^1 \|\gamma'(t)\|_{\mathfrak{X}} \, dt. \tag{2.5}$$

Owing to the fact that $\Gamma_{p,q}$ is nonempty, we see that the CC-metric is an *honest metric* – i.e. $d_{CC}(p,q) < \infty$ for all points $p, q \in \mathbb{G}$.

In the case that $\mathbb{G}$ does satisfy Hörmander's Condition (H), such as in the setting of Examples 2.1 and 2.2, we are able to estimate $d_{CC}(\cdot,\cdot)$ locally as in [4] and [5]. Fixing $p_0 = (x_1^0, x_2^0, \ldots, x_n^0) \in \mathbb{G}$, let $r_i^{p_0}$ denote the minimal length of the Lie bracket iteration (that is, the least integer) such that

$$\Big[X_{j_1}, [X_{j_2}, \cdots [X_{j_{r_i^{p_0}}}, X_i] \cdots]\Big] \neg 0,$$

which is finite since Hörmander's Condition holds. By its definition, $r_i^{p_0}$ is a function of p_0 and is unique – although iterations of length $r_i^{p_0}$ may not be unique. Moreover, $\rho_i(p_0)$ is nonzero precisely when $r_i^{p_0} = 0$. Defining $R_i(p_0) := r_i^{p_0} + 1$, we may apply [2, Theorem 7.34] to obtain the comparison

$$d_{CC}(p_0, p) \sim \sum_{i=1}^{n} |x_i - x_i^0|^{1/R_i(p_0)}$$

for p near p_0; this similarity permits us to define a smooth gauge function which is comparable to the CC-metric:

$$(\mathcal{N}(p_0,p))^{2\mathcal{R}} := \sum_{i=1}^{n} (x_i - x_i^0)^{2\mathcal{R}/R_i(p_0)},$$

where $\mathcal{R} := R_1(p_0) \cdot R_2(p_0) \cdots R_n(p_0)$.

The spaces $\mathbb{G}$ are geodesic metric spaces and, in the case of the Grushin plane (Example 2.1), one may explicitly calculate parametric equations for the geodesics. The following proposition is a summary of two cases presented in [16].

Proposition 2.6 (C.f. [16, pp. 804-807])**.** *Let $n = 2$ so that points of $\mathbb{G}$ are ordered pairs $(x_1, x_2) \in \mathbb{R}^2$ and let*

$$\begin{cases} X_1(x_1,x_2) & := & \dfrac{\partial}{\partial x_1} \\ \\ X_2(x_1,x_2) & := & x_1 \cdot \dfrac{\partial}{\partial x_2} \end{cases}$$

be the vector fields which define $\mathfrak{g}$. Then, writing $p = (x_1^p, x_2^p)$ and $q = (x_1^q, x_2^q)$, we have the following cases:

1. *If $x_2^p = y = x_2^q$, then the curve $\gamma(t) := \Big(x_1(t), x_2(t)\Big)$ connecting p and q given by*

$$\begin{cases} x_1(t) & = & (x_1^q - x_1^p)t + x_1^p \\ x_2(t) & = & y \end{cases} \tag{2.6}$$

is the unique geodesic.

2. *If $x_1^p = 0 = x_1^q$, $x_2^p = 0$, and $x_2^q = h$ for some $h \neq 0$, then the curve $\gamma(t) := \Big(x_1(t), x_2(t)\Big)$ connecting p and q given by*

$$\begin{cases} x_1(t) & := & \dfrac{A}{B}\sin(Bt) \\ \\ x_2(t) & := & \dfrac{A^2}{B}\left(\dfrac{t}{2} - \dfrac{\sin(2Bt)}{4B}\right), \end{cases} \tag{2.7}$$

for some constants A, B, is a geodesic.

The proof is an application of *Pontryagin's Maximum Principle*: By defining the normalized Hamiltonian

$$H\Big((x_1, x_2), (\xi, \eta)\Big) := \frac{1}{2}\left(\xi^2 + x_1^2\eta^2\right),$$

(where ξ, η are the variables dual to x_1, x_2 respectively), the maximum principle asserts that the system of equations

$$\begin{cases} x_1'(t) & = \dfrac{\partial H}{\partial \xi} & = \xi(t) \\ \\ x_2'(t) & = \dfrac{\partial H}{\partial \eta} & = x_1^2(t)\cdot\eta(t) \\ \\ \xi'(t) & = -\dfrac{\partial H}{\partial x_1} & = -x_1(t)\cdot\eta^2(t) \\ \\ \eta'(t) & = -\dfrac{\partial H}{\partial x_2} & = 0 \end{cases} \tag{2.8}$$

holds, as do the initial conditions $x_1(0) = x_1^p, x_1(1) = x_1^q, x_2(0) = x_2^p$, and $x_2(1) = x_2^q$. We may observe at once that η is a constant from (2.8). The proofs of the cases can now be summarized as follows:

1. Since $x_2^p = x_2^q$ implies $\eta = 0$, we infer that ξ is a constant and integrate the first equation of (2.8),

2. Noting the relationship between the first and third lines of (2.8), we recover the ODE

$$x_1''(t) = \xi'(t) = -\eta^2\cdot x_1(t) \iff x_1''(t) + \eta^2\cdot x_1(t) = 0. \tag{2.9}$$

   It is solvable by standard techniques.

It should be noted that the approach outlined above is fragile in the sense that minor changes to the vector fields selected can lead to systems of equations for which solutions are not as easily obtained. Indeed, if we define

$$\begin{cases} X_1(x_1,x_2) & := & \dfrac{\partial}{\partial x_1} \\ \\ X_2(x_1,x_2) & := & x_1^2 \cdot \dfrac{\partial}{\partial x_2}, \end{cases}$$

then the equations in (2.9) are replaced instead by

$$x_1''(t) = \xi'(t) = -\eta^2 \cdot x_1^2(t) \quad \Longleftrightarrow \quad x_1''(t) + \eta^2 \cdot x_1^2(t) = 0.$$

Solving such an equation goes beyond the scope of this dissertation.

Utilizing the CC-metric, we define Grushin-type balls

$$B(p,r) := \{q \in \mathbb{G} \, : \, d_{CC}\,(p,q) < r\}\,.$$

Open subsets $\mathcal{O} \subseteq \mathbb{G}$, domains, and bounded domains (which we shall typically denote by $\Omega \Subset \mathbb{G}$) can then be defined in the expected manner. An example of a ball $B((0,0),r)$ in the setting of Example 2.1 is shown below; the sketch is produced by following the exposition of [2, Subsection 3.4].

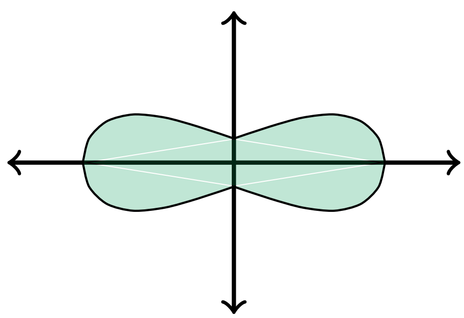

Figure 2: A sketch of a Grushin ball with center at the origin in the case of Example 2.1.

Note that it is (Euclidean) elliptical and non-smooth.

2.3 Calculus in Grushin-Type Spaces

The natural replacement for the Euclidean directional derivatives $\partial/\partial x_k$ are the members $X_k(p)$ of $\mathfrak{X}(p)$. Given a domain $\mathcal{O} \subseteq \mathbb{G}$, $p \in \mathcal{O}$, and function $f : \mathcal{O} \to \mathbb{R}$, the derivative of f at p in the direction of $X_k(p)$ is defined, as in [20], by

$$X_k f(p) := \frac{d}{ds} f\left(\Theta_p(se_k)\right)\bigg|_{s=0} = \lim_{s\to 0} \frac{f(\Theta_p(se_k)) - f(p)}{s}, \tag{2.10}$$

where we have assumed that $X_k(p) \neq 0$ and denoted the k-th coordinate vector by e_k for convenience; should it happen that $X_k(p) = 0$, then we define $X_k f(p) := 0$. If $w : \mathcal{O} \to \mathbb{R}$ is sufficiently smooth (in the Euclidean sense), we define the $\mathbb{G}$ gradient to be

$$\nabla_{\mathbb{G}} w(p) := (X_1 w(p), \ldots, X_n w(p)).$$

Recalling Equation (2.4), we may observe that since $[X_k, X_\ell](p) \neq 0$ for $k < \ell$ and some $p \in \mathbb{G}$, the second-order partial derivatives $X_k X_\ell w$ and $X_\ell X_k w$ are not necessarily equal. Consequently, we introduce a *symmetrized Hessian for w at p*, $\left(D^2 w\right)^\star(p)$, whose (i,j)-th entry (where $1 \leq i, j \leq n$) is given by

$$\left[\left(D^2 w\right)^\star(p)\right]_{ij} := \frac{1}{2}\Big(X_i X_j w(p) + X_j X_i w(p)\Big).$$

These notions of derivatives relative to the geometry of $\mathbb{G}$ also allow for notions of regularity in $\mathbb{G}$.

Definition 2.7. A function $u : \mathcal{O} \to \mathbb{R}$ is said to belong to $C^1_{\mathbb{G}}(\mathcal{O})$ if each first-order partial $X_k w$ exists and is continuous in $\mathcal{O}$ for all $k \leq n$. If the second-order partials $X_i X_j u$ exist and are continuous in $\mathcal{O}$ for all $1 \leq i, j \leq n$, then we say that u belongs to $C^2_{\mathbb{G}}(\mathcal{O})$.

Comparison of the vector fields $\partial/\partial x_k$ and X_k shows that if u is continuously differentiable in the Euclidean sense α times for $\alpha = 1, 2$, which we write as $u \in C^\alpha_{\text{eucl}}(\mathcal{O})$, then $u \in C^\alpha_{\mathbb{G}}(\mathcal{O})$ as well. Hence we arrive at the containment $C^\alpha_{\text{eucl}}(\mathcal{O}) \subseteq C^\alpha_{\mathbb{G}}(\mathcal{O})$. The reverse containment, however, is not guaranteed.

Example 2.8. Let $n = 2$ and X_1, X_2 be as in Example 2.1; consider $u(x_1, x_2) := x_2^{1/3}$. Clearly, $u(x_1, x_2)$ does not possess a Euclidean derivative at the origin $\overline{0} := (0,0)$ – we will show that u possesses a Grushin derivative at $\overline{0}$.

Notice that for all points on the x_2-axis, we have $X_2 u = 0$. To calculate $X_1 u(\overline{0})$ from (2.10), write $\xi = s e_1 = s \cdot \partial/\partial x_1$ and observe Equation (2.3) becomes

$$\begin{cases} \gamma'(t) &= s X_1(\gamma(t)) \\ \gamma(0) &= \overline{0}. \end{cases} \tag{2.11}$$

Denoting the solution of Equation (2.11) by $\gamma_{\overline{0}}(t) = \Big(x_1(t), x_2(t)\Big)$, the uniqueness of $\gamma_{\overline{0}}$ and a straightforward integration implies that $x_1(t) = st$ and $x_2 \equiv 0$. From this and our definition of the partial derivatives we obtain

$$X_1 u(\overline{0}) = \frac{d}{ds} u\left(\Theta_{\overline{0}}(s e_1)\right)\Big|_{s=0} = \frac{d}{ds} u(s,0)\Big|_{s=0} = \frac{d}{ds} 0\Big|_{s=0} = 0. \quad \blacksquare$$

If $Y = y_1X_1 + y_2X_2 + \cdots + y_nX_n \in \mathfrak{g}$ is a smooth vector field, the divergence of Y relative to $\mathfrak{X}$ is given by

$$\operatorname{div}(Y(p)) := \sum_{i=1}^{n} X_i y_i(p).$$

Hence, for example, if $f : \mathcal{O} \to \mathbb{R}$ is a smooth function, then

$$\operatorname{div}(\nabla_{\mathbb{G}} w(p)) = \sum_{i=1}^{n} X_i X_i w(p) = \operatorname{tr}\left(\left(D^2 w\right)^{\star}(p)\right).$$

These definitions permit us to define the operators which are of primary significance to the forgoing sections: For smooth $w : \mathcal{O} \to \mathbb{R}$, the $\mathtt{p}$-Laplace operator

$$\begin{aligned}
\Delta_{\mathfrak{X},\mathtt{p}} w(p) &:= -\operatorname{div}\left(\|\nabla_{\mathbb{G}} w(p)\|^{\mathtt{p}-2} \nabla_{\mathbb{G}} w(p)\right) \\
&= -\Big(\|\nabla_{\mathbb{G}} w(p)\|^{\mathtt{p}-2} \operatorname{tr}\left(\left(D^2 w\right)^{\star}(p)\right) \\
&\quad + (\mathtt{p}-2)\|\nabla_{\mathbb{G}} w(p)\|^{\mathtt{p}-4} \left\langle \left(D^2 w\right)^{\star}(p) \cdot \nabla_{\mathbb{G}} w(p), \nabla_{\mathbb{G}} w(p) \right\rangle \Big)
\end{aligned}$$

for $1 < \mathtt{p} < \infty$, and the ∞-Laplace operator

$$\Delta_{\mathfrak{X},\infty} w(p) := -\left\langle \left(D^2 w\right)^{\star}(p) \cdot \nabla_{\mathbb{G}} w(p), \nabla_{\mathbb{G}} w(p) \right\rangle.$$

Although the focus of this thesis is on solutions of Dirichlet and Cauchy-Dirichlet problems involving the ∞-Laplacian, the former class of problems will utilize solutions of the $\mathtt{p}$-Laplace equations to produce the desired solutions of the ∞-Laplace equations. Exact notions of solutions to equations involving these operators will be discussed in detail in upcoming sections, but we will have need of the function spaces $L^{\mathtt{r}}, L^{\mathtt{r}}_{\text{loc}}, W^{1,\mathtt{r}}, W^{1,\mathtt{r}}_{\text{loc}}$, and $W^{1,\mathtt{r}}_0$. Their definitions mimic those of their Euclidean counterparts, replacing the Euclidean gradient Dw with the Grushin-type gradient $\nabla_{\mathbb{G}} w$.

CHAPTER 3:
SUBELLIPTIC VISCOSITY SOLUTIONS

Recalling the (non-divergence form) definitions of the $\mathtt{p}$- and ∞-Laplacian, it should be noted that both may be treated as functions whose inputs are triplets $(p, \eta, X) \in \mathbb{G} \times \mathfrak{g} \times \mathcal{S}^n$, where we have used the symbol $\mathcal{S}^n$ to denote the collection of all $n \times n$ symmetric matrices with real entries. That is, we may define

$$\mathcal{F}_{\mathtt{p}}(p, \eta, X) := -\left(\|\eta\|^{\mathtt{p}-2}\mathrm{tr}\,(X) + (\mathtt{p}-2)\|\eta\|^{\mathtt{p}-4}\langle X \cdot \eta, \eta\rangle\right)$$

for each $1 < \mathtt{p} < \infty$ and

$$\mathcal{F}_{\infty}(p, \eta, X) := -\langle X \cdot \eta, \eta\rangle\,.$$

Both operators satisfy a property which is called *degenerate elliptic* in [18]: Specifically, if $X, Y \in \mathcal{S}^n$ so that $X \leq Y$ (that is, so that $Y - X$ is *positive semidefinite*), then

$$\mathcal{F}_{\mathtt{p}}(p, \eta, Y) \leq \mathcal{F}_{\mathtt{p}}(p, \eta, X)$$

and

$$\mathcal{F}_{\infty}(p, \eta, Y) \leq \mathcal{F}_{\infty}(p, \eta, X)$$

for all $(p, \eta) \in \mathbb{G} \times \mathfrak{g}$.

For technical reasons, it will be necessary to ensure that certain gradients appearing in forthcoming existence arguments are nonzero. For this reason, we introduce two *auxiliary operators* which are due to [24]: Given $\kappa \in \mathbb{R}$, we define

$$\mathcal{F}^{\kappa}(p, \eta, X) := \min\left\{\|\eta\|^2 - \kappa^2,\, \mathcal{F}_{\infty}(p, \eta, X)\right\}$$

and

$$\mathcal{G}^{\kappa}(p, \eta, X) := \max\left\{\kappa^2 - \|\eta\|^2,\, \mathcal{F}_{\infty}(p, \eta, X)\right\}.$$

Clearly, $\mathcal{F}^\kappa$ and $\mathcal{G}^\kappa$ map $\mathbb{G}\times\mathfrak{g}\times S^n$ into $\mathbb{R}$ and we may treat them as operators via the definitions

$$\begin{cases} \mathcal{F}^\kappa w(p) & := & \mathcal{F}^\kappa(p, \nabla_{\mathbb{G}}\, w, \left(D^2 w\right)^\star) \\ \mathcal{G}^\kappa w(p) & := & \mathcal{G}^\kappa(p, \nabla_{\mathbb{G}}\, w, \left(D^2 w\right)^\star). \end{cases}$$

In light of the properties of $\mathcal{F}_\infty$, we may also infer that both operators are degenerate elliptic. To simplify notation, we will therefore write $\mathcal{H}:\mathbb{G}\times\mathfrak{g}\times S^n\to\mathbb{R}$ to denote any of the above four operators in what follows. We also record the below homogeneous subelliptic equation, which is the focus of this and the forthcoming section.

$$\mathcal{H}w(p) := \mathcal{H}(p, \nabla_{\mathbb{G}}\, w, \left(D^2 w\right)^\star) = 0. \tag{3.1}$$

To begin, fixing some open $\mathcal{O}\subseteq\mathbb{G}$ and any $p_0=(x_1^0, x_2^0,\ldots,x_n^0)\in\mathcal{O}$, define $N(p_0)$ to be the collection of indices j such that $\rho_j(p_0)=0$. Assume that $w:\mathcal{O}\to\mathbb{R}$ is a given smooth function and define

$$\begin{aligned} T_{p_0,w}(p) \;\; := \;\; & w(p_0) + \sum_{k\notin N(p_0)} \frac{1}{\rho_k(p_0)} X_k w(p_0)\cdot(x_k - x_k^0) + \frac{1}{2}\sum_{k\notin N(p_0)} \frac{1}{\rho_k^2(p_0)} X_k X_k w(p_0)\cdot(x_k-x_k^0)^2 \\ & + \sum_{\substack{k,\ell\notin N(p_0)\\ k<\ell}} \left(\frac{1}{(\rho_k\rho_\ell)(p_0)}\cdot\frac{X_\ell X_k w + X_k X_\ell w}{2}(p_0)\right. \\ & \left. - \frac{1}{2\rho_\ell^2(p_0)}\cdot\frac{\partial\rho_\ell}{\partial x_k}(p_0)\cdot X_\ell w(p_0)\right)\cdot(x_k-x_k^0)(x_\ell-x_\ell^0) + \sum_{j\in N(p_0)}\frac{\partial w}{\partial x_j}(p_0)\cdot(x_j-x_j^0). \end{aligned}$$

The function $T_{p_0,w}$ is the equivalent notion of Taylor Polynomials in the Grushin-type setting and we employ it in the two following results. We present proofs which emulate those presented in [4].

Proposition 3.1 (C.f. [4, Proposition 2.1])**.** *If $w\in C^2_{\mathbb{G}}(\mathcal{O})$ and $p_0\in\mathcal{O}$, then*

$$w(p) = T_{p_0,w}(p) + o\left(d_{CC}^2(p_0,p)\right) \text{ as } p\to p_0$$

and the equations $X_a w(p_0) = X_a T_{p_0,w}(p_0)$ and $X_b X_a w(p_0) = X_b X_a T_{p_0,w}(p_0)$ hold for all $a,b\le n$.

Proof. Let $a,b\notin N(p_0)$. A direct calculation yields

$$\begin{aligned} X_a T_{p_0,w}(p) \;\; = \;\; & \rho_a(p)\left(\frac{1}{\rho_a(p_0)}X_a w(p_0) + \rho_a(p)\frac{1}{\rho_a^2(p_0)}X_a X_a w(p_0)\cdot(x_a-x_a^0)\right. \\ & + \sum_{k<a}\left(\frac{1}{(\rho_k\rho_a)(p_0)}\cdot\frac{X_a X_k w + X_k X_a w}{2}(p_0) - \frac{1}{2\rho_a^2(p_0)}\cdot\frac{\partial\rho_a}{\partial x_k}(p_0)\cdot X_a w(p_0)\right)\cdot(x_k-x_k^0) \\ & \left. + \sum_{a<\ell}\left(\frac{1}{(\rho_a\rho_\ell)(p_0)}\cdot\frac{X_\ell X_a w + X_a X_\ell w}{2}(p_0) - \frac{1}{2\rho_\ell^2(p_0)}\cdot\frac{\partial\rho_\ell}{\partial x_a}(p_0)\cdot X_\ell w(p_0)\right)\cdot(x_\ell-x_\ell^0)\right). \end{aligned}$$

Letting $p = p_0$, the right-hand side of the above equation reduces to $X_a w(p_0)$. Similarly, the calculation

$$X_a X_a T_{p_0,w}(p) = \rho_a^2(p) \frac{1}{\rho_a^2(p_0)} X_a X_a w(p_0)$$

shows that $X_a X_a T_{p_0,w}(p_0) = X_a X_a w(p_0)$. Splitting the remaining second-order partials into separate cases, consider first the case where $b < a$. Then from our previous calculations we obtain

$$\begin{aligned} X_b X_a T_{p_0,w}(p) \quad &= \quad (\rho_b \rho_a)(p) \left(\frac{1}{(\rho_b \rho_a)(p_0)} \cdot \frac{X_a X_b w + X_b X_a w}{2}(p_0) - \frac{1}{2\rho_a^2(p_0)} \cdot \frac{\partial \rho_a}{\partial x_b}(p_0) \cdot X_a w(p_0) \right) \\ &\quad + \rho_b(p) \frac{\partial \rho_a}{\partial x_b}(p) \cdot \frac{X_a T_{p_0,w}(p)}{\rho_a(p)}, \end{aligned}$$

which, together with Equation (2.4), implies

$$\begin{aligned} X_b X_a T_{p_0,w}(p_0) \quad &= \quad \frac{X_a X_b w + X_b X_a w}{2}(p_0) - \frac{1}{2} \cdot \frac{\partial \rho_a}{\partial x_b}(p_0) \cdot \frac{\partial w}{\partial x_a}(p_0) + \rho_b(p_0) \frac{\partial \rho_a}{\partial x_b}(p_0) \cdot \frac{\partial w}{\partial x_a}(p_0) \\ &= \quad \frac{1}{2} \left(X_a X_b w + X_b X_a w \right)(p_0) + \frac{1}{2}[X_b, X_a] w(p_0) \\ &= \quad X_b X_a w(p_0). \end{aligned}$$

If we now take $a < b$, observe that $\partial \rho_a / \partial x_b \equiv 0$ and hence

$$X_b X_a T_{p_0,w}(p) = (\rho_b \rho_a)(p) \left(\frac{1}{(\rho_b \rho_a)(p_0)} \cdot \frac{X_a X_b w + X_b X_a w}{2}(p_0) - \frac{1}{2\rho_a^2(p_0)} \cdot \frac{\partial \rho_b}{\partial x_a}(p_0) \cdot X_b w(p_0) \right).$$

From the previous equation,

$$\begin{aligned} X_b X_a T_{p_0,w}(p_0) \quad &= \quad \frac{X_a X_b w + X_b X_a w}{2}(p_0) - \frac{1}{2} \cdot \frac{\partial \rho_b}{\partial x_a}(p_0) \cdot \frac{\partial w}{\partial x_b}(p_0) \\ &= \quad \frac{1}{2} \left(X_a X_b w + X_b X_a w \right)(p_0) - \frac{1}{2}[X_b, X_a] w(p_0) \\ &= \quad X_b X_a w(p_0). \end{aligned}$$

Now let $r, s \in N(p_0)$ and notice

$$X_r T_{p_0,w}(p) = \rho_r(p) \frac{\partial w}{\partial x_r}(p_0) \text{ and } X_s X_r T_{p_0,w}(p) = \rho_s(p) \frac{\partial X_r T_{p_0,w}}{\partial x_s}(p),$$

from which $X_r T_{p_0,w}(p_0) = X_s X_r T_{p_0,w}(p_0) = 0$. If a is, once again, an index failing to belong to $N(p_0)$, we have the mixed second-order partials

$$\begin{cases} X_a X_r T_{p_0,w}(p) &= \left(\rho_a \cdot \dfrac{\partial \rho_r}{\partial x_a}\right)(p) \cdot \dfrac{\partial w}{\partial x_r}(p_0) \\ \\ X_r X_a T_{p_0,w}(p) &= \rho_r(p) \dfrac{\partial}{\partial x_r}\left(X_a w(p)\right). \end{cases} \tag{3.2}$$

The assumption $r \in N(p_0)$ and the second line of (3.2) immediately imply $X_r X_a T_{p_0,w}(p_0) = 0 = X_r X_a w(p_0)$; since we also have

$$\begin{aligned} X_a X_r w(p_0) &= \left(\rho_a \cdot \frac{\partial \rho_r}{\partial x_a}\right)(p) \cdot \frac{\partial w}{\partial x_r}(p_0) + (\rho_a \rho_r)(p_0) \cdot \frac{\partial^2 w}{\partial x_a x_r}(p_0) \\ &= \left(\rho_a \cdot \frac{\partial \rho_r}{\partial x_a}\right)(p) \cdot \frac{\partial w}{\partial x_r}(p_0), \end{aligned}$$

from which we have $X_a X_r T_{p_0,w}(p_0) = X_a X_r w(p_0)$. The error term $o\left(d_{CC}^2(p_0, p)\right)$ as $p \to p_0$ results from applying [2, Proposition 4.10]. □

Proposition 3.2 (C.f. [4, Proposition 3.1]). *If $j \in N(p_0)$, then*

$$\frac{\partial w}{\partial x_j}(p_0) = \frac{1}{\beta_j(p_0)} \sum_{k=1}^{n} \frac{2}{\left(\frac{\partial \rho_j}{\partial x_k} \rho_k\right)(p_0)} \cdot \frac{X_k X_j w + X_j X_k w}{2}(p_0). \tag{3.3}$$

In the above, we have utilized the convention that if $\rho_k(p_0) = 0$ then the associated term is also 0; moreover, we denote by $\beta_j(p_0)$ the number of nonzero terms in the sum (which is necessarily at least 1).

Proof. With j as above, let $i \leq n$ be any index. The work of our previous proof actually shows that

$$\begin{cases} X_j X_i w(p_0) &= 0 \\ \\ X_i X_j w(p_0) &= \left(\rho_i \cdot \dfrac{\partial \rho_j}{\partial x_i}\right)(p_0) \cdot \dfrac{\partial w}{\partial x_j}(p_0); \end{cases}$$

by averaging these two lines, we obtain

$$\frac{X_j X_i w + X_i X_j w}{2}(p_0) = \frac{1}{2}\left(\rho_i \cdot \frac{\partial \rho_j}{\partial x_i}\right)(p_0) \cdot \frac{\partial w}{\partial x_j}(p_0). \tag{3.4}$$

Letting $i_1, \ldots, i_m$ be the indices for which the first factor of the right-hand side of Equation (3.4) are nonzero, Equation (3.3) is obtained by solving for $\partial w / \partial x_j(p_0)$ in (3.4) for $i_1, \ldots, i_m$, then summing the results and dividing by $\beta_j(p_0)$. If no such indices exist, then the right-hand sides of (3.3) and (3.4) are both 0. □

We may now define Grushin-type jets in a manner similar to the definition presented in [18], replacing the Euclidean Taylor expansion by the Grushin-type Taylor Expansion.

Definition 3.3. Let $u : \mathcal{O} \to \mathbb{R}$ for some open $\mathcal{O} \subseteq \mathbb{G}$ and let $p_0 \in \mathcal{O}$. Then we say that $(\eta, X) \in \mathbb{R}^n \times \mathcal{S}^n$ is a member of the *upper subelliptic jet* for u at p_0, written $(\eta, X) \in J^{2,+}u(p_0)$, if

$$\begin{aligned} u(p) \quad &\leq \quad T_{p_0,\eta,X}(p) + o\left(d_{CC}^2(p_0,p)\right) \\ &:= \quad u(p_0) + \sum_{k \notin N(p_0)} \frac{1}{\rho_k(p_0)}\eta_k \cdot (x_k - x_k^0) + \frac{1}{2}\sum_{k \notin N(p_0)} \frac{1}{\rho_k^2(p_0)} X_{kk} \cdot (x_k - x_k^0)^2 \\ &\quad + \sum_{\substack{k,\ell \notin N(p_0) \\ k<\ell}} \left(\frac{1}{(\rho_k\rho_\ell)(p_0)} \cdot X_{k\ell} - \frac{1}{2\rho_\ell^2(p_0)} \cdot \frac{\partial \rho_\ell}{\partial x_k}(p_0) \cdot \eta_\ell \cdot (x_k - x_k^0)(x_\ell - x_\ell^0) \right) \\ &\quad + \sum_{j \in N(p_0)} \left(\frac{1}{\beta_j(p_0)} \sum_{k=1}^{n} \frac{2}{\left(\frac{\partial \rho_j}{\partial x_k}\rho_k\right)(p_0)} \cdot X_{kj} \right) \cdot (x_j - x_j^0) + o\left(d_{CC}^2(p_0,p)\right), \end{aligned} \tag{3.5}$$

where we assume the above holds as $p \to p_0$. The case where the pair (η, X) belongs to the *lower subelliptic jet* for u at p_0, written $(\eta, X) \in J^{2,-}u(p_0)$, is defined similarly, reversing the inequality in (3.5); alternatively, we may observe that $J^{2,-}u(p_0) = -J^{2,+}(-u)(p_0)$.

The *upper subelliptic jet closure* $\overline{J}^{2,+}u(p_0)$ for u at p_0 is defined to be all those $(\eta, X) \in \mathbb{R}^n \times \mathcal{S}^n$ for which there are $(p_n) \subset \mathcal{O}$ with $(\eta_n, X_n) \in J^{2,+}u(p_n)$ satisfying

$$\Big(p_n, u(p_n), \eta_n, X_n\Big) \to \Big(p_0, u(p_0), \eta, X\Big) \text{ as } n \to \infty.$$

A similar definition is made for the *lower subelliptic jet closure* $\overline{J}^{2,-}u(p_0)$.

This leads us to our first definition of the notion of viscosity (sub-/super-)solutions, which is similar to the definition given in [18].

Definition 3.4. Let $\Omega \Subset \mathbb{G}$ be a bounded domain and suppose that $u : \Omega \to \mathbb{R}$ is *upper semicontinuous* – written $u \in \text{USC}(\Omega)$. We say that u is a *viscosity subsolution* of Equation (3.1) if for each $p_0 \in \Omega$ and every pair $(\eta, X) \in J^{2,+}u(p_0)$ we have

$$\mathcal{H}(p_0, \eta, X) \leq 0.$$

If $v : \Omega \to \mathbb{R}$ is *lower semicontinuous* (briefly, $v \in \text{LSC}(\Omega)$), then we say that v is a *viscosity supersolution* to Equation (3.1) if for each $p_0 \in \Omega$ and every pair $(\chi, Y) \in J^{2,-}v(p_0)$ we have

$$\mathcal{H}(p_0, \chi, Y) \geq 0$$

– or more succinctly, due to the relationship between the subelliptic jets, v is a viscosisty supersolution if the upper semicontinuous function $-v$ is a viscosity subsolution.

A continuous function $w : \Omega \to \mathbb{R}$ is a *viscosity solution* to Equation (3.1) if it is both a viscosity sub- and supersolution.

Remark 3.5. In the case that $\mathcal{H} = \mathcal{F}_\infty$, we shall use the term ∞-(sub-/super-)harmonic to refer to the viscosity (sub-/super-)solutions of Equation (3.1).

Remark 3.6. In the case that $\mathcal{H} = \mathcal{F}_{\mathtt{p}}$, care needs to be taken in the $\mathtt{p} < 2$ case due to the singularity which occurs when $\|\nabla_{\mathbb{G}} w\| = 0$; however, since our aim is to use viscosity solutions of the $\mathtt{p}$-Laplacian to produce an ∞-harmonic function, we only concern ourselves with the case $\mathtt{p} \geq 2$.

One can also define viscosity solutions via a class of test functions. Given $u : \mathcal{O} \to \mathbb{R}$, we define the classes of *touching above functions* and *touching below functions* for u as follows: Fixing any point $p_0 \in \mathcal{O}$, we say that the $C^2_{\mathbb{G}}(\mathcal{O})$ function ψ touches u from above at p_0, written $\psi \in \mathcal{TA}(u, p_0)$, if

$$0 = \psi(p_0) - u(p_0) \leq \psi(p) - u(p) \text{ for } p \text{ near } p_0;$$

similarly, we say that the $C^2_{\mathbb{G}}(\mathcal{O})$ function ϕ touches u from below at p_0, written $\phi \in \mathcal{TB}(u, p_0)$, if

$$0 = u(p_0) - \phi(p_0) \leq u(p) - \phi(p) \text{ for } p \text{ near } p_0.$$

These touching functions can be thought of as those $C^2_{\mathbb{G}}$ functions which touch u at a given point and nearby behave in a "locally parabolic" manner.

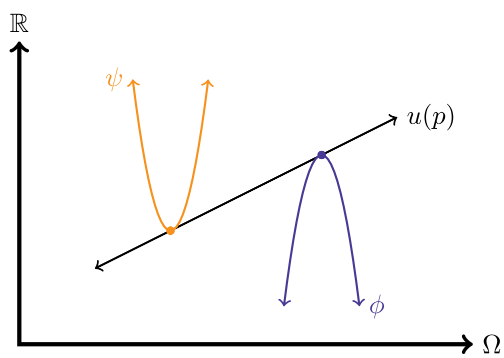

Figure 3: Visualizing the relationship between u and touching functions.

In the case that u is $C^2_{\mathbb{G}}$, a straightforward application of Calculus implies at once that if $\psi \in \mathcal{TA}(u, p_0)$, then

$$\begin{cases} \nabla_{\mathbb{G}}\, u(p_0) &= \nabla_{\mathbb{G}}\, \psi(p_0) \\ \left(D^2 u\right)^{\star}(p_0) &\geq \left(D^2 \psi\right)^{\star}(p_0); \end{cases} \tag{3.6}$$

similarly, if $\phi \in \mathcal{TB}(u, p_0)$, then

$$\begin{cases} \nabla_{\mathbb{G}}\, u(p_0) &= \nabla_{\mathbb{G}}\, \phi(p_0) \\ \left(D^2 u\right)^{\star}(p_0) &\leq \left(D^2 \phi\right)^{\star}(p_0). \end{cases} \tag{3.7}$$

These comparisons, together with the degenerate ellipticity of the operator $\mathcal{H}$ lead to the definition of viscosity solutions via touching functions.

Definition 3.7. Let $\Omega \Subset \mathbb{G}$ be a bounded domain and suppose that $u : \Omega \to \mathbb{R}$ is *upper semicontinuous* – written $u \in \mathrm{USC}(\Omega)$. We say that u is a *viscosity subsolution* of Equation (3.1) if for each $p_0 \in \Omega$ and every $\psi \in \mathcal{TA}(u, p_0)$ the inequality

$$\mathcal{H}\psi(p_0) \leq 0$$

is satisfied. If $v : \Omega \to \mathbb{R}$ is *lower semicontinuous* (briefly, $v \in \mathrm{LSC}(\Omega)$), then we say that v is a *viscosity supersolution* to Equation (3.1) if for each $p_0 \in \Omega$ and every $\phi \in \mathcal{TB}(v, p_0)$ we have

$$\mathcal{H}\phi(p_0) \geq 0.$$

A continuous function $w : \Omega \to \mathbb{R}$ is called a *viscosity solution* of Equation (3.1) if it is both a viscosity sub- and supersolution.

As stated above, Definitions 3.4 and 3.7 are equivalent; this equivalence is a consequence of the following lemma, the proof of which mimics the methods of [17] and [3].

Lemma 3.8. *Given any function $u : \mathcal{O} \to \mathbb{R}$ and a point $p_0 \in \mathcal{O}$,*

$$J^{2,+}u(p_0) = \left\{ \left(\nabla_{\mathbb{G}}\, \psi(p_0), \left(D^2\psi\right)^{\star}(p_0) \right) \,:\, \psi \in \mathcal{TA}(u, p_0) \right\}.$$

Proof. Owing to Equation (3.6) and Proposition 3.1, one containment is easily obtained from the following relation: For p near p_0,

$$
\begin{aligned}
u(p) &\leq \psi(p) \\
&= T_{p_0,\psi}(p) + o\left(d^2_{CC}(p_0,p)\right) \text{ as } p \to p_0 \\
&= T_{p_0,\nabla_{\mathbb{G}}\psi,(D^2\psi)^\star}(p) + o\left(d^2_{CC}(p_0,p)\right) \text{ as } p \to p_0.
\end{aligned}
$$

This implies $\left(\nabla_{\mathbb{G}}\psi(p_0), \left(D^2\psi\right)^\star(p_0)\right) \in J^{2,+}u(p_0)$ for every touching above function ψ at p_0.

To obtain the second containment result, assume that $(\eta, X) \in J^{2,+}u(p_0)$ and begin by defining

$$
g(r) := \sup\left\{\left(T_{p_0,\eta,X}(p) - u(p)\right)^+ \;:\; d_{CC}(p_0,p) \leq r\right\}.
$$

By its definition, g is a nonnegative increasing function; moreover, since (η, X) is a jet entry, the definition of the jets implies $g(r) = o(r^2)$ as $r \downarrow 0$. Selecting some continuous nonnegative, increasing $\tilde{g}$ such that $g(r) \leq \tilde{g}(r)$ and $\tilde{g}(r) = o(r^2)$ as $r \downarrow 0$, we define

$$
\begin{cases}
a_{p_0}(p) := \dfrac{1}{4}\displaystyle\sum_{k=1}^{n}(x_k - x_k^0)^4 \\[2ex]
G(s) := \dfrac{1}{s}\displaystyle\int_s^{2s}\tilde{g}(r)\,dr \\[2ex]
H(t) := \dfrac{1}{t}\displaystyle\int_t^{2t}G(s)\,ds.
\end{cases}
$$

As in [3], the definitions of the functions above imply G is C^1_{eucl} and H, a_{p_0} are C^2_{eucl}, and L'hôspital's Rule establishes

$$
G'(0) = G(0) = 0 = H(0) = H'(0) = H''(0). \tag{3.8}
$$

Define

$$
\psi(p) := T_{p_0,\eta,X}(p) - H\Big(a_{p_0}(p)\Big) - a_{p_0}(p).
$$

Equation (3.8) together with calculations similar to those used to prove Proposition 3.1 show

$$
\begin{cases}
\psi(p_0) = u(p_0) \\
\nabla_{\mathbb{G}}\psi(p_0) = \eta \\
\left(D^2\psi\right)^\star(p_0) = X.
\end{cases} \tag{3.9}
$$

For given, small $r \geq 0$ and p so that $r \leq a_{p_0}(p)$,

$$\psi(p) - u(p) + r \leq \Big(T_{p_0,\eta,X}(p) - u(p)\Big) - H\Big(a_{p_0}(p)\Big) - a_{p_0}(p) + r;$$

since $r \leq g\left(a_{p_0}(p)\right) \leq H\Big(a_{p_0}(p)\Big)$, we conclude that

$$\psi(p) - u(p) + r \leq 0 = \psi(p_0) - u(p_0). \tag{3.10}$$

Equations (3.9) and (3.10) show that ψ is a member of $\mathcal{TA}(u, p_0)$. Since (η, X) was an arbitrary jet entry, the second containment is proven. □

Although the jet and touching function definitions for viscosity solutions are equivalent, the advantage of the first definition is that we can easily state the following result which, for a given function $u : \mathcal{O} \to \mathbb{R}$ and $p_0 \in \mathcal{O}$, relates the Euclidean upper jet $J^{2,+}_{\text{eucl}}u(p_0)$ to $J^{2,+}u(p_0)$. The proof presented below is an adaptation of the one presented for [10, Lemma 3.1], which is an application of [8, Lemma 3.1]. A similar result was obtained in [4, Main Lemma]; its proof relies upon producing Grushin second-order Taylor Polynomials for $C^2_{\mathbb{G}}$ functions and then utilizing the twisting terms and factors in these polynomials to deduce the twisting necessary for jet entries. Given the properties of the collections $J^{2,+}u(p_0)$ and $J^{2,-}u(p_0)$, a similar relationship holds for $J^{2,-}_{\text{eucl}}u(p_0)$ and $J^{2,-}u(p_0)$.

Lemma 3.9 (Subelliptic $\mathbb{G}$ Twisting Lemma)**.** *Let $\mathcal{O} \subseteq \mathbb{G}$ be open, let $u : \mathcal{O} \to \mathbb{R}$, and let $p_0 \in \mathcal{O}$. Suppose that $(\eta, X) \in J^{2,+}_{\text{eucl}}u(p_0)$: Then*

$$\big(\boldsymbol{A}(p_0) \cdot \eta, \boldsymbol{A}(p_0) \cdot X \cdot \boldsymbol{A}^{\mathrm{T}}(p_0) + \boldsymbol{M}(\eta, p_0)\big) \in J^{2,+}u(p_0), \tag{3.11}$$

where

$$(\boldsymbol{A}(p_0))_{k\ell} = \begin{cases} 1, & k = 1 = \ell \\ \rho_k(p_0), & 2 \leq k = \ell \leq n \\ 0, & \textit{otherwise} \end{cases} \tag{3.12}$$

and

$$(\boldsymbol{M}(\eta, p_0))_{k\ell} = \begin{cases} \frac{1}{2} \cdot \frac{\partial \rho_k}{\partial x_\ell}(p)\rho_\ell(p)\eta_k, & \ell < k \\ \\ \frac{1}{2} \cdot \frac{\partial \rho_\ell}{\partial x_k}(p)\rho_k(p)\eta_\ell, & k < \ell \\ \\ \quad 0, & \textit{otherwise.} \end{cases} \tag{3.13}$$

Proof. The result in (3.11) is known (see [8, Corollary 3.2] and [1, Lemma 3]); we shall restrict our attention to verifying Equations (3.12) and (3.13). The $n \times n$ matrix $\boldsymbol{A}$ is defined by [8] as $\boldsymbol{A}(p) := (A_{k\ell}(p))$ where

$$X_k(\cdot) = \sum_{\ell=1}^{n} A_{k\ell}(\cdot)\frac{\partial}{\partial x_\ell}.$$

The definitions (2.1) and (2.2) for the members of $\mathfrak{X}$ imply:

1. $A_{k\ell} \equiv 0$ if $k \neq \ell$;

2. $A_{kk} \equiv 1$ if $k = 1$ and $A_{kk} = \rho_k$ if $2 \leq k \leq n$.

This justifies (3.12). To verify (3.13), recall the definition of $\boldsymbol{M}(\eta, p_0)$ in [8]:

$$(\boldsymbol{M}(\eta, p_0))_{k\ell} := \begin{cases} \frac{1}{2}\sum_{r=1}^{n}\sum_{s=1}^{n}\left(A_{ks}(p_0)\frac{\partial A_{\ell r}}{\partial x_s}(p_0) + A_{\ell s}(p_0)\frac{\partial A_{kr}}{\partial x_s}(p_0)\right)\eta_r, & k \neq \ell \\ \sum_{r=1}^{n}\sum_{s=1}^{n} A_{ks}(p_0)\frac{\partial A_{kr}}{\partial x_s}(p_0)\eta_r, & k = \ell. \end{cases}$$

Because $A_{rs} \equiv 0$ whenever $r \neq s$ we may simplify the equation above:

$$\begin{aligned} (\boldsymbol{M}(\eta, p_0))_{k\ell} =& \frac{1}{2}\sum_{r=1}^{n}\left(\left(A_{kk}(p_0)\frac{\partial A_{\ell r}}{\partial x_k}(p_0) + 0\right) + \left(0 + A_{\ell\ell}(p_0)\frac{\partial A_{kr}}{\partial x_\ell}(p_0)\right)\right)\eta_r \\ =& \frac{1}{2}\left(A_{kk}(p_0)\frac{\partial A_{\ell\ell}}{\partial x_k}(p_0)\eta_\ell + A_{\ell\ell}(p_0)\frac{\partial A_{kk}}{\partial x_\ell}(p_0)\eta_k\right) \text{ if } k \neq \ell, \end{aligned} \tag{3.14}$$

and

$$(\boldsymbol{M}(\eta, p_0))_{kk} = \sum_{r=1}^{n} A_{kk}(p_0)\frac{\partial A_{kr}}{\partial x_k}(p_0)\eta_r = A_{kk}(p_0)\frac{\partial A_{kk}}{\partial x_k}(p_0)\eta_k \text{ if } k = \ell. \tag{3.15}$$

Considering Equation (3.15), note that $\partial A_{kk}/\partial x_k \equiv 0$ for all $k \leq n$: Indeed, $A_{kk} = \rho_k$ is independent of the variables $x_k, \ldots, x_n$. Hence $(\boldsymbol{M}(\eta, p_0))_{kk} = 0$. We now reduce Equation (3.14) as follows:

- Suppose $k = \ell$: Then $A_{kk} = \rho_k = A_{\ell\ell}$ and $\partial A_{kk}/\partial x_\ell = 0 = A_{\ell\ell}/\partial x_k$. Hence

$$(\boldsymbol{M}(\eta, p_0))_{k\ell} = \frac{1}{2}\left(1 \cdot 0 \cdot \eta_\ell + 1 \cdot 0 \cdot \eta_k\right) = 0.$$

- Suppose $k < \ell$ and recall $A_{kk} = \rho_k$ and $A_{\ell\ell} = \rho_\ell$. Since ρ_k is constant with respect to $x_k, \dots, x_n$,

$$\begin{aligned}(\boldsymbol{M}(\eta, p_0))_{k\ell} &= \frac{1}{2}\left(\rho_k(p_0) \cdot \frac{\partial \rho_\ell}{\partial x_k}(p_0)\eta_\ell + \rho_\ell(p_0) \cdot 0 \cdot \eta_k\right) \\ &= \frac{1}{2} \cdot \frac{\partial \rho_\ell}{\partial x_k}(p_0)\rho_k(p_0)\eta_\ell.\end{aligned}$$

- Supposing $\ell < k$, then work similar to the above shows

$$(\boldsymbol{M}(\eta, p_0))_{k\ell} = \frac{1}{2} \cdot \frac{\partial \rho_k}{\partial x_\ell}(p_0)\rho_\ell(p_0)\eta_k.$$

We conclude from the above that the matrix given by (3.13) is indeed $\boldsymbol{M}(\eta, p_0)$. $\square$

CHAPTER 4:
SOLUTIONS TO DIRICHLET PROBLEMS INVOLVING THE SUBELLIPTIC INFINITY LAPLACIAN

Note to Reader

Portions of this chapter have been previously published in [10, pp.77-89] and [11, pp. 41-54], and have been reproduced with permission from their respective publishers.

The following sections investigate existence and uniqueness of solutions to the Dirichlet problem

$$\begin{cases} \Delta_{\mathfrak{X},\infty} w &= 0 \quad \text{in } \Omega \\ w &= g \quad \text{on } \partial\Omega. \end{cases} \tag{DP}$$

As in previous sections, $\Omega \Subset \mathbb{G}$ represents a bounded, Grushin-type domain; the function $g : \partial\Omega \to \mathbb{R}$ is assumed to be continuous. Following the presentation of [10], Section 4.1 assumes the Conditions 1 and 2 of Chapter 2 and that, additionally, the set of zeros for ρ_k is discrete. For some fixed $1 \leq m < n$, we define

$$\begin{cases} \rho_i \equiv 1, \quad 1 \leq i \leq m \\ \rho_j(p) = \sigma(p) = \sigma(x_1, \ldots, x_m), \quad m+1 \leq j \leq n; \end{cases} \tag{4.1}$$

spaces $\mathbb{G}$ defined by frames $\mathfrak{X}$ for which Equation (4.1) hold will be called *σ-Spaces.* (In such spaces, we will denote the set of zeroes for $\rho_j = \sigma$ by $Z \times \mathbb{R}^{n-m+1}$.) Section 4.2, which follows [11], generalizes the previous section, assuming only Conditions 1 and 2 on the frame $\mathfrak{X}$ – we call such spaces *General Triangular Spaces.* In both sections, we establish estimates necessary for Section 4.3, in which we conclude the chapter with a discussion of uniqueness of viscosity solutions to Problem (DP). In all three sections sections we will utilize the following definition, which extends our previous notion of viscosity solutions.

Definition 4.1. Consider Dirichlet Problem

$$\begin{cases} \mathcal{H}w &= 0 \quad \text{in } \Omega \\ w &= g \quad \text{on } \partial\Omega, \end{cases} \tag{4.2}$$

where $\mathcal{H}$ represents $\mathcal{F}^\kappa, \mathcal{G}^\kappa$, or $\Delta_{\mathfrak{X},\infty}$; let Ω, g be as above. A *viscosity subsolution* of Problem (4.2) is a viscosity subsolution u of the first line which also satisfies $u \le g$ on $\partial\Omega$; *viscosity supersolutions* and *viscosity solutions* of Problem (4.2) are defined similarly.

Remark 4.2. When $\mathcal{H} = \mathcal{F}_\infty$, we shall continue to refer to viscosity (sub-/super-)solutions of Problem (4.2) as ∞-(sub-/super-)harmonic functions.

In both Sections 4.1 and 4.2, we will rely upon several common results, the truth of which are not dependent upon the choice of frame $\mathfrak{X}$. The first is an existence statement; it is standard for the theory and, following the approach of [9, Theorem 4.1], we have condensed the results supporting this finding into one theorem. As in [9], the proof follows the layout of [3, Section 4].

Theorem 4.3 (Existence of ∞-Harmonic Functions)**.** *The following are true:*

1. *Let $\kappa \in \mathbb{R}$ and $\mathtt{p} \ge 2$. If $u_\mathtt{p} \in C(\Omega) \cap W^{1,\mathtt{p}}_{\mathrm{loc}}(\Omega)$ is a weak (sub-/super)solution to the $\mathtt{p}$-Laplace problem*

$$\begin{cases} \Delta_\mathtt{p} w = 0 & \text{in } \Omega \\ w = g & \text{on } \partial\Omega, \end{cases} \tag{4.3}$$

   then $u_\mathtt{p}$ is a viscosity (sub-/super)solution to (4.3).

2. *Let $u_\mathtt{p}$ be as before. Passing to a subsequence of $(u_\mathtt{p})_{\mathtt{p}\ge 2}$ as necessary, there exists $u_\infty \in C(\Omega) \cap W^{1,\infty}_{\mathrm{loc}}(\Omega)$ so that*

$$u_\mathtt{p} \to u_\infty \text{ uniformly in } \Omega$$

   as $\mathtt{p} \to \infty$.

3. *The function u_∞ from the previous item is a viscosity solution of one of (4.2) with one of the operators $\mathcal{F}^\kappa, \mathcal{G}^\kappa$, or $\Delta_{\mathfrak{X},\infty}$, the choice of operator depending only upon κ:*

   (a) *If $\kappa > 0$, then u_∞ is a viscosity solution to Problem (4.2) with $\mathcal{H} = \mathcal{F}^\kappa$.*

   (b) *If $\kappa < 0$, then u_∞ is a viscosity solution to Problem (4.2) with $\mathcal{H} = \mathcal{G}^\kappa$.*

   (c) *If $\kappa = 0$, then u_∞ is a viscosity solution to (4.2) with $\mathcal{H} = \Delta_{\mathfrak{X},\infty}$.*

Remark 4.4. Theorem 4.3 was recently proved for general sub-Riemannian spaces in more generality in [15].

The next collection of common results concern an Iterated Maximum Principle, which gives conditions under which we may find points possessing nonempty jet closures for viscosity sub- and supersolutions; this will enable us to produce necessary estimates on the jet entries. As in [18], we will have need for a "penalty function"; specifically, we make use of the function

$$\varphi_{\tau_1,\tau_2,\tau_3,\ldots,\tau_n}(p,q) = \varphi_{\vec{\tau}}(p,q) := \frac{1}{2}\sum_{k=1}^{n} \tau_k (x_k - y_k)^2 \tag{4.4}$$

where the entries of $\vec{\tau} = (\tau_1, \tau_2, \tau_3, \ldots, \tau_n)$ are positive real numbers. The use of n real parameters as opposed to the one employed by [18] allows us to take into account the fact that our functions ρ_k can possibly vanish.

Lemma 4.5 (C.f. [4, Lemma 4.3]). *Let $\Omega \Subset \mathbb{G}$ be a domain, $u \in \mathrm{USC}(\Omega)$, and $v \in \mathrm{LSC}(\Omega)$; assume that there exists some $p_0 \in \Omega$ so that*

$$u(p_0) - v(p_0) > 0.$$

Let $\vec{\tau} = (\tau_1, \tau_2, \tau_3, \ldots, \tau_n) \in \mathbb{R}^n$ have positive coordinates and, for each pair of points $p = (x_1, x_2, x_3, \ldots, x_n)$ and $q = (y_1, y_2, y_3, \ldots, y_n)$ in $\mathbb{G}$, define the functions

$$\begin{aligned}
\varphi_{\tau_1,\tau_2,\tau_3,\ldots,\tau_n}(p,q) &:= \frac{1}{2}\sum_{k=1}^{n} \tau_k (x_k - y_k)^2 \\
\varphi_{\tau_2,\tau_3,\ldots,\tau_n}(p,q) &:= \frac{1}{2}\sum_{k=2}^{n} \tau_k (x_k - y_k)^2 \\
\varphi_{\tau_3,\ldots,\tau_n}(p,q) &:= \frac{1}{2}\sum_{k=3}^{n} \tau_k (x_k - y_k)^2 \\
&\vdots \\
\varphi_{\tau_n}(p,q) &:= \frac{1}{2}\tau_n (x_n - y_n)^2.
\end{aligned}$$

Appealing to the compactness of $\overline{\Omega}$ *and to upper semicontinuity, we may also define*

$$\begin{aligned}
M_{\tau_1,\tau_2,\tau_3,\ldots,\tau_n} &:= \sup_{\overline{\Omega}\times\overline{\Omega}} \{u(p) - v(q) - \varphi_{\tau_1,\tau_2,\tau_3,\ldots,\tau_n}(p,q)\} \\
&= u(p_{\tau_1,\tau_2,\tau_3,\ldots,\tau_n}) - v(q_{\tau_1,\tau_2,\tau_3,\ldots,\tau_n}) - \varphi_{\tau_1,\tau_2,\tau_3,\ldots,\tau_n}(p_{\tau_1,\tau_2,\tau_3,\ldots,\tau_n}, q_{\tau_1,\tau_2,\tau_3,\ldots,\tau_n}) \\
M_{\tau_2,\tau_3,\ldots,\tau_n} &:= \sup_{\overline{\Omega}\times\overline{\Omega}} \{u(p) - v(q) - \varphi_{\tau_2,\tau_3,\ldots,\tau_n}(p,q) \,:\, x_1 = y_1\} \\
&= u(p_{\tau_2,\tau_3,\ldots,\tau_n}) - v(q_{\tau_2,\tau_3,\ldots,\tau_n}) - \varphi_{\tau_2,\tau_3,\ldots,\tau_n}(p_{\tau_2,\tau_3,\ldots,\tau_n}, q_{\tau_2,\tau_3,\ldots,\tau_n}) \\
M_{\tau_3,\ldots,\tau_n} &:= \sup_{\overline{\Omega}\times\overline{\Omega}} \{u(p) - v(q) - \varphi_{\tau_3,\ldots,\tau_n}(p,q) \,:\, x_k = y_k,\ k = 1,2\} \\
&= u(p_{\tau_3,\ldots,\tau_n}) - v(q_{\tau_3,\ldots,\tau_n}) - \varphi_{\tau_3,\ldots,\tau_n}(p_{\tau_3,\ldots,\tau_n}, q_{\tau_3,\ldots,\tau_n}) \\
&\ \vdots \\
M_{\tau_n} &:= \sup_{\overline{\Omega}\times\overline{\Omega}} \{u(p) - v(q) - \varphi_{\tau_3,\ldots,\tau_n}(p,q) \,:\, x_k = y_k,\ k = 1,\ldots,n-1\} \\
&= u(p_{\tau_n}) - v(q_{\tau_n}) - \varphi_{\tau_n}(p_{\tau_n}, q_{\tau_n}).
\end{aligned}$$

Then

$$\lim_{\tau_n\to\infty} \cdots \lim_{\tau_3\to\infty} \lim_{\tau_2\to\infty} \lim_{\tau_1\to\infty} M_{\tau_1,\tau_2,\tau_3,\ldots,\tau_n} = u(p_0) - v(p_0)$$

and

$$\lim_{\tau_n\to\infty} \cdots \lim_{\tau_3\to\infty} \lim_{\tau_2\to\infty} \lim_{\tau_1\to\infty} \varphi_{\tau_1,\tau_2,\tau_3,\ldots,\tau_n}(p_{\tau_1,\tau_2,\tau_3,\ldots,\tau_n}, q_{\tau_1,\tau_2,\tau_3,\ldots,\tau_n}) = 0.$$

Additionally, the first ℓ *coordinates of* $p_{\tau_{\ell+1},\ldots,\tau_n}$ *and* $q_{\tau_{\ell+1},\ldots,\tau_n}$ *are identical – that is,*

$$x_k^{\tau_{\ell+1},\ldots,\tau_n} = y_k^{\tau_{\ell+1},\ldots,\tau_n},\ k = 1,\ldots,\ell.$$

The proof of the Iterated Maximum Principle proceeds precisely as in [4], and leads immediately to the following results which permit us to take the parameters $\tau_k \to \infty$ in any order, and to speak of the full limit as $\tau_{k_1}, \tau_{k_2}, \ldots, \tau_{k_n} \to \infty$.

Corollary 4.6 (C.f. [4, Corollary 4.4])**.** *Under the conditions of Lemma 4.5, each iterated limit of* $M_{\tau_1,\tau_2,\tau_3,\ldots,\tau_n}$ *exists and is equal to* $u(p_0) - v(p_0)$ *– in other words,*

$$\lim_{\tau_{k_1}\to\infty} \cdots \lim_{\tau_{k_{n-2}}\to\infty} \lim_{\tau_{k_{n-1}}\to\infty} \lim_{\tau_{k_n}\to\infty} M_{\tau_1,\tau_2,\tau_3,\ldots,\tau_n} = u(p_0) - v(p_0).$$

Consequently,

$$\lim_{\tau_{k_1}\to\infty} \cdots \lim_{\tau_{k_{n-2}}\to\infty} \lim_{\tau_{k_{n-1}}\to\infty} \lim_{\tau_{k_n}\to\infty} \varphi_{\tau_1,\tau_2,\tau_3,\ldots,\tau_n}(p_{\tau_1,\tau_2,\tau_3,\ldots,\tau_n}, q_{\tau_1,\tau_2,\tau_3,\ldots,\tau_n}) = 0.$$

Lemma 4.7 (C.f. [4, Corollary 4.5])**.** *Under the conditions of Lemma 4.5, the full limit of* $M_{\tau_1,\tau_2,\tau_3,\ldots,\tau_n}$ *exists and is equal to* $u(p_0)-v(p_0)$ *– more precisely,*

$$\lim_{\tau_n,\ldots,\tau_3,\tau_2,\tau_1\to\infty} M_{\tau_1,\tau_2,\tau_3,\ldots,\tau_n} = u(p_0)-v(p_0).$$

In addition,

$$\lim_{\tau_n,\ldots,\tau_3,\tau_2,\tau_1\to\infty} \varphi_{\tau_1,\tau_2,\tau_3,\ldots,\tau_n}(p_{\tau_1,\tau_2,\tau_3,\ldots,\tau_n}, q_{\tau_1,\tau_2,\tau_3,\ldots,\tau_n}) = 0.$$

Remark 4.8. Owing to Lemma 4.7, there is no ambiguity in relabeling the intermediate points $p_{\tau_1,\tau_2,\tau_3,\ldots,\tau_n}$, $q_{\tau_1,\tau_2,\tau_3,\ldots,\tau_n}$, and function $\varphi_{\tau_1,\tau_2,\tau_3,\ldots,\tau_n}$ as $p_{\vec{\tau}}, q_{\vec{\tau}}$, and $\varphi_{\vec{\tau}}$. We will also denote the coordinates of $p_{\vec{\tau}}, q_{\vec{\tau}}$ as $x_k^{\vec{\tau}}, y_k^{\vec{\tau}}$ respectively and, in accordance with Lemma 4.5, denote

$$\begin{cases} p_{\tau_1,\ldots,\tau_k} := \lim\limits_{\tau_k\to\infty}\cdots\lim\limits_{\tau_1\to\infty} p_{\vec{\tau}} & = & (x_1^0,\ldots,x_k^0,x_{k+1}^{\vec{\tau}},\ldots,x_n^{\vec{\tau}}) \\ \\ q_{\tau_1,\ldots,\tau_k} := \lim\limits_{\tau_k\to\infty}\cdots\lim\limits_{\tau_1\to\infty} q_{\vec{\tau}} & = & (x_1^0,\ldots,x_k^0,y_{k+1}^{\vec{\tau}},\ldots,y_n^{\vec{\tau}}) \end{cases} \tag{4.5}$$

for each $1\le k\le n$.

It therefore remains for us to state and prove necessary estimates on the jet entries at the critical points $p_{\vec{\tau}}$ and $q_{\vec{\tau}}$; these estimates lead to comparison principles for the operators $\mathcal{F}^\kappa$ and $\mathcal{G}^\kappa$, which we may then leverage to establish uniqueness of ∞-harmonic functions.

4.1 Estimates in the Case of Sigma Spaces

By applying Lemma 4.5, Corollary 4.6, Lemma 4.7, Equation (4.5), and [18, Theorem 3.2], we have the following estimates. The lemma below (and Lemma 4.10) require that at least one of the viscosity sub- or supersolutions is locally $\mathbb{G}$-Lipschitz; it should be observed that by Theorem 4.3, this assumption will be satisifed once the lemma is applied to u_∞.

Lemma 4.9 (C.f. [10, Lemma 4.4])**.** *Let* $u, v, \varphi_{\vec{\tau}}$, *and* $(p_{\vec{\tau}}, q_{\vec{\tau}})$ *be as in Lemma 4.5. Assume that* $\mathbb{G}$ *is a* σ*-Space and that at least one of the functions* u,v *is locally* $\mathbb{G}$*-Lipschitz. Then:*

1. *There exist* $(\eta_{\vec{\tau}}^+, \mathcal{X}_{\vec{\tau}}) \in \overline{J}^{2,+}u(p_{\vec{\tau}})$ *and* $(\eta_{\vec{\tau}}^-, \mathcal{Y}_{\vec{\tau}}) \in \overline{J}^{2,-}v(q_{\vec{\tau}})$.
2. *Define* $(p\diamond q)_k$ *to be the point whose* k*-th coordinate coincides with* q *and whose other coordinates coincide with* p *– in other words,*

$$(p\diamond q)_k = (x_1,\ldots,x_{k-1},y_k,x_{k+1},\ldots,x_n).$$

Then for each index k,

$$\tau_k(x_k^{\vec{\tau}} - y_k^{\vec{\tau}})^2 \lesssim d_{CC}\left(p_{\vec{\tau}}, (p_{\vec{\tau}} \diamond q_{\vec{\tau}})_k\right). \tag{4.6}$$

For the indices $i \leq m$,

$$\tau_i \left|x_i^{\vec{\tau}} - y_i^{\vec{\tau}}\right| = O(1) \text{ as } \tau_i \to \infty. \tag{4.7}$$

3. *The vector estimate*

$$\lim_{\tau_n \to \infty} \cdots \lim_{\tau_1 \to \infty} \left| \left\|\eta_{\vec{\tau}}^+\right\|^2 - \left\|\eta_{\vec{\tau}}^-\right\|^2 \right| = 0. \tag{4.8}$$

holds.

4. *The matrix estimate*

$$\lim_{\tau_n \to \infty} \cdots \lim_{\tau_1 \to \infty} \left(\left\langle \mathcal{X}^{\vec{\tau}} \eta_{\vec{\tau}}^+, \eta_{\vec{\tau}}^+ \right\rangle - \left\langle \mathcal{Y}^{\vec{\tau}} \eta_{\vec{\tau}}^-, \eta_{\vec{\tau}}^- \right\rangle\right) = 0. \tag{4.9}$$

holds.

Proof. For clarity, we split the proof between the items above.

Item 1.

[18, Theorem 3.2] guarantees the existence of elements in the Euclidean jet closures at $p_{\vec{\tau}}$ and $q_{\vec{\tau}}$: In particular,

$$\left(D_{\mathrm{eucl}(p)} \varphi_{\vec{\tau}}(p_{\vec{\tau}}, q_{\vec{\tau}}), X^{\vec{\tau}}\right) \in \overline{J}^{2,+}_{\mathrm{eucl}} u(p_{\vec{\tau}}) \text{ and } \left(-D_{\mathrm{eucl}(q)} \varphi_{\vec{\tau}}(p_{\vec{\tau}}, q_{\vec{\tau}}), Y^{\vec{\tau}}\right) \in \overline{J}^{2,-}_{\mathrm{eucl}} v(q_{\vec{\tau}}).$$

Applying the $\mathbb{G}$ Twisting Lemma (Lemma 3.9) produces $(\eta_{\vec{\tau}}^+, \mathcal{X}_{\vec{\tau}}) \in \overline{J}^{2,+} u(p_{\vec{\tau}})$ and $(\eta_{\vec{\tau}}^-, \mathcal{Y}_{\vec{\tau}}) \in \overline{J}^{2,-} v(q_{\vec{\tau}})$.

Item 2.

By the definition of $p_{\vec{\tau}}, q_{\vec{\tau}}$, for all points $p, q \in \Omega$ the inequality

$$u(p) - v(q) - \varphi_{\vec{\tau}}(p, q) \leq u(p_{\vec{\tau}}) - v(q_{\vec{\tau}}) - \varphi_{\vec{\tau}}(p_{\vec{\tau}}, q_{\vec{\tau}})$$

is satisfied. Hence assuming (without loss of generality) that u is $\mathbb{G}$-Lipschitz, decreeing $p := (p_{\vec{\tau}} \diamond q_{\vec{\tau}})_k$ and $q := q_{\vec{\tau}}$, and recollecting terms, we obtain

$$\begin{aligned} \tau_k(x_k^{\vec{\tau}} - y_k^{\vec{\tau}})^2 &= \varphi_{\vec{\tau}}(p_{\vec{\tau}}, q_{\vec{\tau}}) - \varphi_{\vec{\tau}}\left((p_{\vec{\tau}} \diamond q_{\vec{\tau}})_k, q_{\vec{\tau}}\right) \\ &\leq u(p_{\vec{\tau}}) - u\left((p_{\vec{\tau}} \diamond q_{\vec{\tau}})_k\right) \\ &\leq K d_{CC}\left(p_{\vec{\tau}}, (p_{\vec{\tau}} \diamond q_{\vec{\tau}})_k\right), \end{aligned} \tag{4.10}$$

where K is the Lipschitz constant for u. This is Inequality (4.6), so to complete Item 2 we turn our attention to to the expression $\tau_k \left|x_k^{\vec{\tau}} - y_k^{\vec{\tau}}\right|$. If $x_k^{\vec{\tau}} \neq y_k^{\vec{\tau}}$ then (4.10) shows

$$\tau_i \left|x_k^{\vec{\tau}} - y_k^{\vec{\tau}}\right| = \tau_k (x_k^{\vec{\tau}} - y_k^{\vec{\tau}})^2 \cdot \frac{1}{\left|x_k^{\vec{\tau}} - y_k^{\vec{\tau}}\right|} \leq \frac{K d_{CC}\left(p_{\vec{\tau}}, (p_{\vec{\tau}} \diamond q_{\vec{\tau}})_k\right)}{\left|x_k^{\vec{\tau}} - y_k^{\vec{\tau}}\right|}. \tag{4.11}$$

Note that because $\rho_k(p_0) \neq 0$ for $1 \leq k \leq m$, we have that x_k^0 has a locally Riemannian neighborhood along the k-th coordinate axis. Thus,

$$d_{CC}\left(p_{\vec{\tau}}, (p_{\vec{\tau}} \diamond q_{\vec{\tau}})_k\right) \lesssim \left|x_k^{\vec{\tau}} - y_k^{\vec{\tau}}\right|. \tag{4.12}$$

Combining (4.11) and (4.12) proves Equation (4.7) and completes the proof of Item 2.

Item 3.

Observe that

$$\frac{\partial}{\partial x_k}\varphi(p_{\vec{\tau}}, q_{\vec{\tau}}) = \tau_k (x_k^{\vec{\tau}} - q_k^{\vec{\tau}}) = -\frac{\partial}{\partial y_k}\varphi(p_{\vec{\tau}}, q_{\vec{\tau}});$$

consequently, referring back to the definition of the matrix $\boldsymbol{A}$, the coordinates of $\eta_{\vec{\tau}}^+$ and $\eta_{\vec{\tau}}^-$ are

$$\left[\eta_{\vec{\tau}}^+\right]_k = \begin{cases} \tau_k (x_k^{\vec{\tau}} - y_k^{\vec{\tau}}), \text{ if } k \leq m \\ \tau_k (x_k^{\vec{\tau}} - y_k^{\vec{\tau}})\sigma(p_{\vec{\tau}}), \text{ if } m+1 \leq k \leq n \end{cases}$$

and

$$\left[\eta_{\vec{\tau}}^-\right]_k = \begin{cases} \tau_k (x_k^{\vec{\tau}} - y_k^{\vec{\tau}}), \text{ if } k \leq m \\ \tau_k (x_k^{\vec{\tau}} - y_k^{\vec{\tau}})\sigma(q_{\vec{\tau}}), \text{ if } m+1 \leq k \leq n. \end{cases}$$

Fixing $\vec{\tau}$ for the moment, this leads to the estimate

$$\left|\left\|\eta_{\vec{\tau}}^+\right\|^2 - \left\|\eta_{\vec{\tau}}^-\right\|^2\right| \leq \sum_{k=m+1}^{n} \left|\sigma^2(p_{\vec{\tau}}) - \sigma^2(q_{\vec{\tau}})\right| \cdot \tau_k^2 \left(x_k^{\vec{\tau}} - y_k^{\vec{\tau}}\right)^2. \tag{4.13}$$

The values τ_i for $i \leq m$ are not present in Inequality (4.13). Taking the iterated limits of (4.13) as $\tau_i \to \infty$, recalling that $\sigma(p)$ depends only upon the first m coordinates of p, and applying the Iterated Maximum Principle yields

$$\lim_{\tau_m \to \infty} \cdots \lim_{\tau_1 \to \infty} \left|\left\|\eta_{\vec{\tau}}^+\right\|^2 - \left\|\eta_{\vec{\tau}}^-\right\|^2\right| = 0.$$

The above implies

$$\lim_{\tau_n \to \infty} \cdots \lim_{\tau_{m+1} \to \infty} \lim_{\tau_m \to \infty} \cdots \lim_{\tau_1 \to \infty} \left|\left\|\eta_{\vec{\tau}}^+\right\|^2 - \left\|\eta_{\vec{\tau}}^-\right\|^2\right| = 0,$$

concluding Item 3.

Item 4.

[18, Theorem 3.2] and the Twisting Lemma imply

$$\left\langle \mathcal{X}^{\vec{\tau}}\eta^{+}_{\vec{\tau}}, \eta^{+}_{\vec{\tau}} \right\rangle - \left\langle \mathcal{Y}^{\vec{\tau}}\eta^{-}_{\vec{\tau}}, \eta^{-}_{\vec{\tau}} \right\rangle = I_1 + I_2,$$

where we define

$$I_1 := \left\langle (\boldsymbol{A}(p_{\vec{\tau}}) \cdot X^{\vec{\tau}} \cdot \boldsymbol{A}^{\mathrm{T}}(p_{\vec{\tau}})) \cdot \eta^{+}_{\vec{\tau}}, \eta^{+}_{\vec{\tau}} \right\rangle - \left\langle (\boldsymbol{A}(q_{\vec{\tau}}) \cdot Y^{\vec{\tau}} \cdot \boldsymbol{A}^{\mathrm{T}}(q_{\vec{\tau}})) \cdot \eta^{-}_{\vec{\tau}}, \eta^{-}_{\vec{\tau}} \right\rangle$$

and

$$I_2 := \left\langle \boldsymbol{M}(D_{\mathrm{eucl}(p)}\varphi_{\vec{\tau}}(p_{\vec{\tau}}, q_{\vec{\tau}}), p_{\vec{\tau}}) \cdot \eta^{+}_{\vec{\tau}}, \eta^{+}_{\vec{\tau}} \right\rangle - \left\langle \boldsymbol{M}(D_{\mathrm{eucl}(q)}\varphi_{\vec{\tau}}(p_{\vec{\tau}}, q_{\vec{\tau}}), q_{\vec{\tau}}) \cdot \eta^{-}_{\vec{\tau}}, \eta^{-}_{\vec{\tau}} \right\rangle. \tag{4.14}$$

Writing $\widetilde{\epsilon} := \boldsymbol{A}(p_{\vec{\tau}}) \cdot \epsilon, \widehat{\chi} := \boldsymbol{A}(q_{\vec{\tau}}) \cdot \chi$ to mean the twisting of $\epsilon, \chi \in \mathbb{R}^n$ according to the Twisting Lemma,

$$\begin{aligned} \left\langle \boldsymbol{A}(p_{\vec{\tau}}) \cdot X^{\vec{\tau}} \cdot \boldsymbol{A}^{\mathrm{T}}(p_{\vec{\tau}})\epsilon, \epsilon \right\rangle - \left\langle \boldsymbol{A}(q_{\vec{\tau}}) \cdot Y^{\vec{\tau}} \cdot \boldsymbol{A}^{\mathrm{T}}(q_{\vec{\tau}})\chi, \chi \right\rangle &= \left\langle X^{\vec{\tau}} \cdot \widetilde{\epsilon}, \widetilde{\epsilon} \right\rangle - \left\langle Y^{\vec{\tau}} \cdot \widehat{\chi}, \widehat{\chi} \right\rangle \\ &\leq \langle \mathcal{C} \cdot \Upsilon, \Upsilon \rangle \end{aligned}$$

where $\Upsilon := \widetilde{\epsilon} \oplus (-\widehat{\chi})$ and $\mathcal{C}$ is a $2n \times 2n$ block matrix resulting from [18, Theorem 3.2] of the form

$$\begin{pmatrix} B & -B \\ -B & B \end{pmatrix}$$

and

$$[B]_{ab} = \begin{cases} \tau_a + 2\delta\tau_a^2, & a = b \\ 0, & a \neq b. \end{cases}$$

(Recall that δ is a consequence of [18, Theorem 3.2].) Choosing $\epsilon := \eta^{+}_{\vec{\tau}}$ and $\chi := \eta^{-}_{\vec{\tau}}$, the above shows

$$\begin{aligned} I_1 &\leq \left\langle B \cdot \left(\widetilde{\eta^{+}_{\vec{\tau}}} - \widehat{\eta^{-}_{\vec{\tau}}}\right), \widetilde{\eta^{+}_{\vec{\tau}}} - \widehat{\eta^{-}_{\vec{\tau}}} \right\rangle \\ &= \sum_{k=m+1}^{n} (\tau_k + 2\delta\tau_k^2)(\sigma^2(p_{\vec{\tau}}) - \sigma^2(q_{\vec{\tau}}))^2 \cdot \tau_k^2(x_k^{\vec{\tau}} - y_k^{\vec{\tau}})^2. \end{aligned} \tag{4.15}$$

The right-hand side of Relation (4.15) is free of the τ_i for $i \leq m$, so proceeding as in the proof of Item 3 we find

$$\lim_{\tau_m \to \infty} \cdots \lim_{\tau_1 \to \infty} I_1 = 0$$

so that

$$\lim_{\tau_n\to\infty}\cdots\lim_{\tau_{m+1}\to\infty}\lim_{\tau_m\to\infty}\cdots\lim_{\tau_1\to\infty} I_1 = 0. \tag{4.16}$$

For the term I_2, let us begin by simplifying the notation for the matrix $\boldsymbol{M}(\cdot,\cdot)$. Appealing to Equation (5.6) in the twisting lemma, we see that

$$\boldsymbol{M}(D_p\varphi_{\vec{\tau}}(p_{\vec{\tau}}, q_{\vec{\tau}}), p_{\vec{\tau}}) = \begin{pmatrix} 0 & S(p_{\vec{\tau}}) \\ S(p_{\vec{\tau}})^T & 0 \end{pmatrix}$$

and

$$\boldsymbol{M}(D_q\varphi_{\vec{\tau}}(p_{\vec{\tau}}, q_{\vec{\tau}}), q_{\vec{\tau}}) = \begin{pmatrix} 0 & S(q_{\vec{\tau}}) \\ S(q_{\vec{\tau}})^T & 0 \end{pmatrix},$$

where, permitting t to represent either the point $p_{\vec{\tau}}$ or $q_{\vec{\tau}}$, the $m\times(n-m)$ matrix $S(t)$ is defined by

$$[S(t)]_{rs} := \frac{1}{2}\cdot\frac{\partial\sigma}{\partial x_r}(t)\cdot\tau_s(x_s^{\vec{\tau}} - y_s^{\vec{\tau}}).$$

Calculations with (4.14) show

$$\begin{aligned} I_2 \quad = \quad & \sum_{\ell=m+1}^{n}\sum_{r=1}^{m}\frac{\partial\sigma}{\partial x_r}(p_{\vec{\tau}})\cdot\tau_r(x_r^{\vec{\tau}} - y_r^{\vec{\tau}})\cdot\tau_\ell^2(x_\ell^{\vec{\tau}} - y_\ell^{\vec{\tau}})^2\sigma(p_{\vec{\tau}}) \\ & -\sum_{\ell=m+1}^{n}\sum_{r=1}^{m}\frac{\partial\sigma}{\partial x_r}(q_{\vec{\tau}})\cdot\tau_r(x_r^{\vec{\tau}} - y_r^{\vec{\tau}})\cdot\tau_\ell^2(x_\ell^{\vec{\tau}} - y_\ell^{\vec{\tau}})^2\sigma(q_{\vec{\tau}}). \end{aligned}$$

We adopt the notation

$$T_{r\ell} := \tau_r(x_r^{\vec{\tau}} - y_r^{\vec{\tau}})\tau_\ell^2(x_\ell^{\vec{\tau}} - y_\ell^{\vec{\tau}})^2\left(\frac{\partial\sigma}{\partial x_r}\cdot\sigma\right)(p_{\vec{\tau}}) - \tau_r(x_r^{\vec{\tau}} - y_r^{\vec{\tau}})\tau_\ell^2(x_\ell^{\vec{\tau}} - y_\ell^{\vec{\tau}})^2\left(\frac{\partial\sigma}{\partial x_r}\cdot\sigma\right)(q_{\vec{\tau}})$$

for the (r,ℓ)-term of I_2. Since $p_{\vec{\tau}} \to p_{\tau_1,\cdots,\tau_i}$ and $q_{\vec{\tau}} \to q_{\tau_1,\cdots,\tau_i}$ as $\tau_1,\ldots\tau_i \to \infty$ $(i \le m)$, and since $1 \le r \le m < \ell \le n$ and $\sigma \in C^2_{\text{eucl}}$, we obtain the iterated limit

$$\begin{aligned} \lim_{\tau_i\to\infty}\cdots\lim_{\tau_1\to\infty} T_{r\ell} \quad = \quad & \tau_r(x_r^{\vec{\tau}} - y_r^{\vec{\tau}})\tau_\ell^2(x_\ell^{\vec{\tau}} - y_\ell^{\vec{\tau}})^2\left(\frac{\partial\sigma}{\partial x_r}\cdot\sigma\right)(p_{\tau_1,\cdots,\tau_i}) \\ & -\tau_r(x_r^{\vec{\tau}} - y_r^{\vec{\tau}})\tau_\ell^2(x_\ell^{\vec{\tau}} - y_\ell^{\vec{\tau}})^2\left(\frac{\partial\sigma}{\partial x_r}\cdot\sigma\right)(q_{\tau_1,\cdots,\tau_i}) \end{aligned}$$

if $i < r$; if $r \le i$ we may apply Item 2, Inequality (4.7), and arrive at

$$\lim_{\tau_i \to \infty} \cdots \lim_{\tau_1 \to \infty} T_{r\ell} \approx \tau_\ell^2 (x_\ell^{\vec{\tau}} - y_\ell^{\vec{\tau}})^2 \left(\frac{\partial \sigma}{\partial x_r} \cdot \sigma \right) (p_{\tau_1, \cdots, \tau_i}) \\ - \tau_\ell^2 (x_\ell^{\vec{\tau}} - y_\ell^{\vec{\tau}})^2 \left(\frac{\partial \sigma}{\partial x_r} \cdot \sigma \right) (q_{\tau_1, \cdots, \tau_i}).$$

This second limit in particular implies that

$$\lim_{\tau_m \to \infty} \cdots \lim_{\tau_1 \to \infty} T_{r\ell} \approx \tau_\ell^2 (x_\ell^{\vec{\tau}} - y_\ell^{\vec{\tau}})^2 \left(\frac{\partial \sigma}{\partial x_r} \cdot \sigma \right) (p_{\tau_1, \cdots, \tau_m}) \\ - \tau_\ell^2 (x_\ell^{\vec{\tau}} - y_\ell^{\vec{\tau}})^2 \left(\frac{\partial \sigma}{\partial x_r} \cdot \sigma \right) (q_{\tau_1, \cdots, \tau_m}) \tag{4.17}$$

for all $r \le m$. Since $\sigma, \partial\sigma/\partial x_r$ depend only upon the first m coordinates of points p, (4.17) implies

$$\lim_{\tau_m \to \infty} \cdots \lim_{\tau_1 \to \infty} I_2 = 0$$

and hence

$$\lim_{\tau_n \to \infty} \cdots \lim_{\tau_{m+1} \to \infty} \lim_{\tau_m \to \infty} \cdots \lim_{\tau_1 \to \infty} I_2 = 0. \tag{4.18}$$

Equation (4.9) then follows from (4.16) and (4.18). □

4.2 Estimates in the Case of General Triangular Spaces

A similar result to Lemma 4.9 can be proven even in the case that the weight functions ρ_k, ρ_j are not equal for $2 \le j, k$. Additional care must be taken with the iterated limits, as we shall show.

Lemma 4.10 (C.f. [11, Lemma 4.2])**.** *Let $u, v, \varphi_{\vec{\tau}}$ and $(p_{\vec{\tau}}, q_{\vec{\tau}})$ be as above. Assume that $\mathbb{G}$ is a General Triangular Space and that at least one of the functions u, v is locally $\mathbb{G}$-Lipschitz. Then:*

1. *There exist $(\eta^+_{\vec{\tau}}, \mathcal{X}_{\vec{\tau}}) \in \overline{J}^{2,+} u(p_{\vec{\tau}})$ and $(\eta^-_{\vec{\tau}}, \mathcal{Y}_{\vec{\tau}}) \in \overline{J}^{2,-} v(q_{\vec{\tau}})$.*

2. *Define $(p \diamond q)_k$ to be the point whose k-th coordinate coincides with q and whose other coordinates coincide with p, in other words,*

   $$(p \diamond q)_k = (x_1, \ldots, x_{k-1}, y_k, x_{k+1}, \ldots, x_n).$$

   Then for each index $1 \le k \le n$,

   $$\tau_k \left(x_k^{\vec{\tau}} - y_k^{\vec{\tau}} \right)^2 \lesssim d_{CC}(p_{\vec{\tau}}, (p_{\vec{\tau}} \diamond q_{\vec{\tau}})_k) \text{ as } \tau_k \to \infty. \tag{4.19}$$

In particular, when $\rho_k(p_0) \neq 0$, we have

$$\tau_k |x_k^{\vec{\tau}} - y_k^{\vec{\tau}}| = O(1) \quad as \quad \tau_k \to \infty. \tag{4.20}$$

3. *The vector estimate*

$$\lim_{\tau_n \to \infty} \cdots \lim_{\tau_1 \to \infty} \left| \|\eta_{\vec{\tau}}^+\|^2 - \|\eta_{\vec{\tau}}^-\|^2 \right| = 0 \tag{4.21}$$

holds.

4. *The matrix estimate*

$$\lim_{\tau_n \to \infty} \cdots \lim_{\tau_1 \to \infty} \left(\langle \mathcal{X}_{\vec{\tau}} \cdot \eta_{\vec{\tau}}^+, \eta_{\vec{\tau}}^+ \rangle - \langle \mathcal{Y}_{\vec{\tau}} \cdot \eta_{\vec{\tau}}^-, \eta_{\vec{\tau}}^- \rangle \right) = 0 \tag{4.22}$$

holds.

Proof. The proof of the first two items proceeds precisely as in the proof of Lemma 4.9. We will instead focus on the crucial differences in our proof of Items 3 and 4 arising from the frame $\mathfrak{X}$.

Item 3.

Owing to [18, Theorem 3.2] and Lemma 3.9, we have that

$$\begin{cases} \eta_{\vec{\tau}}^+ &= \boldsymbol{A}(p_{\vec{\tau}}) \cdot D_{\text{eucl}(p)} \varphi_{\vec{\tau}}(p_{\vec{\tau}}, q_{\vec{\tau}}) \\ \eta_{\vec{\tau}}^- &= \boldsymbol{A}(q_{\vec{\tau}}) \cdot -D_{\text{eucl}(q)} \varphi_{\vec{\tau}}(p_{\vec{\tau}}, q_{\vec{\tau}}). \end{cases}$$

Direct calculation shows

$$\frac{\partial}{\partial x_k} \varphi_{\vec{\tau}}(p_{\vec{\tau}}, q_{\vec{\tau}}) = \tau_k (x_k^{\vec{\tau}} - y_k^{\vec{\tau}}) = -\frac{\partial}{\partial y_k} \varphi_{\vec{\tau}}(p_{\vec{\tau}}, q_{\vec{\tau}}),$$

so we conclude that

$$[\eta_{\vec{\tau}}^+]_k = \begin{cases} \tau_k (x_k^{\vec{\tau}} - y_k^{\vec{\tau}}), & k = 1 \\ \tau_k (x_k^{\vec{\tau}} - y_k^{\vec{\tau}}) \rho_k(p_{\vec{\tau}}), & 2 \leq k \end{cases}$$

and

$$[\eta_{\vec{\tau}}^-]_k = \begin{cases} \tau_k (x_k^{\vec{\tau}} - y_k^{\vec{\tau}}), & k = 1 \\ \tau_k (x_k^{\vec{\tau}} - y_k^{\vec{\tau}}) \rho_k(q_{\vec{\tau}}), & 2 \leq k. \end{cases}$$

This leads us to:

$$\left| \|\eta_{\vec{\tau}}^+\|^2 - \|\eta_{\vec{\tau}}^-\|^2 \right| \leq \sum_{k=2}^{n} \tau_k^2 (x_k^{\vec{\tau}} - y_k^{\vec{\tau}})^2 \left| \rho_k^2(p_{\vec{\tau}}) - \rho_k^2(q_{\vec{\tau}}) \right|. \tag{4.23}$$

Fixing any $2 \leq k \leq n$, observe that by Equation (4.5) we must have

$$\begin{aligned}\lim_{\tau_{k-1}\to\infty}\cdots\lim_{\tau_1\to\infty}\tau_k^2(x_k^{\vec{\tau}}-y_k^{\vec{\tau}})^2\left|\rho_k^2(p_{\vec{\tau}})-\rho_k^2(q_{\vec{\tau}})\right| &= \left|\rho_k^2(x_1^0,\ldots,x_{k-1}^0)-\rho_k^2(x_1^0,\ldots,x_{k-1}^0)\right| \\ &\quad \times \tau_k^2(x_k^{\vec{\tau}}-y_k^{\vec{\tau}})^2 \\ &=0.\end{aligned}$$

Applying the above to Inequality (4.23) and utilizing the terminology of Equation (4.5),

$$\begin{aligned}\lim_{\tau_1\to\infty}\left|\|\eta_{\vec{\tau}}^+\|^2-\|\eta_{\vec{\tau}}^-\|^2\right| &\leq \sum_{k=3}^{n}\tau_k^2(x_k^{\vec{\tau}}-y_k^{\vec{\tau}})^2\left|\rho_k^2(p_{\tau_1})-\rho_k^2(q_{\tau_1})\right| \\ \lim_{\tau_2\to\infty}\lim_{\tau_1\to\infty}\left|\|\eta_{\vec{\tau}}^+\|^2-\|\eta_{\vec{\tau}}^-\|^2\right| &\leq \sum_{k=4}^{n}\tau_k^2(x_k^{\vec{\tau}}-y_k^{\vec{\tau}})^2\left|\rho_k^2(p_{\tau_1,\tau_2})-\rho_k^2(q_{\tau_1,\tau_2})\right| \\ &\vdots \\ \lim_{\tau_{n-2}\to\infty}\cdots\lim_{\tau_1\to\infty}\left|\|\eta_{\vec{\tau}}^+\|^2-\|\eta_{\vec{\tau}}^-\|^2\right| &\leq \tau_n^2(x_n^{\vec{\tau}}-y_n^{\vec{\tau}})^2\left|\rho_n^2(p_{\tau_1,\ldots,\tau_{n-2}})-\rho_n^2(q_{\tau_1,\ldots,\tau_{n-2}})\right| \\ \lim_{\tau_{n-1}\to\infty}\cdots\lim_{\tau_1\to\infty}\left|\|\eta_{\vec{\tau}}^+\|^2-\|\eta_{\vec{\tau}}^-\|^2\right| &\leq \tau_n^2(x_n^{\vec{\tau}}-y_n^{\vec{\tau}})^2\left|\rho_n^2(x_1^0,\ldots,x_{n-1}^0)-\rho_n^2(x_1^0,\ldots,x_{n-1}^0)\right| \\ &= 0.\end{aligned}$$

From this, the limit

$$\lim_{\tau_n\to\infty}\cdots\lim_{\tau_1\to\infty}\left|\|\eta_{\vec{\tau}}^+\|^2-\|\eta_{\vec{\tau}}^-\|^2\right| = 0$$

is clear.

Item 4.

We begin by decomposing the left-hand side of the Estimate (4.22) into two terms as before:

$$\left\langle \mathcal{X}_{\vec{\tau}}\cdot\eta_{\vec{\tau}}^+,\eta_{\vec{\tau}}^+\right\rangle - \left\langle \mathcal{Y}_{\vec{\tau}}\cdot\eta_{\vec{\tau}}^-,\eta_{\vec{\tau}}^-\right\rangle = I_1 + I_2$$

where we have defined

$$I_1 := \left\langle \left(\boldsymbol{A}(p_{\vec{\tau}})\cdot X_{\vec{\tau}}\cdot\boldsymbol{A}^{\mathrm{T}}(p_{\vec{\tau}})\right)\cdot\eta_{\vec{\tau}}^+,\eta_{\vec{\tau}}^+\right\rangle - \left\langle \left(\boldsymbol{A}(q_{\vec{\tau}})\cdot Y_{\vec{\tau}}\cdot\boldsymbol{A}^{\mathrm{T}}(q_{\vec{\tau}})\right)\cdot\eta_{\vec{\tau}}^-,\eta_{\vec{\tau}}^-\right\rangle$$

(recall that $X_{\vec{\tau}}, Y_{\vec{\tau}}$ are a result of [18, Theorem 3.2]), and

$$I_2 := \left\langle \boldsymbol{M}(D_{\mathrm{eucl}(p)}\varphi_{\vec{\tau}}(p_{\vec{\tau}},q_{\vec{\tau}}),p_{\vec{\tau}})\cdot\eta_{\vec{\tau}}^+,\eta_{\vec{\tau}}^+\right\rangle - \left\langle \boldsymbol{M}(D_{\mathrm{eucl}(q)}\varphi_{\vec{\tau}}(p_{\vec{\tau}},q_{\vec{\tau}}),q_{\vec{\tau}})\cdot\eta_{\vec{\tau}}^-,\eta_{\vec{\tau}}^-\right\rangle.$$

Writing $\tilde{\epsilon} := \boldsymbol{A}(p_{\vec{\tau}}) \cdot \epsilon$ and $\hat{\chi} := \boldsymbol{A}(q_{\vec{\tau}}) \cdot \chi$ to represent twisting according to Lemma 3.9,

$$\begin{aligned} I_1 &= \left\langle X_{\vec{\tau}} \cdot \widetilde{\eta_{\vec{\tau}}^{+}}, \widetilde{\eta_{\vec{\tau}}^{+}} \right\rangle - \left\langle Y_{\vec{\tau}} \cdot \widehat{\eta_{\vec{\tau}}^{-}}, \widehat{\eta_{\vec{\tau}}^{-}} \right\rangle \\ &\leq \langle \mathcal{C} \cdot \zeta, \zeta \rangle . \end{aligned} \tag{4.24}$$

Here, $\zeta := \widetilde{\eta_{\vec{\tau}}^{+}} \oplus \left(-\widehat{\eta_{\vec{\tau}}^{-}}\right) \in \mathbb{R}^{2n}$ and $\mathcal{C}$ is a $2n \times 2n$ matrix resulting from [18, Theorem 3.2] which can be represented in block form as

$$\begin{pmatrix} B & -B \\ -B & B \end{pmatrix},$$

where we define

$$[B]_{k\ell} := \begin{cases} \tau_k + 2\delta\tau_k^2, & k = \ell \\ 0, & k \neq \ell \end{cases}$$

and $\delta > 0$ is an arbitrary parameter resulting from the theorem of [18]. The definition of $\mathcal{C}$ and B and Inequality (4.24) together yield

$$\begin{aligned} I_1 &\leq \left\langle B \cdot (\widetilde{\eta_{\vec{\tau}}^{+}} - \widehat{\eta_{\vec{\tau}}^{-}}), (\widetilde{\eta_{\vec{\tau}}^{+}} - \widehat{\eta_{\vec{\tau}}^{-}}) \right\rangle \\ &= \sum_{k=2}^{n} (\tau_k + 2\delta\tau_k^2) \cdot \left(\rho_k^2(p_{\vec{\tau}}) - \rho_k^2(q_{\vec{\tau}})\right)^2 \cdot \tau_k^2 (x_k^{\vec{\tau}} - y_k^{\vec{\tau}})^2. \end{aligned} \tag{4.25}$$

Since the terms on the right-hand side of (4.25) contain no factors τ_ℓ for $\ell \leq k-1$,

$$\lim_{\tau_{k-1} \to \infty} \cdots \lim_{\tau_1 \to \infty} (\tau_k + 2\delta\tau_k^2) \cdot \left(\rho_k^2(p_{\vec{\tau}}) - \rho_k^2(q_{\vec{\tau}})\right)^2 \cdot \tau_k^2 (x_k^{\vec{\tau}} - y_k^{\vec{\tau}})^2 = 0; \tag{4.26}$$

Equation (4.26), work similar to what was employed in Item 3, and Inequality (4.25) therefore show that

$$\lim_{\tau_n \to \infty} \cdots \lim_{\tau_1 \to \infty} I_1 = 0. \tag{4.27}$$

It remains to show that I_2 tends to 0 as $\tau_k \to \infty$ for all $1 \leq k \leq n$.

Recalling the definition of the matrix $\boldsymbol{M}(\cdot, \cdot)$ from Equation (5.6), we may calculate directly the first entry in both of the inner-products defining I_2. Writing $\boldsymbol{M}_p$ and $\boldsymbol{M}_q$ to refer to the matrices resulting from

$\boldsymbol{M}(\cdot,\cdot)$ evaluated at $(D_{\text{eucl}(p)}\varphi_{\vec{\tau}}(p_{\vec{\tau}},q_{\vec{\tau}}),p_{\vec{\tau}}),(D_{\text{eucl}(q)}\varphi_{\vec{\tau}}(p_{\vec{\tau}},q_{\vec{\tau}}),q_{\vec{\tau}})$ respectively:

$$[\boldsymbol{M}_p\cdot\eta^{+}_{\vec{\tau}}]_h=\begin{cases}\dfrac{1}{2}\displaystyle\sum_{\ell=2}^{n}\left(\frac{\partial\rho_\ell}{\partial x_1}\rho_\ell\right)(p_{\vec{\tau}})\cdot\tau_\ell^2(x_\ell^{\vec{\tau}}-y_\ell^{\vec{\tau}})^2 & h=1\\[2em] \dfrac{1}{2}\displaystyle\sum_{\ell=1}^{h-1}\left(\frac{\partial\rho_h}{\partial x_\ell}\rho_\ell^2\right)(p_{\vec{\tau}})\cdot\tau_\ell(x_\ell^{\vec{\tau}}-y_\ell^{\vec{\tau}})\cdot\tau_h(x_h^{\vec{\tau}}-y_h^{\vec{\tau}}) & \\ +\dfrac{1}{2}\displaystyle\sum_{\ell=h+1}^{n}\left(\frac{\partial\rho_\ell}{\partial x_h}\rho_\ell\rho_h\right)(p_{\vec{\tau}})\cdot\tau_\ell^2(x_\ell^{\vec{\tau}}-y_\ell^{\vec{\tau}})^2 & h\geq 2\end{cases}\tag{4.28}$$

and

$$[\boldsymbol{M}_q\cdot\eta^{-}_{\vec{\tau}}]_h=\begin{cases}\dfrac{1}{2}\displaystyle\sum_{\ell=2}\left(\frac{\partial\rho_\ell}{\partial x_1}\rho_\ell\right)(q_{\vec{\tau}})\cdot\tau_\ell^2(x_\ell^{\vec{\tau}}-y_\ell^{\vec{\tau}})^2 & h=1\\[2em] \dfrac{1}{2}\displaystyle\sum_{\ell=1}^{h-1}\left(\frac{\partial\rho_h}{\partial x_\ell}\rho_\ell^2\right)(q_{\vec{\tau}})\cdot\tau_\ell(x_\ell^{\vec{\tau}}-y_\ell^{\vec{\tau}})\cdot\tau_h(x_h^{\vec{\tau}}-y_h^{\vec{\tau}}) & \\ +\dfrac{1}{2}\displaystyle\sum_{\ell=h+1}^{n}\left(\frac{\partial\rho_\ell}{\partial x_h}\rho_\ell\rho_h\right)(q_{\vec{\tau}})\cdot\tau_\ell^2(x_\ell^{\vec{\tau}}-y_\ell^{\vec{\tau}})^2 & h\geq 2.\end{cases}\tag{4.29}$$

Owing to Equations (4.28) and (4.29) and the observation that $\boldsymbol{M}(\cdot,\cdot)$ is symmetric, we may calculate I_2 as follows:

$$\begin{aligned}I_2=&\left\langle\boldsymbol{M}_p\cdot\eta^{+}_{\vec{\tau}},\eta^{+}_{\vec{\tau}}\right\rangle-\left\langle\boldsymbol{M}_q\cdot\eta^{-}_{\vec{\tau}},\eta^{-}_{\vec{\tau}}\right\rangle\\ =&\frac{1}{2}\sum_{\ell=2}^{n}\left(\frac{\partial\rho_\ell}{\partial x_1}\rho_\ell\right)(p_{\vec{\tau}})\cdot\tau_\ell^2(x_\ell^{\vec{\tau}}-y_\ell^{\vec{\tau}})^2\cdot\tau_1(x_1^{\vec{\tau}}-y_1^{\vec{\tau}})\\ &+\frac{1}{2}\sum_{h=2}^{n}\sum_{\ell=1}^{h-1}\left(\frac{\partial\rho_h}{\partial x_\ell}\rho_h\rho_\ell^2\right)(p_{\vec{\tau}})\cdot\tau_\ell(x_\ell^{\vec{\tau}}-y_\ell^{\vec{\tau}})\cdot\tau_h^2(x_h^{\vec{\tau}}-y_h^{\vec{\tau}})^2\\ &+\frac{1}{2}\sum_{h=2}^{n-1}\sum_{\ell=h+1}^{n}\left(\frac{\partial\rho_\ell}{\partial x_h}\rho_\ell\rho_h^2\right)(p_{\vec{\tau}})\cdot\tau_h(x_h^{\vec{\tau}}-y_h^{\vec{\tau}})\cdot\tau_\ell^2(x_\ell^{\vec{\tau}}-y_\ell^{\vec{\tau}})^2\\ &-\frac{1}{2}\sum_{\ell=2}^{n}\left(\frac{\partial\rho_\ell}{\partial x_1}\rho_\ell\right)(q_{\vec{\tau}})\cdot\tau_\ell^2(x_\ell^{\vec{\tau}}-y_\ell^{\vec{\tau}})^2\cdot\tau_1(x_1^{\vec{\tau}}-y_1^{\vec{\tau}})\\ &-\frac{1}{2}\sum_{h=2}^{n}\sum_{\ell=1}^{h-1}\left(\frac{\partial\rho_h}{\partial x_\ell}\rho_h\rho_\ell^2\right)(q_{\vec{\tau}})\cdot\tau_\ell(x_\ell^{\vec{\tau}}-y_\ell^{\vec{\tau}})\cdot\tau_h^2(x_h^{\vec{\tau}}-y_h^{\vec{\tau}})^2\\ &-\frac{1}{2}\sum_{h=2}^{n-1}\sum_{\ell=h+1}^{n}\left(\frac{\partial\rho_\ell}{\partial x_h}\rho_\ell\rho_h^2\right)(q_{\vec{\tau}})\cdot\tau_h(x_h^{\vec{\tau}}-y_h^{\vec{\tau}})\cdot\tau_\ell^2(x_\ell^{\vec{\tau}}-y_\ell^{\vec{\tau}})^2\end{aligned}$$

The sums above may be combined as follows.

$$\begin{aligned}
2I_2 \quad &= \quad \sum_{\ell=2}^{n} \tau_\ell^2 (x_\ell^{\vec{\tau}} - y_\ell^{\vec{\tau}})^2 \cdot \tau_1 (x_1^{\vec{\tau}} - y_1^{\vec{\tau}}) \cdot \left(\left(\frac{\partial \rho_\ell}{\partial x_1} \rho_\ell \right) (p_{\vec{\tau}}) - \left(\frac{\partial \rho_\ell}{\partial x_1} \rho_\ell \right) (q_{\vec{\tau}}) \right) \\
&\quad + \sum_{h=2}^{n} \sum_{\ell=1}^{h-1} \tau_\ell (x_\ell^{\vec{\tau}} - y_\ell^{\vec{\tau}}) \cdot \tau_h^2 (x_h^{\vec{\tau}} - y_h^{\vec{\tau}})^2 \cdot \left(\left(\frac{\partial \rho_h}{\partial x_\ell} \rho_h \rho_\ell^2 \right) (p_{\vec{\tau}}) - \left(\frac{\partial \rho_h}{\partial x_\ell} \rho_h \rho_\ell^2 \right) (q_{\vec{\tau}}) \right) \\
&\quad + \sum_{h=2}^{n-1} \sum_{\ell=h+1}^{n} \tau_h (x_h^{\vec{\tau}} - y_h^{\vec{\tau}}) \cdot \tau_\ell^2 (x_\ell^{\vec{\tau}} - y_\ell^{\vec{\tau}})^2 \cdot \left(\left(\frac{\partial \rho_\ell}{\partial x_h} \rho_\ell \rho_h^2 \right) (p_{\vec{\tau}}) - \left(\frac{\partial \rho_\ell}{\partial x_h} \rho_\ell \rho_h^2 \right) (q_{\vec{\tau}}) \right) \\
&=: \quad T_1 + T_2 + T_3.
\end{aligned}$$

We examine each of the terms T_1, T_2, and T_3 individually.

Term T_1.

By Equation (4.20) and the definition of X_1, we have

$$\tau_1 (x_1^{\vec{\tau}} - y_1^{\vec{\tau}}) = O(1) \text{ as } \tau_1 \to \infty$$

Thus, for $2 \leq \ell \leq n$,

$$\begin{aligned}
&\lim_{\tau_{\ell-1} \to \infty} \cdots \lim_{\tau_1 \to \infty} \tau_\ell^2 (x_\ell^{\vec{\tau}} - y_\ell^{\vec{\tau}})^2 \cdot \tau_1 (x_1^{\vec{\tau}} - y_1^{\vec{\tau}}) \cdot \left(\left(\frac{\partial \rho_\ell}{\partial x_1} \rho_\ell \right) (p_{\vec{\tau}}) - \left(\frac{\partial \rho_\ell}{\partial x_1} \rho_\ell \right) (q_{\vec{\tau}}) \right) \\
&\sim \tau_\ell^2 (x_\ell^{\vec{\tau}} - y_\ell^{\vec{\tau}})^2 \cdot \left(\left(\frac{\partial \rho_\ell}{\partial x_1} \rho_\ell \right) (p_0) - \left(\frac{\partial \rho_\ell}{\partial x_1} \rho_\ell \right) (p_0) \right) = 0.
\end{aligned}$$

We then conclude that

$$\lim_{\tau_n \to \infty} \lim_{\tau_{n-1} \to \infty} \cdots \lim_{\tau_1 \to \infty} \tau_\ell^2 (x_\ell^{\vec{\tau}} - y_\ell^{\vec{\tau}})^2 \cdot \tau_1 (x_1^{\vec{\tau}} - y_1^{\vec{\tau}}) \cdot \left(\left(\frac{\partial \rho_\ell}{\partial x_1} \rho_\ell \right) (p_{\vec{\tau}}) - \left(\frac{\partial \rho_\ell}{\partial x_1} \rho_\ell \right) (q_{\vec{\tau}}) \right) = 0.$$

Term T_2.

Fix ℓ and h with $\ell < h$. We have

$$\begin{aligned}
&\lim_{\tau_{\ell-1} \to \infty} \cdots \lim_{\tau_1 \to \infty} \tau_\ell (x_\ell^{\vec{\tau}} - y_\ell^{\vec{\tau}}) \cdot \tau_h^2 (x_h^{\vec{\tau}} - y_h^{\vec{\tau}})^2 \cdot \left(\left(\frac{\partial \rho_h}{\partial x_\ell} \rho_h \rho_\ell^2 \right) (p_{\vec{\tau}}) - \left(\frac{\partial \rho_h}{\partial x_\ell} \rho_h \rho_\ell^2 \right) (q_{\vec{\tau}}) \right) \\
&= \tau_\ell (x_\ell^{\vec{\tau}} - y_\ell^{\vec{\tau}}) \cdot \tau_h^2 (x_h^{\vec{\tau}} - y_h^{\vec{\tau}})^2 \cdot \rho_\ell^2 (p_0) \left(\left(\frac{\partial \rho_h}{\partial x_\ell} \rho_h \right) (p_{\vec{\tau}}) - \left(\frac{\partial \rho_h}{\partial x_\ell} \rho_h \right) (q_{\vec{\tau}}) \right).
\end{aligned}$$

If $\rho_\ell(p_0) = 0$, we can easily conclude

$$\lim_{\tau_n \to \infty} \lim_{\tau_{n-1} \to \infty} \cdots \lim_{\tau_1 \to \infty} \tau_\ell (x_\ell^{\vec{\tau}} - y_\ell^{\vec{\tau}}) \tau_h^2 (x_h^{\vec{\tau}} - y_h^{\vec{\tau}})^2 \left(\left(\frac{\partial \rho_h}{\partial x_\ell} \rho_h \rho_\ell^2 \right) (p_{\vec{\tau}}) - \left(\frac{\partial \rho_h}{\partial x_\ell} \rho_h \rho_\ell^2 \right) (q_{\vec{\tau}}) \right) = 0.$$

If $\rho_\ell(p_0) \neq 0$, then $x_\ell^{\vec{\tau}}$ and $y_\ell^{\vec{\tau}}$ lie in a locally Riemannian neighborhood of x_ℓ^0. By Equation (4.20)

$$\lim_{\tau_\ell \to \infty} \tau_\ell (x_\ell^{\vec{\tau}} - y_\ell^{\vec{\tau}}) = O(1).$$

Thus

$$\begin{aligned} &\lim_{\tau_\ell \to \infty} \cdots \lim_{\tau_1 \to \infty} \tau_\ell (x_\ell^{\vec{\tau}} - y_\ell^{\vec{\tau}}) \cdot \tau_h^2 (x_h^{\vec{\tau}} - y_h^{\vec{\tau}})^2 \cdot \left(\left(\frac{\partial \rho_h}{\partial x_\ell} \rho_h \rho_\ell^2 \right)(p_{\vec{\tau}}) - \left(\frac{\partial \rho_h}{\partial x_\ell} \rho_h \rho_\ell^2 \right)(q_{\vec{\tau}}) \right) \\ &\sim \tau_h^2 (x_h^{\vec{\tau}} - y_h^{\vec{\tau}})^2 \cdot \rho_\ell^2(p_0) \left(\left(\frac{\partial \rho_h}{\partial x_\ell} \rho_h \right)(p_{\vec{\tau}}) - \left(\frac{\partial \rho_h}{\partial x_\ell} \rho_h \right)(q_{\vec{\tau}}) \right). \end{aligned}$$

We then have

$$\begin{aligned} &\lim_{\tau_{h-1} \to \infty} \cdots \lim_{\tau_1 \to \infty} \tau_\ell (x_\ell^{\vec{\tau}} - y_\ell^{\vec{\tau}}) \cdot \tau_h^2 (x_h^{\vec{\tau}} - y_h^{\vec{\tau}})^2 \cdot \left(\left(\frac{\partial \rho_h}{\partial x_\ell} \rho_h \rho_\ell^2 \right)(p_{\vec{\tau}}) - \left(\frac{\partial \rho_h}{\partial x_\ell} \rho_h \rho_\ell^2 \right)(q_{\vec{\tau}}) \right) \\ &\sim \tau_h^2 (x_h^{\vec{\tau}} - y_h^{\vec{\tau}})^2 \cdot \rho_\ell^2(p_0) \left(\left(\frac{\partial \rho_h}{\partial x_\ell} \rho_h \right)(p_0) - \left(\frac{\partial \rho_h}{\partial x_\ell} \rho_h \right)(p_0) \right) = 0. \end{aligned}$$

In this case, we then have

$$\lim_{\tau_n \to \infty} \cdots \lim_{\tau_1 \to \infty} \tau_\ell (x_\ell^{\vec{\tau}} - y_\ell^{\vec{\tau}}) \cdot \tau_h^2 (x_h^{\vec{\tau}} - y_h^{\vec{\tau}})^2 \cdot \left(\left(\frac{\partial \rho_h}{\partial x_\ell} \rho_h \rho_\ell^2 \right)(p_{\vec{\tau}}) - \left(\frac{\partial \rho_h}{\partial x_\ell} \rho_h \rho_\ell^2 \right)(q_{\vec{\tau}}) \right) = 0.$$

Term T_3.

This term is symmetric with respect to Term T_2; the proof that

$$\lim_{\tau_n \to \infty} \lim_{\tau_{n-1} \to \infty} \cdots \lim_{\tau_1 \to \infty} \tau_h (x_h^{\vec{\tau}} - y_h^{\vec{\tau}}) \tau_\ell^2 (x_\ell^{\vec{\tau}} - y_\ell^{\vec{\tau}})^2 \left(\left(\frac{\partial \rho_\ell}{\partial x_h} \rho_\ell \rho_h^2 \right)(p_{\vec{\tau}}) - \left(\frac{\partial \rho_\ell}{\partial x_h} \rho_\ell \rho_h^2 \right)(q_{\vec{\tau}}) \right) = 0$$

is similar and therefore omitted.

Our work with each of the three terms implies

$$\lim_{\tau_n \to \infty} \lim_{\tau_{n-1} \to \infty} \lim_{\tau_{n-2} \to \infty} \cdots \lim_{\tau_1 \to \infty} 2I_2 = 0.$$

This and the iterated limit (4.27) together prove Item 4. $\square$

4.3 Uniqueness of Infinity-Harmonic Functions

The Lemmas 4.9 and 4.10 provide similar estimates, and so we combine comparison results for both cases into the following theorem. Its aim is to provide a comparison principle for the Dirichlet Problem (4.2) in the case that $\mathcal{H} = \mathcal{F}^\kappa$ – that is, for the problem

$$\begin{cases} \mathcal{F}^\kappa w &= 0 \quad \text{in } \Omega \\ w &= g \quad \text{on } \partial\Omega. \end{cases} \tag{FDP}$$

A corollary result will prove that a similar comparison can be made in the case of the Dirichlet Problem

$$\begin{cases} \mathcal{G}^\kappa w &= 0 \quad \text{in } \Omega \\ w &= g \quad \text{on } \partial\Omega. \end{cases} \tag{GDP}$$

Theorem 4.11. *Let $\mathbb{G}$ be either a σ-Space or a General Triangular Space. Assume that u_∞ is the viscosity solution to* (FDP) *proven to exist by Theorem 4.3; assume also that v is a viscosity subsolution to Problem* (FDP). *Then $v \leq u_\infty$ on $\overline{\Omega}$.*

Proof. Suppose to the contrary and recall that, since u_∞ is both a viscosity sub- and supersolution to (FDP), we will have $v \leq g \leq u_\infty$ on $\partial\Omega$ by our definitions. It must be that

$$\sup_\Omega (v - u_\infty) = v(p_0) - u_\infty(p_0) > 0. \tag{4.30}$$

The results [3, Lemma 5.1, Theorem 5.3] permit us to assume that there exists $\mu(\cdot) > 0$ so that

$$\mathcal{F}^\kappa u_\infty(p) = \mu(p) > 0.$$

Taking the difference of $\mathcal{F}^\kappa u_\infty$ and $\mathcal{F}^\kappa v$ on the sequence $(p_{\vec{\tau}}, q_{\vec{\tau}}) \subset \Omega \times \Omega$,

$$\begin{aligned} 0 < \mu(q_{\vec{\tau}}) &< \mathcal{F}^\kappa u_\infty(q_{\vec{\tau}}) - \mathcal{F}^\kappa v(p_{\vec{\tau}}) \\ &= \min\left\{\|\eta_{\vec{\tau}}^-\|^2 - \kappa^2, -\left\langle \mathcal{Y}_{\vec{\tau}} \cdot \eta_{\vec{\tau}}^-, \eta_{\vec{\tau}}^- \right\rangle\right\} - \min\left\{\|\eta_{\vec{\tau}}^+\|^2 - \kappa^2, -\left\langle \mathcal{X}_{\vec{\tau}} \cdot \eta_{\vec{\tau}}^+, \eta_{\vec{\tau}}^+ \right\rangle\right\} \\ &\leq \max\left\{\|\eta_{\vec{\tau}}^-\|^2 - \|\eta_{\vec{\tau}}^+\|^2, \left\langle \mathcal{X}_{\vec{\tau}} \cdot \eta_{\vec{\tau}}^+, \eta_{\vec{\tau}}^+ \right\rangle - \left\langle \mathcal{Y}_{\vec{\tau}} \cdot \eta_{\vec{\tau}}^-, \eta_{\vec{\tau}}^- \right\rangle\right\}. \end{aligned} \tag{4.31}$$

Since $u_\infty \in C(\Omega) \cap W^{1,\infty}_{\text{loc}}(\Omega)$, the assumptions of Lemmas 4.9 or 4.10 are satisfied (the choice of lemma dependent upon $\mathbb{G}$) – so we may apply the appropriate lemma from the previous sections, [3, Lemma 5.1, Theorem 5.3], and notice

$$\mu(q_{\vec{\tau}}) \to \mu(p_0) > 0 \tag{4.32}$$

and

$$\max\left\{\|\eta_{\vec{\tau}}^-\|^2 - \|\eta_{\vec{\tau}}^+\|^2, \left\langle \mathcal{X}_{\vec{\tau}} \cdot \eta_{\vec{\tau}}^+, \eta_{\vec{\tau}}^+ \right\rangle - \left\langle \mathcal{Y}_{\vec{\tau}} \cdot \eta_{\vec{\tau}}^-, \eta_{\vec{\tau}}^- \right\rangle\right\} \to 0 \tag{4.33}$$

as $\tau_1, \ldots, \tau_n \to \infty$. Regardless of whether $\mathbb{G}$ is a σ-Space or a General Triangular Space, we arrive at a contradiction by applying (4.31), (4.32), and (4.33). $\square$

In the same manner we can prove a similar result for the operator $\mathcal{G}^\kappa$.

Corollary 4.12. *Let $\mathbb{G}$ be either a σ-Space or a General Triangular Space. Assume that u_∞ is the viscosity solution to* (GDP) *proven to exist by Theorem 4.3; assume also that v is a viscosity supersolution to Problem* (GDP). *Then $u_\infty \leq v$ on $\overline{\Omega}$.*

The following properties of of solutions to (FDP) and (GDP) are evident from the definition of the operators $\mathcal{F}^\kappa$ and $\mathcal{G}^\kappa$:

- If u is a viscosity solution to Problem (FDP), then it is a viscosity *supersolution* to Problem (DP) – that is, u is ∞-superharmonic.
- If u is a viscosity solution to Problem (GDP), then it is a viscosity *subsolution* to Problem (DP) – that is, u is ∞-subharmonic.

We now state a lemma which relates solutions of (FDP) and (GDP). In light of the comparisons above, the uniqueness of the ∞-harmonic function u_∞ follows as a corollary.

Lemma 4.13 (C.f. [3, Lemma 5.6])**.** *Let u^κ and u_κ represent the solutions to Problems* (FDP) *and* (GDP) *respectively. Given $\delta > 0$, there exists $\kappa > 0$ so that*

$$u_\kappa \leq u^\kappa \leq u_\kappa + \delta.$$

CHAPTER 5:
SUBPARABOLIC VISCOSITY SOLUTIONS

Having addressed the existence and uniqueness of ∞-harmonic functions in the subelliptic environment for $\mathbb{G}$, we now turn our attention to the related subparabolic environment. Our objective, once again, is to establish existence and uniqueness of solutions to problems for the ∞-Laplacian, this time in the case of Cauchy-Dirichlet type problems. Unlike the subelliptic case, the current chapter and Chapter 6 will work exclusively in what we called General Triangular Spaces in Chapter 4.

In addition to the function spaces identified previously, we will have need for *parabolic* sets, sometimes referred to as *cylinders*, which consider both time and space. For a given $\mathcal{O} \subseteq \mathbb{G}$ which is open and interval $(t_1, t_2) \subset \mathbb{R}$, we define the parabolic set $\mathcal{O}_{t_1,t_2} := \mathcal{O} \times (t_1, t_2)$ and write $\mathcal{O}_{t_2}$ whenever $t_1 = 0$. Its parabolic boundary is

$$\partial_{\text{par}}\, \mathcal{O}_{t_1,t_2} := (\overline{\mathcal{O}} \times \{t_1\}\,) \cup (\partial\mathcal{O} \times (t_1, t))$$

which contains the lower cap and sides of the parabolic set but not its upper cap. Given a function $u : \mathcal{O}_{t_1,t_2} \to X$ for some metric space X, we say that $u \in C(t_1, t_2; X)$ if $u \in C(\mathcal{O}_{t_1,t_2})$ and $\max_{t_1 \le s \le t_2} \|u(\cdot, s)\|_X < \infty$. We say u is a member of $L^{\mathtt{r}}(t_1, t_2; X)$ if

$$\left(\int_{t_1}^{t_2} \|u(\cdot, s)\|_X^{\mathtt{r}}\, ds\right)^{1/\mathtt{r}} < \infty.$$

Finally, we follow the convention of [27] and define the space $V^{\mathtt{r}}(t_1, t_2; \mathcal{O})$:

$$V^{\mathtt{r}}(t_1, t_2; \mathcal{O}) := C(t_1, t_2; L^2(\mathcal{O})) \cap L^{\mathtt{r}}(t_1, t_2; W^{1,\mathtt{r}}(\mathcal{O})).$$

Functions belonging to this space possess the necessary temporal and spatial regularity for the existence arguments invoked in Chapter 6.

As in the subelliptic case, we desire to solve problems involving the ∞-Laplace operator $\Delta_{\mathfrak{X},\infty}$. To that end, introduce the subparabolic equation

$$w_t(p, t) + \mathcal{H}w(p) = 0 \text{ in } \Omega_T \tag{5.1}$$

for bounded domains $\Omega \Subset \mathbb{G}$ and some $T > 0$ and seek to extend our viscosity theory to such subparabolic equations.

Similarly to the subelliptic environment, one can introduce a Grushin-type Taylor expansion each $(p_0, t_0) \in \Omega_T$ for functions $w \in C^2_{\mathbb{G}}(\Omega) \cap C^1_{\text{eucl}}([0,T))$:

$$\begin{aligned} w(p) &= w(p_0,t_0) + w_t(p_0,t_0)\cdot(t-t_0) + \sum_{k\notin N(p_0)} \frac{1}{\rho_k(p_0)} X_k w(p_0,t_0)\cdot(x_k - x_k^0) \\ &+\frac{1}{2}\sum_{k\notin N(p_0)} \frac{1}{\rho_k^2(p_0)} X_kX_k w(p_0,t_0)\cdot (x_k-x_k^0)^2 \\ &+ \sum_{\substack{k,\ell\notin N(p_0)\\ k<\ell}} \frac{1}{(\rho_k\rho_\ell)(p_0)}\cdot\Bigg(\frac{X_\ell X_k w + X_k X_\ell w}{2}(p_0,t_0) \\ &-\frac{1}{2\rho_\ell^2(p_0)}\cdot\frac{\partial \rho_\ell}{\partial x_k}(p_0)\cdot X_\ell w(p_0)\Bigg)\cdot (x_k-x_k^0)(x_\ell - x_\ell^0) \\ &+\sum_{j\in N(p_0)}\left(\frac{1}{\beta_j(p_0)}\sum_{k=1}^{n}\frac{2}{\left(\frac{\partial\rho_j}{\partial x_k}\rho_k\right)(p_0)}\cdot\frac{X_jX_kw + X_kX_jw}{2}(p_0,t_0)\right)\cdot(x_j-x_j^0) \end{aligned}$$

plus an error term $o\left(d_{CC}^2(p_0,p) + |t-t_0|\right)$ as $(p,t)\to(p_0,t_0)$. One could then define subparabolic jets for a (not necessarily regular) function via inequalities like those presented in Definition 3.3.

One can also define a notion of "subparabolic touching functions". Given $u:\Omega_T\to\mathbb{R}$, the set $\mathcal{A}(u,(p_0,t_0))$ denotes the set of "subparabolic touching above functions" for u: That is, the collection containing $\phi \in C^2_{\mathbb{G}}(\Omega)\cap C^1_{\text{eucl}}([0,T))$ such that

$$0 = \phi(p_0,t_0) - u(p_0,t_0) < \phi(p,t) - u(p,t) \text{ near } (p_0,t_0). \tag{5.2}$$

The set $\mathcal{B}(u,(p_0,t_0))$ of "subparabolic touching below functions" contains $\psi \in C^2_{\mathbb{G}}(\Omega)\cap C^1_{\text{eucl}}([0,T))$ which satisfy

$$0 = u(p_0,t_0) - \psi(p_0,t_0) < u(p,t) - \psi(p,t) \text{ near } (p_0,t_0). \tag{5.3}$$

Similar to the work of Chapter 3, one may show that the subparabolic jets are precisely the collection of derivatives for touching functions, and so we we may define the jets in this manner.

Definition 5.1. Given a function $u:\Omega_T\to\mathbb{R}$ and any point $(p_0,t_0)\in\Omega_T$, the *subparabolic superjet* $P^{2,+}\,u(p_0,t_0)$ is the collection

$$\left\{(\phi_t, \nabla_{\mathbb{G}}\,\psi, \left(D^2\phi\right)^\star)(p_0,t_0) \,:\, \phi\in\mathcal{A}(u,(p_0,t_0))\right\}.$$

The *subparabolic subjet* $P^{2,-}\,u(p_0,t_0)$ is the collection

$$\Big\{(\psi_t, \nabla_{\mathbb{G}}\,\psi, \left(D^2\psi\right)^\star)(p_0,t_0) \,:\, \psi \in \mathcal{A}(u,(p_0,t_0))\Big\}\,.$$

Note that $P^{2,-}\,u(p_0,t_0) = -\,P^{2,+}\,[-u(p_0,t_0)]$.

We say that $(a,\eta,X) \in \overline{P}^{2,+}\,u(p_0,t_0)$ if there exist points $(p_n,t_n) \in \Omega_T$ and jet entries $(a_n,\eta_n,X_n) \in P^{2,+}\,u(p_n,t_n)$ such that

$$(p_n,t_n,u(p_n,t_n),a_n,\eta_n,X_n) \to (p_0,t_0,u(p_0,t_0),a,\eta,X)$$

as $n \to \infty$. A similar definition holds for $\overline{P}^{2,-}\,u(p_0,t_0)$.

Definition 5.2. Assume $\mathcal{H} : \mathbb{G}\times\mathbb{R}\times\mathfrak{g}_n\times\mathcal{S}^n \to \mathbb{R}$ is a continuous and proper operator. Let $u \in \mathrm{USC}(\Omega_T)$: We say u is a *parabolic viscosity subsolution* to (5.1) if for each $(p_0,t_0) \in \Omega_T$ and each $(a,\eta,X) \in P^{2,+}\,u(p_0,t_0)$ we have

$$a + \mathcal{H}(p_0,t_0,\eta,X) \le 0.$$

A function $v \in \mathrm{LSC}(\Omega_T)$ is a *parabolic viscosity supersolution* to (5.1) if for each $(p_0,t_0) \in \Omega_T$ and each $(b,\nu,Y) \in P^{2,-}\,v(p_0,t_0)$ we have

$$b + \mathcal{H}(p_0,t_0,\nu,Y) \ge 0$$

– or equivalently, if $-v$ is a parabolic viscosity subsolution. A function $w \in C(\Omega_T)$ is called a *parabolic viscosity solution* if w is both a parabolic viscosity sub- and supersolution.

Remark 5.3. Following the conventions of the previous subelliptic sections, we will refer to parabolic viscosity (sub-/super-)solutions to (5.1) as *parabolic* ∞-(sub-/super-)harmonic functions.

The Lemma 3.9 which we previously utilized has a parabolic analogue which we state below. It provided an explicit relationship between the Euclidean upper jet (denoted as $P^{2,+}_{\mathrm{eucl}}u(p_0,t_0)$) and the Grushin-type upper jet $P^{2,+}\,u(p_0,t_0)$. Because the temporal coordinate a of the ordered triple (a,η,X) requires no twisting, the lemma is proven almost identically to Lemma 3.9

Lemma 5.4 (Parabolic $\mathbb{G}$ Twisting Lemma)**.** *Let $\mathcal{O} \subseteq \mathbb{G}$ be open, let $u : \mathcal{O}_{t_1,t_2} \to \mathbb{R}$, and let $(p_0,t_0) \in \mathcal{O}_{t_1,t_2}$. Suppose that $(a,\eta,X) \in P^{2,+}_{\mathrm{eucl}}u(p_0,t_0)$: Then*

$$\big(a, \boldsymbol{A}(p_0)\cdot\eta, \boldsymbol{A}(p_0)\cdot X\cdot\boldsymbol{A}^{\mathrm{T}}(p_0) + \boldsymbol{M}(\eta,p_0)\big) \in \overset{2,+}{P}\,u(p_0,t_0), \tag{5.4}$$

where

$$(\boldsymbol{A}(p_0))_{k\ell} = \begin{cases} 1, & k = 1 = \ell \\ \\ \rho_k(p), & 2 \leq k = \ell \leq n \\ \\ 0, & \textit{otherwise} \end{cases} \tag{5.5}$$

and

$$(\boldsymbol{M}(\eta, p_0))_{k\ell} = \begin{cases} \frac{1}{2} \cdot \frac{\partial \rho_\ell}{\partial x_k}(p)\rho_k(p)\eta_\ell, & k < \ell \\ \\ \frac{1}{2} \cdot \frac{\partial \rho_k}{\partial x_\ell}(p)\rho_\ell(p)\eta_k, & \ell < k \\ \\ 0, & \textit{otherwise.} \end{cases} \tag{5.6}$$

We will close this section with an examination of what [26] calls parabolic viscosity solutions. Consider the set $\mathcal{A}^-(u(p_0,t_0))$ which contains all $\phi \in C^2_{\mathbb{G}}(\mathcal{O}) \cap C^1_{\text{eucl}}([0,T))$ satisfying

$$0 = \phi(p_0,t_0) - u(p_0,t_0) < \phi(p,t) - u(p,t) \text{ near } (p_0,t_0) \text{ for all } t < t_0.$$

This collection corresponds physically to those test functions where the past alone plays a role in determining the present; evidently, it is a larger collection than $\mathcal{A}\,(u(p_0,t_0))$. We can define the collection $\mathcal{B}^-\,(u(p_0,t_0))$ similarly, and are then able to present a notion of *past* solutions.

Definition 5.5. Let $u \in \text{USC}(\Omega_T)$. The function u is a *past parabolic viscosity subsolution* to (5.1) if for each $(p_0,t_0) \in \Omega_T$ and each $\phi \in \mathcal{A}^-\,(u(p_0,t_0))$ the inequality

$$\phi_t(p_0,t_0) + \mathcal{H}(p_0,t_0,\nabla_{\mathbb{G}}\,\phi(p_0,t_0),(D^2\phi)^\star(p_0,t_0)) \leq 0$$

is satisfied. A function $v \in \text{LSC}(\Omega_T)$ is a *past parabolic viscosity supersolution* to (5.1) if for each $(p_0,t_0) \in \Omega_T$ and each $\psi \in \mathcal{B}^-\,(u(p_0,t_0))$ the inequality

$$\psi_t(p_0,t_0) + \mathcal{H}(p_0,t_0,\nabla_{\mathbb{G}}\,\psi(p_0,t_0),(D^2\psi)^\star(p_0,t_0)) \geq 0$$

is satisfied. A continuous function w is a *past parabolic viscosity solution* to (5.1) if it is both a past parabolic viscosity sub- and supersolution.

A proposition whose proof is obvious follows. We will study the converse in the forthcoming chapter.

Proposition 5.6. *Past parabolic viscosity sub-/supersolutions to* (5.1) *are parabolic viscosity sub-/supersolutions to* (5.1)*.*

CHAPTER 6:
SOLUTIONS TO CAUCHY-DIRICHLET PROBLEMS INVOLVING THE SUBPARABOLIC INFINITY LAPLACIAN

In the current chapter we will present a litany of results which we divide into sections. The first of these sections treats a maximum principle for the parabolic setting which, as in the subelliptic setting, is essential for uniqueness results. Next we prove a subparabolic comparison principle for Cauchy-Dirichlet problems involving the ∞-Laplace equation and show as a corollary that past parabolic viscosity solutions and parabolic viscosity solutions are identical. We then present a proof of the existence of parabolic ∞-harmonic functions which, unlike in the subelliptic setting, requires the use of uniqueness results in our proof. We then conclude by considering Cauchy-Dirichlet problems over infinite cylinders $\Omega \times (0, \infty)$ and show that, under certain restrictions, solutions to these problems will stabilize to the unique subelliptic ∞-harmonic function u_∞ in Ω.

6.1 A Subarabolic Iterated Maximum Principle

Our first step in establishing a comparison principle is proving a parabolic maximum principle. This is the content of the following lemma, which is similar in structure to [14, Lemma 3.6] and [5, Theorem 4.1] and imitates certain steps and calculations found in Chapter 4.

Lemma 6.1. *Suppose that $u \in \mathrm{USC}(\Omega_T)$ is a parabolic viscosity subsolution to* (5.1) *and that $v \in \mathrm{LSC}(\Omega_T)$ is a parabolic viscosity supersolution to* (5.1). *Denoting $\vec{\tau} := (\tau_1, \ldots, \tau_n)$, where each τ_k is a positive, real parameter, define*

$$\varphi_{\vec{\tau}}(p, q, t) := \frac{1}{2} \sum_{k=1}^{n} \tau_k (x_k - y_k)^2$$

and suppose that for each such n-tuple $\vec{\tau}$ the maximum

$$M_{\vec{\tau}} := \sup_{\overline{\Omega} \times \overline{\Omega} \times [0,T)} \Big(u(p, t) - v(q, t) - \varphi_{\vec{\tau}}(p, q, t) \Big) \tag{6.1}$$

occurs at an interior point $(p_{\vec\tau}, q_{\vec\tau}, t_{\vec\tau}) \in \Omega \times \Omega \times (0,T)$. *If*

$$\lim_{\tau_1,\ldots,\tau_n\to\infty} \varphi_{\vec\tau}(p_{\vec\tau}, q_{\vec\tau}, t_{\vec\tau}) = 0,$$

then there exist triples

$$(a, \eta^+_{\vec\tau}, \mathcal{X}_{\vec\tau}) \in \overline{P}^{2,+} u(p_{\vec\tau}, t_{\vec\tau}) \text{ and } (b, \eta^-_{\vec\tau}, \mathcal{Y}_{\vec\tau}) \in \overline{P}^{2,-} v(q_{\vec\tau}, t_{\vec\tau}). \tag{6.2}$$

so that:

1. *We have the equation* $a - b = 0$.

2. *The vector coordinates* $\eta^+_{\vec\tau}$ *and* $\eta^-_{\vec\tau}$ *are given by*

$$\begin{cases} \eta^+_{\vec\tau} &= \boldsymbol{A}(p_{\vec\tau}) \cdot D_{\mathrm{eucl}(p)} \psi_{\vec\tau}(p_{\vec\tau}, q_{\vec\tau}, t_{\vec\tau}) \\ \eta^-_{\vec\tau} &= \boldsymbol{A}(q_{\vec\tau}) \cdot \left(-D_{\mathrm{eucl}(q)} \psi_{\vec\tau}(p_{\vec\tau}, q_{\vec\tau}, t_{\vec\tau})\right). \end{cases} \tag{6.3}$$

3. *The matrices* $\mathcal{X}_{\vec\tau}$ *and* $\mathcal{Y}_{\vec\tau}$ *and vector entries satisfy the inner-product estimate*

$$\lim_{\tau_n\to\infty} \cdots \lim_{\tau_1\to\infty} \Big(\big\langle \mathcal{X}_{\vec\tau} \cdot \eta^+_{\vec\tau}, \eta^+_{\vec\tau} \big\rangle - \big\langle \mathcal{Y}_{\vec\tau} \cdot \eta^-_{\vec\tau}, \eta^-_{\vec\tau} \big\rangle \Big) = 0. \tag{6.4}$$

Before we present the proof of the lemma, we make some necessary observations and comments. First, recall the results of the Iterated Maximum Principle and its corollary results from Chapter 4, which we will need in our work below: The equations

$$\begin{cases} \lim_{\tau_n\to\infty} \cdots \lim_{\tau_1\to\infty} \left(u(p_{\vec\tau}) - v(q_{\vec\tau}) - \varphi_{\vec\tau}(p,q)\right) &= u(p_0) - v(p_0) \\ \\ \lim_{\tau_n\to\infty} \cdots \lim_{\tau_1\to\infty} \varphi_{\vec\tau}(p,q) = \lim_{\tau_1,\ldots,\tau_n\to\infty} \varphi_{\vec\tau}(p,q) &= 0 \end{cases} \tag{6.5}$$

and, writing $p_{\vec\tau} = (x_1^{\vec\tau}, \ldots, x_n^{\vec\tau})$ and $q_{\vec\tau} = (y_1^{\vec\tau}, \ldots, y_n^{\vec\tau})$,

$$\begin{cases} p_{\tau_1,\ldots,\tau_k} &:= \lim_{\tau_k\to\infty} \cdots \lim_{\tau_1\to\infty} p_{\vec\tau} = \quad (x_1^0, \ldots, x_k^0, x_{k+1}^{\vec\tau}, \ldots, x_n^{\vec\tau}) \\ \\ q_{\tau_1,\ldots,\tau_k} &:= \lim_{\tau_k\to\infty} \cdots \lim_{\tau_1\to\infty} q_{\vec\tau} = \quad (x_1^0, \ldots, x_k^0, y_{k+1}^{\vec\tau}, \ldots, y_n^{\vec\tau}) \end{cases} \tag{6.6}$$

for each $1 \leq k \leq n$ are true. Additionally, these limits will hold true even if the order of the iterated limits is changed (although the sequence $(p_{\vec{\tau}}, q_{\vec{\tau}})$ may change). Because $\varphi_{\vec{\tau}}(p,q,t) = \varphi_{\vec{\tau}}(p,q) = \varphi_{\vec{\tau}}(p,q,s)$ for all $s, t \in [0,T)$, Equations (6.5) and (6.6) may be used without alteration in the proof of the lemma.

Second, in the elliptic setting one may show that ∞-harmonic functions belong to $W^{1,\infty}$ by appealing to [28] and exploiting standard techniques of the theory (see, for example, [25].) This fact permits us to assume that either the (elliptic) viscosity sub- or supersolution is locally Lipschitz – from which the Limit (6.4) can be proven. In the current setting, such an assumption is not so clearly founded in the theory. As a consequence of [19], one may show that certain nonlinear parabolic PDE (such as the parabolic p-Laplacian) possess locally Hölder solutions, but it is not known whether such functions belong to $W^{1,\infty}$. To circumvent this difficulty, we introduce the following condition on the functions ρ_k and ρ_ℓ:

$$\lim_{\tau_{k-1}\to\infty} \cdots \lim_{\tau_1\to\infty} \left(\left(\frac{\partial \rho_\ell}{\partial x_k} \rho_\ell \rho_k^2 \right)(p_{\vec{\tau}}) - \left(\frac{\partial \rho_\ell}{\partial x_k} \rho_\ell \rho_k^2 \right)(q_{\vec{\tau}}) \right) \sim (x_k^{\vec{\tau}} - y_k^{\vec{\tau}}) \tag{6.7}$$

for all $1 \leq k < \ell \leq n$.

Remark 6.2. The condition is easily shown to be satisfied in the case that $n = 2$ and ρ_2 is a polynomial. As an example in which (6.7) holds and the functions ρ_k, ρ_ℓ need not be polynomials, let $n = 2$ and define:

$$\rho_2(p) := \frac{1}{2} \sin\left(x_1^2\right).$$

Since we must have $k = 1$ and $\ell = 2$ in (6.7) by our assumptions, and since $\rho_1 \equiv 1$, direct calculation shows

$$\begin{aligned} \left(\frac{\partial \rho_\ell}{\partial x_k} \rho_\ell \rho_k^2 \right)(p_{\vec{\tau}}) - \left(\frac{\partial \rho_\ell}{\partial x_k} \rho_\ell \rho_k^2 \right)(q_{\vec{\tau}}) =& \frac{1}{2} x_1^{\vec{\tau}} \sin\left((x_1^{\vec{\tau}})^2\right) \cos\left((x_1^{\vec{\tau}})^2\right) \\ &- \frac{1}{2} y_1^{\vec{\tau}} \sin\left((y_1^{\vec{\tau}})^2\right) \cos\left((y_1^{\vec{\tau}})^2\right). \end{aligned} \tag{6.8}$$

Applying the Mean Value Theorem, the right-hand side of (6.8) becomes

$$\left(x_1^{\vec{\tau}} - y_1^{\vec{\tau}}\right) \cdot \left(\frac{1}{4} \sin\left(2(z^{\vec{\tau}})^2\right) + (z^{\vec{\tau}})^2 \right)$$

for some $z^{\vec{\tau}}$ in the open interval whose endpoints are $x_1^{\vec{\tau}}$ and $y_1^{\vec{\tau}}$. For τ_1, τ_2 sufficiently large, since $x_1^{\vec{\tau}} \to x_0 \leftarrow y_1^{\vec{\tau}}$ as $\tau_1, \tau_2 \to \infty$, the second factor in the above is bounded and we have satisfied our desired condition.

Remark 6.3. As an example of a frame which *fails* to satisfy Condition (6.7), consider the case where $n = 3$ and $\rho_3(p) = \rho_3(x_1, x_2) := x_1 + x_2$ as on [4, p. 21]. Then

$$\left(\frac{\partial \rho_3}{\partial x_1}\rho_3\rho_1^2\right)(p_{\vec{\tau}}) - \left(\frac{\partial \rho_3}{\partial x_1}\rho_3\rho_1^2\right)(q_{\vec{\tau}}) = (x_1^{\vec{\tau}} + x_2^{\vec{\tau}}) - (y_1^{\vec{\tau}} + y_2^{\vec{\tau}}),$$

which does not uphold the desired relationship.

Proof of Lemma 6.1. Items (A) and (B) above require [18, Theorem 8.3] to find members of the Euclidean parabolic jet closures, after which we apply Lemma 5.4. This necessitates that we show [18, Condition 8.5] is satisfied first, similar to the proof of [13, Lemma 3.4]. Once this is done, we may focus on the proof of Item (C).

We recall [18, Condition 8.5] for convenience: There exists an $r > 0$ so that for every $M > 0$ there is a $C > 0$ so that if $(a, \eta, X) \in P^{2,+}_{\text{eucl}} u(p, t)$ then $a < C$ whenever

$$|p_{\vec{\tau}} - p|_{\text{eucl}} + |t_{\vec{\tau}} - t| < r \text{ and } |u(p,t)| + \|\eta\| + \|X\| < M. \tag{6.9}$$

Suppose that this condition doesn't hold: That is, assume that for every $r > 0$ there is an $M > 0$ such that $a > C$ for all $C > 0$ despite the inclusion $(a, \eta, X) \in P^{2,+}_{\text{eucl}} u(p, t)$ and the Inequalities (6.9) holding. The Lemma 5.4 (the Parabolic $\mathbb{G}$ Twisting Lemma) now implies

$$(a, \boldsymbol{A}(p) \cdot \eta, \boldsymbol{A}(p) \cdot X \cdot \boldsymbol{A}^T(p) + \boldsymbol{M}(\eta, p)) \in P^{2,+} u(p, t). \tag{6.10}$$

The membership (6.10), however, contradicts our assumption that u is a parabolic viscosity subsolution of (5.1). Similar work with $-v$ produces a similar contradiction. Therefore, [18, Condition 8.5] must hold. Utilizing [18, Theorem 8.3], we obtain the (Euclidean) jet memberships

$$\begin{cases} (a, D_{\text{eucl}(p)}\psi_{\vec{\tau}}(p_{\vec{\tau}}, q_{\vec{\tau}}, t_{\vec{\tau}}), X_{\vec{\tau}}) & \in & P^{2,+}_{\text{eucl}} u(p_{\vec{\tau}}, t_{\vec{\tau}}) \\ (b, -D_{\text{eucl}(q)}\psi_{\vec{\tau}}(p_{\vec{\tau}}, q_{\vec{\tau}}, t_{\vec{\tau}}), Y_{\vec{\tau}}) & \in & P^{2,-}_{\text{eucl}} v(q_{\vec{\tau}}, t_{\vec{\tau}}), \end{cases} \tag{6.11}$$

where $X_{\vec{\tau}}, Y_{\vec{\tau}}$ are symmetric $n \times n$ matrices. The Parabolic $\mathbb{G}$ Twisting Lemma implies the desired Grushin-type parabolic jet memberships; Item (A) is a direct consequence of [18, Theorem 8.3], and the Equations (6.3) of Item (B) follow from the twisting lemma.

We now turn to Item (C). Write

$$\left\langle \mathcal{X}_{\vec{\tau}} \cdot \eta^+_{\vec{\tau}}, \eta^+_{\vec{\tau}} \right\rangle - \left\langle \mathcal{Y}_{\vec{\tau}} \cdot \eta^-_{\vec{\tau}}, \eta^-_{\vec{\tau}} \right\rangle = I_1 + I_2$$

by defining the terms

$$I_1 := \left\langle (\boldsymbol{A}(p_{\vec{\tau}}) \cdot X_{\vec{\tau}} \cdot \boldsymbol{A}^T(p_{\vec{\tau}})) \cdot \eta^+_{\vec{\tau}}, \eta^+_{\vec{\tau}} \right\rangle - \left\langle (\boldsymbol{A}(q_{\vec{\tau}}) \cdot Y_{\vec{\tau}} \cdot \boldsymbol{A}^T(q_{\vec{\tau}})) \cdot \eta^-_{\vec{\tau}}, \eta^-_{\vec{\tau}} \right\rangle$$

and

$$I_2 := \left\langle \boldsymbol{M}(D_{\mathrm{eucl}(p)}\psi_{\vec{\tau}}(p_{\vec{\tau}}, q_{\vec{\tau}}, t_{\vec{\tau}}), p_{\vec{\tau}}) \cdot \eta^+_{\vec{\tau}}, \eta^+_{\vec{\tau}} \right\rangle - \left\langle \boldsymbol{M}(D_{\mathrm{eucl}(q)}\psi_{\vec{\tau}}(p_{\vec{\tau}}, q_{\vec{\tau}}, t_{\vec{\tau}}), q_{\vec{\tau}}) \cdot \eta^-_{\vec{\tau}}, \eta^-_{\vec{\tau}} \right\rangle.$$

The proof that

$$\lim_{\tau_n \to \infty} \cdots \lim_{\tau_1 \to \infty} I_1 = 0 \tag{6.12}$$

proceeds precisely as in the proof of Lemma 4.10. Denoting by $\boldsymbol{M}_p$ and $\boldsymbol{M}_q$ the matrices resulting from $\boldsymbol{M}(\cdot,\cdot)$ evaluated at $(D_{\mathrm{eucl}(p)}\varphi_{\vec{\tau}}(p_{\vec{\tau}}, q_{\vec{\tau}}), p_{\vec{\tau}})$ and $(D_{\mathrm{eucl}(q)}\varphi_{\vec{\tau}}(p_{\vec{\tau}}, q_{\vec{\tau}}), q_{\vec{\tau}})$ respectively, the work presented in Chapter 4 for Lemma 4.10 again permits us to show that

$$[\boldsymbol{M}_p \cdot \eta^+_{\vec{\tau}}]_h = \begin{cases} \dfrac{1}{2}\displaystyle\sum_{\ell=2}^{n} \left(\frac{\partial \rho_\ell}{\partial x_1}\rho_\ell\right)(p_{\vec{\tau}}) \cdot \tau_\ell^2 (x_\ell^{\vec{\tau}} - y_\ell^{\vec{\tau}})^2 & h = 1 \\[2em] \dfrac{1}{2}\displaystyle\sum_{\ell=1}^{h-1} \left(\frac{\partial \rho_h}{\partial x_\ell}\rho_\ell^2\right)(p_{\vec{\tau}}) \cdot \tau_\ell (x_\ell^{\vec{\tau}} - y_\ell^{\vec{\tau}}) \cdot \tau_h (x_h^{\vec{\tau}} - y_h^{\vec{\tau}}) & \\ +\dfrac{1}{2}\displaystyle\sum_{\ell=h+1}^{n} \left(\frac{\partial \rho_\ell}{\partial x_h}\rho_\ell\rho_h\right)(p_{\vec{\tau}}) \cdot \tau_\ell^2 (x_\ell^{\vec{\tau}} - y_\ell^{\vec{\tau}})^2 & h \geq 2 \end{cases}$$

and

$$[\boldsymbol{M}_q \cdot \eta^-_{\vec{\tau}}]_h = \begin{cases} \dfrac{1}{2}\displaystyle\sum_{\ell=2} \left(\frac{\partial \rho_\ell}{\partial x_1}\rho_\ell\right)(q_{\vec{\tau}}) \cdot \tau_\ell^2 (x_\ell^{\vec{\tau}} - y_\ell^{\vec{\tau}})^2 & h = 1 \\[2em] \dfrac{1}{2}\displaystyle\sum_{\ell=1}^{h-1} \left(\frac{\partial \rho_h}{\partial x_\ell}\rho_\ell^2\right)(q_{\vec{\tau}}) \cdot \tau_\ell (x_\ell^{\vec{\tau}} - y_\ell^{\vec{\tau}}) \cdot \tau_h (x_h^{\vec{\tau}} - y_h^{\vec{\tau}}) & \\ +\dfrac{1}{2}\displaystyle\sum_{\ell=h+1}^{n} \left(\frac{\partial \rho_\ell}{\partial x_h}\rho_\ell\rho_h\right)(q_{\vec{\tau}}) \cdot \tau_\ell^2 (x_\ell^{\vec{\tau}} - y_\ell^{\vec{\tau}})^2 & h \geq 2. \end{cases}$$

Direct computations with these matrices results in the equation

$$\begin{aligned} 2I_2 = &\sum_{\ell=2}^{n} \tau_\ell^2 (x_\ell^{\vec{\tau}} - y_\ell^{\vec{\tau}})^2 \cdot \tau_1 (x_1^{\vec{\tau}} - y_1^{\vec{\tau}}) \cdot T_\ell(p_{\vec{\tau}}, q_{\vec{\tau}}) \\ &+ \sum_{h=2}^{n}\sum_{\ell=1}^{h-1} \tau_\ell (x_\ell^{\vec{\tau}} - y_\ell^{\vec{\tau}}) \cdot \tau_h^2 (x_h^{\vec{\tau}} - y_h^{\vec{\tau}})^2 \cdot S^1_{h\ell}(p_{\vec{\tau}}, q_{\vec{\tau}}) \\ &+ \sum_{h=2}^{n-1}\sum_{\ell=h+1}^{n} \tau_h (x_h^{\vec{\tau}} - y_h^{\vec{\tau}}) \cdot \tau_\ell^2 (x_\ell^{\vec{\tau}} - y_\ell^{\vec{\tau}})^2 \cdot S^2_{h\ell}(p_{\vec{\tau}}, q_{\vec{\tau}}), \end{aligned} \tag{6.13}$$

where we have used the definitions

$$\begin{cases} T_\ell(p,q) & := & \left(\dfrac{\partial\rho_\ell}{\partial x_1}\rho_\ell\right)(p) - \left(\dfrac{\partial\rho_\ell}{\partial x_1}\rho_\ell\right)(q) \\ \\ S^1_{h\ell}(p,q) & := & \left(\dfrac{\partial\rho_\ell}{\partial x_\ell}\rho_h\rho_\ell^2\right)(p) - \left(\dfrac{\partial\rho_\ell}{\partial x_\ell}\rho_h\rho_\ell^2\right)(q) \\ \\ S^2_{h\ell}(p,q) & := & \left(\dfrac{\partial\rho_\ell}{\partial x_h}\rho_\ell\rho_h^2\right)(p) - \left(\dfrac{\partial\rho_\ell}{\partial x_h}\rho_\ell\rho_h^2\right)(q). \end{cases}$$

Since we *do not* have $\tau_k(x_k^{\vec{\tau}} - {}_k^{\vec{\tau}}) = O(1)$ as $\tau_k \to \infty$, the remainder of our work will be dedicated to showing

$$\lim_{\tau_n\to\infty}\cdots\lim_{\tau_1\to\infty} I_2 = 0$$

and will differ from the exposition of Chapter 4.

From the Condition (6.7) we may deduce:

- $T_\ell(p_{\vec{\tau}}, q_{\vec{\tau}}) \sim (x_1^{\vec{\tau}} - y_1^{\vec{\tau}})$, from which

$$\tau_\ell^2(x_\ell^{\vec{\tau}} - y_\ell^{\vec{\tau}})^2 \cdot \tau_1(x_1^{\vec{\tau}} - y_1^{\vec{\tau}}) \cdot T_\ell(p_{\vec{\tau}}, q_{\vec{\tau}}) \sim \tau_\ell^2(x_\ell^{\vec{\tau}} - y_\ell^{\vec{\tau}})^2 \cdot \tau_1(x_1^{\vec{\tau}} - y_1^{\vec{\tau}})^2. \tag{6.14}$$

- As $\tau_1, \ldots, \tau_{\ell-1} \to \infty$,

$$\tau_\ell(x_\ell^{\vec{\tau}} - y_\ell^{\vec{\tau}}) \cdot \tau_h^2(x_h^{\vec{\tau}} - y_h^{\vec{\tau}})^2 \cdot S^1_{h\ell}(p_{\vec{\tau}}, q_{\vec{\tau}}) \sim \tau_\ell(x_\ell^{\vec{\tau}} - y_\ell^{\vec{\tau}})^2 \cdot \tau_h^2(x_h^{\vec{\tau}} - y_h^{\vec{\tau}})^2. \tag{6.15}$$

- As $\tau_1, \ldots, \tau_{h-1} \to \infty$,

$$\tau_h(x_h^{\vec{\tau}} - y_h^{\vec{\tau}}) \cdot \tau_\ell^2(x_\ell^{\vec{\tau}} - y_\ell^{\vec{\tau}})^2 \cdot S^2_{h\ell}(p_{\vec{\tau}}, q_{\vec{\tau}}) \sim \tau_h(x_h^{\vec{\tau}} - y_h^{\vec{\tau}})^2 \cdot \tau_\ell^2(x_\ell^{\vec{\tau}} - y_\ell^{\vec{\tau}})^2. \tag{6.16}$$

Recalling (6.5) as it applies to $\varphi_{\vec{\tau}}$, we also have:

$$\begin{cases} \lim\limits_{\tau_1\to\infty} \tau_1(x_1^{\vec{\tau}} - y_1^{\vec{\tau}})^2 \cdot \tau_\ell^2(x_\ell^{\vec{\tau}} - y_\ell^{\vec{\tau}})^2 & = & 0 \\ \\ \lim\limits_{\tau_\ell\to\infty}\cdots\lim\limits_{\tau_1\to\infty} \tau_\ell(x_\ell^{\vec{\tau}} - y_\ell^{\vec{\tau}})^2 \cdot \tau_h^2(x_h^{\vec{\tau}} - y_h^{\vec{\tau}})^2 & = & 0, \text{ when } \ell < h \\ \\ \lim\limits_{\tau_h\to\infty}\cdots\lim\limits_{\tau_1\to\infty} \tau_h(x_h^{\vec{\tau}} - y_h^{\vec{\tau}})^2 \cdot \tau_\ell^2(x_\ell^{\vec{\tau}} - y_\ell^{\vec{\tau}})^2 & = & 0, \text{ when } h < \ell. \end{cases} \tag{6.17}$$

Iterated limits of I_2 are now calculated from Equations (6.14), (6.15), (6.16), and (6.17):

$$\begin{aligned}
\lim_{\tau_1\to\infty} 2I_2 \quad \sim \quad & 0+\sum_{h=3}^{n}\tau_2(x_2^{\vec{\tau}}-y_2^{\vec{\tau}})^2\cdot\tau_h^2(x_h^{\vec{\tau}}-y_h^{\vec{\tau}})^2 \\
& +\sum_{h=4}^{n}\sum_{\ell=3}^{h-1}\tau_\ell(x_\ell^{\vec{\tau}}-y_\ell^{\vec{\tau}})\cdot\tau_h^2(x_h^{\vec{\tau}}-y_h^{\vec{\tau}})^2\cdot S_{h\ell}^1(p_{\tau_1},q_{\tau_1}) \\
& +\sum_{\ell=3}^{n}\tau_2(x_2^{\vec{\tau}}-y_2^{\vec{\tau}})^2\cdot\tau_\ell^2(x_\ell^{\vec{\tau}}-y_\ell^{\vec{\tau}})^2 \\
& +\sum_{h=3}^{n-1}\sum_{\ell=h+1}^{n}\tau_h(x_h^{\vec{\tau}}-y_h^{\vec{\tau}})\cdot\tau_\ell^2(x_\ell^{\vec{\tau}}-y_\ell^{\vec{\tau}})^2\cdot S_{h\ell}^2(p_{\tau_1},q_{\tau_1}),
\end{aligned}$$

$$\begin{aligned}
\lim_{\tau_2\to\infty}\lim_{\tau_1\to\infty} 2I_2 \quad \sim \quad & 0+\sum_{h=4}^{n}\tau_3(x_3^{\vec{\tau}}-y_3^{\vec{\tau}})^2\cdot\tau_h^2(x_h^{\vec{\tau}}-y_h^{\vec{\tau}})^2 \\
& +\sum_{h=5}^{n}\sum_{\ell=4}^{h-1}\tau_\ell(x_\ell^{\vec{\tau}}-y_\ell^{\vec{\tau}})\cdot\tau_h^2(x_h^{\vec{\tau}}-y_h^{\vec{\tau}})^2\cdot S_{h\ell}^1(p_{\tau_1,\tau_2},q_{\tau_1,\tau_2}) \\
& +\sum_{\ell=4}^{n}\tau_3(x_3^{\vec{\tau}}-y_3^{\vec{\tau}})^2\cdot\tau_\ell^2(x_\ell^{\vec{\tau}}-y_\ell^{\vec{\tau}})^2 \\
& +\sum_{h=4}^{n-1}\sum_{\ell=h+1}^{n}\tau_h(x_h^{\vec{\tau}}-y_h^{\vec{\tau}})\cdot\tau_\ell^2(x_\ell^{\vec{\tau}}-y_\ell^{\vec{\tau}})^2\cdot S_{h\ell}^2(p_{\tau_1,\tau_2},q_{\tau_1,\tau_2}),
\end{aligned}$$

$$\begin{aligned}
\lim_{\tau_3\to\infty}\lim_{\tau_2\to\infty}\lim_{\tau_1\to\infty} 2I_2 \quad \sim \quad & 0+\sum_{h=5}^{n}\tau_4(x_4^{\vec{\tau}}-y_4^{\vec{\tau}})^2\cdot\tau_h^2(x_h^{\vec{\tau}}-y_h^{\vec{\tau}})^2 \\
& +\sum_{h=6}^{n}\sum_{\ell=5}^{h-1}\tau_\ell(x_\ell^{\vec{\tau}}-y_\ell^{\vec{\tau}})\cdot\tau_h^2(x_h^{\vec{\tau}}-y_h^{\vec{\tau}})^2\cdot S_{h\ell}^1(p_{\tau_1,\tau_2,\tau_3},q_{\tau_1,\tau_2,\tau_3}) \\
& +\sum_{\ell=5}^{n}\tau_4(x_4^{\vec{\tau}}-y_4^{\vec{\tau}})^2\cdot\tau_\ell^2(x_\ell^{\vec{\tau}}-y_\ell^{\vec{\tau}})^2 \\
& +\sum_{h=5}^{n-1}\sum_{\ell=h+1}^{n}\tau_h(x_h^{\vec{\tau}}-y_h^{\vec{\tau}})\cdot\tau_\ell^2(x_\ell^{\vec{\tau}}-y_\ell^{\vec{\tau}})^2\cdot S_{h\ell}^2(p_{\tau_1,\tau_2,\tau_3},q_{\tau_1,\tau_2,\tau_3}), \\
& \vdots \\
\lim_{\tau_{n-3}\to\infty}\cdots\lim_{\tau_1\to\infty} 2I_2 \quad \sim \quad & 0+\sum_{h=n-1}^{n}\tau_{n-2}(x_{n-2}^{\vec{\tau}}-y_{n-2}^{\vec{\tau}})^2\cdot\tau_h^2(x_h^{\vec{\tau}}-y_h^{\vec{\tau}})^2 \\
& +\tau_{n-1}(x_{n-1}^{\vec{\tau}}-y_{n-1}^{\vec{\tau}})\cdot\tau_n^2(x_n^{\vec{\tau}}-y_n^{\vec{\tau}})^2\cdot S_{n(n-1)}^1(p_{\tau,\ldots,\tau_{n-3}},q_{\tau_1,\ldots,\tau_{n-3}}) \\
& +\sum_{\ell=n-1}^{n}\tau_{n-2}(x_{n-2}^{\vec{\tau}}-y_{n-2}^{\vec{\tau}})^2\cdot\tau_\ell^2(x_\ell^{\vec{\tau}}-y_\ell^{\vec{\tau}})^2 \\
& +\tau_{n-1}(x_{n-1}^{\vec{\tau}}-y_{n-1}^{\vec{\tau}})\cdot\tau_n^2(x_n^{\vec{\tau}}-y_n^{\vec{\tau}})^2\cdot S_{(n-1)n}^2(p_{\tau,\ldots,\tau_{n-3}},q_{\tau_1,\ldots,\tau_{n-3}}).
\end{aligned}$$

This pattern and the last of the limits shown above imply that

$$\lim_{\tau_{n-1}\to\infty}\lim_{\tau_{n-2}\to\infty}\cdots\lim_{\tau_1\to\infty}2I_2\sim\lim_{\tau_{n-1}\to\infty}\left(0+2\tau_{n-1}(x_{n-1}^{\vec{\tau}}-y_{n-1}^{\vec{\tau}})^2\cdot\tau_n^2(x_n^{\vec{\tau}}-y_n^{\vec{\tau}})^2\right)=0,$$

and hence we infer

$$\lim_{\tau_n \to \infty} \lim_{\tau_{n-1} \to \infty} \cdots \lim_{\tau_1 \to \infty} I_2 = 0. \tag{6.18}$$

Equations (6.12) and (6.18) together result in the limit (6.4). □

6.2 The Parabolic Comparison Principle

The Cauchy-Dirichlet Problem which is under investigation is the following:

$$\begin{cases} w_t + \Delta_{\mathfrak{X},\infty} w &= 0 \quad \text{in } \Omega_T \\ w &= g \quad \text{on } \partial_{\text{par}} \Omega_T, \end{cases} \tag{CDP}$$

where we assume that $\Omega \Subset \mathbb{G}$ is a domain, $T > 0$, and $g \in C(\partial_{\text{par}} \Omega_T)$. Similarly to the terminology used for our previous Dirichlet problems, we say: A parabolic viscosity subsolution u of (CDP) is a parabolic ∞-subharmonic function u which also satisfies $u \leq g$ on $\partial_{\text{par}} \Omega$; parabolic viscosity supersolutions and solutions of (CDP). As before, we call a parabolic viscosity (sub-/super-)solution w a parabolic ∞ (sub-/super-)harmonic function.

Theorem 6.4. *Let $\Omega \Subset \mathbb{G}$ be a domain, $T > 0$, and $g \in C(\partial_{\text{par}} \Omega)$ be given; suppose that the functions ρ_k, ρ_ℓ satisfy Equation* (6.7). *If u is a parabolic viscosity subsolution and v is a parabolic viscosity supersolution of Problem* (CDP), *then $u \leq v$ on $\overline{\Omega}_T$.*

Proof. Observe that u may be assumed to be a strict ∞-subharmonic function without loss of generality: Indeed, if not then, similarly to [18], we note that

$$u_\varepsilon(p,t) := u(p,t) - \frac{\varepsilon}{T-t}$$

is ∞-subharmonic and satisfies

$$\begin{cases} (u_\varepsilon)_t + \Delta_{\mathfrak{X},\infty} u_\varepsilon &\leq -\dfrac{\varepsilon}{(T-t)^2} < 0 \\ \lim\limits_{t \uparrow T} u_\varepsilon &= -\infty \text{ uniformly in } \Omega. \end{cases} \tag{6.19}$$

Since $u_\varepsilon \leq v$ implies $u \leq v$ as $\varepsilon \to 0$, we may proceed by proving the theorem under the assumption that u satisfies the Relation (6.19).

Assume that there exists some $(p_0,t_0)\in\Omega_T$ such that

$$\delta := \sup_{\Omega_T}(u-v) = u(p_0,t_0)-v(p_0,t_0) > 0.$$

Clearly, $(p_0,t_0)\notin \partial_{\text{par}}\,\Omega_T$ since $u\le g\le v$ on the parabolic boundary. Employing $\varphi_{\vec{\tau}}$ as in Lemma 6.1, we obtain triples $(p_{\vec{\tau}},q_{\vec{\tau}},t_{\vec{\tau}})$ for each $\vec{\tau}=(\tau_1,\dots,\tau_n)$ so that

$$M_{\vec{\tau}} := \sup_{\overline{\Omega}\times\overline{\Omega}\times(0,T)} \left(u(p,t)-v(q,t)-\varphi_{\vec{\tau}}(p,q,t)\right) = u(p_{\vec{\tau}},t_{\vec{\tau}})-v(q_{\vec{\tau}},t_{\vec{\tau}})-\varphi_{\vec{\tau}}(p_{\vec{\tau}},q_{\vec{\tau}},t_{\vec{\tau}}).$$

and $(p_{\vec{\tau}},t_{\vec{\tau}})\to(p_0,t_0)\leftarrow(q_{\vec{\tau}},t_{\vec{\tau}})$ as $\tau_1,\tau_2,\dots,\tau_n\to\infty$. Since $(p_0,t_0)\in\Omega_T$, we can infer that $p_{\vec{\tau}},q_{\vec{\tau}}\in\Omega$ for sufficiently large $\tau_1,\tau_2,\dots,\tau_n$. Additionally, $0<t_{\vec{\tau}}$ for large $\tau_1,\tau_2,\dots,\tau_n$ since otherwise

$$\begin{aligned}\delta < M_{\vec{\tau}} =& u(p_{\vec{\tau}},0)-v(q_{\vec{\tau}},0)-\varphi_{\vec{\tau}}((p_{\vec{\tau}},0),(q_{\vec{\tau}},0))\\ \le& g(p_{\vec{\tau}},0)-g(q_{\vec{\tau}},0)-\varphi_{\vec{\tau}}((p_{\vec{\tau}},0),(q_{\vec{\tau}},0)),\end{aligned}$$

but the right-hand side is continuous and must tend to 0 as $\tau_1,\tau_2,\dots,\tau_n\to\infty$. We may therefore treat $(p_{\vec{\tau}},t_{\vec{\tau}})$ and $(q_{\vec{\tau}},t_{\vec{\tau}})$ as interior points of the parabolic domain and may now apply Lemma 6.1.

By Lemma 6.1, there exist $(a,\eta^+_{\vec{\tau}},\mathcal{X}_{\vec{\tau}})\in\overline{P}^{2,+}u(p_{\vec{\tau}},t_{\vec{\tau}})$ and $(b,\eta^-_{\vec{\tau}},\mathcal{Y}_{\vec{\tau}})\in\overline{P}^{2,-}v(q_{\vec{\tau}},t_{\vec{\tau}})$ satisfying

$$\begin{cases} a-b &= 0\\ a-\left\langle \mathcal{X}_{\vec{\tau}}\cdot\eta^+_{\vec{\tau}},\eta^+_{\vec{\tau}}\right\rangle &\le -\dfrac{\varepsilon}{T^2}\\ b-\left\langle \mathcal{Y}_{\vec{\tau}}\cdot\eta^-_{\vec{\tau}},\eta^-_{\vec{\tau}}\right\rangle &\ge 0.\end{cases} \tag{6.20}$$

Utilizing the relations above,

$$\begin{aligned} 0<\frac{\varepsilon}{T^2} &\le b-\left\langle \mathcal{Y}_{\vec{\tau}}\cdot\eta^-_{\vec{\tau}},\eta^-_{\vec{\tau}}\right\rangle - a + \left\langle \mathcal{X}_{\vec{\tau}}\cdot\eta^+_{\vec{\tau}},\eta^+_{\vec{\tau}}\right\rangle\\ &= \left\langle \mathcal{X}_{\vec{\tau}}\cdot\eta^+_{\vec{\tau}},\eta^+_{\vec{\tau}}\right\rangle - \left\langle \mathcal{Y}_{\vec{\tau}}\cdot\eta^-_{\vec{\tau}},\eta^-_{\vec{\tau}}\right\rangle. \end{aligned} \tag{6.21}$$

However, the right-hand side of (6.21) tends to 0 as $\tau_1,\tau_2,\dots,\tau_n\to\infty$ by Lemma 6.1, Item (C), and this is a contradiction. Therefore, $u\le v$ in $\overline{\Omega}_T$ as desired. □

The following corollary completes our examination of past parabolic viscosity solutions begun in Chapter 5; together with Proposition 5.6, it implies the equivalence of parabolic ∞-harmonic functions and past parabolic solutions to (5.1). Under the assumption that $\mathbb{G}$ satisfies Hörmander's Condition (H), it is precisely

[5, Corollary 4.4]. However, in the more general setting of this dissertation, we are unable to utilize comparisons between the CC-metric and a smooth gauge $\mathcal{N}(\cdot,\cdot)$; the fundamental issue lies with [2, Theorem 7.34], which requires Hörmander's Condition among its assumptions. To circumvent this difficulty, we impose a more general assumption that encompasses, for example, the spaces under consideration in [29] and utilizes the observation that $\|\cdot\|_{\text{eucl}}^m$ is smooth for sufficiently large $m > 1$.

Corollary 6.5. *Suppose that Ω and T are as in Theorem 6.4, that the functions ρ_k, ρ_ℓ satisfy* (6.7)*, and that there exists $m_0 \in \mathbb{N}$ so that for every $p_0 \in \Omega$,*

$$\|p - p_0\|_{\text{eucl}}^m \leq d_{CC}(p_0, p) \tag{6.22}$$

for $m_0 \leq m$ and p near p_0. Then:

1. *A parabolic ∞-subharmonic function u is a past parabolic viscosity subsolution of Equation* (5.1)*.*
2. *A parabolic ∞-superharmonic function v is a past parabolic viscosity supersolution of Equation* (5.1)*.*
3. *A parabolic ∞-harmonic function w is a past parabolic viscosity solution of Equation* (5.1)*.*

Proof. We need only prove Item 1 due to the similarity of the other items. Since this proof is similar in layout to the proofs of [26, Theorem 1], [13, Corollary 3.7], and [14, Corollary 4.4], we will restrict our attention to the essential details.

Assume that u is *not* a past parabolic viscosity subsolution and let $(p_0, t_0) \in \Omega_T$ be a point such that there is $\phi \in \mathcal{A}^-(u(p_0, t_0))$ satisfying

$$\phi_t(p_0, t_0) + \Delta_{\mathfrak{X},\infty}\phi(p_0, t_0) \geq \varepsilon > 0$$

for a small parameter ε. Select even $m \geq m_0$ so that $\|\cdot\|_{\text{eucl}}^m$ is smooth and fix $r > 0$ so small that $B(p_0, 4r) \Subset \Omega$. Denoting $B := \{p \in \Omega : \|p - p_0\|_{\text{eucl}}^m < r\}$, we have $B \Subset \Omega$ by (6.22); define the parabolic ball $S_r := B \times (t_0 - r, t_0)$. Then the function

$$\phi_r(p, t) := \phi(r, t) + \|p - p_0\|_{\text{eucl}}^m + (t - t_0)^m - r^m$$

is a classical supersolution to (5.1) in S_r, provided that r is chosen sufficiently small. Observe that $u \leq \phi_r$ on $\partial_{\text{par}} S_r$, but $u(p_0, t_0) > \phi_r(p_0, t_0)$. This contradicts Theorem 6.4, and therefore u must be a past parabolic viscosity subsolution. □

As in [14], we may also utilize Theorem 6.4 to obtain some estimates resulting from the boundary data; the proofs of the following corollaries mimic those of [14, Corollary 4.6] and [14, Corollary 4.7].

Corollary 6.6. *Let $\Omega_T \Subset \mathbb{G}, T > 0$, and ρ_ℓ, ρ_k be as in Theorem 6.4. If w_1, w_2 are parabolic ∞-harmonic functions with given boundary data $g_1, g_2 \in C(\partial_{\text{par}} \Omega_T)$ respectively, then*

$$\sup_{\Omega_T} |w_1 - w_2| \leq \sup_{\partial_{\text{par}} \Omega_T} |g_1 - g_2| \,. \tag{6.23}$$

Proof. We define

$$\overline{W}(p,t) := w_2(p,t) + \sup_{\partial_{\text{par}} \Omega_T} |g_1 - g_2|$$

and

$$\underline{W}(p,t) := w_2(p,t) - \sup_{\partial_{\text{par}} \Omega_T} |g_1 - g_2| \,.$$

By these definitions, we infer that $\underline{W} \leq w_1 \leq \overline{W}$ on $\partial_{\text{par}} \Omega_T$; applying Theorem 6.4, $\underline{W} \leq w_1 \leq \overline{W}$ in $\overline{\Omega}_T$. Subtracting $\underline{W}$ from each member of this inequality, we have

$$0 \leq w_1(p,t) - w_2(p,t) + \sup_{\partial_{\text{par}} \Omega_T} |g_1 - g_2| \leq 2 \sup_{\partial_{\text{par}} \Omega_T} |g_1 - g_2|$$

– from which Inequality (6.23) follows. □

Corollary 6.7. *If $\Omega_T \Subset \mathbb{G}, T > 0$, and ρ_ℓ, ρ_k are as in Theorem 6.4, then every ∞-harmonic function w satisfies the inequality*

$$\sup_{\Omega_T} |w| \leq \sup_{\partial_{\text{par}} \Omega_T} |g|$$

with respect to its boundary data $g \in C(\partial_{\text{par}} \Omega_T)$.

Proof. The results is proven by following the methods of Corollary 6.6, but defining

$$\overline{W}(p) := \sup_{\partial_{\text{par}} \Omega_T} |g| \text{ and } \underline{W}(p) := - \sup_{\partial_{\text{par}} \Omega_T} |g|.$$

□

6.3 Existence of Parabolic Infinity-Harmonic Functions

Having established uniqueness of ∞-harmonic functions in Grushin-type spaces $\mathbb{G}$ whose frames $\mathfrak{X}$ are weighted by functions ρ_ℓ, ρ_k satisfying Condition (6.7), it now remains for us to show that ∞-harmonic functions exist for each cylinder Ω_T over a bounded domain $\Omega_T \Subset \mathbb{G}$. As in [14], we will accomplish this by employing Perron's Method and utilizing our comparison principle, Theorem 6.4; our results and their proofs are also similar to those presented in [21, Chapter 2].

Lemma 6.8. *Let Ω, T, and m_0 be as in Corollary 6.5. Let $\mathcal{L}$ denote a family of parabolic ∞-superharmonic functions and define*

$$u_{\mathcal{L}}(p,t) := \inf \{u(p,t) \,:\, u \in \mathcal{L}\}.$$

Then if $u_{\mathcal{L}}$ is finite in a dense subset of Ω_T, it is also a parabolic ∞-superharmonic function.

Proof. Firstly, since each member of $\mathcal{L}$ is lower semicontinuous, standard techniques establish $u_{\mathcal{L}}$ is also lower semicontinuous. Fixing any point $(p_0,t_0) \in \Omega_T$, any $\psi \in \mathcal{B}^-(u_{\mathcal{L}}, (p_0,t_0))$, and selecting even $m \geq 4$ larger than m_0, define

$$\hat{\psi}(p,t) := \psi(p,t) - \|p-p_0\|^m_{\text{eucl}} - (t-t_0)^m$$

and notice

$$\begin{aligned} u_{\mathcal{L}}(p,t) - \hat{\psi}(p,t) - \|p-p_0\|^m_{\text{eucl}} - (t-t_0)^m &= u_{\mathcal{L}}(p,t) - \psi(p,t) \\ &\geq u_{\mathcal{L}}(p_0,t_0) - \psi(p_0,t_0) \\ &= u_{\mathcal{L}}(p_0,t_0) - \hat{\psi}(p_0,t_0) \\ &= 0. \end{aligned}$$

This implies

$$u_{\mathcal{L}}(p,t) - \hat{\psi}(p,t) \geq \|p-p_0\|^m_{\text{eucl}} + (t-t_0)^m \tag{6.24}$$

near (p_0,t_0). The lower semicontinuity of $u_{\mathcal{L}}$ permits us to find a sequence $(p_k,t_k) \to (p_0,t_0)$ with $t_k < t_0$ so that

$$u_{\mathcal{L}}(p_k,t_k) - \hat{\psi}(p_k,t_k) \to u_{\mathcal{L}}(p_0,t_0) - \hat{\psi}(p_0,t_0) = 0. \tag{6.25}$$

For each k, since $u_{\mathcal{L}}$ is an infimum there exists some $v_k \in \mathcal{L}$ so that

$$u_{\mathcal{L}}(p_k,t_k) \leq v_k(p_k,t_k) \leq u_{\mathcal{L}}(p_k,t_k) + \frac{1}{k}; \tag{6.26}$$

(6.24) and the left-hand side of (6.26) together imply

$$v_k(p,t) - \hat{\psi}(p,t) \geq u_{\mathcal{L}}(p,t) - \hat{\psi}(p,t) \geq \|p-p_0\|^m_{\text{eucl}} + (t-t_0)^m. \tag{6.27}$$

Taking any compact neighborhood $B \Subset \Omega_T$ containing (p_0, t_0), the lower semicontinuous $v_k - \hat{\psi}$ attains its minimum at some point $(q_k, s_k) \in B$. From (6.26) and (6.27),

$$\begin{aligned} u_{\mathcal{L}}(p_k,t_k) - \hat{\psi}(p_k,t_k) + \frac{1}{k} &\geq v_k(p_k,t_k) - \hat{\psi}(p_k,t_k) \\ &\geq v_k(q_k,s_k) - \hat{\psi}(q_k,s_k) \\ &\geq \|q_k - p_0\|^m_{\text{eucl}} + (s_k - t_0)^m \geq 0 \end{aligned} \tag{6.28}$$

when k is so large that $(p_k, t_k) \in B$. The limit (6.25) and Relation (6.28) together imply $(q_k, s_k) \to (p_0, t_0)$ as $k \to \infty$.

Claim 6.9. Defining

$$\phi_k(p,t) := \hat{\psi}(p,t) + \|p - q_k\|^m_{\text{eucl}} + (t - s_k)^m, \tag{6.29}$$

we have $\phi_k \in \mathcal{B}^-(v_k, (q_k, s_k))$.

Proof of Claim 6.9. The definitions of ψ and $\hat{\psi}$, together with Equation (6.29), imply that ϕ_k is $C^2_{\mathbb{G}}$; Equation (6.29) also implies

$$\phi_k(q_k,s_k) = \hat{\psi}(q_k,s_k) = v_k(q_k,s_k). \tag{6.30}$$

For (p,t) near (q_k, s_k), Inequality (6.27) shows

$$v_k(p,t) - \phi_k(p,t) = v_k(p,t) - \hat{\psi}(p,t) - \Big(\|p-q_k\|^m_{\text{eucl}} + (t-s_k)^m\Big) \geq 0. \tag{6.31}$$

Equations (6.30) and (6.31) complete the claim. ■

Claim 6.10. With ϕ_k defined as in Claim 6.9, the following equations hold:

$$\begin{cases} (\phi_k)_t(q_k,s_k) &= \hat{\psi}_t(q_k,s_k) \\ \nabla_{\mathbb{G}}\,\phi_k(q_k,s_k) &= \nabla_{\mathbb{G}}\,\hat{\psi}(q_k,s_k) \\ \left(D^2\phi_k\right)^\star(q_k,s_k) &= \left(D^2\hat{\psi}\right)^\star(q_k,s_k). \end{cases} \tag{6.32}$$

Proof of Claim 6.10. Explicit calculation yields the temporal derivative

$$(\phi_k)_t(p,t) = \hat{\psi}_t(p,t) + m(t-s_k)^{m-1}$$

and, writing $q_k = (y_1^k, y_2^k, \ldots, y_n^k)$, the spatial derivatives

$$\begin{cases} X_i\phi_k(p,t) &= X_i\hat{\psi}(p,t) + m\rho_i(p)(x_i - y_i^k)\|p-q_k\|_{\text{eucl}}^{m-2} \\ \\ X_jX_i\phi_k(q_k,s_k) &= X_jX_i\hat{\psi}(q_k,s_k) \\ &\quad +m(m-2)\rho_i(p)(x_i-y_i^k)(x_j-y_j^k)\|p-q_k\|_{\text{eucl}}^{m-4}, \quad i \le j \\ \\ X_jX_i\phi_k(q_k,s_k) &= X_jX_i\hat{\psi}(q_k,s_k) \\ &\quad +m\left(\dfrac{\partial\rho_i}{x_j}\rho_j\right)(p)\cdot(x_i-y_i^k)\|p-q_k\|_{\text{eucl}}^{m-2} \\ &\quad +m(m-2)\rho_i(p)(x_i-y_i^k)(x_j-y_j^k)\|p-q_k\|_{\text{eucl}}^{m-4}, \quad j < i. \end{cases}$$

Replacing p by q_k and t by s_k in the results above, we recover the desired equations. ■

Since each v_k is a parabolic ∞-superharmonic function, Claim 6.9 now implies

$$(\phi_k)_t(q_k,s_k) + \Delta_{\mathfrak{X},\infty}\phi_k(q_k,s_k) \ge 0. \tag{6.33}$$

Replacing derivatives in (6.33) via the equations in (6.32), we obtain

$$\hat{\psi}_t(q_k,s_k) + \Delta_{\mathfrak{X},\infty}\hat{\psi}(q_k,s_k) \ge 0;$$

allowing $k \to \infty$, we obtain

$$\psi_t(p_0,t_0) + \Delta_{\mathfrak{X},\infty}\psi(p_0,t_0) \ge 0,$$

as desired. □

A similar result can be proven for collections of parabolic ∞-subharmonic functions.

Lemma 6.11. *Let $\mathcal{L}$ denote a family of parabolic ∞-subharmonic functions and define*

$$v_{\mathcal{L}}(p,t) := \sup\{v(p,t) \,:\, v \in \mathcal{L}\}.$$

Then if $v_{\mathcal{L}}$ is finite in a dense subset of Ω_T, it is also a parabolic ∞-subharmonic function.

We wish to utilize families of ∞-superharmonic and ∞-subharmonic functions to produce a continuous solution to the parabolic ∞-Laplace Equation (5.1), and this is the content of the forthcoming results. We require a definition.

Definition 6.12. Given a function $u : \mathcal{O}_{t_1,t_2} \subseteq \mathbb{G} \times \mathbb{R} \to \mathbb{R}$, the *upper* and *lower semicontinuous envelopes* for u are

$$u^*(p,t) := \limsup_{r\downarrow 0} \{u(q,s) \,:\, d_{CC}(q,p) + |t-s| \leq r\}$$

and

$$u_*(p,t) := \liminf_{r\downarrow 0} \{u(q,s) \,:\, d_{CC}(q,p) + |t-s| \leq r\}$$

respectively.

Lemma 6.13. *Let v be a given parabolic ∞-superharmonic function on Ω_T, and let $\mathcal{L}(v)$ be the family of all parabolic ∞-subharmonic functions u satisfying $u \leq v$. If $\hat{u} \in \mathcal{L}(v)$ but $\hat{u}_*$ is not a parabolic ∞-superharmonic function, then there exists some $(\hat{p},\hat{t}) \in \Omega_T$ and some $w \in \mathcal{L}(v)$ such that $\hat{u}(\hat{p},\hat{t}) < w(\hat{p},\hat{t})$.*

Proof. Let $(p_0,t_0) \in \Omega_T$ and $\psi \in \mathcal{B}^-(\hat{u}_*, (p_0,t_0))$ be a point and touching below function for which $\hat{u}_*$ fails to be a parabolic ∞-superharmonic function: That is,

$$\psi_t(p_0,t_0) + \Delta_{\mathfrak{X},\infty}\psi(p_0,t_0) < 0. \tag{6.34}$$

Define

$$\hat{\psi}(p,t) := \psi(p,t) - \|p-p_0\|_{\text{eucl}}^m - (t-t_0)^m$$

as in the proof of Lemma 6.8, observe that $\hat{\psi} \in \mathcal{B}(\hat{u}_*, (p_0,t_0))$; the equations

$$\begin{cases} \hat{\psi}_t(p_0,t_0) &= \psi_t(p_0,t_0) \\ \nabla_{\mathbb{G}}\,\hat{\psi}_t(p_0,t_0) &= \nabla_{\mathbb{G}}\,\psi(p_0,t_0) \\ \left(D^2\hat{\psi}\right)^\star(p_0,t_0) &= \left(D^2\psi\right)^\star(p_0,t_0) \end{cases}$$

hold; and, finally,

$$\hat{u}_*(p,t) - \hat{\psi}(p,t) \geq \|p-p_0\|_{\text{eucl}}^m + (t-t_0)^m,$$

similar to Inequality (6.24). Given a compact neighborhood $B \Subset \Omega_T$ of (p_0,t_0) and some integer m, we also introduce the useful notation

$$B_{m\varepsilon} := B \cap \{(p,t) \,:\, \|p-p_0\|_{\text{eucl}}^m, (t-t_0)^m \leq m\varepsilon\}.$$

Now $\hat{u} \in \mathcal{L}(v)$ implies $\hat{u} \leq v$ and $\hat{\psi}(p_0, t_0) = \hat{u}_*(p_0, t_0) \leq \hat{u}(p_0, t_0) \leq v(p_0, t_0)$. Moreover, we see that $\hat{\psi}(p_0, t_0) < v(p_0, t_0)$, since otherwise $\hat{\psi} \in \mathcal{B}^-(v, (p_0, t_0))$ and then the relationship between the derivatives of ψ and $\hat{\psi}$ and Inequality (6.34) would contradict the assumption that v is a parabolic ∞-superharmonic function in Ω_T. Hence there exists some $\varepsilon > 0$ so that

$$\hat{\psi}(p, t) + 4\varepsilon \leq v(p, t)$$

in $B_{2\varepsilon}$. The definition of $B_{2\varepsilon}$ also suggests

$$\hat{u}(p, t) \geq \hat{u}_*(p, t) \geq \hat{\psi}(p, t) + 4\varepsilon \tag{6.35}$$

in $B_{2\varepsilon} \setminus B_\varepsilon$. If we define

$$w(p, t) := \begin{cases} \max\left\{\hat{\psi}(p, t) + 4\varepsilon,\, \hat{u}(p, t)\right\}, & (p, t) \in B_\varepsilon \\ \hat{u}(p, t), & (p, t) \in \Omega_T \setminus B_\varepsilon, \end{cases}$$

then the Inequality (6.35) implies

$$w = \max\left\{\hat{\psi} + 4\varepsilon,\, \hat{u}\right\} = \sup\left\{\hat{\psi} + 4\varepsilon,\, \hat{u}\right\}$$

so that by Lemma 6.8 we conclude w is a parabolic ∞-subharmonic function in Ω_T – hence $w \in \mathcal{L}(v)$. Additionally, because

$$0 = \hat{u}_*(p_0, t_0) - \hat{\psi}(p_0, t_0) = \liminf_{r \downarrow 0} \left\{\hat{u}(p, t) - \hat{\psi}(p, t) \,:\, d_{CC}\,(p, p_0) + |t - t_0| \leq r\right\},$$

there exists $(\hat{p}, \hat{t}) \in B_\varepsilon$ so that $\hat{u}(\hat{p}, \hat{t}) - \hat{\psi}(\hat{p}, \hat{t}) < 4\varepsilon$; we may infer that $w(\hat{p}, \hat{t}) = \hat{\psi}(\hat{p}, \hat{t}) + 4\varepsilon > \hat{u}(\hat{p}, \hat{t})$, and this completes the proof. □

Theorem 6.14. *Let $\Omega \Subset \mathbb{G}, T > 0$, and assume that the functions ρ_ℓ, ρ_k satisfy Condition* (6.7). *Suppose u is a parabolic ∞-subharmonic function, v is a parabolic ∞-superharmonic function, and that these functions satisfy*

$$\begin{cases} u \leq v & \text{in } \Omega_T \\ u_* = v^* & \text{on } \partial_{\text{par}}\,\Omega_T. \end{cases}$$

Then there exists a parabolic ∞-harmonic function $u_{\infty,T}$ satisfying $u_{\infty,T} \in C(\Omega_T)$.

Proof. Notice that the collection $\mathcal{L}(v)$ of parabolic ∞-harmonic functions in Ω_T which are bounded above by v is nonempty – in particular, $u \in \mathcal{L}(v)$. Define

$$u_{\infty,T}(p,t) := \sup_{\Omega_T} \{ f(p,t) \, : \, f \in \mathcal{L}(v) \} ,$$

and note that $u \leq u_{\infty,T} \leq v$ in Ω_T. By Lemma (6.11), we know that $u_{\infty,T}$ is a parabolic ∞-subharmonic function. Also, we must have $(u_{\infty,T})_*$ is a parabolic ∞-superharmonic function: Otherwise there is some $(\hat{p},\hat{t}) \in \Omega_T$ and $w \in \mathcal{L}(v)$ so that $u_{\infty,T}(\hat{p},\hat{t}) < w(\hat{p},\hat{t})$, and this is contradictory to the construction of $u_{\infty,T}$.

Now

$$u_{\infty,T} = (u_{\infty,T})^* \leq v^* = u_* \leq (u_{\infty,T})_* \text{ on } \partial_{\text{par}} \Omega_T$$

by our assumptions on u and v. We may therefore apply our comparison principle, Theorem 6.4 to conclude $u_{\infty,T} \leq (u_{\infty,T})_*$. Since $(u_{\infty,T})_* \leq u_{\infty,T}$ by definition, we have found that the function $u_{\infty,T} = (u_{\infty,T})_* = (u_{\infty,T})^*$ is continuous and parabolic ∞-harmonic. □

6.4 Asymptotic Limits With Respect to Time

With our elliptic and parabolic existence results complete, we now have interest in the asymptotic limits of parabolic viscosity solutions to the (parabolic) ∞-Laplacian

$$w_t + \Delta_{\mathfrak{X},\infty} w = 0. \tag{6.36}$$

more specifically, we wish to prove the following theorem.

Theorem 6.15. *Let $\Omega \Subset \mathbb{G}$ be a bounded domain; let $U \in C(\Omega \times [0,\infty))$ be a solution to*

$$\begin{cases} w_t(p,t) + \Delta_{\mathfrak{X},\infty} w(p,t) = 0 \; \text{in } \Omega \times (0,\infty) \\ \qquad\qquad\qquad w(p,t) = g(p) \; \text{on } \partial_{\text{par}} (\Omega \times (0,\infty)) \end{cases} \tag{6.37}$$

with continuous boundary data g. Let u_∞ be an (elliptic) ∞-harmonic function in Ω. Assume also that $\partial\Omega$ satisfies the condition of positive geometric density*; that the functions ρ_k, ρ_ℓ satisfy* (6.7)*; and that there exists m_0 as in the statement of Corollary 6.5. Then we will have $U \to u_\infty$ uniformly in Ω as $t \to \infty$.*

For convenience, we state the definition of *positive geometric density* as it appears on [19, p. 1].

Definition 6.16. Given a bounded domain $\Omega \Subset \mathbb{R}^n$, the boundary $\partial\Omega$ is said to satisfy the condition of *positive geometric density* if: There exists $\alpha \in (0,1)$ and $\varepsilon > 0$ such that for every $p \in \partial\Omega$ and each ball

$B_{\mathrm{eucl}}(p,\delta)$ with $\delta \le \varepsilon$, the inequality

$$m\left(\Omega \cap B_{\mathrm{eucl}}(p,\delta)\right) \le (1-\alpha)\cdot m\left(B_{\mathrm{eucl}}(p,\delta)\right),$$

where we have used $m(\cdot)$ to denoted n-dimensional Lebesgue measure.

To prove Theorem 6.15 we shall construct parabolic test functions from elliptic ones as in [26]. To accomplish this, we will need two observations, the first of which is a homogeneity property. Specifically, direct calculation shows that if $u \in C^2_{\mathbb{G}}$ is a solution to (6.37), then the function $v(p,t) := k^{1/2}u(p,kt)$ is a $C^2_{\mathbb{G}}$ solution; this homogeneity property also holds for parabolic viscosity solutions to Equation (6.37). The second observation is the lemma below whose proof appears on [19, p. 170] and in [26].

Lemma 6.17. *Let U be as in Theorem 6.15. Then for every $(p,t) \in \Omega \times (0,\infty)$ and for $0 < s < t$,*

$$|U(p,t-s) - U(p,t)| \le \frac{\|g\|_{\Omega,\infty}}{(1-s/t)^{3/2}} \cdot \frac{s}{t}.$$

Proof of Theorem 6.15. Suppose that $U : \Omega \times (0,\infty) \to \mathbb{R}$ is a parabolic viscosity solution to Problem (6.37). By the results of [19, Chapter III], the family $\{U(\cdot,t) : t \in (0,\infty)\}$ is equicontinuous and, moreover, bounded uniformly above by $\|g\|_{\Omega,\infty} < \infty$. The Arzela-Ascoli Theorem therefore implies the existence of some subsequence (t_k) so that $t_k \to \infty$ and $U(\cdot,t_k)$ uniformly to some function $u(\cdot) \in C(\Omega)$ as $k \to \infty$ which satisfies $u = g$ on $\partial\Omega$. In light of Theorem 4.3 and because proving u is a viscosity supersolution will proceed similarly to the subsolution case, we will only prove that u is a viscosity subsolution of (6.36).

Let $p_0 \in \Omega$ be given and let $\phi \in C^2_{\mathbb{G}}(\Omega)$ be any function satisfying

$$0 = \phi(p_0) - u(p_0) < \phi(p) - u(p) \text{ near } p_0.$$

By uniform convergence, there exist $p_k \to p_0$ as $k \to \infty$ such that $U(\cdot,t_k) - \phi(\cdot)$ possesses as minimum at p_k. Define

$$\phi_k(p,t) := \phi(p) + \|g\|_{\Omega,\infty}\left(\frac{t}{t_k}\right)^{-3/2}\frac{t_k - t}{t_k}.$$

Notice that $\phi_k \in C^2_{\mathbb{G}}(\Omega) \cap C^1_{\mathrm{eucl}}((0,\infty))$. By Lemma 6.17,

$$\begin{aligned} U(p_k,t_k) - \phi_k(p_k,t_k) = U(p_k,t_k) - \phi(p_k) &\ge U(p,t_k) - \phi(p) \\ &\ge U(p,t) - \phi(p) - \|g\|_{\Omega,\infty}\left(\frac{t}{t_k}\right)^{-3/2}\frac{t_k - t}{t_k} \\ &= U(p,t) - \phi_k(p,t) \end{aligned}$$

for all $p \in \Omega$ and $t \in (0, t_k)$. This shows that $\phi_k \in \mathcal{A}^- (u(p_k, t_k))$ and, invoking Corollary 6.5,

$$(\phi_k)_t (p_k, t_k) + \Delta_{\mathfrak{X},\infty} \phi_k(p_k, t_k) \leq 0.$$

Calculating $(\phi_k)_t (p_k, t_k)$ directly from the definition of ϕ_k and recollecting terms,

$$\Delta_{\mathfrak{X},\infty} \phi_k(p_k, t_k) \leq \frac{\|g\|_{\Omega,\infty}}{t_k}.$$

Now we may let $k \to \infty$ and conclude the desired result. □

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APPENDIX A:

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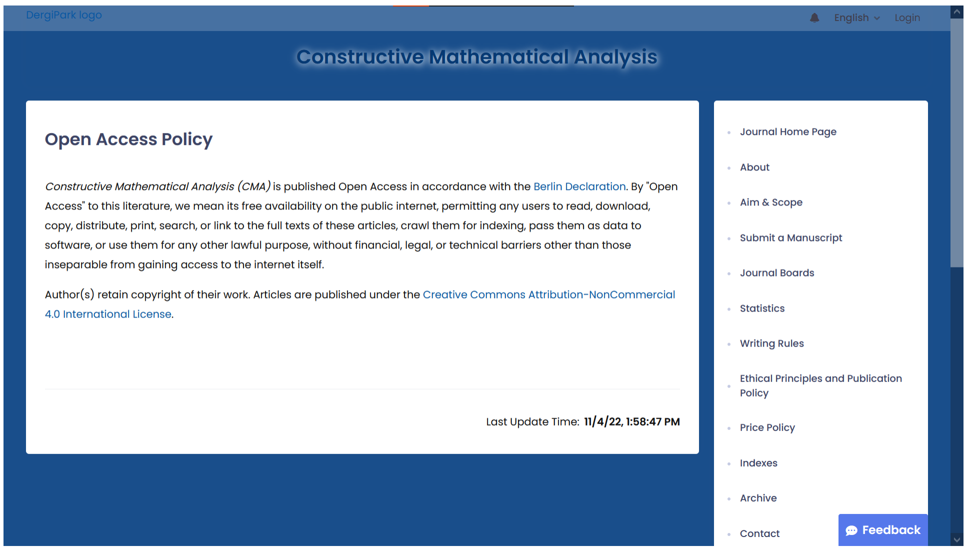
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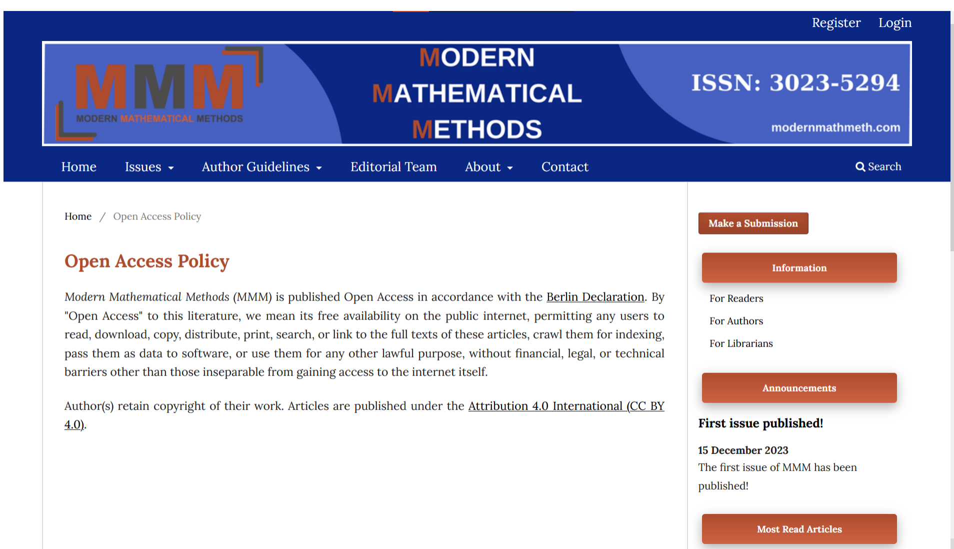
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