April 2024

The Effect of Fixed Time Delays on the Synchronization Phase Transition

Shaizat Bakhytzhan
University of South Florida

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by

Shaizat Bakhytzhan

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Master of Arts
Department of Mathematics and Statistics
College of Arts and Sciences
University of South Florida

Major Professor: Razvan Teodorescu, Ph.D.
Sherwin Kouchekian, Ph.D.
Seung-Yeop Lee, Ph.D.

Date of Approval:
April 12, 2024

Keywords: Delay differential equations, Coupled oscillators, Kuromoto model

DEDICATION

To my family,

Akmarzhan, Ibrakhim and Mariyam.

ACKNOWLEDGMENTS

I would like to thank my advisors Dr.Teodorescu and Dr.Kouchekian for their insight, knowledge, and guidance, without which this paper would not have been written. I would also like to thank all the professors I have had throughout my studies here, especially Dr.Curtin, whose classes introduced me to the basic graduate student workload. And of course, Dr.Kouchekian in his classes, which I very much enjoyed being in as well. I offer my regards and blessings to all of those who supported me in any respect during the completion of the project. Last but not least, I am heartily thankful to my wife and my children for their unconditional love and support.

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ABSTRACT

Nature is full of synchronization phenomena, which are essential to many scientific fields like biology, chemistry, physics, and neuroscience. The Kuramoto model is a well-known theoretical model that helps explain the fundamental ideas behind synchronization dynamics [6]. Nevertheless, in practical situations, systems frequently display intrinsic latency, which can greatly impact their behavior during synchronization. This insight inspired our work, which looks at the results of adding temporal delays to the Kuramoto model. In particular, we investigate how the system's synchronization dynamics are affected by delays. We shed light on the mechanisms underpinning synchronization in the face of temporal delays and clarify how these delays affect the system's emergent collective behavior through research and numerical simulations.

CHAPTER 1:
THE KURAMOTO MODEL

1.1 Kuramoto's Model

Kuramoto's model below enables us to represent the motions of feebly linked systems of oscillators whose cylces are limited

$$\dot{\theta}_i = \omega_i + \sum_{j=1}^{N} \Gamma_{ij}(\theta_j - \theta_i), i = 1, 2, \ldots, N \tag{1.1}$$

where θ_i are the phases and ω_i are the limit cycle natural frequencies of the oscillators. Kuramoto took further steps to make the model less complicated. Moreover,the ordinary differential equation, $\frac{d\theta_i}{dt}$ describes how quickly the phase of oscillator i is advancing or retarding over time. We will use the following simple system called mean field model [7]

$$\Gamma_{ij}(\theta) = \frac{K}{N}\sin(\theta) \tag{1.2}$$

where $i, j = 1, 2, \ldots, N$. Since in this model, the individual oscillators interact with the other $N-1$ oscillators with uniform strength, how they are distributed in real space is completely irrelevant. The coupling constant K is assumed to be positive, so that any pair of oscillators may favor minimizing their phase difference rather than maximizing it. This simplified the analysis of the model as shown below [6]:

$$\dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N} \sin(\theta_j - \theta_i) \tag{1.3}$$

The factor $\frac{1}{N}$ is introduced to ensure that the model behaves properly in the thermodynamic limit,$N \to \infty$, ω_i stands for the natural frequency of oscillator i.The frequencies ω_i are dispersed using a function $g(\omega)$, which is typically unimodal and symmetric around the mean frequency Ω.

Due to the model's rotational symmetry, we may utilize a rotating frame to redefine $\omega_i \rightarrow \omega_i + \Omega$ for all i and set $\Omega = 0$, indicating deviations from the mean frequency [2].

Equation (1.3) can be expressed more simply by entering the order parameter $r(t)$ given as

$$r(t) \exp i\psi(t) = \frac{1}{N} \sum_{j=1}^{N} \exp(i\theta_j) \tag{1.4}$$

where the modulus $0 \leq r(t) \leq 1$ represents the phase-coherence of the population of oscillators and ψ indicates the average phase. The values of $r \simeq 1$ and $r \simeq 0$ indicate that all oscillators are phase-locked or move incoherently. Multiplying both parts of eq(1.4) by $\exp(-i\theta_i)$, we get

$$r(t) \exp i(\psi(t) - \theta_i) = \frac{1}{N} \sum_{j=1}^{N} \exp(i(\theta_j - \theta_i)), \tag{1.5}$$

and only considering the imaginary part gives

$$r(t) \sin(\psi(t) - \theta_i) = \frac{1}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i), \tag{1.6}$$

Thus, we can rewrite Eq(1.3) as

$$\dot{\theta}_i = \omega_i + Kr(t) \sin(\psi - \theta_i) \tag{1.7}$$

where the coherency of the oscillators is proportional to r, and Kr is the effective coupling. Since we may assume that the average phase, ψ, is equal to zero without losing generality, we can express Eq(1.7) as

$$\dot{\theta}_i = \omega_i - Kr(t) \sin(\theta_i) \tag{1.8}$$

The model's mean-field nature is made clear in this form. It seems as though every oscillator is isolated from the others, even though they are in fact interacting through the mean-field values ψ and r. To be more specific, the phase θ_i moves toward the mean phase ψ compared to the phase of any particular oscillator. Furthermore, there exists a corresponding relationship between the coupling's effective strength and coherence r. A positive feedback loop between coupling and coherence is established by this proportionality. More oscillators tend to join the synchronized pack as r increases and the effective coupling Kr rises as the population gets more coherent. The process will continue if the new hires continue to raise the coherence; if not, it will become self-limiting [8].

1.2 Kuramoto's Analysis

Equation(1.4) order parameter equation can be written as

$$r\exp(i\psi) = \int_{-\pi}^{\pi} \exp(i\theta) \left(\frac{1}{N} \sum_{j=1}^{N} \delta(\theta - \theta_j) \right) d\theta \tag{1.9}$$

The arithmetic mean in Eq(1.4) now becomes an average across phase and frequency as, in the limit of infinitely many oscillators, they may be expected to be distributed with a probability density $\rho(\theta, \omega, t)$, namely,

$$r\exp(i\psi) = \int_{-\pi}^{\pi} \int_{-\infty}^{+\infty} \exp(i\theta)\rho(\theta, \omega, t)g(\omega)\, d\theta d\omega \tag{1.10}$$

This formula demonstrates how oscillator synchronization can be measured using the order parameter. As $K \to 0$, the oscillators rotate at angular frequencies specified by their own natural frequencies, as shown in Eq (1.7) : $\theta_i \approx \omega_i t + \theta_i(0)$. As a result, using the Riemann-Lebesgue lemma to define $\theta = \omega t$ in Eq(1.10), we can infer that $r \to 0$ as $t \to 0$ and the oscillators are not synchronized. The oscillators synchronize to their average phase, $\theta_i \approx \psi$, in the case of high coupling, $K \to \infty$, and Eq(1.10) implies $r \to 1$. For intermediate couplings with $K_c < K < \infty$, some oscillators are phase-locked ($\dot{\theta}_i = 0$), while others rotate out of sync with the locked oscillators [1].

By noting that each oscillator in Eq (1.3) moves with an angular or drift velocity $v_i = \omega_i + Kr\sin(\psi - \theta_i)$, we can derive a continuity equation for the oscillator density. Therefore, with following normalization condition

$$\int_{-\pi}^{\pi} \rho(\theta, \omega, t) d\theta = 1, \quad \forall \omega, t \tag{1.11}$$

the continuity equation must be satisfied by the one-oscillator density

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial \theta}[\omega + Kr\sin(\psi - \theta)]\rho = 0. \tag{1.12}$$

The system of equations $(1.10)-(1.12)$ has a simple stationary solution, $\rho = \frac{1}{2\pi}, r = 0$, which corresponds to an angular distribution of oscillators with equal probability in the interval $[-\pi, +\pi]$. Such a stationary solution is also called an incoherent solution or just incoherent because the oscillators work incoherently. Now, let's attempt to identify a straightforward solution that corresponds to oscillator synchronization. Global synchronization, or phase locking, occurs at the strong coupling limit, resulting in $r = 1$ for all oscillators with the same phase, $\theta_i = \psi[= \omega_i t + \theta_i(0)]$. Lower levels of synchronization with a stationary amplitude, $0 < r < 1$, are possible for a finite connection. Why would r have a lower value? At a certain angle $Kr\sin(\theta - \psi) = \omega$, a standard oscillator with velocity $\nu = \omega - Kr\sin(\theta - \psi)$ will stabilize and $-\frac{\pi}{2} \leq (\theta - \psi) \leq \frac{\pi}{2}$. The natural laboratory frame of reference is locked for all such oscillators. Frequencies $|\omega|> Kr$ prohibit the locking of oscillators. Equation (1.12) states that their stationary density obeys $\nu\rho = C$ (constant) when they lose synchronization with the locked oscillators. We have achieved a partially synchronized stationary state, wherein a portion of the oscillators remain locked at a constant phase, and the remaining oscillators rotate out of lock with each other.

After certain period of time, the system finally reaches a steady state. Consequently, the initial term in Eq(1.9) disappears, leaving us with

$$\rho(\theta, \omega) = \frac{C}{|\omega - Kr\sin(\theta)|}, \tag{1.13}$$

which represents the density of incoherent oscillators, commonly referred to as the drift group. Normalization constant can be obtained with Eq(1.10)

$$C = \frac{1}{2\pi}\sqrt{\omega^2 - (Kr)^2}. \tag{1.14}$$

Furthermore, as $t \to \infty$, where $|\theta_i| \leq \frac{\pi}{2}$, it follows from Eq(1.8) that the dynamics of oscillators with $|\omega| \leq Kr$ approach $\omega_i = Kr\sin(\theta_i)$. This oscillator group is "locked," or synchronized and has distribution

$$\rho(\theta, \omega) = \delta[Kr\sin(\theta) - \omega]H(\cos(\theta)) \qquad \textit{where} \quad |\omega| \leq Kr \tag{1.15}$$

where $H(x)$ is Heaviside step function. The order parameter r can be computed using Eq ((1.10), (1.13), (1.15)) :

$$
\begin{aligned}
r = &\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\infty}^{\infty} e^{i(\theta-\psi)} \delta \left[\theta - \psi - \sin^{-1}\left(\frac{\omega}{Kr}\right)\right] g(\omega) d\theta d\omega + \\
&\int_{-\pi}^{\pi} \int_{|\omega|>Kr} e^{i(\theta-\psi)} \frac{Cg(\omega)}{|\omega - Kr\sin(\theta-\psi)|} d\theta d\omega.
\end{aligned}
\tag{1.16}
$$

The drift group term vanishes, since $g(\omega) = g(-\omega)$. Out of Eq(1.13), we have $\rho(\theta,\omega) = \rho(\theta+\pi,-\omega)$. In the lock term the imaginary part disappears; since $\rho(\theta,\omega) = \rho(-\theta,-\omega)$ and $g(\omega) = g(-\omega)$,

$$
\begin{aligned}
r = &\int_{|\omega|>Kr} \cos\left[\sin^{-1}\left(\frac{\omega}{Kr}\right)\right] g(\omega) d\omega \\
= &\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(\theta) g(Kr\sin(\theta)) Kr\cos(\theta) d\theta \\
= &Kr \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2(\theta) g(Kr\sin(\theta)) d\theta
\end{aligned}
\tag{1.17}
$$

this equation has the trivial solution at $r = 0$ valid for any value of K, corresponding to incoherent phase with

$$
\rho(\theta,\omega) = \frac{1}{2\pi} \qquad \forall\theta,\omega \tag{1.18}
$$

which, when $r \neq 0$, corresponding to the partially synchronized phase of Eq(1.15), contains a second branch of solutions. As a result

$$
1 = K \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2(\theta) g(Kr\sin(\theta)) d\theta. \tag{1.19}
$$

After setting $r \to 0^+$ in Eq(1.19) this solution breaks down consistently from $r = 0$ at the poin t when $K = K_c$. Therefore

$$
1 = Kg(0) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2(\theta) d\theta. \tag{1.20}
$$

and

$$
K_c = \frac{2}{\pi g(0)} \tag{1.21}
$$

Kuramoto [6] proposed this formula and the reasoning behind it. This formula and the arguments leading to it were suggested by Kuramoto [6].

The system when $K < K_c$, the system is in an incoherent state where the oscillators show separate oscillations, and when is in incoherent state in which the oscillators exhibit independent oscillations, and when $K > K_c$, is in coherent state in which part of oscillators population is synchronized. By expanding the integral in Eq(1.19) in relation to r,

$$1 = K \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2(\theta) \left(g(0) + \frac{g'(0)Kr\sin(\theta)}{1!} + \frac{g''(0)(Kr\sin(\theta))^2}{2!} + \dots \right) d\theta, \tag{1.22}$$

after taking the integral,

$$1 \simeq K \left(\frac{1}{K_c} + \frac{g''(0)(K_c r)^2 \pi}{16} \right), \tag{1.23}$$

Terms can be rearranged as,

$$\frac{K_c - K}{K_c} = \mu \simeq \frac{g''(0)K_c^3 r^2 \pi}{16}. \tag{1.24}$$

Regarding the smooth, unimodal, and even density of the Lorentzian distribution, $g(\omega) = \frac{\gamma^2}{\pi(\omega^2+\gamma^2)}$, $g'(0) = 0$ and $g''(0) = -\frac{16}{\pi K_c^3} < 0$. For all $K > K_c = 2\gamma$ we get

$$r \simeq \sqrt{\mu} = \sqrt{\frac{K - K_c}{K_c}}. \tag{1.25}$$

Thus, the system bifurcation is supper-critical for $K > K_c$ if $g''(0) < 0$ and sub-critical for $K < K_c$ if $g''(0) > 0$.

1.3 Stability of Solutions and Open Problems

It may be seen that Kuramoto's computations for the partially synchronized phase do not specify whether or not this phase is stable on a local or global. Strogatz [8] has studied the linear stability theory of incoherence.

1.4 Synchronization As N Approaches Infinity

Strogatz conducted the first comprehensive stability analysis of the incoherent solution for the infinite oscillators system [9], [11], [10]. When the coupling parameter $r = 0$, the system becomes incoherent. This state has linear stability, which means that tiny perturbations do not result in exponential growth. The answer is not unique, as numerous solutions of K_0 fulfill an equation (1.17). When K is less than the crucial coupling value, K_c, the state is considered neutrally stable. Neutrally stable indicates that the system does neither diverge or converge with time; it retains its state. Furthermore, the oscillators in this condition have equiprobability (1.18), which means that every potential state is equally likely. When the coupling parameter K hits the critical value ($K = K_c$), a new stationary solution arises from the neutrally stable one. This new solution is known as the partly synchronized state. If the coupling strength above the critical value ($K > K_c$), the incoherent state becomes unstable. In this situation, a synchronization state separates from the unstable incoherent state. This suggests that, above a given amount of coupling, the oscillators begin to synchronize their behavior.

1.5 Synchronization at Finite N

A problem with Kuramoto's model's kinetic equation—which models populations with an unlimited nu mber of elements—is the finite size effect. As $t \to \infty$ a population of finitely numerous Kuramoto oscillators reaches a stationary state, as demonstrated by the Lyapunov function argument [12]. In this work, we present a rigorous analysis for large finite -N of Eq(1.3), and then prove the convergence as $N \to \infty$. However, [[7], [5], [3]] have investigated the problem using computer simulation and physical arguments. The deviations seem to be $O(N^{-\frac{1}{2}})$ except in close proximity to K_c.

CHAPTER 2:
MINIMAL PRESENTATION ON DYNAMICAL SYSTEMS WITH TIME DELAYS

2.1 Time-Delayed Systems

Time delay is a common occurrence in nature, science, and technology, particularly in control engineering, dynamical systems, communication, and biology. Typically, a dynamical system is represented by a model using an ordinary differential equation of this structure:

$$\dot{x} = f(t, x(t)), \tag{2.1}$$

where the variables $x(t) \in R^n$ are the state variables. According to the Cauchy-Kowaleskaya initial-value theorem, if the function f is continuous in both its variables, then given the initial condition $x(t_0) = x_0$ at $t = t_0$, the state variables $x(t)$ for $t_0 \leq t \leq T$ can be found uniquely, for some $T > t_0$. Therefore, the change of states with time is defined by these differential equations and only knowledge of the value at $t = t_0$ is needed. However, in practice, the future values of the state variables of $x(t)$ depend on also their past values for many systems. Then these systems are called time-delay systems. Hence ordinary differential equations can not satisfactorily describe behavior of the such dynamic systems, in that case, *delay differential equations* are used [4] . In general, a delay systems can be described by the delay differential equation (DDE)

$$\dot{x} = f(x(t), x(t - \tau)), \tag{2.2}$$

where $\tau > 0$ is the delay. To understand the impact of time delays, it is necessary to consider a description that accounts for the system's history. This entails using a dynamical system with a phase space of infinite dimensions.Indeed, it is found that the state of the Delay Differential Equation (DDE) at time t is determined by the function $x(\theta)$, $\theta \in [t-\tau, t]$. For analytical functions f, x, it can be seen by Taylor-expanding the right-hand side with respect to the delay τ, that the problem (2.2) is equivalent to an ordinary differential equation of *infinite* order, instead of first-order (2.1). Consequently, equations (2.2) can generate highly intricate behaviors and bifurcation scenarios, including hysteresis, self-focusing, amplification, etc.

2.2 General Stability Analysis of the Synchronized Kuramoto System Under Time Delays

In this section, we consider the Kuramoto system without time delays, in the limit of the completely synchronized state, to which we then apply the perturbation of one delayed oscillator. The stability analysis (with and without time delays) is meant to identify relevant timescales of the system and therefore to predict orders of magnitude for the parameter τ at which the presence of time delays would be likely to induce qualitatively novel behavior of the system.

Let $N \in \mathbb{N}$ be the number of identical oscillators of phases $y_j(t)$, $j = 1, 2, \ldots, N$. We denote by $z_j = e^{iy_j}$ the uni-modular oscillating variable and by $\lambda > 0$ the *coupling constant* and $K = \frac{\lambda N}{2}$ the *rescaled* coupling constant of the system, such that the Kuramoto equations of motions read:

$$\frac{dy_j}{dt} = \lambda \sum_{l=1}^{N} \sin(y_l - y_j) + \omega_j,$$

where ω_j is the natural frequency of the oscillator z_j. For a general analysis, we may take ω_j from an unimodal distribution centered on the common frequency Ω, with variance σ^2. Taking first $\sigma \to 0$ and shifting the phases by the common terms Ωt, the equations of motion become

$$\dot{z}_j = z_j \frac{1}{N} \sum_{l=1}^{N} (z_l \overline{z_j} - z_j \overline{z_l})$$

Introducing the collective variables

$$\phi_n = \frac{1}{N} \sum_{j=1}^{N} z_j^n,$$

we can write

$$\dot{\phi}_1 = K[\phi_1 - \phi_2 \overline{\phi_1}]$$

2.2.1 The Synchronized State

An obvious solution (modulo the overall rotation $\theta_j = \Omega t$) is that of constant equal phases, or $z_j = 1$. Then

$$\phi_n = 1,\ n = 1, 2, \ldots,$$

and the state is one of equilibrium. To classify it further regarding stability, we consider a perturbation around the $z_j(t) = 1$ state, taking

$$y_j(t) = \epsilon_j(t),\ z_j(t) \simeq 1 + i\epsilon_j - \frac{\epsilon_j^2}{2} + \ldots,$$

where $|\epsilon_j| \ll 1$. The equation satisfied by the collective modes follows from their $\epsilon-$expansion

$$\phi_n \simeq 1 + in\langle\epsilon\rangle - \frac{n^2\langle\epsilon^2\rangle}{2} + \ldots,$$

where we have introduced the notation

$$\langle\Psi\rangle = \frac{1}{N}\sum_{j=1}^{N}\Psi_j$$

representing the sample average of the quantity Ψ, taken with discrete uniform measure. The equation of motion for the collective modes becomes (at order 2)

$$\frac{d}{dt}\left[i\langle\epsilon\rangle - \frac{\langle\epsilon^2\rangle}{2}\right] = 2K\left[\langle\epsilon^2\rangle - (\langle\epsilon\rangle)^2\right],$$

or upon separating the real and imaginary parts,

$$\frac{d\langle\epsilon\rangle}{dt} = 0,$$

$$\frac{d\langle\epsilon^2\rangle}{dt} = -4K\langle(\epsilon - \langle\epsilon\rangle)^2\rangle,$$

with the solution (taking the average perturbation to equal zero)

$$\langle \epsilon^2 \rangle(t) = \langle \epsilon^2 \rangle_0 e^{-4Kt},$$

$$\phi_1(t) \simeq 1 - \frac{\langle \epsilon^2 \rangle_0}{2} e^{-4Kt}$$

We conclude that the fully synchronized state of the Kuramoto model is stable under perturbations, which are described by a martingale with exponentially decreasing variance. The timescale on which such perturbations are dissipated is of the order of K^{-1}.

2.2.1.1 The effect of a single delayed oscillator Let one of the oscillators be characterized by a time delay $\tau > 0$, such that its phase can be expanded perturbatively as

$$y(t-\tau) = y(t) - \tau \dot{y}(t) + \ldots,$$

in the limit $\tau \to 0^+$. Then making use of the Kuramoto equations without delay, we have for the delayed oscillator

$$y(t-\tau) \simeq y(t) - 2K\tau \frac{1}{N} \sum_{j=1}^{N} \sin(y_j - y),$$

while for all the others

$$\dot{y}_j = 2K \frac{1}{N} \left[\sum_{i \neq j=1}^{N} \sin(y_i - y_j) + \sin(y(t-\tau) - y_j) \right]$$

Around the fully synchronized state, $y_j = 0$, this gives

$$y(t) = y(t-\tau) + 2K\tau \sin y$$

$$\dot{y}_j = \frac{2K}{N} \sin\left(2K\tau \sin(y)\right)$$

2.2.1.2 Case 1: short delays, $\tau \ll K^{-1}$

In this case, the equation becomes

$$\dot{y}_j = \frac{2K}{N} \sin\left(2K\tau \sin(y)\right) \simeq \frac{4K^2\tau}{N} \sin(y),$$

and we observe that the average perturbation $\langle \dot{\epsilon} \rangle \neq 0$ has the same behavior as that of $\sin(y)$, it is no longer a martingale, so the fully synchronized state stops being one of stable equilibrium and acquires a driving term. However, in the macroscopic limit $N \to \infty$, this effect is negligibly small.

2.2.1.3 Case 2: delays of order $\tau = O(K^{-1})$

For longer delays, it becomes possible for the argument of $\sin\left(2K\tau \sin(y)\right)$ to exceed $\frac{\pi}{2}$ in modulus, which means that the perturbation stops being driven by the delayed oscillator and can experience cycle-like behavior, in the form of hysteresis. The threshold value separating this case from the short delay case is

$$2K\tau = \frac{\pi}{2} \Rightarrow \tau = \frac{\pi}{4K}$$

Therefore, in the case of delays longer than $\frac{\pi}{4K}$, we predict that the long-time limit of the system will display distinct phases alternating between positive and negative drift, with a modulation of the order K^{-1}.

2.3 Numerical Solutions For System of DDE

In this section, we will investigate numerically the theoretical predictions made earlier. For simplicity of coding, the system of equations is written in a modified form. Let us consider the following system of equations :

$$\begin{cases} \dot{y}_1(t) = -sin(y_1(t) - y_2(t)) - sin(y_1(t) - y_3(t)) \\ \dot{y}_2(t) = -sin(y_2(t) - y_1(t)) - sin(y_2(t) - y_3(t)) \\ \dot{y}_3(t) = -sin(y_3(t) - y_1(t)) - sin(y_3(t) - y_2(t)) \end{cases} \tag{2.3}$$

on $[0, 2\pi]$ with history $y_1(0) = \frac{\pi}{6}$,$y_2(0) = \frac{2\pi}{3}$, $y_3(0) = \frac{5\pi}{3}$ for $t \leq 0$. Where $K = -1$ is the effective coupling constant. This value will be utilized in subsequent experiments as well.

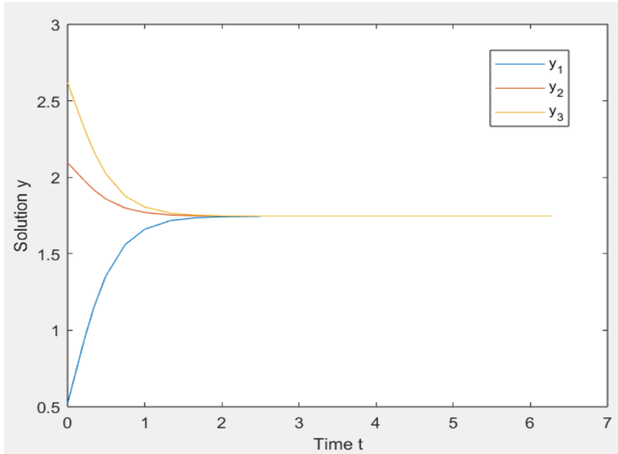


Figure 1. System (2.3) with given initial conditions

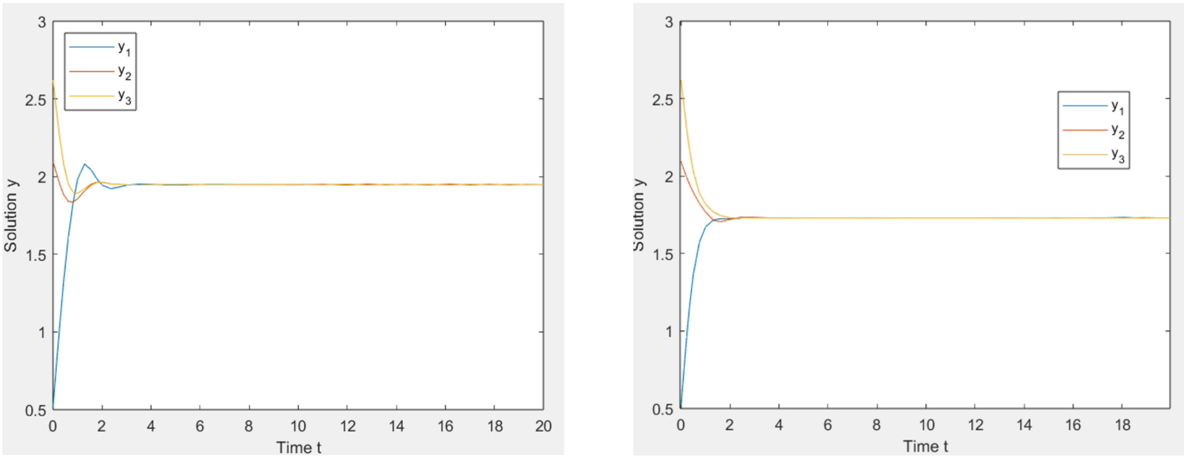
Once we implemented this system in the MatLab program, we captured images (fig. 1) and observed that, over time, the graphical representations of each equation tended to overlap quickly.

2.3.1 Adding Delays Into One Equation in the System

Introducing delays into one of the equations of the system results in the following set of equations.

$$\begin{cases} \dot{y}_1(t) = -sin(y_1(t) - y_2(t-\tau)) - sin(y_1(t) - y_3(t-\tau)) \\ \dot{y}_2(t) = -sin(y_2(t) - y_1(t)) - sin(y_2(t) - y_3(t)) \\ \dot{y}_3(t) = -sin(y_3(t) - y_1(t)) - sin(y_3(t) - y_2(t)) \end{cases} \tag{2.4}$$

With initial conditions identical to those of system (2.3), over the time interval [0, 20], we introduced $\tau = 1$ into the first equation and then into the second equation. Subsequently, we captured Fig. 2 (a) and 2(b) respectively. Upon analysis, it becomes evident that there is minimal deviation compared to the case without delay; the lines overlap at approximately the same temporal distance.



(a) **(b)**

Figure 2. System (2.4) a) delay in the first equation b) delay in the second equation

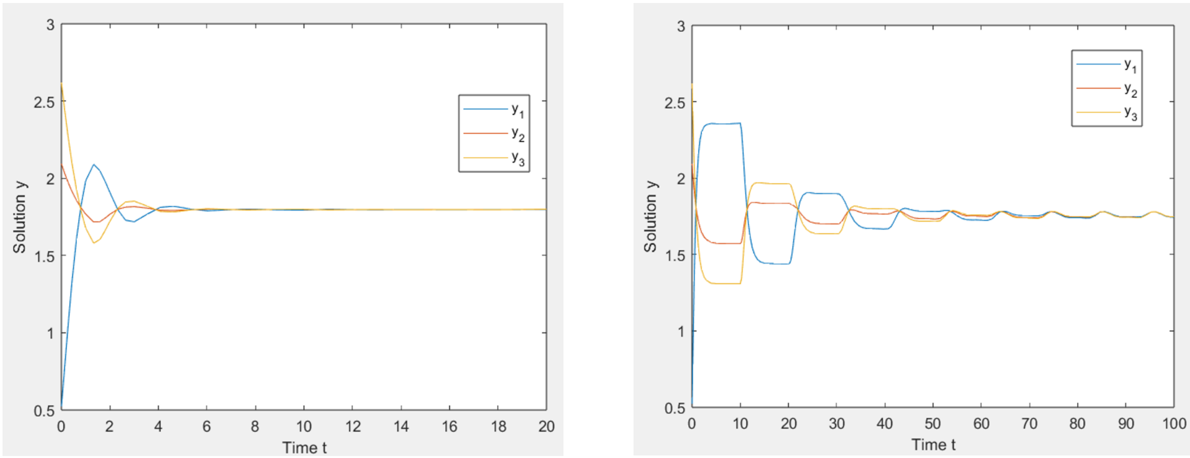
2.3.2 Adding Delays Into Every Equation in the System

To evaluate the impact of incorporating delays into every equation within the system, we consider the following formulation. Let's examine the graphical representations of the system while varying τ.

$$\begin{cases} \dot{y}_1(t) = -sin(y_1(t) - y_2(t-\tau)) - sin(y_1(t) - y_3(t-\tau)) \\ \dot{y}_2(t) = -sin(y_2(t) - y_1(t-\tau)) - sin(y_2(t) - y_3(t-\tau)) \\ \dot{y}_3(t) = -sin(y_3(t) - y_1(t-\tau)) - sin(y_3(t) - y_2(t-\tau)) \end{cases} \tag{2.5}$$

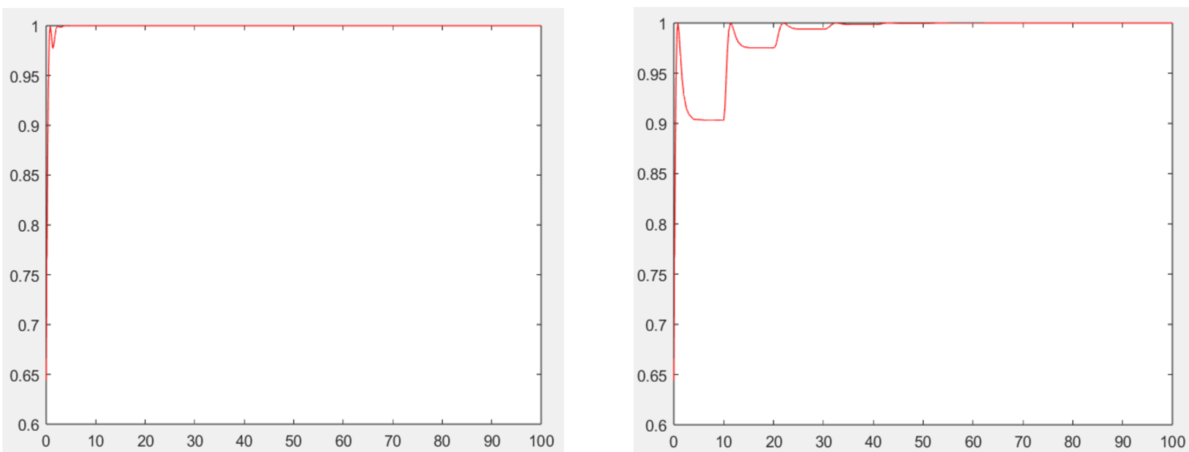
The initial conditions and observation time interval remain consistent with previous systems. When $\tau = 1$ (Fig.3a), it becomes evident that the overlapping time extends compared to the scenario with delay in only one equation, accompanied by slightly larger fluctuations. However, when $\tau = 10$ (Fig.3b), the fluctuations become significantly pronounced, indicating severe system instability due to substantial delay. Furthermore, achieving overlap for such systems becomes challenging. It may take longer than depicted in Figure a) for overlap to occur. Having observed the graphical disparities between systems with delays of $\tau = 1$ and $\tau = 10$, let's now investigate their synchronization or coherence behavior $r(t)$, as depicted in Figure 3(a) and (b).

Indeed, the distinction is evident concerning the delays. In both cases, where $\tau = 1$ and $\tau = 10$, we can confirm that the system exhibits coherence, although with slight temporal discrepancies. Therefore, both systems demonstrate coherence characteristics.



(a) **(b)**

Figure 3. System (2.5) a) $\tau = 1$ b) $\tau = 10$



(a) **(b)**

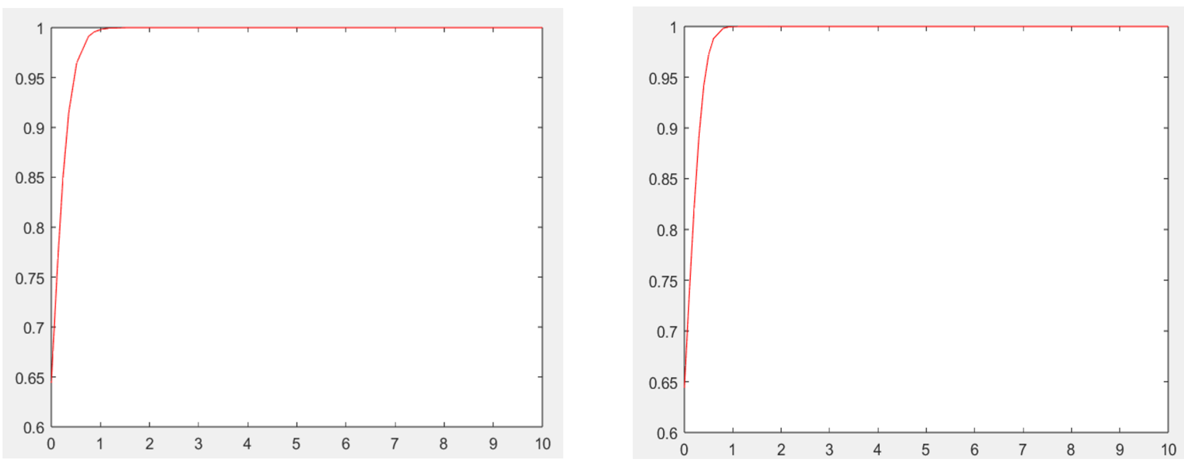
Figure 4. Solution for the function $r(t)$ corresponding to Figure 3(a) and (b)

2.3.3 Adding Delays Into Everywhere in the System

Let's introduce delays into each equation throughout the entirety of the system according to the following formulation.

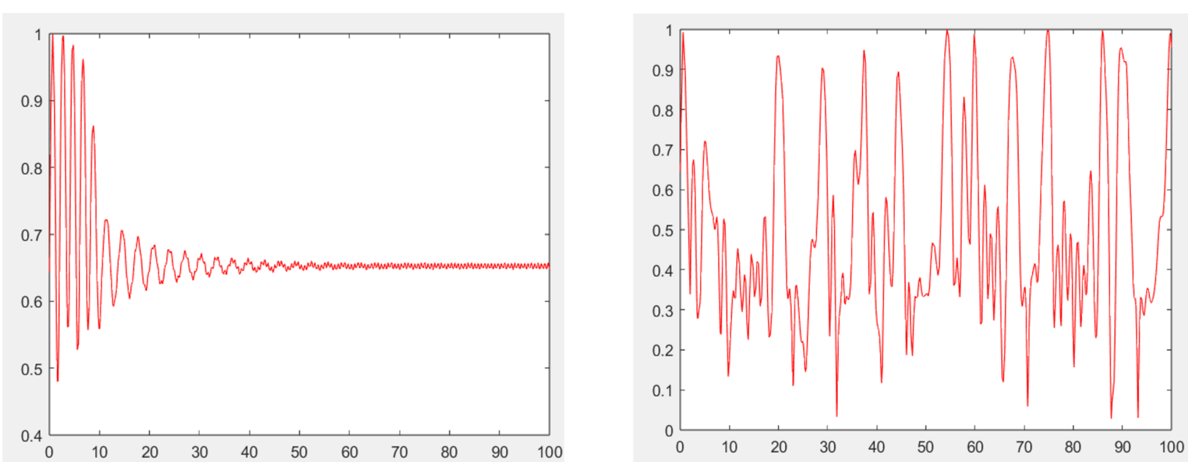
$$\begin{cases} \dot{y}_1(t) = -sin(y_1(t-\tau) - y_2(t-\tau)) - sin(y_1(t-\tau) - y_3(t-\tau)) \\ \dot{y}_2(t) = -sin(y_2(t-\tau) - y_1(t-\tau)) - sin(y_2(t-\tau) - y_3(t-\tau)) \\ \dot{y}_3(t) = -sin(y_3(t-\tau) - y_1(t-\tau)) - sin(y_3(t-\tau) - y_2(t-\tau)) \end{cases} \tag{2.6}$$

With this given system, let's observe the coherence of the system. We will vary and analyze their graphical solutions. As observed, the delay terms in this system are more complex than in previous systems. We will keep the initial conditions unchanged but adjust the time interval corresponding to the values of τ. This adjustment is necessary because when we use large values for τ, it becomes crucial to observe the graphics over a larger interval, as it's apparent that fluctuations exhibit more severe behavior. The graphics obtained when $\tau = 0.01$ (Fig.5 a)) and $\tau = 0.1$ (Fig.5 b)) exhibit notable similarity, indicating system synchronization. However, when $\tau = 1$ (Fig.6 a)), the system becomes partially synchronized, while for $\tau = 2$ (Fig.6 b)), the system does not synchronize.



(a) **(b)**

Figure 5. System (2.6) a) $\tau = 0.001$ and b) $\tau = 0.1$



(a) **(b)**

Figure 6. System (2.6) a) $\tau = 1$ and b) $\tau = 2$

Now, let's investigate the effect of initial conditions on the system using the exact formulation provided. In Figure 7 and Figure 8, we present four graphs obtained for initial conditions with three points that are equally distant from each other. Specifically, $y_1(t) = 0$, $y_2(t) = 0.1$, $y_3(t) = -0.1$ and $y_1(t) = 0$, $y_2(t) = 0.01$, $y_3(t) = -0.01$. The x-axis represents the time interval, while the y-axis denotes the function $r(t)$. It's noteworthy that after $\tau = 0.5$, achieving synchronization becomes challenging in both figures. Therefore, the effect of initial conditions on synchronization becomes more pronounced as τ increases beyond 0.5, suggesting a critical threshold for synchronization in the system.

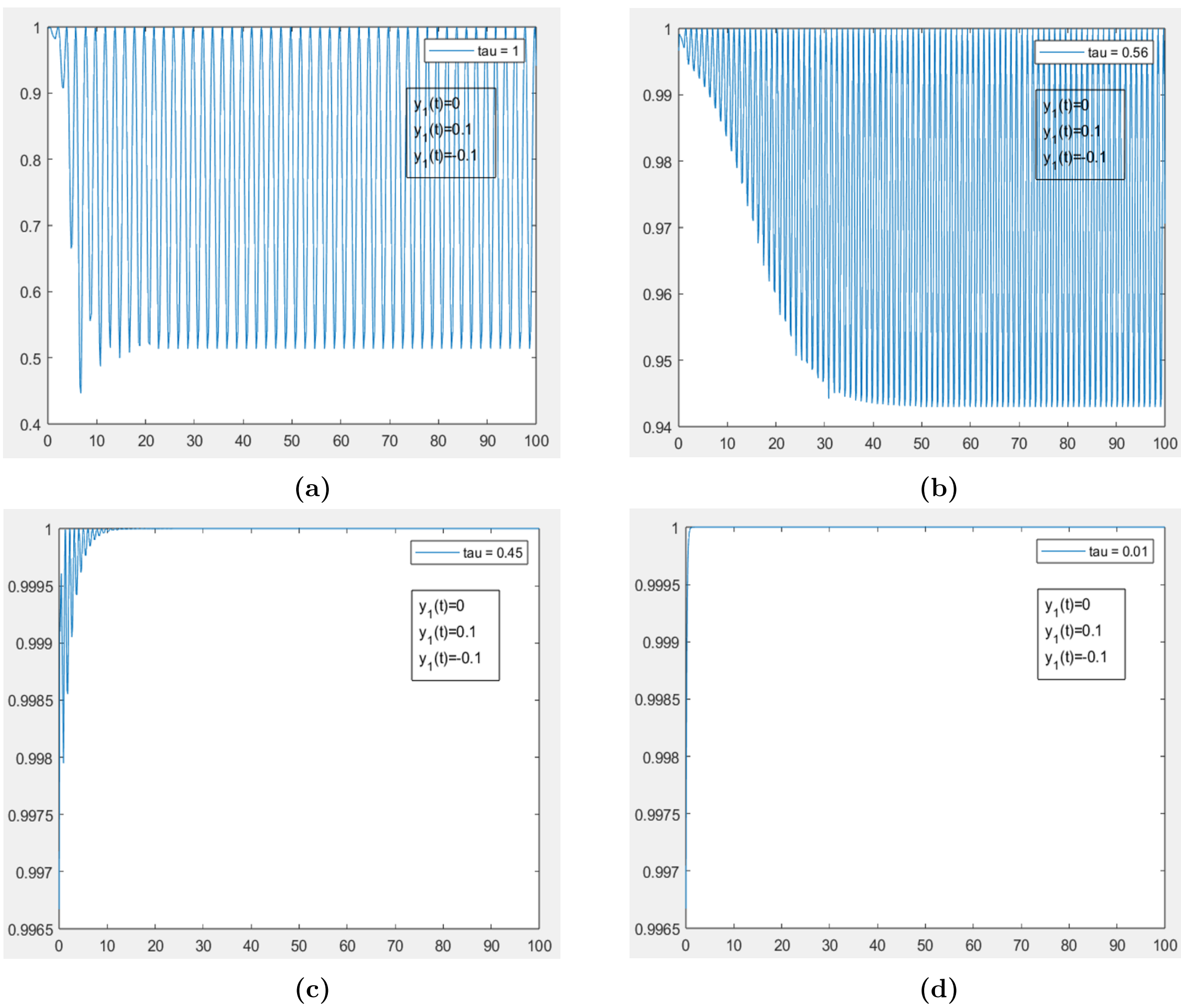
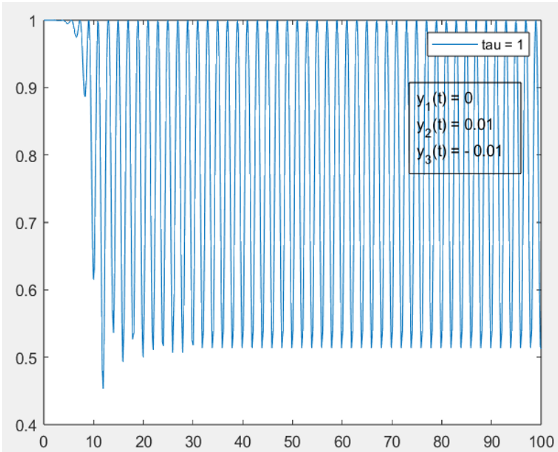
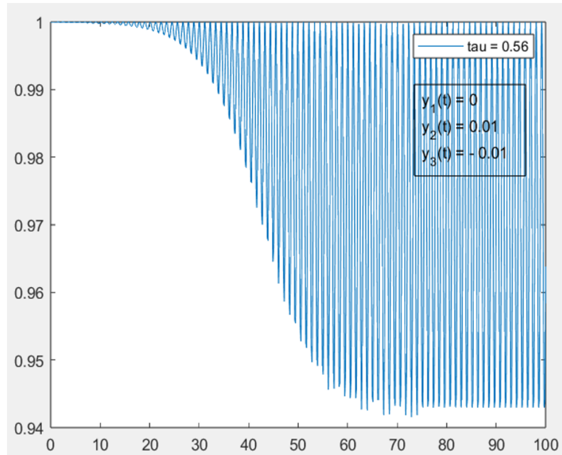


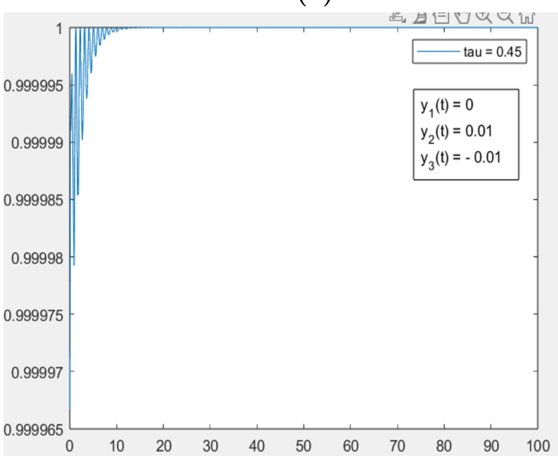
Figure 7. Graphics for $r(t)$ function with system (2.6) for the given initial conditions



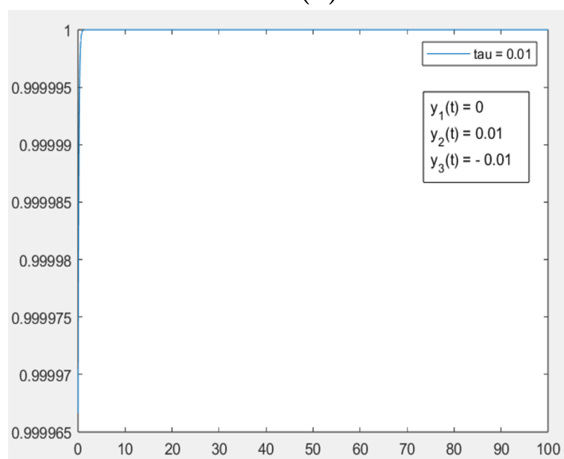
(a)



(b)



(c)



(d)

Figure 8. Graphics for $r(t)$ function with system (2.6) for the given initial conditions

2.3.4 Considering Synchronization For a Large Number of Oscillators With Delay

We intend to extend the previous system described in Equation 2.6 to accommodate a larger number of oscillators. As a result, we arrive at the following formulation:

$$\dot{y}_k(t) = -\sum_{j=1, j\neq k}^{N} sin(y_k(t-\tau) - y_j(t-\tau)) \tag{2.7}$$

where $j = 1, 2, ..., N$. For the initial conditions, we use the formula $y_j = \frac{n}{N}\epsilon$ for $j = 1, 2, ..., N$, where the value of ϵ remains fixed. In Figure 9, results were obtained for $N = 50$ and $\epsilon = 0.3$. The values of τ vary as follows: $\tau = [0.01, 0.1, 0.25, 0.5, 1]$, observed over the time interval $t = [0, 100]$. The result indicates that only when τ is small can a large number of oscillators synchronize.

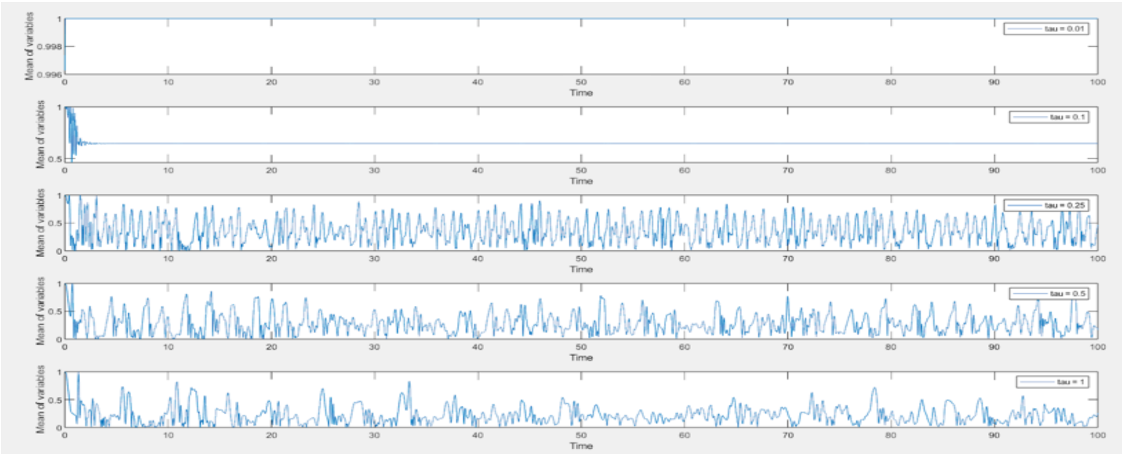


Figure 9. System (2.7) with N=50

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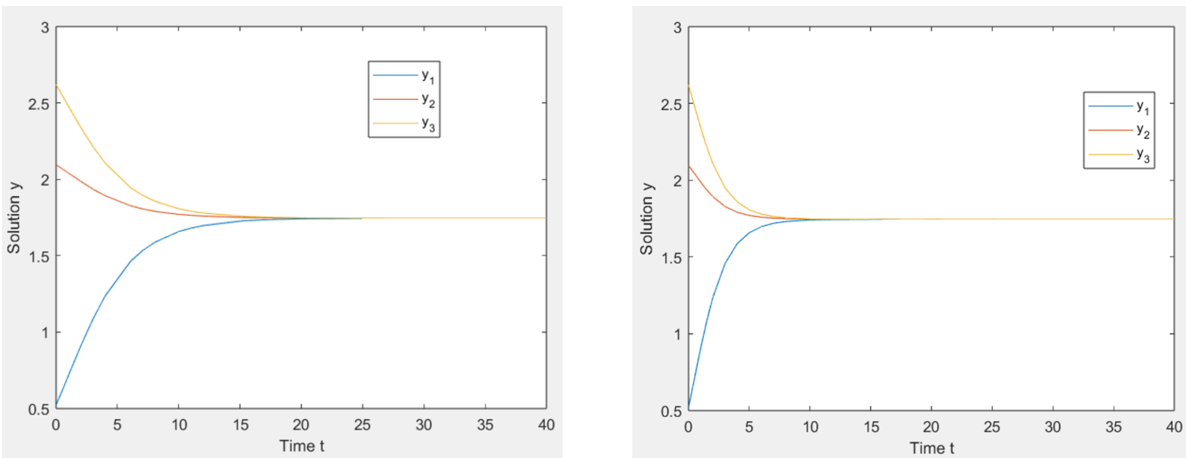
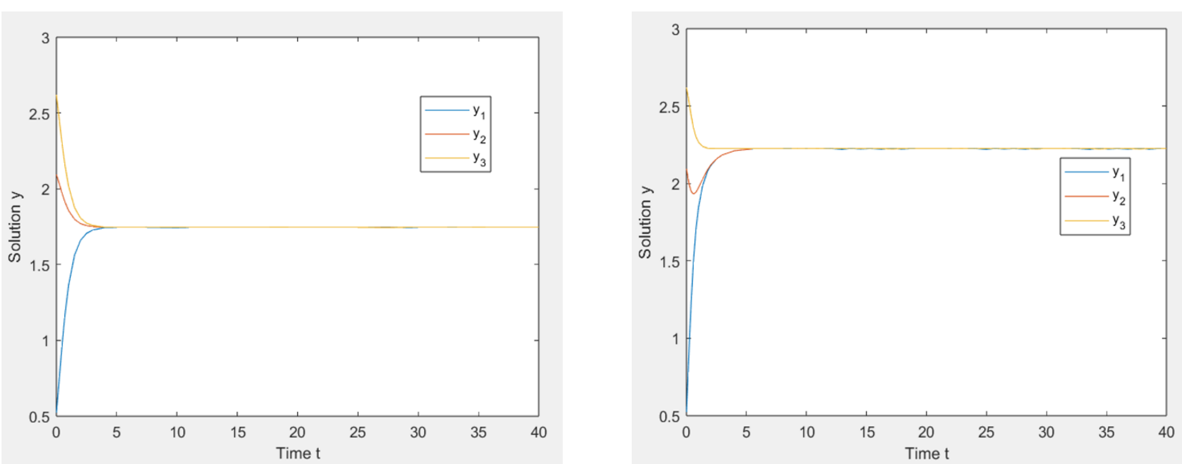
APPENDIX A:
ADDITIONAL FIGURES

A.1 Additional Findings



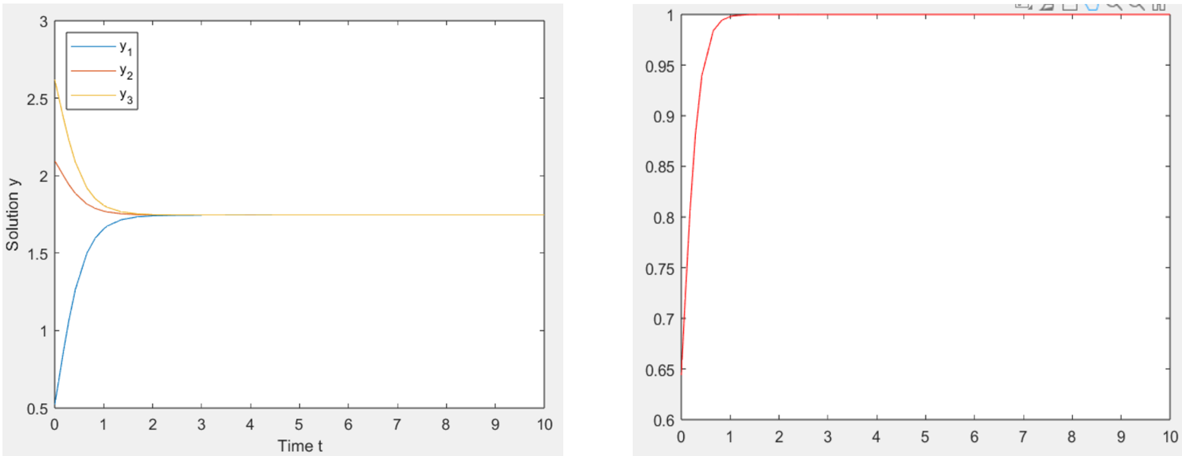
(a) **(b)**

Figure 10. System 2.3, t=[0 , 40] , a) $K = -0.1$, b) $K = -0.2$



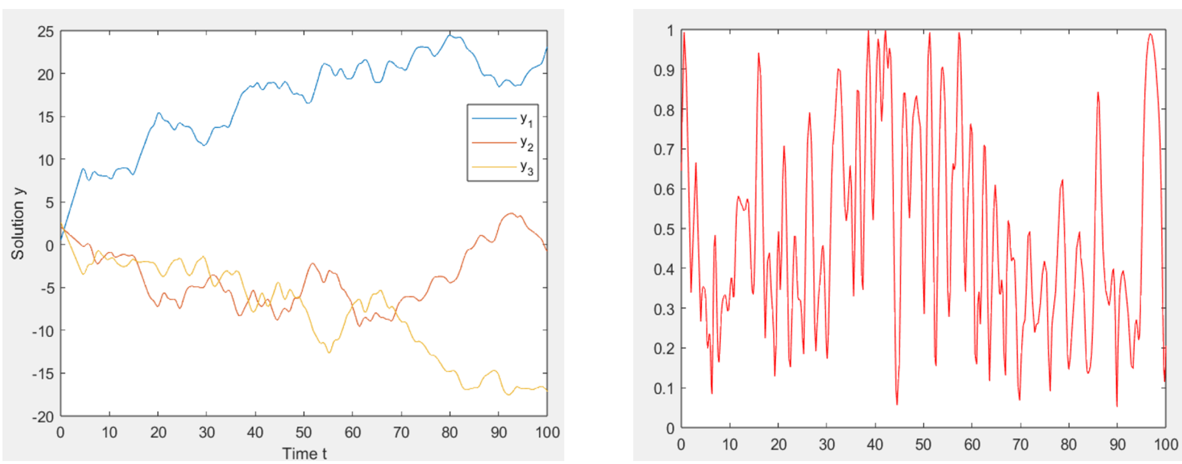
(a) **(b)**

Figure 11. System 2.3, t=[0 , 40] , a) $K = -0.5$, b) $K = -1$



(a) **(b)**

Figure 12. System 2.7 with initial conditions $y_1(0) = \pi/6, y_2(0) = 2\pi/3, y_3(0) = 5\pi/3$, t=[0 , 100] , $\tau = 0.0001$, a) y-axis y solutions , x-axis time t , b) y-axis r(t), x-axis time t



(a) **(b)**

Figure 13. System 2.7 with initial conditions $y_1(0) = \pi/6, y_2(0) = 2\pi/3, y_3(0) = 5\pi/3$, t=[0 , 100] , $\tau = 4$, a) y-axis y solutions , x-axis time t , b) y-axis r(t), x-axis time t

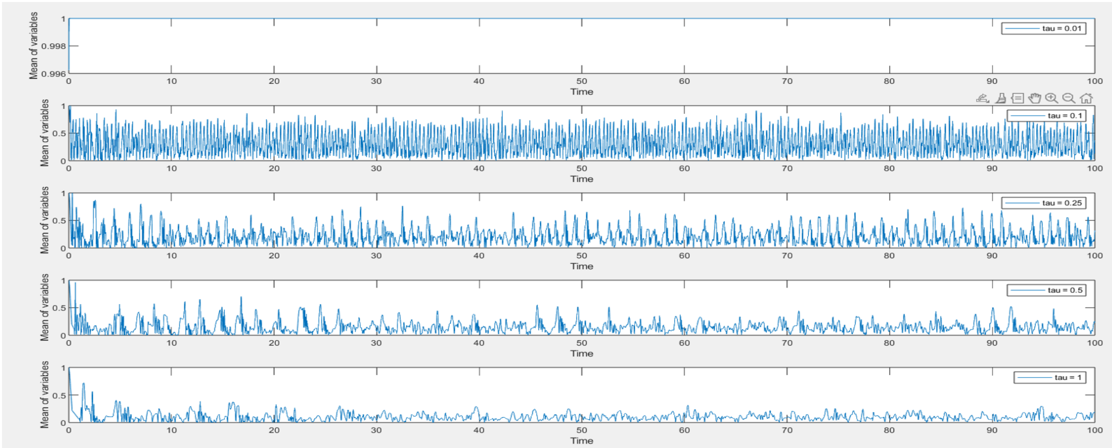


Figure 14. System (2.7) with N=100

APPENDIX B:
MATLAB CODES

B.1 Small N

```
lags = [ 0.5 1 2 4]
n=length(lags);
tspan = [0 100];
sol=zeros(3,1000);
for i=1:n
T=lags(i);
sol= dde23(@ddefun, lags(i), @history, tspan) ;
sol.f=abs((exp(1i*sol.y(1,:))+exp(1i*sol.y(2,:))+exp(1i*sol.y(3,:)))/3);
figure
txt = ['tau = ',num2str(T) ];
plot( sol.x, sol.f, '-' ,'DisplayName',txt);
dim = [.7 .2 .5 .6];
str = 'y__1(t) = 0.5', 'y__2(t) = 0.7' ,'y__3(t) = 10';
annotation('textbox',dim,'String',str,'FitBoxToText','on');
legend show
end
function v = ddefun(x,y,Z)
T = Z(:,1);
v = zeros(3,1);
v(1) = -sin(T(1)-T(2))-sin(T(1)-T(3));
v(2) = -sin(T(2)-T(1))-sin(T(2)-T(3));
v(3) =-sin(T(3)-T(2))-sin(T(3)-T(1));
```

end function s = history(t, r,N,epsilon)

N = 3;

epsilon=0.3;

s = zeros(N, 1);

for j = 1:N

s(j) = (j/N) * epsilon ;

end

end

B.2 MatLab Codes For General System and Large N With For Loop

N = 100;

epsilon=0.3;

lags = [0.01 0.1 0.25 0.5 1];

n = length(lags);

tspan = [0 100];

$num_{p}oints = 1000$;

for i = 1:n

T = lags(i);

sol = zeros(N, $num_{p}oints$);

$sol_{f}unc = @(t)dde23(@(t, y, Z)ddefun(t, y, Z, T, N), T, @history, tspan)$;

$sol_{s}truct = sol_{f}unc(tspan)$;

$f = abs(mean(exp(1i * sol_{s}truct.y), 1))$;

subplot(n, 1, i);

txt = ['tau = ', num2str(T)];

$plot(sol_{s}truct.x, f,' -',' DisplayName', txt)$;

dim = [.6 .2 .5 .6];

str = 'y_j(t) = n/N * epsilon', 'j=1,2,...,N' ;

annotation('textbox',dim,'String',str,'FitBoxToText','on');

```
xlabel('Time');
ylabel('Mean of variables');
legend('show');
end
function v = ddefun(t, y, Z, T, N)
T = Z(:, 1);
v = zeros(N, 1);
for k = 1:N
for j = 1:N
if k  = j
v(k) = v(k) - sin(T(k) - T(j));
end
end
end
end

function s = history(t, r, N, epsilon)
N = 100;
epsilon = 0.3;
s = zeros(N, 1);
for j = 1:(N/2)
s(j) = (j/N) * epsilon;
end
for k = (N/2)+1:N
s(k) = -(k/N) * epsilon;
end
end
```