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Mathematical Modeling and Analysis of Options with Jump-Diffusion Volatility

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Mathematical Modeling and Analysis of Options
with Jump-Diffusion Volatility

by

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A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
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DEDICATION

To my sister, who taught me how to learn and love math, and to my husband for all the love, support and encouragement he has given me while working on my dissertation.

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MATHEMATICAL MODELING AND ANALYSIS OF OPTIONS
WITH JUMP-DIFFUSION VOLATILITY

Irena Andreevska

ABSTRACT

Several existing pricing models of financial derivatives as well as the effects of volatility risk are analyzed. A new option pricing model is proposed which assumes that stock price follows a diffusion process with square-root stochastic volatility. The volatility itself is mean-reverting and driven by both diffusion and compound Poisson process. These assumptions better reflect the randomness and the jumps that are readily apparent when the historical volatility data of any risky asset is graphed. The European option price is modeled by a homogeneous linear second-order partial differential equation with variable coefficients. The case of underlying assets that pay continuous dividends is considered and implemented in the model, which gives the capability of extending the results to American options. An American option price model is derived and given by a non-homogeneous linear second order partial integro-differential equation. Using Fourier and Laplace transforms an exact closed-form solution for the price formula for European call/put options is obtained.

1 INTRODUCTION

The initial focus of the dissertation is to analyze the current pricing models of financial derivatives, including the effects of volatility risk and uncertainty. Most option pricing schemes formulated to date have been based on the classical Black-Scholes theory (1973). Black and Scholes have modeled the stock price with a stochastic differential equation driven by a geometric Brownian motion and have quantified the risk by a constant volatility parameter. The constant volatility assumption is frequently invalid in the world markets. There are a number of extensions of the original Black-Scholes pricing model that have been pursued in practice and in the literature which consider other forms of volatility: time dependent, time and state dependent, and stochastic - discrete or continuous. Popular continuous stochastic volatility models are offered by Hull and White (1987) who model the volatility as a square-root function that follows geometric Brownian motion, Scott (1987) and Stein and Stein (1991) use a mean-reverting OU process to describe the volatility function (the first one with exponential and the second with an absolute value function). All of these researchers assumed that the price of the underlying asset and its volatility are uncorrelated. Heston (1993) releases this assumption when offered a model that uses a square-root volatility function and a volatility parameter that follows the CIR process and allows a non-zero correlation between the stock and the volatility. All of the mentioned extensions assume continuous paths of the stock prices. In practice, asset prices occasionally jump. A typical example is the 1987 market crash. A daily move of 20% as in the S&P 500 is unlikely in the lognormal model (see Figure 1.1 a)). Even before the "Black Monday" (the day of the market crash), Merton in 1976 accounts for nonlognormal behavior by adding discrete jumps to the asset price and

keeping the volatility constant. However, after the market crash, there were more attempts among researchers to build models that allow large market movements, also known as returns with "fat tails", and at the same time keep the randomness of the volatility. Bates in 1996 offered a jump-diffusion option price model with a stock price that follows a jump-diffusion process and stochastic volatility driven by a Brownian motion. Jumps in returns can explain large movements to some extent, however the impact of these jumps is momentary; today's jump in returns has nothing to do with the future distribution of returns, and large movements were present both before and after October 19, 1987. Also, a negative drop of 65% in one day in the stock price of Company XYZ requires really high volatility that the pure diffusive volatility model cannot produce (see Figure 1.1 b)). Thus, the proposed model in this dissertation

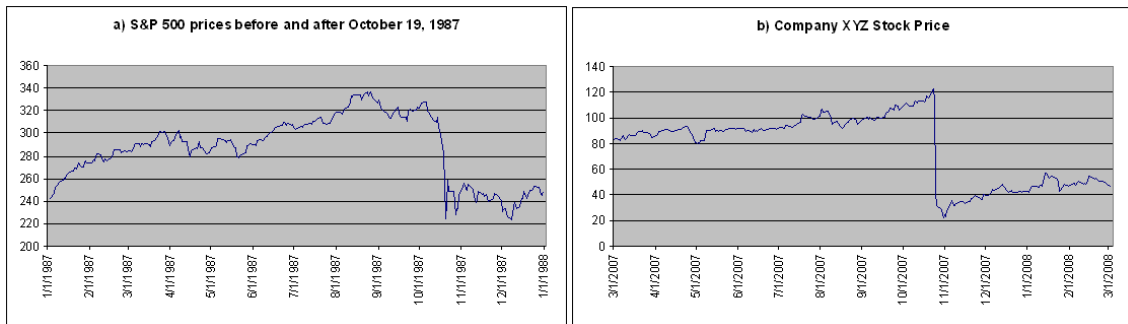


Figure 1.1: a) High return of 39.1% of S&P 500 during the first 10 months of 1987 followed by a drop of 20.4% on October 19, 1987 leaving the index price almost unchanged from its level at the beginning of the year
b) Drop of 82% in Company XYZ's stock price in just one week

accounts for jumps in volatility. It is based on a stock price stochastic differential equation driven by a Brownian motion and a volatility that follows stochastic differential equation driven by both Brownian motion and a compound Poisson process, in order to better reflect the randomness and the jumps that are readily apparent when the historical volatility data of any stock or risky asset is graphed or when looking at the behavior over time of implied volatilities. In the Black-Scholes model the market is complete so the derivatives can be perfectly hedged with the underlying asset, only. However, in stochastic volatility models there is more than one source of randomness and so perfect hedging is impossible. Thus, the use of a benchmark option to hedge

the intended option is necessary. Using this hedging technique in Chapter 3, pricing models for both European and American options are derived. The pricing models are given by a linear second-order partial integro-differential equation, the first one homogeneous and the second one with a nonhomogeneous term that accounts for the extra privileges that the American options offer. In Chapter 4 we derive an exact solution of the homogeneous PDE for the pure diffusion case or the so-called Heston's model by using Fourier and Laplace transforms. The solution obtained is identical to the one that Heston provides, to which he arrives by guessing its form. In Chapter 5 the homogeneous PDE in the jump-diffusion case is solved. The calculation of the Greeks and their application to few investment strategies are given in Chapter 6. Useful stochastic calculus definitions and theorems as well as a brief introduction to the Black-Scholes pricing model are given in Chapter 2.

2.1 Stochastic Calculus Definitions, Notions and Theorems

In this section we discuss several stochastic processes and their properties, widely used in the next chapter. We also define quadratic variation and covariation processes for semimartingales and their properties, as well as the multi-dimensional Itô formula.

Definition 2.1.1 *A real-valued stochastic process B_t is a **standard Brownian motion** if it satisfies the following properties:*

- (i) $B_0 = 0$;
- (ii) B_t is a continuous function of t almost surely;
- (iii) B_t has independent normally distributed increments:

$$B_t - B_s \sim N(0, t - s), \text{ for } s < t.$$

Definition 2.1.2 *A **compound Poisson process** C_t with rate λ and jump size distribution G is a continuous time stochastic process given by*

$$C_t = \sum_{k=1}^{N(t)} J_k, \quad t > 0$$

where $N(t)$ is a counting process with intensity λ , and $\{J_k, k = 1, 2, \dots\}$ are independent identically distributed random variables, with distribution G , which are also independent of $N(t)$.

In the rest of this section assume that X and Y are semimartingales such that $X(0-) = Y(0-) = 0$.

Definition 2.1.3 The *quadratic variation* process of X , denoted by $[X, X]$, is defined by

$$[X, X](t) = X(t)^2 - 2 \int_0^t X(s) dX(s). \quad (2.1.1)$$

Example 2.1.1 Using the definition above and the definition of the Itô integral it can be shown that $[t, t] = 0$ and $[B, B](t) = t$ where B is a Brownian motion.

Definition 2.1.4 The *path by path continuous part* $[X, X]^c$ of $[X, X]$ is defined by

$$[X, X](t) = [X, X]^c(t) + \sum_{0 \leq s \leq t} (\Delta X(s))^2. \quad (2.1.2)$$

Example 2.1.2 For the semimartingale $X = B + C$, where B is a Brownian motion and C is a compound Poisson process, the quadratic variation is given by $[X, X](t) = t + \sum_{k=1}^{N(t)} J_k^2$, since the continuous part is $[X, X]^c = [B, B] = t$ and the jump part is

$$\sum_{0 \leq s \leq t} (\Delta X(s))^2 = \sum_{0 \leq s \leq t} (\Delta C(s))^2 = \sum_{k=1}^{N(t)} J_k^2.$$

Definition 2.1.5 The *covariation* process of X and Y is defined by the following *polarization identity*

$$[X, Y] = \frac{1}{2} ([X + Y, X + Y] - [X, X] - [Y, Y]). \quad (2.1.3)$$

Theorem 2.1.6 If X is a quadratic pure jump semimartingale, that is $[X, X]^c = 0$ and Y is an arbitrary semimartingale, then

$$[X, Y](t) = X(0)Y(0) + \sum_{0 \leq s \leq t} \Delta X(s) \Delta Y(s).$$

Example 2.1.3 The theorem above implies that the covariation of a Brownian motion B and a compound Poisson process C is zero, $[B, C] = 0$, since B has no jump term $\Delta B(s) = 0$, for $0 < s \leq t$, and $B(0) = 0$.

The Itô formula extends the change of variable formula of the classical calculus to stochastic integrals with semimartingale integrators. We will use the one-dimensional Itô formula to develop the stock price model in the Black-Scholes setting, as well as the multi-dimensional Itô formula to develop the stock price model with jump-diffusive volatility.

Theorem 2.1.7 Multi-dimensional Itô formula. *If \mathbf{X} is a vector of d semimartingales and $g : \mathbb{R}^d \rightarrow \mathbb{R}$ has continuous second order partial derivatives, then*

- (i) $g(\mathbf{X})$ is a semimartingale, and
- (ii) the integral form of the Itô formula is

$$\begin{aligned} g(\mathbf{X}(t)) - g(\mathbf{X}(0)) &= \sum_{i=1}^d \int_{0+}^t \frac{\partial g}{\partial x_i}(\mathbf{X}(s-)) dX_i(s) \\ &+ \frac{1}{2} \sum_{i,j=1}^d \int_{0+}^t \frac{\partial^2 g}{\partial x_i \partial x_j}(\mathbf{X}(s-)) d[X_i, X_j]^c(s) \\ &+ \sum_{0 < s \leq t} \left[g(\mathbf{X}(s)) - g(\mathbf{X}(s-)) - \sum_{i=1}^d \frac{\partial g}{\partial x_i}(\mathbf{X}(s-)) \Delta X_i(s) \right]. \end{aligned}$$

In the next chapters we will deal with increments of Brownian motions and compound Poisson processes, so it is important to mention what is the meaning of a Stochastic Differential Equations (SDE) used to describe the evolution of a state vector \mathbf{X} .

The meaning of the **Stochastic Differential Equation**

$$d\mathbf{X}(t) = \mathbf{a}(\mathbf{X}(t), t) dt + \mathbf{b}(\mathbf{X}(t), t) d\mathbf{Y}(t), t \geq 0,$$

is given by the **Stochastic Integral Equation**

$$\mathbf{X}(t) = \mathbf{X}(0) + \int_0^t \mathbf{a}(\mathbf{X}(s), s) ds + \int_0^t \mathbf{b}(\mathbf{X}(s), s) d\mathbf{Y}(s), t \geq 0, \quad (2.1.4)$$

where \mathbf{a}, \mathbf{b} are $(d, 1), (d, d')$ -dimensional matrices whose entries are real valued Borel measurable functions, the state \mathbf{X} is an \mathbb{R}^d -valued stochastic process and the input

Y is a vector in $\mathbb{R}^{d'}$ consisting of d' real-valued semimartingales.

Example 2.1.4 The stochastic differential equation

$$dX(t) = cX(t)dt + \sigma X(t)dB(t), \quad t \in [0, \tau], \quad (2.1.5)$$

has a unique solution

$$X(t) = X(0)\exp\left[(c - \sigma^2/2)t + \sigma B(t)\right],$$

called **geometric Brownian Motion** process.

2.2 Black-Scholes Derivative Pricing Model

Suggested by Samuelson and used by Black and Scholes, the price of a risk-free asset (bond) β_t at time t can be described by an ordinary differential equation

$$d\beta_t = r\beta_t dt,$$

where r is the interest rate for lending or borrowing money. The price of a risky asset (stock) S_t at time t with constant rate of return μ , constant volatility σ and infinitesimal increments of Brownian motion dW_t , is modeled by a stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dW_t. \quad (2.2.6)$$

We can justify and financially interpret this SDE simply by dividing the equation above by the stock price S_t . Then the right hand side of the infinitesimal return dS_t/S_t has a return term μdt and a risky term σdW_t . Using Example 2.1.4 we obtain that the price of the risky asset S_t at time t is given by

$$S(t) = S(0)\exp\left[(\mu - \sigma^2/2)t + \sigma W(t)\right],$$

where S_0 is the initial stock price.

Derivatives or *contingent claims* are contracts related to an underlying asset. We are mainly interested in European and American options.

A *European Call Option* is a contract that gives the holder the right, but not the obligation, to buy one unit of an underlying asset for a predetermined *strike price* K on the *maturity* date T . The payoff function of the European call option is

$$h(S_T) = \max(S_T - K, 0) = (S_T - K)^+$$

where S_T is the underlying asset price at maturity T .

A *European Put Option* is a contract that gives the holder the right, but not the obligation, to sell one unit of an underlying asset for a predetermined *strike price* K on the *maturity* date T . The payoff function of the European put option is

$$h(S_T) = \max(K - S_T, 0) = (K - S_T)^+.$$

An *American Call (Put) Option* is a contract that gives the holder the right, but not the obligation, to buy (sell) one unit of an underlying asset for a predetermined *strike price* K at any time of one's choice before the option's expiration date T . The time τ at which the option is exercised is called the *exercise time*.

Black and Scholes have derived a partial differential equation that holds for the price of any derivative on a non-dividend paying stock. The derivation is based on the main economic principles: no-arbitrage and the creation of riskless portfolio. The principle of *no-arbitrage* says that in a perfectly liquid market (it is possible to buy and sell any finite quantity of the underlying asset at any time) there exist no opportunities to earn a risk-free profit (free lunch). Also, they have assumed that the trading is continuous in time, and there are no transaction costs or taxes.

Suppose there exists a pricing function $P(t, S_t)$ of a European option with maturity T and a payoff function $h(S_T)$ with enough regularity that we can apply the one-

dimensional Itô formula and obtain:

$$dP(t, S_t) = \left(\mu S_t \frac{\partial P}{\partial S} + \frac{\partial P}{\partial t} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 P}{\partial S^2} \right) dt + \sigma S_t \frac{\partial P}{\partial S} dW_t. \quad (2.2.7)$$

Construct a portfolio by holding one option and selling a_t units of the risky asset S_t . The value Π of this portfolio at time t is:

$$\Pi = P - aS.$$

The change of the value of the portfolio in a small time interval dt is given by:

$$d\Pi = dP - adS.$$

Note that we do not differentiate $a = a(t, S)$ because it is being fixed during this time interval. Substituting (2.2.6) and (2.2.7) in the equation above, we obtain:

$$dP - adS = \left(\mu S \frac{\partial P}{\partial S} + \frac{\partial P}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} - a\mu S \right) dt + \left(\sigma S \frac{\partial P}{\partial S} - a\sigma S \right) dW.$$

Choosing $a = \frac{\partial P}{\partial S}$ (called delta-hedge ratio) we eliminate the risky part that comes from the presence of the Brownian motion increment, as a hedging strategy. Since we have assumed that there is no arbitrage opportunity, the portfolio must grow at a risk-free rate, hence $d\Pi = r\Pi dt$. Now the equation above results in the Black-Scholes PDE

$$\frac{\partial P}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS \frac{\partial P}{\partial S} - rP = 0, \quad 0 \leq t \leq T, \quad 0 \leq S < \infty, \quad (2.2.8)$$

with a terminal condition $P(T, S) = h(S_T)$. The literature presents two approaches in solving the Black-Scholes PDE: the martingale approach and the approach of reduction to a heat equation. In this section we are solving this equation using a Fourier transform. First, set $S = e^x$ and $\tau = T - t$, then the PDE (2.2.8) becomes

$$\frac{\partial P}{\partial \tau} = \frac{1}{2} \sigma^2 \frac{\partial^2 P}{\partial x^2} + \left(r - \frac{1}{2} \sigma^2 \right) \frac{\partial P}{\partial x} - rP, \quad 0 \leq \tau \leq T, \quad -\infty < x < \infty, \quad (2.2.9)$$

with initial condition

$$P(0, x) = h(x) = \begin{cases} \max \{e^x - K, 0\} & \text{for call options} \\ \max \{K - e^x, 0\} & \text{for put options.} \end{cases}$$

Define the Fourier transform $\mathcal{F}[f(x)]$ to be

$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)e^{i\omega x} dx.$$

Then the following properties hold

$$\mathcal{F} \left[\frac{\partial f}{\partial t} \right] = \frac{\partial F}{\partial t}, \quad \mathcal{F} \left[\frac{\partial f}{\partial x} \right] = -i\omega F, \quad \mathcal{F} \left[\frac{\partial^2 f}{\partial x^2} \right] = -\omega^2 F.$$

Applying a Fourier transform with respect to x to equation (2.2.9) and using the properties above, the following linear PDE is obtained:

$$\frac{\partial \hat{P}}{\partial \tau} = \left(\frac{1}{2}i\omega\sigma^2 - \frac{1}{2}\omega^2\sigma^2 - i\omega r - r \right) \hat{P}$$

whose solution is given by

$$\hat{P}(\tau, \omega) = C_1 e^{\tau \left(\frac{1}{2}i\omega\sigma^2 - \frac{1}{2}\omega^2\sigma^2 - i\omega r - r \right)}. \quad (2.2.10)$$

The constant C_1 can be determined using the initial condition

$$C_1 = \mathcal{F}[h(x)].$$

Now applying the inverse Fourier Transform to (2.2.10) we get

$$P(\tau, x) = \mathcal{F}^{-1} \left[\mathcal{F}[h(x)] e^{\tau \left(\frac{1}{2}i\omega\sigma^2 - \frac{1}{2}\omega^2\sigma^2 - i\omega r - r \right)} \right].$$

Setting $G(\omega) = e^{\tau \left(\frac{1}{2}i\omega\sigma^2 - \frac{1}{2}\omega^2\sigma^2 - i\omega r - r \right)}$, the inverse Fourier transform $g(x) = \mathcal{F}^{-1}[G(\omega)]$

can be calculated using the property

$$\mathcal{F}^{-1} \left[e^{-\beta \omega^2} \right] = \sqrt{\frac{\pi}{\beta}} e^{-x^2/4\beta}$$

and completing the square in $G(\omega)$

$$G(\omega) = e^{-\frac{\tau \sigma^2}{2} \left[\omega + \frac{\frac{1}{2}i(2r-\sigma^2)}{\sigma^2} \right]^2} e^{-\tau \frac{(\sigma^2+2r)^2}{8\sigma^2}},$$

yielding

$$g(x) = \sqrt{\frac{2\pi}{\sigma^2 \tau}} e^{-\frac{\left(x - \frac{\sigma^2 \tau - 2r\tau}{2}\right)^2}{2\sigma^2 \tau} - r\tau}.$$

By the convolution theorem, for a European call option, we have

$$\begin{aligned} P(\tau, x) &= \mathcal{F}^{-1} [\mathcal{F} [h(x)] \mathcal{F} [g(x)]] \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} h(x-w) g(w) dw \\ &= \frac{1}{\sigma \sqrt{2\pi \tau}} \int_{-\infty}^{x-\ln K} (e^{x-w} - K) e^{-\frac{\left(w - \frac{\sigma^2 \tau - 2r\tau}{2}\right)^2}{2\sigma^2 \tau} - r\tau} dw \\ &= \frac{e^x}{\sigma \sqrt{2\pi \tau}} \int_{-\infty}^{x-\ln K} e^{-\frac{\left(w + \frac{\sigma^2 \tau + 2r\tau}{2}\right)^2}{2\sigma^2 \tau}} dw - \frac{e^{-r\tau}}{\sigma \sqrt{2\pi \tau}} \int_{-\infty}^{x-\ln K} K e^{-\frac{\left(w - \frac{\sigma^2 \tau - 2r\tau}{2}\right)^2}{2\sigma^2 \tau}} dw \end{aligned}$$

since $h(x-w) = 0$ when $w \leq x - \ln K$. Setting

$$\begin{aligned} d_1 &= \frac{1}{\sigma \sqrt{\tau}} \left[x - \ln K + \left(r + \frac{1}{2} \sigma^2 \right) \tau \right], \\ d_2 &= d_1 - \sigma \sqrt{\tau} \end{aligned}$$

and substituting $x = \ln S$ and $\tau = T - t$ we obtain a closed form solution of the Black-Scholes PDE that represents a price of a European call option

$$P_{call}(t, S) = S\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2), \quad (2.2.11)$$

where Φ is the cumulative standard normal distribution function. Using the *put-call*

parity (a relationship between put and call options with the same maturity T and the same strike price K)

$$P_{call}(t, S) - P_{put}(t, S) = S - Ke^{-r(T-t)}, \quad (2.2.12)$$

the European put option pricing formula can be obtained

$$P_{put}(t, S) = Ke^{-r(T-t)}\Phi(-d_2) - S\Phi(-d_1).$$

It is worth pointing out that the drift term μ doesn't appear in the B-S pricing formula, which means the value of the option doesn't depend on the investors' risk preferences. The reason for this is the perfect hedging strategy which allows complete elimination of the risk. This observation of risk-neutrality is a major breakthrough in the option pricing theory.

The Black-Scholes formula can also be derived using the risk-neutral pricing method, taking the price of the option to be the risk-neutral expected value discounted at the risk-free interest rate:

$$P(t, S) = e^{-rT}\mathbb{E}_{\mathbb{Q}}[h(S_T)]$$

where \mathbb{Q} is the so called *equivalent martingale measure*, a probability measure equivalent to the objective probability \mathbb{P} , under which (i) the discounted price $\tilde{S}_t = e^{-rt}S_t$ is a martingale, and (ii) the expected value of the discounted payoff of a derivative gives its no-arbitrage price. Next, we will find the risk-neutral measure \mathbb{Q} :

$$\begin{aligned} d\tilde{S}_t &= d(e^{-rt}S_t) = -re^{-rt}S_t dt + e^{-rt}dS_t \\ &= (\mu - r)\tilde{S}_t dt + \sigma\tilde{S}_t dW_t. \end{aligned}$$

For \tilde{S}_t to be a martingale we will absorb the drift term into the martingale term:

$$d\tilde{S}_t = \sigma\tilde{S}_t \left[dW_t + \frac{\mu - r}{\sigma} dt \right]$$

where $\theta = \frac{\mu-r}{\sigma}$ is called the *market price of asset risk*¹ or the *Sharpe ratio* (the ratio of the risk premium to volatility). Define

$$dW_t^* = dW_t + \theta t,$$

then $d\tilde{S}_t = \sigma\tilde{S}_t dW_t^*$ is a martingale. Using the Girsanov's theorem, a unique equivalent martingale measure \mathbb{Q} will be obtained:

$$\begin{aligned} \frac{d\mathbb{Q}}{d\mathbb{P}} &= \exp\left(-\frac{1}{2}\int_0^T \theta^2 dt - \int_0^T \theta dW_t\right) \\ &= \exp\left(-\frac{1}{2}\theta^2 T - \theta W_T\right). \end{aligned}$$

The stock price SDE under the risk-neutral measure is obtained by replacing the real world rate of return μ with the risk-free interest rate r :

$$dS_t = rS_t dt + \sigma S_t dS_t. \quad (2.2.13)$$

The rate of return under the risk-neutral measure should equal the real rate of return minus the total asset risk

$$\mu - \frac{\mu-r}{\sigma}\sigma = r.$$

2.3 Pricing American Options In the Black-Scholes Settings

In 1973, Merton relaxed one of the assumptions in the Black-Scholes model by considering an asset paying continuous dividends at rate q . The dividend payment reduces the growth of the stock price from S_t to $S_t e^{-q(T-t)}$ so that the pricing model of *options on dividend-paying stock* becomes

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + (r-q)S \frac{\partial P}{\partial S} - rP = 0, \quad 0 \leq t \leq T. \quad (2.3.14)$$

¹If the state variable X_t follows the process $dX_t = \mu^{\mathbb{P}}(X_t)dt + \sigma(X_t)dB_t^{\mathbb{P}}$ where $W_t^{\mathbb{P}}$ is a Brownian motion under the objective probability measure \mathbb{P} and there exists equivalent probability measure \mathbb{Q} such that X_t under \mathbb{Q} is given by $dX_t = \mu^{\mathbb{Q}}(X_t)dt + \sigma(X_t)dB_t^{\mathbb{Q}}$, then the market price of risk process is defined by $\Gamma(X_t) = [\sigma(X_t)]^{-1}[\mu^{\mathbb{P}}(X_t) - \mu^{\mathbb{Q}}(X_t)]$.

Using the appropriate boundary conditions, the price of a European call option on a dividend-paying asset can be obtained

$$P_{div.call}(t, S, K) = Se^{-q(T-t)}\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2)$$

The price of a European put option can be easily determined using the put-call parity for options on a dividend-paying stock

$$P_{call}(t, S, K) - P_{put}(t, S, K) = Se^{-q(T-t)} - Ke^{-r(T-t)}.$$

This model becomes extremely useful when extending the pricing results to American options. When the underlying asset pays no dividends, an early exercise of the American call option is always undesirable, and here is why: first, the privilege of an early exercise of the American options, in addition to all the rights that the European options have, makes the American options worth at least their European counterpart,

$$P^A(T, S, K) \geq P^E(T, S, K). \quad (2.3.15)$$

This extra cost is called an *early exercise premium*. Second, the Put-Call parity implies

$$P_{call}^E(T-t, S, K) = \underbrace{S_t - K}_{\text{exercise value}} + \underbrace{P_{put}^E(T-t, S, K)}_{\text{insurance against } S_T < K} + \underbrace{K(1 - e^{-r(T-t)})}_{\text{time value of money on K}} > S_t - K, \quad (2.3.16)$$

since both, the value of the put option and the time value on the strike K , are positive for all $t < T$. Combining inequalities (2.3.15) and (2.3.16) we obtain

$$P_{call}^A(T-t, S, K) > S_t - K,$$

which means that if we exercise the American call prior to time T we will receive $S_t - K$ which is less than $P_{call}^A(T-t, S, K)$, the amount that we would receive if we just sell the American call. However, early exercise of an American put option on

a non-dividend paying asset might be preferable. Once again, we demonstrate that using the put-call parity implication

$$P_{put}(T-t, S, K) = P_{call}(T-t, S, K) - S_t + Ke^{-r(T-t)}. \quad (2.3.17)$$

Arguing as before, the put will never be exercised as long as $P_{put}^A(T-t, S, K) > K - S_t$. This inequality and relationship (2.3.17) imply

$$P_{call}(T-t, S, K) > K(1 - e^{-r(T-t)}).$$

This means that when the time value of K exceeds the insurance value of the put (when a company is going bankrupt the call value becomes almost valueless) we cannot rule out early exercise.

We would like to exercise options early because we want to receive something sooner rather than later. When we exercise call options we receive stock so when the stock pays dividends it is normal that we would prefer early exercise. When exercising put options we receive an amount K , so when exercising puts early we can earn interest on this amount. If we look at the interest as a dividend on cash, we may say dividends are the only reason for early exercise. It has been shown that early

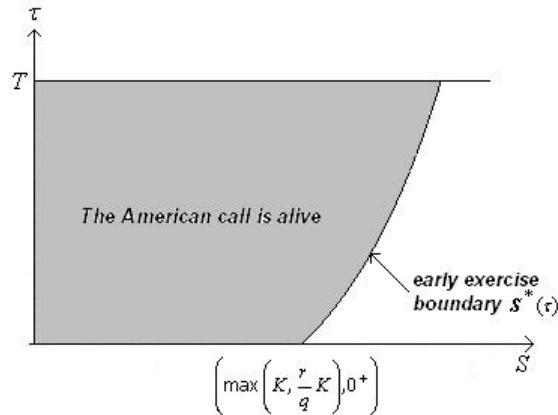


Figure 2.1: An American call on an asset paying continuous dividends is alive only within the domain $\{(S, \tau) : S \in [0, S_C^*(\tau)], \tau \in (0, T]\}$

exercise of options on a dividend paying asset is optimal only when the asset price S_τ ,

at a given time to expiry τ , rises above (for calls) or falls below (for puts) some critical asset value $S^*(\tau)$ called *optimal exercise price* (Figures 2.1 and 2.2). The collection of these critical asset values forms a curve known as *optimal exercise boundary*, which we will denote by $a(\tau)$. If $S^*(\tau)$ is a known function, the American option pricing problem becomes a boundary value problem with time dependent boundary.

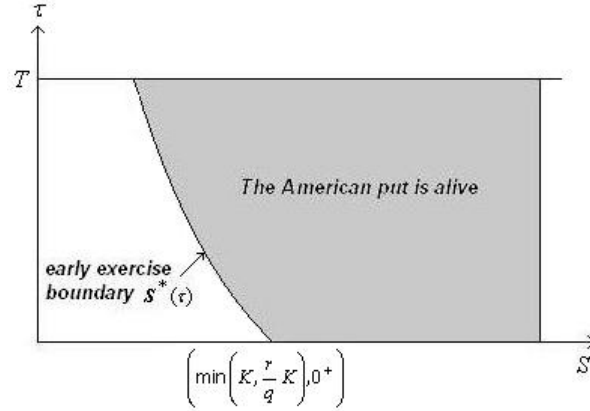


Figure 2.2: An American put on an asset paying continuous dividends is alive only within the domain $\{(S, \tau) : S \in (S_P^*(\tau), \infty), \tau \in (0, T]\}$

The put-call parity doesn't hold for American options, however a useful put-call symmetry relation for the prices of the American call and put options as well as a relation for their optimal exercise prices have been established. They are given with the expressions

$$P_A(\tau, S; K, r, q) = P_A(\tau, K; S, q, r), \quad (2.3.18)$$

$$S_C^*(\tau; r, q) = \frac{K^2}{S_P^*(\tau; q, r)}.$$

To derive pricing formulas for American options on a dividend-paying asset we will consider the effects of a continuous¹ dividend yield at a constant rate q . The price of an American option is modeled with the boundary value problem

$$\frac{\partial P}{\partial \tau} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} - (r - q)S \frac{\partial P}{\partial S} + rP = 0, \quad (2.3.19)$$

in the bounded region $0 \leq S \leq a(\tau)$ for calls and $S \geq a(\tau)$ for puts, and $0 \leq \tau \leq T$,

¹For discrete dividends check [14]

subject to the initial and boundary conditions

$$\begin{aligned}
P_{call}(0, S) &= \max(S - K), & P_{put}(0, S) &= \max(K - S) \\
P_{call}(\tau, 0) &= 0, & P_{put}(\tau, 0) &= 0 \\
P_{call}(\tau, a(\tau)) &= a(\tau) - K, & P_{put}(\tau, a(\tau)) &= K - a(\tau) \\
\lim_{S \rightarrow a(\tau)} \frac{\partial P_{call}}{\partial S} &= 1, & \lim_{S \rightarrow a(\tau)} \frac{\partial P_{put}}{\partial S} &= -1.
\end{aligned}$$

Whenever $S > a(\tau)$, the American call value is simply its intrinsic value $S - K$, and in the case of the American put, for $S < a(\tau)$, the put equals $K - S$. If we substitute $P_{call} = S - K$ and $P_{put} = K - S$ in the PDE above, the equation becomes $qS - rK$ for American call and $rK - qS$ for American put option. Then the American call/put pricing model is given by the following nonhomogeneous PDE

$$\frac{\partial P}{\partial \tau} - \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} - (r - q)S \frac{\partial P}{\partial S} + rP = g(S), \quad (2.3.20)$$

in the unbounded region $0 \leq S < \infty$ and $0 \leq \tau \leq T$, where

$$g(S) = \begin{cases} 0, & \text{if } S < a(\tau) \\ qS - rK, & \text{if } S \geq a(\tau) \end{cases}$$

for a call option, and

$$g(S) = \begin{cases} 0, & \text{if } S > a(\tau) \\ rK - qS, & \text{if } S \leq a(\tau) \end{cases}$$

for a put option.

The solution of the American option pricing PDE in Black-Scholes setting can be obtained by first substituting S by e^x , then taking Fourier transform of the PDE above with respect to the spatial variable and finally applying Duhamel's principle. The details will not be presented here, since the extension of this result in a jump-diffusion volatility setting is left for a future work.

2.4 Several Existing Extensions of the Black-Scholes Model

The Black-Scholes model is based on a geometric Brownian motion and it is very useful as a first approximation of the price change. The problems with this model are the assumptions that the volatility is kept constant, trading takes place continuously in time and that the stock price dynamics has a continuous sample path with probability one. There have been several attempts among the researchers to relax these assumptions, by defining alternative stochastic processes for the stock price and/or specifying deterministic or stochastic models for the stock price volatility. A well known discrepancy between the Black-Scholes option prices and the market traded option prices is the *smile/skew curve* obtained when the *implied volatility* I is graphed against the strike price K . The *implied volatility* I is a quantity used to compare certain model predictions and observations, and is defined to be the value of the volatility parameter which, when plugged in the BS formula, the observed market price and the BS option price coincide:

$$P_{BS}(t, S; K, T, I) = P^{obs}.$$

The smile effect shows that the implied volatilities of market prices are not constant

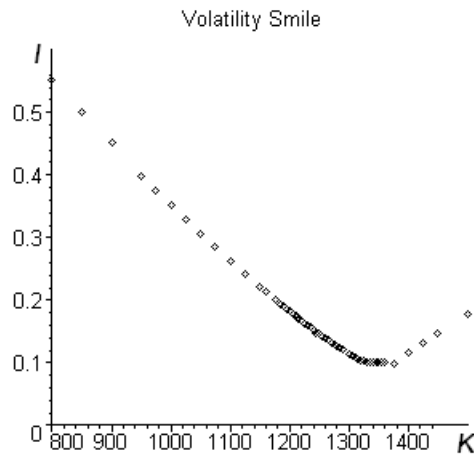


Figure 2.3: Volatility smile for S&P 500 call options. The S&P 500 index on June 21, 2006 is \$1252.20, with rate of return $r = 0.97\%$, and maturity date August 23, 2006

but depend on the strike price and the maturity time, and is illustrated in Figure 2.3 with data taken from S&P 500 index options. Another assumption made by the Black-Scholes model is that the stock returns are normally distributed. However, empirical studies are showing that the true distribution is more skewed than the normal distribution and it has fatter tails (see Figure 2.4).

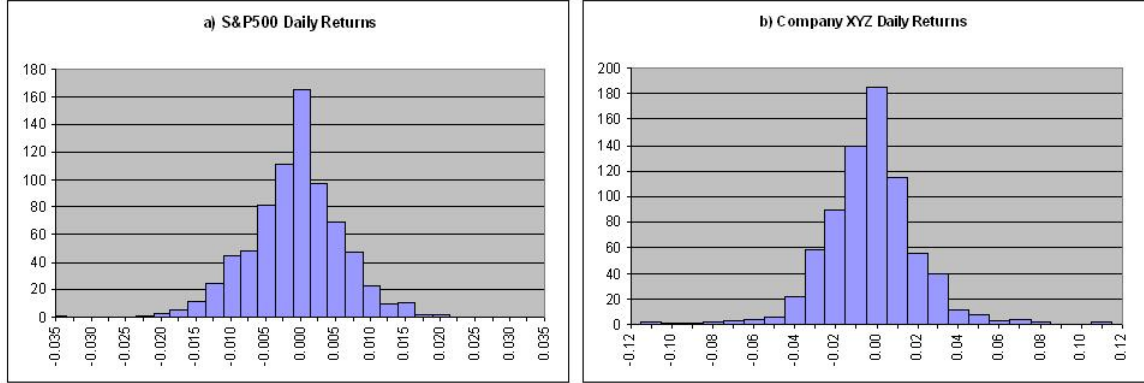


Figure 2.4: a) Daily returns for S&P 500 index prices between 07/01/04 and 06/30/07.
 b) Daily returns of Company XYZ stock prices between 07/01/04 and 06/30/07.

One attempt to modify the Black-Scholes model so it reflects the real market behavior is done by assuming that volatility is a positive deterministic function of time and stock price, where the SDE modeling the stock price is

$$dS_t = \mu S_t dt + \sigma(t, S_t) S_t dW_t.$$

Different choices of $\sigma(t, S_t)$ have provided several models in the literature. Worth mentioning is the *Constant Elasticity of Variance* model by Hull (2000) (initially suggested by Cox and Ross (1976)), in which $\sigma(t, S_t) = \sigma S_t^{-\alpha}$, $0 \leq \alpha \leq 1$. The model achieves the volatility smile effect, but the disadvantage is that the stock price and the volatility changes are perfectly correlated. Another special case is the *Time-Dependent Volatility* model, so that $\sigma(t, S_t) = \sigma(t)$. In this case the option price can be computed using the Black-Scholes formula with volatility parameter $\sqrt{\bar{\sigma}^2}$, where

$$\bar{\sigma}^2 = \frac{1}{T-t} \int_t^T \sigma^2(\tau) d\tau$$

is the time averaged volatility. Under this model, all options with the same maturity and fixed time t have the same volatility so the smile effect is not present. However, the implied volatility varies with time to maturity, since $\bar{\sigma}^2$ is different for different maturity times. In general, the problem with all deterministic volatility function models is the stability of the function over time: it may work with this week's data but the next week's data will suggest completely different volatility function.

The empirical studies of stock prices also reveal that the estimated volatility has "random" behavior. A fix for this as well as the fat-tailed returns is the stochastic volatility (SV) modeling. In the SV models the asset price is given by the SDE

$$dS_t = \mu S_t dt + \sigma_t S_t dW_t, \quad (2.4.21)$$

where σ_t is a positive function of a stochastic process Y_t . By letting Y_t be driven by a second Brownian motion Z_t we are achieving the "not perfect" correlation of the volatility and the stock price. However, this makes the market incomplete and thus, more complicated calculations and model derivation are required. Monitoring the behavior of historical and implied volatilities suggest that the volatility tends to go up for a certain period of time, then drops down for similar time span, then goes up again, and so on. In other words, the volatility reverts around its mean, which is modeled by

$$dY_t = \alpha(m - Y_t)dt + \dots dZ_t.$$

In the equation above α is called the rate of mean reversion, m is the long-run mean of Y , and Z_t is a Brownian motion such that $corr(W_t, Z_t) = \rho$. There are economic arguments for a negative correlation between volatility and asset prices. In most models it is taken to be a constant between -1 and 1 . There are several SV models studied in detail in the literature and used in practice. All models mentioned below use equation (2.4.21) for the asset price, but use different volatility functions driven by different stochastic processes:

1. *Hull-White* (1987). This is the first SV model in the literature, in which the

volatility function is $f(y) = \sqrt{y}$ and it is driven by a geometric Brownian motion

$$dY_t = c_1 Y_t dt + c_2 Y_t dZ_t.$$

Assuming uncorrelated Brownian motions, Hull and White have derived a closed-form solution for this model.

2. *Scott*(1987) and *Stein-Stein* (1991). Both models assume a volatility driven by a mean-reverting Ornstein-Uhlenbeck process

$$dY_t = \alpha(m - Y_t)dt + \beta dZ_t,$$

but different volatility functions; Scott has used $f(y) = e^y$ and Stein and Stein have used $f(y) = \sqrt{y}$. In both cases a closed-form solution has been derived, assuming uncorrelated Brownian motions, $\text{corr}(W_t, Z_t) = 0$.

3. *Heston*(1993) and *Ball-Roma*(1994). The Cox-Ingersoll-Ross (CIR) process is used as a volatility driving process

$$dY_t = \alpha(m - Y_t)dt + \beta\sqrt{Y_t}dZ_t,$$

and $f(y) = \sqrt{y}$ as a volatility function. This formulation has the advantage of strictly positive volatility as long as $\alpha m - \frac{\beta^2}{2} \geq 0$ (Buff [7]). Ball and Roma have considered the case of uncorrelated Brownian motions. The Heston's model is the most popular in the literature and in practice, since it assumes a correlation of the Brownian motions, $\text{corr}(W_t, Z_t) = \rho$ where $\rho \in [-1, 1]$ and gives a closed-form solution by guessing the form of the pricing formula and the form of the characteristic functions of the risk-neutral probabilities used in the pricing formula. Models that offer closed form solutions are more attractive for the market makers since they need less computation time. Although Heston's pricing formula in its closed form leaves an infinite integral to be solved by a numerical method, it is still much faster than other SV models.

The stochastic volatility models correct the constant volatility assumption in the Black-Scholes model and generate leptokurtic return distribution, but they don't account for possible jumps in asset price and volatility. Large price changes are present in the financial markets. They can be caused by speculations, companies earnings call, expectation of a new product, or the recent bad judgment of real estate lenders. These large price movements cannot be generated by pure diffusion processes. Bates [18] finds that pure diffusion models will need implausibly high volatility levels to explain them. Eraker [20] concludes that adding jumps in the stock price SDE can explain some but not all of the market movements present before and after the 1987 market crash. In his joint work with Johannes and Polson (2000) he investigates the performance of models with jumps in prices and volatility pointing out the significance of jumps in volatility.

All of the above initiated the necessity of incorporating jumps in the pricing model and deriving a closed form pricing formula. The derivation process and the results are presented in the next three chapters.

3 MODELING OPTIONS WRITTEN ON STOCKS WITH JUMP-DIFFUSION VOLATILITY

3.1 Formulation of the European Options Pricing Model

The market "jump" phenomena are often best modeled as volatility jump processes [2]. Considering this fact as well as the random characteristic of the volatility, it seems natural to propose a pricing model with asset price driven by a geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma_t S_t dW_t \quad (3.1.1)$$

where the volatility is a positive real function $\sigma_t = f(Y_t)$ driven by a mean-reverting jump-diffusion process Y_t :

$$dY_t = \alpha(m - Y_t)dt + \zeta \sqrt{Y_t} dZ_t + dC_t. \quad (3.1.2)$$

The processes W_t and Z_t are correlated Brownian motions, $\text{corr}(W_t, Z_t) = \rho$, α is the rate of mean-reversion, m is the long-term volatility mean, ζ is the volatility of the volatility process. The process $C_t = \sum_{k=1}^{N(t)} J_k$, $t > 0$, is a compound Poisson process with intensity λ , and J_k , $k = 1, 2, \dots$ are independent identically distributed random variables with distribution $\phi(J_k)$. The sum of dN_t jumps is the compound Poisson process dC_t that usually has symbolic notation

$$JdN_t = \sum_{k=1}^{dN_t} J_k.$$

The infinitesimal moments of the jump process are $E[JdN_t] = \lambda E[J]dt$ and $Var[JdN_t] = \lambda E[J^2]dt$. Also, we assume that the diffusion processes are uncorrelated to the jump process, that is $corr(W_t, C_t) = 0$ and $corr(Z_t, C_t) = 0$.

The price of the option P is a function of t , S_t and Y_t . Assuming that $P(t, S_t, Y_t)$ has continuous second order partial derivatives and using the differential form of the multi-dimensional Itô formula given in Theorem 2.1.7, we have:

$$\begin{aligned} dP(t, S_t, Y_t) &= \frac{\partial P}{\partial t}dt + \frac{\partial P}{\partial S}dS_t + \frac{\partial P}{\partial y}dY_t + \frac{1}{2}\frac{\partial^2 P}{\partial S^2}d[S, S]_t^c + \frac{\partial^2 P}{\partial S \partial y}d[S, Y]_t^c \\ &+ \frac{1}{2}\frac{\partial^2 P}{\partial y^2}d[Y, Y]_t^c + \left[P(Y + J) - P(Y) - \frac{\partial P}{\partial y}\Delta Y \right] dN_t, \end{aligned} \quad (3.1.3)$$

where

$$dN_t = \begin{cases} 0 & \text{with probability } p = 1 - \lambda dt, \text{ (jump doesn't occur)} \\ 1 & \text{with probability } p = \lambda dt, \text{ (jump occurs).} \end{cases}$$

Proposition 3.1.1 *The continuous part of the quadratic variation of S , where S is given by SDE (3.1.1) is*

$$[S, S]^c = S^2(0) + \int_0^t S_\tau^2 f^2(Y_\tau) d\tau. \quad (3.1.4)$$

Proof. First, note that S_t has no jump term, hence $[S, S]_t^c = [S, S]_t$. Next, the stochastic integral equation that corresponds to SDE (3.1.1) is given by

$$S(t) = \underbrace{S(0)}_D + \underbrace{\int_0^t \mu S_\tau d\tau}_{A_t} + \underbrace{\int_0^t \sigma_\tau S_\tau dW_\tau}_{M_t}, \quad t \geq 0$$

By the linearity of quadratic variation, we have

$$[S, S] = [D, D] + [D, A] + [D, M] + [A, D] + [A, A] + [A, M] + [M, D] + [M, A] + [M, M].$$

All covariations with arguments D and A or D and M are zero because $D = S(0)$ is a constant and $A(0) = 0 = M(0)$. The quadratic variation $[A, A] = 0$ because $d[t, t] = 0$

and $[A, M] = [M, A] = 0$ since $d[t, W] = d[W, t] = 0$ (see Example 2.1.1). Thus

$$\begin{aligned} [S, S]^c &= [D, D] + [M, M], \text{ equivalent to} \\ [S, S]^c &= S^2(0) + \int_0^t S_\tau^2 f^2(Y_\tau) \underbrace{d[W, W]_\tau}_{d\tau} \\ &= S^2(0) + \int_0^t S_\tau^2 f^2(Y_\tau) d\tau. \end{aligned}$$

■

Proposition 3.1.2 *The continuous part of the quadratic variation of Y , where Y is given by the SDE (3.1.2) is*

$$[Y, Y]^c = Y^2(0) + \zeta^2 \int_0^t Y_\tau d\tau. \quad (3.1.5)$$

Proof. The Stochastic Integral Equation that corresponds to SDE (3.1.2) is given by

$$Y(t) = \underbrace{Y(0)}_D + \underbrace{\int_0^t \alpha(m - Y_\tau) d\tau}_{A_t} + \zeta \underbrace{\int_0^t \sqrt{Y_\tau} dZ_\tau}_{M_t} + \underbrace{\int_0^t dC_\tau}_{F_t}. \quad (3.1.6)$$

Since D is a constant and $A(0) = M(0) = F(0) = 0$, the covariations $[D, A]$, $[D, M]$, $[D, F]$, $[A, D]$, $[M, D]$, $[F, D]$ equal 0. The result obtained in Example 2.1.1 implies $[A, A] = 0$ ($d[t, t] = 0$) and $[A, M] = [M, A] = 0$ ($d[t, Z] = d[Z, t] = 0$). In Example 2.1.3 we have shown that the covariation of a Brownian motion and a compound Poisson process is zero, thus $[M, F] = [F, M] = 0$. From Theorem 2.1.6 it follows that $[A, F] = [F, A] = 0$. Finally, using the results above as well as the linearity of quadratic covariation we obtain

$$[Y, Y] = [D, D] + [M, M] + [F, F].$$

However, the process F_t is a pure jump so $[F, F]^c = 0$. Therefore

$$[Y, Y]^c = Y^2(0) + \zeta^2 \int_0^t Y_\tau d[Z, Z]_\tau = Y^2(0) + \zeta^2 \int_0^t Y_\tau d\tau.$$

■

Proposition 3.1.3 *The continuous part of the covariation of S and Y , where S and Y are given by (3.1.1) and (3.1.2), respectively, is*

$$[S, Y]^c = S(0)Y(0) + \int_0^t \rho \zeta S_\tau \sqrt{Y_\tau} f(Y_t) d\tau. \quad (3.1.7)$$

Proof. The SIE of the sum of S and Y is given by

$$\begin{aligned} (S + Y)(t) &= \underbrace{S(0) + Y(0)}_D + \underbrace{\int_0^t [\mu S_\tau + \alpha(m - Y_\tau)] d\tau}_{A_t} + \underbrace{\int_0^t \sigma_\tau S_\tau dW_\tau}_{M_t^1} \\ &\quad + \zeta \underbrace{\int_0^t \sqrt{Y_\tau} dZ_\tau}_{M_t^2} + \underbrace{\int_0^t dC_\tau}_{F_t}. \end{aligned}$$

Using the same arguments as in the previous two propositions, we have

$$[D, A] = [D, M^{1,2}] = [D, F] = [A, D] = [M^{1,2}, D] = [F, D] = [A, A] = [A, M^{1,2}] = [M^{1,2}, A] = [M^{1,2}, F] = [F, M^{1,2}] = [A, F] = [F, A] = 0, \text{ giving}$$

$$\begin{aligned} [S + Y, S + Y] &= [D, D] + [M^1, M^1] + [M^2, M^2] + 2[M^1, M^2] + [F, F] \\ &= (S(0) + Y(0))^2 + \int_0^t S_\tau^2 f^2(Y_\tau) \underbrace{d[W, W]_\tau}_{d\tau} + \zeta^2 \int_0^t Y_\tau \underbrace{d[Z, Z]_\tau}_{d\tau} \\ &\quad + 2\zeta \int_0^t S_\tau f(Y_\tau) \sqrt{Y_\tau} \underbrace{d[W, Z]_\tau}_{\rho d\tau} + \int_0^t d[C, C]_\tau, \end{aligned}$$

Since F_t is a pure jump process, the continuous part of its quadratic variation is 0. Hence

$$\begin{aligned} [S + Y, S + Y]^c &= (S(0) + Y(0))^2 + \int_0^t S_\tau^2 f^2(Y_\tau) d\tau \\ &\quad + 2\zeta \int_0^t \rho S_\tau f(Y_\tau) \sqrt{Y_\tau} d\tau + \zeta^2 \int_0^t Y_\tau d\tau. \end{aligned}$$

This result, the results of the previous two propositions and the polarization identity, imply

$$[S, Y]^c = S(0)Y(0) + \int_0^t \rho\zeta S_\tau f(Y_\tau) \sqrt{Y_\tau} d\tau.$$

■

Differentiating equations (3.1.4), (3.1.5) and (3.1.7) we obtain the following expressions for $d[S, S]_t^c$, $d[Y, Y]_t^c$ and $d[S, Y]_t^c$:

$$d[S, S]_t^c = S^2 f^2(y) dt, \quad d[Y, Y]_t^c = y dt \quad \text{and} \quad d[S, Y]_t^c = \rho\zeta S \sqrt{y} f(y) dt.$$

Equation (3.1.3) then becomes

$$\begin{aligned} dP(t, S_t, Y_t) &= \frac{\partial P}{\partial t} dt + \frac{\partial P}{\partial S} dS_t + \frac{\partial P}{\partial y} dY_t + \frac{1}{2} S^2 f^2(y) \frac{\partial^2 P}{\partial S^2} dt + \rho\zeta S \sqrt{y} f(y) \frac{\partial^2 P}{\partial S \partial y} dt \\ &\quad + \frac{1}{2} \zeta^2 y \frac{\partial^2 P}{\partial y^2} dt + \left[P(y + J) - P(y) - J \frac{\partial P}{\partial y} \right] dN_t. \end{aligned} \quad (3.1.8)$$

In the Black-Scholes case it is sufficient to hedge with the underlying asset, only, because there is a single source of randomness. However, in the jump-diffusion volatility case we try to hedge with the underlying asset and another (benchmark) option written on the same underlying asset just with either later expiration date or different strike price. Hence, we create a portfolio Π with one call option P , a_t units of stock and b_t units of the benchmark option Q with same payoff function as P , and consider its value

$$\Pi = P - a_t S_t - b_t Q.$$

Assuming the portfolio is self-financing, the change in the portfolio value in a small time interval dt is

$$d\Pi = dP - a_t dS_t - b_t dQ. \quad (3.1.9)$$

Using equation (3.1.8) once for dP and then for dQ we obtain

$$\begin{aligned} d\Pi &= \frac{\partial P}{\partial t} dt + \frac{\partial P}{\partial S} dS_t + \frac{\partial P}{\partial y} dY_t + \frac{1}{2} S^2 f^2(y) \frac{\partial^2 P}{\partial S^2} dt + \rho \zeta S \sqrt{y} f(y) \frac{\partial^2 P}{\partial S \partial y} dt \\ &+ \frac{1}{2} \zeta^2 y \frac{\partial^2 P}{\partial y^2} dt + \left[P(y+J) - P(y) - J \frac{\partial P}{\partial y} \right] dN_t - a_t dS_t - b_t \frac{\partial Q}{\partial t} dt \\ &- b_t \frac{\partial Q}{\partial S} dS_t - b_t \frac{\partial Q}{\partial y} dY_t - \frac{1}{2} b_t S^2 f^2(y) \frac{\partial^2 Q}{\partial S^2} dt - b_t \rho \zeta S \sqrt{y} f(y) \frac{\partial^2 Q}{\partial S \partial y} dt \\ &- \frac{1}{2} b_t \zeta^2 y \frac{\partial^2 Q}{\partial y^2} dt - b_t \left[Q(y+J) - Q(y) - J \frac{\partial Q}{\partial y} \right] dN_t. \end{aligned}$$

Substituting for dS_t and dY_t and grouping the terms we obtain:

$$\begin{aligned} d\Pi &= \left[\frac{\partial P}{\partial t} + \mu S \frac{\partial P}{\partial S} + \alpha(m-y) \frac{\partial P}{\partial y} + \frac{1}{2} S^2 f^2(y) \frac{\partial^2 P}{\partial S^2} + \rho \zeta S \sqrt{y} f(y) \frac{\partial^2 P}{\partial S \partial y} \right. \\ &+ \frac{1}{2} \zeta^2 y \frac{\partial^2 P}{\partial y^2} - a_t \mu S - b_t \frac{\partial Q}{\partial t} - b_t \mu S \frac{\partial Q}{\partial S} - b_t \alpha(m-y) \frac{\partial Q}{\partial y} - \frac{1}{2} b_t S^2 f^2(y) \frac{\partial^2 Q}{\partial S^2} \\ &- \left. b_t \rho \zeta S \sqrt{y} f(y) \frac{\partial^2 Q}{\partial S \partial y} - \frac{1}{2} b_t \zeta^2 y \frac{\partial^2 Q}{\partial y^2} \right] dt + S f(y) \left[\frac{\partial P}{\partial S} - a_t - b_t \frac{\partial Q}{\partial S} \right] dW_t \\ &+ \zeta \sqrt{y} \left[\frac{\partial P}{\partial y} - b_t \frac{\partial Q}{\partial y} \right] dZ_t + \left[\frac{\partial P}{\partial y} - b_t \frac{\partial Q}{\partial y} \right] dC_t \\ &+ \left[P(y+J) - P(y) - J \frac{\partial P}{\partial y} - b_t Q(y+J) - b_t Q(y) + b_t J \frac{\partial Q}{\partial y} \right] dN_t. \quad (3.1.10) \end{aligned}$$

To eliminate the risk that comes from the diffusion terms we choose suitable values for a_t and b_t

$$b_t = \frac{\partial P}{\partial y} \left(\frac{\partial Q}{\partial y} \right)^{-1} \quad \text{and} \quad a_t = \frac{\partial P}{\partial S} - \frac{\partial Q}{\partial S} \frac{\partial P}{\partial y} \left(\frac{\partial Q}{\partial y} \right)^{-1}. \quad (3.1.11)$$

By the principle of no arbitrage $d\Pi = r\Pi dt = r(P - a_t S_t - b_t Q) dt$. Substituting the right hand side of this equation in (3.1.10) and the expressions for a_t and b_t given by (3.1.11), we obtain

$$\begin{aligned}
& r \left[P - S \frac{\partial P}{\partial S} - S \frac{\partial Q}{\partial S} \frac{\partial P}{\partial y} \left(\frac{\partial Q}{\partial y} \right)^{-1} - \frac{\partial P}{\partial y} \left(\frac{\partial Q}{\partial y} \right)^{-1} Q \right] dt \\
&= \left[\frac{\partial P}{\partial t} + \mu S \frac{\partial P}{\partial S} + \alpha(m-y) \frac{\partial P}{\partial y} + \frac{1}{2} S^2 f^2(y) \frac{\partial^2 P}{\partial S^2} + \rho \zeta S \sqrt{y} f(y) \frac{\partial^2 P}{\partial S \partial y} + \frac{1}{2} \zeta^2 y \frac{\partial^2 P}{\partial y^2} \right. \\
&\quad - \mu S \left(\frac{\partial P}{\partial S} - \frac{\partial Q}{\partial S} \frac{\partial P}{\partial y} \left(\frac{\partial Q}{\partial y} \right)^{-1} \right) - \frac{\partial Q}{\partial t} \frac{\partial P}{\partial y} \left(\frac{\partial Q}{\partial y} \right)^{-1} - \mu S \frac{\partial Q}{\partial S} \frac{\partial P}{\partial y} \left(\frac{\partial Q}{\partial y} \right)^{-1} \\
&\quad - \alpha(m-y) \frac{\partial Q}{\partial y} \frac{\partial P}{\partial y} \left(\frac{\partial Q}{\partial y} \right)^{-1} - \frac{1}{2} S^2 f^2(y) \frac{\partial^2 Q}{\partial S^2} \frac{\partial P}{\partial y} \left(\frac{\partial Q}{\partial y} \right)^{-1} \\
&\quad \left. - \rho \zeta S \sqrt{y} f(y) \frac{\partial^2 Q}{\partial S \partial y} \frac{\partial P}{\partial y} \left(\frac{\partial Q}{\partial y} \right)^{-1} - \frac{1}{2} \zeta^2 y \frac{\partial^2 Q}{\partial y^2} \frac{\partial P}{\partial y} \left(\frac{\partial Q}{\partial y} \right)^{-1} \right] dt \\
&+ \left[P(y+J) - P(y) - J \frac{\partial P}{\partial y} - \left(Q(y+J) - Q(y) - J \frac{\partial Q}{\partial y} \right) \frac{\partial P}{\partial y} \left(\frac{\partial Q}{\partial y} \right)^{-1} \right] dN_t.
\end{aligned}$$

After canceling out some of the terms, the equation above becomes

$$\begin{aligned}
& r \left[P - S \frac{\partial P}{\partial S} - S \frac{\partial Q}{\partial S} \frac{\partial P}{\partial y} \left(\frac{\partial Q}{\partial y} \right)^{-1} - \frac{\partial P}{\partial y} \left(\frac{\partial Q}{\partial y} \right)^{-1} Q \right] dt \\
&= \left[\frac{\partial P}{\partial t} + \frac{1}{2} S^2 f^2(y) \frac{\partial^2 P}{\partial S^2} + \rho \zeta S \sqrt{y} f(y) \frac{\partial^2 P}{\partial S \partial y} + \frac{1}{2} \zeta^2 y \frac{\partial^2 P}{\partial y^2} - \frac{\partial Q}{\partial t} \frac{\partial P}{\partial y} \left(\frac{\partial Q}{\partial y} \right)^{-1} \right. \\
&\quad - \frac{1}{2} S^2 f^2(y) \frac{\partial^2 Q}{\partial S^2} \frac{\partial P}{\partial y} \left(\frac{\partial Q}{\partial y} \right)^{-1} - \rho \zeta S \sqrt{y} f(y) \frac{\partial^2 Q}{\partial S \partial y} \frac{\partial P}{\partial y} \left(\frac{\partial Q}{\partial y} \right)^{-1} \\
&\quad \left. - \frac{1}{2} \zeta^2 y \frac{\partial^2 Q}{\partial y^2} \frac{\partial P}{\partial y} \left(\frac{\partial Q}{\partial y} \right)^{-1} \right] dt \\
&+ \left[P(y+J) - P(y) - (Q(y+J) - Q(y)) \frac{\partial P}{\partial y} \left(\frac{\partial Q}{\partial y} \right)^{-1} \right] dN_t.
\end{aligned}$$

Multiplying the last equation by $\left(\frac{\partial P}{\partial y} \right)^{-1}$ and moving all terms containing P on the left and all terms containing Q on the right hand side, we obtain

$$\begin{aligned}
& \left[\frac{\partial P}{\partial t} + rS \frac{\partial P}{\partial S} + \frac{1}{2} S^2 f^2(y) \frac{\partial^2 P}{\partial S^2} + \rho \zeta S \sqrt{y} f(y) \frac{\partial^2 P}{\partial S \partial y} + \frac{1}{2} \zeta^2 y \frac{\partial^2 P}{\partial y^2} - rP \right] \left(\frac{\partial P}{\partial y} \right)^{-1} dt \\
& + [P(y+J) - P(y)] \left(\frac{\partial P}{\partial y} \right)^{-1} dN_t \\
= & \left[\frac{\partial Q}{\partial t} + rS \frac{\partial Q}{\partial S} + \frac{1}{2} S^2 f^2(y) \frac{\partial^2 Q}{\partial S^2} + \rho \zeta \sqrt{y} S f(y) \frac{\partial^2 Q}{\partial S \partial y} + \frac{1}{2} \zeta^2 y \frac{\partial^2 Q}{\partial y^2} - rQ \right] \left(\frac{\partial Q}{\partial y} \right)^{-1} dt \\
& + [Q(y+J) - Q(y)] \left(\frac{\partial Q}{\partial y} \right)^{-1} dN_t.
\end{aligned}$$

Taking expectation over the probability distribution of jumps we obtain

$$\begin{aligned}
& \left[\frac{\partial P}{\partial t} + rS \frac{\partial P}{\partial S} + \frac{1}{2} S^2 f^2(y) \frac{\partial^2 P}{\partial S^2} + \rho \zeta S \sqrt{y} f(y) \frac{\partial^2 P}{\partial S \partial y} + \frac{1}{2} \zeta^2 y \frac{\partial^2 P}{\partial y^2} - rP \right] \left(\frac{\partial P}{\partial y} \right)^{-1} dt \\
& + \lambda E [P(t, S, y+J) - P(t, S, y)] \left(\frac{\partial P}{\partial y} \right)^{-1} dt \\
= & \left[\frac{\partial Q}{\partial t} + rS \frac{\partial Q}{\partial S} + \frac{1}{2} S^2 f^2(y) \frac{\partial^2 Q}{\partial S^2} + \rho \zeta \sqrt{y} S f(y) \frac{\partial^2 Q}{\partial S \partial y} + \frac{1}{2} \zeta^2 y \frac{\partial^2 Q}{\partial y^2} - rQ \right] \left(\frac{\partial Q}{\partial y} \right)^{-1} dt \\
& + \lambda E [Q(t, S, y+J) - Q(t, S, y)] \left(\frac{\partial Q}{\partial y} \right)^{-1} dt, \tag{3.1.12}
\end{aligned}$$

where

$$E [P(t, S, y+J) - P(t, S, y)] = \int_0^\infty [P(t, S, y+J) - P(t, S, y)] \phi(J) dJ. \tag{3.1.13}$$

The left and right hand sides of equation (3.1.12) are identical, except that the first one is a function of P , only, and the other one is a function of Q , only. Then, there must be a function $k(t, S, y)$ such that

$$\begin{aligned}
& \frac{\partial P}{\partial t} + rS \frac{\partial P}{\partial S} + \frac{1}{2} S^2 f^2(y) \frac{\partial^2 P}{\partial S^2} + \rho \zeta S \sqrt{y} f(y) \frac{\partial^2 P}{\partial S \partial y} + \frac{1}{2} \zeta^2 y \frac{\partial^2 P}{\partial y^2} - rP \\
& + \lambda E [P(t, S, y+J) - P(t, S, y)] = -k(t, S, y) \frac{\partial P}{\partial y}.
\end{aligned}$$

The term in front of $\frac{\partial P}{\partial S}$ is the drift term of the stock price SDE under the risk-neutral probability measure. So the term in front of $\frac{\partial P}{\partial y}$ should be the drift term of the SDE

that corresponds to the volatility function in the risk-neutral world. This implies that the function $k(t, S, y)$ is a difference of the real world drift term and the total market price of volatility risk $\Gamma(y)$:

$$k(t, S, y) = \alpha(m - y) - \Gamma(y)\sqrt{y}.$$

Cheridito, Filipovic and Kimmel (for details see [5]) have modeled the market price of risk associated with the volatility driven by a process given by equation (3.1.2) as $\Gamma(y) = \gamma\sqrt{y}$ for some constant γ (with $\gamma = 0$ possible) and in theory is determined by the benchmark option $Q(t, S_t, Y_t)$. All of the above proves the following theorem:

Theorem 3.1.4 *The price of a European call option written on a stock driven by SDE (3.1.1) and volatility that follows SDE (3.1.2), with strike price K and maturity T is modeled with the following terminal-boundary value problem*

$$\begin{aligned} \frac{\partial P}{\partial t} + rS \frac{\partial P}{\partial S} + [\alpha(m - y) - \gamma y] \frac{\partial P}{\partial y} + \frac{1}{2} S^2 f^2(y) \frac{\partial^2 P}{\partial S^2} + \rho \zeta S \sqrt{y} f(y) \frac{\partial^2 P}{\partial S \partial y} + \frac{1}{2} \zeta^2 y \frac{\partial^2 P}{\partial y^2} \\ - rP + \lambda E [P(t, S, y + J) - P(t, S, y)] = 0, \end{aligned} \quad (3.1.14)$$

$$P(T, S, y) = \max(S - K, 0),$$

$$\lim_{S \rightarrow \infty} [P(t, S, y) - S] = 0,$$

$$P(t, 0, y) = 0,$$

$$P(t, S, 0) = 0, \quad (3.1.15)$$

where $0 \leq S < \infty, 0 \leq y < \infty, 0 \leq t \leq T$, and

$$E [P(t, S, y + J) - P(t, S, y)] = \int_{-\infty}^{\infty} [P(t, S, y + J) - P(t, S, y)] \phi(J) dJ$$

is the expected value of the change in the option price with respect to the jump probability distribution function.

When solving this boundary value problem, we will find it convenient to express it in the following form:

Theorem 3.1.5 *The terminal-boundary value problem in Theorem 3.1.4 for European call option is equivalent to the following initial-boundary value problem:*

$$\begin{aligned} \frac{\partial P}{\partial \tau} = & \left(r - \frac{1}{2}f^2(y) \right) \frac{\partial P}{\partial x} + [\alpha(m - y) - \gamma y] \frac{\partial P}{\partial y} + \frac{1}{2}f^2(y) \frac{\partial^2 P}{\partial x^2} + \rho\zeta\sqrt{y}f(y) \frac{\partial^2 P}{\partial x \partial y} \\ & + \frac{1}{2}\zeta^2 y \frac{\partial^2 P}{\partial y^2} - rP + \lambda E [P(\tau, x, y + J) - P(\tau, x, y)], \end{aligned} \quad (3.1.16)$$

$$P(0, x, y) = \max(e^x - K, 0), \quad (3.1.17)$$

$$\lim_{x \rightarrow -\infty} P(\tau, x, y) = 0,$$

$$\lim_{x \rightarrow \infty} [P(\tau, x, y) - e^x] = 0,$$

$$P(\tau, x, 0) = 0, \quad (3.1.18)$$

where $-\infty < x < \infty, 0 \leq y < \infty, 0 \leq \tau \leq T$, and

$$E [P(\tau, x, y + J) - P(\tau, x, y)] = \int_{-\infty}^{\infty} [P(\tau, x, y + J) - P(\tau, x, y)] \phi(J) dJ.$$

Proof. Define $S = e^x$ and $t = T - \tau$. Then for the partial derivatives in PIDE (3.1.14) we have

$$\frac{\partial P}{\partial S} = e^{-x} \frac{\partial P}{\partial x}, \quad \frac{\partial^2 P}{\partial S^2} = e^{-2x} \left(\frac{\partial^2 P}{\partial x^2} - \frac{\partial P}{\partial x} \right), \quad \text{and} \quad \frac{\partial P}{\partial t} = -\frac{\partial P}{\partial \tau}.$$

Substituting the variables and the partial derivatives appropriately, the jump-diffusion pricing model (3.1.14) becomes (3.1.16) in the region $-\infty < x < \infty, 0 < y < \infty$ and $0 < \tau < T$, and the formerly terminal now initial condition is defined as

$$P(0, x, y) = \max(e^x - K, 0).$$

■

3.2 Formulation of the American Options Pricing Model

When discussing the American call options in the Black-Scholes setting we mentioned that for a non-dividend paying asset they can be treated as European options. Thus, when modeling American options it makes sense to consider dividend paying underlying assets.

Assume that the underlying asset S_t pays dividends at a continuous rate q ,

$$dS_t = (\mu - q)S_t dt + f(Y_t)S_t dW_t \quad (3.2.19)$$

with volatility that follows a jump-diffusion process

$$dY_t = \alpha(m - Y_t)dt + \zeta\sqrt{Y_t}dZ_t + J_t dN_t, \quad (3.2.20)$$

where W_t and Z_t are correlated diffusion processes with correlation ρ and N_t is a Poisson process not correlated to the previous two. Define the dividend process to be

$$dD_t = qS_t dt, \quad (3.2.21)$$

then the time change in the portfolio, initially given by equation (3.1.9), becomes

$$d\Pi = dP - a_t dS_t - a_t dD_t - b_t dQ. \quad (3.2.22)$$

If we own the underlying asset, we receive dividends, and vice versa, if we short the asset we need to pay dividends. The $a_t dD_t$ term accounts for this.

Applying Itô's lemma for both, dP and dQ , we obtain:

$$\begin{aligned}
d\Pi = & \left[\frac{\partial P}{\partial t} + (\mu - q)S \frac{\partial P}{\partial S} + \alpha(m - y) \frac{\partial P}{\partial y} + \frac{1}{2}S^2 f^2(y) \frac{\partial^2 P}{\partial S^2} + \rho\zeta S \sqrt{y} f(y) \frac{\partial^2 P}{\partial S \partial y} \right. \\
& + \frac{1}{2}\zeta^2 y \frac{\partial^2 P}{\partial y^2} - a_t(\mu - q)S - a_t q S_t - b_t \frac{\partial Q}{\partial t} - b_t(\mu - q)S \frac{\partial Q}{\partial S} - b_t \alpha(m - y) \frac{\partial Q}{\partial y} \\
& \left. - \frac{1}{2}b_t S^2 f^2(y) \frac{\partial^2 Q}{\partial S^2} - b_t \rho \zeta S \sqrt{y} f(y) \frac{\partial^2 Q}{\partial S \partial y} - \frac{1}{2}b_t \zeta^2 y \frac{\partial^2 Q}{\partial y^2} \right] dt \\
& + S f(y) \left[\frac{\partial P}{\partial S} - a_t - b_t \frac{\partial Q}{\partial S} \right] dW_t + \sqrt{y} \left[\frac{\partial P}{\partial y} - b_t \frac{\partial Q}{\partial y} \right] dZ_t \\
& + [P(y + J) - P(y) - b_t Q(y + J) - b_t Q(y)] dN_t. \tag{3.2.23}
\end{aligned}$$

We can eliminate the risk that comes from the diffusion terms by purchasing/selling a_t and b_t shares of stock and options, respectively, where these parameters are given by

$$b_t = \frac{\partial P}{\partial y} \left(\frac{\partial Q}{\partial y} \right)^{-1} \quad \text{and} \quad a_t = \frac{\partial P}{\partial S} - \frac{\partial Q}{\partial S} \frac{\partial P}{\partial y} \left(\frac{\partial Q}{\partial y} \right)^{-1}. \tag{3.2.24}$$

By the principle of no arbitrage, which mathematically is given by $d\Pi = r\Pi dt$, and substituting the expressions for a_t and b_t in (3.2.23) we obtain

$$\begin{aligned}
r \left[P - S \frac{\partial P}{\partial S} - S \frac{\partial Q}{\partial S} \frac{\partial P}{\partial y} \left(\frac{\partial Q}{\partial y} \right)^{-1} - \frac{\partial P}{\partial y} \left(\frac{\partial Q}{\partial y} \right)^{-1} Q \right] dt = \\
\left[\frac{\partial P}{\partial t} + \frac{1}{2}S^2 f^2(y) \frac{\partial^2 P}{\partial S^2} + \rho\zeta S \sqrt{y} f(y) \frac{\partial^2 P}{\partial S \partial y} + \frac{1}{2}\zeta^2 y \frac{\partial^2 P}{\partial y^2} - qS \frac{\partial P}{\partial S} \right. \\
- qS \frac{\partial Q}{\partial S} \frac{\partial P}{\partial y} \left(\frac{\partial Q}{\partial y} \right)^{-1} - \frac{\partial Q}{\partial t} \frac{\partial P}{\partial y} \left(\frac{\partial Q}{\partial y} \right)^{-1} - \frac{1}{2}S^2 f^2(y) \frac{\partial^2 Q}{\partial S^2} \frac{\partial P}{\partial y} \left(\frac{\partial Q}{\partial y} \right)^{-1} \\
\left. - \rho\zeta S \sqrt{y} f(y) \frac{\partial^2 Q}{\partial S \partial y} \frac{\partial P}{\partial y} \left(\frac{\partial Q}{\partial y} \right)^{-1} - \frac{1}{2}\zeta^2 y \frac{\partial^2 Q}{\partial y^2} \frac{\partial P}{\partial y} \left(\frac{\partial Q}{\partial y} \right)^{-1} \right] dt \\
+ \left[P(y + J) - P(y) - (Q(y + J) - Q(y)) \frac{\partial P}{\partial y} \left(\frac{\partial Q}{\partial y} \right)^{-1} \right] dN_t.
\end{aligned}$$

Rearranging the terms in the equation above and taking expectation over the probability distribution of jumps, we obtain

$$\begin{aligned}
& \left\{ \frac{\partial P}{\partial t} + (r - q)S \frac{\partial P}{\partial S} + \frac{1}{2}S^2 f^2(y) \frac{\partial^2 P}{\partial S^2} + \rho \zeta S \sqrt{y} f(y) \frac{\partial^2 P}{\partial S \partial y} + \frac{1}{2} \zeta^2 y \frac{\partial^2 P}{\partial y^2} - rP \right. \\
& \left. + \lambda E [P(y + J) - P(y)] \right\} \left(\frac{\partial P}{\partial y} \right)^{-1} dt \\
& = \left\{ \frac{\partial Q}{\partial t} + (r - q)S \frac{\partial Q}{\partial S} + \frac{1}{2}S^2 f^2(y) \frac{\partial^2 Q}{\partial S^2} + \rho \zeta \sqrt{y} S f(y) \frac{\partial^2 Q}{\partial S \partial y} + \frac{1}{2} \zeta^2 y \frac{\partial^2 Q}{\partial y^2} - rQ \right. \\
& \left. + \lambda E [Q(y + J) - Q(y)] \right\} \left(\frac{\partial Q}{\partial y} \right)^{-1} dt.
\end{aligned}$$

Using same arguments as in the European option case, we can show the following proposition

Proposition 3.2.1 *The price of an option written on a stock with continuous yield q described by SDE (3.2.19) and volatility driven by SDE (3.2.20), is modeled with the partial integro-differential equation*

$$\begin{aligned}
\frac{\partial P}{\partial t} + (r - q)S \frac{\partial P}{\partial S} + [\alpha(m - y) - \gamma y] \frac{\partial P}{\partial y} + \frac{1}{2}S^2 f^2(y) \frac{\partial^2 P}{\partial S^2} + \rho \zeta S \sqrt{y} f(y) \frac{\partial^2 P}{\partial S \partial y} \\
+ \frac{1}{2} \zeta^2 y \frac{\partial^2 P}{\partial y^2} - rP + \lambda E [P(t, S, y + J) - P(t, S, y)] = 0, \quad (3.2.25)
\end{aligned}$$

where

$$E [P(t, S, y + J) - P(t, S, y)] = \int_0^\infty [P(t, S, y + J) - P(t, S, y)] \phi(J) dJ$$

is the expected value of the change in the option price with respect to the jump probability distribution function.

We can define a model for the price of an American option by adding boundary conditions to the proposition above. Analogous to the constant volatility setting (Section 2.3), the function $P(t, S, y)$ satisfies a free boundary problem. In the jump-diffusion volatility case the free boundary becomes a surface because it depends on additional spatial variable y . Let $a(y, t)$ be the early exercise boundary at time t and volatility level y (the path of critical stock prices at which early exercise occurs), then

the following proposition holds:

Proposition 3.2.2 *The price of an American option written on a stock that pays dividends at a continuous rate q and given by SDE (3.2.19), with volatility driven by SDE (3.2.20), is modeled with PIDE (3.2.25) in the region $0 \leq t \leq T$, $0 \leq S \leq a(y, t)$ and $0 \leq y < \infty$. The boundary conditions in the American options case are*

$$\begin{aligned} P(T, S, y) &= \begin{cases} \max(S - K, 0), & \text{for a call option} \\ \max(K - S, 0), & \text{for a put option,} \end{cases} \\ P(t, a(y, t), y) &= \begin{cases} a(y, t) - K, & \text{for a call option} \\ K - a(y, t), & \text{for a put option} \end{cases} \\ \lim_{S \rightarrow a(y, t)} \frac{\partial P}{\partial S} &= \begin{cases} 1, & \text{for a call option} \\ -1, & \text{for a put option.} \end{cases} \end{aligned} \quad (3.2.26)$$

The last two boundary conditions are provided by Fouque et al. [6] to ensure continuity of P and $\frac{\partial P}{\partial S}$.

Let $S = e^x$ and $\tau = T - t$, then it is easy to show the following result:

Proposition 3.2.3 *The free boundary value problem in Proposition (3.2.2) for $P(t, S, y)$ is equivalent to*

$$\begin{aligned} \frac{\partial P}{\partial \tau} &= \left(r - q - \frac{1}{2}f^2(y) \right) \frac{\partial P}{\partial x} + [\alpha(m - y) - \gamma y] \frac{\partial P}{\partial y} + \frac{1}{2}f^2(y) \frac{\partial^2 P}{\partial x^2} + \rho\zeta\sqrt{y}f(y) \frac{\partial^2 P}{\partial x \partial y} \\ &+ \frac{1}{2}\zeta^2 y \frac{\partial^2 P}{\partial y^2} - rP + \lambda \int_{-\infty}^{\infty} [P(\tau, x, y + J) - P(\tau, x, y)] \phi(J) dJ, \end{aligned} \quad (3.2.27)$$

in the bounded domain $-\infty < x < \ln a(y, \tau)$, $0 < y < \infty$ and $0 < \tau < T$. The initial and boundary conditions are

$$P(0, x, y) = \begin{cases} \max(e^x - K, 0), & \text{for a call option} \\ \max(K - e^x, 0), & \text{for a put option,} \end{cases}$$

$$\begin{aligned}
P(\tau, a(y, \tau), y) &= \begin{cases} \ln a(y, \tau) - K, & \text{for a call option} \\ K - \ln a(y, \tau), & \text{for a put option,} \end{cases} \\
\lim_{x \rightarrow \ln a(y, \tau)} \frac{\partial P}{\partial x} &= \begin{cases} a(y, \tau), & \text{for a call option} \\ -a(y, \tau), & \text{for a put option.} \end{cases} \tag{3.2.28}
\end{aligned}$$

Theorem 3.2.4 *The homogeneous PIDE in equation (3.2.27) in the bounded domain $-\infty < x < \ln a(y, \tau)$ is equivalent to the nonhomogeneous PIDE*

$$\begin{aligned}
\frac{\partial P}{\partial \tau} - \left(r - q - \frac{1}{2} f^2(y) \right) \frac{\partial P}{\partial x} - [\alpha(m - y) - \gamma y] \frac{\partial P}{\partial y} - \frac{1}{2} f^2(y) \frac{\partial^2 P}{\partial x^2} - \rho \zeta \sqrt{y} f(y) \frac{\partial^2 P}{\partial x \partial y} \\
- \frac{1}{2} \zeta^2 y \frac{\partial^2 P}{\partial y^2} + rP - \lambda E[P(\tau, x, y + J) - P(\tau, x, y)] = H(x - \ln a(y, \tau))(qe^x - rK)
\end{aligned} \tag{3.2.29}$$

for an American call option, and

$$\begin{aligned}
\frac{\partial P}{\partial \tau} - \left(r - q - \frac{1}{2} f^2(y) \right) \frac{\partial P}{\partial x} - [\alpha(m - y) - \gamma y] \frac{\partial P}{\partial y} - \frac{1}{2} f^2(y) \frac{\partial^2 P}{\partial x^2} - \rho \zeta \sqrt{y} f(y) \frac{\partial^2 P}{\partial x \partial y} \\
- \frac{1}{2} \zeta^2 y \frac{\partial^2 P}{\partial y^2} + rP - \lambda E[P(\tau, x, y + J) - P(\tau, x, y)] = H(\ln a(y, \tau) - x)(rK - qe^x)
\end{aligned} \tag{3.2.30}$$

for an American put option, both solved in the unbounded domain $-\infty < x < \infty$, $0 \leq y < \infty$ and $0 \leq \tau \leq T$.

The function $H(x)$ is the Heaviside step function, defined by

$$H(x) = \begin{cases} 1, & \text{for } x \geq 0 \\ 0, & \text{for } x < 0. \end{cases}$$

Proof. Take an incomplete Fourier transform of eq. (3.2.27) with respect to x

$$\int_{-\infty}^{\ln a(y, \tau)} e^{i\omega x} \left\{ \frac{\partial P}{\partial \tau} - \left(r - q - \frac{1}{2} f^2(y) \right) \frac{\partial P}{\partial x} - [\alpha(m - y) - \gamma y] \frac{\partial P}{\partial y} - \frac{1}{2} f^2(y) \frac{\partial^2 P}{\partial x^2} \right. \\ \left. - \rho \zeta \sqrt{y} f(y) \frac{\partial^2 P}{\partial x \partial y} - \frac{1}{2} \zeta^2 y \frac{\partial^2 P}{\partial y^2} + rP \right. \\ \left. - \lambda \int_{-\infty}^{\infty} [P(\tau, x, y + J) - P(\tau, x, y)] \phi(J) dJ \right\} dx = 0.$$

Since $\int_{-\infty}^{\ln a(y, \tau)} \dots dx = \int_{-\infty}^{\infty} \dots dx - \int_{\ln a(y, \tau)}^{\infty} \dots dx$ we can write the expression above as follows

$$\mathcal{F} \left\{ \frac{\partial P}{\partial \tau} - \left(r - q - \frac{1}{2} f^2(y) \right) \frac{\partial P}{\partial x} - [\alpha(m - y) - \gamma y] \frac{\partial P}{\partial y} - \frac{1}{2} f^2(y) \frac{\partial^2 P}{\partial x^2} \right. \\ \left. - \rho \zeta f(y) \sqrt{y} \frac{\partial^2 P}{\partial x \partial y} - \frac{1}{2} \zeta^2 y \frac{\partial^2 P}{\partial y^2} + rP - \lambda \int_{-\infty}^{\infty} [P(\tau, x, y + J) - P(\tau, x, y)] \phi(J) dJ \right\} \\ = \int_{\ln a(y, \tau)}^{\infty} e^{i\omega x} \left\{ \frac{\partial P}{\partial \tau} - \left(r - q - \frac{1}{2} f^2(y) \right) \frac{\partial P}{\partial x} - [\alpha(m - y) - \gamma y] \frac{\partial P}{\partial y} - \frac{1}{2} f^2(y) \frac{\partial^2 P}{\partial x^2} \right. \\ \left. - \rho \zeta f(y) \sqrt{y} \frac{\partial^2 P}{\partial x \partial y} - \frac{1}{2} \zeta^2 y \frac{\partial^2 P}{\partial y^2} + rP - \lambda \int_{-\infty}^{\infty} [P(\tau, x, y + J) - P(\tau, x, y)] \phi(J) dJ \right\} dx.$$

When $x > \ln a(y, \tau)$ the price of the American call option is simply the payoff, $P(\tau, x, y) = e^x - K$, and so $\frac{\partial P}{\partial \tau} = \frac{\partial P}{\partial y} = \frac{\partial^2 P}{\partial y^2} = \frac{\partial^2 P}{\partial x \partial y} = 0$, $\frac{\partial P}{\partial x} = e^x$ and $\frac{\partial^2 P}{\partial x^2} = e^x$. Thus, the integral on the right hand side of the equation above becomes

$$\int_{\ln a(y, \tau)}^{\infty} e^{i\omega x} (qe^x - rK) dx = \int_{-\infty}^{\infty} e^{i\omega x} H(x - \ln a(y, \tau)) (qe^x - rK) dx. \quad (3.2.31)$$

Taking the inverse Fourier transform we will obtain (3.2.29).

The American put option nonhomogeneous PDE (3.2.30) can be derived similarly.

Note that the non-homogeneous term $H(x - \ln a(y, \tau)) (qe^x - rK)$ in PIDE (3.2.29) arises only when the call option is exercised early, that is $x > \ln a(y, \tau)$, in which case the holder receives dividends by owning the stock S , qe^x , and pays interest of rK for borrowing K . ■

4 CLOSED FORM SOLUTION OF A PURE DIFFUSION OPTION PRICING MODEL

Before we move to determining a closed form solution to the jump diffusion model described in the previous chapter, let's focus on the Heston's model in which the underlying asset price follows the diffusion process

$$dS_t = \mu S_t dt + \sqrt{Y_t} S_t dW_t \quad (4.0.1)$$

and the volatility is driven by mean-reverting diffusion process Y_t :

$$dY_t = \alpha(m - Y_t)dt + \zeta \sqrt{Y_t} dZ_t. \quad (4.0.2)$$

Setting the jump amplitude J equal to 0 in the derivation process given in the previous chapter, we will obtain a PDE that describes the price of an option written on an asset driven by SDEs (4.0.1) and (4.0.2):

$$\frac{\partial P}{\partial \tau} = \left(r - \frac{1}{2}y \right) \frac{\partial P}{\partial x} + [\alpha(m - y) - \gamma y] \frac{\partial P}{\partial y} + \frac{1}{2}y \frac{\partial^2 P}{\partial x^2} + \rho \zeta y \frac{\partial^2 P}{\partial x \partial y} + \frac{1}{2} \zeta^2 y \frac{\partial^2 P}{\partial y^2} - rP, \quad (4.0.3)$$

where $-\infty < x < \infty$, $0 \leq y < \infty$, $0 \leq \tau \leq T$ and the option price at maturity is $P(0, x, y) = \max(e^x - K, 0)$ in the case of a call option, or $P(0, x, y) = \max(K - e^x, 0)$ in the case of a put option. Heston solves this PDE by guessing the form of the solution and Heston's solution formula is widely used in the financial arena currently. The method that we use in the following section will give us a complete solution of PDE (4.0.3) in a closed form without any guessing. The result offered here substantially improves Heston's approach.

4.1 Integral Transforms of the Pure Diffusion Pricing Model

Assume that $\int_{-\infty}^{\infty} |P(\tau, x, y)| dx < \infty$, and that the function $P(\tau, x, y)$ is of exponential order, that is $\lim_{y \rightarrow \infty} |P(\tau, x, y)e^{-By}| = 0$ for some real number B .

Define $\hat{P}(\tau, \omega, y)$ to be the Fourier transform of $P(\tau, x, y)$ with respect to x :

$$\hat{P}(\tau, \omega, y) \equiv \mathcal{F}[P(\tau, x, y)] = \int_{-\infty}^{\infty} P(\tau, x, y)e^{i\omega x} dx.$$

Then the following properties hold

$$\begin{aligned} \mathcal{F}\left[\frac{\partial P}{\partial x}\right] &= -i\omega\hat{P}(\tau, \omega, y), \quad \mathcal{F}\left[\frac{\partial^2 P}{\partial x^2}\right] = -\omega^2\hat{P}(\tau, \omega, y), \quad \mathcal{F}\left[\frac{\partial P}{\partial \tau}\right] = \frac{\partial \hat{P}}{\partial \tau}, \\ \mathcal{F}\left[\frac{\partial P}{\partial y}\right] &= \frac{\partial \hat{P}}{\partial y}, \quad \text{and} \quad \mathcal{F}\left[\frac{\partial^2 P}{\partial y^2}\right] = \frac{\partial^2 \hat{P}}{\partial y^2}. \end{aligned}$$

Also,

$$\mathcal{F}\left[\frac{\partial^2 P}{\partial x \partial y}\right] = \int_{-\infty}^{\infty} e^{i\omega x} \frac{\partial^2 P}{\partial x \partial y} dx = \frac{\partial}{\partial y} \left(\int_{-\infty}^{\infty} e^{i\omega x} \frac{\partial P}{\partial x} dx \right) = \frac{\partial}{\partial y} (-i\omega\hat{P}) = -i\omega \frac{\partial \hat{P}}{\partial y}. \quad (4.1.4)$$

Applying the Fourier transform to equation (4.0.3) and using the above properties, we obtain:

$$\frac{\partial \hat{P}}{\partial \tau} = i\omega \left(\frac{1}{2}y - r \right) \hat{P} + [\alpha(m - y) - \gamma y] \frac{\partial \hat{P}}{\partial y} - \frac{1}{2}\omega^2 y \hat{P} - i\rho\omega\zeta y \frac{\partial \hat{P}}{\partial y} + \frac{1}{2}\zeta^2 y \frac{\partial^2 \hat{P}}{\partial y^2} - r\hat{P}$$

which is equivalent to the PDE

$$\frac{\partial \hat{P}}{\partial \tau} = \frac{1}{2}\zeta^2 y \frac{\partial^2 \hat{P}}{\partial y^2} + [\alpha m - (\alpha + \gamma + i\rho\omega\zeta) y] \frac{\partial \hat{P}}{\partial y} - [r + i\omega r + \frac{1}{2}(\omega^2 - i\omega) y] \hat{P}. \quad (4.1.5)$$

The domain of y is $[0, \infty)$, so we will apply the Laplace transform to equation (4.1.5) with respect to y . Let \tilde{P} be the Laplace transform of \hat{P}

$$\tilde{P} \equiv \mathcal{L} \left[\hat{P}(\tau, \omega, \psi) \right] = \int_0^\infty \hat{P}(\tau, \omega, y) e^{-\psi y} dy,$$

provided that the integral converges when $y \rightarrow \infty$. We will find useful the following properties of the Laplace transform:

$$\begin{aligned} \mathcal{L} \left[\frac{\partial \hat{P}}{\partial \tau} \right] &= \frac{\partial \tilde{P}}{\partial \tau}, \quad \mathcal{L} \left[\frac{\partial \hat{P}}{\partial y} \right] = \psi \tilde{P}(\tau, \omega, \psi) - \hat{P}(\tau, \omega, 0), \quad \text{and} \\ \mathcal{L} \left[\frac{\partial^2 \hat{P}}{\partial y^2} \right] &= \psi^2 \tilde{P}(\tau, \omega, \psi) - \psi \hat{P}(\tau, \omega, 0) - \frac{\partial \hat{P}}{\partial y}(\tau, \omega, 0). \end{aligned}$$

However, when $y = 0$, the price of the underlying asset loses the risky term and it becomes completely deterministic. We buy options in order to eliminate or reduce the risk associated with buying/selling risky assets. Thus, in the case of no risk it is natural to assume that the price of the option is 0, $P(\tau, x, 0) = 0$. Then, the Fourier transform of $P(\tau, x, 0)$ is $\hat{P}(\tau, \omega, 0) = 0$. Now, the properties above are simplified to

$$\mathcal{L} \left[\frac{\partial \hat{P}}{\partial y} \right] = \psi \tilde{P}(\tau, \omega, \psi), \quad \text{and} \quad \mathcal{L} \left[\frac{\partial^2 \hat{P}}{\partial y^2} \right] = \psi^2 \tilde{P}(\tau, \omega, \psi) - \frac{\partial \hat{P}}{\partial y}(\tau, \omega, 0).$$

Also, it is easy to see that $\mathcal{L} \left[y \hat{P}(\tau, \omega, y) \right] = -\frac{\partial \tilde{P}}{\partial \psi}$ by differentiating

$$\tilde{P} = \int_0^\infty \hat{P}(\tau, \omega, y) e^{-\psi y} dy,$$

with respect to ψ :

$$\begin{aligned} \frac{\partial \tilde{P}}{\partial \psi} &= \frac{\partial}{\partial \psi} \int_0^\infty \hat{P}(\tau, \omega, y) e^{-\psi y} dy = \int_0^\infty \hat{P}(\tau, \omega, y) \frac{\partial}{\partial \psi} e^{-\psi y} dy \\ &= - \int_0^\infty y e^{-\psi y} \hat{P}(\tau, \omega, y) dy = -\mathcal{L} \left[y \hat{P} \right]. \end{aligned}$$

The last property will be very helpful in calculating the Laplace transforms of $y \frac{\partial \hat{P}}{\partial y}$ and $y \frac{\partial^2 \hat{P}}{\partial y^2}$. Using integration by parts, we obtain:

$$\begin{aligned}
\mathcal{L} \left[y \frac{\partial \hat{P}}{\partial y} \right] &= \int_0^\infty y e^{-\psi y} \frac{\partial \hat{P}}{\partial y} dy \\
&= y e^{-\psi y} \hat{P} \Big|_0^\infty - \int_0^\infty e^{-\psi y} \hat{P} dy + \psi \int_0^\infty y e^{-\psi y} \hat{P} dy \\
&= \psi \mathcal{L} \left[y \hat{P} \right] - \tilde{P} = -\psi \frac{\partial \tilde{P}}{\partial \psi} - \tilde{P},
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{L} \left[y \frac{\partial^2 \hat{P}}{\partial y^2} \right] &= \int_0^\infty y e^{-\psi y} \frac{\partial^2 \hat{P}}{\partial y^2} dy \\
&= y e^{-\psi y} \frac{\partial \hat{P}}{\partial y} \Big|_0^\infty - \int_0^\infty e^{-\psi y} \frac{\partial \hat{P}}{\partial y} dy + \psi \int_0^\infty y e^{-\psi y} \frac{\partial \hat{P}}{\partial y} dy \\
&= -\mathcal{L} \left[\frac{\partial \hat{P}}{\partial y} \right] + \psi \mathcal{L} \left[y \frac{\partial \hat{P}}{\partial y} \right] = -2\psi \tilde{P} - \psi^2 \frac{\partial \tilde{P}}{\partial \psi}.
\end{aligned}$$

Having these results, when applying Laplace transform to equation (4.1.5), we obtain the first-order linear PDE:

$$\begin{aligned}
\frac{\partial \tilde{P}}{\partial \tau} &= -\frac{1}{2} \zeta^2 \left(\psi^2 \frac{\partial \tilde{P}}{\partial \psi} + 2\psi \tilde{P} \right) + \alpha m \psi \tilde{P} + (\alpha + \gamma + i\omega \rho \zeta) \left(\psi \frac{\partial \tilde{P}}{\partial \psi} + \tilde{P} \right) \\
&\quad - (r + i\omega r) \tilde{P} + \frac{1}{2} (\omega^2 - i\omega) \frac{\partial \tilde{P}}{\partial \psi}
\end{aligned}$$

which simplifies to

$$\begin{aligned}
\frac{\partial \tilde{P}}{\partial \tau} &+ \left[\frac{1}{2} \zeta^2 \psi^2 - (\alpha + \gamma + i\omega \rho \zeta) \psi - \frac{1}{2} (\omega^2 - i\omega) \right] \frac{\partial \tilde{P}}{\partial \psi} \\
&= [(\alpha m - \zeta^2) \psi + (\alpha + \gamma + i\omega \rho \zeta - r - i\omega r)] \tilde{P}. \tag{4.1.6}
\end{aligned}$$

Recall that at maturity, when $\tau = 0$, the price of the option is $P(0, x, y) = h(x)$, and once we take the Fourier transform with respect to x and the Laplace transform with respect to y we obtain

$$\tilde{P}(0, \omega, \psi) = \mathcal{L}[\mathcal{F}[h(x)]] = \mathcal{L}[\hat{h}(\omega)] = \hat{h}(\omega)\mathcal{L}[1] = \frac{\hat{h}(\omega)}{\psi}.$$

4.2 Solution of the Transformed Pure Diffusion Pricing Model

Proposition 4.2.1 *The general solution of the partial differential equation*

$$\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} = f(x, y)w, \quad (4.2.7)$$

is given by

$$w = \exp\left[\int^x f(t, y - x + t)dt\right] \Phi(y - x)$$

where Φ is an arbitrary continuously differentiable function, and the lower limit of the integral can be chosen arbitrarily.

Proof. Using the Leibniz integral rule we calculate $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$

$$\begin{aligned} \frac{\partial w}{\partial x} &= \frac{\partial}{\partial x} \left[\exp\left(\int^x f(t, y - x + t)dt\right) \Phi(y - x) \right] \\ &= \left[f(x, y) - \int^x f'(t, y - x + t)dt \right] e^{\int^x f(t, y - x + t)dt} \Phi(y - x) - e^{\int^x f(t, y - x + t)dt} \Phi'(y - x) \\ \frac{\partial w}{\partial y} &= \frac{\partial}{\partial y} \left[\exp\left(\int^x f(t, y - x + t)dt\right) \Phi(y - x) \right] \\ &= \int^x f'(t, y - x + t)dt e^{\int^x f(t, y - x + t)dt} \Phi(y - x) + e^{\int^x f(t, y - x + t)dt} \Phi'(y - x) \end{aligned}$$

Adding up the partial derivatives, we have

$$\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} = f(x, y) \underbrace{\exp\left(\int^x f(t, y - x + t)dt\right) \Phi(y - x)}_w.$$

■

Proposition 4.2.2 *The closed-form solution of the partial differential equation*

$$\begin{aligned} \frac{\partial \tilde{P}}{\partial \tau} + \left[\frac{1}{2} \zeta^2 \psi^2 - (\alpha + \gamma + i\omega\rho\zeta)\psi - \frac{1}{2}(\omega^2 - i\omega) \right] \frac{\partial \tilde{P}}{\partial \psi} \\ = [(\alpha m - \zeta^2)\psi + (\alpha + \gamma + i\omega\rho\zeta - r - i\omega r)] \tilde{P} \end{aligned} \quad (4.2.8)$$

with initial condition

$$\tilde{P}(0, \omega, \psi) = \frac{\hat{h}(\omega)}{\psi}$$

is given by

$$\begin{aligned} \tilde{P}(\tau, \omega, \psi) = \frac{\hat{h}(\omega) [2(\xi + i\theta)]^{2 - \frac{\alpha m}{\zeta^2}} e^{\tau \left[\frac{\alpha m(\alpha + \gamma - \theta)}{\zeta^2} - r - \theta + i \left(\frac{\alpha m(\rho\zeta\omega + \xi)}{\zeta^2} - r\omega - \xi \right) \right]}{\left[\xi + i\theta - i(\alpha + \gamma + i\rho\zeta\omega) + e^{-i\tau(\xi + i\theta)} (\xi + i\theta + i(\alpha + \gamma + i\rho\zeta\omega)) \right]} \\ \cdot \frac{f(\psi; \tau, \omega)}{\psi - \psi_0}, \end{aligned} \quad (4.2.9)$$

where ψ_0 , $f(\psi; \tau, \omega)$, $\xi = \xi(\omega)$ and $\theta = \theta(\omega)$ are defined below.

$$\begin{aligned} f(\psi; \tau, \omega) = \{ i\zeta^2\psi (e^{-i\tau(\xi + i\theta)} - 1) + (\xi + i\theta) + i(\alpha + \gamma + i\rho\zeta\omega) \\ + e^{-i\tau(\xi + i\theta)} [(\xi + i\theta) - i(\alpha + \gamma + i\rho\zeta\omega)] \}^{\frac{2\alpha m}{\zeta^2} - 1}, \end{aligned} \quad (4.2.10)$$

$$\psi_0 = A(\tau, \omega) = \frac{i [e^{-i\tau(\xi + i\theta)} - 1] [(\xi + i\theta)^2 + (\alpha + \gamma + i\rho\zeta\omega)^2]}{\zeta^2 \{ \xi + i\theta - i(\alpha + \gamma + i\rho\zeta\omega) + e^{-i\tau(\xi + i\theta)} [\xi + i\theta + i(\alpha + \gamma + i\rho\zeta\omega)] \}}, \quad (4.2.11)$$

$$\begin{aligned} \xi &= \sqrt{\frac{1}{2} \left\{ \zeta^2 \omega^2 (\rho^2 - 1) - (\alpha + \gamma)^2 + \sqrt{G(\omega)} \right\}}, \\ \theta &= \frac{\zeta^2 \omega - 2\rho\zeta\omega(\alpha + \gamma)}{\sqrt{2 \left\{ \zeta^2 \omega^2 (\rho^2 - 1) - (\alpha + \gamma)^2 + \sqrt{G(\omega)} \right\}}}, \end{aligned} \quad (4.2.12)$$

and

$$G(\omega) = [\zeta^2 \omega^2 (\rho^2 - 1) - (\alpha + \gamma)^2]^2 + [\zeta^2 \omega - 2\rho\zeta\omega(\alpha + \gamma)]^2. \quad (4.2.13)$$

Proof. The following transformation will lead to a first order PDE with coefficients 1 in front of the partial derivatives of PDE (4.2.8)

$$\begin{aligned}\eta &= \int \frac{d\psi}{\frac{1}{2}\zeta^2\psi^2 - (\alpha + \gamma + i\omega\rho\zeta)\psi - \frac{1}{2}(\omega^2 - i\omega)} \\ &= \frac{i}{\sqrt{a+ib}} \ln \frac{\sqrt{a+ib} - i(\zeta^2\psi - \alpha - \gamma - i\omega\rho\zeta)}{\sqrt{a+ib} + i(\zeta^2\psi - \alpha - \gamma - i\omega\rho\zeta)}\end{aligned}\quad (4.2.14)$$

where $a = a(\omega) = \zeta^2\omega^2(\rho^2 - 1) - (\alpha + \gamma)^2$ and $b = b(\omega) = \zeta^2\omega - 2\rho\zeta\omega(\alpha + \gamma)$. Let

$$\sqrt{a+ib} = \xi + i\theta, \quad (4.2.15)$$

then we can find $\sqrt{a+ib}$ explicitly by squaring (4.2.15) and then solving the system of equations

$$\begin{cases} \xi^2 - \theta^2 = a(\omega) = \zeta^2\omega^2(\rho^2 - 1) - (\alpha + \gamma)^2 \\ 2\xi\theta = b(\omega) = \zeta^2\omega - 2\rho\zeta\omega(\alpha + \gamma) \end{cases}$$

The solution of this system is:

$$\begin{aligned}\xi &= \pm \sqrt{\frac{1}{2} \left(a \pm \sqrt{a^2 + b^2} \right)} \\ \theta &= \frac{b}{\pm \sqrt{2(a \pm \sqrt{a^2 + b^2})}}.\end{aligned}\quad (4.2.16)$$

or, in terms of ω as follows:

$$\begin{aligned}\xi &= \sqrt{\frac{1}{2} \left\{ \zeta^2\omega^2(\rho^2 - 1) - (\alpha + \gamma)^2 + \sqrt{G(\omega)} \right\}}, \\ \theta &= \frac{\zeta^2\omega - 2\rho\zeta\omega(\alpha + \gamma)}{2\xi},\end{aligned}\quad (4.2.17)$$

$$G(\omega) = [\zeta^2\omega^2(\rho^2 - 1) - (\alpha + \gamma)^2]^2 + [\zeta^2\omega - 2\rho\zeta\omega(\alpha + \gamma)]^2 \quad (4.2.18)$$

where the \pm signs have been replaced with +'s in (4.2.17) and are appropriately chosen according to available information that we need $\xi(\omega)$ and $\theta(\omega)$ to be real valued functions.

Going back to simplifying PDE (4.2.8) using the η -transformation, we calculate ψ

and $\frac{d\psi}{d\eta}$ in terms of η , obtaining the following expressions:

$$\begin{aligned}\psi &= \frac{i(\xi + i\theta)(1 - e^{i\eta(\xi+i\theta)})}{\zeta^2(1 + e^{i\eta(\xi+i\theta)})} + \frac{\alpha + \gamma + i\rho\zeta\omega}{\zeta^2}, \\ \frac{d\psi}{d\eta} &= \frac{2(\xi + i\theta)^2 e^{i\eta(\xi+i\theta)}}{\zeta^2(1 + e^{i\eta(\xi+i\theta)})^2}.\end{aligned}\quad (4.2.19)$$

Then PDE (4.2.8) becomes

$$\frac{\partial \tilde{P}}{\partial \tau} + \frac{\partial \tilde{P}}{\partial \eta} = \left[i(\alpha m - \zeta^2) \frac{(\xi + i\theta)(1 - e^{i\eta(\xi+i\theta)})}{\zeta^2(1 + e^{i\eta(\xi+i\theta)})} + \frac{\alpha m}{\zeta^2}(\alpha + \gamma + i\omega\rho\zeta) - r - ir\omega \right] \tilde{P}.\quad (4.2.20)$$

This PDE is of the same form as PDE (4.2.7) so to obtain its solution we can apply Proposition 4.2.1. Choosing the lower bound of the integral in this proposition to be 0 will simplify the determination of the arbitrary function $\Phi(\eta - \tau)$. Thus, we need to calculate the integral

$$\begin{aligned}& \int_0^\tau \left(\frac{i(\alpha m - \zeta^2)(\xi + i\theta)(1 - e^{i(\eta-\tau+t)(\xi+i\theta)})}{\zeta^2(1 + e^{i(\eta-\tau+t)(\xi+i\theta)})} + \frac{\alpha m(\alpha + \gamma + i\omega\rho\zeta)}{\zeta^2} - r - ir\omega \right) dt \\ &= \left[\frac{\alpha m(\alpha + \gamma + i\omega\rho\zeta)}{\zeta^2} - r - ir\omega + \frac{i(\alpha m - \zeta^2)(\xi + i\theta)}{\zeta^2} \right] \tau \\ &+ \ln \left(\frac{1 + e^{i(\eta-\tau)(\xi+i\theta)}}{1 + e^{i\eta(\xi+i\theta)}} \right)^{\frac{2\alpha m}{\zeta^2} - 1}.\end{aligned}$$

Setting $\tau = 0$ makes the integral above equal to 0 and using the initial condition in terms of η , the arbitrary function $\Phi(\eta)$ becomes easy to determine:

$$\Phi(\eta) = \frac{\hat{h}(\omega)\zeta^2(1 + e^{i\eta(\xi+i\theta)})}{i(\xi + i\theta)(1 - e^{i\eta(\xi+i\theta)}) + (\alpha + \gamma + i\rho\zeta\omega)(1 + e^{i\eta(\xi+i\theta)})}.\quad (4.2.21)$$

Finally, the exact solution of PDE (4.2.8) is given by

$$\begin{aligned}\tilde{P}(\tau, \omega, \eta) &= \frac{\hat{h}(\omega)\zeta^2 e^{\tau \left[\frac{\alpha^2 m + \alpha\gamma m - \theta(\alpha m - \zeta^2)}{\zeta^2} - r + i \left(\frac{\alpha m \rho \zeta \omega + \xi(\alpha m - \zeta^2)}{\zeta^2} - r\omega \right) \right]}{i(\xi + i\theta)(1 - e^{i(\eta-\tau)(\xi+i\theta)}) + (\alpha + \gamma + i\rho\zeta\omega)(1 + e^{i(\eta-\tau)(\xi+i\theta)})} \times \\ &\quad \left(\frac{1 + e^{i(\eta-\tau)(\xi+i\theta)}}{1 + e^{i\eta(\xi+i\theta)}} \right)^{\frac{2(\alpha m - \zeta^2)}{\zeta^2}},\end{aligned}\quad (4.2.22)$$

and substituting the expression (4.2.14) for η in the expression above, we obtain formula (4.2.9), which proves the proposition. ■

4.3 Inverse Integral Transforms of the Pure Diffusion Pricing Formula

In the previous section we found the solution of the first order PDE (4.2.8). However, this solution doesn't give us the price of the option. To find the price of a European call (put) options we will need to apply an inverse Laplace transform with respect to ψ and an inverse Fourier transform with respect to ω in the pricing formula (4.2.9). The inverse Laplace transform is given by

$$f(t) = \frac{1}{2\pi i} \int_{\epsilon-i\infty}^{\epsilon+i\infty} e^{st} \hat{f}(s) ds, \quad (4.3.23)$$

where ϵ is such that the contour of integration is to the right-hand side of any singularities of $\hat{f}(s)$, or translated in terms of $\hat{P}(\tau, \omega, y)$ the inversion formula above will read:

$$\begin{aligned} \hat{P}(\tau, \omega, y) = & \frac{\hat{h}(\omega) [2(\xi + i\theta)]^{2-\frac{2\alpha m}{\zeta^2}} e^{\tau \left[\frac{\alpha m(\alpha + \gamma - \theta)}{\zeta^2} - r - \theta + i \left(\frac{\alpha m(\rho \zeta \omega + \xi)}{\zeta^2} - r \omega - \xi \right) \right]}}{[\xi + i\theta - i(\alpha + \gamma + i\rho \zeta \omega) + e^{-i\tau(\xi + i\theta)} (\xi + i\theta + i(\alpha + \gamma + i\rho \zeta \omega))]} \times \\ & \frac{1}{2\pi i} \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{e^{y\psi} f(\psi; \tau, \omega)}{\psi - \psi_0} d\psi. \end{aligned} \quad (4.3.24)$$

The number of singularities in the function above will depend on the value of $\frac{2\alpha m}{\zeta^2} - 1$, the power of the function defined by $f(\psi; \tau, \omega)$. Buff [7] explains that in order for the volatility process \sqrt{y} with y driven by 3.1.2 to be positive, for the parameters α, m and ζ the inequality $\alpha m \geq \frac{\zeta^2}{2}$ must hold. This makes the exponent of $f(\psi; \tau, \omega)$ nonnegative. This implies that the function $\tilde{P}(\tau, \omega, \psi)$ has one singularity that is also a simple pole at $\psi = \psi_0$. To calculate the integral in (4.3.24) we use the *Cauchy Integral Formula* that states

Lemma 4.3.1 *If $f(s)$ is analytic function within and on a simple closed curve C and*

s_0 is any point interior to C , then

$$f(s_0) = \frac{1}{2\pi i} \oint_C \frac{f(s)}{s - s_0} ds$$

where C is traversed in the positive (counterclockwise) sense.

Corollary 4.3.2 *The inverse Laplace Transform of $\tilde{P}(\tau, \omega, \psi)$ given by (4.2.9) is equal to*

$$\hat{P}(\tau, \omega, y) = \hat{h}(\omega) e^{A(\tau, \omega)y + \tau[B(\omega) + iC(\omega)]} \times \left\{ \frac{2(\xi + i\theta)}{\xi + i\theta - i(\alpha + \gamma + i\rho\zeta\omega) + e^{-i\tau(\xi + i\theta)} [\xi + i\theta + i(\alpha + \gamma + i\rho\zeta\omega)]} \right\}^{\frac{2\alpha m}{\zeta^2}}, \quad (4.3.25)$$

where $A(\tau, \omega)$ is defined as in (4.2.11) and $B(\omega)$ and $C(\omega)$ are given by:

$$\begin{aligned} B(\omega) &= \frac{\alpha m(\alpha + \gamma + \theta(\omega))}{\zeta^2} - r \\ C(\omega) &= \frac{\alpha m(\rho\zeta\omega - \xi(\omega))}{\zeta^2} - r\omega, \end{aligned} \quad (4.3.26)$$

and $\hat{h}(\omega)$ is the Fourier transform of the boundary condition defined throughout the section.

Proof. The function $f(\psi; \tau, \omega)$ is analytic everywhere and so is any exponential function. Thus the product $e^{y\psi} f(\psi; \tau, \omega)$ is analytic everywhere. Then, applying the *Cauchy Integral Formula*, the integral in (4.3.24) becomes:

$$\frac{1}{2\pi i} \int_{\epsilon - i\infty}^{\epsilon + i\infty} \frac{e^{y\psi} f(\psi; \omega)}{\psi - \psi_0} d\psi = e^{y\psi_0} f(\psi_0; \tau, \omega).$$

An expression for $f(\psi_0; \tau, \omega)$ can be easily calculate by plugging in ψ_0 given by (4.2.11) in (4.2.10)

$$f(\psi_0; \tau, \omega) = \left\{ \frac{4(\xi + i\theta)^2 e^{-i\tau(\xi + i\theta)}}{\xi + i\theta - i(\alpha + \gamma + i\rho\zeta\omega) + e^{-i\tau(\xi + i\theta)} [\xi + i\theta + i(\alpha + \gamma + i\rho\zeta\omega)]} \right\}^{\frac{2\alpha m}{\zeta^2} - 1}.$$

The new, simplified form of $\hat{P}(\tau, \omega, y)$ in (4.3.24) is given by (4.3.25). ■

Now, we present the exact solution of the partial differential equation (4.0.3):

Theorem 4.3.3 *The price of a European Call option on an underlying asset that follows the diffusion process (4.0.1) driven by mean-reverting diffusion volatility (4.0.2) can be calculated using the formula*

$$P(\tau, x, y) = e^x P_1(\tau, x, y) - K P_2(\tau, x, y), \quad (4.3.27)$$

where $x = \ln S_t$,

$$P_1(\tau, x, y) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \Re \left[\frac{e^{-i\omega(x - \ln K)} \hat{g}(\omega + i; \tau)}{-i\omega} \right] d\omega, \quad (4.3.28)$$

$$P_2(\tau, x, y) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \Re \left[\frac{e^{-i\omega(x - \ln K)} \hat{g}(\omega; \tau, y)}{-i\omega} \right] d\omega, \quad (4.3.29)$$

$$\hat{g}(\omega; \tau, y) = e^{A(\tau, \omega)y + \tau[B(\omega) + iC(\omega)]} \times$$

$$\left\{ \frac{2(\xi + i\theta)}{\xi + i\theta - i(\alpha + \gamma + i\rho\zeta\omega) + e^{-i\tau(\xi + i\theta)} [\xi + i\theta + i(\alpha + \gamma + i\rho\zeta\omega)]} \right\}^{\frac{2\alpha m}{\zeta^2}}. \quad (4.3.30)$$

$\xi = \xi(\omega)$ and $\theta = \theta(\omega)$ are defined as in (4.2.17), $A(\omega; \tau)$ is defined as in (4.2.11) and $B(\omega)$ and $C(\omega)$ are as in (4.3.26).

Proof. Using the definition of $\hat{g}(\omega; \tau, y)$ we can write (4.3.25) as

$$\hat{P}(\tau, \omega, y) = \hat{h}(\omega) \hat{g}(\omega; \tau, y)$$

Then, to find the price of the option we need to take inverse Fourier transform of this equation with respect to ω . Let $g(x; \tau)$ be the inverse Fourier transform of $\hat{g}(\omega; \tau, y)$

$$g(x; \tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega x} \hat{g}(\omega; \tau) d\omega. \quad (4.3.31)$$

Note that, if $g(x; \tau)$ is a probability density function, then $\hat{g}(\omega; \tau, y)$ will be its characteristic function. Also, it is easy to check that at maturity, i.e. when $\tau = 0$, $\hat{g}(\omega; 0) = 1$, and so

$$g(x; 0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega x} = \delta(x),$$

where $\delta(x)$ is the Dirac delta function. By definition, $\delta(x) \geq 0$ and $\int_{-\infty}^{\infty} \delta(x) dx = 1$, so it can be interpreted as a probability density function, and so can $g(x)$.

Next, using the convolution theorem we obtain:

$$\begin{aligned} P(\tau, x, y) &= \mathcal{F}^{-1} \left[\hat{h}(\omega) \hat{g}(\omega; \tau, y) \right] \\ &= \int_{-\infty}^{\infty} h(x - \bar{x}) g(\bar{x}; \tau) d\bar{x} \end{aligned}$$

where $h(x)$ is the terminal condition. Thus, for the European call option case, we have:

$$\begin{aligned} P(\tau, x, y) &= \int_{-\infty}^{\infty} \max \{ e^{x - \bar{x}} - K, 0 \} g(\bar{x}; \tau) d\bar{x} \\ &= \frac{e^x}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{x - \ln K} e^{-\bar{x} - i\omega \bar{x}} \hat{g}(\omega; \tau) d\bar{x} d\omega - \frac{K}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{x - \ln K} e^{-i\omega \bar{x}} \hat{g}(\omega; \tau) d\bar{x} d\omega. \end{aligned}$$

Denote

$$\begin{aligned} P_1(\tau, x, y) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{x - \ln K} e^{-\bar{x} - i\omega \bar{x}} \hat{g}(\omega; \tau) d\bar{x} d\omega, \text{ and} \\ P_2(\tau, x, y) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{x - \ln K} e^{-i\omega \bar{x}} \hat{g}(\omega; \tau) d\bar{x} d\omega, \end{aligned}$$

and calculate these integrals. For the first integral, we will change the integration variable from ω to $\omega + i$ and denote $\hat{g}_1(\omega) = \hat{g}(\omega + i)$ where needed, so that integral

P_1 becomes

$$\begin{aligned}
P_1 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{g}(\omega + i) \lim_{\chi \rightarrow -\infty} \frac{e^{-i\omega\chi} - e^{-i\omega(x-\ln K)}}{i\omega} d\omega \\
&= \frac{1}{2\pi} \lim_{\chi \rightarrow -\infty} \int_0^{\infty} \frac{e^{-i\omega\chi} \hat{g}_1(\omega) - e^{i\omega\chi} \hat{g}_1(-\omega)}{i\omega} d\omega \\
&\quad + \frac{1}{2\pi} \int_0^{\infty} \frac{e^{-i\omega(x-\ln K)} \hat{g}(\omega + i) - e^{i\omega(x-\ln K)} \hat{g}(-\omega + i)}{-i\omega} d\omega.
\end{aligned}$$

Similarly, for the second integral P_2 , we have:

$$\begin{aligned}
P_2 &= \frac{1}{2\pi} \left[\lim_{\chi \rightarrow -\infty} \int_0^{\infty} \frac{e^{-i\omega\chi} \hat{g}(\omega) - e^{i\omega\chi} \hat{g}(-\omega)}{i\omega} d\omega \right. \\
&\quad \left. - \int_0^{\infty} \frac{e^{-i\omega(x-\ln K)} \hat{g}(\omega) - e^{i\omega(x-\ln K)} \hat{g}(-\omega)}{i\omega} d\omega \right]
\end{aligned}$$

In order to move on with the integration, we need the following facts:

1. $\hat{g}(-\omega) = \overline{\hat{g}(\omega)}$

$$\xi(-\omega) = \xi(\omega), \text{ since all } \omega\text{-terms in } \xi \text{ are squared,}$$

$$\theta(-\omega) = \frac{-\omega [\zeta^2 - 2\rho\zeta(\alpha + \gamma)]}{\xi(-\omega)} = -\theta(\omega),$$

$$\begin{aligned}
A(-\omega; \tau) &= \frac{i(e^{-i\tau(\xi-i\theta)} - 1) [(\xi - i\theta)^2 + (\alpha + \gamma - i\rho\zeta\omega)^2]}{\zeta^2 \{ \xi - i\theta - i(\alpha + \gamma - i\rho\zeta\omega) + e^{-i\tau(\xi-i\theta)} [\xi - i\theta + i(\alpha + \gamma - i\rho\zeta\omega)] \}} \\
&= \frac{-i(e^{i\tau(\xi-i\theta)} - 1) [(\xi - i\theta)^2 + (\alpha + \gamma - i\rho\zeta\omega)^2]}{\zeta^2 \{ \xi - i\theta + i(\alpha + \gamma - i\rho\zeta\omega) + e^{i\tau(\xi-i\theta)} [\xi - i\theta - i(\alpha + \gamma - i\rho\zeta\omega)] \}} \\
&= \overline{A(\omega; \tau)}
\end{aligned}$$

$$B(-\omega) = \frac{\alpha^2 m + \alpha\gamma m + \alpha m\theta(-\omega)}{\zeta^2} - r = B(\omega) - \frac{2\alpha m\theta(\omega)}{\zeta^2}$$

$$C(-\omega) = \frac{-\alpha m\rho\zeta\omega - \alpha m\xi(-\omega)}{\zeta^2} + r\omega = -C(\omega) - \frac{2\alpha m\xi(\omega)}{\zeta^2}$$

Combining all of these we get

$$\begin{aligned}
\hat{g}(-\omega) &= \frac{e^{-\frac{2i\alpha m\tau(\xi-i\theta)}{\zeta^2}} e^{A(-\omega)y+\tau[B(\omega)-iC(\omega)]} [2(\xi-i\theta)]^{\frac{2\alpha m}{\zeta^2}}}{\{\xi-i\theta-i(\alpha+\gamma-i\rho\zeta\omega) + e^{-i\tau(\xi-i\theta)}[\xi-i\theta+i(\alpha+\gamma-i\rho\zeta\omega)]\}^{\frac{2\alpha m}{\zeta^2}}} \\
&= \frac{e^{\overline{A(\omega)}y+\tau[B(\omega)-iC(\omega)]} [2(\xi-i\theta)]^{\frac{2\alpha m}{\zeta^2}}}{\{\xi-i\theta+i(\alpha+\gamma-i\rho\zeta\omega) + e^{i\tau(\xi-i\theta)}[\xi-i\theta-i(\alpha+\gamma-i\rho\zeta\omega)]\}^{\frac{2\alpha m}{\zeta^2}}} \\
&= \overline{\hat{g}(\omega)}
\end{aligned}$$

2. $\hat{g}(-\omega+i) = \overline{\hat{g}(\omega+i)}$

Using the fact that complex conjugate of a composition of functions is a composition of the complex conjugate of the functions and using the previous fact, we get

$$\overline{\hat{g}(\omega+i)} = \overline{\hat{g}(\omega+i)} = \hat{g}(\omega-i) = \hat{g}(-\omega+i)$$

3. Now it is clear that

$$\frac{\overline{e^{-i\omega(x-\ln K)}g(\omega)}}{-i\omega} = \frac{e^{i\omega(x-\ln K)}g(-\omega)}{i\omega},$$

and

$$\frac{\overline{e^{-i\omega(x-\ln K)}g(\omega+i)}}{-i\omega} = \frac{e^{i\omega(x-\ln K)}g(-\omega+i)}{i\omega}$$

4. If the characteristic function $\phi(t)$ is known, Shephard [16] gives a formula to compute the distribution function $F(x)$ by using an inversion theorem. The result is

$$F(x) = \frac{1}{2} - \frac{1}{2\pi} \int_0^\infty \frac{\phi(t)e^{-itx} - \phi(-t)e^{itx}}{it} dt$$

Using facts 3. and 4. we can continue the computation of P_1 and P_2

$$\begin{aligned}
P_1(\tau, x, y) &= \frac{1}{2} - \lim_{\chi \rightarrow -\infty} G(\chi) + \frac{1}{2\pi} \int_0^\infty 2\Re \left[\frac{e^{-i\omega(x-\ln K)}\hat{g}(\omega+i)}{-i\omega} \right] d\omega \\
&= \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \Re \left[\frac{e^{-i\omega(x-\ln K)}\hat{g}(\omega+i)}{-i\omega} \right] d\omega,
\end{aligned}$$

$$\begin{aligned}
P_2(\tau, x, y) &= \frac{1}{2} - \lim_{\chi \rightarrow -\infty} G(\chi) + \frac{1}{2\pi} \int_0^\infty 2\Re \left[\frac{e^{-i\omega(x-\ln K)} \hat{g}(\omega)}{-i\omega} \right] d\omega \\
&= \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \Re \left[\frac{e^{-i\omega(x-\ln K)} \hat{g}(\omega)}{-i\omega} \right] d\omega,
\end{aligned}$$

where $G(x)$ is the cumulative distribution function that corresponds to the characteristic function $\hat{g}(\omega)$. ■

Note: The price of the put option can be calculated using the put-call parity (2.2.12).

When $\tau = 0$, the integrals in the expressions of P_1 and P_2 in the theorem above are

$$\int_0^\infty \Re \left[\frac{e^{-i\omega(x-\ln K)}}{-i\omega} \right] d\omega = \mathbf{sgn}(x - \ln K) \frac{\pi}{2},$$

where $\mathbf{sgn}(x)$ is the signum function defined as follows

$$\mathbf{sgn}(x) = \begin{cases} 1 & , x > 0 \\ 0 & , x = 0 \\ -1 & , x < 0. \end{cases}$$

This implies

$$P_1(0, x, y) = P_2(0, x, y) = \begin{cases} 1 & , e^x - K > 0 \\ \frac{1}{2} & , e^x = K \\ 0 & , e^x - K < 0. \end{cases}$$

which confirms that at maturity the price of a European call option is exactly the payoff, $\max(e^x - K, 0)$.

The problem is well-posed and we believe the solution derived above is unique, but we haven't shown its uniqueness.

The integrals (4.3.28) and (4.3.29) in the theorem above require numerical integration. The discussion forums about the Heston's model on "Wilmott Forums" indicate that programmers are in favor of the Fast Fourier Transform algorithm or the Simpson's rule when it comes to evaluating these integrals.

Heston arrives at the same option pricing formula as we do, but using different solution technique. He guesses the functional form of the characteristic function to be

$$f(\tau, \omega, y) = e^{C_1(\tau, \omega) + D_1(\tau, \omega)y + i\omega x},$$

and then he solves for $C_1(\tau, \omega)$ and $D_1(\tau, \omega)$. He states that this guess exploits the linearity of the coefficients in the pricing PDE. The advantage of our solution technique is that it doesn't require any guessing, which means that other volatility functions and/or different boundary conditions can be used. Our solution approach includes Heston's solution as a particular case.

5 CLOSED FORM SOLUTION OF A JUMP DIFFUSION MODEL FOR EUROPEAN OPTIONS

To determine the option price on a stock that follows a diffusion process with jump-diffusion volatility described by SDEs (3.1.1) and (3.1.2) we use an approach similar to the pure diffusion case.

5.1 Integral Transforms of the Jump-Diffusion Pricing Model

In Section 3.1 the jump-diffusion pricing model for European options was derived:

$$\begin{aligned} \frac{\partial P}{\partial \tau} = & \left(r - \frac{1}{2}y \right) \frac{\partial P}{\partial x} + [\alpha(m - y) - \gamma y] \frac{\partial P}{\partial y} + \frac{1}{2}y \frac{\partial^2 P}{\partial x^2} + \rho \zeta y \frac{\partial^2 P}{\partial x \partial y} + \frac{1}{2} \zeta^2 y \frac{\partial^2 P}{\partial y^2} - rP \\ & + \lambda \int_{-\infty}^{\infty} [P(\tau, x, y + J) - P(\tau, x, y)] \phi(J) dJ. \end{aligned} \quad (5.1.1)$$

In this section we want to solve this PIDE in the region $-\infty < x < \infty$, $0 < y < \infty$ and $0 < \tau < T$, under the initial condition $P(\tau, x, y) = h(x)$, where $h(x)$ is the payoff function.

Assume that $P(\tau, x, y)$ is measurable in (τ, x, y) , absolutely integrable with respect to x , and that the function $P(\tau, x, y)$ is of exponential order with respect to y . We have calculated the Fourier transform with respect to x of all parts but the integral of PIDE (5.1.1) in the previous section. Using those results and

$$\begin{aligned}
& \mathcal{F} \left[\int_{-\infty}^{\infty} [P(\tau, x, y + J) - P(\tau, x, y)] \phi(J) dJ \right] \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\omega x} [P(\tau, x, y + J) - P(\tau, x, y)] \phi(J) dJ dx \\
&= \int_{-\infty}^{\infty} [\hat{P}(\tau, \omega, y + J) - \hat{P}(\tau, \omega, y)] \phi(J) dJ,
\end{aligned}$$

we obtain

$$\begin{aligned}
\frac{\partial \hat{P}}{\partial \tau} &= \frac{1}{2} \zeta^2 y \frac{\partial^2 \hat{P}}{\partial y^2} + [\alpha m - (\alpha + \gamma + i\rho\omega\zeta) y] \frac{\partial \hat{P}}{\partial y} - [r + i\omega r + \frac{1}{2} (\omega^2 - i\omega) y] \hat{P} \\
&+ \lambda \int_{-\infty}^{\infty} [\hat{P}(\tau, \omega, y + J) - \hat{P}(\tau, \omega, y)] \phi(J) dJ.
\end{aligned} \tag{5.1.2}$$

As in the previous section, we want to take Laplace transform of PIDE (5.1.2) with respect to y . We can use the previously calculated Laplace transforms of most of the terms in this equation except for the integral terms. The Laplace transform of the second integral is straight forward

$$\begin{aligned}
\mathcal{L} \left[\int_{-\infty}^{\infty} \hat{P}(\tau, \omega, y) \phi(J) dJ \right] &= \int_0^{\infty} \int_{-\infty}^{\infty} e^{-\psi y} \hat{P}(\tau, \omega, y) \phi(J) dJ d\psi \\
&= \tilde{P}(\tau, \omega, \psi) \int_{-\infty}^{\infty} \phi(J) dJ = \tilde{P}(\tau, \omega, \psi),
\end{aligned}$$

since $\phi(J)$ is the probability density function of J and thus $\int_{-\infty}^{\infty} \phi(J) dJ = 1$.

Setting $P(\tau, x, y) = 0$ for all $y \leq 0$ and $z = y + J$ in the first integral, its Laplace transform becomes easy to calculate:

$$\begin{aligned}
\mathcal{L} \left[\int_{-\infty}^{\infty} \hat{P}(\tau, \omega, y + J) \phi(J) dJ \right] &= \int_J^{\infty} \int_{-\infty}^{\infty} e^{-\psi(z-J)} \hat{P}(\tau, \omega, z) \phi(J) dJ d\psi \\
&= \int_{-\infty}^{\infty} e^{\psi J} \phi(J) \left(\int_0^{\infty} e^{-\psi z} \hat{P}(\tau, \omega, z) d\psi \right) dJ, \text{ since } \hat{P}(\tau, \omega, y) = 0, \forall y \leq 0 \\
&= \tilde{P}(\tau, \omega, \psi) \int_{-\infty}^{\infty} e^{\psi J} \phi(J) dJ.
\end{aligned}$$

The last integral is nothing else but the Moment Generating function of J . Let it be

denoted by $M(\psi)$. Now, the transformed equation becomes first order PDE

$$\begin{aligned} \frac{\partial \tilde{P}}{\partial \tau} + \left[\frac{1}{2} \zeta^2 \psi^2 - (\alpha + \gamma + i\omega\rho\zeta)\psi - \frac{1}{2}(\omega^2 - i\omega) \right] \frac{\partial \tilde{P}}{\partial \psi} \\ = [\lambda M(\psi) + (\alpha m - \zeta^2)\psi + (\alpha + \gamma + i\omega\rho\zeta - r - i\omega r - \lambda)] \tilde{P}. \end{aligned} \quad (5.1.3)$$

5.2 Solution of the transformed Jump-Diffusion Pricing PIDE

The difference between the PDE of the pure diffusion model (4.2.8) and the PDE of the jump-diffusion model (5.1.3) is in the expression in the right hand side, only, so when solving PDE (5.1.3) we take the same approach as when solving PDE (4.2.8). This is demonstrated in the following Proposition.

Proposition 5.2.1 *The solution of PDE (5.1.3) with initial condition*

$$\tilde{P}(0, \omega, \psi) = \frac{\hat{h}(\omega)}{\psi}$$

is given by

$$\begin{aligned} \tilde{P}(\tau, \omega, \psi) = \frac{\hat{h}(\omega) [2(\xi + i\theta)]^{2(1-\alpha m)} e^{\tau[\alpha^2 m + \alpha\gamma m - r - \theta(\alpha m - \zeta^2) + i(\alpha m\rho\omega - r\omega + \xi(\alpha m - \zeta^2))]} \\ [\xi + i\theta - i(\alpha + \gamma + i\rho\zeta\omega) + e^{-i\tau(\xi + i\theta)} (\xi + i\theta + i(\alpha + \gamma + i\rho\zeta\omega))] \times \\ \frac{f(\psi; \tau, \omega)}{\psi - \psi_0} \cdot e^{\frac{-i\lambda}{\xi + i\theta}(I_1 - I_2 - I_3 + I_4)}, \end{aligned} \quad (5.2.4)$$

where ψ_0 , $f(\psi; \tau, \omega)$, $\xi = \xi(\omega)$ and $\theta = \theta(\omega)$ are defined as in Proposition 4.2.2, and

$$\begin{aligned} I_{1,2}(\tau, \omega, \psi) &= \int_{-\infty}^{\infty} e^{\frac{iJ[(\xi + i\theta) - i(\alpha + \gamma + i\rho\zeta\omega)]}{\zeta^2}} \mathbf{E}_1(a_{1,2}) \phi(J) dJ, \text{ and} \\ I_{3,4}(\tau, \omega, \psi) &= \int_{-\infty}^{\infty} e^{\frac{-iJ[(\xi + i\theta) + i(\alpha + \gamma + i\rho\zeta\omega)]}{\zeta^2}} \mathbf{E}_1(a_{3,4}) \phi(J) dJ. \end{aligned}$$

The function \mathbf{E}_1 is the Exponential integral, defined by

$$\mathbf{E}_1(z) = \int_z^{\infty} \frac{e^{-u}}{u} du, \quad |\arg(z)| < \pi,$$

and the variables a_1, a_2, a_3 and a_4 are defined by

$$\begin{aligned}
a_1 &= \frac{2iJ(\xi + i\theta)[\xi + i\theta - i(\alpha + \gamma + i\rho\zeta\omega - \zeta^2\psi)]e^{-i\tau(\xi+i\theta)}}{\zeta^2 \{ \xi + i\theta + i(\alpha + \gamma + i\rho\zeta\omega - \zeta^2\psi) + [\xi + i\theta - i(\alpha + \gamma + i\rho\zeta\omega - \zeta^2\psi)]e^{-i\tau(\xi+i\theta)} \}}, \\
a_2 &= \frac{iJ}{\zeta^2} [(\xi + i\theta) - i(\alpha + \gamma + i\rho\zeta\omega - \zeta^2\psi)], \\
a_3 &= \frac{-2iJ(\xi + i\theta)[\xi + i\theta + i(\alpha + \gamma + i\rho\zeta\omega - \zeta^2\psi)]}{\zeta^2 \{ \xi + i\theta + i(\alpha + \gamma + i\rho\zeta\omega - \zeta^2\psi) + [\xi + i\theta - i(\alpha + \gamma + i\rho\zeta\omega - \zeta^2\psi)]e^{-i\tau(\xi+i\theta)} \}}, \\
a_4 &= -\frac{iJ}{\zeta^2} [(\xi + i\theta) + i(\alpha + \gamma + i\rho\zeta\omega - \zeta^2\psi)].
\end{aligned}$$

Proof. We use the η -transformation calculated in (4.2.14)

$$\eta = \frac{i}{\xi + i\theta} \ln \frac{\xi + i\theta - i(\zeta^2\psi - \alpha - \gamma - i\omega\rho\zeta)}{\xi + i\theta + i(\zeta^2\psi - \alpha - \gamma - i\omega\rho\zeta)}.$$

Then, equation (5.1.3) can be rewritten as

$$\begin{aligned}
\frac{\partial \tilde{P}}{\partial \tau} + \frac{\partial \tilde{P}}{\partial \eta} &= \left\{ \frac{i(\alpha m - \zeta^2)(\xi + i\theta)(1 - e^{i\eta(\xi+i\theta)})}{\zeta^2(1 + e^{i\eta(\xi+i\theta)})} + \frac{\alpha m}{\zeta^2}(\alpha + \gamma + i\rho\zeta\omega) \right. \\
&\quad \left. - r - i\omega r - \lambda + \lambda M \left(\frac{i(\xi + i\theta)(1 - e^{i\eta(\xi+i\theta)})}{\zeta^2(1 + e^{i\eta(\xi+i\theta)})} + \frac{\alpha + \gamma + i\rho\zeta\omega}{\zeta^2} \right) \right\} \tilde{P}. \quad (5.2.5)
\end{aligned}$$

By proposition (4.2.1) the solution of this equation can be calculated as

$$\tilde{P}(\tau, \omega, \eta) = \exp \left(\int_0^\tau RHS(\eta - \tau + t) dt \right) \Phi(\eta - \tau),$$

where RHS is the function on the right hand side of PDE (5.2.5) without \tilde{P} and Φ is an arbitrary function that can be determined using the initial condition. Since when $\tau = 0$, $\exp \left(\int_0^\tau RHS(\eta - \tau + t) dt \right) = 1$, for $\Phi(\eta)$ we obtain

$$\Phi(\eta) = \frac{\hat{h}(\omega)\zeta^2(1 + e^{i\eta(\xi+i\theta)})}{i(\xi + i\theta)(1 - e^{i\eta(\xi+i\theta)}) + (\alpha + \gamma + i\rho\zeta\omega)(1 + e^{i\eta(\xi+i\theta)})}.$$

For the integral term we have

$$\begin{aligned} \int_0^\tau RHS(\eta - \tau + t) dt &= \left[\frac{\alpha m(\alpha + \gamma + i\omega\rho\zeta) + i(\alpha m - \zeta^2)(\xi + i\theta)}{\zeta^2} - r - ir\omega - \lambda \right] \tau \\ &\quad + \ln \left(\frac{1 + e^{i(\eta-\tau)(\xi+i\theta)}}{1 + e^{i\eta(\xi+i\theta)}} \right)^{\frac{2(\alpha m - \zeta^2)}{\zeta^2}} \\ &\quad + \lambda \int_0^\tau M \left(\frac{i(\xi + i\theta)(1 - e^{i(\eta-\tau+t)(\xi+i\theta)})}{\zeta^2(1 + e^{i(\eta-\tau+t)(\xi+i\theta)})} + \frac{\alpha + \gamma + i\rho\zeta\omega}{\zeta^2} \right) dt, \end{aligned}$$

thus, the solution of equation (5.2.5) is given by

$$\begin{aligned} \tilde{P}(\tau, \omega, \eta) &= \frac{\hat{h}(\omega)\zeta^2 e^{\tau \left[\frac{\alpha^2 m + \alpha\gamma m - \theta(\alpha m - \zeta^2)}{\zeta^2} - r - \lambda + i \left(\frac{\alpha m \rho \omega + \xi(\alpha m - \zeta^2)}{\zeta^2} - r\omega \right) \right]}{i(\xi + i\theta)(1 - e^{i(\eta-\tau)(\xi+i\theta)}) + (\alpha + \gamma + i\rho\zeta\omega)(1 + e^{i(\eta-\tau)(\xi+i\theta)})} \\ &\quad \times \left(\frac{1 + e^{i(\eta-\tau)(\xi+i\theta)}}{1 + e^{i\eta(\xi+i\theta)}} \right)^{\frac{2(\alpha m - \zeta^2)}{\zeta^2}} e^{\lambda \int_0^\tau M \left(\frac{i(\xi+i\theta)(1 - e^{i(\eta-\tau+t)(\xi+i\theta)})}{\zeta^2(1 + e^{i(\eta-\tau+t)(\xi+i\theta)})} + \frac{\alpha + \gamma + i\rho\zeta\omega}{\zeta^2} \right) dt}. \end{aligned}$$

Using the definition of the function M , for the integral above we have

$$\begin{aligned} &\int_0^\tau M \left(\frac{i(\xi + i\theta)}{\zeta^2} \frac{1 - e^{i(\xi+i\theta)(\eta-\tau+t)}}{1 + e^{i(\xi+i\theta)(\eta-\tau+t)}} + \frac{\alpha + \gamma + i\rho\zeta\omega}{\zeta^2} \right) dt \\ &= \int_{-\infty}^\infty \phi(J) e^{\frac{J(\alpha + \gamma + i\rho\zeta\omega)}{\zeta^2}} \int_0^\tau e^{\frac{iJ(\xi+i\theta)}{\zeta^2} \frac{1 - e^{i(\xi+i\theta)(\eta-\tau+t)}}{1 + e^{i(\xi+i\theta)(\eta-\tau+t)}}} dt dJ \\ &= \int_{-\infty}^\infty e^{\frac{J(\alpha + \gamma + i\rho\zeta\omega)}{\zeta^2}} \phi(J) \int_{lb(J)}^{ub(J)} \frac{2\zeta^2 J e^z}{(\zeta^2 z + iJ(\xi + i\theta))(\zeta^2 z - iJ(\xi + i\theta))} dz dJ, \quad (5.2.6) \end{aligned}$$

where

$$ub(J) = \frac{iJ(\xi + i\theta)}{\zeta^2} \frac{1 - e^{i\eta(\xi+i\theta)}}{1 + e^{i\eta(\xi+i\theta)}}, \quad \text{and} \quad lb(J) = \frac{iJ(\xi + i\theta)}{\zeta^2} \frac{1 - e^{i(\xi+i\theta)(\eta-\tau)}}{1 + e^{i(\xi+i\theta)(\eta-\tau)}}. \quad (5.2.7)$$

The last integral is obtained by substituting the whole expression in the exponent by

$$z = \frac{iJ(\xi + i\theta)}{\zeta^2} \frac{1 - e^{i(\xi+i\theta)(\eta-\tau+t)}}{1 + e^{i(\xi+i\theta)(\eta-\tau+t)}},$$

and calculating dt to be

$$dt = \frac{-2\zeta^2 J}{(iJ(\xi + i\theta) + \zeta^2 z)(iJ(\xi + i\theta) - \zeta^2 z)} dz.$$

Also, for the last integral in (5.2.6) we have

$$\begin{aligned} & \int_{lb(J)}^{ub(J)} \frac{2\zeta^2 J e^z}{(\zeta^2 z + iJ(\xi + i\theta))(\zeta^2 z - iJ(\xi + i\theta))} dz \\ &= \int_{lb(J)}^{ub(J)} \frac{i\zeta^2 e^z}{(\xi + i\theta)(\zeta^2 z + iJ(\xi + i\theta))} dz - \int_{lb(J)}^{ub(J)} \frac{i\zeta^2 e^z}{(\xi + i\theta)(\zeta^2 z - iJ(\xi + i\theta))} dz \\ &= \frac{e^{\frac{iJ(\xi+i\theta)}{\zeta^2}}}{i(\xi + i\theta)} \int_{lb_1(J)}^{ub_1(J)} \frac{e^{-z}}{z} dz - \frac{e^{-\frac{iJ(\xi+i\theta)}{\zeta^2}}}{i(\xi + i\theta)} \int_{lb_2(J)}^{ub_2(J)} \frac{e^{-z}}{z} dz, \end{aligned} \quad (5.2.8)$$

where

$$\begin{aligned} ub_1(J) &= \frac{2iJ(\xi + i\theta)e^{i\eta(\xi+i\theta)}}{\zeta^2(1 + e^{i\eta(\xi+i\theta)})}, & lb_1(J) &= \frac{2iJ(\xi + i\theta)e^{i(\xi+i\theta)(\eta-\tau)}}{\zeta^2(1 + e^{i(\xi+i\theta)(\eta-\tau)})}, \\ ub_2(J) &= \frac{-2iJ(\xi + i\theta)}{\zeta^2(1 + e^{i\eta(\xi+i\theta)})}, & lb_2(J) &= \frac{-2iJ(\xi + i\theta)}{\zeta^2(1 + e^{i(\xi+i\theta)(\eta-\tau)})}. \end{aligned}$$

Using the definition of Exponential integral, the last expression in (5.2.8) can be written as

$$\begin{aligned} & \frac{e^{\frac{iJ(\xi+i\theta)}{\zeta^2}}}{i(\xi + i\theta)} \left[\mathbf{E}_1 \left(\frac{2iJ(\xi + i\theta)e^{i(\xi+i\theta)(\eta-\tau)}}{\zeta^2(1 + e^{i(\xi+i\theta)(\eta-\tau)})} \right) - \mathbf{E}_1 \left(\frac{2iJ(\xi + i\theta)e^{i\eta(\xi+i\theta)}}{\zeta^2(1 + e^{i\eta(\xi+i\theta)})} \right) \right] \\ & - \frac{e^{-\frac{iJ(\xi+i\theta)}{\zeta^2}}}{i(\xi + i\theta)} \left[\mathbf{E}_1 \left(\frac{-2iJ(\xi + i\theta)}{\zeta^2(1 + e^{i(\xi+i\theta)(\eta-\tau)})} \right) - \mathbf{E}_1 \left(\frac{-2iJ(\xi + i\theta)}{\zeta^2(1 + e^{i\eta(\xi+i\theta)})} \right) \right]. \end{aligned} \quad (5.2.9)$$

This result will give us an expression for $\tilde{P}(\tau, \omega, \eta)$,

$$\begin{aligned} \tilde{P}(\tau, \omega, \eta) &= \frac{\hat{h}(\omega)\zeta^2 e^{\tau \left[\frac{\alpha m(\alpha+\gamma-\theta)}{\zeta^2} - r - \lambda + \theta + i \left(\frac{\alpha m(\rho\zeta\omega+\xi)}{\zeta^2} - r\omega - \xi \right) \right]}{i(\xi + i\theta)(1 - e^{i(\eta-\tau)(\xi+i\theta)}) + (\alpha + \gamma + i\rho\zeta\omega)(1 + e^{i(\eta-\tau)(\xi+i\theta)})} \times \\ & \left(\frac{1 + e^{i(\eta-\tau)(\xi+i\theta)}}{1 + e^{i\eta(\xi+i\theta)}} \right)^{\frac{2(\alpha m - \zeta^2)}{\zeta^2}} e^{\frac{-i\lambda}{\xi+i\theta}(I_1 - I_2 - I_3 + I_4)}, \end{aligned} \quad (5.2.10)$$

with

$$\begin{aligned}
I_1(\tau, \omega, \eta) &= \int_{-\infty}^{\infty} e^{\frac{iJ[(\xi+i\theta)-i(\alpha+\gamma+i\rho\zeta\omega)]}{\zeta^2}} \mathbf{E}_1 \left(\frac{2iJ(\xi+i\theta)e^{i(\xi+i\theta)(\eta-\tau)}}{\zeta^2(1+e^{i(\xi+i\theta)(\eta-\tau)})} \right) \phi(J)dJ, \\
I_2(\tau, \omega, \eta) &= \int_{-\infty}^{\infty} e^{\frac{iJ[(\xi+i\theta)-i(\alpha+\gamma+i\rho\zeta\omega)]}{\zeta^2}} \mathbf{E}_1 \left(\frac{2iJ(\xi+i\theta)e^{i\eta(\xi+i\theta)}}{\zeta^2(1+e^{i\eta(\xi+i\theta)})} \right) \phi(J)dJ, \\
I_3(\tau, \omega, \eta) &= \int_{-\infty}^{\infty} e^{-\frac{iJ[(\xi+i\theta)+i(\alpha+\gamma+i\rho\zeta\omega)]}{\zeta^2}} \mathbf{E}_1 \left(\frac{-2iJ(\xi+i\theta)}{\zeta^2(1+e^{i(\xi+i\theta)(\eta-\tau)})} \right) \phi(J)dJ, \text{ and} \\
I_4(\tau, \omega, \eta) &= \int_{-\infty}^{\infty} e^{-\frac{iJ[(\xi+i\theta)+i(\alpha+\gamma+i\rho\zeta\omega)]}{\zeta^2}} \mathbf{E}_1 \left(\frac{-2iJ(\xi+i\theta)}{\zeta^2(1+e^{i\eta(\xi+i\theta)})} \right) \phi(J)dJ. \quad (5.2.11)
\end{aligned}$$

When (5.2.9) and (5.2.11) are expressed in terms of ψ we obtain exactly (5.2.4). ■

5.3 Inverse Integral Transforms of the Jump-Diffusion Pricing Formula

Proposition 5.2.1 gives the solution of the transformed jump-diffusion pricing model. In order to obtain the option pricing formula of the jump-diffusion model in terms of x and y we need to apply the inverse Laplace transform with respect to ψ and the inverse Fourier transform with respect to ω .

Theorem 5.3.1 *The inverse Laplace transform of $\tilde{P}(\tau, \omega, \psi)$ given by (5.2.4) is*

$$\begin{aligned}
\hat{P}(\tau, \omega, y) &= \hat{h}(\omega) e^{A(\tau, \omega)y + \tau[B(\omega) + iC(\omega)]} e^{-\frac{i\lambda}{\xi+i\theta}(I'_1 - I'_2 - I'_3 + I'_4)} \times \\
&\quad \left\{ \frac{2(\xi+i\theta)}{\xi+i\theta - i(\alpha+\gamma+i\rho\zeta\omega) + e^{-i\tau(\xi+i\theta)} [\xi+i\theta + i(\alpha+\gamma+i\rho\zeta\omega)]} \right\}^{\frac{2\alpha m}{\zeta^2}}, \quad (5.3.12)
\end{aligned}$$

where $A(\tau, \omega)$, $B(\omega)$ and $C(\omega)$ are given by:

$$\begin{aligned}
A(\tau, \omega) &= \frac{i [e^{-i\tau(\xi+i\theta)} - 1] [(\xi+i\theta)^2 + (\alpha+\gamma+i\rho\zeta\omega)^2]}{\zeta^2 \{ \xi+i\theta - i(\alpha+\gamma+i\rho\zeta\omega) + e^{-i\tau(\xi+i\theta)} [\xi+i\theta + i(\alpha+\gamma+i\rho\zeta\omega)] \}}, \\
B(\omega) &= \frac{\alpha m(\alpha+\gamma+\theta(\omega))}{\zeta^2} - r \\
C(\omega) &= \frac{\alpha m(\rho\zeta\omega - \xi(\omega))}{\zeta^2} - r\omega. \quad (5.3.13)
\end{aligned}$$

Also, I'_1, I'_2, I'_3 and I'_4 are all functions of ω and/or τ and are given by:

$$I'_{1,2} = \int_{-\infty}^{\infty} e^{\frac{iJ}{\zeta^2}[(\xi+i\theta)-i(\alpha+\gamma+i\rho\zeta\omega)]} \mathbf{E}_1(a'_{1,2})\phi(J)dJ, \text{ and}$$

$$I'_{3,4} = \int_{-\infty}^{\infty} e^{-\frac{iJ}{\zeta^2}[(\xi+i\theta)+i(\alpha+\gamma+i\rho\zeta\omega)]} \mathbf{E}_1(a'_{3,4})\phi(J)dJ,$$

with a'_1, a'_2, a'_3 and a'_4 defined as

$$a'_1(\omega) = \frac{iJ}{\zeta^2}[(\xi+i\theta)-i(\alpha+\gamma+i\rho\zeta\omega)],$$

$$a'_2(\tau, \omega) = \frac{2iJ(\xi+i\theta)[\xi+i\theta-i(\alpha+\gamma+i\rho\zeta\omega)]}{\zeta^2 \{ \xi+i\theta-i(\alpha+\gamma+i\rho\zeta\omega) + [\xi+i\theta+i(\alpha+\gamma+i\rho\zeta\omega)]e^{-i\tau(\xi+i\theta)} \}},$$

$$a'_3(\omega) = -\frac{iJ}{\zeta^2}[(\xi+i\theta)+i(\alpha+\gamma+i\rho\zeta\omega)],$$

$$a'_4(\tau, \omega) = \frac{-2iJ(\xi+i\theta)[\xi+i\theta+i(\alpha+\gamma+i\rho\zeta\omega)]e^{-i\tau(\xi+i\theta)}}{\zeta^2 \{ \xi+i\theta-i(\alpha+\gamma+i\rho\zeta\omega) + [\xi+i\theta+i(\alpha+\gamma+i\rho\zeta\omega)]e^{-i\tau(\xi+i\theta)} \}}.$$

Proof. Applying the Laplace inversion formula to (5.2.4) we obtain:

$$\hat{P}(\tau, \omega, y) = \frac{\hat{h}(\omega) [2(\xi+i\theta)]^{2-\frac{2\alpha m}{\zeta^2}} e^{\tau \left[\frac{\alpha m(\alpha+\gamma-\theta)}{\zeta^2} - r - \theta + i \left(\frac{\alpha m(\rho\zeta\omega+\xi)}{\zeta^2} - r\omega - \xi \right) \right]}}{[\xi+i\theta-i(\alpha+\gamma+i\rho\zeta\omega) + e^{-i\tau(\xi+i\theta)} (\xi+i\theta+i(\alpha+\gamma+i\rho\zeta\omega))]} \times$$

$$\int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{e^{y\psi} f(\psi; \tau, \omega) e^{\frac{-i\lambda}{\xi+i\theta}(I_1-I_2-I_3+I_4)}}{\psi - \psi_0} d\psi, \quad (5.3.14)$$

where ϵ is such that the contour of integration is to the right side of any singularities of $\frac{f(\psi; \tau, \omega) e^{\frac{-i\lambda}{\xi+i\theta}(I_1-I_2-I_3+I_4)}}{\psi - \psi_0}$ and I_1, I_2, I_3 and I_4 are functions of ψ and are defined as in Proposition 5.2.1. Using the same argument as in the pure diffusion case, the exponent $\frac{2\alpha m}{\zeta^2} - 1 \geq 0$, thus the function

$$f(\psi; \tau, \omega) = \{ i\zeta^2\psi (e^{-i\tau(\xi+i\theta)} - 1) + (\xi+i\theta) + i(\alpha+\gamma+i\rho\zeta\omega) + e^{-i\tau(\xi+i\theta)} [(\xi+i\theta) - i(\alpha+\gamma+i\rho\zeta\omega)] \}^{\frac{2\alpha m}{\zeta^2} - 1} \quad (5.3.15)$$

is analytic. The exponential function $e^{y\psi}$ is also analytic. A potential singularity point for $e^{\frac{-i\lambda}{\xi+i\theta}(I_1-I_2-I_3+I_4)}$ is the value for ψ for which the denominator in a_2 and a_4

is 0, that is

$$\psi_1 = \frac{(\xi + i\theta)(1 + e^{-i\tau(\xi+i\theta)})}{i\zeta^2(1 - e^{-i\tau(\xi+i\theta)})} - \frac{(\alpha + \gamma + i\rho\zeta\omega)}{\zeta^2},$$

however, $f(\psi_1; \tau, \omega) = 0$. Hence, we can apply the *Cauchy Integral Formula* (4.3.1) to calculate the integral in (5.3.14) with

$$\psi_0 = \frac{i [e^{-i\tau(\xi+i\theta)} - 1] [(\xi + i\theta)^2 + (\alpha + \gamma + i\rho\zeta\omega)^2]}{\zeta^2 \{ \xi + i\theta - i(\alpha + \gamma + i\rho\zeta\omega) + e^{-i\tau(\xi+i\theta)} [\xi + i\theta + i(\alpha + \gamma + i\rho\zeta\omega)] \}} \quad (5.3.16)$$

as a simple pole. This implies that

$$\begin{aligned} & \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{f(\psi; \tau, \omega) e^{y\psi} e^{\frac{-i\lambda}{\xi+i\theta}(I_1(\psi)-I_2(\psi)-I_3(\psi)+I_4(\psi))}}{\psi - \psi_0} d\psi \\ &= 2\pi i f(\psi_0; \tau, \omega) e^{y\psi_0} e^{\frac{-i\lambda}{\xi+i\theta}(I_1(\psi_0)-I_2(\psi_0)-I_3(\psi_0)+I_4(\psi_0))}, \end{aligned}$$

with

$$f(\psi_0; \tau, \omega) = \left\{ \frac{4(\xi + i\theta)^2 e^{-i\tau(\xi+i\theta)}}{\xi + i\theta - i(\alpha + \gamma + i\rho\zeta\omega) + e^{-i\tau(\xi+i\theta)} [\xi + i\theta + i(\alpha + \gamma + i\rho\zeta\omega)]} \right\}^{\frac{2\alpha m}{\zeta^2} - 1}.$$

When we plug ψ_0 's expression above in $I_1(\psi)$, $I_2(\psi)$, $I_3(\psi)$ and $I_4(\psi)$, these four will become functions of τ and ω . Denoting them by $I_j(\psi_0) = I'_j(\omega)$, for $j = 1, 3$ and $I_j(\psi_0) = I'_j(\tau, \omega)$, for $j = 2, 4$, we obtain the required result. ■

Theorem 5.3.2 *The price of a European Call option on an underlying asset that follows the diffusion process (4.0.1) driven by mean-reverting jump-diffusion volatility (3.1.2) can be calculated using the formula*

$$P(\tau, x, y) = e^x P_1(\tau, x, y) - K P_2(\tau, x, y), \quad (5.3.17)$$

where $x = \ln S_t$,

$$\begin{aligned}
P_1(\tau, x, y) &= \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \Re \left[\frac{e^{-i\omega(x-\ln K)} \hat{g}(\omega + i; \tau)}{-i\omega} \right] d\omega, \\
P_2(\tau, x, y) &= \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \Re \left[\frac{e^{-i\omega(x-\ln K)} \hat{g}(\omega; \tau, y)}{-i\omega} \right] d\omega, \\
\hat{g}(\omega; \tau, y) &= A(\tau, \omega) y + \tau [B(\omega) + iC(\omega)] e^{-\frac{i\lambda}{\xi+i\theta}(I'_1 - I'_2 - I'_3 + I'_4)} \times \\
&\quad \left\{ \frac{2(\xi + i\theta)}{\xi + i\theta - i(\alpha + \gamma + i\rho\zeta\omega) + e^{-i\tau(\xi+i\theta)} [\xi + i\theta + i(\alpha + \gamma + i\rho\zeta\omega)]} \right\}^{\frac{2\alpha m}{\zeta^2}},
\end{aligned}$$

$\xi = \xi(\omega)$ and $\theta = \theta(\omega)$ are defined as in (4.2.17), $A(\tau, \omega)$, $B(\omega)$ and $C(\omega)$ are defined as in (5.3.13) and $I'_1(\omega)$, $I'_2(\tau, \omega)$, $I'_3(\omega)$ and $I'_4(\tau, \omega)$ are defined as in Proposition 5.3.1.

Proof. The only difference between this theorem and theorem 4.3.3 is the definition of the function $\hat{g}(\omega; \tau, y)$. Thus, we can use the same proof for proving this theorem as long as we show that $\hat{g}(\omega; 0) = 1$ and that $\hat{g}(-\omega) = \overline{\hat{g}(\omega)}$.

First, note that at time $\tau = 0$

$$I'_2(0, \omega) = I'_1(\omega) \text{ and } I'_4(0, \omega) = I'_3(\omega)$$

which makes

$$e^{-\frac{i\lambda}{\xi+i\theta}(I'_1(\omega) - I'_2(0, \omega) - I'_3(\omega) + I'_4(0, \omega))} = 1.$$

and so $\hat{g}(\omega; 0) = 1$.

Next, recall that $\xi(-\omega) = \xi(\omega)$, $\theta(-\omega) = -\theta(\omega)$. Then, it is easy to show that

$$I'_1(-\omega) = \overline{I'_3(\omega)}, \overline{I'_1(\omega)} = I'_3(-\omega), I'_2(-\omega; \tau) = \overline{I'_4(\omega; \tau)} \text{ and } I'_4(-\omega; \tau) = \overline{I'_2(\omega; \tau)}.$$

Using these results as well as few results that we proved in Theorem 4.3.3,

$$A(-\omega; \tau) = \overline{A(\omega; \tau)}, B(-\omega) = B(\omega) - \frac{2\alpha m \theta(\omega)}{\zeta^2}, \text{ and } C(-\omega) = -C(\omega) - \frac{2\alpha m \xi(\omega)}{\zeta^2},$$

we obtain $\hat{g}(-\omega) = \overline{\hat{g}(\omega)}$.

The rest of the proof follows that of Theorem 4.3.3. ■

6 OPTION GREEKS

When managing a portfolio of options, both market makers and option investors are interested in assessing the change in the option price and its sensitivity when change in the price of the underlying asset, volatility and interest rate occurs. The Option Greeks are formulas that express the change in the option price when one of the parameters changes; thus they are considered to be measures of risk exposure. Mathematically, they are simply the derivative of the option price with respect to one input only while the other variables are kept constant.

6.1 Delta

Delta (Δ) is the most well known and the most important of the option greeks. It measures the change in the option price when the stock price increases/decreases by \$1, that is

$$\Delta = \frac{\partial P}{\partial S}.$$

Knowing the Delta value of the option is important for option traders. If you believe that the price of the underlying asset will go up one dollar within a few days and bought call options in order to prepare for that move, the delta of your call options will tell you exactly how much money you will make with that \$1 surge. The option delta therefore helps you plan how much call options to buy if you are planning to capture a certain profit.

We used this parameter when deriving the Black-Scholes equation in section 2.2 where we used Δ shares of stock to hedge the risk associated with the randomness in the stock price. We start the calculation of the Δ of a European call option in the

jump-diffusion model by differentiating formula (5.3.17) with respect to x , and thus we have

$$\frac{\partial P}{\partial S} = \frac{\partial P}{\partial x} \frac{\partial x}{\partial S} = P_1 + \frac{\partial P_1}{\partial x} - K e^{-x} \frac{\partial P_2}{\partial x},$$

where P_1 and P_2 are defined in Theorem 5.3.2. To calculate $\frac{\partial P_1}{\partial x}$, change the variable $\omega + i$ with ω , in which case $-i\omega$ should be replaced by $-1 - i\omega$. Then we have:

$$P_1(\tau, x, y) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \Re \left[\frac{-e^{-(x-\ln K)} e^{-i\omega(x-\ln K)} \hat{g}(\omega; \tau, y)}{1 + i\omega} \right] d\omega,$$

and

$$\begin{aligned} \frac{\partial P_1}{\partial x} &= \frac{K}{\pi e^x} \int_0^\infty \Re \left[\frac{e^{-i\omega(x-\ln K)} \hat{g}(\omega; \tau, y)}{1 + i\omega} \right] d\omega + \frac{K}{\pi e^x} \int_0^\infty \Re \left[\frac{i\omega e^{-i\omega(x-\ln K)} \hat{g}(\omega; \tau, y)}{1 + i\omega} \right] d\omega \\ &= \frac{K}{\pi e^x} \int_0^\infty \Re [e^{-i\omega(x-\ln K)} \hat{g}(\omega; \tau, y)] d\omega. \end{aligned}$$

The calculation of $\frac{\partial P_2}{\partial x}$ doesn't require any substitution and, similar to the previous derivative, we obtain

$$\frac{\partial P_2}{\partial x} = \frac{1}{\pi} \int_0^\infty \Re [e^{-i\omega(x-\ln K)} \hat{g}(\omega; \tau, y)] d\omega.$$

Thus,

$$\frac{\partial P_1}{\partial x} = \frac{K}{e^x} \frac{\partial P_2}{\partial x},$$

which proves the following result.

Proposition 6.1.1 *The Delta of a European call option in a jump-diffusion volatility setting is*

$$\Delta = \frac{\partial P}{\partial S} = P_1, \tag{6.1.1}$$

where P_1 and its components are defined in Theorem 5.3.2.

Considering the fact that P_1 is a probability function we have that the Delta for a European call is between 0 and 1, which confirms two very logical things from a financial perspective - the price of the call increases as the stock price increases and the change in the call price can not be greater than the change in the stock price.

Using the put-call parity, the Delta of the European put is

$$\Delta_P = \Delta_C - 1, \quad (6.1.2)$$

and hence, Δ_P is always negative. This means that the price of the put decreases as the price of the underlying asset increases, and that the change in the price of the put is less than the change of the stock price.

6.2 Gamma

Gamma measures the change in Delta as the stock price changes, thus it is defined as the derivative of Δ with respect to S . This parameter is important because it shows how fast our Delta position changes in relation to the price of the underlying asset, however, it is not normally needed for the calculation of most option trading strategies. Gamma is particularly important for Delta neutral traders who want to predict how to reset their Delta neutral positions as the price of the underlying stock changes.

The relationship between the call and the put Deltas defined in equation (6.1.2) implies that the call and put Gammas are equal. Below we demonstrate how they can be calculated in the jump-diffusion volatility case:

$$\frac{\partial^2 P}{\partial S^2} = \frac{\partial P_1}{\partial S} = \frac{\partial P_1}{\partial x} \frac{\partial x}{\partial S} = e^{-x} \frac{K}{e^x} \frac{1}{\pi} \int_0^\infty \Re [e^{-i\omega(x-\ln K)} \hat{g}(\omega; \tau, y)] d\omega.$$

Proposition 6.2.1 *The Gamma of a European call/put option in a jump-diffusion volatility setting is*

$$\Gamma_C = \Gamma_P = \frac{\partial^2 P}{\partial S^2} = \frac{K}{e^{2x}} \frac{1}{\pi} \int_0^\infty \Re [e^{-i\omega(x-\ln K)} \hat{g}(\omega; \tau, y)] d\omega, \quad (6.2.3)$$

where P_1 and its components are defined in Theorem 5.3.2.

6.3 Theta

Theta measures how fast the premium of a stock option decays with time. This parameter is measured in days and is active even on the weekends, when the markets are closed. Some option trading strategies that are particularly Theta sensitive are Calendar Call Spread and Calendar Put Spread where traders need to maintain a net positive Theta in order to ensure a profit.

Mathematically it is defined as a partial derivative of the option value with respect to the time to expiration $\tau = T - t$. It's easy to see that

$$\begin{aligned}\frac{\partial P_1}{\partial \tau} &= \frac{K}{e^x} \frac{1}{\pi} \int_0^\infty \Re \left[\frac{-e^{-i\omega(x - \ln K)} \frac{\partial \hat{g}(\omega; \tau, y)}{\partial \tau}}{1 + i\omega} \right] d\omega, \\ \frac{\partial P_2}{\partial \tau} &= \frac{1}{\pi} \int_0^\infty \Re \left[\frac{-e^{-i\omega(x - \ln K)} \frac{\partial \hat{g}(\omega; \tau, y)}{\partial \tau}}{i\omega} \right] d\omega,\end{aligned}$$

and so, for $\frac{\partial P}{\partial \tau}$ we have:

$$\begin{aligned}\frac{\partial P}{\partial \tau} &= e^x \frac{\partial P_1}{\partial \tau} - K \frac{\partial P_2}{\partial \tau} \\ &= \frac{K}{\pi} \int_0^\infty \Re \left[\frac{e^{-i\omega(x - \ln K)} \frac{\partial \hat{g}(\omega; \tau, y)}{\partial \tau}}{i\omega - \omega^2} \right] d\omega.\end{aligned}$$

To calculate $\frac{\partial \hat{g}(\omega; \tau, y)}{\partial \tau}$ we need to calculate $\frac{\partial A(\tau; \omega)}{\partial \tau}$, $\frac{\partial I'_2(\tau, \omega)}{\partial \tau}$ and $\frac{\partial I'_4(\tau, \omega)}{\partial \tau}$. The first derivative can be shown to be

$$\frac{\partial A(\tau, \omega)}{\partial \tau} = \frac{2(\xi + i\theta)^2 [(\xi + i\theta)^2 + (\alpha + \gamma + i\rho\zeta\omega)^2] e^{-i\tau(\xi + i\theta)}}{\zeta^2 \{ \xi + i\theta - i(\alpha + \gamma + i\rho\zeta\omega) + e^{-i\tau(\xi + i\theta)} [\xi + i\theta + i(\alpha + \gamma + i\rho\zeta\omega)] \}^2},$$

and for the other two we can apply the Leibniz integral rule to the exponential integrals and obtain

$$\begin{aligned}
\frac{\partial I'_2}{\partial \tau} &= \int_{-\infty}^{\infty} e^{\frac{iJ[(\xi+i\theta)-i(\alpha+\gamma+i\rho\zeta\omega)]}{\zeta^2}} \frac{\partial a'_2}{\partial \tau} \mathbf{E}_1(a'_2) \phi(J) dJ \\
&= \frac{-i(\xi+i\theta)[\xi+i\theta+i(\alpha+\gamma+i\rho\zeta\omega)]^{-i\tau(\xi+i\theta)}}{\xi+i\theta-i(\alpha+\gamma+i\rho\zeta\omega)+[\xi+i\theta+i(\alpha+\gamma+i\rho\zeta\omega)]e^{-i\tau(\xi+i\theta)}} \times \\
&\int_{-\infty}^{\infty} \exp\left(\frac{iJ[(\xi+i\theta)^2+(\alpha+\gamma+i\rho\zeta\omega)^2][e^{-i\tau(\xi+i\theta)}-1]}{\zeta^2\{\xi+i\theta-i(\alpha+\gamma+i\rho\zeta\omega)+[\xi+i\theta+i(\alpha+\gamma+i\rho\zeta\omega)]e^{-i\tau(\xi+i\theta)}\}}\right) \phi(J) dJ, \\
\frac{\partial I'_4}{\partial \tau} &= \int_{-\infty}^{\infty} \phi(J) e^{\frac{-iJ[(\xi+i\theta)+i(\alpha+\gamma+i\rho\zeta\omega)]}{\zeta^2}} \frac{\partial a'_4}{\partial \tau} \mathbf{E}_1(a'_4) dJ \\
&= \frac{i(\xi+i\theta)[\xi+i\theta-i(\alpha+\gamma+i\rho\zeta\omega)]}{\xi+i\theta-i(\alpha+\gamma+i\rho\zeta\omega)+[\xi+i\theta+i(\alpha+\gamma+i\rho\zeta\omega)]e^{-i\tau(\xi+i\theta)}} \times \\
&\int_{-\infty}^{\infty} \exp\left(\frac{iJ[(\xi+i\theta)^2+(\alpha+\gamma+i\rho\zeta\omega)^2][e^{-i\tau(\xi+i\theta)}-1]}{\zeta^2\{\xi+i\theta-i(\alpha+\gamma+i\rho\zeta\omega)+[\xi+i\theta+i(\alpha+\gamma+i\rho\zeta\omega)]e^{-i\tau(\xi+i\theta)}\}}\right) \phi(J) dJ.
\end{aligned}$$

Note that the integral function can be expressed in terms of $A(\tau, \omega)$ as $e^{JA(\tau, \omega)}$. Then for the partial derivative with respect to τ of the exponent of $\hat{g}(\omega; \tau, y)$ we have:

$$\begin{aligned}
&\frac{\partial}{\partial \tau} \left[A(\tau, \omega)y + \tau(B(\omega) + iC(\omega)) - \frac{i\lambda}{\xi+i\theta}(I'_1 - I'_2 - I'_3 + I'_4) \right] \\
&= \frac{2(\xi+i\theta)^2 [(\xi+i\theta)^2 + (\alpha+\gamma+i\rho\zeta\omega)^2] e^{-i\tau(\xi+i\theta)}}{\zeta^2 \{\xi+i\theta-i(\alpha+\gamma+i\rho\zeta\omega) + e^{-i\tau(\xi+i\theta)} [\xi+i\theta+i(\alpha+\gamma+i\rho\zeta\omega)]\}^2 y} \\
&+ B(\omega) + iC(\omega) + \lambda \int_{-\infty}^{\infty} e^{JA(\tau, \omega)} \phi(J) dJ, \tag{6.3.4}
\end{aligned}$$

and hence

$$\begin{aligned}
\frac{\partial \hat{g}}{\partial \tau} &= \hat{g}(\omega; \tau, y) \left\{ B(\omega) + iC(\omega) + \lambda \int_{-\infty}^{\infty} e^{JA(\tau, \omega)} \phi(J) dJ \right. \\
&+ \frac{2(\xi+i\theta)^2 [(\xi+i\theta)^2 + (\alpha+\gamma+i\rho\zeta\omega)^2] e^{-i\tau(\xi+i\theta)}}{\zeta^2 \{\xi+i\theta-i(\alpha+\gamma+i\rho\zeta\omega) + e^{-i\tau(\xi+i\theta)} [\xi+i\theta+i(\alpha+\gamma+i\rho\zeta\omega)]\}^2 y} \\
&\left. - \frac{2i\alpha m(\xi+i\theta) [\xi+i\theta+i(\alpha+\gamma+i\rho\zeta\omega)] e^{-i\tau(\xi+i\theta)}}{\zeta^2 \{\xi+i\theta-i(\alpha+\gamma+i\rho\zeta\omega) + [\xi+i\theta+i(\alpha+\gamma+i\rho\zeta\omega)] e^{-i\tau(\xi+i\theta)}\}} \right\}. \tag{6.3.5}
\end{aligned}$$

Proposition 6.3.1 *The Theta of a European call option in a jump-diffusion volatility*

setting is given by

$$\Theta_C = \frac{\partial P}{\partial \tau} = \frac{K}{\pi} \int_0^\infty \Re \left[\frac{e^{-i\omega(x - \ln K)} \frac{\partial \hat{g}(\omega; \tau, y)}{\partial \tau}}{i\omega - \omega^2} \right] d\omega, \quad (6.3.6)$$

where $\frac{\partial \hat{g}(\omega; \tau, y)}{\partial \tau}$ is given by expression (6.3.5).

Using the put-call parity, the Theta for European put options is found to be

$$\Theta_P = \Theta_C - rKe^{-r\tau}.$$

6.4 Vega

Though not a Greek letter, this measure falls under the "Greeks". Vega measures the sensitivity of the option price to the volatility of the underlying asset. Vega is quoted to show the theoretical price change for every *one percentage point* (0.01) change in implied volatility. An increase in volatility raises the price of both, call and put, options. One option trading strategy that is particularly Vega sensitive is Straddle (buying a call and a put with the same strike price and time to expiration). This strategy is used when the buyer believes the volatility will be pretty high, and thus the stock price will move up or down. If the stock price rises, the trader will make a profit on the call and if the stock price declines, the trader will make a profit on the purchased put.

Vega is defined as a partial derivative of the option price with respect to the volatility σ which, in the model described in this paper, is $\sigma = \sqrt{y}$. Then

$$\frac{\partial P}{\partial \sigma} = 2\sqrt{y} \frac{\partial P}{\partial y}.$$

Since

$$\frac{\partial P_1}{\partial y} = \frac{K}{e^x} \frac{1}{\pi} \int_0^\infty \Re \left[\frac{-e^{-i\omega(x-\ln K)} \frac{\partial \hat{g}(\omega; \tau, y)}{\partial y}}{1 + i\omega} \right] d\omega, \text{ and}$$

$$\frac{\partial P_2}{\partial y} = \frac{1}{\pi} \int_0^\infty \Re \left[\frac{-e^{-i\omega(x-\ln K)} \frac{\partial \hat{g}(\omega; \tau, y)}{\partial y}}{i\omega} \right] d\omega,$$

we have:

$$\frac{\partial P}{\partial y} = \frac{K}{\pi} \int_0^\infty \Re \left[\frac{e^{-i\omega(x-\ln K)} \frac{\partial \hat{g}(\omega; \tau, y)}{\partial y}}{i\omega - \omega^2} \right] d\omega.$$

It is easy to see that

$$\frac{\partial \hat{g}}{\partial y} = A(\tau, \omega) \hat{g}(\omega; \tau, y),$$

which gives us the following:

Proposition 6.4.1 *The Vega of a European call/put option in a jump-diffusion volatility setting is given by*

$$V_C = V_P = \frac{\partial P}{\partial \sigma} = \frac{2K\sqrt{y}}{\pi} \int_0^\infty \Re \left[\frac{A(\omega, \tau) e^{-i\omega(x-\ln K)} \hat{g}(\omega; \tau, y)}{i\omega - \omega^2} \right] d\omega. \quad (6.4.7)$$

Note: Because of put-call parity the Vega is the same for European call and put options with same strike price and time to expiration.

7 RESULTS AND CONCLUSION

Initially, this paper introduces the Black-Scholes models for pricing European and American options written on non-dividend and dividend paying underlying assets. The constant volatility and the lognormal distribution of the returns assumptions are argued to be invalid in comparison to real financial market data. The necessity of having randomly changing volatility is presented. Moreover, European and American option pricing models are derived that allow volatility with jump-diffusive behavior as stated and proved in Theorem 3.1.4 and Theorem 3.2.4. The European option pricing model is given by a homogeneous second-order linear partial differential equation with variable coefficients, and the American options pricing model is given by a nonhomogeneous form of the European options PDE, reduced from a free-boundary value problem.

The fast pace of the changes in financial markets requires fast derivative price computation methods. With that in mind, we seek a closed form solution for the European option pricing model. Heston arrived at a closed form solution for the pure diffusion volatility case using the method of characteristic functions by guessing their form. His guess is exploiting the linearity of the coefficients in the pricing PDE. This means that for different choices of the volatility function or different boundary conditions, this solution technique can not be used. We use a rigorous PDE approach to determine a closed form solution of the pricing PDE (3.1.14), an approach that has more flexibility to accommodate different determining conditions. First, setting the jump frequency to 0, the jump-diffusion pricing model becomes pure diffusion, identical to the one that Heston derives. Then applying a Fourier transform with respect to the logarithm of the stock price variable, x , and a Laplace transform with respect to

the volatility variable, y , the initial PDE is reduced to a first-order linear PDE. Next, a solution of this PDE is derived and finally an exact closed-form pricing formula is obtained. Using a similar approach, a closed-form solution of the jump-diffusion volatility model has also been derived. This is stated and proved in Theorem 5.3.2. Both, pure diffusion and jump-diffusion, option pricing formulas involve integrals that need numerical evaluation. The challenge in evaluating these integrals is what method will be the fastest.

Market makers and investors are not interested in determining the option price only. They are also interested in knowing the hedging parameters. The last chapter gives formulas for calculating the option Greeks that are widely used when investment strategies are made.

There are several problems that we would like to address in some future work. The first one is to release the constant risk rate assumption and use a stochastic risk rate. Another problem is to determine a solution of the nonhomogeneous PDE and so obtain a closed-form pricing formula for American options. Allowing jumps in the stock price while keeping the jumps in the volatility process and finding closed-form solution for this model is another challenge that we would like to consider. However, we need to keep in mind that even though adding more assumptions and parameters to the pricing models might show a more realistic picture of the behavior of the financial markets; this also increases the pricing complexity as well as the time required for estimation of the parameters and calculation of the price itself. Proving the uniqueness of the solutions obtained in this dissertation will follow as well.

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