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Classification of Finite Topological Quandles and Shelves via Posets

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by

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Abstract

The objective of this dissertation is to investigate finite topological quandles and topological shelves. Precisely, we give a classification of both finite topological quandles and topological shelves using the theory of posets. For quandles with more than one orbit, we prove the following Theorem.

Proposition 0.0.1. *Let X be a finite quandle with n orbits $X_1, \ldots, X_n$. Then any right continuous poset on X is n-partite with vertex sets $X_1, \ldots, X_n$.*

For connected quandles, we prove the following Theorem.

Theorem 0.0.2. *There is no T_0-topology on a finite connected quandle X that makes X into a right topological quandle.*

We used computer programming to help in exploring and analyzing various finite topological quandles and shelves. Since right multiplications in topological quandles are homeomorphisms, this implies that there is no T_0-topology on a finite connected quandle X that makes X into a right topological quandle. By ignoring this condition and investigating topological shelves in which right multiplications are *not homeomorphisms*, we were able to obtain T_0-topology on a finite connected shelves. For example we obtained the following Theorem

Proposition 0.0.3. *Let S be a shelf such that $R_{a_1} = R_{a_2} = R_{a_3} = \ldots = R_{a_m}$ and $L_{a_1}(x) = a_1, L_{a_2}(x) = a_2, \ldots, L_{a_m}(x) = a_m$, for some $a_1, \ldots, a_m \in S$ and for all $x \in S$. Then S has at least $\gamma(m)$ partial orders continuous on S, where $\gamma(m)$ is number of T_o topologies on m symbols.*

The results of this study reveal the significance of topological quandles and shelves as important mathematical structures that can be used to study a wide range of problems in topology and knot theory. One of the key findings is that there is no connected topological quandle, while there exist connected topological shelves. Additionally, an enumeration of topological quandles and shelves was provided, which has important implications for the study of these structures.

Keywords: Topological quandles, topological shelves, computer programming, topology, knot theory.

Chapter 1: Review of Knot Theory

In this chapter, we will cover some fundamental concepts of knot theory that will be necessary for understanding later chapters. Specifically, we will review knot diagrams, knot invariants, Reidemeister moves, and n-colorability. Additionally, we will examine some frequently used knots, such as the trefoil knot. In subsequent chapters, we will delve into the study of quandles, which are algebraic structures that are modeled on the three Reidemeister moves. These structures are employed to construct invariants of knots and links.

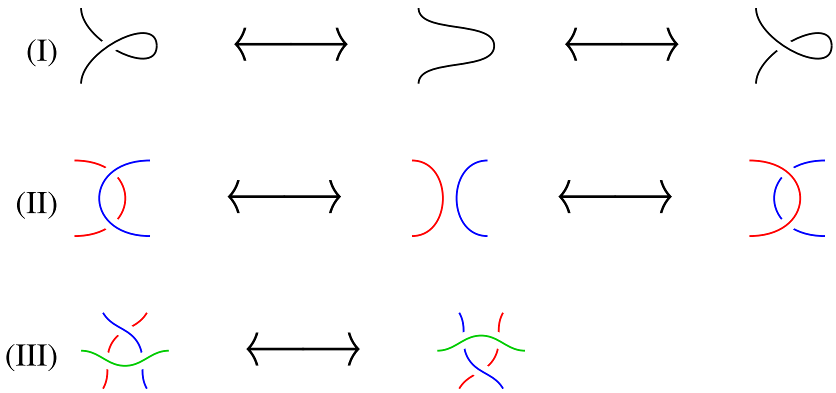
1.1 Introduction

Knot theory is the study and classifications of embeddings of $\mathbb{S}^1$, the unit circle, into the three dimensional Euclidean space $\mathbb{R}^3$ or its compactification $\mathbb{S}^3$. Such an embedding is what we refer to as a knot. A link is a collection of knots that are intertwined or linked together. More precisely, a link is a finite set of disjoint, nontrivial knots embedded in the three-dimensional Euclidean space. Knot theory has been applied, for example, in the study of how certain enzymes act on DNA molecules. One of the main topics of knot theory is the classification of knots. Two knots are considered equivalent if one can be continuously deformed to obtain the other one. This is an equivalence relation on the set of knots and thus we obtain equivalence classes of knots. An invariant of knots is a function on the set of knots such that it is constant on each equivalence class. An invariant thus can be thought of as a tool for distinguishing knots. If the value of a knot invariant is different for two knots, then the knots are not equivalent. A knot invariant can be a well-defined algebraic object such as a number, a polynomial, or a group. In other words, given two knots K_1 and K_2, we say that they are equivalent, and write $K_1 \cong K_2$, if there exists a continuous map $F : [0,1] \times \mathbb{R}^3 \to \mathbb{R}^3$ such that $F(0, K_1) = K_1, F(1, K_1) = K_2$ and $F(t,)$ is required to be a homeomorphism for all $t \in [0, 1]$. Invariants provide a good way to demonstrate the non-equivalence of certain links, since if a given invariant is different for two links, they cannot be equivalent. While invariants are useful for distinguishing non-equivalent links, a set of operations called Reidemeister moves provide a powerful method for proving that two links are equivalent. Kurt Reidemeister showed that two knots are equivalent if and only if they are connected by a finite number of Reidemeister moves and ambient isotopy of plane.

This Thesis is organized as follows: In the first chapter we will introduce knot theory and define equivalent knots. In the second chapter, we review the notion of quandles and how it is related to knots, in the third chapter we will study finite topological quandles and we will give a list of topological quandles upto order 5. Furthermore, we will prove that there is no connected finite topological quandle. In chapter 4 we study infinite topological quandles and see if we get more topological quandles, finally in chapter 5 we will study topological shelves.

1.2 Knot Diagrams

Knot diagrams are useful for studying knots because they allow us to visualize the knot and its properties. We can also use knot diagrams to study the properties of specific knots. For example, we can count the number of crossings in a knot diagram to determine the knot's crossing number, which is a measure of its complexity. We can also use the diagram to study the knot's orientation, chirality, and other properties. One important result in the theory of knot diagrams is the Reidemeister moves. In 1926, Kurt Reidemeister proved that if we have different representations (or projections) of the same knot, we can get one to look like the other using just three types of moves called RI, RII, and RIII, he did this by showing that all knot deformations can be reduced to a sequence of three types of "moves," shown below.



Type I move is just twist where the string twists over itself, Type II move is poke in which one string slides under or over another string, Type III move is one where string slides over another crossing and knot remains same. So, two diagrams are isomorphic if one can be obtained from another by applying series of Redmeister moves.

Reidemeister moves are important because they can be used if two given diagrams we want to determine whether they represent the same knot/link or not. Consider for instance the "U" shaped and the "α shaped diagrams. It is intuitively clear that they both correspond to the same embedding of the circle since it is possible to untwist the α .

Theorem 1.2.1. *Two link diagrams correspond to isotopic links if and only if one can be obtained from the other via plane isotopies and finitely many applications of Reidemeister moves.*

Note. [16] gives an informal argument for this proof. Rigorous proof was given by [3].

1.3 Knot Invariants

One of the fundamental questions in knot theory is how to distinguish knots. There are an infinite number of possible knots, so it is not feasible to simply list all the possible knots and study each one individually. Instead, knot theorists have developed various techniques to study knots, such as using diagrams to represent knots and operations like connected sum of knots and knot mutations etc.

One particularly useful tool in knot theory is the concept of knot invariants. A knot invariant is a quantity that is associated with a knot and does not change when the knot is deformed or transformed in certain ways. In other words, it is a property of the knot that is invariant under Reidemeister moves.

There are many different types of knot invariants, each with its own strengths and weaknesses. Some examples of knot invariants include the knot polynomial such as Alexander polynomial, the Jones polynomial, and the HOMFLY polynomial. These polynomials are functions that take a knot as input and output a polynomial, which can be used to distinguish one knot from another.

Overall, knot invariants are a powerful tool in knot theory that allows us to study the properties of knots in a rigorous and systematic way. They have many applications in mathematics and physics, such as in the study of DNA, fluid dynamics, and quantum field theory. By developing new and more powerful knot invariants, we can continue to deepen our understanding of knots and their properties.

Before we study knot invariant linking number let's see what is positive and negative crossing. In the diagram below the upper one is positive and the lower one is negative crossing.

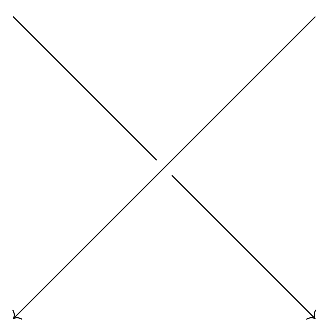


Figure 1.1: Positive crossing

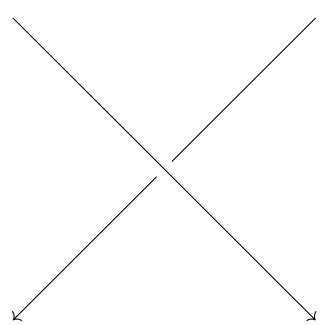


Figure 1.2: Negative crossing

The linking number of a link is an invariant we can use to distinguish between oriented links. The total number of positive crossings minus the total number of negative crossings is equal to twice the linking number. Any possible crossing can be found by rotating one of these diagrams. For example, consider the Hopf link it is a simplest non-trivial ring, and it has the linking number +1.

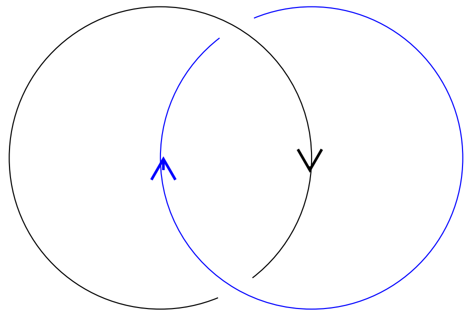


Figure 1.3: Hopf link

Proposition 1.3.1. *Linking Number is invariant of the link diagram.*

Proof. Type I move does not contribute the linking number because the crossing in question involves only one component. For Reidemeister move II, the signs of the crossings are +1 in one of them and it is -1 on the other giving contribution zero so the linking number doesn't change. For Reidemeister move III also number of positive and negative crossings are same so linking number will remain the same. □

Next, we will study tricolorability, it is one of the method to distinguish unknot from other knots Although the linking number is an interesting idea, It is defined only for links, not knots.

1.4 Fox Coloring of Knots

Definition 1.4.1. A link is tricolorable if it has a diagram, such that we can assign either 0, 1, or 2 (these numbers are called colors) to its arcs in such a way that the following conditions are satisfied:

- each arc is assigned a single color,
- at least two colors are used, and
- at each crossing, either all arcs have the same color or all of the three colors meet.

To understand tricolorability, it is useful to consider how it can be used to distinguish knots. Suppose we have a diagram of a knot K. We color the arcs by the elements $0, 1, 2$ in such a way that at each crossing, the sum of the colors of the under arcs equal twice the color of the over arc modulo 3. A coloring which uses at least two different colors is called non-trivial coloring.

If we are able to color all the components of the knot in this way, such that no two adjacent components have the same color, then the knot is said to be tricolorable. Otherwise, it is not tricolorable.

Remark: Tricolorability is invariant under Reidemeister moves.

First, consider the unknot, it is a simple loop with no crossings. It cannot be tricolored since only one color can be assigned to its one arc as shown below, hence violating requirement (1). Therefore, every tricolorable knot is nontrivial. If we can tricolor a given diagram, we can confidently say that it represents a knot other than the unknot. And since we can certainly tricolor some diagrams, that proves existence of nontrivial knots.

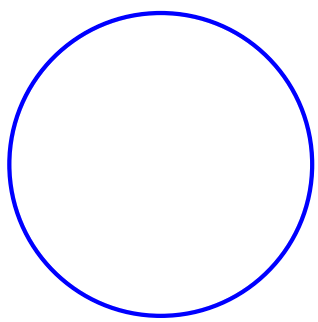


Figure 1.4: Unknot

One of the common nontrivial knot trefoil is 3-colorable as illustrated below.

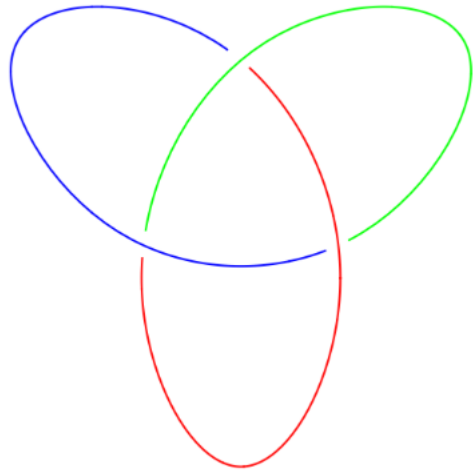


Figure 1.5: Trefoil

Let us see what the above remarks and theorems do and what are their limitations. Tricolorability can differentiate between the trefoil knot and the unknot, and between any tricolorable and any non-tricolorable knots. So, we can see trefoil is tricolorable but figure 8 is not so we can conclude both are not equivalent to each other.

However, we are not able, for example, to demonstrate that the figure-eight knot is different from the unknot using the above methods. The main difference between the unknot and the figure 8 knot is their level of complexity and topological properties. The unknot is the simplest possible knot and has no nontrivial topological features, while the figure 8 knot is a more complex knot with nontrivial topological properties that cannot be untangled into a straight line without cutting or crossing the knot.

In the forthcoming chapter, we are going to delve into a new invariant quandle and explore its unique properties. Following this, we will move on to the next chapter, where we will study topologies. By the end of these two chapters, we will have gained a deep understanding of both quandles and topologies and their interrelationships, which will equip you with the tools to analyze and solve complex problems in knot theory.

Chapter 2: Quandles and Knots

This thesis in some way, is combination of two different topics, quandles and continuity in topology. The concept of quandle is derived from Reidemeister moves. Quandles are algebraic structures that arise naturally in the study of knot theory, and they have important connections to other areas of mathematics such as group theory, algebraic topology, and category theory. Quandles have been used extensively to construct invariant of knots and links. In next chapter, we will be exploring two related topics: posets and quandles. Posets, or partially ordered sets, are sets equipped with a binary relation that reflects a partial ordering among their elements. Posets are a fundamental concept in mathematics, and they have many applications in a variety of fields.

2.1 Introduction and definitions

Definition 2.1.1. A Shelf is a non empty set with binary operation $(a, b) \to a * b$ satisfying For any $a, b, c \in X$, we have $(a * b) * c = (a * c) * (b * c)$.

Definition 2.1.2. Racks: A rack, X , is a non-empty set with a binary operation $(a, b) \to a * b$ satisfying the following conditions:

1. For any $a, b \in X$, there is a unique $c \in X$ such that $a = c * b$.

2. For any $a, b, c \in X$, we have $(a * b) * c = (a * c) * (b * c)$.

Just as groups are an algebraic structure motivated by the symmetries, quandles are algebraic structures motivated by knots. We get the term quandle from [15].

Definition 2.1.3. Quandles:

A quandle is a set X with a binary operation $*$ satisfying the following three axioms:

1. For all x in X, $x * x = x$,
2. For all $y, z \in X$, there exists a unique x such that $x * y = z$,
3. For all $x, y, z \in X$, $(x * y) * z = (x * z) * (y * z)$.

If we define a map $R_x : y \rightarrow y * x$, then for quandles, second axiom is equivalent to saying R_x is a bijective.

Remark: $L_x : y \rightarrow x * y$, might not be bijective.

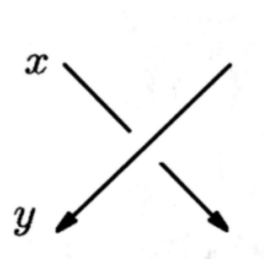
By definition a shelf is a rack and a rack is a quandle: $\{Quandles\} \subset \{Racks\} \subset \{Shelves\}$

Typical examples of quandles include the following.

1. Any non-empty set X with the operation $a * b = a$ for any $a, b \in X$ is a quandle called as a trivial quandle.
2. A conjugacy class X of a group G with the quandle operation $a * b = b^{-1}ab$. We call this a conjugation quandle.
3. Quandle defined by $a * b = 2b - a$ on an abelian group is called as Takasaki quandle.

2.2 Quandles and its relevance to Knot Theory

We will show how quandle is related with knot theorey. If we denote move as x*y illustrated below



Then all three moves represent three properties of quandles.

If we follow the quandle crossing rule after performing the I move, since a is crossing under a, the bottom arc must be labelled $a * a$. If we want Quandle to be invariant under I, we need the labels at the bottom on either side of the R1 move to match. Since a is on the left and $a * a$ is on the right, this means that $a * a = a$. Note that a was arbitrary, so this must hold for all labels. Hence we should require $a*a = a$ for all a which is first axiom of quandle definition.

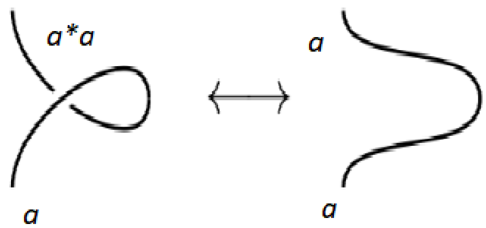


Figure 2.1: Reidemeister Move I

Similarly, the left and right sides of the II move, for any a and b we require $b = c * a$., there is a color c such that $b = c * a$ by the second axiom.

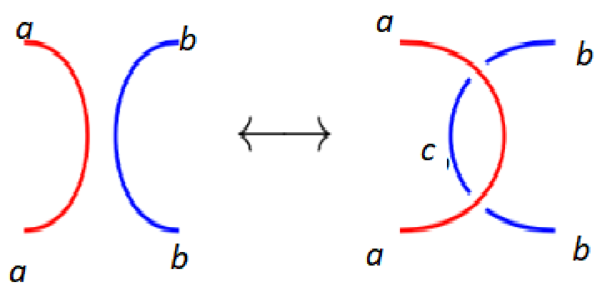


Figure 2.2: Reidemeister Move II

We require that the labels at the top and bottom match on either side of the III move. At the top we start with a, b, c from left to right. On the bottom, note that the left and middle strand labels match already. All that's left is to require $(a * b) * c = (a * c) * (c * b)$

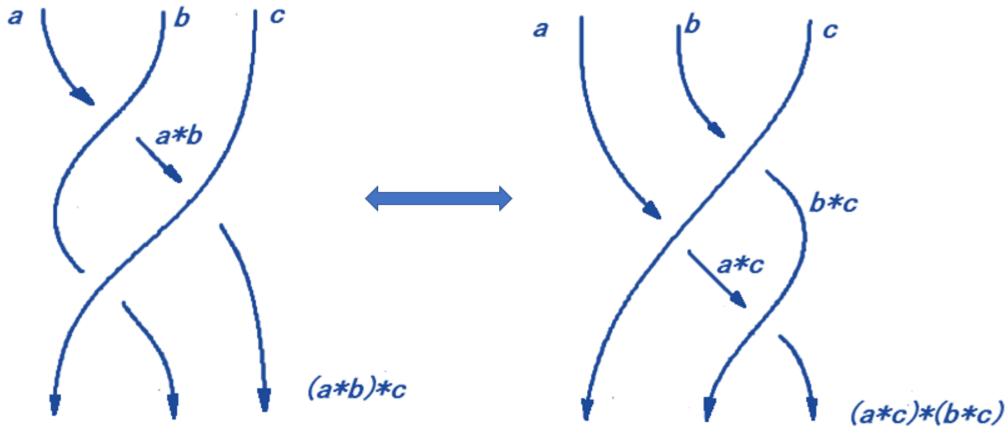
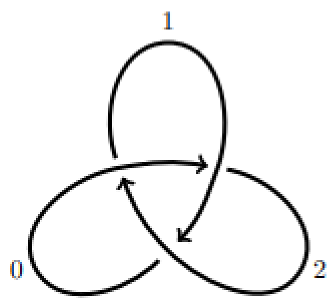


Figure 2.3: Reidemeister Move III

Quandle operation tables, called the quandle matrix, can be used to color knots as we see below for the trefoil.



*	0	1	2
0	0	2	1
1	2	1	0
2	1	0	2

Definition 2.2.1. The quandle matrix associated with a finite quandle Q with n elements, M_Q, is the $n \times n$ matrix whose (i, j)-th entry is given by $x_i * x_j$,

$$M_Q = \begin{pmatrix} x_1 * x_1 & \dots & x_1 * x_n \\ . & & \\ . & . & \\ . & . & . \\ x_n * x_1 & \dots. & x_n * x_n \end{pmatrix}$$

It is helpful to look at the operation table. To find $x * y$ in the operation table, look in the row labeled with x and the column labeled with y ;

For example, if we take $X = \mathbb{Z}_4$ with $x * y = 2y - x$ we get following table:

$*$	0	1	2	3
0	0	2	0	2
1	3	1	3	1
2	2	0	2	0
3	1	3	1	3

Definition 2.2.2. A function $\phi : (X_1, *_1) \to (X_2, *_2)$ is a quandle homomorphisim if

$$\phi(a *_1 b) = \phi(a) *_2 \phi(b)$$

for all a, b in X_1

Further if ϕ is bijective then it is called an isomorphism.

We will denote by $Aut(X)$ the automorphism group of X. The subgroup of $Aut(X)$, generated by the automorphisms R_x , is called the inner automorphism group of X and denoted by $Inn(X)$.

Definition 2.2.3. The orbit of an element x in X denoted by Orb(x) is a set of elements y in X such that there exists element $f \in Inn(X)$ that maps x to y.

Let us consider the dihedral quandle R_8 and the element $1 \in R_8$. Recall that the dihedral quandle is defined by by $x * y = 2y - x (mod\ 8)$ on the set of integers $1, 2, 3, 4, 5, 6, 7, 8$.

Under this action, we can determine the orbit of 1 as follows:

The element 1 itself belongs to its orbit. We have $2 * 1 = 3$, $3 * 1 = 2$, $4 * 1 = 8$, $5 * 1 = 7$, $6*1 = 5$, $7*1 = 6$, $8*1 = 4$, so the orbit of 1 contains the elements $1, 2, 3, 4, 5, 6, 7, 8$. Therefore,

$$Orb(1) := \{1, 2, 3, 4, 5, 6, 7, 8\}.$$

Note that in general, the orbit of an element x under the action of a group G on a quandle Q is the set of all elements $y \in Q$ such that there exists a group element $g \in G$ with $g(x) = y$.

In other words, it is the set of all elements that can be obtained from x by applying group elements to it.

2.3 Some Classes of Quandles

- A quandle X is involutory, or a kei, if, $\forall x \in X \;\; R_x$ is an involution.
- A quandle is connected if Inn(X) acts transitively on X.
- A quandle is faithful if $x \rightarrow R_x$ is an injective mapping from X to $Inn(X)$.
- A Latin quandle is a quandle such that for each $x \in X$, the left multiplication L_x by x is a bijection. That is, the multiplication table of the quandle is a Latin square.

These are just a few of the many classes of quandles that have been studied in mathematics. Each class has its own set of properties and applications, and many interesting open problems remain to be solved in the study of quandles.

Theorem 2.3.1. *A dihedral quandle R_{2n} has 2 distinct orbits and R_{2n+1} has unique orbit.*

Proof. For a dihedral quandle of odd order : $x * y = 2y - x$ implies $\forall z \in X$, there exists $y = \frac{z+x}{2}$ shows it has unique orbit.
For an even order orb(x)={$x, x+2, x+4, ..x+2n-2$}, so if we consider orbit of 0 and 1 $Orb(0)$ will be $\{0, 2, 4, ...2n-2\}$ and $orb(1) = \{1, 3, 5, ...2n-1\}$. □

Remark 2.3.1. If Q is a quandle then any permutation on Q generates a isomorphic quandle.

Example 2.3.2. If $M = [[0, 0, 0, 1], [1, 1, 1, 2], [2, 2, 2, 0], [3, 3, 3, 3]]$. then by appling permutation (3,2) that is transposition we get $[[0, 0, 1, 0], [1, 1, 3, 1], [2, 2, 2, 2], [3, 3, 0, 3]]$ and by appling transposition (1,2) we get quandle $[[0, 0, 0, 2], [1, 1, 1, 0], [2, 2, 2, 1], [3, 3, 3, 3]]$.

In the fourth chapter of our work, we will introduce the concept of a topological quandle, which is a mathematical structure that is closely related to knot theory. This structure provides a way to study the properties of knots and links in a topological setting, which is essential for understanding their behavior and characteristics.

To investigate the properties of topological quandles, we will use computer calculations to test all possible quandles that can be formed by applying permutations on a set of n elements.

Chapter 3: Posets and T_0-topology

Here we will introduce some relevant definitions. We will start with partial and total order and in section 3.3 we will review T_0-topology. One important result that we will explore in this chapter is the equivalence between posets and T_0-topologies. A T_0-topology is a type of topological space where distinct points have distinct neighborhoods. We will show how every poset can be associated with a T_0-topology, and conversely, how every T_0-topology can be viewed as a poset. We will end this chapter by giving enumeration of distinct topologies and distinct T_0-topologies up to order 6.

3.1 Partial and Total orders

Definition 3.1.1. A poset is a pair $P = (A, \leq)$ where A is a set and $\leq$ is a relation such that:

- $(A, \leq)$ is reflexive: $x \leq x$ for all $x \in A$.
- $(A, \leq)$ is antisymmetric: if $x \leq y$ and $y \leq x$ then $x = y$.
- $(A, \leq)$ is transitive: if $x \leq y$ and $y \leq z$, then $x \leq z$.

Examples:

1. $P = \{1, 2.., \}$, $a \leq b$ where $\leq$ denotes the usual inequality on positive integers.
2. $P = \{A_1, A_2, ...\}$ where A_j are sets and $\leq = \subseteq$

Given a poset $P = (A, B)$, $(x, y) \in B$ will often be denoted as $x \leq y$. In this way a poset gives a comparison of elements by $\leq$ on A that behaves in the way one expects. The only potential exception is that it is possible for $x, y \in A$ to be incomparable, which means that neither $x \leq y$ nor $y \leq x$. One says that x covers y in P if $x \neq y$ and $x \leq z \leq y$ implies that either $z = x$ or

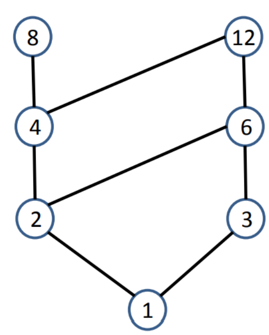
$z = y$.

Definition 3.1.2. A poset $(X, \leq)$ is connected if for all $x, y \in X$, there exists sequence of elements $x = x_1, x_2, \ldots, x_n = y$ such that every two consecutive elements x_i and x_{i+1} are comparable (meaning $x_i < x_{i+1}$ or $x_{i+1} < x_i$).

Definition 3.1.3. An element a is an immediate predecessor of b in P if $(a, b) \in P$ and a is a maximal such element which means there does not exist an element $c \neq a$ with $(a, c) \in P$ and $(c, b) \in P$.

The Hasse diagram of a poset is a pictorial way of looking at a poset. By the Hasse diagram of a (finite) partially ordered set $(A, <)$ we mean a directed graph with set of nodes A, the edges of which are all pairs (x, y) such that x is an immediate predecessor of y. Hasse diagrams are usually drawn with their edges directed upwards.

Consider the poset on $\{1, 2, 3, 4, 6, 8, 12\}$ with divisibility operation. then $1 < 2, 2 < 4$ since 2 divides 4, 2 $<$ 6 since 2 divides 6 and To draw a hasse diagram we will draw edge between 1 and 2 since $1 < 2$.



Definition 3.1.4. Total Order on a set: A binary relation R on a set A is a total order on A iff $\forall x, y \in A$, either xRy or yRx.

Definition 3.1.5. A preorder on a set X is a reflexive and transitive relation $\leq$; thus $x \leq x$ and if $x \leq y$ and $y \leq z$, then $x \leq z$. A preorder is a partial order if it is antisymmetric, which means that $x \leq y$ and$y \leq x$ implies $x = y$.

Example 3.1.6. Consider $(\mathbb{Z}, <)$ defined as $a < b$ if a divides b then it is reflextive, transitive but not antisymmetric since 1 divides -1 and -1 divides 1.

It is an example of infinite preorder which is not a poset,

Example 3.1.7. Relation $\leq$ is a total order on the set of real numbers $\mathbb{R}$

Example 3.1.8. $R = \{(a, a), (b, b), (c, c), (d, d)\}$ is trivially, a partial order on $S = \{a, b, c, d\}$. But it is not the case that it is a total order, since we do not have that for every pair of elements in S, (a, b) or $(b, a) \in R$.

Definition 3.1.9. A *directed* graph G is a pair (V, E) where V is the set of vertices and E is a list of directed line segments called edges between pairs of vertices.

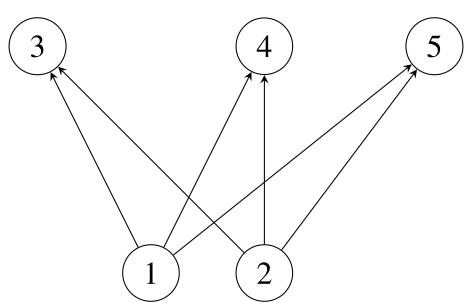
An edge from a vertex x to a vertex y will be denoted symbolically by $x < y$ and we will say that x and y are *adjacent*.

Definition 3.1.10. An *independent set* in a graph is a set of pairwise non-adjacent vertices.

Definition 3.1.11. A (directed) graph $G = (V, E)$ is called biparatite if V is the union of two disjoint independent sets V_1 and V_2.

Definition 3.1.12. A (directed) graph G is called *complete biparatite* if G is bipartite and for every $v_1 \in V_1$ and $v_2 \in V_2$ there is an edges in G that joins v_1 and v_2.

Example 3.1.13. Let $V = V_1 \cup V_2$ where $V_1 = \{1, 2\}$ and $V_2 = \{3, 4, 5\}$. Then the directed graph $G = (V, E)$ is complete biparatite graph.



As the union of an empty family of subsets of a set X is the empty subset, the empty subset is open with respect to any topology on X. As the intersection of an empty family of subsets of X is the whole set X, the whole set X is open with respect to any topology on X.

Definition 3.1.14. Let X be a finite space. A basis for a topology on X is a collection $\mathcal{B}$ of subsets of X (called basis elements) such that

- For each $x \in X$, there is at least one basis element $B \in \mathcal{B}$ such that $x \in B$.
- If $x \in B_1 \cap B_2$ where $B_1, B_2 \in \mathcal{B}$ then there is $B_3 \in \mathcal{B}$ such that $x \in B_3$ and $B_3 \subset B_1 \cap B_2$.

For $x \in X$, define U_x to be the intersection of the open sets that contain x. Then U_x is an open set and the set of open sets U_x is a basis for X. Infact, it is the unique minimal basis of X

Definition 3.1.15. A *continuous map* $f\colon X \to Y$ *from a topological space* X *to a topological space* Y is a mapping $f\colon X \to Y$ such that for every open set $U \in Y$, $f^{-1}(V)$ is open in Y.

Example 3.1.16. If X and Y are two topological spaces, the two projection maps $\phi_1\colon X \times Y \to X, (x, y) \mapsto x$ and $\phi_2\colon X \times Y \to Y, (x, y) \mapsto y$ are continuous maps.

3.2 Hausdorff Spaces, T_1 Spaces and T_0 Spaces

Definition 3.2.1. A *Hausdorff space* X is a topological space X such that for every pair p and q of points of X with $p \neq q$, there are neighborhoods G of p and H of q, respectively, such that $G \cap H = \emptyset$.

A subspace of a Hausdorff space is again a Hausdorff space. Any finite subset of a Hausdorff space is a closed subset. The topological product of two Hausdorff spaces is again a Hausdorff space. Cartesian space $\mathbb{R}^n$ is an example for a Hausdorff space.

The importance of Hausdorff spaces partially stems from the fact that solutions sets of equations with values in Hausdorff spaces are closed: Let f and $g\colon X \to Y$ be two continuous maps from an arbitrary topological space into a Hausdorff space. Then

$$\{x \in X : f(x) = g(x)\}$$

is a closed subset of X.

Definition 3.2.2. A topological space X is *connected* if

$$X = \coprod_{i \in I} X_i \implies \exists i \in I : X = X_i$$

whenever $(X_i)_{i \in I}$ is a family of open subsets of X.

To show that a subset Y of a connected space X is all of X, it is enough to show that Y is non-empty and both an open and closed subset of X.

A subspace of $\mathbb{R}$ is connected if and only it is an interval.

An open cover $(U_i)_{i \in I}$ of a topological space X is *locally finite* if every point of X possesses a neighborhood G such that exists only finitely many $i \in I$ with $G \cap U_i \neq \emptyset$.

A subset K of the underlying set of a Hausdorff space X is *compact* if it is a compact space when endowed with the subspace topology. It is *relatively compact in* X if its closure in X is compact.

A compact subspace of a Hausdorff space X is always a closed subset of X. A closed subset of a compact space is again compact. The product of two compact spaces is again a compact space.

Definition 3.2.3. T_0 space: A topological space X is T_0 space if for any two points x, y in X, there is an open set U such that $x \in U$ and $y \notin U$.

T_0 space is also called as Kolmogorov space.

The following table gives a list of the number of inequivalent T_0 topologies, along with all inequivalent topologies upto order 6, which means all topological sets up to order 6 including empty set.

Table 3.1: Number of topologies on a set with n points.

n	Inequivalent topologies	Inequivalent T_0 topologies
0	1	1
1	1	1
2	3	2
3	9	5
4	33	16
5	139	66
6	718	318

Above table only counts topologies upto isomorphism. For example if $X = \{1, 2, 3\}$ be a set with 3 elements. There are 29 distinct topologies on X but only 9 inequivalent topologies:

1. {∅, {1,2,3}}
2. {∅, {3}, {1,2,3}}
3. {∅, {1,2}, {1,2,3}}
4. {∅, {3}, {1,2}, {1,2,3}}
5. {∅, {3}, {2,3}, {1,2,3}} (T_0)
6. {∅, {3}, {1,3}, {2,3}, {1,2,3}} (T_0)
7. {∅, {1}, {2}, {1,2}, {1,2,3}} (T_0)
8. {∅, {2}, {3}, {1,2}, {2,3}, {1,2,3}} (T_0)
9. {∅, {1}, {2}, {3}, {1,2}, {1,3}, {2,3}, {1,2,3}} (T_0)

3.3 Relating T_0-Topologies to Posets

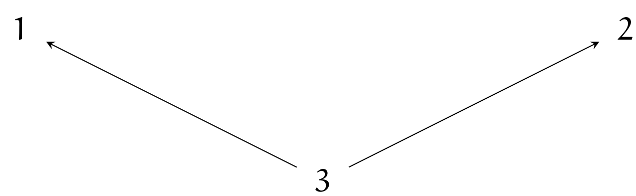
Let X be a finite space. For $x \in X$, define U_x to be the intersection of the open sets that contain x, hence it will be smallest open set containing x. Define a relation $\leq$ on the set X by $x \leq y$ if $x \in U_y$ or, equivalently, $U_x \subset U_y$. If we are given poset we can form topology formed by $U_x = \{u \in X | u \leq x\}$ hence $U_x \subset U_y$ implies $x < y$.

The relation $\leq$ is a partial order if and only if X is T_0. From each T_0 topology we can generate poset and vice versa.

Theorem 3.3.1. *Every finite poset is equivalent to a T_0 topology.*

Formal proof for this was given by [**?**].

Example 3.3.2. Consider T_0 topology on $n = 3$, $\tau = \{\emptyset, \{3\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$ we can generate poset as follows: since smallest open set containing 3 is $\{3\}$, 2 is $\{2,3\}$ and 1 is $\{1,3\}$. also, $\{3\} \subset \{1,3\}$ $\{3\} \subset \{2,3\}$ hence $3 < 1$ and $3 < 2$.



and if we have above poset then smallest open set containing 1 will be $U_1 = \{1,2\}$, similarly, $U_2 = \{2,3\}$ and $U_3 = \{3\}$. In simple words, given two distinct points x and y in the space, we say that $x \leq y$ if every open set containing x also contains y.

Since we have various tools available for posets we will study for a given quandle do we have any poset that makes it continuous quandle.

Chapter 4: Topological quandles

In this chapter we develop a bridge between topological spaces and quandles. We will start by exploring possibility of a quandle to be a topological quandle and we will give enumeration of various topological quandles. To carry out this enumeration, we will use a combination of analytical and computational methods. We will show there is no connected topological quandle, some explicit computer programs and outputs are given.

4.1 Topological Quandles

Definition 4.1.1. A *topological quandle* is a quandle X is a topological space such that the map $X \times X \ni (x, y) \longmapsto x * y \in X$ is continuous.

It is clear that any finite quandle is automatically a topological quandle with respect to the discrete topology.

In a topological quandle, the right multiplication $R_x : y \to y * x \in X$ is a homeomorphism.

In other words, right topological quandle means that for all $x, y, z \in X$,

$$x < y \implies x * z < y * z.$$

and, since left multiplications are not necessarly bijective maps, left topological quandle means that for all $x, y, z \in X$,

$$x < y \implies z * x \leq z * y.$$

where $x < y$ is relation in poset representing T_0 topology as described in 3.3

4.2 Topologies on Non-connected Quandles

As we mentioned earlier, since T_1-topologies on a finite set are discrete, we will focus on T_0-topologies on *finite quandles*. A map on finite spaces is continuous if and only if it preserves the order. It turned out that on a finite quandle with a T_0-topology, left multiplications can not be continuous as can be seen in the following theorem.

Theorem 4.2.1. *Let X be a finite quandle endowed with a T_0-topology. Assume that for all $z \in X$, the map L_z is continuous, then $x \leq y$ implies $L_z(x) = L_z(y)$.*

Proof. We prove this theorem by contradiction. Let X be a finite quandle endowed with a T_0-topology.
Assume that $x \leq y$ and $L_z(x) \neq L_z(y)$. If $x = y$, then obviously $L_z(x) = L_z(y)$. Now assume $x < y$, then for all $a \in X$, the continuity of L_a implies that $a * x \leq a * y$. Assume that there exist $a_1 \in X$ such that, $z_1 := a_1 * x = L_{a_1}(x) < a_1 * y = L_{a_1}(y)$.
The invertibility of right multiplications in a quandle implies that there exist unique a_2 such that $a_2 * x = a_1 * y$ hence $a_1 * x < a_2 * x$ which implies $a_1 \neq a_2$. Now we have $a_1 * x < a_2 * x \leq a_2 * y = z_2$. We claim that $a_2 * x < a_2 * y$.
if $a_2 * y = a_2 * x$ and since $a_2 * x = a_1 * y$ we will have $a_2 * y = a_2 * x = a_1 * y$ hence $a_2 * y = a_1 * y$ but $a_1 \neq a_2$, thus contradiction.
Now that we have proved $a_2 * x < a_2 * y$, then there exists a_3 such that $a_2 * y = a_3 * x$ we get, $a_2 * x < a_3 * x$ repeating the above argument we get, $a_3 * x < a_3 * y$. Notice that a_1, a_2 and a_3 are all pairwise disjoint elements of X. Similarly, we construct an *infinite* chain, $a_1 * x < a_2 * x < a_3 * x < \cdots$, which is impossible since X is a finite quandle. Thus we obtain a contradiction. □

We have the following Corollary.

Corollary 4.2.2. Let X be a finite quandle endowed with a T_0-topology. If C is a chain of X as a poset then any left continuous function L_x on X is a constant function on C.

4.3 Topologies on Connected Quandles

Theorem 4.3.1. *There is no T_0-topology on a finite connected quandle X that makes X into a right topological quandle.*

Proof. Let $x < y$. Since X is connected quandle, there exists $\phi \in Inn(X)$ such that $y = \phi(x)$. Since X is finite, ϕ has a finite order m in the group $Inn(X)$. Since ϕ is a continous automorphism then $x < \phi(x)$ implies $x < \phi^m(x)$ giving a contradiction. □

Corollary 4.3.2. There is no T_0-topology on any latin quandle that makes it into a right topological quandle.

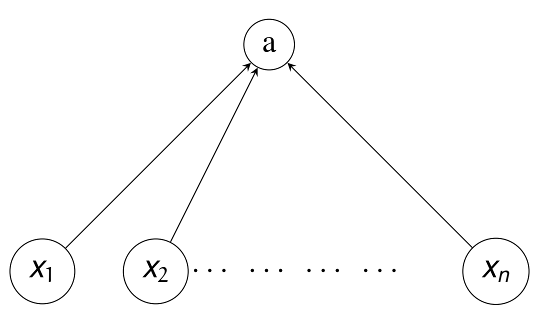
Thus Theorem 4.3.1 leads us to consider quandles X that are not connected, that is $X = X_1 \cup X_2 \cup \ldots \cup X_k$ as orbit decomposition, search for T_0-topology on X and investigate the continuity of the binary operation.

4.4 Revisiting Non-connected Quandles

Proposition 4.4.1. *Let X be a finite quandle with orbit decomposition $X = X_1 \cup \{a\}$, then there exist non trivial T_0-topology which makes X right continuous.*

Proof. Let $X = X_1 \cup \{a\}$ be the orbit decomposition of the quandle X. For any $x, y \in X_1$, there exits $\phi \in Inn(X)$ such that $\phi(x) = y$ and $\phi(a) = a$. Declare that $x < a$, then $\phi(x) < a$. Thus for any $z \in X_1$ we have $z < a$. □

The T_0-topology in Proposition 4.4.1 is precisely given by $x < a$ for all $x \in X_1$, illustrated by Hasse diagram below.



Remark 4.4.2. One can generate topology by $a < x_i$ for $i \in \{1, 2, ..n\}$

Proposition 4.4.3. *Let X be a finite quandle with two orbits X_1 and X_2. Then any right continuous poset on X is biparatite with vertex sets X_1 and X_2.*

Proof. We prove this proposition by contradiction. For every $x_1, y_1 \in X_1$ such that $x_1 < y_1$. We know that there exist $\phi \in Inn(X)$ such that $\phi(x_1) = y_1$. Hence, $x_1 < \phi(x_1)$ implies $x_1 < \phi^m(x_1) = x_1$, where m is the order of ϕ in $Inn(X)$. Thus we have a contradiction.

□

Proposition 4.4.4. *Let X be a finite quandle with two orbits X_1 and X_2. Then the complete bipartite graph with vertex set X_1 and X_2 forms a right continuous poset.*

Proof. Let X be a finite quandle with two orbits X_1 and X_2. If $x \in X_1$ and $y \in X_2$ then for every $\phi \in Inn(X)$ we have $\phi(x) \in X_1$ and $\phi(y) \in X_2$. Proposition 4.4.3 gives that the graph is bipartite and thus $x < y$. We then obtain $\phi(x) < \phi(y)$ giving the result. □

Remark 4.4.5. By Proposition 4.4.4 and Theorem 4.2.1, there is a non-trivial T_0-topology making X right continuous if and only if the quandle has more than one orbit.

Notice that Proposition 4.4.4 can be generalized to n-paratite complete graph.

4.5 Dihedral Quandles and Topologies

In this section we will study special case of non connected shelves that is Dihedral quandles of even order, we will analyze how many posets will make it topological quandle.

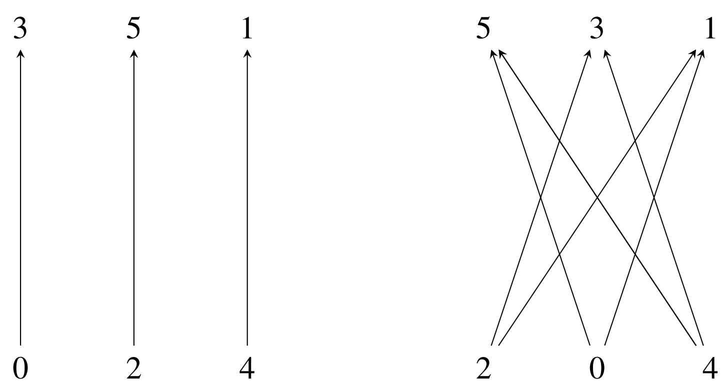
To begin, we present a table summarizing right continuous posets.

The following table gives the list of right continuous posets on some even dihedral quandles. In the table, the notation (a, b) on the right column means $a < b$,, where a and b are elements of the poset.

Table 4.1: Right continuous posets(representing T_0 topology) on dihedral quandles

Quandle	Posets
R_4	((0,1),(2,1),(0,3),(2,3))
R_6	((0, 1), (0, 5), (2, 1), (2, 3), (4, 3), (4, 5)) ; ((0,3), (2, 5), (4, 1)).
R_8	((2, 7), (4, 7), (6, 1), (6, 3), (0, 5), (2, 5), (4, 1), (0, 3)) ; ((0, 1), (6, 7), (4, 5), (0, 7), (2, 1), (2, 3), (4, 3), (6, 5)).
R_{10}	((0, 1), (6, 7), (4, 5), (2, 1), (8, 9), (2, 3), (4, 3), (8, 7), (0, 9), (6, 5)) ; ((4, 7), (6, 9), (2, 9), (8, 1), (8, 5), (0, 7), (6, 3), (2, 5), (4, 1), (0, 3)) ; ((2, 7), (8, 3), (0, 5), (4, 9), (6, 1)).

Notice that in table 4.1, the dihedral quandle R_4 has only one right continuous poset $((0,1),(2,1),(0,3),(2,3))$ which is complete biparatite. While the dihedral quandle R_6 has two continuous posets $((0,1),(0,5),(2,1),(2,3),(4,3),(4,5))$ and $((0,3),(2,5),(4,1))$ illustrated below.



Moreover, in table 4.1, for R_8 the bijection f given by $f(k) = 3k - 2$ makes the two posets isomorphic. The same bijection gives isomorphism between the first two posets of R_{10}. The following Theorem characterizes non complete biparatite posets on dihedral quandles.

Theorem 4.5.1. *Let R_{2n} be a dihedral quandle of even order. Then R_{2n} has $s+1$ right continuous posets, where s is number of odd natural numbers less than n and relatively non coprime with n.*

Proof. Let $X = R_{2n}$ be the dihedral quandle and we know dihedral quandle has two orbits $X_1 = \{0, 2, \dots, 2n-2\}$ and $X_2 = \{1, 3, \dots, 2n-1\}$. For every $x \in X_2$, we construct a partial order $<_x$ on R_{2n}, such that for all $y \in X$, we have $2y <_x 2y - x$ and $2y <_x 2y + x$. Then $<_x$ is clearly right continuous partial order since $2y < 2y - x$ and $2y < 2y + x$ for all y imply that $2z - 2y < 2z - (2y - x)$. In other words we obtain $2y * z < (2y - x) * z$. From the definition of the order $<_x$ it is clear the two partial orders $<_x$ and $<_{2n-x}$ are the same. Hence we obtain the following distinct partial orders $<_1, <_3, \dots$. Now we check which ones are isomorphic. If m is odd and $gcd(n, m) = 1$ then $f(k) = mk - 2$ is a bijective function making $<_1$ and $<_m$ isomorphic. Now let m be odd and $gcd(m, n) = k > 1$. The two posets $<_1$ and $<_m$ are non isomorphic since $<_1$ is connected poset, as in Definition 3.1.2, and $<_m$ is not connected poset. We show that these are the only right continuous posets. Given a right continuous poset on R_{2n} then $a < b$ can be written as $a < a - (a - b)$ which implies that $a <_x b$ where $x = a - b$. Now if $a < b$ then by Proposition 4.4.3, we have $a \in X_1$, $b \in X_2$. Now let $a = 2\alpha$ and $b = 2\beta + 1$ then $a - b = 2(\alpha - \beta) - 1 \in X_2$. This ends the proof.

□

Corollary 4.5.2. For the dihedral quandle R_{2^n} with 2^n elements, there is a unique right continuous poset.

4.6 Some Computer Calculations

In this section we give non-trivial right and left continuous posets on the finite quandles of order up to 5 based on Maple and Python computations. In the following tables we have excluded the trivial and connected quandles.

For n=3 we have only one non-trivial quandle.

Table 4.2: Continuous posets on quandles of order 3

Quandle for n = 3	Right continuous Posets	Left continuous poset
$\begin{bmatrix} 0\ 0\ 1 \\ 1\ 1\ 0 \\ 2\ 2\ 2 \end{bmatrix}$	((0,2),(1,2))	((0,1))

As seen in Table 4.2 for $n = 3$, there exist a unique right continuous poset and a unique left continuous poset.

Table 4.3: Continuous posets on quandles of order 4

Quandles for n = 4	Right continuous poset	Left continuous poset
$\begin{bmatrix} 0\,0\,0\,0 \\ 1\,1\,1\,2 \\ 2\,2\,2\,1 \\ 3\,3\,3\,3 \end{bmatrix}$	((0, 3)) ; ((0, 1), (0, 2), (0, 3)) ; ((0, 1), (0, 3), (1, 2)) ; ((0, 1), (0, 2), (1, 3), (2, 3)) ; ((2, 3), (1, 3)) ; ((2, 3), (1, 3), (0, 3)).	((0,1),(1,2)) and ((1,2))
$\begin{bmatrix} 0\,0\,0\,1 \\ 1\,1\,1\,2 \\ 2\,2\,2\,0 \\ 3\,3\,3\,3 \end{bmatrix}$	((0,3),(1,3),(2,3))	((0,1),(1,2)) and ((1,2))
$\begin{bmatrix} 0\,0\,1\,1 \\ 1\,1\,0\,0 \\ 2\,2\,2\,2 \\ 3\,3\,3\,3 \end{bmatrix}$	((0,2),(1,2),(0,3),(1,3),(2,3)) ; ((0,2),(1,2),(0,3),(1,3)) ; ((0,2),(1,2)) ; ((2,3)).	((0,1),(2,3)) and ((2,3))
$\begin{bmatrix} 0\,0\,0\,0 \\ 1\,1\,3\,2 \\ 2\,3\,2\,1 \\ 3\,2\,1\,3 \end{bmatrix}$	((0,1),(0,2),(0,3))	None
$\begin{bmatrix} 0\,0\,1\,1 \\ 1\,1\,0\,0 \\ 3\,3\,2\,2 \\ 2\,2\,3\,3 \end{bmatrix}$	((0,2),(0,3),(1,2),(1,3))	((0,1),(2,3))

Table 4.4: Continuous posets on quandles of order 5.

Quandles for n = 5	Right continuous poset	Left continuous poset
$\begin{bmatrix} 0&0&0&0&0 \\ 1&1&1&1&1 \\ 2&2&2&2&3 \\ 3&3&3&3&2 \\ 4&4&4&4&4 \end{bmatrix}$	((0,1),(1,2),(1,3),(0,4)) ; ((0,2),(0,3),(1,2),(1,3),(4,2),(4,3)) ; ((0,2),(0,3),(1,2),(1,3),(2,4),(3,4)) ; ((0,1),(1,4),(4,2),(4,3)).	((0,1),(1,2),(2,3)) ; ((0,1),(1,2)) ; ((1,2)).
$\begin{bmatrix} 0&0&0&0&0 \\ 1&1&1&1&2 \\ 2&2&2&2&3 \\ 3&3&3&3&1 \\ 4&4&4&4&4 \end{bmatrix}$	((0,1), (0,2), (0,3), (2,4), (3,4), (1,4)) ; ((0,4)) ; ((0,1), (0,2), (0,3)) ; ((0,4),(4,1),(4,2),(4,3)).	((0,1), (1,2), (2,3)) ; ((0,1), (1,2)) ; ((2,3)).
$\begin{bmatrix} 0&0&0&0&1 \\ 1&1&1&1&0 \\ 2&2&2&2&3 \\ 3&3&3&3&2 \\ 4&4&4&4&4 \end{bmatrix}$	((1,2),(0,3),(2,4),(3,4)) ; ((1,2),(0,2),(1,3),(0,3),(2,4),(3,4)) ; ((1,4),(0,4)) ; ((1,2),(0,2),(1,3),(0,3)).	((1,2),(0,1),(2,3)) ; ((0,1),(0,2)) ; ((0,2),(1,2)).
$\begin{bmatrix} 0&0&0&0&1 \\ 1&1&1&1&2 \\ 2&2&2&2&3 \\ 3&3&3&3&0 \\ 4&4&4&4&4 \end{bmatrix}$	((0,4),(1,4),(2,4),(3,4)).	((1,2),(0,1),(2,3)) ; ((0,1),(0,2)) ; ((0,2),(1,2)).

Table 4.4 : Continued

Quandles for n=5	Right continuous poset	Left continuous poset
$\begin{bmatrix} 0&0&0&1&1 \\ 1&1&1&0&0 \\ 2&2&2&2&2 \\ 3&3&4&3&3 \\ 4&4&3&4&4 \end{bmatrix}$	((0,2), (1,2) (2,3),(2,4)) ; ((0,2), (1,2)) ; ((2,3),(2,4)) ; ((0,3),(0,4),(1,3),(1,4)).	((0,1),(1,2),(3, 4)) ; ((3,4)) ; ((0,1),(1,2)).
$\begin{bmatrix} 0&0&0&1&1 \\ 1&1&1&2&2 \\ 2&2&2&0&0 \\ 3&3&3&3&3 \\ 4&4&4&4&4 \end{bmatrix}$	((0, 3),(1,3), (2,3), (3,4)) ; ((0,3),(1,3),(2, 3)) ; ((3,4)).	((0,1),(1,2),(3,4)) ; ((3,4)) ; ((0,1),(1,2)).
$\begin{bmatrix} 0&0&0&1&2 \\ 1&1&1&2&0 \\ 2&2&2&0&1 \\ 3&3&3&3&3 \\ 4&4&4&4&4 \end{bmatrix}$	((0,3),(1,3),(2,3),(0,4),(1,4),(2,4)) ; ((0,3),(1,3),(2,3)) ; ((3,4)).	((0,1),(1,2)) ; ((0,1)).
$\begin{bmatrix} 0&0&0&0&0 \\ 1&1&1&1&1 \\ 2&2&2&2&2 \\ 4&4&4&3&3 \\ 3&3&3&4&4 \end{bmatrix}$	((0,1),(0,2)) ; ((0,1),(1,3),(1,4)) ; (0,1),(1,2),(2,3),(2,4)).	((0,1),(0,2)) ; ((0,1),(1,2),(3,4)) ; ((0,1),(3,4)) ; ((3,4)).

Table 4.4 : Continued

Quandles for n=5	Right continuous poset	Left continuous poset
$\begin{bmatrix} 0&0&1&1&1 \\ 1&1&0&0&0 \\ 2&2&2&4&3 \\ 3&3&4&3&2 \\ 4&4&3&2&4 \end{bmatrix}$	((0,2),(0,3),(0,4),(1,2),(1,3),(1,4)).	((0,1),(2,3)) ; ((0,1)).
$\begin{bmatrix} 0&0&1&1&1 \\ 1&1&0&0&0 \\ 2&2&2&2&2 \\ 4&4&4&3&3 \\ 3&3&3&4&4 \end{bmatrix}$	((0,2),(0,3),(0,4),(1,2),(1,3),(1,4)) ; ((0,2),(1,2)) ; ((2,3),(2,4)) ; ((0,3),(0,4),(1,3),(1,4)).	((0,1),(2,3)) ; ((0,1)).
$\begin{bmatrix} 0&0&1&1&1 \\ 1&1&0&0&0 \\ 3&4&2&4&3 \\ 4&2&4&3&2 \\ 2&3&3&2&4 \end{bmatrix}$	((0,2),(0,3),(0,4),(1,2),(1,3),(1,4)).	None
$\begin{bmatrix} 0&0&0&0&0 \\ 1&1&1&2&2 \\ 2&2&2&1&1 \\ 3&4&4&3&3 \\ 4&3&3&4&4 \end{bmatrix}$	((0,1),(0,2),(0,3),(0,4)) ; ((0,1),(0,2)) ; ((0,1),(0,2),(2,3),(2,4),(1,3),(1,4)) ; ((1,3),(1,4),(2,3),(2,4)).	((0,1),(1,2),(3,4)) ; ((0,1),(1,2)) ; ((0,1),(3,4)) ; ((0,1)).

Table 4.4 : Continued

Quandles for n=5	Right continuous poset	Left continuous poset
$\begin{bmatrix} 0\,0\,0\,0\,0 \\ 1\,1\,1\,1\,1 \\ 2\,2\,2\,4\,3 \\ 3\,3\,4\,3\,2 \\ 4\,4\,3\,2\,4 \end{bmatrix}$	((0,2),(0,3),(0,4)) ; ((0,2),(0,3),(0,4),(1,2),(1,3),(1,4)) ; ((0,1)) ; ((0,2),(0,3),(0,4),(0,1),(1,2),(1,3),(1,4)).	((0,1),(0,2),(0,3)) ; ((0,1),(0,2)) ; ((0,1)).
$\begin{bmatrix} 0\,0\,0\,1\,1 \\ 1\,1\,1\,2\,2 \\ 2\,2\,2\,0\,0 \\ 4\,4\,4\,3\,3 \\ 3\,3\,3\,4\,4 \end{bmatrix}$	((0,4),(1,4),(2,4),(0,3),(1,3),(2,3)).	((0,1),(0,2)) ; ((0,1),(1,2),(3,4)) ; ((0,1),(3,4)) ; ((3,4)).
$\begin{bmatrix} 0\,0\,0\,0\,0 \\ 1\,1\,4\,2\,3 \\ 2\,3\,2\,4\,1 \\ 3\,4\,1\,3\,2 \\ 4\,2\,3\,1\,4 \end{bmatrix}$	((0,1),(0,2),(0,3),(0,4)).	None
$\begin{bmatrix} 0\,0\,1\,1\,1 \\ 1\,1\,0\,0\,0 \\ 2\,2\,2\,2\,3 \\ 3\,3\,3\,3\,2 \\ 4\,4\,4\,4\,4 \end{bmatrix}$	((1,2),(0,2),(1,3),(0,3),(2,4),(3,4)) ; ((1,4),(0,4)) ; ((1,2),(0,2),(1,3),(0,3)).	((0,1),(2,3)) ; ((0,1)).

Table 4.4 : Continued

Quandles for n=5	Right continuous poset	Left continuous poset
$\begin{bmatrix} 0\,0\,0\,0\,0 \\ 1\,1\,1\,2\,2 \\ 2\,2\,2\,1\,1 \\ 4\,4\,4\,3\,3 \\ 3\,3\,3\,4\,4 \end{bmatrix}$	((0,1),(0,2)) ; ((0,1),(1,3),(1,4)) ; ((1,3),(1,4),(2,3),(2,4)).	((0,1),(0,2)) ; ((0,1),(1,2),(3,4)) ; ((0,1),(3,4)) ; ((3,4)).
$\begin{bmatrix} 0\,0\,0\,0\,0 \\ 1\,1\,1\,2\,2 \\ 2\,2\,2\,1\,1 \\ 3\,3\,3\,3\,3 \\ 4\,4\,4\,4\,4 \end{bmatrix}$	((1,3),(2,3)) ; ((1,3),(2,3),(1,4),(2,4)) ; ((0,4)) ; ((3,2),(3,1)) ; ((1,3),(2,3),(4,1),(4,2)).	((0,1),(1,2),(3,4)) ; ((0,1),(1,2)) ; ((0,1),(3,4)) ; ((0,1)).
$\begin{bmatrix} 0\,0\,0\,0\,0 \\ 1\,1\,1\,1\,1 \\ 2\,2\,2\,4\,3 \\ 3\,3\,4\,3\,2 \\ 4\,4\,3\,2\,4 \end{bmatrix}$	((0,2),(0,3),(0,4)) ; ((0,2),(0,3),(0,4),(1,2),(1,3),(1,4)) ; ((0,1)) ; ((0,2),(0,3),(0,4),(0,1),(1,2),(1,3),(1,4)).	((0,1),(0,2),(0,3)) ; ((0,1),(0,2)) ; ((0,1)).

Chapter 5: Topological Shelves

5.1 Introduction

In this chapter we analyze similar results proved for quandles and try to generalize them for shelves. These algebraic structures are derived from the axiomatization of Reidemeister move III in classical knot theory. We have seen in chapter 3 that quandles are subset of shelves. In this chapter we will expand our research, since quandle is a stronger condition than shelf, we will come across various connected and non connected topological shelves.

Definition 5.1.1. A Shelf is a non empty set with binary operation $(a, b) \rightarrow a * b$ satisfying following condition:
For any $a, b, c \in X$, we have $(a * b) * c = (a * c) * (b * c)$.

In other words, operation distributes over itself from right.

Example 5.1.2. In $\mathbb{Z}_n$, binary operation defined by $x * y = ax + by$, with $b(a + b - 1) = 0$.

If shelf has a an identity element. That is, there exists an element 1 in S such that $1 * x = x = x * 1$ for all $x \in S$. then we call it as a unital shelf. Like connected quandles we will study connected shelves and its orbits.

5.2 Connected Shelves

In this section, we classify connected shelves of order up to 5.

Definition 5.2.1. *A connected shelf* $(X, *)$ *is a shelf such that for all* $x, y \in X$*, there exists a finite* *number of elements $x_1, x_2, \dots, x_m$ such that*

$$y = ((((x * x_1) * x_2) * x_3) \dots) * x_m$$

The definition of a connected shelf simply means that we can go from one element to any other element by finite number of steps. In other words, the orbit of each element must equal the shelf itself. In the following table, we give the list of all connected shelves, racks and quandles.

Table 5.1: Number of inequivalent connected shelves, rack and quandles on a set with n points

n	# of connected shelves	# of connected racks	# of connected quandles
1	1	1	1
2	2	1	0
3	5	2	1
4	18	2	1
5	165	4	3
6	3987	4	2

We refer the reader to Appendix A for the complete list of connected shelves of order less than or equal to 5, up to isomorphism. Within this list, we search for shelves that support topologies by giving poset to make it continuous toological shelve.

Our computer search results support the following conjecture for order up to 5.

Conjecture 5.2.2. *Let S be a shelf. Then there exists a shortest cycle that covers all the elements in S exactly once if and only if S is connected.*

A cycle from 0 to 0 that covers all elements in the shelf may contain an element more than once. For example, consider the following connected shelf of order 4.

$*$	0	1	2	3
0	0	1	1	3
1	0	1	2	0
2	0	1	2	0
3	0	1	1	3

A cycle from 0 to 0 in this shelf is

$$0 \to 1 \to 2 \to 1 \to 3 \to 0.$$

However the shortest cycle from 0 to 0 is

$$0 \to 3 \to 1 \to 2 \to 0.$$

Definition 5.2.1. Let S be a finite shelf. For each element $x \in S$, let $r(x)$ be the number of elements of S which act trivially on x, i.e. the set

$$r(x) = |y \in S| \; x * y = x|$$

and let $c(x)$ be the number of elements of S on which x acts trivially, i.e. the set

$$c(x) = |y \in S| \; y * x = y|$$

In terms of the shelf's operation table, $r(x)$ counts the number of times x appear in row x and $c(x)$ counts the number of entries in the column of x equal their row number.

5.2.1 Latin Shelves

In this subsection, we explore the groups generated by rows in Latin shelves. Recall the left multiplication map. For each $x \in S$, the left multiplication by x is the map denoted by

$$L_x : S \to S$$

and given by

$$L_x(y) := x * y.$$

Definition 5.2.3. *A shelf is Latin or strongly connected if the shelf operation is left-invertible. This means the rows of a Latin shelf are permutations of* $\{1, 2, ..., n\}$.

Clearly, Latin shelves are connected, and a Latin quandle is always a Latin shelf.

Let $z \in S$, where S is a Latin shelf.

$$L_{x*y}(z) = (x * y) * z = (x * z) * (y * z) = L_x(z) * L_y(z)$$

In a Latin shelf, each L_x, where $x \in S$, is a bijection.

Let X be the set of all L_x, i.e.

$$X = \{L_x : x \in S\}.$$

Clearly, we have $X \subset S_n$.

Let $t \in S$. Define an operation $\triangleright$ on the set X as follows:

$$(L_x \triangleright L_y)(t) := L_x(t) * L_y(t).$$

Then

$$(L_x \triangleright L_y)(t) = L_{x*y}(t).$$

Now consider

$$
\begin{aligned}
((L_x \triangleright L_y) \triangleright L_z)(t) &= (L_x \triangleright L_y)(t) * L_z(t) \\
&= (L_x(t) * L_y(t)) * L_z(t) \\
&= (L_x(t) * L_z(t)) * (L_y(t) * L_z(t)) \\
&= (L_x \triangleright L_z)(t) * (L_y \triangleright L_z)(t) \\
&= L_{x*z}(t) * L_{y*z}(t) \\
&= (L_{x*z} \triangleright L_{y*z})(t) \\
&= ((L_x \triangleright L_z) \triangleright (L_y \triangleright L_z))(t)
\end{aligned}
$$

Since t is arbitrary, we have

$$(L_x \triangleright L_y) \triangleright L_z = (L_x \triangleright L_z) \triangleright (L_y \triangleright L_z),$$

i.e.

$$L_{x*y} \triangleright L_z = L_{x*z} \triangleright L_{y*z}.$$

Thus, $(X, \triangleright)$ is a shelf.

Note that $L_x \triangleright L_x = L_{x*x}$. So, if the elements in the shelf S are idempotent, so are the elements in X.

Now we show that the elements in X do not satisfy the second axiom of a quandle.

Let $t \in S$.

$$
\begin{aligned}
((L_x \triangleright L_y) \triangleright L_y)(t) &= (L_x \triangleright L_y)(t) * L_y(t) \\
&= (L_x(t) * L_y(t)) * L_y(t) \\
&= (L_x(t) * L_y(t)) * (L_y(t) * L_y(t)) \\
&= (L_x \triangleright L_y)(t) * (L_y \triangleright L_y)(t) \\
&= L_{x*y}(t) * L_{y*y}(t) \\
&= L_{(x*y)*(y*y)}(t) \neq L_x(t)
\end{aligned}
$$

For each $x \in S$, define a map

$$\phi : S \to X$$

given by

$$x \mapsto L_x$$

The map ϕ is a homomorphism because

$$\phi(x * y) = L_{x*y} = L_x \triangleright L_y = \phi(x) \triangleright \phi(y)$$

In fact, it is an epimorphism.

Now let G be the group generated by L_x, where $x \in S$, i.e. $G = \langle L_x : x \in S \rangle$. Since we are only considering finite shelves, G is a finitely generated group.

Define an operation on G as

$$L_x \diamond L_y := L_y^{-1} L_x L_y$$

Then we have

$$
\begin{aligned}
(L_x \diamond L_y) \diamond L_z &= L_z^{-1}(L_x \diamond L_y)L_z \\
&= L_z^{-1}(L_y^{-1}L_xL_y)L_z \\
&= (L_yL_z)^{-1}L_x(L_yL_z)
\end{aligned}
$$

and

$$
\begin{aligned}
(L_x \diamond L_z) \diamond (L_y \diamond L_z) &= (L_z^{-1}L_xL_z) \diamond (L_z^{-1}L_yL_z) \\
&= (L_z^{-1}L_yL_z)^{-1}(L_z^{-1}L_xL_z)(L_z^{-1}L_yL_z) \\
&= (L_z^{-1}L_y^{-1}L_z)(L_z^{-1}L_xL_z)(L_z^{-1}L_yL_z) \\
&= (L_z^{-1}L_y^{-1})L_x(L_yL_z) \\
&= (L_yL_z)^{-1}L_x(L_yL_z),
\end{aligned}
$$

which imply

$$(L_x \diamond L_y) \diamond L_z = (L_x \diamond L_z) \diamond (L_y \diamond L_z)$$

Hence the conjugation in G turns it into a shelf.

We are now interested in seeing whether the conjugation in G satisfies Axiom 1 and Axiom 2 of a quandle. Consider $L_x \diamond L_x$.

$$L_x \diamond L_x = L_x^{-1}L_xL_x = L_x$$

Thus the elements in G are idempotent.

Let's consider Axiom 2. We show that under certain conditions on left multiplication, elements in G satisfy Axiom 2. In other words, under certain conditions on left multiplication, G is a quandle.

$$
\begin{aligned}
(L_x \diamond L_y) \diamond L_y &= L_y^{-1}(L_x \diamond L_y)L_y \\
&= L_y^{-1}(L_y^{-1}L_xL_y)L_y \\
&= (L_y^{-1})^2 L_x (L_y)^2 \\
&= (L_y^2)^{-1} L_x (L_y^2)
\end{aligned}
$$

If rows are either 2-cycles or the identity permutation, then $(L_x \diamond L_y) \diamond L_y = L_x$, and thus $(G, \diamond)$ is a quandle. If cycles are disjoint, $(L_x \diamond L_y) \diamond L_y = L_x$, and thus $(G, \diamond)$ is still a quandle.

We have more connected shelves than quandles. Here is a list of connected shelves and continuous posets on it:

Order 2: There are two connected shelves.

Table 5.2: Continuous posets on connected shelves of order 2

Shelf for $n = 2$	Continuous Posets
$\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$	((0,1))
$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$	None

Order 3: There are five connected shelves.

Table 5.3: Continuous posets on shelves of order 3

Shelves for $n = 3$	Continuous poset
$\begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$	((0, 1)). ((0, 1), (1, 2)) ((0, 1), (0, 2))
$\begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ 1 & 0 & 2 \end{bmatrix}$	None
$\begin{bmatrix} 0 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$	None
$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix}$	None
$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$	((1,2)) ((0,1),(1,2))

Order : There are 18 connected shelves.

Table 5.4: Continuous posets on connected shelves of order 4.

Connected shelves of order 4.	Continuous Poset
$\begin{bmatrix} 0&1&2&3 \\ 0&1&2&3 \\ 0&1&2&3 \\ 0&1&2&3 \end{bmatrix}$	All 16 Posets of order 4
$\begin{bmatrix} 0&2&3&1 \\ 3&1&0&2 \\ 1&3&2&0 \\ 2&0&1&3 \end{bmatrix}$	None
$\begin{bmatrix} 0&1&3&2 \\ 0&1&3&2 \\ 1&0&2&3 \\ 1&0&2&3 \end{bmatrix}$	None
$\begin{bmatrix} 0&1&2&3 \\ 0&1&2&3 \\ 1&0&2&3 \\ 1&0&2&3 \end{bmatrix}$	None
$\begin{bmatrix} 1&1&1&1 \\ 2&2&2&2 \\ 3&3&3&3 \\ 0&0&0&0 \end{bmatrix}$	None

Table 5.4: Continued

Connected Shelves for n=4	Continuous poset
$\begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 3 & 3 \\ 1 & 1 & 2 & 2 \end{bmatrix}$	((1,2),(3,0))
$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \\ 1 & 2 & 0 & 3 \end{bmatrix}$	None
$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{bmatrix}$	((2,3)) ((1,2),(1,3)) ((0,1), (0,3),(1,2)) ((0,1), (1,2), (1,3)) ((2,3),(1,3)) ((0,1),(1,2),(2,3))
$\begin{bmatrix} 0 & 1 & 1 & 3 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 1 & 3 \end{bmatrix}$	((1,2)) ((0,1),(1,2),(3,0))

Table 5.4: Continued

Connected Shelves for n=4	Continuous poset
$\begin{bmatrix} 0&1&1&3 \\ 0&1&2&3 \\ 0&1&2&3 \\ 0&1&1&3 \end{bmatrix}$	((0,1)) ((0,1),(1,2)) ((2,1),(1,0),(1,3)) ((0,1),(0,2),(3,2),(3,1))
$\begin{bmatrix} 0&1&1&3 \\ 0&1&2&3 \\ 0&1&2&3 \\ 0&2&2&3 \end{bmatrix}$	((1,2)) ((0, 1), (0, 3), (1, 2)) ((1,2),(2,3)) ((1, 2), (1, 3), (0, 2), (0, 3]))
$\begin{bmatrix} 0&1&1&3 \\ 3&1&2&0 \\ 3&1&2&0 \\ 0&1&1&3 \end{bmatrix}$	((1,2))
$\begin{bmatrix} 0&1&2&3 \\ 0&1&2&3 \\ 0&1&2&3 \\ 0&2&1&3 \end{bmatrix}$	((1,3),(1,2)) ((0,2),(1,2))
$\begin{bmatrix} 0&1&1&1 \\ 0&1&2&2 \\ 0&1&2&3 \\ 0&1&2&3 \end{bmatrix}$	((1,2)) ((0,1),(1,2),(3,0))

Table 5.4: Continued

Connected Shelves for n=4	Continuous poset
$\begin{bmatrix} 0\,1\,1\,2 \\ 0\,1\,2\,3 \\ 0\,1\,2\,3 \\ 0\,1\,2\,3 \end{bmatrix}$	((1,2)) ((1,3),(1,2)) ((2,3),(3,4))
$\begin{bmatrix} 0\,1\,1\,3 \\ 0\,1\,2\,3 \\ 0\,1\,2\,3 \\ 0\,2\,1\,3 \end{bmatrix}$	None
$\begin{bmatrix} 0\,1\,2\,3 \\ 0\,1\,2\,3 \\ 0\,1\,2\,3 \\ 1\,0\,0\,3 \end{bmatrix}$	((1,2))

We can see for order 2 $[[1,1],[0,0]]$ is a connected shelf which is not a quandle.

5.3 Topological shelves

In this section we introduce the notion of a *topological shelf.*

Definition 5.3.1. A topological shelf is a shelf $(X,*)$ which is a topological space such that the map $X \times X \to X$ sending (x,y) to $x*y$ is a continuous map.

Remark: Like quandles any finite shelf is a topological shelf with respect to the discrete topology.

We will check if theorem 4.3.1 holds true here, if I will take $[[1,1],[0,0]]$ is a connected right topological quandle. So, in case of shelfs there exist connected right topological shelves.

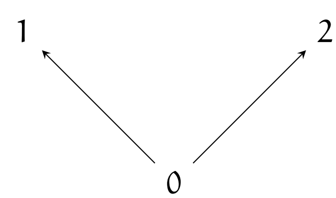
Theorem 5.3.2. *Shelve defined by* $x*a = x+1 \pmod{n}$ *for all* a *doesnt have any right continuous poset.*

Proof. If we consider there exist some poset with $x < y$ implies $a*x \leq a*y$ hence $x+1 \leq y+1$. proceeding this way we get $x+k \leq y+k$. for all k.
if $k = y - x \pmod{n}$, then we will get $y \leq x$. which gives contradiction. □

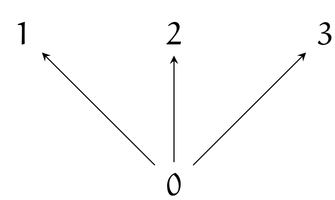
Theorem 5.3.3. *If there exist* $a_1, a_2, .., a_k$ *such that* $L_{a_i} = L_{a_j}$ *for* $1 \leq i, j \leq k$ *then any poset on the subset* $\{a_1, ..., a_k\}$ *gives rise to a right continuous shelf on* X.

Proof. If $L_{a_i} = L_{a_j}$ and if $a_i \leq a_j$ then $a_i * x = a_j * x$ for all x, making it right continuous.
We can obtain explicit finite shelves that are not rack by considering $S = \mathbb{Z}_n$ and choose $\alpha, \beta \in \mathbb{Z}_n$ with the condition $\beta^2 + \alpha\beta - \beta$ and α non-invertible. □

Example 5.3.4. Consider $S = [[0,1,2,3],[0,1,2,3],[0,1,2,3],[0,2,1,3]]$ then S is topological shelf, since there exists following poset which makes it continous.



Example 5.3.5. Consider $S = [[0,1,2,3,4],[0,1,2,3,4],[0,1,2,3,4],[0,1,2,3,4],[0,2,3,1,4]]$ and poset $\{0 < 2, 0 < 3, 0 < 1\}$ then S is topological shelf on $n = 5$, since there exists following poset which makes it continuous.



Theorem 5.3.6. *If S is shelf on set X, and if $x, y \in X$ such that $L_x = L_y$ and R_x= R_y for $x \neq y$ then the poset $\{x < y\}$ makes S into a topological shelf.*

Proof. For $x < y$ we have $x * z = y * z$ and $z * x = z * y$ making it continuous. □

Remark 5.3.7. We can also consider poset $y < x$, which will be isomorphic to poset $x < y$.

Proposition 5.3.8. *If S is shelf on set X, and if $x, y \in X$ such that $L_x \cap L_y = \emptyset$ and $R_{L_x(i)} = R_{L_x(j)}$ then poset $x < y$ is continuous.*

Proposition 5.3.9. *Let the shelf X be given by the binary operation $x * y = y$ for all $x, y \in X$. Then any poset structure on X makes it a topological shelf.*

Proof. For all $x, y, z, w \in X$, we have $x \leq y$ and $z \leq w$ implies automatically that $x * z \leq y * w$ making the binary operation *continuous* map. □

Remark 5.3.10. Notice here that the cardinality of X wasn't used in the proof. Thus we have examples in both cases finite and infinite cardinalities.

One can generalize the previous proposition 5.3.9 to obtain the following proposition

Proposition 5.3.11. *Let the shelf X be given by the binary operation $x * y = \phi(y)$ for all $x, y \in X$, where the map $\phi : X \to X$ satisfies $\phi^2 = \phi$ (a projection). Then if furthermore ϕ is an endomorphism of a poset structure on X, then X becomes a topological shelf.*

Proof. Assume that we have $x \leq y$ and $z \leq w$ for $x, y, z, w \in X$. Then the condition $x*z \leq y*w$ is equivalent to $\phi(z) \leq \phi(w)$ meaning that ϕ is a poset homomorphism. □

Proposition 5.3.12. *Let S be a shelf such that $R_{a_1} = R_{a_2} = R_{a_3} = ... = R_{a_m}$ and $L_{a_1}(x) = a_1, L_{a_2}(x) = a_2, \ldots, L_{a_m}(x) = a_m$, for some $a_1, \ldots, a_m \in S$ and for all $x \in S$. Then S has at least $\gamma(m)$ posets continuous on S, where $\gamma(m)$ is number of T_o topologies on m symbols.*

Proof. For any poset on $a_1, a_2 .... a_k$ so, in poset $x < y$ if $x, y \in \{a_1, a_2, ..a_k\}$ now if $x <= y$ and $z <= w$ let us consider three cases.

Case I) If $x = y$ and $z = w$ then $x * z = y * w$.

Case II) $x = y$ and $z < w$ then let $z = a_i$ and $w = a_j$. $x * z = x * a_i = a_i$ and $y * w = y * a_j = a_j$ implies $x * z < y * w$. and $z * x = a_i * x = R_{a_i}(x) = R_{a_j}(x) = R_{a_j}(y) = a_j * y = w * y$.

Case III) if $x < y$ and $z < w$ then $x = a_i, y = a_j, z = a_k$ and $w = a_l$.

hence, we have $a_i < a_j$ and $a_k < a_l$, which implies $a_i * a_k = a_k$ and $a_j * a_l = a_l$ giving $x * z < y * w$. □

The above preposition can be illustrated using following example:

Example 5.3.13. If $S = [[0, 1, 2, 3, 4], [0, 1, 2, 3, 4], [1, 0, 2, 3, 4], [1, 0, 2, 3, 4], [1, 0, 2, 3, 4]]$ then continuous posets are all of $\{3 < 4\}, \{2 < 3, 2 < 4\}, \{2 < 3, 3 < 4\}, \{3 < 4, 2 < 4\}$ (posets are listed upto isomorphisim.)

Let's relax condition of left continuity to only one variable.

Proposition 5.3.14. *If for a shelve* $R_{a_1} = R_{a_2} = R_{a_3}, \dots, = R_{a_m}$ *and* $L_{a_1} = a_1$ *and Range of* L_{a_2}*=Range of* L_{a_3}, $\dots$*,=Range of* $L_{a_m} = \{a_2, a_3, \dots, a_m\}$ *then poset* $\{a_1 < a_2, a_1 < a_3, \dots, a_1 < a_m\}$ *will be continuous.*

Proof. Let's consider above poset then $a_1 < a_i$ and $a_1 < a_j$ then $a_1 * a_1 = a_1$ and $a_i * a_j = a_k$ hence $a_1 * a_1 < a_i * a_j$. For any z, $a_1 * z = a_i * z$ since $R_{a_1} = R_{a_j}$ and $z * a_1 = a_1$ and $z * a_i = a_j$ for some $j \in \{2, 3, \dots, m\}$, hence we get $z * a_1 < z * a_i$ □

5.4 Infinite Shelves

We want to investigate infinite topological shelves that are not racks or quandles. For topological racks and topological quandles the reader should consult [11] and [6].

One important concept in infinite posets is that of a chain. A chain is a subset of a poset in which any two elements are comparable (i.e., one is greater than or equal to the other). An infinite poset may have chains of arbitrary length, which can make it difficult to analyze.

Another important concept is that of an anti-chain. An anti-chain is a subset of a poset in which no two elements are comparable. An infinite poset may have an infinite number of anti-chains, which can make it difficult to identify the structure of the poset.

If $(X, <)$ and $(Y, <_1)$ form two partial orders that are continuous then we define partial order on $(X * Y, <")$ as $(a, b) <"(a', b')$ iff $a < a'$ and $b <_1 b'$

Example 5.4.1. Binary operation defined by $x * y = ax + b$ on $\mathbb{R}$ forms a infinite shelf which is neither a quandle nor a rack.

Proposition 5.4.2. *Consider the real line $\mathbb{R}$ with binary operation defined by $x * y = ax + b$, where $a \neq 0$ and b are real numbers. The standard topology on $\mathbb{R}$ makes $(\mathbb{R}, *)$ into a topological shelf.*

Proof. One checks that the preimage by the operation $*$ of an open interval (α, β) is open in $\mathbb{R} \times \mathbb{R}$. This comes from the fact that $\alpha < ax + b < \beta$ gives that the preimage is $(\frac{\alpha-b}{a}, \frac{\beta-b}{a}) \times \mathbb{R}$. This ends the proof. □

Example 5.4.3. We can extend this condition to plane $\mathbb{R}^2$, it can becomes a topological shelf with the operation

$$\vec{x} * \vec{y} := A\vec{x} + B\vec{y}$$

with A and B some 2×2 matrices.

For $(\mathbb{R}^2, *)$ to become a shelf following condition needs to be satisfied

$$(\vec{x} * \vec{y}) * \vec{z} = (\vec{x} * \vec{z}) * (\vec{y} * \vec{z})$$

Solving we get, $(A\vec{x} + B\vec{y}) * \vec{z} = (A\vec{x} + B\vec{z}) + (A\vec{y} + B\vec{z})$

hence, $A * (A\vec{x} + B\vec{y}) + B\vec{z} = (A\vec{x} + B\vec{z}) * (A\vec{y} + B\vec{z}) = A(A\vec{x} + B\vec{z}) + B(A\vec{y} + B\vec{z})$ which gives $AB = BA$ and $B = AB + B^2$.

Considering $\mathbb{R}^2$ with its standard topology, the operation $* : \mathbb{R}^2 \to \mathbb{R}^2$ is clearly continuous function.

Since if we consider $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$, $\vec{X} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $\vec{Y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ then $\vec{X} * \vec{Y} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + b_{11}y_1 + b_{12}y_2 \\ a_{21}x_1 + a_{22}x_2 + b_{21}y_1 + b_{22}y_2 \end{pmatrix}$ which is continuous.

Example 5.4.4. If we will assume last example with matrix A and B in $\mathbb{Z}$, then to form a shelf we just need $AB = BA$ and $B = AB + B^2$.

If we consider $A = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$, and $B = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$ then $AB + B^2 = B$ and $AB = BA$.

For this A and B if we define $(\vec{x} + \vec{m}) * (\vec{y} + \vec{n}) = (\vec{x} * \vec{y}) + A\vec{m} + B\vec{n}$.

If we define $[\vec{x}] * [\vec{y}] := [\vec{x} * \vec{y}]$ where $[\vec{x}], [\vec{y}] \in \mathbb{R}^2/\mathbb{Z}^2$ so this forms a shelf structure on torus $T^2 = S^1 \times S^1$.

Theorem 5.4.5. *If A is an $n \times n$ diagonal integer matrix then binary operation defined by*

$$\vec{x} * \vec{y} := A\vec{x} + (I - A)\vec{y}$$

forms a topological shelf.

Proof. If A is diagonal matrix and let $B = I - A$ then $AB = BA$. and $I = A + B$ hence $B = AB + B^2$, which makes binary operation a shelf, as seen above is a continuous operation.

If we take A as diagonal matrix with at least one of the diagonal entry zero, A becomes non-invertible, so the operation $\vec{x} * \vec{y} := A\vec{x} + B\vec{y}$ gives an idempotent shelf which is not a quandle on the n-torus $(S^1)^n$.

□

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Appendix A: Code to list all topological quandles

The program is designed to analyze Quandle structures and their isomorphic copies. It first takes a Quandle structure as input and calculates all of its isomorphic copies. Once it has determined the isomorphic copies of the original Quandle, it proceeds to analyze each one in order to determine it and check to see if it satisfies left and/or right continuity. This analysis is performed for all posets on n. Automating the analysis of Quandle isomorphisms and poset continuity allows users to more efficiently study these structures more and make new discoveries about their properties.

The following program is executable in SageMath.

Step I: Declare variables P=Permutations(4)

SD={}

for z in range(23):

d=P[z].dict()

N=P[2].to.matrix

e=Q.dict()

d[0]=d[1]-1

d[1]=d[2]-1

d[2]=d[3]-1

d[3]=d[4]-1

Step II: Input quandle here

M=[[1,1,1,1],[2,2,2,2],[3,3,3,3],[0,0,0,0]]

N=[[0,0,1,1],[1,1,0,0],[3,3,2,2],[2,2,3,3]]

Step III: Form isomorphic copies of quandle using permutation

i=0

```
for j in range(4)
N[d[i]][d[j]]=d[M[i][j]]
i=1
for j in range(4):
N[d[i]][d[j]]=d[M[i][j]]
i=2
for j in range(4):
N[d[i]][d[j]]=d[M[i][j]]
i=3
for j in range(4):
N[d[i]][d[j]]=d[M[i][j]]
**Step IV: Take all posets and for each poset test contiuity taking all isomorphic copies of quandle**
Po = Posets(4)
for p in range(1,15):
flag=0
for i in range(4):
for j in range(4):
if Po[p].isgequal (i,j):
for k in range(4):
for l in range(4):
if Po[p].isequal (k,l):
if Po[p].isgequal (N[i][k],N[j][l])==0: flag=flag+1
**Step V:   If Poset satisfies continuity condition print poset and quandle**
if flag<=0: print(p)
print(N)
```

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message from Extracta Mathematicae

JM JESÚS MARÍA FERNÁNDEZ CASTILLO <castillo@unex.es>
To lahrani@usf.edu
Cc Secretary Extracta
5/18/2023

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