

A Study on the Necessary Conditions for Odd Perfect Numbers

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Abstract

A collection of all of the known necessary conditions for an odd perfect number to exist, along with brief descriptions as to how these were discovered. This was done in order to facilitate those who would like to further pursue the necessary conditions for odd perfect numbers, or those who are searching for odd perfect numbers themselves. All past research into odd perfect numbers has been collected, and their methods were briefly detailed. Present day limitations are usually discovered by improving upon algorithms, specifically those used to find the previous limitations. Improving upon algorithms that have been used for this purpose seems to be the appropriate path to take when trying to find new, or stricter, necessary conditions.

This study is on odd perfect numbers, and the conditions which limit their existence. A perfect number is a natural number (an integer greater than zero), whose sigma function equals twice the number itself, i.e. $\sigma(N)=2N$. The sigma function is the sum of all a numbers divisors. As an example, 6 is a perfect number. This is due to $\sigma(6)=12$; this is because $1+2+3+6=12$. The existence of even perfect numbers is easily documented, as shown above. Euclid proved that there is a one-to-one correspondence between even perfect numbers and Mersenne primes (a prime being of the form 2^p-1 , where p is also prime). An even perfect number will always be of the form $(2^p-1) 2^{p-1}$. The proof is as follows.

Theorem: Let n be a positive integer. Then, n is an even perfect number iff $n = 2^{p-1}(2^p-1)$ where 2^p-1 is prime (which implies p is prime, by Mersenne).

Proof: \Leftarrow Suppose $n = 2^{p-1}(2^p-1)$, where 2^p-1 is prime. Since perfect numbers are defined by the multiplicative arithmetic function, σ , we can use this information to reach the desired outcome. By the definition of a multiplicative arithmetic function, $F(xy)=F(x)F(y)$. Then $\sigma(n)=\sigma(2^{p-1})\sigma(2^p-1)$. Recall that $\sigma(p^a)=\frac{p^{a+1}-1}{p-1}$. (proof of this shown below). Since 2^p-1 is prime, it's only divisors are 1 and itself, so $\sigma(2^p-1)=2^p$. Then, $\sigma(n)=\sigma(2^{p-1})\sigma(2^p-1)=\left(\frac{2^p-1}{2-1}\right)2^p=(2^p-1)2^p=2(2^{p-1}(2^p-1))=2n$. Then, n is an even perfect number.

[Proof of $\sigma(p^a)=\frac{p^{a+1}-1}{p-1}$: Since the only divisors of p^a are $1, p, p^2, \dots, p^a$, $\sigma(p^a)=1+p+\dots+p^a$, in other words, the geometric series with ratio p . Therefore, $\sigma(p^a)=\frac{p^{a+1}-1}{p-1}$]

\Rightarrow Suppose that n is an even perfect number. Let $n=2^s t$, where s is a positive integer and t is odd.

Then, $\sigma(n)=2n=2^{s+1}t$. Also, note, $\sigma(n)=\sigma(2^s)\sigma(t)=(2^{s+1}-1)\sigma(t)$. Since the $\gcd(2^{s+1}-1, 2^{s+1})=1$, obviously, we have that $2^{s+1} | (2^{s+1}-1)\sigma(t) \Rightarrow 2^{s+1} | \sigma(t)$. $\Rightarrow \sigma(t)=k 2^{s+1}$ where k is some positive integer. We also have that $2^{s+1}t=(2^{s+1}-1)(k 2^{s+1})$. Then $t=(2^{s+1}-1)k$ and $k>1$. Then, since $k|t$ and k isn't 1 or t , $\sigma(t) \geq 1+t+k = 1 +$

$k2^{s+1} = 1 + \sigma(t)$. This is a contradiction. Hence, $k=1$ with $t=2^{s+1}-1$. And $\sigma(t)=t+1 \Rightarrow t$ is prime. Thus, $n=2^s(2^{s+1}-1)$ where $2^{s+1}-1$ is prime.

There are currently 47 Mersenne primes known to exist, and, therefore, 47 even perfect numbers have been discovered. [19] The question becomes whether or not odd perfect numbers exist. There have been no discoveries of odd perfect numbers, although many people have been working on this problem for centuries. There have been many studies leading to a new necessary condition, or limitations, for an odd perfect number to exist. That is the main purpose of this paper: to detail all of the necessary conditions for odd perfect numbers, in order to facilitate the further study of the subject. While it would be better, in a lot of cases, to detail the actual proofs for these studies, many of them are numerous pages long, or proofs by algorithms, which cannot be shown here. Some proofs of the most recent discoveries will be shown, but some will be omitted due to space constraints. A portion of these will be detailed in the appendix.

Lower Bound for Odd Perfect Numbers

There are many restrictions regarding the existence of odd perfect numbers, and a lower bound is just one of them. New proofs throughout history have further strengthened this lower bound. At first, Kanold proved that if an odd perfect number existed, it had to be greater than 10^{20} . [17] Kanold used the results from Ore about Harmonic Numbers. In order to come up with this condition. [24] This was further strengthened by Tuckerman, who proved that an odd perfect number would have to be greater than 10^{36} . [26] This was done with an infinite tree-generating algorithm, which, if allowed to execute indefinitely, would enumerate all odd perfect numbers. A partial execution led Tuckerman to show that any odd perfect number had to have a sufficiently large component, which led him to his result. This was again raised by Hagis, who studied different cases for odd perfect numbers, and then, combining the results, raised the bound to 10^{50} , followed by Brent and Cohen, who brought the lower bound up to

10^{160} . [6] [2]. Brent and Cohen used an algorithm, with an elliptic curve method to limit the number of factorization, to compute their findings, and, after improving upon their algorithm, later were able to raise the lower bound to 10^{300} . [3] A very recently published paper by Ochem and Rao improved upon the algorithm used by Cohen and Brent to obtain a lower bound of 10^{1500} . [23] This is the latest known information on the lower bound of odd perfect numbers, and can be found in the appendix.

Number of prime Divisors

Studies have shown that in order for an odd perfect number to exist, there have to be a minimum number of prime divisors. There have been studies on the number of distinct prime divisors it must have, and on the total number of prime divisors. It was originally conjectured, and subsequently proven by Gradshteyn, that an odd perfect number has to have at least 6 distinct prime divisors [1] Hagi later showed that it was necessary to have a minimum of 8 distinct prime divisors. [7]. This was obtained by assuming an odd perfect number exists that has 6 or 7 distinct divisors, and resulting in a contradiction. The most recent information available on the number of distinct prime divisors comes from Nielsen, who came up with an algorithm to show that there has to be at least 9 distinct prime divisors. He did this by showing a contradiction in any case where there are only 8 prime divisors. In his study, Nielsen also improved upon earlier results about the number of distinct prime divisors an odd perfect number must have if it isn't divisible by 3. Kishore and Hagi had stated that there must be 10 and 11, respectively, but Nielsen showed that this number must be at least 12. [18] [8] [21] Norton was able to prove that if an odd perfect number was not divisible by 3 or 5, then it must have at least 15 distinct prime divisors, and if it also wasn't divisible by 7, it must have at least 27 distinct prime divisors. [22] These are the largest known limitations for distinct prime divisors.

There are also, of course, limitations in regards to the number of prime factors, in general, not just distinct prime factors. Iannucci and Sorli showed that an odd perfect number must have at least 37 prime factors, while Hare improved upon this to show that there must be at least 47. [15] [11] Hare

used an algorithm that resulted in a partition, allowing him to work on a smaller base, and for the algorithm to verify his result. Hare later improved upon his own algorithm, by eliminating a previous roadblock, showing that there must be at least 75 prime divisors for an odd perfect number to exist.

[12] Ochem and Rao, in the recently published paper, increase this to 101 prime factors. They also go on to state that the largest component must be greater than 10^{62} . [23]

Largest Prime Factor

There are also conditions about the largest prime factors for a number to be an odd perfect number. Hagis showed that every odd perfect number must be divisible by a prime greater than 10^4 . He later improved upon this himself, to show that they must be divisible by a prime greater than 10^5 . [10] This was done by assuming that an odd perfect number existed with no prime divisor greater than his conjectured result, and reaching a contradiction. Cohen and Hagis worked together to improve upon his results even further, limiting this to numbers with their largest prime divisor being greater than 10^6 . [9]. Jenkins then improved upon this to say that all odd perfect numbers must have a prime factor exceeding 10^7 . [16] Goto has, very recently, improved upon this with an algorithm, to show that 10^8 is an appropriate lower bound for the largest prime factor. [5] Iannucci also shows that the second largest and third largest prime divisors must exceed 10000 and 100 respectively [13] [14]

Other Conditions

There are many varying restrictions on the existence of odd perfect numbers as well. Euler showed that an odd perfect number N must have the form $N = q^e a_1^{2g_1} a_2^{2g_2} \dots a_n^{2g_n}$ where q and all a_i are distinct prime numbers and $q \equiv e \equiv 1 \pmod{4}$. The proof for this is shown here

Proof: Let N be an odd perfect number. That is, $\sigma(N) = 2N$. We need to first show that if N is an odd perfect number, it must have exactly one prime to an odd power.

Suppose, for contradiction, that N has either two or more primes to an odd power, or no primes to an odd power.

Case 1: N has no primes to an odd power. Then, $\sigma(N) = \prod(\sigma(p_i^{n_i}))$ since all n_i are even, $\sigma(p_i^{n_i})$ is a sum of n_i+1 odd terms (since $\sigma(p_i^{n_i}) = p_i^{n_i} + p_i^{n_i-1} + \dots + p_i + 1$) $\Rightarrow \sigma(p_i^{n_i})$ is odd $\Rightarrow \sigma(N)$ is odd. A contradiction, since $\sigma(N) = 2N$ if N is perfect.

Case 2: N has multiple primes to an odd power. Suppose n_j and n_k are both odd, where j does not equal k . Since $\sigma(p_j^{n_j})$ and $\sigma(p_k^{n_k})$ are even, $4|\sigma(N)$. This contradicts $\sigma(N) = 2N$ once again.

Therefore, N has exactly one prime divisor to an odd power.

We now know that $N = q^e a_1^{2g_1} a_2^{2g_2} \dots a_n^{2g_n}$. It remains to be shown that $q \equiv e \equiv 1 \pmod{4}$. Since q and e are odd, it is sufficient to show that q and e are not $\equiv 3 \pmod{4}$.

Suppose $q \equiv 3 \pmod{4}$. Then $(q+1)|\sigma(q^e)|\sigma(N)$. This is a contradiction, since $q+1 \equiv 0 \pmod{4}$ and 4 does not divide $\sigma(N)$.

Suppose $e \equiv 3 \pmod{4}$. Then $\sigma(q^e) \equiv q^e + q^{e-1} + \dots + q + 1 \equiv 1 + 1 + \dots + 1 + 1 \equiv e + 1 \equiv 0 \pmod{4}$, a contradiction.

Therefore $q \equiv e \equiv 1 \pmod{4}$.

Stuyvaert remarked that an odd perfect number must be the sum of two squares. [4] Touchard showed that if an odd perfect number N exists then $N \equiv 1 \pmod{12}$ or $N \equiv 9 \pmod{36}$. His proof is below:

Proof: Let N be an odd perfect number. We must first show that if n is an odd integer and $n \equiv 5 \pmod{6}$ then n is not perfect.

Assume $n \equiv 5 \pmod{6}$. Then $n \equiv 2 \pmod{3}$. Note that n is not a square, since n is not $\equiv 1 \pmod{3}$. Note also that if $d|n$, either $d \equiv -1 \pmod{3}$ and $n/d \equiv 1 \pmod{3}$, or $d \equiv 1 \pmod{3}$ and $n/d \equiv -1 \pmod{3}$. Then $\sigma(n) = \sum d + (n/d) \equiv 0 \pmod{3}$. We can see from here, that n is not perfect.

Theorem: For any odd perfect number, N , $N \equiv 1 \pmod{12}$ or $N \equiv 9 \pmod{36}$.

Proof: N is not of the form $6k+5$ as shown above. Then, N is of the form $6k+1$ or $6k+3 \Rightarrow N \equiv 1 \pmod{6}$ and $N \equiv 1 \pmod{4}$, or $N \equiv 3 \pmod{6}$ and $N \equiv 1 \pmod{4}$. The first set of congruences shows that $N \equiv 1 \pmod{12}$. The second set shows that $N \equiv 9 \pmod{12}$, or $N = 12k + 9$. Note, that in the second case, if 3 does not divide k , $\sigma(n) \equiv 0 \pmod{4}$, and n is not perfect. So, $3|m$, and $N \equiv 9 \pmod{36}$.

[25]

It is known because of Nielsen that odd perfect numbers also have an upper bound. That is, if an odd perfect number N has r distinct prime factors then $N < 2^{4^r}$. If some of the smaller prime factors of N are known, this bound can be improved upon.[20] This proof can be found in the appendix.

These are the known results for the necessary conditions involving odd perfect numbers. Further developments are being worked upon, by improving the algorithms used in many modern day proofs. Using and improving these algorithms is the best way to further the search for odd perfect numbers.

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Appendix

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