November 2021

Zeros of Harmonic Polynomials and Related Applications

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Zeros of Harmonic Polynomials and Related Applications

by

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A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
with a concentration in Pure and Applied Mathematics
Department of Mathematics and Statistics
College of Arts and Sciences
University of South Florida

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Date of Approval:
October, 2021

Keywords: Bézout's theorem,maximum number of zeros,Green theorem,rational functions,Shwarz function,gravitational lensing

Dedication

To the soul of my Mom and my Dad.

Acknowledgments

There have been many people who have profoundly impacted my journey, but I would like to begin by acknowledging the person who never gave up on me and stayed by my side until I received my degree. First and foremost, I would like to thank Dr. Catherine Bénéteau for her profound kindness and support throughout the past few years. Since the very beginning, she has been steadfast in her support. She immediately stepped forward when I found myself without an advisor. When life took a turn, she made everything a little easier. When I suddenly had to return to Saudi Arabia, Dr. Catherine stayed in constant contact with me. She continued to work with me via Skype. Dr. Catherine never giving up is what gave me the hope and desire to continue my studies, despite hardship.

Also, I would like to thank Dr. Omar Alharbi for his support and always finding the time for me, despite his busy schedule, and Nathan Hayford for assisting me with the history of harmonic polynomials. Without their help, this dissertation may not have been completed.

In addition, I would also like to thank the other faculty at the University of South Florida who all played a vital role in my academic journey. I am indebted to Dr. Dmitry Khavinson, Dr. Mohamed Elhamdadi, Dr. Seung-Yeop Lee, and Dr. Myrto Manolaki. I am very grateful for their dedication and willingness to serve on my committee. I would also like to thank the King Abdul Aziz University. Without them, I would not have been able to come to the United States and get my Ph.D. and be part of the USF community.

I will never have the words to adequately express how thankful I am for my friends and family who continue to support me in overcoming any obstacles I have faced throughout the years. If it were not for their love and unwavering faith in me, I may not have been able to continue what I started.

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Abstract

In this thesis, we study topics related to harmonic functions, where we are interested in the maximum number of solutions of a harmonic polynomial equation and how it is related to gravitational lensing. In Chapter 2, we study the conditions that we should have on the real or complex coefficients of a polynomial p to get the maximum number of distinct solutions for the equation $p(z) - \bar{z}^2 = 0$, where deg $p = 2$. In Chapter 3, we discuss the lens equation when the lens is an ellipse, a limaçon, or a Neumann Oval. Also, we give a counterexample to a conjecture by C. Bénéteau and N. Hudson in [2]. We also discuss estimates related to the maximum number of solutions for the lens equation for the Neumann Oval.

Chapter 1

Introduction

In this thesis, we are interested in studying the maximum number of zeros of a harmonic polynomial and applications such as gravitational lensing. We begin by defining harmonic polynomials and discussing known results about their zeros. We then turn to a discussion of rational functions and gravitational lensing. In Chapter 2, we will consider a simple case when the polynomials involved have degree 2 and will examine conditions on the coefficients that lead to maximal numbers of solutions. In Chapter 3, we examine numbers of images obtained for different gravitational lenses.

1.1 Harmonic polynomials

The fundamental theorem of algebra (FTA) states that every complex polynomial of degree n has precisely n complex roots counting multiplicities. This theorem has been looked at from analytical, topological, and algebraic points of view, see [7, 23, 12, 32]. In 1992, T. Sheil-Small wanted to examine what happens in the context of harmonic polynomials, specifically in regards to an upper bound on the number of zeros. A complex function h in z and $\bar{z}$ is considered to be harmonic if it satisfies the Laplace equation, i.e, $h_{xx} + h_{yy} = 0$ where $z = x + iy$ and $\bar{z} = x - iy$, where the first and second partial derivatives of h are continuous. Note that, by considering the change of variables $z = x + iy$ and $\bar{z} = x - iy$, where x and y are real variables, we can write any complex harmonic polynomial as follows

$$h(z) := p(z) - \overline{q(z)}, \tag{1.1}$$

where p and q are analytic polynomials.

Having terms in $\bar{z}$ and z in a harmonic polynomial will play a big role in the maximum number of solution of $h(z) = 0$. In fact, it is even possible to have an infinite number of solutions. Consider the following example.

Example 1. Let $h(z) = z^n - \bar{z}^n$. Writing $z = re^{i\theta}$, where $0 \leq \theta < 2\pi$, $h(z) = 0$ means that $r^n e^{in\theta} - r^n e^{-in\theta} = 0$, which implies that $r^n e^{2in\theta} = r^n$, thus $r^n \left(e^{2in\theta} - 1\right) = 0$. This occurs when $\theta = \left(\frac{k}{n}\right)\pi$ for any $k \in \mathbb{Z}$. Thus we obtain n equally spaced lines through the origin, which gives rise to an infinite number of solutions.

Sheil-Small and his student Wilmshurst extended the FTA to harmonic polynomials and showed that $h(z) = p(z) - \overline{q(z)}$ has n^2 zeros at most if n, the degree p, is strictly greater than the degree of q. In 1994, Wilmshurst's [45] findings included a more general sufficient condition for h to have a finite number of zeros and settled this conjecture by using Bézout's theorem from algebraic geometry:

Theorem 1. Bézout's Theorem *[39]*
Let $P(x, y) = u(x, y) + iv(x, y)$ (with u and v real) be a complex-valued real analytic polynomial, where u has degree m and v has degree n, and suppose that u and v are relatively prime (i.e, contain no nontrivial common factors). Then P has at most mn zeros in $\mathbb{C}$.

In other words, Wilmshurst rewrote the complex equations by using real variables, which produced a system of real equations. He then counted the number of intersections of those curves, which is at most n^2. Wilmshurst [46] continued on in 1998 to prove that h will have at most n^2 zeros when deg $p = n$ and deg $q = m = n - 1$. He conjectured , when $1 \leq m < n - 1$, $h(z) = p(z) - \overline{q(z)} = 0$ gives $3n - 2 + m(m - 1)$ zeros of h at maximum.

In particular, if $m = 1$ the conjecture of Wilmshurst states that $p(z) - \bar{z}$, where p is an (analytic) polynomial of degree $n > 1$, has $3n - 2$ zeros at most. This conjecture was studied by many people, such as Bshouty, Crofoot, Lizzaik, Sarason, and others who worked on this problem and other questions related to harmonic mappings (see, e.g., [5, 35]). The tool of complex dynamics was used in 2001 by D. Khavinson and G. Świątek [22] to show that Wilmshurst's conjecture is true for $f(z) = p(z) - \bar{z}$. In other words, Khavinson and Świątek proved that the equation $\bar{z} - p(z) = 0$, where p is an analytic polynomial of degree $n > 1$, has at most $3n - 2$ complex zeros. They noticed that if z is a fixed point of $\overline{p(z)} = z$, then it is also a fixed point of $Q(z) = \overline{p(\overline{p(z)})}$, and $Q(z)$ is an analytic polynomial of degree n^2. Because of this, they were able to apply some facts from complex dynamics, together with the argument principle for harmonic functions, to estimate the number of fixed points of $Q(z)$ which led them to the maximum number of zeros of $\bar{z} - p(z)$. Additionally, they proved the sharpness for $n = 2$ by using quadratic polynomials.

L. Geyer's findings in 2008 [15] show that the mentioned bound is sharp for all n by using the work of Sarason and Crofoot [35] regarding the existence of certain extremal polynomials p.

In 2016, D.Khavinson, S.Y. Lee and A.Saez [20] got a sharper version of Wilmshurst's theorem, when the harmonic polynomial $h(z) = p(z) - \overline{q(z)}$ has real coefficients. They showed that the equation $h(z) = 0$ has at most $n^2 - n$ solutions that satisfy $(Rez)(Imz) \neq 0$. They also showed that, for all $n > m$, there exists a harmonic polynomial $h(z) = p_n(z) - \overline{q_m(z)}$ with at least $3n - 2$ or $m^2 + m + n$ roots.

Other researchers took a probabilistic approach to answering questions about numbers of roots of (1.1). That is, how many roots does equation (1.1) have on average? In 2009, Li and Wei [44] considered the probabilistic distribution of zeros of random harmonic polynomials. They were able to produce an equation for the expected number of zeros of (1.1). The authors showed that if the degree p equals the degree $q = n$, then the expected number of zeros is asymptotically $\frac{\pi}{4}n^{3/2}$ as $n \to \infty$. Furthermore they showed that if $m =$deg q and $n =$deg p, then when $m = \alpha n + o(n)$ with $0 \leq \alpha < 1$ and n goes to infinity, the expected number of zeros is asymptotically n.

On the other hand, others were trying to prove or disprove Wilmshurst's Conjecture for $m > 1$. In 2013, by using algebraic geometry, Lee, Lerario, and Lundberg [24] proved that there exist analytic polynomials p and q of degree n and m respectively where $n \geq 4$ and $m = n - 3$ such that the number of zeros of $p(z) - \overline{q(z)}$ exceeds

$$n^2 - 4n + 4\left\lfloor \frac{n-2}{\pi}\text{arctan}\frac{\sqrt{n^2-2n}}{n} \right\rfloor + 2,$$

which produces an unlimited number of counterexamples to Wilmshurst's Conjecture.

In 2014, Hauenstein, Lerario, Lundberg, and Mehta [17] used an experimental, certified-counting approach, in order to generalize those counterexamples. In addition, the authors of [17] conjecture that there exist polynomials p and q such that (1.1) has $n^2/2 - n + 12$ zeros, where degree $p = n$ and degree $q = n/2$.

In 2016, Lerario and Lundberg proved and sharpened the numerical results obtained in [29]. Most recently, A. Thomack [42] showed, again using probabilistic techniques, that if the degree of the anti-analytic polynomial $\overline{q(z)}$ is fixed, then the expected number of zeros is asymptotically equal to n as $n \to \infty$. This is to be compared with the following observation using the argument principle that if the degree p equals $n > 1$, then the number of zeros of the equation $p(z) = \bar{z}$ is at least n.

This does not close the open problem posed in the 2011 survey [21] regarding a sharp upper bound on the number of zeros of the harmonic polynomial $h(z) = p(z) - \overline{q(z)}$.

Open Problem. If p and q are analytic polynomials of degree n and m respectively where $n > 3$ and $1 < m < n-1$, how many zeros at most will the equation $p(z) - \overline{q(z)} = 0$ have?

1.2 Rational Harmonic Polynomials and Gravitational Lensing

In 2006, D. Khavinson and G. Neumann [19] studied what happens if you replace the polynomial p by a rational function $r(z)$. That is, how many solutions does the equation

$$r(z) - \bar{z} = 0$$

have, with $r(z) = p(z)/q(z)$, where p and q are relatively prime, analytic polynomials such that $n = \deg r$ = max (deg p, deg q) > 1. Moreover, by using the same technique that Khavinson and Świątek used in [22], they were able to prove that the upper bound on the number of zeros of $r(z) - \bar{z} = 0$ cannot exceed $5n - 5$. In fact, this problem is connected to a problem in mathematical physics known as gravitational lensing, making the problem interesting and valuable to a much wider audience. At the same time, S. H. Rhie, who is an astrophysicist, proved that this upper bound is sharp for every $n > 1$ by using a simple geometric construction.

Gravitational lensing is a phenomenon that describes how the lens plane (L), which consists of objects such as galaxies, can affect the number of images an observer sees from a source plane. Assume the lens plane and the source plane are parallel, and the observer's line of vision is perpendicular to these planes, as the following picture illustrates (Figure [1]).

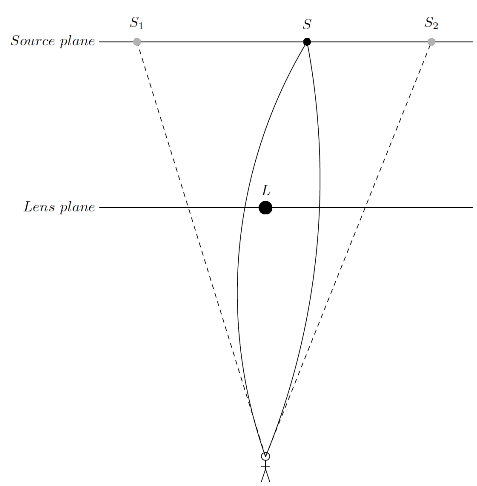

Figure 1. Gravitational lensing, the lens and source plane.

The light is deflected due to the gravitational force of the lens, and thus the observer will see multiple images of the same light source. Astrophysicists have been studying the equation which connect the position of a light source and the mass distribution of a gravitational lens (such as, a galaxy) to the number of images that can be observed from the source. This equation is called the lens equation (see Figure [2]).

Suppose the lens plane is taken to contain n point masses. Let w denote the source position, z the position of the lensed image, and z_j the position of the point mass j, where all $w, z, z_j \in \mathbb{C}$. Let m_j be the mass of the j^{th} point mass, then the lens equation is given by:

$$w = z + \gamma\bar{z} - sign(\sigma)\sum_{j=1}^{n}\frac{m_j}{\bar{z} - \overline{z_j}},$$

where γ is a real constant representing the normalized shear, and $\sigma \neq 0$ is a real constant representing the optical depth. Therefore the number of roots of this rational equation involving z and $\bar{z}$ is the same as the number of images of the lens equation.

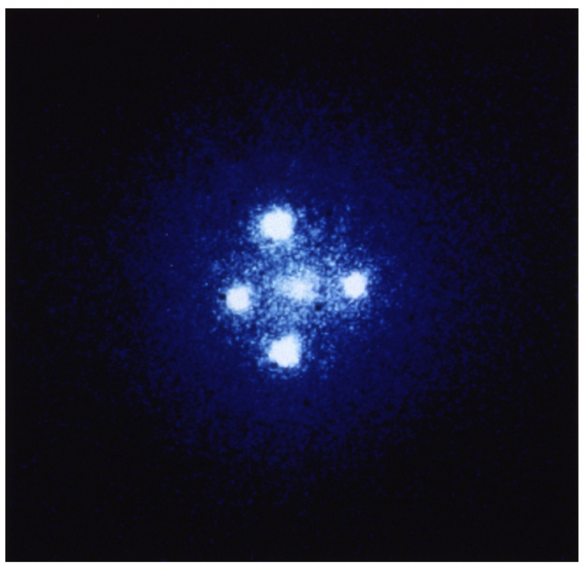

Figure 2. Four separate images of a single quasar located behind a galaxy, with an additional image too dim to see. [2]

It is also possible to consider multiple lens planes such as in [31]. In addition, the authors of [19] discuss replacing the point mass distribution by a continuous mass distribution, (see, e.g, [27] and [41]). The lens equation becomes:

$$w = z - \int_{\Omega} \frac{d\mu(\zeta)}{\bar{z} - \bar{\zeta}} - \gamma \bar{z}, \tag{1.2}$$

where Ω is the lens. This generalization led to the study the number of roots of the lens equation for continuous densities that have different shapes, see [1, 4, 11, 2].

The phenomenon of gravitational lensing was examined since the work of Einstein, as a consequence of his theory of general relativity and was verified experimentally by Arthur Eddington in 1919. Since then, the theory of gravitational lensing has been widely studied in the astrophysics community by people such as Witt, Mao, Peters, Rhie, and Burke. Part of their work was estimating the maximal number of solutions of the lens equation, among other subjects. For further details on the historical aspects of the lens equation including estimates made much earlier by astrophysicists on the maximal number of solutions, see [11, 21].

A good approximation for modeling galaxies is to pick either a uniform distribution or a collection of points masses.

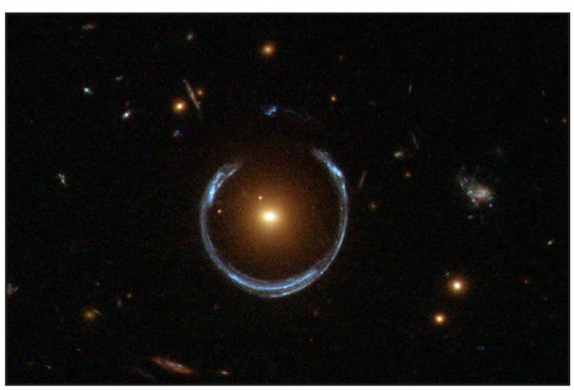

Figure 3. Light from a blue galaxy distorted by a red galaxy.[2]

Figure 4. An elliptical galaxy which is lensing light from two different background galaxies.[2]

The "Einstein Ring" solutions pictured in Figure [3] and [4] can be obtained from either the discrete or continuous distribution of masses. For the discrete case, when $n = 1$, setting $w = \gamma = \bar{z}_1 = 0$, we obtain as the solutions a circle with radius $\sqrt{\sigma.m_1}$. The continuous case is more involved; for the details, we refer to [11].

Recently, the authors of [37] have studied the number of pre-images of $f(z) = \eta$ for given but arbitrary $\eta \in \mathbb{C}$ for more general harmonic mappings f.

Note that the maximal number of solutions to the equation $r(z) - \bar{z} = 0$ for a rational function $r(z)$ are known. However, it is still an open problem to find the maximal numbers of solution to $r(z) = \bar{z}^m$ for $m > 1$.

Chapter 2

Harmonic Polynomials of Degree 2

In Chapter 1, we discussed known results regarding the maximum number of zeros for a harmonic polynomial $h(z) = p(z) - \overline{q(z)}$ where p and q are analytic polynomials of degree n and m respectively. We saw that results are known for $n = m$ or $n = m-1$, and also for any n with $m = 1$. In the case that $n = m$, it is possible to have an infinite number of solutions, but if the number of solutions is finite, there can be at most n^2. Similarly, if $n = m - 1$, the maximum number of solutions is n^2. These follow from Bézout's theorem. It would be interesting to understand which coefficients of p and q give rise to this maximum number of coefficients.

In this chapter, we will study that question in the simplest possible case that $n = m = 2$, and we will see that already in this situation, the answer is complicated. Notice that we considered only the case that $p(z) = \bar{z}^2$ here, where p is an analytic polynomial of degree 2. If we have a more general equation $p(z) = \overline{q(z)}$, where p and q are analytic polynomials of degree 2, by completing the square for q and using a change of variables, we can reduce it to the case $p(z) = \bar{z}^2$. In other words, we assume that $p(z) = az^2 + bz + c$ and $q(z) = dz^2 + ez + f$, where $a, b, c, d, f, e \in \mathbb{C}$, $a \neq 0$ and $d \neq 0$, we can rewrite $q(z)$ as $q(z) = d(z + m)^2 + \frac{4fd-e^2}{4d}$, where $m = \frac{e}{2d}$. Using the change of variables $u = z + m$ gives $q(u-m) = du^2 + \frac{4fd-e^2}{4d}$. By subtracting the constant $\frac{4fd-e^2}{4d}$, dividing by d, and rewriting $\tilde{p}(u) = p(u-m)$, we get an equation of the form $\tilde{p}(u) = \bar{u}^2$, as desired.

We will present two theorems which examine the conditions on the coefficients of $p(z) - \bar{z}^2$ that gives rise to four distinct solutions. We begin by considering the case that p has real coefficients.

Notice that if $a = \pm 1$ and $b = c = 0$ then $az^2 + bz + c = \bar{z}^2$ has an infinite number of solutions. So we will only examine the case when the equation has a finite number of solutions.

Theorem 2. *The equation $az^2 + bz + c = \bar{z}^2$ where $a \neq 0$ and $a, b, c \in \mathbb{R}$ has 4 solutions if and only if one of the following conditions is satisfied:*

- $a > 1,\ c > 0,$ *and* $2\sqrt{c(a-1)} < |b| < 2\sqrt{\frac{c}{a+3}}\ (a+1)$;
- $-3 < a < 1,\ a \neq -1$, $c > 0$, *and* $|b| > 2\sqrt{\frac{c}{a+3}}\ |a+1|$;
- $-3 < a < 1,\ a \neq -1$, $c < 0$, *and* $|b| > 2\sqrt{c(a-1)}$;
- $a < -3,\ c < 0$ *and* $2\sqrt{c(a-1)} < |b| < 2\sqrt{\frac{c}{a+3}}\ |a+1|$.

Proof

Consider the equation

$$az^2 + bz + c = \bar{z}^2 \tag{2.1}$$

and let $z = x + iy$ where $x, y \in \mathbb{R}$. Substituting into (2.1) gives:

$$\begin{aligned} & a(x+iy)^2 + b(x+iy) + c = (x-iy)^2 \\ \Leftrightarrow\ & ax^2 + 2aixy - ay^2 - x^2 + 2ixy + y^2 + bx + iby + c = 0 \\ \Leftrightarrow\ & (a-1)x^2 + (1-a)y^2 + bx + c = 0 \qquad (2.2) \\ & \text{and} \qquad 2(a+1)xy + by = 0. \qquad (2.3) \end{aligned}$$

From (2.3)

$$y(2(a+1)x + b) = 0 \Rightarrow y = 0 \quad \text{or} \quad 2(a+1)x + b = 0.$$

So, we will have two cases.

Case I:

$$y = 0 \overset{(2.2)}{\Rightarrow} (a-1)x^2 + bx + c = 0.$$

In order to achieve the maximum number of distinct solutions, we need this quadratic equation to have two solutions and thus, we have to have the following conditions:

$$a \neq 1 \qquad \text{and} \qquad b^2 - 4(a-1)c > 0.$$

In that case,

$$x = \frac{-b \pm \sqrt{b^2 - 4(a-1)c}}{2(a-1)}.$$

Now, if $a \neq 1$ then $b^2 - 4c(a-1) > 0$ in the following cases:

1. If $b \neq 0$, then:

   - If $a > 1$ and $c \leq 0$ or if $a < 1$ and $c \geq 0$, then $b^2 > 4c(a-1)$ is always satisfied since $4c(a-1) \leq 0$.
   - If $a > 1$ and $c \geq 0$, then we have to have $|b| > 2\sqrt{c(a-1)}$.
   - If $a < 1$ and $c \leq 0$, then we have to have $|b| > 2\sqrt{c(a-1)}$.

2. If $b = 0$, then:

   - If $a > 1$ and $c < 0$ or if $a < 1$ and $c > 0$, then $b^2 > 4c(a-1)$ is always satisfied since $4c(a-1) < 0$.

So therefore, we see that in Case I ($y = 0$), we have 2 distinct solutions under any of the following conditions on the coefficients:

I(i) $b \neq 0$, $a > 1$ and $c \leq 0$.

I(ii) $b \neq 0$, $a < 1$ and $c \geq 0$.

I(iii) $b = 0$, $a > 1$ and $c < 0$.

I(iv) $b = 0$, $a < 1$ and $c > 0$.

I(v) $b \neq 0$, $a > 1$ and $c \geq 0$, and $|b| > 2\sqrt{c(a-1)}$.

I(vi) $b \neq 0$, $a < 1$ and $c \leq 0$, and $|b| > 2\sqrt{c(a-1)}$.

Case II:

$$y \neq 0 \overset{from (2.3)}{\Rightarrow} 2(a+1)x + b = 0.$$

If $a \neq -1$ then

$$x = \frac{-b}{2(a+1)}. \tag{2.4}$$

Substituting (2.4) into (2.2) gives:

$$(a-1)\frac{b^2}{4(a+1)^2} - (a-1)y^2 + b\left(\frac{-b}{2(a+1)}\right) + c = 0$$

$$
\begin{aligned}
\Leftrightarrow \quad & (a-1)\frac{b^2}{4(a+1)^2} - (a-1)y^2 - \frac{b^2}{2(a+1)} + c = 0 \\
\Leftrightarrow \quad & (a-1)y^2 = \frac{(a-1)}{(a+1)^2}\frac{b^2}{4} - \frac{b^2}{2(a+1)} + c \\
\Leftrightarrow \quad & y^2 = \frac{(a-1)^2}{(a+1)^2(a-1)^2}\frac{b^2}{4} - \frac{2(a+1)(a-1)}{(a+1)^2(a-1)^2}\frac{b^2}{4} + \frac{4(a+1)^2(a-1)c}{4(a+1)^2(a-1)^2} \\
\Leftrightarrow \quad & y^2 = \frac{-(a^2+2a-3)b^2 + 4(a-1)(a+1)^2c}{4(a+1)^2(a-1)^2} \\
\Leftrightarrow \quad & y^2 = \frac{-(a^2+2a-3)b^2 + 4(a-1)(a+1)^2c}{4(a^2-1)^2} \\
\Leftrightarrow \quad & y^2 = \frac{(a-1)\left[4c(a+1)^2 - b^2(a+3)\right]}{4(a^2-1)^2}.
\end{aligned}
$$

In order to achieve the maximum number of distinct solutions we need $(a-1)\left[4c(a+1)^2 - b^2(a+3)\right]$ to be positive, which would lead to

$$4c(a+1)^2 > b^2(a+3) \quad \text{and} \quad a > 1,$$

or

$$4c(a+1)^2 < b^2(a+3), \quad a < 1 \quad \text{and} \quad a \neq -1.$$

In that case,

$$y = \pm\frac{1}{2}\frac{\sqrt{(a-1)\left[4c(a+1)^2 - b^2(a+3)\right]}}{a^2-1}.$$

If $4c(a+1)^2 > b^2(a+3)$ and $a > 1$ then we conclude that $a+3 > 0$ since $a > 1$ which implies $c > 0$. Therefore our conditions on a, b and c are as follows:

$$a > 1, \quad c > 0, \quad \text{and} \quad |b| < \frac{2\sqrt{c}\,(a+1)}{\sqrt{a+3}}.$$

However if $4c(a+1)^2 < b^2(a+3), a < 1$, and $a \neq -1$ we need to consider the sign of $a+3$.

(i) If $-3 < a < 1$ and $a \neq -1$ then $a+3 > 0$.

- If $c < 0$ then the inequality $4c(a+1)^2 < b^2(a+3)$ is always satisfied.
- If $c \geq 0$ then the following inequality $|b| > 2\sqrt{\frac{c}{a+3}}\ |a+1|$ must be satisfied.

(ii) If $a < -3$ then $a+3 < 0$, which implies that c must be negative and $|b| < 2\sqrt{\frac{c}{a+3}}\ |a+1|$.

So therefore, we see that in Case II ($y \neq 0$), we have two distinct solutions under any of the following conditions on the coefficients:

II(i) $a > 1,\ c > 0,$ and $|b| < \frac{2\sqrt{c}}{\sqrt{a+3}}\ (a+1)$;

II(ii) $-3 < a < 1,\ a \neq -1$, and $c < 0$.

II(iii) $b \neq 0$, $-3 < a < 1,\ a \neq -1$, $c \geq 0$, and $|b| > 2\sqrt{\frac{c}{a+3}}\ |a+1|$;

II(iv) $a < -3,\ c < 0$ and $|b| < 2\sqrt{\frac{c}{a+3}}\ |a+1|$.

To get four distinct solutions, we have to satisfy the conditions in Cases I and II.

Note that Conditions I(i), I(iii), and I(iv) are not compatible with any of the conditions II.

Conditions II(i) and I(v) are compatible when $a > 1$, $b \neq 0$, $c > 0$, and $2\sqrt{c(a-1)} < |b| < \frac{2\sqrt{c}}{\sqrt{a+3}}\ (a+1)$.

Conditions II(ii) and I(vi) are compatible when $-3 < a < 1,\ a \neq -1$, $b \neq 0$, $c < 0$, and $|b| > 2\sqrt{c(a-1)}$.

Conditions II(iv) and I(vi) are compatible when $a < -3$, $b \neq 0$, $c < 0$ and $2\sqrt{c(a-1)} < |b| < 2\sqrt{\frac{c}{a+3}}\ |a+1|$.

Conditions II(iii) and I(ii) are compatible when $-3 < a < 1$, $a \neq -1$, $b \neq 0$, $c > 0$, and $|b| > 2\sqrt{\frac{c}{a+3}}\ |a+1|$;

Therefore, Equation (2.1) will have four distinct solutions if and only if $b \neq 0$ and one of the following conditions is satisfied:

- $a > 1,\ c > 0,$ and $2\sqrt{c(a-1)} < |b| < \frac{2\sqrt{c}}{\sqrt{a+3}}\ (a+1)$;
- $-3 < a < 1,\ a \neq -1$, $c > 0$, and $|b| > 2\sqrt{\frac{c}{a+3}}\ |a+1|$;
- $-3 < a < 1,\ a \neq -1$, $c < 0$, and $|b| > 2\sqrt{c(a-1)}$;
- $a < -3,\ c < 0$ and $2\sqrt{c(a-1)} < |b| < 2\sqrt{\frac{c}{a+3}}\ |a+1|$, which is what we wanted to prove. ■

Notice that each of these 4 compatibility conditions leading to the maximum number of distinct solutions can occur, as illustrated by the following 4 figures.

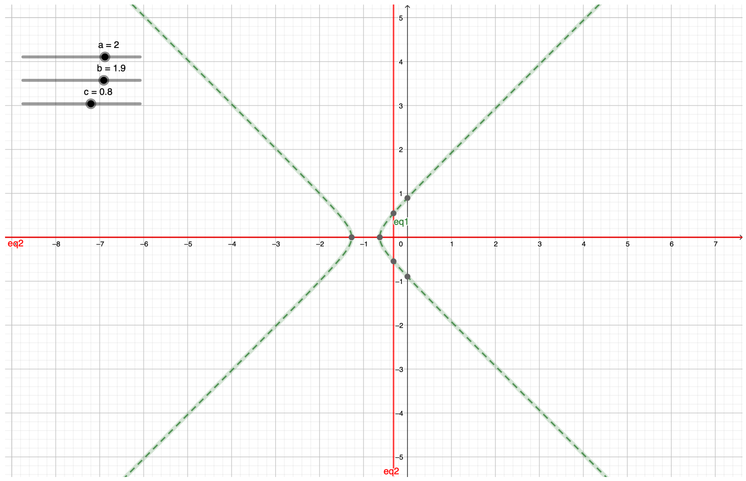

Figure 5. Four solutions when $a > 1$, $c > 0$ and $2\sqrt{c(a-1)} < |b| < \frac{2\sqrt{c}}{\sqrt{a+3}}\ (a+1)$, located at the intersection of the dotted curve and solid lines.

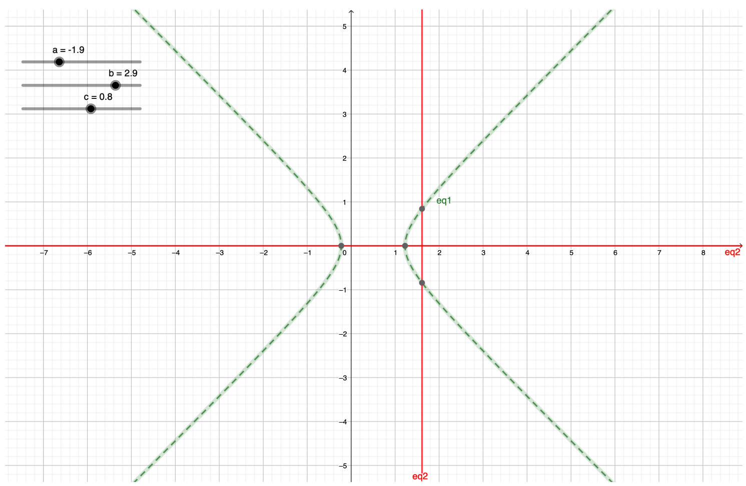

Figure 6. Four solutions when $-3 < a < 1$, $|b| > 2\sqrt{\frac{c}{a+3}}\ |a+1|$, and $c > 0$, located at the intersection of the dotted curve and solid lines.

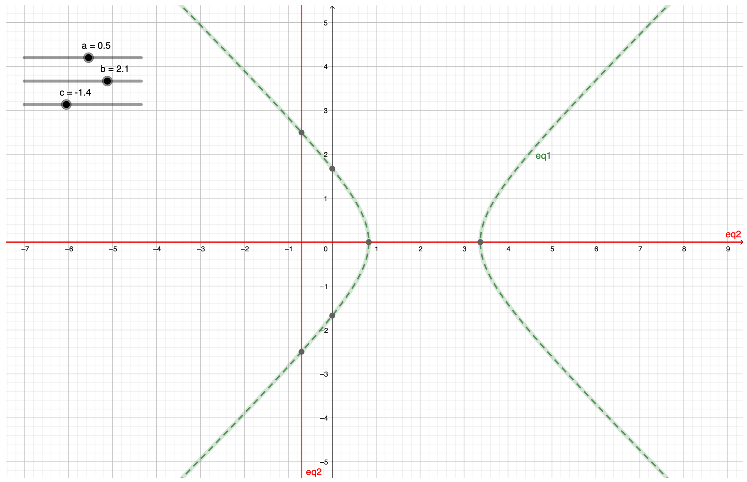

Figure 7. Four solutions when $-3 < a < 1$,$|b| > 2\sqrt{c(a-1)}$, and $c < 0$, located at the intersection of the dotted curve and solid lines.

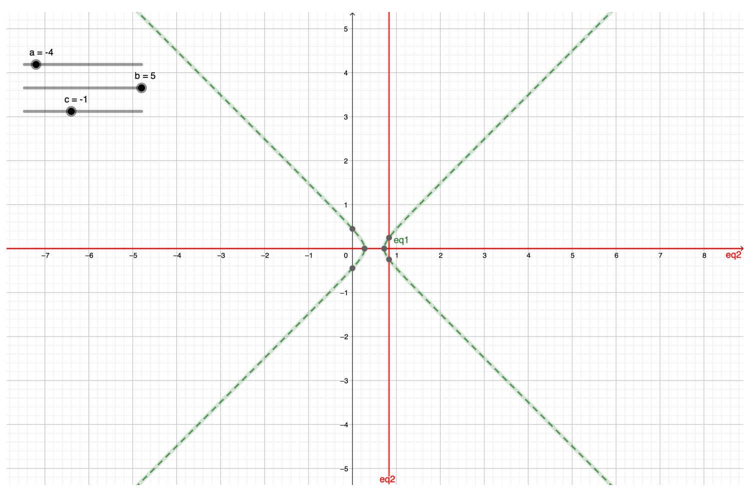

Figure 8. Four solutions when $a < -3$, $c < 0$ and $2\sqrt{c(a-1)} < |b| < 2\sqrt{\frac{c}{a+3}}\ |a+1|$, located at the intersection of the dotted curve and solid lines.

Remark. We can notice from the proof that the cases $a = 1$ and $a = -1$ are different:

- If $a = 1$, $b \neq 0$, and $b^2 \neq 4c$, we will have one solution to Equation (2.1), while if $a = 1$, $b \neq 0$, and $b^2 = 4c$, we will have an infinite numbers of solutions. That is because the real curve arising from Equation (2.2) gives a line parallel to the y-axis and intersects the x-axis at $\frac{-c}{b}$, while the real curves arising from Equation (2.3) are the x-axis and a line parallel to the y-axis, intersecting x-axis at $x = \frac{-b}{4}$. If $-\frac{c}{b} \neq -\frac{b}{4}$, i.e., $b^2 \neq 4c$, then we obtain one solution, while if $b^2 = 4c$, we obtain an infinite number of solutions.

- If $a = -1$ and $b \neq 0$, we will have two distinct solutions to Equation (2.1) under the condition $b^2 > -8c$, because Equation (2.2) gives rise to a hyperbola and Equation (2.3) gives rise to the x-axis, and those intersect in two points under the given condition. Otherwise, these curves do not intersect. If $a = -1$ and $b = 0$, we will obtain an infinite number of solutions.

The following figures illustrate the previous remark.

Figure 9. One solution when $a = 1$, $b \neq 0$, and $b^2 \neq 4c$, located at the intersection of the dotted curve and solid lines.

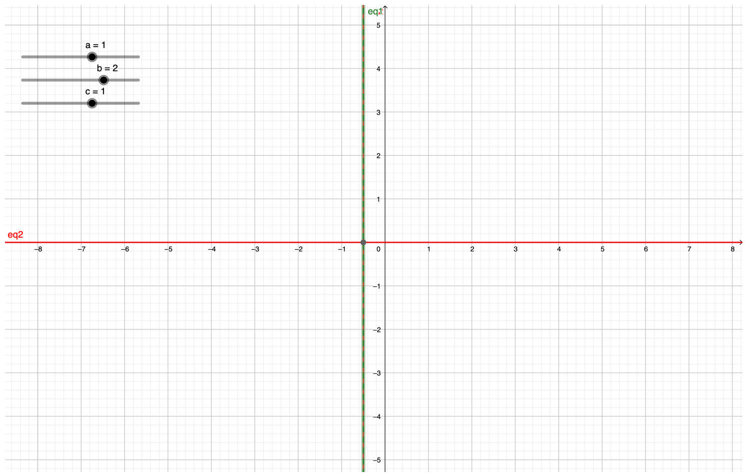

Figure 10. Infinite number of solution when $a = 1$, $b \neq 0$, and $b^2 = 4c$.

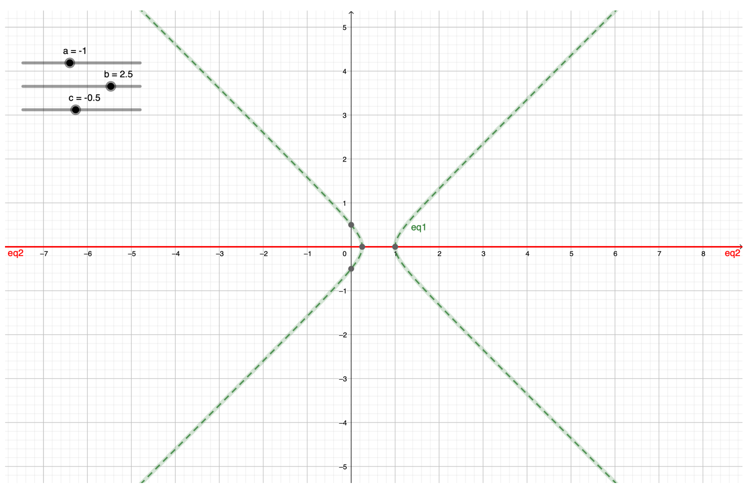

Figure 11. Two solutions when $a = -1$, $b \neq 0$, and $b^2 > -8c$, located at the intersection of the dotted curve and solid lines.

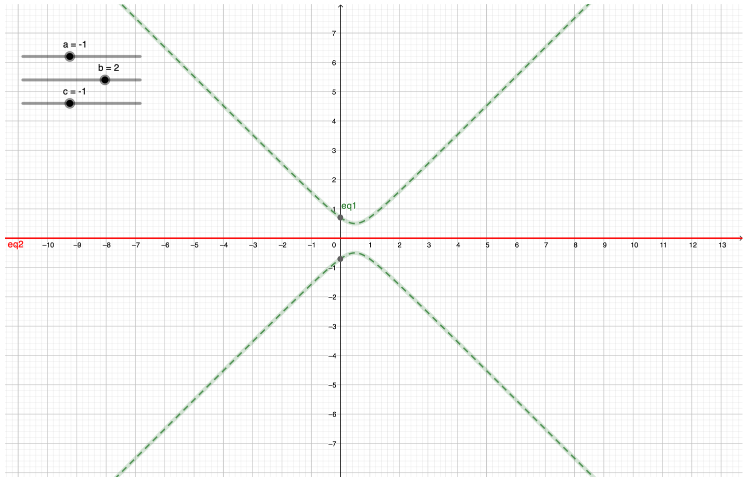

Figure 12. There is no solution when $a = -1$, $b \neq 0$, and $b^2 < -8c$.

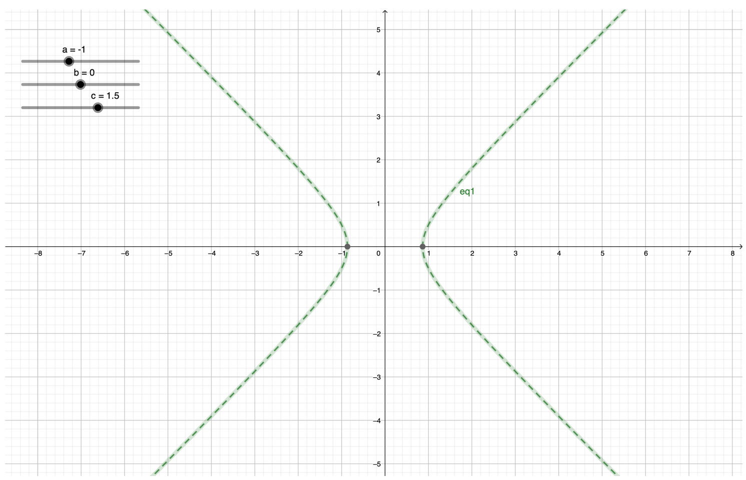

Figure 13. Infinite number of solutions when $a = -1$ and $b = 0$.

As we can see, even the case where p has real coefficients is somewhat complicated. Let us now consider the case that p has complex coefficients.

Note. If $a = \pm 1$ and $b = c = 0$ then $(a + ib)z^2 + (c + id)z + e + if = \bar{z}^2$ has an infinite number of solutions. So, we will only examine the case when the equation has a finite number of solutions.

Theorem 3. *The equation* $(a+ib)z^2+(c+id)z+e+if = \bar{z}^2$ *where* $a \neq 0$ *or* $b \neq 0$ *and* $a, b, c, d, e, f \in \mathbb{R}$ *has four distinct solutions if any of the following conditions are satisfied:*

- $a = 1$, $b \neq 0$, $d \neq 0$, $e = f = c = 0$, *and* $|b| < 2$.

- $a = 1$, $b \neq 0$, $c \neq 0$, $e = f = d = 0$, *and* $|b| < 2$.
- $a = 1$, $b \neq 0$, $f \neq 0$, $c = e = 0$, $d \neq 0$, $d^2 > 4bf$, *and* $4d^2 + 4fb^3 > b^2d^2$.
- $a = 1$, $b \neq 0$, $f \neq 0$, $d = e = 0$, $c \neq 0$, $c^2 + 4bf > 0$, *and* $4c^2 - b^2c^2 > 4fb^3$.
- $a \neq 1$, $c = d \neq 0$ $b = e = f = 0$, *and* $a \in (-\infty, -1) \cup (\frac{1}{3}, \infty)$.
- $b = e = f = c = 0$, $d \neq 0$, *and* $a \in (-3, -1) \cup (-1, 1)$.
- $b = e = f = d = 0$, $c \neq 0$, *and* $a \in (-3, -1) \cup (-1, 1)$.
- $a \neq \pm 1$, $b = e = 0$, $f \neq 0$, $c = d \neq 0$, $c^2 > 2(a+1)f$, *and* $(a+1)^2c^2 > -2(c^2 - f(a-1))(a^2-1)$.
- $a > 1$, $b = f = c = 0$, $d \neq 0$, $e < 0$ *and* $-4(a-1)e < d^2 < \frac{-4(a+1)^2e}{(a+3)}$.
- $a \neq -1$, $-3 < a < 1$, $b = f = c = 0$, $d \neq 0$, $e < 0$ *and* $d^2(a+3) > -4(a+1)^2e$.
- $a \neq -1$, $-3 < a < 1$, $b = f = c = 0$, $d \neq 0$, $e > 0$ *and* $d^2 > -4(a-1)e$.
- $a < -3$, $b = f = c = 0$, $d \neq 0$, $e > 0$, $d^2(a+3) > -4(a+1)^2e$ *and* $d^2 > -4(a-1)e$.
- $b = f = d = 0$, $c \neq 0$, $a > 1$, $e > 0$, $c^2 - 4(a-1)e > 0$, *and* $c^2(a+3) < 4(a+1)^2e$.
- $b = f = d = 0$, $c \neq 0$, $a < 1$, $a \neq -1$, $e < 0$, $c^2 - 4(a-1)e > 0$, *and* $c^2(a+3) > 4(a+1)^2e$.
- $b = f = d = 0$, $c \neq 0$, $-3 < a < 1$, $a \neq -1$, $e > 0$, *and* $c^2(a+3) > 4(a+1)^2e$.

Proof

Consider the equation

$$(a+ib)z^2 + (c+id)z + e + if = \bar{z}^2. \tag{2.5}$$

Letting $z = x + iy$ and $\bar{z} = x - iy$. and substituting into (2.5) gives:

$$(a+ib)(x+iy)^2 + (c+id)(x+iy) + e + if = (x-iy)^2$$
$$\Rightarrow ax^2 + 2iaxy - ay^2 + ibx^2 - 2bxy - iby^2 + cx + icy + idx - dy + e + if = x^2 - 2ixy - y^2$$

$$\Rightarrow (a-1)x^2 + (1-a)y^2 - 2bxy + cx - dy + e = 0$$
$$\text{and} \quad bx^2 - by^2 + (2a+2)xy + cy + dx + f = 0$$

$$\Rightarrow (a-1)(x^2-y^2) - 2bxy + cx - dy + e = 0 \tag{2.6}$$

$$\text{and} \quad b(x^2-y^2) + 2(a+1)xy + dx + cy + f = 0. \tag{2.7}$$

Case I : $a = 1$.

- If $b \neq 0$:
  - If $e = 0$ and $f = 0$:
    - If $c = 0$ and $d \neq 0$, substituting into (2.6) gives:

$$\begin{aligned} &-2bxy - dy = 0 \\ &\Rightarrow y(-2bx - d) = 0 \\ &\Rightarrow y = 0 \quad \text{or} \quad x = \frac{-d}{2b}. \end{aligned}$$

Substituting $y = 0$ into (2.7) gives:

$$bx^2 + dx = 0 \Rightarrow x = 0 \quad \text{or} \quad x = \frac{-d}{b}.$$

Now substituting $x = \frac{-d}{2b}$ into (2.7), we get:

$$4b^2y^2 + 8dy + d^2 = 0.$$

In order to achieve the maximum number of distinct solutions, we need this quadratic equation to have two solutions and thus, we have to have the following condition:

$$64d^2 - 16b^2d^2 > 0 \Leftrightarrow 16d^2(4-b^2) > 0 \Leftrightarrow -2 < b < 2.$$

In that case,

$$y = \frac{-8d \pm \sqrt{64d^2 - 16b^2d^2}}{8b^2}.$$

So, for this case we will have four solutions if $-2 < b < 2$ and $b \neq 0$.

* If $c \neq 0$ and $d = 0$, substituting into (2.6) gives:

$$\begin{aligned} & -2bxy + cx = 0 \\ & \Rightarrow x(-2by + c) = 0 \\ & \Rightarrow y = \frac{c}{2b} \quad \text{or} \quad x = 0. \end{aligned}$$

Substituting $x = 0$ into (2.7) gives:

$$-by^2 + cy = 0 \Rightarrow y = 0 \quad \text{or} \quad y = \frac{c}{b}.$$

Now substituting $y = \frac{c}{2b}$ into (2.7), we get:

$$4b^2x^2 + 8cx + c^2 = 0.$$

In order to achieve the maximum number of solutions, we need this quadratic equation to have two solutions and thus, we have to have the following condition:

$$64c^2 - 16b^2c^2 > 0 \Leftrightarrow 16c^2(4 - b^2) > 0 \Leftrightarrow -2 < b < 2.$$

In that case,

$$x = \frac{-8c \pm \sqrt{64c^2 - 16b^2c^2}}{8b^2}.$$

So, for this case we will have four solutions if $-2 < b < 2$ and $b \neq 0$.

– If $e = 0$ and $f \neq 0$:

* If $c = 0$ and $d \neq 0$, substituting into (2.6) gives:

$$\begin{aligned} &-2bxy - dy = 0 \\ &\Rightarrow y(-2bx - d) = 0 \\ &\Rightarrow y = 0 \quad \text{or} \quad x = \frac{-d}{2b}. \end{aligned}$$

Substituting $y = 0$ into (2.7) gives:

$$bx^2 + dx + f = 0.$$

In order to have two solutions, we have to have the following condition:

$$d^2 - 4bf > 0.$$

In that case,

$$x = \frac{-d \pm \sqrt{d^2 - 4bf}}{2b}.$$

Now substituting $x = \frac{-d}{2b}$ into (2.7), we get:

$$4b^2y^2 + 8dy - 4fb + d^2 = 0.$$

In order to have two solutions, we have to have the following condition:

$$64d^2 + 16(4fb^3 - b^2d^2) > 0 \Leftrightarrow 4d^2 + 4fb^3 > b^2d^2.$$

In that case,

$$y = \frac{-8d \pm \sqrt{64d^2 + 16(4fb^3 - b^2d^2)}}{8b^2}.$$

So, for this case we will have four solutions if $d^2 > 4bf$ and $4d^2 + 4fb^3 > b^2d^2$.

* If $c \neq 0$ and $d = 0$, substituting into (2.6) gives:

$$\begin{aligned} &-2bxy + cx = 0 \\ &\Rightarrow x(-2by + c) = 0 \\ &\Rightarrow y = \frac{c}{2b} \quad \text{or} \quad x = 0. \end{aligned}$$

Substituting $x = 0$ into (2.7) gives:

$$-by^2 + cy + f = 0.$$

In order to have two solutions, we have to have the following condition:

$$c^2 + 4bf > 0.$$

In that case,

$$y = \frac{c \pm \sqrt{c^2 + 4bf}}{2b}.$$

Now substituting $y = \frac{c}{2b}$ into (2.7), we get:

$$4b^2x^2 + 8cx + 4fb + c^2 = 0.$$

In order to have two solutions, we have to have the following condition:

$$64c^2 - 16(4fb^3 + b^2c^2) > 0 \Leftrightarrow 4c^2 - b^2c^2 > 4fb^3.$$

In that case,

$$x = \frac{-8c \pm \sqrt{64c^2 - 16(4fb^3 + b^2c^2)}}{8b^2}.$$

So, for this case we will have four solutions if $c^2 + 4bf > 0$ and $4c^2 - b^2c^2 > 4fb^3$.

Case II : $a \neq 1$.

- If $b = 0$:
  - If $e = 0$ and $f = 0$:
    - If $c = d \neq 0$, substituting into (2.6) gives:

$$\begin{aligned}
&(a-1)(x^2 - y^2) + c(x - y) = 0 \\
\Rightarrow &(x - y)\left[(a-1)(x + y) + c\right] = 0 \\
\Rightarrow &x = y \quad \text{or} \quad y = \frac{-c}{a-1} - x.
\end{aligned}$$

By substituting $x = y$ into (2.7) we get:

$$2(a+1)x^2 + 2cx = 0.$$

In order to have two solutions, we have to have $a \neq -1$.

In that case,

$$x = 0 \quad \text{or} \quad x = \frac{-c}{a+1}.$$

When $x = 0$, we get $y = 0$.

When $x = \frac{-c}{a+1}$, we get $y = \frac{-c}{a+1}$.

Now substituting $y = \frac{-c}{a-1} - x$ into (2.7), we get:

$$2(a^2 - 1)x^2 + 2(a+1)cx - c^2 = 0.$$

In order to have two solutions, we have to have $a \neq \pm 1$ and $4(a+1)^2c^2 + 8c^2(a^2-1) > 0$.

In that case,

$$x = \frac{-2(a+1)c \pm \sqrt{4(a+1)^2c^2 + 8c^2(a^2-1)}}{4(a^2-1)}.$$

Now, if $a \neq \pm 1$ then $4(a+1)^2c^2 + 8c^2(a^2-1) > 0$ in the following cases:

$$\begin{aligned}
&4(a+1)^2c^2 + 8c^2(a^2-1) > 0 \Leftrightarrow 4(a+1)^2c^2 > -8c^2\left(a^2-1\right) \\
\Leftrightarrow &(a+1)^2 > -2\left(a^2-1\right) \Leftrightarrow a^2 + 2a + 1 > -2a^2 + 2 \\
\Leftrightarrow &3a^2 + 2a - 1 > 0 \Leftrightarrow (a+1)\left(a - \frac{1}{3}\right) > 0 \\
\Leftrightarrow &a > -1 \text{ and } a > \frac{1}{3} \quad \text{or} \quad a < -1 \text{ and } < \frac{1}{3} \\
\Leftrightarrow &a > \frac{1}{3} \quad \text{or} \quad a < -1.
\end{aligned}$$

So, we will have four solutions if $a \in (-\infty, -1) \cup (\frac{1}{3}, \infty) \quad \forall c, d \in \mathbb{R} - \{0\}$ where $c = d$.

* If $c = 0$ and $d \neq 0$, substituting into (2.7) gives:

$$2(a+1)xy + dx = 0.$$

In order to have two solutions, we have to have $a \neq -1$.

In that case,

$$x = 0 \quad \text{or} \quad y = \frac{-d}{2(a+1)}.$$

By substituting $x = 0$ into (2.6) we get:

$$\begin{aligned}
&-(a-1)y^2 - dy = 0 \\
\Rightarrow\ &y = 0 \quad \text{or} \quad y = \frac{-d}{a-1}.
\end{aligned}$$

Now substituting $y = \frac{-d}{2(a+1)}$ into (2.6), we get:

$$(a-1)x^2 + T = 0 \quad \text{where} \quad T = \frac{d^2(a+3)}{4(a+1)^2}$$
$$\Rightarrow x^2 = \frac{T}{1-a} \quad \text{since} \quad a \neq 1.$$

In order to achieve the maximum number of solutions, we need $a \neq \pm 1$ and $\frac{d^2(a+3)}{4(a+1)^2(1-a)} > 0$.

In that case,

$$x = \pm \frac{\sqrt{-4(a-1)T}}{2(a-1)}.$$

Now, if $a \neq \pm 1$ then $\frac{d^2(a+3)}{4(a+1)^2(1-a)} > 0$ in the following case:

$$\frac{d^2(a+3)}{4(a+1)^2(1-a)} > 0$$
$$\Leftrightarrow a+3 > 0 \quad \text{and} \quad 1-a > 0 \qquad \text{or} \qquad a+3 < 0 \quad \text{and} \quad 1-a < 0$$
$$\Leftrightarrow a > -3 \quad \text{and} \quad 1 > a \qquad \text{or} \qquad a < -3 \quad \text{and} \quad 1 < a$$
$$\Leftrightarrow a \in (-3,-1) \cup (-1,1) \quad \text{since} \quad a \neq 1.$$

So, we will have four solutions if $a \in (-3,-1) \cup (-1,1) \quad \forall d \in \mathbb{R} - \{0\}$.

* If $c \neq 0$ and $d = 0$, substituting into (2.7) gives:

$$2(a+1)xy + cy = 0.$$

In order to have two solutions, we have to have $a \neq -1$.

In that case,

$$y = 0 \quad \text{or} \quad x = \frac{-c}{2(a+1)}.$$

By substituting $y = 0$ into (2.6) we get:

$$(a-1)x^2 + cx = 0$$
$$\Rightarrow x = 0 \quad \text{or} \quad x = \frac{-c}{a-1}.$$

Now substituting $x = \frac{-c}{2(a+1)}$ into (2.6), we get:

$$(a-1)y^2 + G = 0 \quad \text{where} \quad G = \frac{c^2(a+3)}{4(a+1)^2}$$
$$\Rightarrow y^2 = \frac{G}{1-a} \quad \text{since} \quad a \neq 1.$$

In order to achieve the maximum number of solutions, we need $a \neq \pm 1$ and $\frac{c^2(a+3)}{4(a+1)^2(1-a)} > 0$.

In that case,

$$y = \pm\frac{\sqrt{-4(a-1)G}}{2(a-1)}.$$

Now, if $a \neq \pm 1$ then $\frac{c^2(a+3)}{4(a+1)^2(1-a)} > 0$ in the following case:

$$\frac{c^2(a+3)}{4(a+1)^2(1-a)} > 0$$
$$\Leftrightarrow a+3>0 \quad \text{and} \quad 1-a>0 \qquad \text{or} \qquad a+3<0 \quad \text{and} \quad 1-a<0$$
$$\Leftrightarrow a>-3 \quad \text{and} \quad 1>a \qquad \text{or} \qquad a<-3 \quad \text{and} \quad 1<a$$
$$\Leftrightarrow a \in (-3,-1) \cup (-1,1) \quad \text{since} \quad a \neq 1.$$

So, we will have four solutions if $a \in (-3,-1) \cup (-1,1) \quad \forall c \in \mathbb{R} - \{0\}$.

– If $e = 0$ and $f \neq 0$:

∗ If $c = d \neq 0$, substituting into (2.6) gives:

$$(a-1)(x^2-y^2) + c(x-y) = 0$$
$$\Rightarrow (x-y)\left[(a-1)(x+y)+c\right] = 0$$
$$\Rightarrow x = y \quad \text{or} \quad y = \frac{-c}{a-1} - x.$$

By substituting $x = y$ into (2.7) we get:

$$2(a+1)x^2 + 2cx + f = 0.$$

In order to obtain two solutions, we have to have $a \neq -1$ and $4c^2 - 8(a+1)f > 0$.

In that case,

$$x = \frac{-2c \pm \sqrt{4c^2 - 8(a+1)f}}{4(a+1)}.$$

Now substituting $y = \frac{-c}{a-1} - x$ into (2.7), we get:

$$2(a^2-1)x^2 + 2(a+1)cx - c^2 + f(a-1) = 0.$$

In order to achieve the maximum number of solutions, we need this quadratic equation to have two solutions and thus, we have to have $a \neq \pm 1$ and $(a+1)^2c^2 + 2(c^2 - f(a-1))(a^2-1) > 0$. In that case,

$$x = \frac{-2(a+1)c \pm \sqrt{4(a+1)^2c^2 + 4(c^2 - f(a-1))(2(a^2-1))}}{4(a^2-1)}.$$

So, we will have four solutions if $c^2 > 2(a+1)f$ and $(a+1)^2c^2 + 2(c^2 - f(a-1))(a^2-1) > 0$.

– If $e \neq 0$ and $f = 0$:

  * If $c = 0$ and $d \neq 0$, substituting into (2.7) gives:

$$2(a+1)xy + dx = 0.$$

In order to have two solutions, we have to have $a \neq -1$.

In that case,

$$x = 0 \quad \text{or} \quad y = \frac{-d}{2(a+1)}.$$

By substituting $x = 0$ into (2.6) we get:

$$(a-1)y^2 + dy - e = 0.$$

In order to have two solutions, we have to have $a \neq 1$ and $d^2 + 4(a-1)e > 0$.

In that case,

$$y = \frac{-d \pm \sqrt{d^2 + 4(a-1)e}}{2(a-1)}.$$

Now, if $a \neq 1$ then $d^2 + 4(a-1)e > 0$ in the following cases.

Condition I gives the following:

· $a > 1$ and $e > 0$.

· $a < 1$ and $e < 0$.

· $a > 1$ and $e < 0$ and $d^2 + 4(a-1)e > 0$.

- $a < 1$ and $e > 0$ and $d^2 + 4(a-1)e > 0$.

Now, substituting $y = \frac{-d}{2(a+1)}$ into (2.6), we get:

$$
\begin{aligned}
&(a-1)x^2 - \frac{d^2(a-1)}{4(a+1)^2} + \frac{d^2}{2(a+1)} + e = 0 \\
&x^2 - \frac{d^2}{4(a+1)^2} + \frac{d^2}{2(a+1)(a-1)} + \frac{e}{a-1} = 0 \\
&\Rightarrow x^2 = \frac{d^2(a-1)}{4(a+1)^2(a-1)} - \frac{2(a+1)d^2}{4(a+1)^2(a-1)} - \frac{4(a+1)^2 e}{4(a+1)^2(a-1)} \\
&\Rightarrow x^2 = \frac{d^2(a-1-2a-2) - 4(a+1)^2 e}{4(a+1)^2(a-1)} \\
&\Rightarrow x^2 = \frac{-d^2(a+3) - 4(a+1)^2 e}{4(a+1)^2(a-1)}.
\end{aligned}
$$

In order to achieve the maximum number of solutions, we need $a \neq \pm 1$ and $\frac{-d^2(a+3)-4(a+1)^2 e}{4(a+1)^2(a-1)} > 0$, which would lead to

$$d^2(a+3) + 4(a+1)^2 e > 0, \quad 1 - a > 0 \quad \text{and} \quad a \neq -1,$$

or

$$d^2(a+3) + 4(a+1)^2 e < 0 \quad \text{and} \quad 1 - a < 0.$$

In that case,

$$x = \pm \frac{1}{2(a+1)} \sqrt{\frac{-d^2(a+3) - 4(a+1)^2 e}{a-1}}.$$

If $a \neq \pm 1, d^2(a+3) + 4(a+1)^2 e < 0,$ and $a > 1$ then $e > 0$ is such that $d^2(a+3) + 4(a+1)^2 e < 0$.

If $a \neq \pm 1, d^2(a+3) + 4(a+1)^2 e > 0,$ and $a < 1$ then we have two cases:

- If $-3 < a < 1$ where $a \neq -1$ then $e > 0$ or $e < 0$ is such that $d^2(a+3)+4(a+1)^2 e > 0$.
- If $a < -3$ then $e > 0$ is such that $d^2(a+3) + 4(a+1)^2 e > 0$.

So, Condition II gives the following:

- $a > 1$, $e < 0$ and $d^2(a+3) + 4(a+1)^2 e < 0$.
- $a \neq -1$, $-3 < a < 1$, $e < 0$ and $d^2(a+3) > -4(a+1)^2 e$.
- $a \neq -1$, $-3 < a < 1$ and $e > 0$.
- $a < -3$, $e > 0$ and $d^2(a+3) > -4(a+1)^2 e$.

By combining the first and second conditions to get four solutions we have to have one of the following:

- $a > 1$, $e < 0$ and $-4(a-1)e < d^2 < \frac{-4(a+1)^2 e}{(a+3)}$.
- $a \neq -1$, $-3 < a < 1$, $e < 0$ and $d^2(a+3) > -4(a+1)^2 e$.
- $a \neq -1$, $-3 < a < 1$, $e > 0$ and $d^2 > -4(a-1)e$.
- $a < -3$, $e > 0$, $d^2(a+3) > -4(a+1)^2 e$ and $d^2 > -4(a-1)e$.

∗ If $c \neq 0$ and $d = 0$, substituting into (2.7) gives:

$$2(a+1)xy + cy = 0.$$

In order to have two solutions, we have to have $a \neq -1$.

In that case,

$$y = 0 \quad \text{or} \quad x = \frac{-c}{2(a+1)}.$$

By substituting $y = 0$ into (2.6) we get:

$$(a-1)x^2 + cx + e = 0.$$

In order to have two solutions, we have to have $a \neq -1$ and $c^2 - 4(a-1)e > 0$. In that case,

$$x = \frac{-c \pm \sqrt{c^2 - 4(a-1)e}}{2(a-1)}.$$

Now, if $a \neq 1$ then $c^2 - 4(a-1)e > 0$ in the following cases.

Condition I gives the following:

- $a > 1$ and $e < 0$.
- $a < 1$, $e > 0$ and $a \neq -1$.
- $a > 1$, $e > 0$ and $c^2 - 4(a-1)e > 0$.
- $a < 1$, $e < 0$, $a \neq -1$ and $c^2 - 4(a-1)e > 0$.

Now substituting $x = \frac{-c}{2(a+1)}$ into (2.6), we get:

$$\begin{aligned}
&-(a-1)y^2 + \frac{c^2(a-1)}{4(a+1)^2} - \frac{c^2}{2(a+1)} + e = 0 \\
\Rightarrow &(a-1)y^2 = \frac{c^2(a-1)}{4(a+1)^2} - \frac{c^2}{2(a+1)} + e \\
\Rightarrow \quad & y^2 = \frac{c^2(a-1)}{4(a+1)^2(a-1)} - \frac{c^2}{2(a+1)(a-1)} + \frac{e}{(a-1)} \\
\Rightarrow \quad & y^2 = \frac{c^2(a-1) - 2(a+1)c^2 + 4(a+1)^2 e}{4(a+1)^2(a-1)} \\
\Rightarrow \quad & y^2 = \frac{(a-1-2a-2)c^2 + 4(a+1)^2 e}{4(a+1)^2(a-1)} \\
\Rightarrow \quad & y^2 = \frac{-(a+3)c^2 + 4(a+1)^2 e}{4(a+1)^2(a-1)}.
\end{aligned}$$

In order to achieve the maximum number of solutions, we need $a \neq \pm 1$ and $\frac{-(a+3)c^2+4(a+1)^2 e}{4(a+1)^2(a-1)} > 0$, which would lead to

$$-(a+3)c^2 + 4(a+1)^2 e > 0 \quad \text{and} \quad a - 1 > 0,$$

or

$$-(a+3)c^2 + 4(a+1)^2 e < 0, a - 1 < 0, \quad \text{and} \quad a \neq -1.$$

In that case,

$$y = \pm \frac{1}{2(a+1)} \sqrt{\frac{-c^2(a+3) + 4(a+1)^2 e}{a-1}}.$$

If $a \neq \pm 1, -(a+3)c^2 + 4(a+1)^2 e > 0,$ and $a > 1$ then $e > 0$ such that $-(a+3)c^2 + 4(a+1)^2 e > 0$.

If $a \neq \pm 1, -(a+3)c^2 + 4(a+1)^2 e < 0,$ and $a < 1$ then we have two cases:

- If $-3 < a < 1$ where $a \neq -1$ then$a + 3 > 0$ which implies $e > 0$ such that $(a+3)c^2 > 4(a+1)^2 e$ or $e < 0$.
- If $a < -3$ then $e < 0$ such that $(a+3)c^2 > 4(a+1)^2 e$.

So, Condition II gives the following:

- $a > 1$, $e > 0$ and $c^2(a+3) < 4(a+1)^2 e$.
- $-3 < a < 1$, $e > 0$, $a \neq -1$ and $c^2(a+3) > 4(a+1)^2 e$.
- $a < 1$, $e < 0$, $a \neq -1$ and $c^2(a+3) > 4(a+1)^2 e$.

By combining the first and second conditions to get four solutions we should have one of the following:

- $a > 1$ and $e > 0$ and $c^2 - 4(a-1)e > 0$ and $c^2(a+3) < 4(a+1)^2 e$.
- $-3 < a < 1$ and $e > 0$ and $a \neq -1$ and $c^2(a+3) > 4(a+1)^2 e$.
- $a < 1$ and $e < 0$ and $a \neq -1$ and $c^2 - 4(a-1)e > 0$ and $c^2(a+3) > 4(a+1)^2 e$, which is what we wanted to prove. ■

Remark.

- We will have infinite number of solutions under the following conditions:
  - $a = -1$, $b = f = c = d = 0$ and $e \in \mathbb{R}$, since substituting into (2.6) and (2.7) gives a hyperbola $x^2 - y^2 = \frac{e}{2}$, which means we will have an infinite number of solutions.
  - $a \neq 1$, $b \neq 0$,and $e = f = c = d = 0$, since substituting into (2.6) and (2.7) gives $(x^2 - y^2)(b^2 + 2a^2 - 2) = 0$ which implies that if $b^2 + 2a^2 = 2$ we will have an infinite number of solutions.
- For $a = 1$ and $b = 0$, we get at most one or two solutions under some conditions on the coefficients. For more details, see Appendix A.
- When we studied the remaining cases we could not state nice conditions on the coefficients in order to achieve the maximum number of solutions. But we know that we will have either a hyperbola and a rational functions or two hyperbolas which will give us maximum four intersections, so, in that case we will have at most four solutions. For more details, see Appendix A.

Notice that even if the degree p is equal two, it is complicated to determine which conditions on the coefficients of p give rise to the maximum number of solutions. Given this fact, it is natural to pose the following.

Open Problem. If p has degree n and if the equation $p(z) = \bar{z}^n$ has a finite number of solutions, which conditions on the coefficients of p give rise to the maximum number of distinct solutions?

Chapter 3

Uniform Mass Density lenses and Number of Images

In Chapter 1, we discussed the relationship between rational harmonic functions and the numbers of images for the lens equation

$$w = z - \int_{\Omega} \frac{d\mu(\zeta)}{\bar{z} - \bar{\zeta}} - \gamma \bar{z}, \tag{3.1}$$

where w denote the source position, z the position of the lensed image, γ is a real constant representing the normalized shear, and Ω is the lens, modeled by a domain in $\mathbb{C}$. Many authors have investigated the number of zeros of the lens equation for continuous densities. For more details see [1, 4, 3, 11, 2].

In this chapter, we will see how many images we can get for different kinds of lenses, i.e. for different Ω's. In the first two sections, we will present in detail the techniques that have been utilized to get the number of solutions for the lens equation when the shape of the lens is an ellipse or a limaçon [11, 2]. In particular, we disprove a conjecture of Bénéteau and Hudson in [2]. In the third section, we will use the same technique to study the maximum number of solutions for the lens equation for a Neumann Oval, and we will examine a special case of a modified Neumann Oval that achieves the maximum number of solutions.

3.1 Ellipse with uniform mass density

In 2009, D. Fassnacht, C. R. Keeton and D. Khavinson [11] investigated the number of solutions to the lens equation for an ellipse shaped lens with uniform mass density. In this section, we will explain their technique and results.

3.1.1 The Lens Equation

An ellipse lens which is given by $\Omega := \{z = x + iy | \frac{x^2}{a^2} + \frac{y^2}{b^2} < 1\}$ has area $ab\pi$, where $a > b > 0$ are fixed. Let $\Gamma := \partial\Omega$ be the ellipse. The lens equation for an ellipse-shaped lens Ω with uniform mass density

$d\mu(\zeta) = \frac{1}{\pi} dA(\zeta)$, is:

$$\bar{w} = \bar{z} - \frac{1}{\pi} \int\limits_{\Omega} \frac{dA}{z - \zeta} - \gamma z, \tag{3.2}$$

where $dA := dA(\zeta) = \frac{d\zeta\, d\bar{\zeta}}{2i}$ is area measure, and the integral in the lens equation (3.2) involves a normalizing factor $\frac{1}{\pi}$, in order to have area of ellipse equal to 1 in the case that $ab = 1$.

In order to solve the lens equation (3.2), the authors of [11] rewrite the integral as a line integral by using the complex form of Green's Theorem, which we state as follows.

Theorem 4. Complex Form of Green's Theorem *(GT) [40]*
Let $F(z, \bar{z})$ be continuous and have continuous partial derivatives in a region $\mathcal{R}$ and on its boundary C, where $z = x + iy, \bar{z} = x - iy$ are complex conjugate coordinates. Then Green's theorem can be written in the complex form

$$\oint_C F(z, \bar{z}) dz = 2i \iint_{\mathcal{R}} \frac{\partial F}{\partial \bar{z}} dA,$$

where dA represents the element of area $dxdy$, and C is oriented in the positive direction.

To calculate the integral in the lens equation (3.2), we need to distinguish between z inside or outside the domain Ω, so we will have the image inside the galaxy which called a dim image, or outside the galaxy which called a bright image.

For $z \notin \Omega$,

$$-\frac{1}{\pi} \int\limits_{\Omega} \frac{dA}{z - \zeta} = -\frac{1}{2\pi i} \left(2i \int\limits_{\Omega} \frac{1}{z - \zeta}\, dA \right)$$

Letting $F(\zeta, \bar{\zeta}) = \frac{\bar{\zeta}}{z - \zeta}$ which is well-defined $\forall \zeta \in \Omega$ and C^1. So, by using Green Theorem on Ω where $\frac{\partial F}{\partial \bar{\zeta}} = \frac{1}{z - \zeta}$ we have:

$$\begin{aligned} -\frac{1}{\pi} \int\limits_{\Omega} \frac{dA}{z - \zeta} &= -\frac{1}{2\pi i} \int\limits_{\partial\Omega} \frac{\bar{\zeta}}{z - \zeta}\, d\zeta \\ &= \frac{1}{2\pi i} \int\limits_{\partial\Omega} \frac{\bar{\zeta}\, d\zeta}{\zeta - z}. \end{aligned} \tag{3.3}$$

For $z \in \Omega$,

By applying Green Theorem again in $\Omega - D(z, \epsilon)$ and then take a limit when $\epsilon \to 0$, we get that:

$$-\frac{1}{\pi} \int\limits_{\Omega} \frac{dA}{z - \zeta} = -\bar{z} + \frac{1}{2\pi i} \int\limits_{\partial\Omega} \frac{\bar{\zeta}\, d\zeta}{\zeta - z}. \tag{3.4}$$

3.1.2 The Schwarz Function

As we notice, in equations (3.3) and (3.4), we have the same Cauchy integral

$$\frac{1}{2\pi i}\int_{\Gamma}\frac{\bar{\zeta}\,d\zeta}{\zeta - z}.$$

In order to work with this Cauchy integral, we will replace $\bar{\zeta}$ by a function equal to $\bar{\zeta}$ on Γ but analytic in a neighborhood of Γ. This function is called the Schwarz function. More specifically, assume that Γ is an analytic arc which can be defined in the complex plane by the function $f\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) \equiv F(z,\bar{z}) = 0$, where $F(z,\bar{z})$ is analytic function and $\bigtriangledown f = \frac{\partial F}{\partial \bar{z}} \neq 0$ on Γ. Then, there exists a unique function $S(z)$, called the Schwarz function of Γ, such that $S(z) = \bar{z}$ on Γ and is analytic in a neighborhood of Γ. For more details, see [13, 38].

The Schwarz function $S(\zeta)$ for the ellipse can be calculated as follows (see, e.g., [11]):

$$\begin{aligned}
&\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\\
&b^2x^2 + a^2y^2 = a^2b^2\\
&b^2\left(\frac{\zeta+\bar{\zeta}}{2}\right)^2 + a^2\left(\frac{\zeta-\bar{\zeta}}{2i}\right)^2 = a^2b^2\\
&\frac{b^2}{4}(\zeta+\bar{\zeta})^2 - \frac{a^2}{4}(\zeta-\bar{\zeta})^2 = a^2b^2\\
&b^2\left(\zeta^2 + 2\zeta\bar{\zeta} + \bar{\zeta}^2\right) - a^2\left(\zeta^2 - 2\zeta\bar{\zeta} + \bar{\zeta}^2\right) = 4a^2b^2\\
&\left(b^2 - a^2\right)\zeta^2 + 2\left(b^2 + a^2\right)\zeta\bar{\zeta} + \left(b^2 - a^2\right)\bar{\zeta}^2 = 4a^2b^2\\
&c^2\bar{\zeta}^2 - 2\left(b^2 + a^2\right)\zeta\,\bar{\zeta} + c^2\zeta^2 + 4a^2b^2 = 0, \qquad \text{where} \quad c^2 = a^2 - b^2.
\end{aligned}$$

As we see this equation is a second degree equation in $\bar{\zeta}$, so the authors of [11] used the quadratic formula in order to find $\bar{\zeta}$:

$$\bar{\zeta} = \frac{a^2+b^2}{c^2}\zeta \pm \frac{2ab}{c^2}\left(\sqrt{\zeta^2 - c^2}\right),$$

where they chose the branch cut inside the ellipse between $-c$ and c. Moreover, we need to choose the sign in front of the square root that gives the correct value for $\bar{\zeta}$ when $\zeta \in \Gamma$. For example, when $\zeta = a$, we get

$$\begin{aligned} \frac{a^2+b^2}{c^2}a - \frac{2ab}{c^2}\sqrt{a^2-c^2} &= \frac{a^3+ab^2}{a^2-b^2} - \frac{2ab\sqrt{a^2-a^2+b^2}}{a^2-b^2} \\ &= \frac{a^3+ab^2-2ab^2}{a^2-b^2} \\ &= \frac{a^3-ab^2}{a^2-b^2} = \frac{a\left(a^2-b^2\right)}{a^2-b^2} \\ &= a = \bar{\zeta}. \end{aligned}$$

Therefore, we choose the negative sign, and obtain the Schwarz function of the ellipse:

$$\begin{aligned} \bar{\zeta} &= \frac{a^2+b^2}{c^2}\zeta - \frac{2ab}{c^2}\left(\sqrt{\zeta^2-c^2}\right) \\ &= \frac{a^2+b^2-2ab}{c^2}\zeta - \frac{2ab}{c^2}\left(\sqrt{\zeta^2-c^2}-\zeta\right) \\ &= \psi_1(\zeta) + \psi_2(\zeta), \end{aligned}$$

where $\psi_1(\zeta) = \frac{a^2+b^2-2ab}{c^2}\zeta$ and $\psi_2(\zeta) = -\frac{2ab}{c^2}\left(\sqrt{\zeta^2-c^2}-\zeta\right)$. Note that ψ_1 is analytic in $\overline{\Omega}$ while ψ_2 is analytic in $\mathbb{C} - \overline{\Omega}$ and $\psi_2(\infty) = 0$.

3.1.3 Solving the Lens Equation and Counting Solutions

As we discussed in Section 3.1.1, the authors of [11] used Green's Theorem to evaluate the integral in the lens equation (3.2), which then becomes the following line integral :

$$\int_{\Gamma} \frac{\bar{\zeta}\, d\zeta}{\zeta - z}.$$

This line integral can be rewritten in the following form by using the Schwarz function from Section 3.1.2 as

$$\int_{\partial\Omega} \frac{\psi_1(\zeta)+\psi_2(\zeta)\, d\zeta}{\zeta - z}.$$

Now, for $z \in \mathbb{C} - \overline{\Omega}$ we have $\frac{\psi_1(\zeta)}{\zeta-z}$ is analytic inside $\overline{\Omega}$, thus, by using Cauchy's Theorem, we get that $\frac{1}{2\pi i}\int_{\partial\Omega} \frac{\psi_1(\zeta)}{\zeta-z}\ d\zeta = 0$. On the other hand, $\psi_2(\zeta)$ is analytic outside $\overline{\Omega}$, thus, by using the Cauchy Integral

Formula, we get $\frac{1}{2\pi i}\int\limits_{\partial\Omega}\frac{\psi_2(\zeta)}{\zeta-z}\,d\zeta=\psi_2(z)$. So, from (3.2) and (3.3) for $z\notin\Omega$ the lens equation becomes:

$$\begin{aligned}\bar{w}&=\bar{z}+\frac{2ab}{c^2}\left(-\sqrt{z^2-c^2}+z\right)-\gamma z\\ \bar{z}-\left(\gamma-\frac{2ab}{c^2}\right)z-\bar{w}&=\frac{2ab}{c^2}\sqrt{z^2-c^2}.\end{aligned}\tag{3.5}$$

By squaring both sides, the authors of [11] got a complex quadratic equation. In addition, by replacing z by $x+iy$ and $\bar{z}$ by $x-iy$ where x and y are in $\mathbb{R}$ and $\bar{w}$ by $s-im$ where s and m are constants in $\mathbb{R}$, they obtain a system of two real equations of degree 2, and therefore, by Bézout's Theorem, there are at most 4 images outside the ellipse.

In like manner, for $z\in\overline{\Omega}$, we have $\psi_1(\zeta)$ is analytic inside $\overline{\Omega}$, thus,by using the Cauchy Integral Formula, we get $\frac{1}{2\pi i}\int\limits_{\partial\Omega}\frac{\psi_1(\zeta)}{\zeta-z}\,d\zeta=\psi_1(z)$. On the other hand, we have $\frac{\psi_2(\zeta)}{\zeta-z}$ is analytic outside $\overline{\Omega}$ and $\psi_2(\infty)=0$, thus, by using Cauchy's Theorem, $\frac{1}{2\pi i}\int\limits_{\partial\Omega}\frac{\psi_2(\zeta)}{\zeta-z}\,d\zeta=0$. So, from (3.2) and (3.4) for $z\in\Omega$ the lens equation becomes:

$$\bar{w}=z\left(\frac{a^2+b^2-2ab}{c^2}-\gamma\right),\tag{3.6}$$

which is a linear equation in one parameter, thus, they got one solution. Therefore, the authors of [11] obtained the following theorem:

Theorem 5. *[11] An elliptic lens Ω (say, a galaxy) with a uniform mass density may produce at most four "bright" lensing images of a point light source outside Ω and one ("dim") image inside Ω, i.e., at most 5 lensing images altogether.*

Remark. As we notice from the lens equation (3.5) and (3.6), having the shearing term will not affect the number of solutions.

Let us now turn to a similar discussion with a different shape.

3.2 Limaçon with uniform mass density

In 2018, C. Bénéteau and N. Hudson were interested in the number of solutions of the lens equation if the lens is a limaçon shape [2]. In this section, we will explain their technique and the results they got.

3.2.1 The Lens Equation

A limaçon lens (see Figure [14]), given by $\Omega:=\left\{z=re^{i\theta}\in\mathbb{C}\mid r<a+b\cos\theta\right\}$ has area $\pi\left(a^2+\frac{b^2}{2}\right)$, where $a,\,b\in\mathbb{R}$ are fixed. Let $\Gamma:=\partial\Omega$ be the limaçon. The lens equation for an limaçon-shaped lens Ω

with uniform mass density $d\mu(\zeta) = \frac{2}{3\pi} dA(\zeta)$, is:

$$\bar{w} = \bar{z} - \frac{2}{3\pi} \int\limits_{\Omega} \frac{dA}{z - \zeta} - \gamma z. \tag{3.7}$$

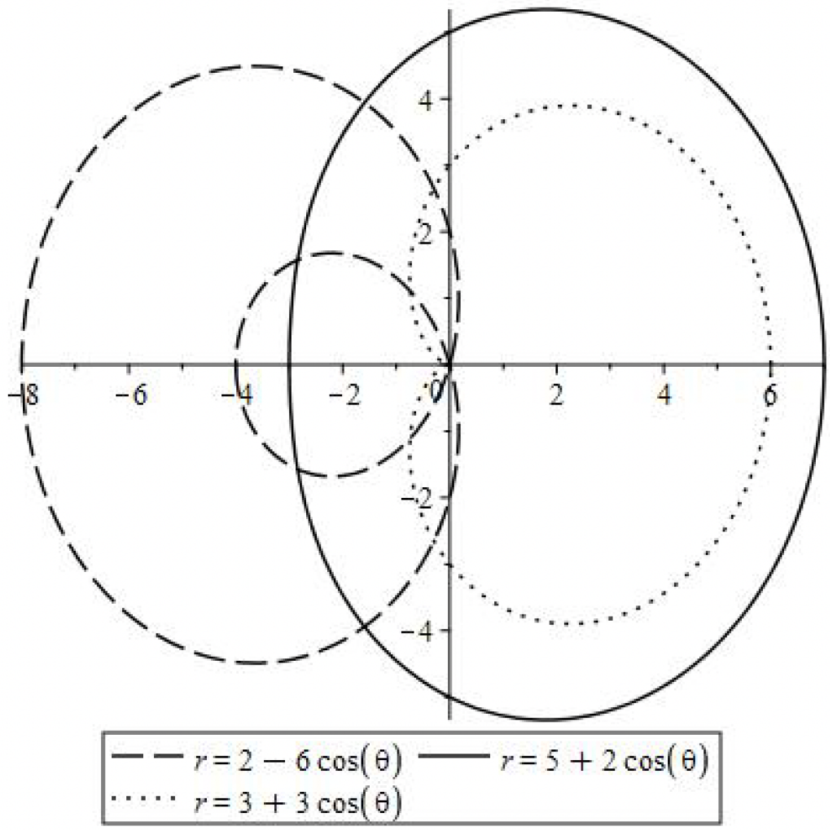

Figure 14. Graph of three different limaçons

where $dA := dA(\zeta) = \frac{d\zeta\, d\bar{\zeta}}{2i}$ is area measure, and the integral involves a normalizing factor of $\frac{2}{3\pi}$ in order to have the area of the limaçon equal to 1, for $\left(a^2 + \frac{b^2}{2}\right) = \frac{3}{2}$. In order to solve the lens equation (3.7), the authors of [2] rewrote the integral as a line integral by using the complex form of Green's Theorem (see Theorem 4). For $z \notin \Omega$, as before,

$$-\frac{2}{3\pi} \int\limits_{\Omega} \frac{dA}{z - \zeta} = -\frac{2}{3} \left(\frac{1}{2\pi i} \iint\limits_{\Omega} \frac{d\zeta\, d\bar{\zeta}}{z - \zeta} \right) = \frac{2}{3} \left(\frac{1}{2\pi i} \int\limits_{\partial\Omega} \frac{\bar{\zeta}\, d\zeta}{\zeta - z} \right).$$

For $z \in \Omega$,

$$-\frac{2}{3\pi}\int_{\Omega}\frac{dA}{z-\zeta} = -\frac{2}{3}\left(\frac{1}{2\pi i}\iint_{\Omega}\frac{d\zeta\, d\bar{\zeta}}{z-\zeta}\right) = -\frac{2}{3}\bar{z} + \frac{2}{3}\left(\frac{1}{2\pi i}\int_{\partial\Omega}\frac{\bar{\zeta}\, d\zeta}{\zeta - z}\right).$$

3.2.2 The Schwarz Function

As before, let us calculate the Schwarz function of the limaçon so that $\bar{\zeta} = S(\zeta)$ on Γ, where S is analytic in a neighborhood of Γ. Since $r = a + b\cos(\theta)$ and $r^2 = \zeta\bar{\zeta}$, we can rewrite the equation of the limaçon as

$$\begin{aligned}
&\zeta\bar{\zeta} - \frac{b}{2}(\zeta + \bar{\zeta}) = a(\zeta\bar{\zeta})^{\frac{1}{2}} \\
&(\zeta\bar{\zeta})^2 + \frac{b^2}{4}\left(\zeta^2 + \bar{\zeta}^2 + 2\zeta\bar{\zeta}\right) - b\left(\zeta^2\bar{\zeta} + \zeta\bar{\zeta}^2\right) = a^2\zeta\bar{\zeta} \\
&\bar{\zeta}^2\left(\zeta - \frac{b}{2}\right)^2 - \bar{\zeta}\zeta\left(b\left(\zeta - \frac{b}{2}\right) + a^2\right) + \frac{b^2}{4}\zeta^2 = 0.
\end{aligned}$$

Since this equation is quadratic in $\bar{\zeta}$, we can apply the quadratic formula to obtain:

$$\begin{aligned}
\bar{\zeta} =& \frac{1}{2\left(\zeta - \frac{b}{2}\right)^2}\zeta\left(b\left(\zeta - \frac{b}{2}\right) + a^2\right) \\
&\pm \frac{1}{2\left(\zeta - \frac{b}{2}\right)^2}\sqrt{\zeta^2\left(b\left(\zeta - \frac{b}{2}\right) + a^2\right)^2 - 4\left(\zeta - \frac{b}{2}\right)^2\frac{b^2}{4}\zeta^2} \\
=& \frac{\zeta}{2\left(\zeta - \frac{b}{2}\right)^2}\left[b\left(\zeta - \frac{b}{2}\right) + a^2 \pm a\sqrt{2b\left(\zeta + \frac{a^2 - b^2}{2b}\right)}\right],
\end{aligned}$$

as long as we suitably define the square root. In order to simplify the above expression, we define $A := \frac{a^2-b^2}{2b}$.

The authors of [2] chose the branch cut along the ray $(-\infty, -A]$ with $a > b > 0$. Moreover, we need to choose the sign in front of the square root that gives the correct value for $\bar{\zeta}$ when $\zeta \in \Gamma$. For example, when $\zeta = ia$, we get

$$\begin{aligned}
&\frac{ia}{2\left(ia - \frac{b}{2}\right)^2}\left[iab - \frac{b^2}{2} + a^2 + a\sqrt{i2ab + a^2 - b^2}\right] \\
&\qquad = \frac{ia}{2\left(ia - \frac{b}{2}\right)^2}\left[iab - \frac{b^2}{2} + a^2 + a(a + ib)\right] \\
&\qquad = \frac{ia}{2\left(ia - \frac{b}{2}\right)^2}\left[iab - \frac{b^2}{2} + 2a^2 + iab\right] \\
&\qquad = \frac{-2ia\left(ia - \frac{b}{2}\right)^2}{2\left(ia - \frac{b}{2}\right)^2} = -ia = \bar{\zeta}.
\end{aligned}$$

Therefore, we choose the positive sign for the equation of the Schwarz function.

$$\begin{aligned}\bar{\zeta} &= \frac{\zeta}{2\left(\zeta - \frac{b}{2}\right)^2}\left[b\left(\zeta - \frac{b}{2}\right) + a^2 + a\sqrt{2b\left(\zeta + \frac{a^2 - b^2}{2b}\right)}\right] \\ &= \psi_1(\zeta) + \psi_2(\zeta),\end{aligned}$$

where $\psi_1(\zeta) = \frac{\zeta}{2\left(\zeta - \frac{b}{2}\right)^2}\left[b\left(\zeta - \frac{b}{2}\right) + a^2\right]$ and $\psi_2(\zeta) = \frac{a\,\zeta\,\sqrt{2b\left(\zeta + \frac{a^2 - b^2}{2b}\right)}}{2\left(\zeta - \frac{b}{2}\right)^2}$. Note that ψ_1 is analytic in $\mathbb{C} - \{\frac{b}{2}\}$ and ψ_2 is analytic in $\Omega - \{\frac{b}{2}\}$, where $\frac{b}{2}$ lies inside Ω.

3.2.3 Solving the Lens Equation and Counting Solutions

As before, to solve the lens equation, we need to calculate the following integral

$$\int_{\partial\Omega} \frac{\bar{\zeta}\, d\zeta}{\zeta - z},$$

which can be rewritten in the following form by using the Schwarz function as:

$$\int_{\partial\Omega} \frac{\psi_1(\zeta) + \psi_2(\zeta)\, d\zeta}{\zeta - z}.$$

Define $f(\zeta) := \frac{\psi_1(\zeta) + \psi_2(\zeta)}{\zeta - z}$. Thus, for $z \notin \Omega$ we will use the Residue Theorem at $\{\frac{b}{2}\}$ since $f(\zeta)$ is analytic in $\Omega - \{\frac{b}{2}\}$. We get

$$\begin{aligned}\frac{1}{2\pi i}\int_{\partial\Omega} f(\zeta)d\zeta &= \text{Res}\left(f(\zeta); \frac{b}{2}\right) \\ &= g_1\left(\frac{b}{2}\right),\end{aligned}$$

where

$$\begin{aligned}f(\zeta) &= \frac{\zeta}{2(\zeta - z)\left(\zeta - \frac{b}{2}\right)^2}\left[b\left(\zeta - \frac{b}{2}\right) + a^2 + a\sqrt{2b\left(\zeta + \frac{a^2 - b^2}{2b}\right)}\right] \\ f(\zeta) &= \frac{g(\zeta)}{\left(\zeta - \frac{b}{2}\right)^2} \Rightarrow g(\zeta) = \frac{\zeta}{2(\zeta - z)}\left[b\left(\zeta - \frac{b}{2}\right) + a^2 + a\left(2b\zeta + a^2 - b^2\right)^{1/2}\right].\end{aligned}$$

$$\begin{aligned}g_1(\zeta) := g'(\zeta) &= \frac{2(\zeta - z) - 2\zeta}{4(\zeta - z)^2}\left[b\left(\zeta - \frac{b}{2}\right) + a^2 + a\left(2b\zeta + a^2 - b^2\right)^{1/2}\right] + \\ &\quad \frac{\zeta}{2(\zeta - z)}\left[b + \frac{a}{2}\left(2b + a^2 - b^2\right)^{-\frac{1}{2}} \cdot 2b\right] \\ g_1\left(\frac{b}{2}\right) &= \frac{-2za^2 + \left(\frac{b}{2} - z\right)b^2}{2\left(\frac{b}{2} - z\right)^2}.\end{aligned}$$

Therefore,

$$\frac{1}{2\pi i}\int\limits_{\partial\Omega} f(\zeta)d\zeta = \frac{-2za^2 + \left(\frac{b}{2} - z\right)b^2}{2\left(\frac{b}{2} - z\right)^2}.$$

On the other hand, for $z \in \Omega$ we will use the Residue Theorem at $\left\{\frac{b}{2}, z\right\}$ since $f(\zeta)$ is analytic in $\Omega - \left\{\frac{b}{2}, z\right\}$, and so:

$$\begin{aligned}\frac{1}{2\pi i}\int\limits_{\partial\Omega} f(\zeta)d\zeta &= \text{Res}\left(f(\zeta), \frac{b}{2}\right) + \text{Res}\left(f(\zeta), z\right) \\ &= g_1\left(\frac{b}{2}\right) + h(z).\end{aligned}$$

where

$$\begin{aligned}f(\zeta) &= \frac{\zeta}{2(\zeta - z)\left(\zeta - \frac{b}{2}\right)^2}\left[b\left(\zeta - \frac{b}{2}\right) + a^2 + a\sqrt{2b\left(\zeta + \frac{a^2 - b^2}{2b}\right)}\right] \\ f(\zeta) &= \frac{h(\zeta)}{\zeta - z} \Rightarrow h(\zeta) = \frac{\zeta}{2\left(\zeta - \frac{b}{2}\right)^2}\left[b\left(\zeta - \frac{b}{2}\right) + a^2 + a\left(2b\zeta + a^2 - b^2\right)^{\frac{1}{2}}\right] \\ h(z) &= \frac{z}{2\left(z - \frac{b}{2}\right)^2}\left[b\left(z - \frac{b}{2}\right) + a^2 + a\left(2bz + a^2 - b^2\right)^{\frac{1}{2}}\right] \\ h(z) &= \frac{z}{2\left(z - \frac{b}{2}\right)^2}\left[-b\left(\frac{b}{2} - z\right) + a^2 + a\sqrt{2b(z + A)}\right].\end{aligned}$$

So:

$$\begin{aligned}\frac{1}{2\pi i}\int\limits_{\partial\Omega} f(\zeta)d\zeta &= \frac{-2za^2 + \left(\frac{b}{2} - z\right)b^2}{2\left(\frac{b}{2} - z\right)^2} + \frac{z}{2\left(z - \frac{b}{2}\right)^2}\left[-b\left(\frac{b}{2} - z\right) + a^2 + a\sqrt{2b(z + A)}\right] \\ &= \frac{b(b/2 - z)(b - z) - a^2 z + az\sqrt{2b(z + A)}}{2(z - b/2)^2}.\end{aligned}$$

Thus, the authors of [2] proved the following theorem:

Theorem 6. *[2] Suppose $\Omega := \{z = re^{i\theta} \in \mathbb{C} \mid r < a + b\ cos(\theta)\}$ is the interior of a limaçon with constants chosen so that $a > b > 0$, with constant mass density $d\mu(\zeta) = \frac{2}{3\pi}dA(\zeta)$. Then for $z \notin \Omega$, the lens equation is:*

$$\bar{w} = \bar{z} + \frac{2}{3}\left[\frac{-2a^2 z + b^2(b/2 - z)}{2(b/2 - z)^2}\right] - \gamma z, \tag{3.8}$$

while for $z \in \Omega$, the lens equation is:

$$\bar{w} = \bar{z} + \frac{2}{3}\left[\frac{b(b/2 - z)(b - z) - a^2 z + az\sqrt{2b(z + A)}}{2(z - b/2)^2}\right] - \gamma z. \tag{3.9}$$

In order to solve the lens equation (3.8), they replace z by $x + iy$ and $\bar{z}$ by $x - iy$ where x and y are in $\mathbb{R}$. Also, by assuming that $\bar{w} = s - im$ where s and m are real constants, they got a system of two real equations of degree 3, and therefore, there are at most 9 solutions to the lens equation (3.8) by Bézout's Theorem. Notice that, the Khavinson and Neumann theorem [19] for rational functions produces $5n - 5$ images so, for $n = 3$ is 10 solutions, which is not improvement in this case. In like manner, if the shearing term vanishes, we get $n = 2$, and thus there are 5 solutions by the Khavinson-Neumann theorem, while Bézout's theorem still gives 9 solutions.

On the other hand, for the lens equation (3.9) they could not utilize the Khavinson-Neumann theorem since Equation (3.9) is not a rational function because it has a square root. By rewriting the complex variables in terms of real variables and using some algebraic manipulation, they could rewrite Equation (3.9) as a system of two real equations. In addition, they mentioned that the shearing term does not impact the degree of real part, which is 4, while it affects the degree of the imaginary part, which is 3 if there is no shearing term ($\gamma = 0$) and 4 otherwise. Therefore, they obtain the following theorem.

Theorem 7. *[2] Suppose* $\Omega := \left\{z = re^{i\theta} \in \mathbb{C} \mid r < a + b\cos\theta\right\}$ *is the interior of a limaçon with constants chosen so that* $a > b > 0$, *with constant mass density* $d\mu(\zeta) = \frac{2}{3\pi} dA(\zeta)$. *Then there are at most 9 solutions to the lens equation (3.8) if the shearing term* $\gamma \neq 0$*, and at most 5 solutions if* $\gamma = 0$*, while there are at most 16 solutions to the lens equation (3.9) if the shearing term* $\gamma \neq 0$, *and at most 12 solutions if* $\gamma = 0$.

The authors of [2] mentioned that the expected number of solutions of the lens equation (3.9) is most likely over estimated. Moreover, based on numerical experiments, they conjectured that there is at most 1 solution to the lens equation (3.8) and at most 4 solutions to the lens equation (3.9).

After reviewing the system of real equations which is produced by Equation (3.8) and redrawing them with the different values of a and b, we produced a counter example to the previous conjecture as you will see in Figures [15] and [16].

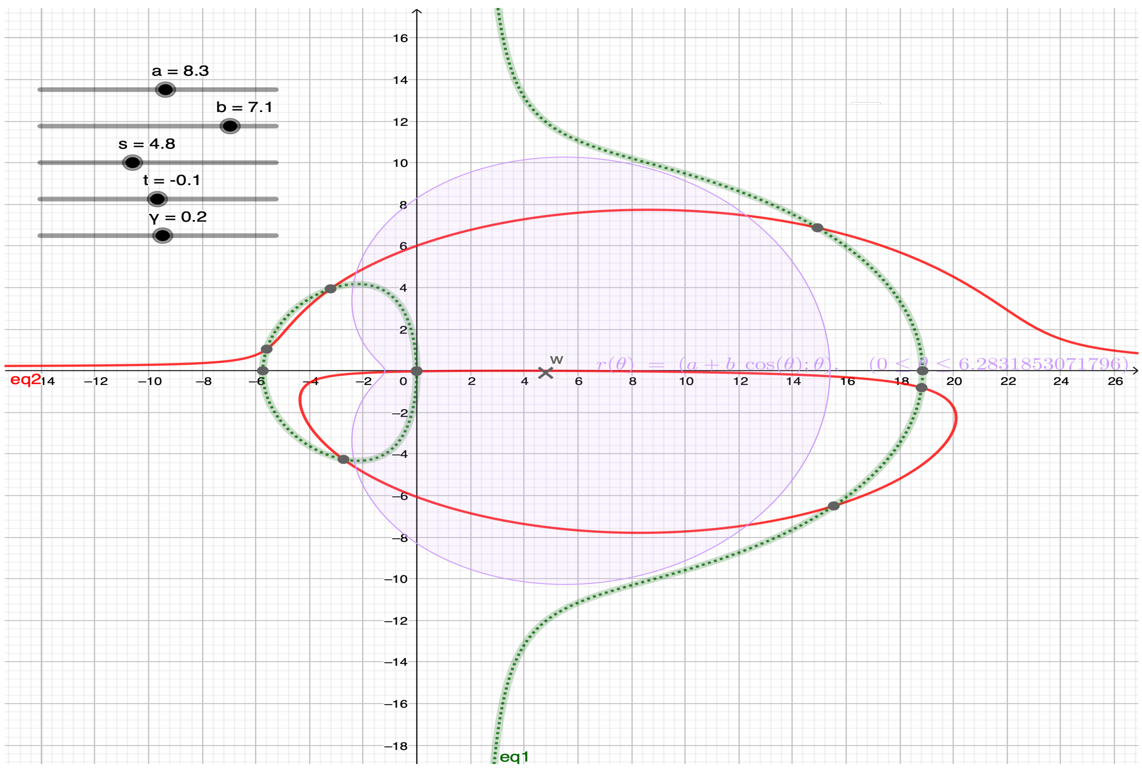

Figure 15. Zeros of lens equation which is a limaçon shaped lens for $r = 8.3 + 7.1\cos(\theta)$ and source position $w = 4.8 - 0.1i$ with shearing term $\gamma \neq 0$. There are 6 images, located at the intersection of the dotted and solid curves that lie outside of the limaçon.

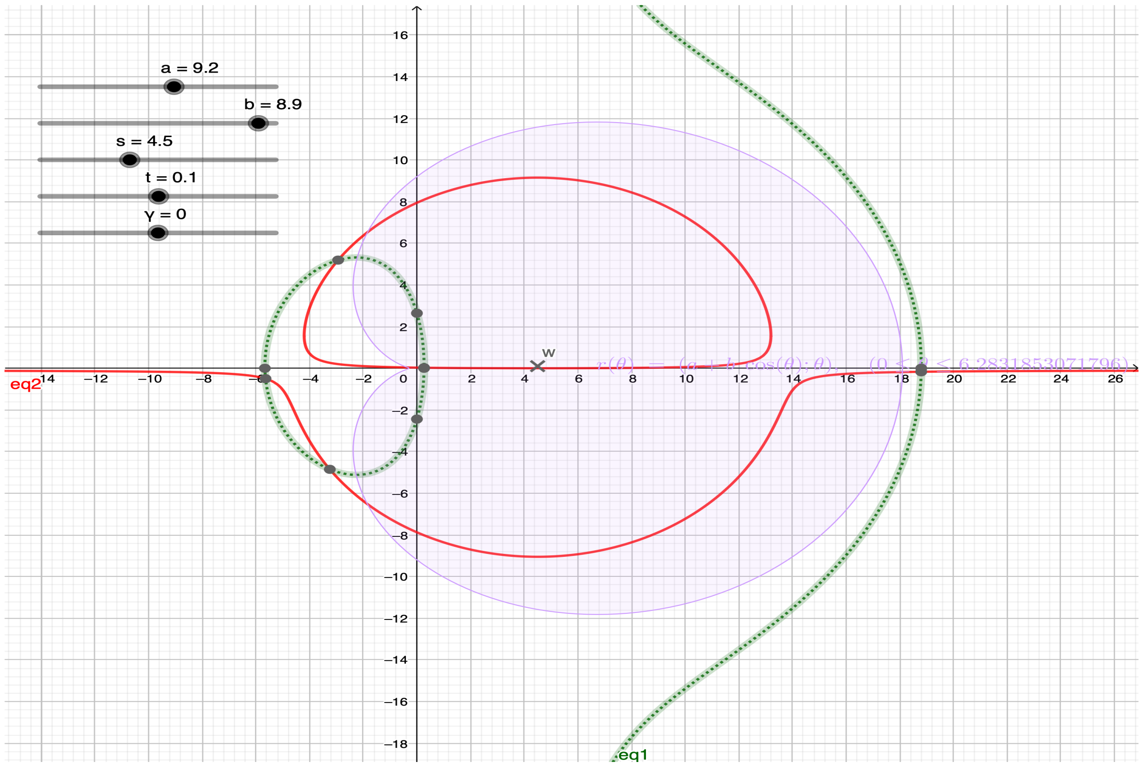

Figure 16. Zeros of lens equation which is a limaçon shaped lens for $r = 9.2 + 8.9\cos(\theta)$ and source position $w = 4.5 + 0.1i$ with shearing term $\gamma = 0$. There are 4 images, located at the intersection of the dotted and solid curves that lie outside of the limaçon.

From Figures [15] and [16] we can see that we will have 6 images which lie outside of the limaçon when $\gamma \neq 0$, while we will have 4 images which lie outside of the limaçon when $\gamma = 0$ which contradicts the conjecture of the authors of [2].

Also, the authors in [2] assumed that $\gamma = 0$ for $z \notin \Omega$ and they got one solution. If we choose $\gamma \neq 0$, we can get more solutions, as in Figure [17].

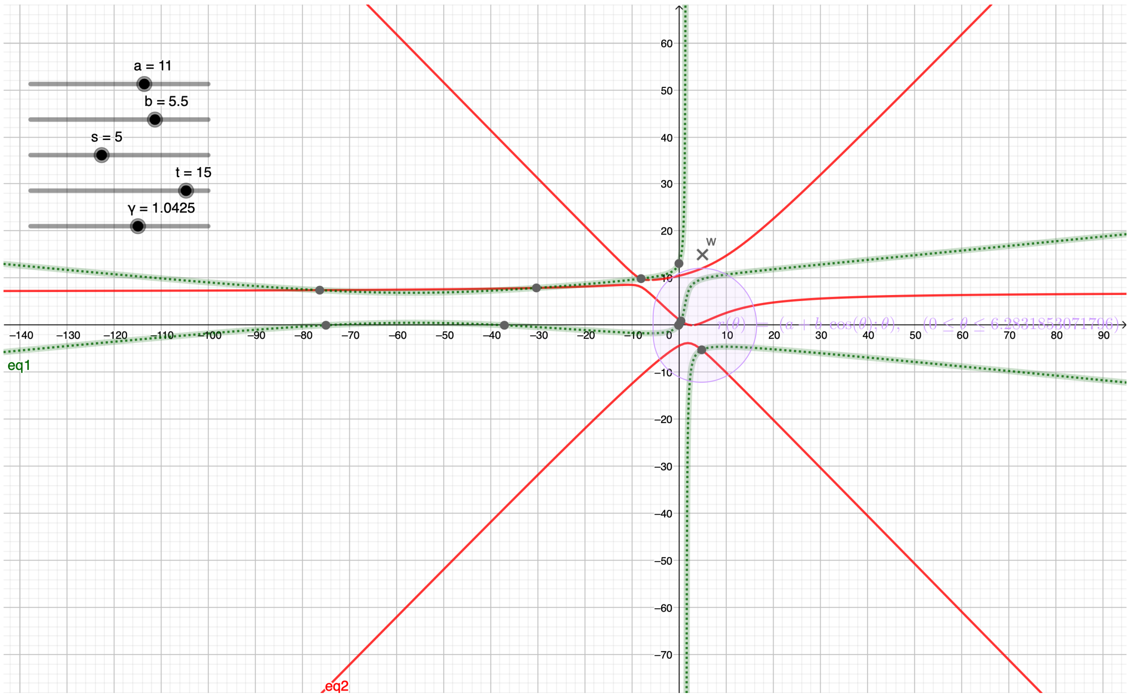

Figure 17. Zeros of lens equation which is a limaçon shaped lens for $r = 11 + 5.5\cos(\theta)$ and source position $w = 5 + 15i$ with shearing term $\gamma \neq 0$. There are 3 images, located at the intersection of the dotted and solid curves that lie outside of the limaçon.

In addition, for the lens equation (3.9),there are 4 solutions when $\gamma \neq 0$ and 2 solutions when $\gamma = 0$. See Figures [18] and [19].

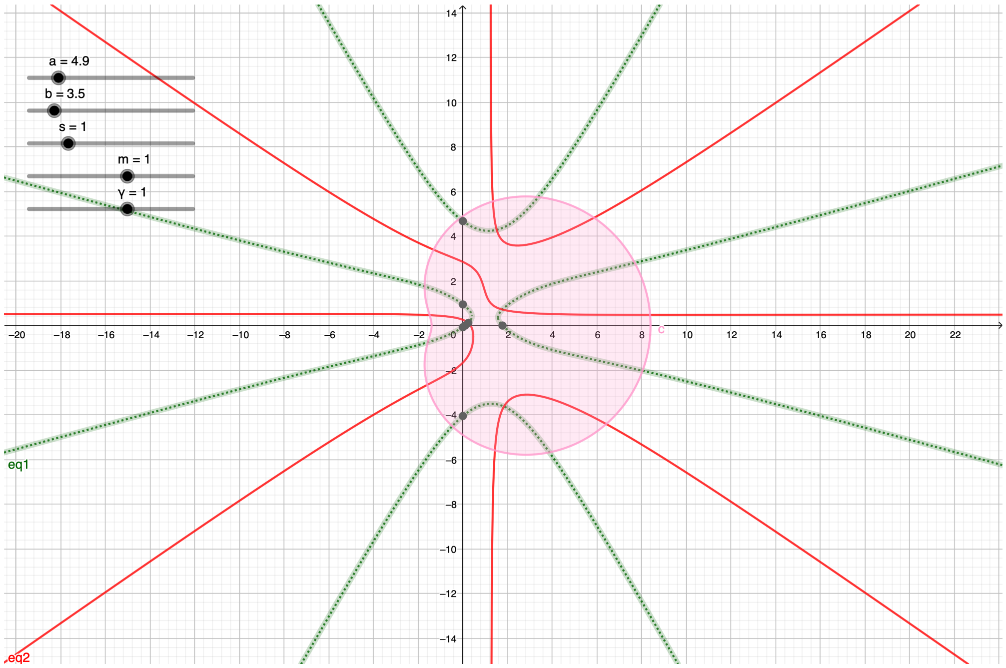

Figure 18. Zeros of lens equation which is a limaçon shaped lens for $r = 4.9 + 3.5\cos(\theta)$ and source position $w = 1 + i$ with shearing term $\gamma \neq 0$. There are 4 images, located at the intersection of the dotted and solid curves that lie inside of the limaçon.

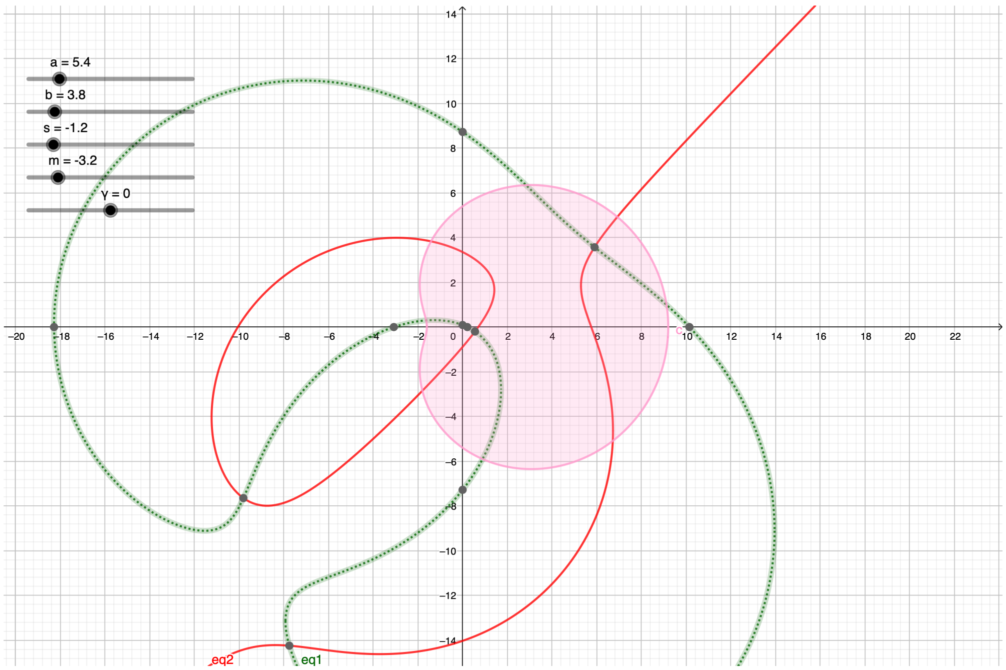

Figure 19. Zeros of lens equation which is a limaçon shaped lens for $r = 5.4 + 3.8\cos(\theta)$ and source position $w = -1.2 - 3.2i$ with shearing term $\gamma = 0$. There are 2 images, located at the intersection of the dotted and solid curves that lie inside of the limaçon.

As a result of the previous discussion we have the following conjecture:

Conjecture 1. Suppose $\Omega := \left\{z = re^{i\theta} \in \mathbb{C} \mid r < a + b\cos\theta\right\}$ is the interior of a limaçon with constants chosen so that $a > b > 0$, with constant mass density $d\mu(\zeta) = \frac{2}{3\pi}dA(\zeta)$. Then there are at most 6 solutions (outside) to the lens equation (3.8) when $\gamma \neq 0$ and 4 solutions (outside) when $\gamma = 0$. Moreover, there is at most 4 solutions (inside) to the lens equation (3.9) when $\gamma \neq 0$ and 2 solutions (inside) when $\gamma = 0$.

3.3 Neumann Oval with uniform mass density

Now, let us look at a different shape called Neumann Oval to see how many images we will get. In this section we will use the same technique that have been used for the limaçon shape.

3.3.1 The Lens Equation

A Neumann Oval (see Figure [20]) is a shape whose polar equation given by $r^2 = a^2 + 4b^2\cos^2(\theta)$, where $a\,, b \in \mathbb{R}$. Note that, the area of Neumann Oval is $2\pi(a^2 + 2b^2)$. Let $\Omega := \{z = re^{i\theta} \in \mathbb{C}\ |r^2 < a^2 + 4b^2\cos^2(\theta)\ \}$, where a and b are fixed. Let $\Gamma = \partial\Omega$ be the Neumann Oval. The lens equation for a Neumann Oval-shaped lens with uniform mass density $d\mu(\zeta) = \frac{1}{3\pi}dA(\zeta)$, is:

$$\bar{w} = \bar{z} - \frac{1}{3\pi}\int\limits_{\Omega}\frac{dA}{z-\zeta} - \gamma z. \tag{3.10}$$

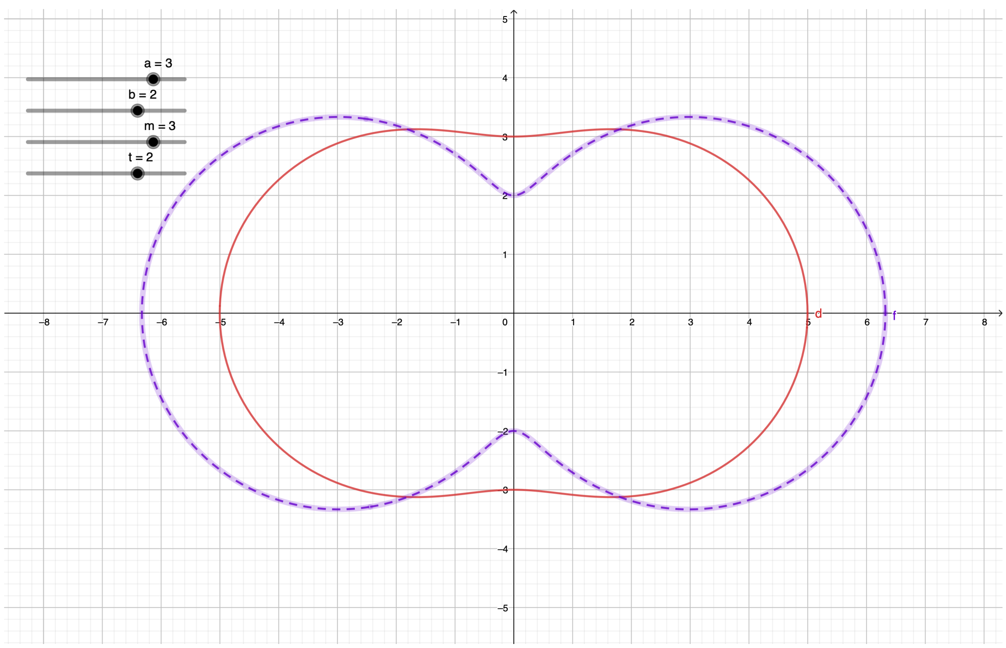

Figure 20. Neumann Oval shapes.

where $dA := dA(\zeta) = \frac{d\zeta\, d\bar{\zeta}}{2i}$ is area measure, and the integral in the lens equation (3.10) involve a normalizing factor $\frac{1}{3\pi}$ in order to have area of Neumann Oval equal 1 when $(a^2 + 2b^2) = \frac{3}{2}$. To be able to calculate the integral in the lens equation (3.10) we will use the complex form of Green's Theorem as we explained before in Section 3.1.1 in order to convert that integral to a line integral. For $z \notin \Omega$,

$$-\frac{1}{3\pi}\int_{\Omega}\frac{dA}{z-\zeta} = -\frac{1}{3}\left(\frac{1}{2\pi i}\iint_{\Omega}\frac{d\zeta\, d\bar{\zeta}}{z-\zeta}\right) = \frac{1}{3}\left(\frac{1}{2\pi i}\int_{\partial\Omega}\frac{\bar{\zeta}\, d\zeta}{\zeta - z}\right).$$

For $z \in \Omega$,

$$-\frac{1}{3\pi}\int_{\Omega}\frac{dA}{z-\zeta} = -\frac{1}{3}\left(\frac{1}{2\pi i}\iint_{\Omega}\frac{d\zeta\, d\bar{\zeta}}{z-\zeta}\right) = -\frac{1}{3}\bar{z} + \frac{1}{3}\left(\frac{1}{2\pi i}\int_{\partial\Omega}\frac{\bar{\zeta}\, d\zeta}{\zeta - z}\right).$$

We can notice that the above equations are different from what we have in the ellipse and limaçon shapes by the normalizing factor $\frac{1}{3}$.

3.3.2 The Schwarz Function

Let us now revisit the Schwarz function of the Neumann Oval (see, e.g., [13]). By knowing that $r^2 = a^2 + 4b^2\cos^2(\theta)$ and $r^2 = \zeta\bar{\zeta}$, we can rewrite the equation of the Neumann Oval as

$$\begin{aligned}
&\left(\zeta\bar{\zeta}\right)^2 = a^2\left(\zeta\bar{\zeta}\right) + b^2(\zeta+\bar{\zeta})^2 \\
&\zeta^2\bar{\zeta}^2 - b^2\left(\zeta+\bar{\zeta}\right)^2 - a^2\zeta\bar{\zeta} = 0 \\
&\zeta^2\bar{\zeta}^2 - b^2\left(\zeta^2 + 2\zeta\bar{\zeta} + \bar{\zeta}^2\right) - a^2\zeta\bar{\zeta} = 0 \\
&\left(\zeta^2 - b^2\right)\bar{\zeta}^2 - \zeta(a^2+2b^2)\bar{\zeta} - b^2\zeta^2 = 0.
\end{aligned}$$

As we see this equation is a second degree equation in $\bar{\zeta}$, so we will utilize the quadratic formula in order to find $\bar{\zeta}$:

$$\bar{\zeta} = \frac{(a^2+2b^2)\zeta \,\pm\, \zeta\sqrt{a^4+4a^2b^2+4b^2\zeta^2}}{2(\zeta^2-b^2)},$$

where we choose an appropriate branch cut. In our case, we will have branch points at $\zeta = \pm ia\sqrt{\frac{1}{4}\left(\frac{a}{b}\right)^2+1}$, so we will choose the branch cut outside the Neumann Oval.

In other words $\zeta = \pm ia\sqrt{\frac{1}{4}\left(\frac{a}{b}\right)^2+1}$, which lies on the imaginary axis.
Letting $\theta = \frac{\pi}{2} \Rightarrow r^2 = a^2 \overset{\text{if } a>0}{\Longrightarrow} r = a$, so since $\sqrt{\frac{1}{4}\left(\frac{a}{b}\right)^2+1} > 1$ and $a > 0$ then $a\sqrt{\frac{1}{4}\left(\frac{a}{b}\right)+1} > a$.

Moreover, we need to choose the sign in front of the square root that gives the correct value for $\bar{\zeta}$ when

$\zeta \in \Gamma$. For example, when $\zeta = ia$, we have

$$\begin{aligned}\frac{ia\left(a^2+2b^2\right)+ia\sqrt{a^4+4a^2b^2-4a^2b^2}}{2\left(-a^2-b^2\right)} &= \frac{ia\left(a^2+2b^2\right)+iaa^2}{-2\left(a^2+b^2\right)} \\ &= -ia\frac{\left(2a^2+2b^2\right)}{2\left(a^2+b^2\right)} \\ &= -ia = \bar{\zeta}.\end{aligned}$$

Therefore, we choose the positive sign for the equation of the Schwarz function, and get

$$S(\zeta) = \psi_1(\zeta) + \psi_2(\zeta),$$

where $\psi_1(\zeta) = \frac{(a^2+2b^2)\zeta}{2(\zeta^2-b^2)}$ and $\psi_2(\zeta) = \frac{\zeta\sqrt{a^4+4a^2b^2+4b^2\zeta^2}}{2(\zeta^2-b^2)}$. In addition, ψ_1 is analytic in $\mathbb{C} - \{\pm b\}$ while ψ_2 is analytic in $\Omega - \{\pm b\}$. Note that the points $\pm 2b$ are on Γ, therefore the points $\pm b$ are inside Ω. (Since $r^2 = a^2 + 4b^2\cos^2(\theta)$ if $a = 0$ and $\theta = 0$ then $r^2 = 4b^2 \Rightarrow r = \pm 2b \Rightarrow \pm b \in \Omega$.)

3.3.3 Solving the Lens Equation and Counting Solutions

As we discussed earlier in Section 3.3.1, to evaluate the integral in (3.10), we rewrite it as a line integral using Green's Theorem and use the Schwarz function we calculated, and need to investigate the following integral:

$$\int_{\partial\Omega} \frac{\psi_1(\zeta) + \psi_2(\zeta)\, d\zeta}{\zeta - z}.$$

Define $f(\zeta) := \frac{\psi_1(\zeta)+\psi_2(\zeta)}{\zeta - z}$. Thus, for $z \notin \Omega$ we will use the Residue Theorem at $\{\pm b\}$ since $f(\zeta)$ is analytic in $\Omega - \{\pm b\}$. We get

$$\begin{aligned}\frac{1}{2\pi i}\int_{\partial\Omega} f(\zeta)d\zeta &= \text{Res}\left(f(\zeta), b\right) + \text{Res}\left(f(\zeta), -b\right) \\ &= g_1(b) + g_2(-b),\end{aligned}$$

where

$$f(\zeta) = \frac{\zeta}{2(\zeta - z)(\zeta + b)(\zeta - b)}\left[a^2 + 2b^2 + \sqrt{a^4 + 4a^2b^2 + 4b^2\zeta^2}\right].$$

so

$$\begin{aligned}
g_1(\zeta) &= \frac{\zeta}{2(\zeta - z)(\zeta + b)} \left[a^2 + 2b^2 + \sqrt{a^4 + 4a^2b^2 + 4b^2\zeta^2}\right] \\
g_1(b) &= \frac{b}{2(b-2)(2b)} \left[a^2 + 2b^2 + \sqrt{(a^2 + 2b^2)^2}\right] \\
&= \frac{1}{4(b-z)} \left[2a^2 + 4b^2\right] = \frac{a^2 + 2b^2}{2(b-z)} \\
g_2(\zeta) &= \frac{\zeta}{2(\zeta - z)(\zeta - b)} \left[a^2 + 2b^2 + \sqrt{a^4 + 4a^2b^2 + 4b^2\zeta^2}\right] \\
g_2(-b) &= \frac{-b}{2(-b-z)(-2b)} \left[a^2 + 2b^2 + \sqrt{(a^2 + 2b^2)^2}\right] \\
&= \frac{1}{-4(b+z)} \left[2a^2 + 4b^2\right] = \frac{a^2 + 2b^2}{-2(b+z)}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{1}{2\pi i}\int_{\partial\Omega} f(\zeta)d\zeta &= \frac{\left(a^2 + 2b^2\right)\left(-2(b+z)\right) + 2(b-z)\left(a^2 + 2b^2\right)}{-4(b-z)(b+z)} \\
&= \frac{-2b\left(a^2 + 2b^2\right) - 2z\left(a^2 + 2b^2\right) + 2b\left(a^2 + 2b^2\right) - 2z\left(a^2 + 2b^2\right)}{4(z-b)(z+b)} \\
&= \frac{-4z\left(a^2 + 2b^2\right)}{4(z-b)(z+b)} = \frac{-2z\left(a^2 + 2b^2\right)}{2(z^2 - b^2)} \\
&= \frac{z(a^2 + 2b^2)}{b^2 - z^2}.
\end{aligned}$$

In like manner, for $z \in \Omega$ we will use the Residue Theorem at $\{\pm b, z\}$ since $f(\zeta)$ is analytic in $\Omega - \{\pm b, z\}$, we will have:

$$\begin{aligned}
\frac{1}{2\pi i}\int_{\partial\Omega} f(\zeta)d\zeta &= \text{Res}\left(f(\zeta), b\right) + \text{Res}\left(f(\zeta), -b\right) + \text{Res}\left(f(\zeta), z\right) \\
&= g_1(b) + g_2(-b) + g_3(z),
\end{aligned}$$

where

$$\begin{aligned}
g_3(\zeta) &= \frac{\zeta}{2(\zeta^2 - b^2)} \left[a^2 + 2b^2 + \sqrt{a^4 + 4a^2b^2 + 4b^2\zeta^2}\right] \\
g_3(z) &= \frac{z}{2(z^2 - b^2)} \left[a^2 + 2b^2 + \sqrt{a^4 + 4a^2b^2 + 4b^2z^2}\right].
\end{aligned}$$

Therefore

$$\frac{1}{2\pi i}\int_{\partial\Omega} f(\zeta)d\zeta = \frac{z(a^2+2b^2)+z\sqrt{a^4+4a^2b^2+4b^2z^2}-2z(a^2+2b^2)}{2(z^2-b^2)}$$
$$= \frac{-z(a^2+2b^2)+z\sqrt{a^4+4a^2b^2+4b^2z^2}}{2(z^2-b^2)}.$$

We therefore get the following theorem.

Theorem 8. *Suppose $\Omega := \{z = re^{i\theta} \in \mathbb{C} \mid r^2 < a^2 + 4b^2\cos^2(\theta)\}$ is the interior of a Neumann Oval shape with constant mass density $d\mu(\zeta) = \frac{1}{3\pi}dA(\zeta)$. Then for $z \notin \Omega$, the lens equation is*

$$\bar{w} = \bar{z} - \frac{1}{3}\left[\frac{z(a^2+2b^2)}{z^2-b^2}\right] - \gamma z, \tag{3.11}$$

while for $z \in \Omega$, the lens equation is

$$\bar{w} = \frac{2}{3}\bar{z} - \frac{1}{3}\left[\frac{z(a^2+2b^2)-z\sqrt{a^4+4a^2b^2+4b^2z^2}}{2(z^2-b^2)}\right] - \gamma z. \tag{3.12}$$

Following the same technique that was used for the Neumann Oval, we rewrite (3.11) as a system of two real equations of degree 3, and therefore, by Bézout's Theorem, there are at most 9 images outside the Neumann Oval, while the Khavinson and Neumann theorem for rational functions produce $5n-5$ for $n=3$ is 10 solutions, which is not improvement in this case. In like manner, if the shearing term vanishes we get $n=2$, and thus the Khavinson-Neumann theorem gives 5 solutions, while Bézout's theorem still gives 9.

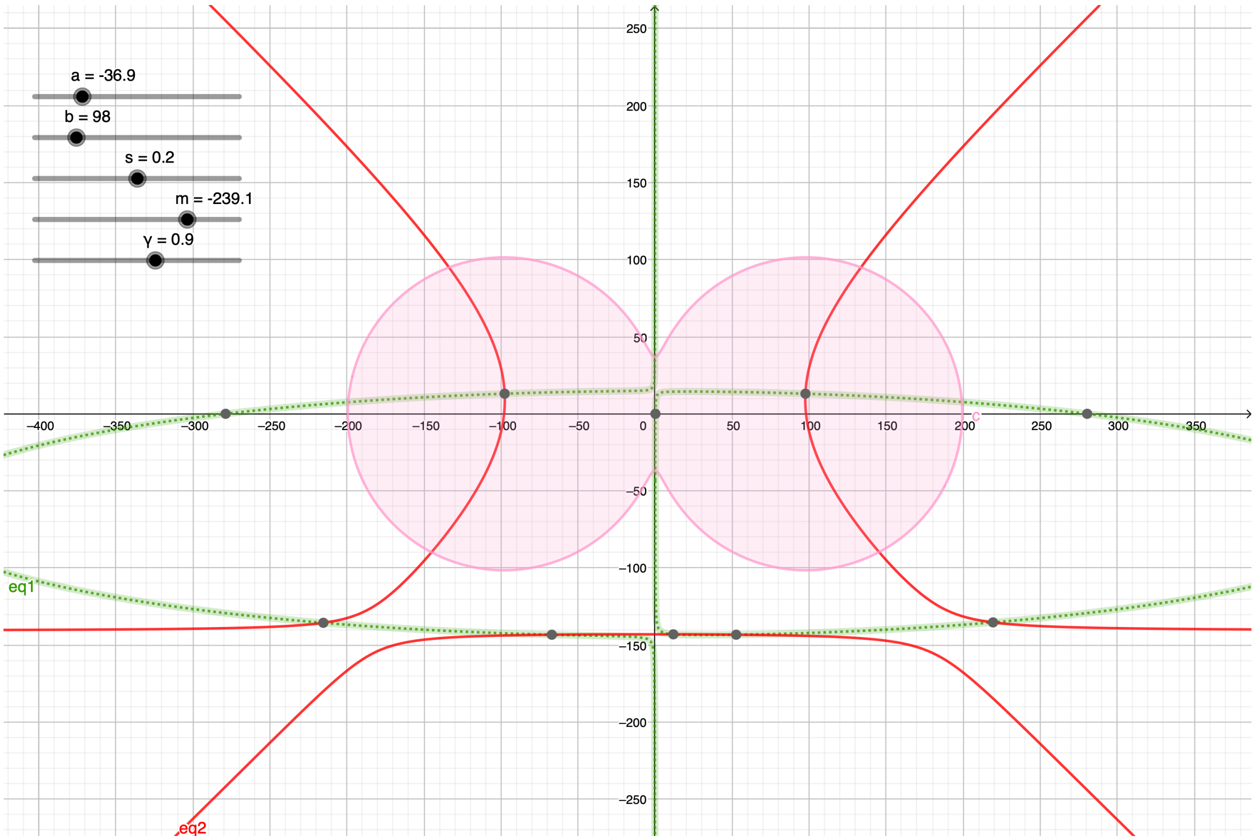

Figure 21. Zeros of lens equation which is a Neumann Oval shaped lens for $r^2 = (-36.9)^2 + 4(98)^2\cos^2(\theta)$ and source position $w = 0.2 - 239.1i$ with shearing term $\gamma \neq 0$. There are 5 images, located at the intersection of the dotted and solid curves that lie outside of the Neumann Oval.

On the other hand, for the lens equation (3.12) the theorem of the Khavinson-Neumann no longer holds since Equation (3.12) is not a rational function. By using real variables to express the complex variables and using some algebraic operations of squaring both sides and canceling common factors of $4(z^2 - b^2)$, we rewrite equation (3.12) as a system of two real equations. Moreover, we notice that the shearing term does not impact the degree of the real part, which is 4, while it affect the degree of the imaginary part, which is 3 if there is no shearing term ($\gamma = 0$) and 4 otherwise. Therefore, we obtain the following theorem:

Theorem 9. *Suppose* $\Omega := \{z = re^{i\theta} \in \mathbb{C} \,|r^2 < a^2 + 4b^2\cos^2(\theta)\}$ *is the interior of a Neumann Oval with constant mass density* $d\mu(\zeta) = \frac{1}{3\pi}dA(\zeta)$*. Then there are at most 9 solutions to the lens equation (3.11) if the shearing term* $\gamma \neq 0$*, and at most 5 solutions if* $\gamma = 0$*, while there are at most 16 solutions to the lens equation (3.12) if the shearing term* $\gamma \neq 0$*, and at most 12 solutions if* $\gamma = 0$.

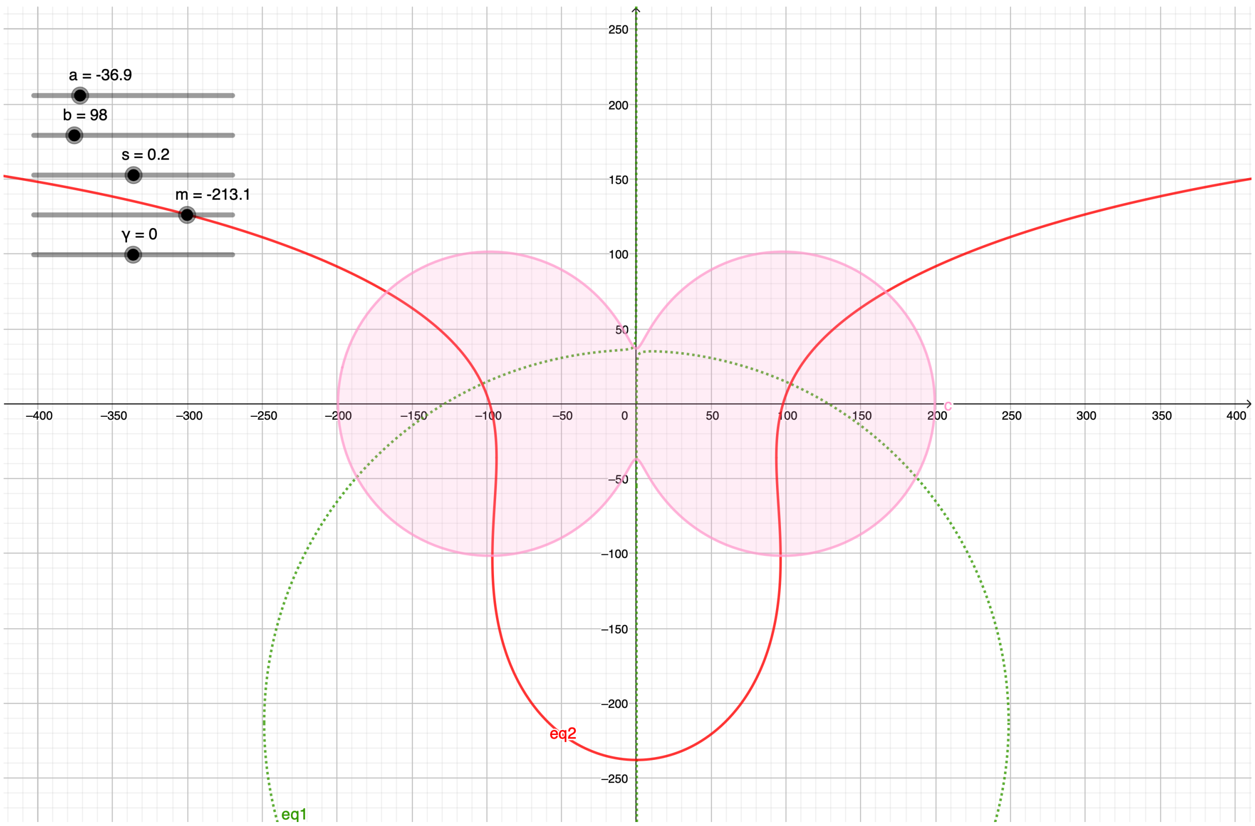

Figure 22. Zeros of lens equation which is a Neumann Oval shaped lens for $r^2 = (-36.9)^2 + 4(98)^2\cos^2(\theta)$ and source position $w = 0.2 - 213.1i$ with shearing term $\gamma = 0$. There is 1 image, located at the intersection of the dotted and solid curves that lie outside of the Neumann Oval.

It is clear that the expected number of solutions of the lens equation (3.12) is over estimated because the original equation has a radical term. Moreover, numerical experiments show that we have 5 images outside when the shearing term is there, and 1 image outside when the shearing term vanishes, while we have 4 solution inside when the shearing term is present and 2 solutions inside when the shearing term vanishes.

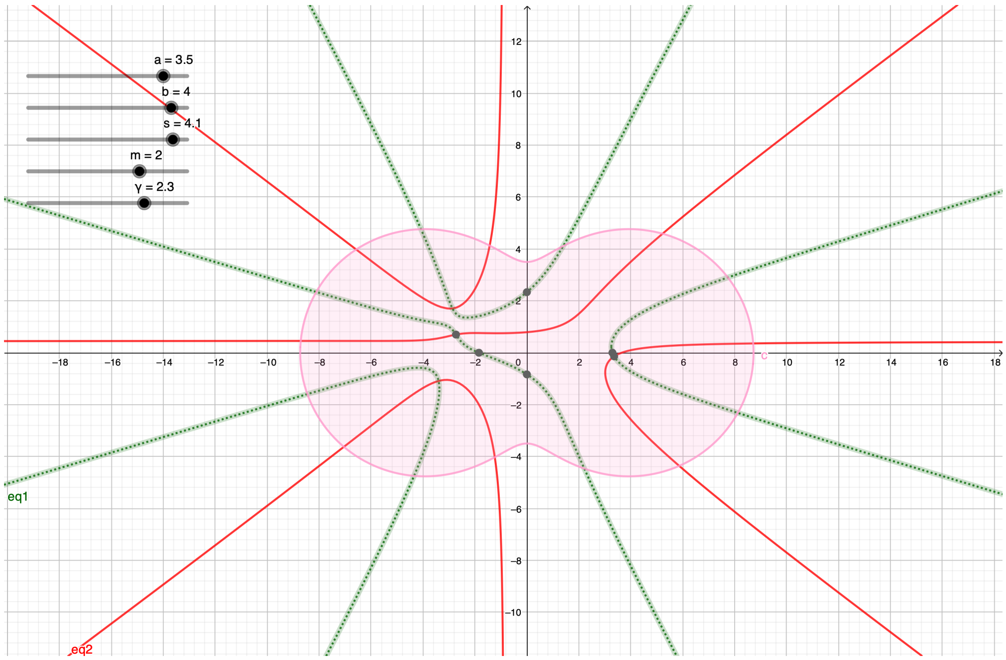

Figure 23. Zeros of lens equation which is a Neumann Oval shaped lens for $r^2 = (3.5)^2 + 4(4)^2\cos^2(\theta)$ and source position $w = 4.1 + 2i$ with shearing term $\gamma \neq 0$. There are 4 images, located at the intersection of the dotted and solid curves that lie inside of the Neumann Oval.

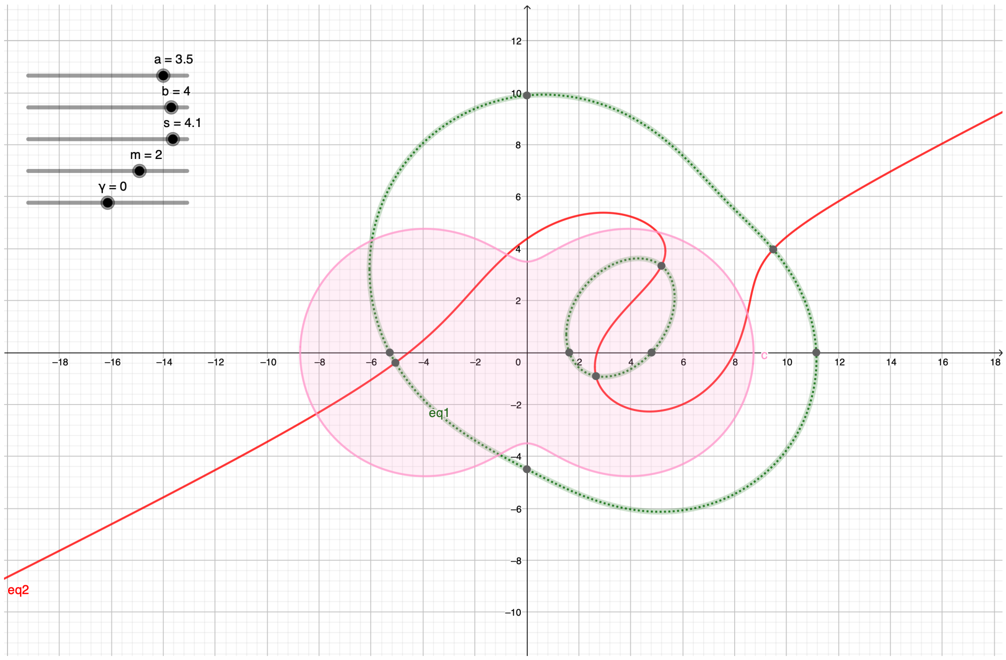

Figure 24. Zeros of lens equation which is a Neumann Oval shaped lens for $r^2 = (3.5)^2 + 4(4)^2\cos^2(\theta)$ and source position $w = 4.1 + 2i$ with shearing term $\gamma = 0$. There are 2 images, located at the intersection of the dotted and solid curves that lie inside of the Neumann Oval.

As a result of the previous discussion we have the following conjecture.

Conjecture 2. Suppose $\Omega := \{z = re^{i\theta} \in \mathbb{C}\ |r^2 < a^2 + 4b^2\cos^2(\theta)\}$ is the interior of a Neumann Oval, with constant mass density $d\mu(\zeta) = \frac{1}{3\pi} dA(\zeta)$. Then there are at most 5 solutions (outside) to the lens equation (3.11) when $\gamma \neq 0$ and 1 solution (outside) when $\gamma = 0$. Also, there is at most 4 solutions (inside) to the lens equation (3.12) when $\gamma \neq 0$ and 2 solutions (inside) when $\gamma = 0$.

3.3.4 Modifying the normalizing factor in a special case of the Neumann Oval

Consider the curve with equation

$$r^2 = c^2 \ \cos^2(\theta) + \sin^2(\theta).$$

Letting $\sin^2(\theta) = 1 - \cos^2(\theta)$, we get

$$r^2 = (c^2 - 1)\cos^2(\theta) + 1. \tag{3.13}$$

When $|c| > 1$, this reduces to a special case of a Neumann Oval. In other words, by assuming that $a^2 = 1$ and $4b^2 = c^2 - 1$, we get (3.13).

Although we were not able to find the extreme case for the Neumann Oval, notice that if we modify the normalizing factor to be $\frac{2}{\pi}$ instead of $\frac{1}{3\pi}$, the same calculation lead to the lens equation

$$\bar{w} = \bar{z} - \frac{2}{\pi}\int_{\Omega} \frac{dA}{z-\zeta} - \gamma z, \tag{3.14}$$

and corresponding Schwarz function

$$\bar{\zeta} = \frac{(c^2+1)\zeta + 2\,\zeta\sqrt{(c^2-1)\zeta^2 + c^2}}{4\zeta^2 - c^2 + 1}.$$

As we discussed earlier, having z inside or outside Ω leads to the following theorem in the case of a modified normalizing factor.

Theorem 10. *Suppose* $\Omega := \{z = re^{i\theta} \in \mathbb{C}\ |r^2 < c^2 \ \cos^2(\theta) + \sin^2(\theta)\ \}$ *is the interior of a Neumann Oval where* $|c| > 1$ *with constant mass density* $d\mu(\zeta) = \frac{1}{\pi}dA(\zeta)$*. Then for* $z \notin \Omega$*, the lens equation is*

$$\bar{w} = \bar{z} + 2\left[\frac{2z(c^2+1)}{c^2 - 4z^2 - 1}\right] - \gamma z, \tag{3.15}$$

while for $z \in \Omega$

$$\bar{w} = -\bar{z} + \frac{2z\left(2\sqrt{(c^2-1)z^2 + c^2} - c^2 - 1\right)}{4z^2 + 1 - c^2} - \gamma z. \tag{3.16}$$

By the same reasoning, this leads to a maximum number of 9 solutions if $z \notin \Omega$, $\gamma \neq 0$ and 5 solutions if $z \notin \Omega$, $\gamma = 0$. Also, we get from the lens equation (3.16), that the maximum number of solutions is 16 if $z \in \Omega$, $\gamma \neq 0$ and 12 if $z \in \Omega$, $\gamma = 0$. In this case, we were able to find an example of the extreme case which proves the sharpness for the number of solutions for the Neumann Oval with modified normalizing factor when $z \notin \Omega$, see Figures [25] and [26].

While, for $z \in \Omega$ by numerical experiments we were able to get 4 solutions inside the modified Neumann Oval, which may not be sharp in this case, (see Figures [27] and [28]) .

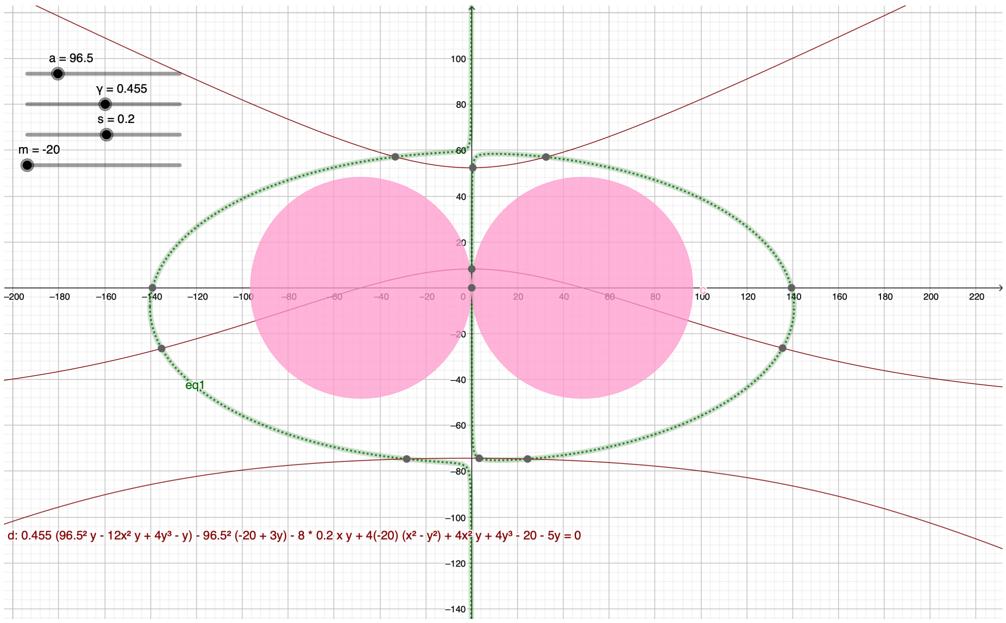

Figure 25. Zeros of lens equation which is a Neumann Oval shaped lens with a modified normalizing factor for $r^2 = ((96.5)^2 - 1)\cos^2\theta + 1$ and source position $w = 0.2 - 20i$ with a shearing term of $\gamma = 0.455$. There are 9 images, located at the intersection of the dotted and solid curves that lie outside of the Neumann Oval.

Theorem 11. *Suppose* $\Omega := \{z = re^{i\theta} \in \mathbb{C}\ |r^2 < (c^2-1)\cos^2(\theta)+1$ *is the interior of a modified Neumann Oval where* $|c| > 1$ *with constant mass density* $d\mu(\zeta) = \frac{2}{\pi}dA(\zeta)$ *Then there are at most 9 solutions to the lens equation (3.15) if the shearing term* $\gamma \neq 0$*, and at most 5 solutions if* $\gamma = 0$*, while there are at most 16 solutions to the lens equation (3.16) if the shearing term* $\gamma \neq 0$*, and at most 12 solutions if* $\gamma = 0$*.*

As we discussed earlier in Section 3.3.3 about the over estimate number of solutions we have the following conjecture

Conjecture 3. Suppose $\Omega := \{z = re^{i\theta} \in \mathbb{C}\ |r^2 < (c^2-1)\cos^2(\theta)+1\}$ is the interior of a Neumann Oval, with constant mass density $d\mu(\zeta) = \frac{2}{\pi}dA(\zeta)$. Then there is at most 8 solutions (inside) to the lens equation (3.16) when $\gamma \neq 0$ and 4 solutions (inside) when $\gamma = 0$.

Note. Note that if $0 < |c| < 1$, then we get a shape looks like a Neumann Oval in different axis. Similar calculations lead to the same theorems. By numerical experiments, the number of images for $0 < |c| < 1$ was less than the case $|c| > 1$.

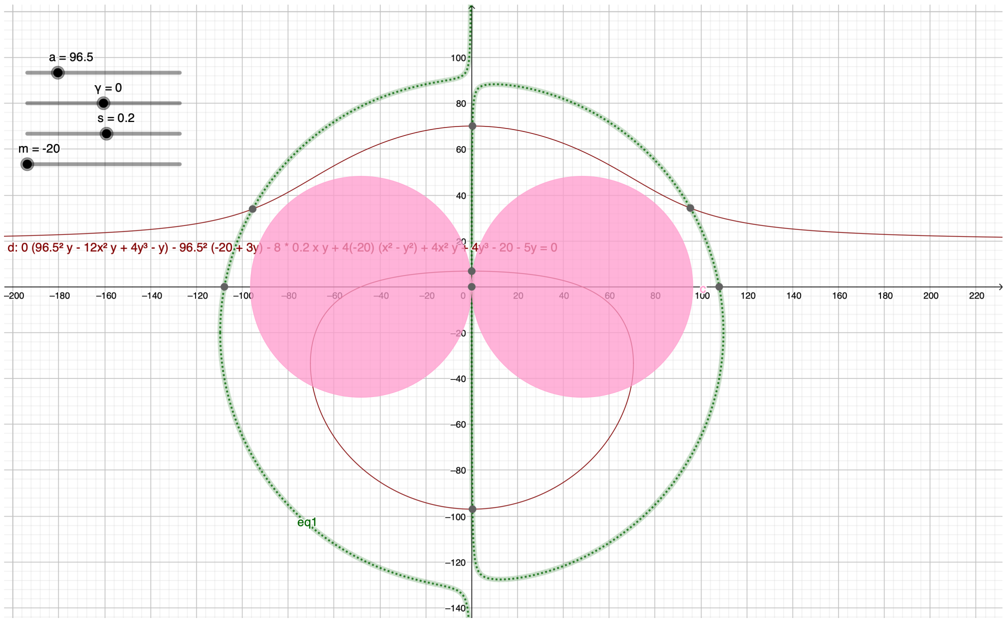

Figure 26. Zeros of lens equation which is a Neumann Oval shaped lens with a modified normalizing factor for $r^2 = ((96.5)^2 - 1)\cos^2\theta + 1$ and source position $w = 0.2 - 20i$ with shearing term $\gamma = 0$. There are 5 images, located at the intersection of the dotted and solid curves that lie outside of the Neumann Oval.

Remark. We investigated several other shapes to try the maximum number of images that we can get, such as Rose curves, Archimedian Spirals, and Cassini Ovals but it was not easy to use the Schwarz function along the boundary of these shapes and calculate the associated Cauchy integral. Further studies are needed for these shapes.

Remark. It would be interesting to study, as we did in Chapter 2, what conditions on the coefficients of the equations considered in this chapter (e.g., the ellipse, the limaçon, and the Neumann Oval) give rise to the maximum number of solutions.

Remark. It would be interesting to understand the impact of the normalizing factor on the maximum number of images one can obtain for a particular lens equation.

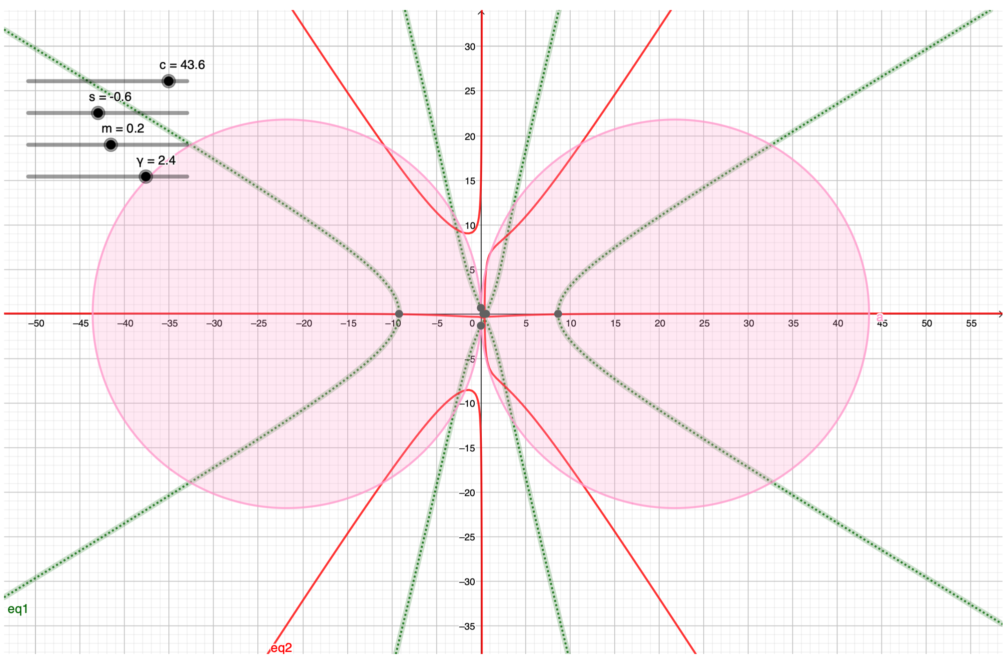

Figure 27. Zeros of lens equation which is a Neumann Oval shaped lens with a modified normalizing factor for $r^2 = ((43.6)^2 - 1)\cos^2\theta + 1$ and source position $w = -0.6 - .2i$ with a shearing term of $\gamma = 2.4$. There are 9 images, located at the intersection of the dotted and solid curves that lie inside of the Neumann Oval.

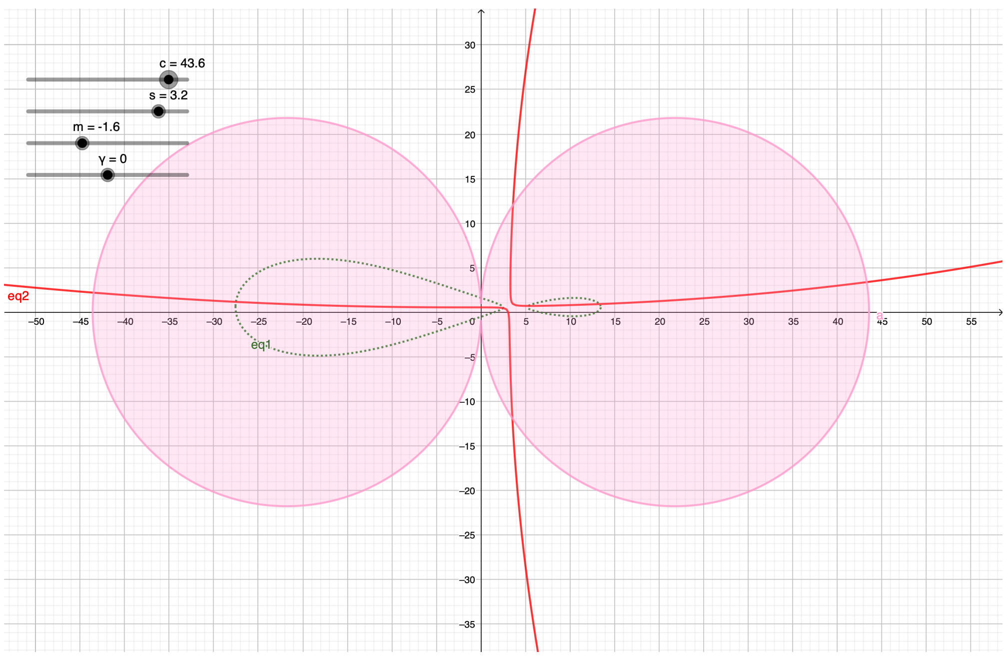

Figure 28. Zeros of lens equation which is a Neumann Oval shaped lens with a modified normalizing factor for $r^2 = ((43.6)^2 - 1)\cos^2\theta + 1$ and source position $w = 3.2 - 1.6i$ with shearing term $\gamma = 0$. There are 5 images, located at the intersection of the dotted and solid curves that lie inside of the Neumann Oval.

References

[1] S. Bell, B. Ernst, S. Fancher, C. Keeton, A. Komanduru, and E. Lundberg. *Spiral galaxy lensing: a model with twist*, Math. Phys. Anal.Geom., **17** no. 3-4, (2014), 305–322.

[2] C. Bénéteau and N. Hudson. *A survey on the maximal number of solutions of equations related to gravitational lensing.* Complex Analysis and Dynamical System, Trends Math. Cham: Birkhäuser/Springer, (2018), 23-38.

[3] W. Bergweiler and A. Eremenko. *On the number of solutions of some transcendental equations*, Anal. Math. Phys., **8** (2018), 185-196.

[4] W. Bergweiler and A. Eremenko. *On the number of solutions of a transcendental equation arising in the theory of gravitational lensing*, Comput.Methods Funct. Theory, **10** no. 1, (2010), 303-324.

[5] D. Bshouty, W. Hengartner, and T. Suez, *The exact bound on the number of zeros of harmonic polynomials*, J. Anal. Math., **67** (1995), 207-218.

[6] W. L. Burke, *Multiple gravitational imaging by distributed masses*, Astrophys. J. 244 (1981), L1.

[7] R. B. Burckel, *A classical proof of the fundamental theorem of algebra dissected*, Mathematics Newsletter of the Ramanujan Mathematical Society, arXiv:1109.1459.

[8] L. Carleson and T. Gamelin, *Complex dynamics*, Springer-Verlag, New York-Berlin-Heidelberg (1993).

[9] P. Duren, *Harmonic mappings in the plane*, Cambridge Tracts in Mathematics **156**, Cambridge University Press (2004).

[10] A. Edelman and E. Kostlan, *How many zeros of a random polynomial are real?*, Bull. Amer. Math. Soc., **32** (1995), 1-37.

[11] C. Fassnacht, C. Keeton, and D. Khavinson. *Gravitational lensing by elliptical galaxies and the Schwarz function*, Analysis and Mathematical Physics. Ed: B. Gustafsson, A. Vasil'ev, (2009), 115-129.

[12] B. Fine and G. Rosenberger, *The fundamental theorem of algebra,Undergraduate Texts in Mathematics*, Springer-Verlag, New York, (1997).

[13] P. Davis. *The Schwarz function and its applications*, Carus mathematical monographs **17**, Math. Assoc. of America, (1960).

[14] A. Gabrielov, D. Novikov and B. Shapiro, *Mystery of point charges*, J. London Math. Soc., **95** no. 2, (2007), 443-472.

[15] L. Geyer. *Sharp bounds for the valence of certain harmonic polynomials*, Proc. Amer. Math. Soc., **136** no. 2, (2008), 549-555.

[16] Ph. Griths and J. Harris. *Principles of algebraic geometry*. Pure and Applied Mathematics. Wiley-Interscience, New York, (1978).

[17] J. D. Hauenstein, A. Lerario, E. Lundberg and D. Mehta, *Experiments on the zeros of harmonic polynomials using certified counting*, Exper. Math., **24** (2015), 133-141.

[18] Ch. Keeton, S. Mao and H. Witt, *Gravitational lenses with more than four images: i. classification of caustics*,The Astophysical J., **537** (2000), 697-727.

[19] D. Khavinson and G. Neumann, *On the number of zeros of certain rational harmonic functions*, Proc. Amer. Math. Soc., **134** (2005), 1077-1085.

[20] D. Khavinson, S.-Y. Lee and A. Saez, *Zeros of harmonic polynomials, critical lemniscates, and caustics*, Complex Analysis and its Synergies, **4** (2018).

[21] D. Khavinson and G. Neumann. *From the fundamental theorem of algebra to astrophysics:a harmonious path.*, Notices Amer. Math. Soc., **55** (2008), 666-675.

[22] D. Khavinson and G.Świątek, *On the number of zeros of certain harmonic polynomials*, Proc. Amer. Math. Soc., **131** (2003), 409-414.

[23] S. G. Krantz and H. R. Parks, *The Implicit Function Theorem.* History,Theory, and Applications, Birkhäuser, Boston (2002).

[24] S.-Y. Lee, A. Lerario, and E. Lundberg *Remarks on Wilmshurt's theorem*, Indiana Univ. Math. J., **64** no. 4, (2015), 1153-1167.

[25] S.-Y. Lee and A. Saez, *Searching for the maximal valence of harmonic polynomials: a new example*, (2015), arXiv:1512.03793.

[26] S. Mao, A. O. Petters, and H. J. Witt, *Properties of point mass lenses on a regular polygon and the problem of maximum number of images*,Jerusalem, Israel, (1997), edited by T. Piran, World Scientific, Singapore (1998), 1494-1496.

[27] R. Narayan and M. Bartelmann, Lectures on gravitational lensing, in *Proceedings of the 1995 Jerusalem Winter School* (1995); online version `http://cfawww.harvard.edu/$$narayan/papers/JeruLect.ps.`

[28] R. Peretz and J. Schmid, *On the zero sets of certain complex polynomials*, Israel Math. Conf. Proc., **11** (1997), 203-208.

[29] A. Lerario, E. Lundberg, *On the zeros of random harmonic polynomials: The truncated model*, Jornal of mathmatical Analysis and applications, **438** (2016), 1041-1054.

[30] A. O. Petters, H. Levine, and J. Wambsganss, *Singularity theory and gravitational lensing*, Birkhäuser, Boston (2001).

[31] A. O. Petters and M. C. Werner, *Mathematics of gravitational lensing: multiple imaging and magnification*, Article in General Relativity and Gravitation, **42** (2010), 2011-2046.

[32] R. Remmert, *"Chapter 4. The fundamental theorem of algebra"* in H.-D. Ebbinghaus, H. Hermes, F. Hirzebruch, M. Koecher, K. Mainzer, J. Neukirch, A. Prestel, and R. Remmert, Numbers, second edition, Graduate Texts in Mathematics (translated from the German by H. L.S. Orde), Springer-Verlag (1991).

[33] S. H. Rhie, *Can a gravitational quadruple lens produce 17 images?*, (2001), arXiv:astro-ph/0103463.

[34] S. H. Rhie, *n-point gravitational lenses with* $5(n-1)$ *images*, (2003), arXiv:astro-ph/0305166.

[35] D. Sarason, written communication to the first author of [21], Feb. 1999, Oct. 2000.

[36] P. Schneider, J. Ehlers, and E. E. Falco, *Gravitational lenses, astronomy and astrophysics library*, Springer-Verlag, New York-Berlin-Heidelberg, (1992).

[37] O. Séte and J. Zur *Number and location of pre-images under harmonic mappings in the plane*, Annales Fennici Mathematici, **46** (2021), 225–247.

[38] H.S. Shapiro. *The Schwarz function and its generalization to higher dimensions*, University of Arkansas Lecture Notes in the Mathematical Sciences **9**, John Wiley and Sons (1992).

[39] T. Sheil-Small, *Complex Polynomials*, Cambridge Studies in Advanced Mathematics **73**, Cambridge University Press, (2002).

[40] M. R. Spiegel, S. Lipschutz, J. J. Schiller, and D. Spellman, *Complex variables with an introduction to conformal mapping and its applications*, (2015), McGraw-Hill.

[41] N. Straumann, *Complex formulation of lensing theory and applications*, Helvetica Physica Acta, **70** (1997), 894-908.

[42] A. Thomack, *On the zeros of random harmonic polynomials: the naive model*, (2016), arXiv:1610.02611.

[43] J. Wambsganss, *Gravitational lensing in astronomy*, Living Rev. Relativity 1 (1998) [Online Article]: cited on January 26, (2004).

[44] W. Li and A. Wei, *On the expected number of zeros of a a random harmonic polynomial.* Proc. Amer. Math. Soc., **137** (2009), no. 1, 195–204.

[45] A. S. Wilmshurst, *Complex harmonic mappings and the valence of harmonic polynomials*, D. Phil. thesis, University of York, U.K. (1994).

[46] A. S. Wilmshurst, *The valence of harmonic polynomials*, Proc. Amer.Math. Soc., **126** (1998), 2077-2081.

[47] H. J. Witt, *Investigation of high amplification events in light curves of gravitationally lensed quasars*, Astron. Astrophys., **236** (1990), 311-322.

Appendix A

Maximum of solutions of $p(z) = \bar{z}^2$ in the case of complex coefficients.

In this Appendix, we will discuss the maximum number of solutions of the equation $(a+ib)z^2+(c+id)z+e+if = \bar{z}^2$ where $a \neq 0$ or $b \neq 0$ and $a, b, c, d, e, f \in \mathbb{R}$, for the cases when we were not able to get precise conditions on the coefficients.

Case I : $a = 1$.

- If $b = 0$:
  - If $e = 0$ and $f = 0$:
    - If $c = d = 0$, substituting into (2.6) gives:

$$xy = 0 \Rightarrow x = 0 \quad or \quad y = 0$$

$$\text{if } x = 0 \overset{sub\ into 2.7}{\Rightarrow} y = 0$$

$$\text{if } y = 0 \overset{subinto 2.7}{\Rightarrow} x = 0$$

      In this case we will have one solution which is the origin.

    - If $c = d \neq 0$, substituting into (2.6) gives:

$$x = y$$

      By substituting $x = y$ into (2.7) we get:

$$4x^2 + 2cx = 0 \Rightarrow x = 0 \quad \text{or} \quad x = \frac{-c}{2}$$

      When $x = 0$, we get $y = 0$.

      When $x = \frac{-c}{2}$, we get $y = \frac{-c}{2}$.

      In this case we will have two solutions.

* If $0 \neq c \neq d \neq 0$, substituting into (2.6) gives:

$$y = \frac{c}{d}x$$

  Now substituting $y = \frac{c}{d}x$ into (2.7), we get:

$$\frac{4c}{d}x^2 + \frac{d^2 + c^2}{d}x = 0$$
$$\Rightarrow x = 0 \quad \text{or} \quad x = -\frac{d^2 + c^2}{4c}$$

  In this case we will have two solutions.

* If $c = 0$ and $d \neq 0$, substituting into (2.6) gives:

$$y = 0$$

  Substituting $y = 0$ into (2.7) gives:

$$x = 0$$

  So, for this case we will have one solution which is the origin.

* If $c \neq 0$ and $d = 0$, substituting into (2.6) gives:

$$x = 0$$

  Substituting $x = 0$ into (2.7) gives:

$$y = 0$$

  So, for this case we will have one solution which is the origin.

– If $e = 0$ and $f \neq 0$:

* If $c = d = 0$, substituting into (2.6) gives:

$$4xy + f = 0 \Rightarrow y = \frac{-f}{4x}$$

  which is a rational function with vertical asymptote $x = 0$ and horizontal asymptote $y = 0$. So, we will have infinite number of solutions.

* If $c = d \neq 0$, substituting into (2.6) gives:

$$x = y$$

By substituting $x = y$ into (2.7) we get:

$$4x^2 + 2cx + f = 0 \Rightarrow x = \frac{-c \pm \sqrt{4c^2 - 16f}}{8}$$

In this case we will have two solutions if $4c^2 - 16f > 0$.

* If $0 \neq c \neq d \neq 0$, substituting into (2.6) gives:

$$y = \frac{c}{d}x$$

Substituting $y = \frac{c}{d}x$ into (2.7) gives:

$$4cx^2 + (d^2 + c^2)x + fd = 0$$
$$\Rightarrow x = \frac{-(d^2 + c^2) \pm \sqrt{(d^2 + c^2)^2 - 16cfd}}{8c}$$

In this case we will have two solutions if $(d^2 + c^2)^2 > 16cfd$.

* If $c = 0$ and $d \neq 0$, substituting into (2.6) gives:

$$y = 0$$

Substituting $y = 0$ into (2.7) gives:

$$dx = -f \Rightarrow x = \frac{-f}{d}$$

So, for this case we will have one solution which is $(\frac{-f}{d}, 0)$.

* If $c \neq 0$ and $d = 0$, substituting into (2.6) gives:

$$x = 0$$

Substituting $x = 0$ into (2.7) gives:

$$cy = -f \Rightarrow y = \frac{-f}{c}$$

So, for this case we will have one solution which is $(0, \frac{-f}{c})$.

– If $e \neq 0$ and $f = 0$:

  * If $c = d \neq 0$, substituting into (2.6) gives:

$$y = x + \frac{e}{c}$$

By substituting $y = x + \frac{e}{c}$ into (2.7) we get:

$$\begin{aligned} 4cx^2 + (4e + 2c^2)x + ec = 0 \\ \Rightarrow x = \frac{-(4e + 2c^2) \pm \sqrt{(4e + 2c^2)^2 - 16c^2 e}}{8c} \end{aligned}$$

In this case we will have two solutions if $(4e + 2c^2)^2 - 16c^2 e > 0 \Rightarrow 16e^2 + 4c^4 > 0$ which is true $\forall c, d, e \in \mathbb{R} - \{0\}$.

  * If $0 \neq c \neq d \neq 0$, substituting into (2.6) gives:

$$y = \frac{c}{d}x + \frac{e}{d}$$

Substituting $y = \frac{c}{d}x + \frac{e}{d}$ into (2.7) gives:

$$\begin{aligned} 4cx^2 + (4e + d^2 + c^2)x + ec = 0 \\ \Rightarrow x = \frac{-(4e + d^2 + c^2) \pm \sqrt{(4e + d^2 + c^2)^2 - 16c^2 e}}{8c} \end{aligned}$$

In this case we will have two solutions if $(4e + d^2 + c^2)^2 > 16c^2 e$.

  * If $c = 0$ and $d \neq 0$, substituting into (2.6) gives:

$$dy = e \Rightarrow y = \frac{e}{d}$$

Substituting $y = \frac{e}{d}$ into (2.7) gives:

$$\frac{4e + d^2}{d}x = 0 \Rightarrow x = 0$$

So, for this case we will have one solution which is $(0, \frac{e}{d})$.

* If $c \neq 0$ and $d = 0$, substituting into (2.6) gives:

$$cx = -e \Rightarrow x = \frac{-e}{c}$$

Substituting $x = \frac{-e}{c}$ into (2.7) gives:

$$\frac{-4e + c^2}{c} y = 0 \Rightarrow y = 0$$

So, for this case we will have one solution which is $(\frac{-e}{c}, 0)$.

- If $e \neq 0$ and $f \neq 0$
  * If $c = d \neq 0$, substituting into (2.6) gives:

$$y = x + \frac{e}{c}$$

By substituting $y = x + \frac{e}{c}$ into (2.7) we get:

$$4cx^2 + (4e + 2c^2)x + ec + fc = 0$$
$$\Rightarrow x = \frac{-(4e + 2c^2) \pm \sqrt{(4e + 2c^2)^2 - 16c^2(e + f)}}{8c}$$

In this case we will have two solutions if $4e^2 + c^4 > 4c^2 f$.

  * If $0 \neq c \neq d \neq 0$, substituting into (2.6) gives:

$$y = \frac{c}{d}x + \frac{e}{d}$$

Substituting $y = \frac{c}{d}x + \frac{e}{d}$ into (2.7) gives:

$$4cx^2 + (4e + d^2 + c^2)x + ec + fd = 0$$
$$\Rightarrow x = \frac{-(4e + d^2 + c^2) \pm \sqrt{(4e + d^2 + c^2)^2 - 16c(ec + fd)}}{8c}$$

In this case we will have two solutions if $(4e + d^2 + c^2)^2 > 16c(ec + fd)$.

  * If $c = 0$ and $d \neq 0$, substituting into (2.6) gives:

$$y = \frac{e}{d}.$$

Substituting $y = \frac{e}{d}$ into (2.7) gives:

$$4ex + d^2x + fd = 0 \Rightarrow x = \frac{-fd}{4e + d^2}$$

So, for this case we will have one solution which is $(\frac{-fd}{4e+d^2}, \frac{e}{d})$ where $4e + d^2 \neq 0$.

∗ If $c \neq 0$ and $d = 0$, substituting into (2.6) gives:

$$x = \frac{-e}{c}.$$

Substituting $x = \frac{-e}{c}$ into (2.7) gives:

$$-4ey + c^2y + cf = 0 \Rightarrow y = \frac{-cf}{c^2 - 4e}$$

So, for this case we will have one solution which is $(\frac{-e}{c}, \frac{-cf}{c^2-4e})$ where $c^2 - 4e \neq 0$.

- If $b \neq 0$:
  - If $e = 0$ and $f = 0$:

∗ If $c = d = 0$, substituting into (2.6) gives:

$$xy = 0 \Rightarrow x = 0 \quad or \quad y = 0$$

$$\text{if } x = 0 \overset{sub\ into 2.7}{\Rightarrow} y = 0$$

$$\text{if } y = 0 \overset{subinto 2.7}{\Rightarrow} x = 0$$

In this case we will have one solution which is the origin.

∗ If $c = d \neq 0$, substituting into (2.6) gives:

$$y = \frac{cx}{2bx + c}$$

which is a rational function with vertical asymptote $x = \frac{-c}{2b}$ and horizontal asymptote $y = \frac{c}{2b}$.

From (2.7) we get:

$$\left(x - \left(\frac{-c}{2b}\right)\right)^2 - \left(y - \frac{c}{2b}\right)^2 + \frac{4}{b}xy = 0$$

which is hyperbola function. In this case the maximum number of solutions we can get is four.

* If $0 \neq c \neq d \neq 0$, substituting into (2.6) gives:

$$y = \frac{cx}{2bx + d}$$

which is a rational function with vertical asymptote $x = \frac{-d}{2b}$ and horizontal asymptote $y = \frac{c}{2b}$.

From (2.7) we get:

$$\frac{\left(x - \left(\frac{-d}{2b}\right)\right)^2}{G} - \frac{\left(y - \frac{c}{2b}\right)^2}{G} + \frac{4}{bG}xy = 1$$
$$G = \frac{d^2 - c^2}{4b^2}$$

which is hyperbola function. In this case the maximum number of solutions we can get is four.

.

- If $e = 0$ and $f \neq 0$

  * If $c = d = 0$, substituting into (2.6) gives:

$$-2bxy = 0 \Rightarrow x = 0 \quad or \quad y = 0$$

$$\text{if } x = 0 \overset{sub\ into (2.7)}{\Rightarrow} y = \pm\sqrt{\frac{f}{b}}$$

In this case we have two solutions if we satisfy one of the following conditions

· $f > 0$ and $b > 0$.

· $f < 0$ and $b < 0$.

$$\text{if } y = 0 \overset{sub\ into (2.7)}{\Rightarrow} x = \pm\sqrt{\frac{-f}{b}}$$

In this case we have two solutions if we satisfy one of the following conditions

· $f < 0$ and $b > 0$.

· $f > 0$ and $b < 0$.

Since we can not combine the above conditions we will have maximum two solutions.

∗ If $c = d \neq 0$, substituting into (2.6) gives:

$$y = \frac{cx}{2bx + c}$$

which is a rational function with vertical asymptote $x = \frac{-c}{2b}$ and horizontal asymptote $y = \frac{c}{2b}$.

From (2.7) we get:

$$\frac{\left(y - \frac{c}{2b}\right)^2}{f} - \frac{\left(x - \left(\frac{-c}{2b}\right)\right)^2}{f} - \frac{4}{bf}xy = 1$$

which is hyperbola function. In this case the maximum number of solutions we can get is four.

∗ If $0 \neq c \neq d \neq 0$, substituting into (2.6) gives:

$$y = \frac{cx}{2bx + d}$$

which is a rational function with vertical asymptote $x = \frac{-d}{2b}$ and horizontal asymptote $y = \frac{c}{2b}$.

From (2.7) we get:

$$\frac{\left(x - \left(\frac{-d}{2b}\right)\right)^2}{Q} - \frac{\left(y - \frac{c}{2b}\right)^2}{Q} - \frac{4}{M - N - f}xy = 1$$

$$Q = \frac{M - N - f}{b}$$

$$M = \frac{d^2}{4b}$$

$$N = \frac{c^2}{4b}$$

which is hyperbola function. In this case the maximum number of solutions we can get is four.

– If $e \neq 0$ and $f = 0$:

∗ If $c = d = 0$, substituting into (2.6) gives:

$$y = \frac{e}{2bx}$$

which is a rational function with vertical asymptote $x = 0$ and horizontal asymptote $y = 0$.

From (2.7) we get:

$$\frac{y^2}{T} - \frac{x^2}{T} = 1$$
$$T = \frac{2e}{b^2}$$

which is hyperbola function. In this case the maximum number of solutions we can get is four.

* If $c = d \neq 0$, substituting into (2.6) gives:

$$y = \frac{cx - e}{2bx + c}.$$

which is a rational function with vertical asymptote $x = \frac{-c}{2b}$ and horizontal asymptote $y = \frac{c}{2b}$.

From (2.7) we get:

$$\left(x - \left(\frac{-c}{2b}\right)\right)^2 - \left(y - \frac{c}{2b}\right)^2 + \frac{4}{b}xy = 0.$$

which is hyperbola function. In this case the maximum number of solutions we can get is four.

* If $0 \neq c \neq d \neq 0$, substituting into (2.6) gives:

$$y = \frac{cx - e}{2bx + d}.$$

which is a rational function with vertical asymptote $x = \frac{-d}{2b}$ and horizontal asymptote $y = \frac{c}{2b}$.

From (2.7) we get:

$$\frac{\left(x - \left(\frac{-d}{2b}\right)\right)^2}{G} - \frac{\left(y - \frac{c}{2b}\right)^2}{G} + \frac{4}{bG}xy = 1$$
$$G = \frac{d^2 - c^2}{4b^2}$$

which is hyperbola function. In this case the maximum number of solutions we can get is four.

* If $c = 0$ and $d \neq 0$, substituting into (2.6) gives:

$$y = \frac{e}{2bx + d}$$

which is a rational function with vertical asymptote $x = \frac{-d}{2b}$ and horizontal asymptote $y = 0$.

From (2.7) we get:

$$\frac{\left(x - \left(\frac{-d}{2b}\right)\right)^2}{L} - \frac{y^2}{L} + \frac{4}{bL}xy = 1$$
$$L = \frac{d^2}{4b^2}.$$

which is hyperbola function. In this case the maximum number of solutions we can get is four.

* If $c \neq 0$ and $d = 0$, substituting into (2.6) gives:

$$x = \frac{-e}{c - 2by}$$

which is a rational function with vertical asymptote $x = 0$ and horizontal asymptote $y = \frac{c}{2b}$.

From (2.7) we get:

$$\frac{\left(y - \frac{c}{2b}\right)^2}{T} - \frac{x^2}{T} - \frac{4}{bT}xy = 1$$
$$T = \frac{c^2}{4b^2}$$

which is hyperbola function. In this case the maximum number of solutions we can get is four.

- If $e \neq 0$ and $f \neq 0$:

  * If $c = d = 0$, substituting into (2.6) gives:

$$y = \frac{e}{2bx}$$

which is a rational function with vertical asymptote $x = 0$ and horizontal asymptote $y = 0$.

From (2.7) we get:

$$\frac{y^2}{f} - \frac{x^2}{f} - \frac{4}{bf}xy = 1$$

which is hyperbola function. In this case the maximum number of solutions we can get is four.

$*$ If $c = d \neq 0$, substituting into (2.6) gives:

$$y = \frac{cx + e}{2bx + c}$$

which is a rational function with vertical asymptote $x = \frac{-c}{2b}$ and horizontal asymptote $y = \frac{c}{2b}$.

From (2.7) we get:

$$\frac{\left(y - \frac{c}{2b}\right)^2}{f} - \frac{\left(x - \left(\frac{-c}{2b}\right)\right)^2}{f} - \frac{4}{bf}xy = 1$$

which is hyperbola function. In this case the maximum number of solutions we can get is four.

$*$ If $0 \neq c \neq d \neq 0$, substituting into (2.6) gives:

$$y = \frac{cx + e}{2bx + d}$$

which is a rational function with vertical asymptote $x = \frac{-d}{2b}$ and horizontal asymptote $y = \frac{c}{2b}$.

From (2.7) we get:

$$\frac{\left(x - \left(\frac{-d}{2b}\right)\right)^2}{Q} - \frac{\left(y - \frac{c}{2b}\right)^2}{Q} - \frac{4}{M - N - f}xy = 1$$

$$Q = \frac{M - N - f}{b}$$

$$M = \frac{d^2}{4b}$$

$$N = \frac{c^2}{4b}$$

which is hyperbola function. In this case the maximum number of solutions we can get is four.

* If $c = 0$ and $d \neq 0$, substituting into (2.6) gives:

$$y = \frac{e}{2bx + d}$$

which is a rational function with vertical asymptote $x = \frac{-d}{2b}$ and horizontal asymptote $y = 0$.

From (2.7) we get:

$$\frac{\left(x - \left(\frac{-d}{2b}\right)\right)^2}{Q_1} - \frac{y^2}{Q_1} + \frac{4}{M - f}xy = 1$$
$$Q_1 = \frac{M - f}{b}$$
$$M = \frac{d^2}{4b}$$

which is hyperbola function. In this case the maximum number of solutions we can get is four.

* If $c \neq 0$ and $d = 0$, substituting into (2.6) gives:

$$y = \frac{cx + e}{2bx}$$

which is a rational function with vertical asymptote $x = 0$ and horizontal asymptote $y = \frac{c}{2b}$.

From (2.7) we get:

$$\frac{\left(y - \frac{c}{2b}\right)^2}{H} - \frac{x^2}{H} - \frac{4}{bH}xy = 1$$
$$H = \frac{c^2 + 4fb}{4b^2}$$

which is hyperbola function. In this case the maximum number of solutions we can get is four.

Case II : $a \neq 1$.

- If $b = 0$:
  - If $e = 0$ and $f = 0$:
    - If $c = d = 0$, substituting into (2.6) gives:

$$
\begin{aligned}
&(a-1)(x^2 - y^2) = 0 \\
&\overset{\text{since } a \neq 1}{\Rightarrow} x^2 - y^2 = 0 \\
&\Rightarrow x^2 = y^2 \\
&\Rightarrow x = y \quad \text{or} \quad x = -y
\end{aligned}
$$

If $x = y$ substitute into (2.7) we get:

$$
\begin{aligned}
&2(a+1)x^2 = 0 \\
\Rightarrow &a = -1 \quad \text{or} \quad x = 0
\end{aligned}
$$

- If $a = -1$ then we will have infinite number of solutions.
- If $x = 0$ then $y = 0$.

If $x = -y$ substitute into (2.7) we get:

$$
\begin{aligned}
&-2(a+1)x^2 = 0 \\
\Rightarrow &a = -1 \quad \text{or} \quad x = 0
\end{aligned}
$$

- If $a = -1$ then we will have infinite number of solutions.
- If $x = 0$ then $y = 0$.

So, if $a = -1$ we will have infinite number of solutions while if $a \neq -1$ we will have one solution which is the origin.

- If $0 \neq c \neq d \neq 0$, substituting into (2.6) gives:

$$
\begin{aligned}
&\frac{\left(x - \left(\frac{-c}{2(a-1)}\right)\right)^2}{G} - \frac{\left(y - \left(\frac{-d}{2(a-1)}\right)\right)^2}{G} = 1 \\
&G = \frac{c^2 - d^2}{4(a-1)^2}
\end{aligned}
$$

which is hyperbola function.

From (2.7) we get:

$$y = \frac{-dx}{2(a+1)x + c}$$

which is a rational function with vertical asymptote $x = \frac{-c}{2(a+1)}$ and horizontal asymptote $y = \frac{-d}{2(a+1)}$. In this case the maximum number of solutions we can get is four where $a \neq -1$.

- If $e = 0$ and $f \neq 0$
  - If $c = d = 0$, substituting into (2.6) gives:

$$(a-1)(x^2 - y^2) = 0$$
$$\overset{\text{since } a\neq 1}{\Rightarrow} x^2 - y^2 = 0$$
$$\Rightarrow x^2 = y^2$$
$$\Rightarrow y = x \quad \text{or} \quad y = -x.$$

From (2.7) we get:

$$2(a+1)xy + f = 0 \Rightarrow xy = \frac{-f}{2(a+1)}$$

So, we have to have $a \neq -1$

  - If $y = x$ then $x = \pm\sqrt{\frac{-f}{2(a+1)}} \Rightarrow f > 0$ and $a < -1$ or $f < 0$ and $a > -1$.
  - If $y = -x$ then $x = \pm\sqrt{\frac{f}{2(a+1)}} \Rightarrow f > 0$ and $a > -1$ or $f < 0$ and $a < -1$.

From the above cases we can notice that we will have two solutions at maximum.

  - If $0 \neq c \neq d \neq 0$, substituting into (2.6) gives:

$$\frac{\left(x - \left(\frac{-c}{2(a-1)}\right)\right)^2}{G} - \frac{\left(y - \left(\frac{-d}{2(a-1)}\right)\right)^2}{G} = 1$$
$$G = \frac{c^2 - d^2}{4(a-1)^2}$$

which is hyperbola function.

From (2.7) we get:

$$y = \frac{-f - dx}{2(a+1)x + c},$$

which is a rational function with vertical asymptote $x = \frac{-c}{2(a+1)}$ and horizontal asymptote $y = \frac{-d}{2(a+1)}$. In this case the maximum number of solutions we can get is four.

* If $c = 0$ and $d \neq 0$, substituting into (2.6) gives:

$$\frac{\left(y - \left(\frac{-d}{2(a-1)}\right)\right)^2}{G_1} - \frac{x^2}{G_1} = 1$$
$$G_1 = \frac{d^2}{4(a-1)^2},$$

  which is hyperbola function.

  From (2.7) we get:

$$y = \frac{-f - dx}{2(a+1)x},$$

  which is a rational function with vertical asymptote $x = 0$ and horizontal asymptote $y = \frac{-d}{2(a+1)}$. In this case the maximum number of solutions we can get is four.

* If $c \neq 0$ and $d = 0$, substituting into (2.6) gives:

$$\frac{\left(x - \left(\frac{-c}{2(a-1)}\right)\right)^2}{G_2} - \frac{y^2}{G_2} = 1$$
$$G_2 = \frac{c^2}{4(a-1)^2}$$

  which is hyperbola function.

  From (2.7) we get:

$$y = \frac{-f}{2(a+1)x + c},$$

  which is a rational function with vertical asymptote $x = \frac{-c}{2(a+1)}$ and horizontal asymptote $y = 0$. In this case the maximum number of solutions we can get is four.

– If $e \neq 0$ and $f = 0$

* If $c = d = 0$, substituting into (2.6) gives:

$$(a-1)(x^2 - y^2) = -e$$
$$\overset{\text{since } a\neq 1}{\Rightarrow} x^2 - y^2 = \frac{-e}{a-1}$$
$$\Rightarrow \frac{y^2}{S_1} - \frac{x^2}{S_1} = 1 \qquad \text{where} \qquad S_1 = \frac{e}{a-1},$$

which is hyperbola function.

From (2.7) we have:

$$2(a+1)xy = 0$$
$$\Rightarrow a = -1 \quad \text{or} \quad x = 0 \quad \text{or} \quad y = 0$$

· If $a = -1$ then we will have infinite number of solutions.

· If $x = 0$ then $y^2 = \frac{e}{a-1} \Rightarrow y = \pm\sqrt{\frac{e}{a-1}} \Rightarrow e > 0$ and $a > 1$ or $e < 0$ and$a < 1$.

· If $y = 0$ then $x^2 = \frac{-e}{a-1} \Rightarrow x = \pm\sqrt{\frac{e}{1-a}} \Rightarrow e > 0$ and $a < 1$ or $e < 0$ and $a > 1$.

From the above cases we can notice that we will have two solutions at maximum when $a \neq -1$. While we will have infinite number of solutions when $a = -1$.

* If $c = d \neq 0$, substituting into (2.6) gives:

$$\frac{\left(y - \left(\frac{-c}{2(a-1)}\right)\right)^2}{G} - \frac{\left(x - \left(\frac{-c}{2(a-1)}\right)\right)^2}{G} = 1$$
$$G = \frac{e}{a-1}$$

which is hyperbola function.

From (2.7) we get:

$$y = \frac{-cx}{2(a+1)x + c},$$

which is a rational function with vertical asymptote $x = \frac{-c}{2(a+1)}$ and horizontal asymptote $y = \frac{-c}{2(a+1)}$. In this case the maximum number of solutions we can get is four.

* If $0 \neq c \neq d \neq 0$, substituting into (2.6) gives:

$$\frac{\left(x - \left(\frac{-c}{2(a-1)}\right)\right)^2}{R} - \frac{\left(y - \left(\frac{-d}{2(a-1)}\right)\right)^2}{R} = 1$$
$$R = \frac{K - L - e}{a - 1}$$
$$K = \frac{c^2}{4(a-1)}$$
$$L = \frac{d^2}{4(a-1)}$$

which is hyperbola function.

From (2.7) we get:

$$y = \frac{-dx}{2(a+1)x + c}$$

which is a rational function with vertical asymptote $x = \frac{-c}{2(a+1)}$ and horizontal asymptote $y = \frac{-d}{2(a+1)}$. In this case the maximum number of solutions we can get is four.

- If $e \neq 0$ and $f \neq 0$

  * If $c = d = 0$, substituting into (2.6) gives:

$$\frac{y^2}{R_3} - \frac{x^2}{R_3} = 1$$
$$R_3 = \frac{e}{a - 1}$$

which is hyperbola function.

From (2.7) we get:

$$y = \frac{-f}{2(a+1)x},$$

which is a rational function with vertical asymptote $x = 0$ and horizontal asymptote $y = 0$. In this case the maximum number of solutions we can get is four.

  * If $c = d \neq 0$, substituting into (2.6) gives:

$$\frac{\left(y - \left(\frac{-c}{2(a-1)}\right)\right)^2}{R_3} - \frac{\left(x - \left(\frac{-c}{2(a-1)}\right)\right)^2}{R_3} = 1$$
$$R_3 = \frac{e}{a - 1}$$

which is hyperbola function.

From (2.7) we get:

$$y = \frac{-f - cx}{2(a+1)x + c},$$

which is a rational function with vertical asymptote $x = \frac{-c}{2(a+1)}$ and horizontal asymptote $y = \frac{-c}{2(a+1)}$. In this case the maximum number of solutions we can get is four.

* If $0 \neq c \neq d \neq 0$, substituting into (2.6) gives:

$$\frac{\left(x - \left(\frac{-c}{2(a-1)}\right)\right)^2}{R} - \frac{\left(y - \left(\frac{-d}{2(a-1)}\right)\right)^2}{R} = 1$$
$$R = \frac{K - L - e}{a-1}$$
$$K = \frac{c^2}{4(a-1)}$$
$$L = \frac{d^2}{4(a-1)},$$

which is hyperbola function.

From (2.7) we get:

$$y = \frac{-f - dx}{2(a+1)x + c}$$

which is a rational function with vertical asymptote $x = \frac{-c}{2(a+1)}$ and horizontal asymptote $y = \frac{-d}{2(a+1)}$. In this case the maximum number of solutions we can get is four.

* If $c = 0$ and $d \neq 0$, substituting into (2.6) gives:

$$\frac{\left(y - \left(\frac{-d}{2(a-1)}\right)\right)^2}{R_1} - \frac{x^2}{R_1} = 1$$
$$R_1 = \frac{L + e}{a-1}$$
$$L = \frac{d^2}{4(a-1)},$$

which is hyperbola function.

From (2.7) we get:

$$y = \frac{-f - dx}{2(a+1)x}$$

which is a rational function with vertical asymptote $x = 0$ and horizontal asymptote $y = \frac{-d}{2(a+1)}$. In this case the maximum number of solutions we can get is four.

* If $c \neq 0$ and $d = 0$, substituting into (2.6) gives:

$$\frac{\left(x - \left(\frac{-c}{2(a-1)}\right)\right)^2}{R_2} - \frac{y^2}{R_2} = 1$$
$$R_2 = \frac{K - e}{a - 1}$$
$$K = \frac{c^2}{4(a-1)},$$

which is hyperbola function.

From (2.7) we get:

$$y = \frac{-f}{2(a+1)x + c}$$

which is a rational function with vertical asymptote $x = \frac{-c}{2(a+1)}$ and horizontal asymptote $y = 0$. In this case the maximum number of solutions we can get is four.

- If $b \neq 0$:
  - If $e = 0$ and $f = 0$:
    * If $c = d = 0$, substituting into (2.6) gives:

$$(a-1)\left(x^2 - y^2\right) - 2bxy = 0$$
$$\Rightarrow xy = \frac{(a-1)\left(x^2 - y^2\right)}{2b} \tag{A.1}$$

Substituting (A.1) into (2.7) gives:

$$b\left(x^2 - y^2\right) + \frac{2(a+1)(a-1)\left(x^2 - y^2\right)}{2b} = 0$$
$$\Rightarrow b^2\left(x^2 - y^2\right) + 2\left(a^2 - 1\right)\left(x^2 - y^2\right) = 0$$
$$\Rightarrow \left(x^2 - y^2\right)\left(b^2 + 2a^2 - 2\right) = 0$$
$$\Rightarrow x = \pm y \quad \text{or} \quad b^2 + 2a^2 = 2.$$

If $x = y \Rightarrow y^2 = 0 \Rightarrow y = 0 = x$

If $x = -y \Rightarrow y^2 = 0 \Rightarrow y = 0 = x$

So, we have one solution which is the origin or infinite number of solutions if $b^2 + 2a^2 = 2$.

∗ If $c = d \neq 0$, substituting into (2.6) gives:

$$\left(x - \left(\frac{-c}{2(a-1)}\right)\right)^2 - \left(y - \left(-\frac{c}{2(a-1)}\right)\right)^2 - \frac{2b}{(a-1)}xy = 0,$$

which is hyperbola function.

Also, from (2.7) we get:

$$\left(x - \left(\frac{-d}{2b}\right)\right)^2 - \left(y - \frac{c}{2b}\right)^2 + \frac{2(a+1)}{b}xy = 0,$$

which is hyperbola function. In this case the maximum number of solutions we can get is four.

∗ If $0 \neq c \neq d \neq 0$, substituting into (2.6) gives:

$$\frac{\left(x - \left(\frac{-c}{2(a-1)}\right)\right)^2}{R} - \frac{\left(y - \left(\frac{-d}{2(a-1)}\right)\right)^2}{R} - \frac{2b}{K-L}xy = 1$$

$$R = \frac{K-L}{a-1}$$

$$K = \frac{c^2}{4(a-1)}$$

$$L = \frac{d^2}{4(a-1)},$$

which is hyperbola function.

From (2.7) we get:

$$\frac{\left(x - \left(\frac{-d}{2b}\right)\right)^2}{Q} - \frac{\left(y - \frac{c}{2b}\right)^2}{Q} + \frac{2(a+1)}{M-N}xy = 1$$

$$Q = \frac{M-N}{b}$$

$$M = \frac{d^2}{4b}$$

$$N = \frac{c^2}{4b},$$

which is hyperbola function. In this case the maximum number of solutions we can get is four.

* If $c = 0$ and $d \neq 0$, substituting into (2.6) gives:

$$\frac{\left(y - \left(\frac{-d}{2(a-1)}\right)\right)^2}{R_1} - \frac{x^2}{R_1} + \frac{2b}{L}xy = 1$$
$$R_1 = \frac{L}{a-1}$$
$$L = \frac{d^2}{4(a-1)},$$

which is hyperbola function.

From (2.7) we get:

$$\frac{\left(x - \left(\frac{-d}{2b}\right)\right)^2}{Q_1} - \frac{y^2}{Q_1} + \frac{2(a+1)}{M}xy = 1$$
$$Q_1 = \frac{M}{b}$$
$$M = \frac{d^2}{4b}$$

which is hyperbola function. In this case the maximum number of solutions we can get is four.

* If $c \neq 0$ and $d = 0$, substituting into (2.6) gives:

$$\frac{\left(x - \left(\frac{-c}{2(a-1)}\right)\right)^2}{R_2} - \frac{y^2}{R_2} - \frac{2b}{K}xy = 1$$
$$R_2 = \frac{K}{a-1}$$
$$K = \frac{c^2}{4(a-1)},$$

which is hyperbola function.

From (2.7) we get:

$$\frac{\left(y - \frac{c}{2b}\right)^2}{Q_2} - \frac{x^2}{Q_2} - \frac{2(a+1)}{N}xy = 1$$
$$Q_2 = \frac{N}{b}$$
$$N = \frac{c^2}{4b},$$

which is hyperbola function. In this case the maximum number of solutions we can get is four.

– If $e = 0$ and $f \neq 0$:

* If $c = d = 0$, substituting into (2.6) gives:

$$x^2 - y^2 - \frac{2b}{(a-1)}xy = 0$$

which is hyperbola function.

From (2.7) we get:

$$\frac{y^2}{B} - \frac{x^2}{B} - \frac{2(a+1)}{f}xy = 1$$
$$B = \frac{f}{b},$$

which is hyperbola function. In this case the maximum number of solutions we can get is four.

* If $c = d \neq 0$, substituting into (2.6) gives:

$$\left(x - \left(\frac{-c}{2(a-1)}\right)\right)^2 - \left(y - \left(\frac{-c}{2(a-1)}\right)\right)^2 - \frac{2b}{(a-1)}xy = 0$$

which is hyperbola function.

From (2.7) we get:

$$\frac{\left(y - \frac{c}{2b}\right)^2}{B} - \frac{\left(x - \left(\frac{-c}{2b}\right)\right)^2}{B} - \frac{2(a+1)}{f}xy = 1$$
$$B = \frac{f}{b},$$

which is hyperbola function. In this case the maximum number of solutions we can get is four.

* If $0 \neq c \neq d \neq 0$, substituting into (2.6) gives:

$$\frac{\left(x - \left(\frac{-c}{2(a-1)}\right)\right)^2}{R} - \frac{\left(y - \left(\frac{-d}{2(a-1)}\right)\right)^2}{R} - \frac{2b}{K-L}xy = 1$$
$$R = \frac{K-L}{a-1}$$
$$K = \frac{c^2}{4(a-1)}$$
$$L = \frac{d^2}{4(a-1)},$$

which is hyperbola function.

From (2.7) we get:

$$\frac{\left(x-\left(\frac{-d}{2b}\right)\right)^2}{Q}-\frac{\left(y-\frac{c}{2b}\right)^2}{Q}+\frac{2(a+1)}{M-N-f}xy=1$$
$$Q=\frac{M-N-f}{b}$$
$$M=\frac{d^2}{4b}$$
$$N=\frac{c^2}{4b},$$

which is hyperbola function. In this case the maximum number of solutions we can get is four.

* If $c=0$ and $d\neq 0$, substituting into (2.6) gives:

$$\frac{\left(y-\left(\frac{-d}{2(a-1)}\right)\right)^2}{R_1}-\frac{x^2}{R_1}+\frac{2b}{L}xy=1$$
$$R_1=\frac{L}{a-1}$$
$$L=\frac{d^2}{4(a-1)}$$

which is hyperbola function.

From (2.7) we get:

$$\frac{\left(x-\left(\frac{-d}{2b}\right)\right)^2}{Q_1}-\frac{y^2}{Q_1}+\frac{2(a+1)}{M-f}xy=1$$
$$Q_1=\frac{M-f}{b}$$
$$M=\frac{d^2}{4b}$$

which is hyperbola function. In this case the maximum number of solutions we can get is four.

* If $c \neq 0$ and $d = 0$, substituting into (2.6) gives:

$$\frac{\left(x - \left(\frac{-c}{2(a-1)}\right)\right)^2}{R_2} - \frac{y^2}{R_2} - \frac{2b}{K}xy = 1$$
$$R_2 = \frac{K}{a-1}$$
$$K = \frac{c^2}{4(a-1)},$$

which is hyperbola function.

From (2.7) we get:

$$\frac{\left(y - \frac{c}{2b}\right)^2}{Q_2} - \frac{x^2}{Q_2} - \frac{2(a+1)}{N+f}xy = 1$$
$$Q_2 = \frac{N+f}{b}$$
$$N = \frac{c^2}{4b}$$

which is hyperbola function. In this case the maximum number of solutions we can get is four.

- If $e \neq 0$ and $f = 0$:

  * If $c = d = 0$, substituting into (2.6) gives:

$$\frac{x^2}{T} - \frac{y^2}{T} + \frac{2b}{e}xy = 1$$
$$T = \frac{e}{a-1}$$

which is hyperbola function.

From (2.7) we get:

$$x^2 - y^2 + \frac{2(a+1)}{b}xy = 0$$

which is hyperbola function. In this case the maximum number of solutions we can get is four.

* If $c = d \neq 0$, substituting into (2.6) gives:

$$\frac{\left(x - \left(\frac{-c}{2(a-1)}\right)\right)^2}{T} - \frac{\left(y - \left(\frac{-c}{2(a-1)}\right)\right)^2}{T} + \frac{2b}{e}xy = 1$$
$$T = \frac{e}{a-1},$$

which is hyperbola function.

From (2.7) we get:

$$\left(x - \left(\frac{-c}{2b}\right)\right)^2 - \left(y - \frac{c}{2b}\right)^2 + \frac{2(a+1)}{b}xy = 0,$$

which is hyperbola function. In this case the maximum number of solutions we can get is four.

* If $0 \neq c \neq d \neq 0$, substituting into (2.6) gives:

$$\frac{\left(x - \left(\frac{-c}{2(a-1)}\right)\right)^2}{R} - \frac{\left(y - \left(\frac{-d}{2(a-1)}\right)\right)^2}{R} - \frac{2b}{K-L-e}xy = 1$$
$$R = \frac{K-L-e}{a-1}$$
$$K = \frac{c^2}{4(a-1)}$$
$$L = \frac{d^2}{4(a-1)}$$

which is hyperbola function.

From (2.7) we get:

$$\frac{\left(x - \left(\frac{-d}{2b}\right)\right)^2}{Q} - \frac{\left(y - \frac{c}{2b}\right)^2}{Q} + \frac{2(a+1)}{M-N}xy = 1$$
$$Q = \frac{M-N}{b}$$
$$M = \frac{d^2}{4b}$$
$$N = \frac{c^2}{4b}$$

which is hyperbola function. In this case the maximum number of solutions we can get is four.

* If $c = 0$ and $d \neq 0$, substituting into (2.6) gives:

$$\frac{\left(y - \left(\frac{-d}{2(a-1)}\right)\right)^2}{R_1} - \frac{x^2}{R_1} + \frac{2b}{L+e}xy = 1$$
$$R_1 = \frac{L+e}{a-1}$$
$$L = \frac{d^2}{4(a-1)},$$

which is hyperbola function. From (2.7) we get:

$$\frac{\left(x - \left(\frac{-d}{2b}\right)\right)^2}{Q_1} - \frac{y^2}{Q_1} + \frac{2(a+1)}{M}xy = 1$$
$$Q_1 = \frac{M}{b}$$
$$M = \frac{d^2}{4b},$$

which is hyperbola function. In this case the maximum number of solutions we can get is four.

* If $c \neq 0$ and $d = 0$, substituting into (2.6) gives:

$$\frac{\left(x - \left(\frac{-c}{2(a-1)}\right)\right)^2}{R_2} - \frac{y^2}{R_2} - \frac{2b}{K-e}xy = 1$$
$$R_2 = \frac{K-e}{a-1}$$
$$K = \frac{c^2}{4(a-1)}$$

which is hyperbola function.

From (2.7) we get:

$$\frac{\left(y - \frac{c}{2b}\right)^2}{Q_2} - \frac{x^2}{Q_2} - \frac{2(a+1)}{N}xy = 1$$
$$Q_2 = \frac{N}{b}$$
$$N = \frac{c^2}{4b}$$

which is hyperbola function. In this case the maximum number of solutions we can get is four.

– If $e \neq 0$ and $f \neq 0$

* If $c = d = 0$, substituting into (2.6) gives:

$$\frac{y^2}{V} - \frac{x^2}{V} + \frac{2b}{e}xy = 1$$
$$V = \frac{e}{a-1}$$

which is hyperbola function.

From (2.7) we get:

$$\frac{y^2}{W} - \frac{x^2}{W} - \frac{2(a+1)}{f}xy = 1$$
$$W = \frac{f}{b}$$

which is hyperbola function. In this case the maximum number of solutions we can get is four.

* If $c = d \neq 0$, substituting into (2.6) gives:

$$\frac{\left(y - \left(\frac{-c}{2(a-1)}\right)\right)^2}{V} - \frac{\left(x - \left(\frac{-c}{2(a-1)}\right)\right)^2}{V} + \frac{2b}{e}xy = 1$$
$$V = \frac{e}{a-1}$$

which is hyperbola function.

From (2.7) we get:

$$\frac{\left(y - \frac{c}{2b}\right)^2}{W} - \frac{\left(x - \left(\frac{-c}{2b}\right)\right)^2}{W} - \frac{2(a+1)}{f}xy = 1$$
$$W = \frac{f}{b}$$

which is hyperbola function. In this case the maximum number of solutions we can get is four.

∗ If $0 \neq c \neq d \neq 0$, substituting into (2.6) gives:

$$\frac{\left(x-\left(\frac{-c}{2(a-1)}\right)\right)^2}{R} - \frac{\left(y-\left(\frac{-d}{2(a-1)}\right)\right)^2}{R} - \frac{2b}{K-L-e}xy = 1$$

$$R = \frac{K-L-e}{a-1}$$

$$K = \frac{c^2}{4(a-1)}$$

$$L = \frac{d^2}{4(a-1)}$$

which is hyperbola function.

From (2.7) we get:

$$\frac{\left(x-\left(\frac{-d}{2b}\right)\right)^2}{Q} - \frac{\left(y-\frac{c}{2b}\right)^2}{Q} + \frac{2(a+1)}{M-N-f}xy = 1$$

$$Q = \frac{M-N-f}{b}$$

$$M = \frac{d^2}{4b}$$

$$N = \frac{c^2}{4b}$$

which is hyperbola function. In this case the maximum number of solutions we can get is four.

∗ If $c = 0$ and $d \neq 0$, substituting into (2.6) gives:

$$\frac{\left(y-\left(\frac{-d}{2(a-1)}\right)\right)^2}{R_1} - \frac{x^2}{R_1} + \frac{2b}{L+e}xy = 1$$

$$R_1 = \frac{L+e}{a-1}$$

$$L = \frac{d^2}{4(a-1)}$$

which is hyperbola function.

From (2.7) we get:

$$\frac{\left(x-\left(\frac{-d}{2b}\right)\right)^2}{Q_1} - \frac{y^2}{Q_1} + \frac{2(a+1)}{M-f}xy = 1$$

$$Q_1 = \frac{M-f}{b}$$

$$M = \frac{d^2}{4b}$$

which is hyperbola function. In this case the maximum number of solutions we can get is four.

* If $c \neq 0$ and $d = 0$, substituting into (2.6) gives:

$$\frac{\left(x - \left(\frac{-c}{2(a-1)}\right)\right)^2}{R_2} - \frac{y^2}{R_2} - \frac{2b}{K-e}xy = 1$$
$$R_2 = \frac{K-e}{a-1}$$
$$K = \frac{c^2}{4(a-1)},$$

which is hyperbola function.

From (2.7) we get:

$$\frac{\left(y - \frac{c}{2b}\right)^2}{Q_2} - \frac{x^2}{Q_2} - \frac{2(a+1)}{N+f}xy = 1$$
$$Q_2 = \frac{N+f}{b}$$
$$N = \frac{c^2}{4b},$$

which is hyperbola function. In this case the maximum number of solutions we can get is four.