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Riemann-Hilbert Problems for Nonlocal Reverse-Time Nonlinear Second-order and Fourth-order AKNS Systems of Multiple Components and Exact Soliton Solutions

Alle Adjiri
University of South Florida

Riemann-Hilbert Problems for Nonlocal Reverse-Time Nonlinear Second-order and Fourth-order AKNS Systems of Multiple Components and Exact Soliton Solutions

by

Alle Adjiri

A dissertation submitted in partial fulfillment
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Department of Mathematics and Statistics
College of Arts and Sciences
University of South Florida

Major Professor: Wen-Xiu Ma, Ph.D.
Catherine A. Bénéteau, Ph.D.
Mohamed Elhamdadi, Ph.D.
Sherwin Kouchekian, Ph.D.
Ivan Rothstein, Ph.D.

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Dedication

To the Lord, to the memory of my father and mother, to my wife and family, I humbly dedicate this dissertation.

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Abstract

We first investigate the solvability of an integrable nonlinear nonlocal reverse-time six-component fourth-order AKNS system generated from a reduced coupled AKNS hierarchy under a reverse-time reduction. Riemann-Hilbert problems will be formulated by using the associated matrix spectral problems, and exact soliton solutions will be derived from the reflectionless case corresponding to an identity jump matrix. Secondly, we present the inverse scattering transform for solving a class of eight-component AKNS integrable equations obtained by a specific reduction associated with a block matrix spectral problem. The inverse scattering transform based on Riemann-Hilbert problems is presented along with a jump matrix taken to be the identity matrix to derive soliton solutions.

Chapter 1

Introduction

1.1 History of the background

Over centuries, local integrable systems of equations have been the main center of research interest by mathematicians and physicists. We mean by integrable equations partial differential equations (PDEs) that can be presented through a Lax pair [42]–[43], and that have infinitely many symmetries and conservation laws. The Korteweg-de Vries (KdV) equation [7, 8, 33, 34], the nonlinear Schrödinger (NLS) equation [6], Sine-Gordon (SG) equation or Kadomtsev-Petviashvili (KP) equation were some of the best well known integrable equations even though the list of integrable evolution equations was not exhaustive [49]. In fact, The KdV equation was discovered by Diederik Korteweg and his student Gustav De Vries in 1895 as a nonlinear partial differential water wave equation. Its solution is the Scott Russell's solitary wave observed on the Edinburgh-Glasgow canal in 1834 [19, 35]. As to the KP equation, it was called a generalization to two spatial dimensions, x and y, of the one-dimensional KdV equation. It was written by Kadomtsev and Petviashvili in 1970s [49]. Both, the KdV and KP evolution equations has described weakly nonlinear shallow water waves. The nonlinear Schrödinger equation is a nonlinear variation of the Schrödinger equation found in 1925 by Erwin Schrödinger. It created a great impact in quantum physics.This evolution equation has described a weakly nonlinear dispersive wave trains in a media. Beyond the mathematics aspect, these integrable equations have described many physical phenomena such as magnetic fields, plasma physics, blood flow in arteries, and nonlinear optics [53], etc.

Some models of theses evolution equations are:

The Korteweg-de Vries equation:

$$p_t(x,t) - 6p(x,t)p_x(x,t) + p_{xxx}(x,t) = 0, \tag{1.1}$$

the nonlinear Schrödinger equation:

$$ip_t(x,t) + p_{xx}(x,t) + 2|p(x,t)|^2 p(x,t) = 0, \tag{1.2}$$

and the Kadomtsev-Petviashvili equation: [49, 48]

$$(p_t(x,y,t) - 6p(x,y,t)p_x(x,y,t) + p_{xxx}(x,y,t))_x - 3p_{yy}(x,y,t) = 0. \tag{1.3}$$

Although those integrable equations were found, explicit soliton solutions were not easily derived. We mean by a soliton, a type of localized solitary wave which maintains its shape after it collides elastically with another wave of the same type. During this period, numerical experiments and observational methods were utilized to find results or solve them. In 1960s, Krustal and Gardner found the inverse scattering transform (IST) to solve exactly those integrable nonlinear evolution equations (the KdV, the NLS, the sine-Gordon and the KP), and derived explicitly soliton solutions. This was a great method that has permitted to solve many integrable equations by then. This finding method was analogous to the method of the Fourier transform for linear partial differential equations (PDEs). The main point of the study and the work over those integrable equations was local.

In the recent years, Mark Ablowitz and Ziad Musslimani discovered that some nonlinear local integrable systems could be reduced under parity-time (PT) symmetric reductions to nonlocal integrable equations [1]–[3]. That is to say, it is invariant under the joint transformation: $x \to -x$, $t \to -t$, and $i \to -i$. This means that the solution of the particles at the location x and $-x$ could be coupled in the space by a suitable reduction of the AKNS. This notion was extended to the time symmetry $t \to -t$, and to the space-time symmetry $x \to -x$ and $t \to -t$.

Ablowitz and Musslimani found that the system of the coupled nonlinear Schrödinger AKNS equations [25, 37]:

$$\begin{cases} ip_t(x,t) = p_{xx}(x,t) - 2rp^2(x,t), \\ -ir_t(x,t) = r_{xx}(x,t) - 2pr^2(x,t), \end{cases} \tag{1.4}$$

under the PT preserving symmetric reduction $r(x,t) = \rho p^*(-x,t)$, for $\rho = \pm 1$, gives the nonlocal NLS equation

$$ip_t(x,t) + p_{xx}(x,t) \pm 2p^2(x,t)p^*(-x,t) = 0. \tag{1.5}$$

Interestingly, this equation (1.5) is invariant under the joint transformation

$$\begin{cases} x \to -x, \\ t \to -t, \\ i \to -i. \end{cases} \tag{1.6}$$

Also, $p(x,t)$ and $p^*(-x,-t)$ are both solutions of the PT symmetric nonlocal Schrödinger equation (1.5).

The system of the coupled mKdV AKNS equations:

$$\begin{cases} p_t(x,t) = -p_{xxx}(x,t) + 6p(x,t)r(x,t)p_x(x,t), \\ r_t(x,t) = -r_{xxx}(x,t) + 6p(x,t)r(x,t)r_x(x,t), \end{cases} \tag{1.7}$$

becomes under the PT symmetric reduction $r(x,t) = \rho p^*(-x,-t)$, for $\rho = \pm 1$, the nonlocal complex mKdV equation:

$$p_t(x,t) + p_{xxx}(x,t) - 6\rho p(x,t)p^*(-x,-t)p_x(x,t) = 0. \tag{1.8}$$

The system of the coupled sine-Gordon AKNS equations:

$$\begin{cases} p_{xt}(x,t) = -2s(x,t)p(x,t), \\ r_{xt}(x,t) = -2s(x,t)r(x,t), \\ s_x(x,t) = -\big(p(x,t)r(x,t)\big)_t, \end{cases} \tag{1.9}$$

becomes under the reverse space-time symmetric reduction $r(x,t) = -p(-x,-t)$, the nonlocal sine-Gordon equation:

$$p_{xt}(x,t) + 2s(x,t)p(x,t) = 0, \text{ for } \quad s(-x,-t) = s(x,t). \tag{1.10}$$

Although those nonlocal evolution equations are integrable, they also posses an infinite number of conservation laws. Besides the PT of the nonlocal nonlinear Schrödinger (NNLS), Ablowitz and Musslimani derived the reverse-time and reverse space-time NNLS equations, but they did not solve them. Recently, Jianke Yang has contributed in helping solve them.

Under the reverse-time symmetric reduction $r(x,t) = \rho p(x,-t)$, for $\rho = \pm 1$, the system (1.4) gives the nonlocal NLS equation:

$$ip_t(x,t) - p_{xx}(x,t) \pm 2p^2(x,t)p(x,-t) = 0, \tag{1.11}$$

and under the reverse space-time symmetric reduction $r(x,t) = \rho p(-x,-t)$, for $\rho = \pm 1$, we obtain the nonlocal NLS equation

$$ip_t(x,t) - p_{xx}(x,t) \pm 2p^2(x,t)p(-x,-t) = 0. \tag{1.12}$$

Many researches were done on NNLS [38, 39], and some remarks came up that solutions of nonlocal integrable equations in finite time could collapse. Such solutions might have singularities at finite time. In contrast, this kind of behavior of the fundamental solitons do not blowup in finite time [47]. Those nonlocal evolution equations were solved by the IST, and explicit solutions were found.

Over five years ago, Jianke Yang has done researches on the nonlocal NLS, and he solved them in the reverse-time, in reverse-space, and in reverse space-time using the inverse scattering based on Riemann-Hilbert problems [9]–[12]. Many other methods like Darboux transformation [4] and Hirota's bilinear method [5] have been used to investigate the solvability of the nonlocal NLS equations.

1.2 Motivation

The main problem is that most investigations in solving integrable local or nonlocal nonlinear evolution equations are based on the use of the inverse scattering transform, Darboux transformation or Hirota's bilinear method. In addition, investigating the dynamics of nonlocal evolution equations is still an active exploration. This perspective leads us to focus the dissertation on the solvability of a nonlinear nonlocal reverse-time six-component higher-order AKNS system, and the inverse scattering for a nonlocal reverse-time eight-component AKNS system via Riemann-Hilbert problems from different spectral matrices. Latterly, we investigate different dynamical behaviours of exact solutions.

1.3 Overview of the dissertation

We are going to lay out the dissertation as follows: in chapter 2, we exhibit certain methods used to solve integrable evolution equations, such as the inverse scatting transform, Darboux transformation, Hirota's bilinear method, and Riemann -Hilbert problems. In chapter 3, we present the AKNS hierarchy of multiple components along with a six-component AKNS hierarchy of coupled fourth-order integrable equations. As to chapter 4, we analyse the Riemann-Hilbert problems associated with the corresponding matrix spectral problems which are closely related to the inverse scattering method. In chapter 5, we generate soliton solutions from reflectionless problems by taking the identity jump matrix [13]–[16]. In chapter 6, we present a few examples of soliton solutions and snoop in the dynamics. Also, in chapter 7, we present the inverse scattering transform of a nonlocal reverse-time nonlinear second-order nonlinear Schrödinger equation, and present exact one-soliton solution. Finally, the last two chapters will be the concluding remarks and references.

1.4 Preliminary

1.4.1 Lax pair

1.4.1.1 Introduction and concept

The Lax pair was introduced by Peter Lax in 1968 [36]. It consists of finding a pair of linear differential operators L and M such that if they can represent a corresponding evolution equation

$$K(x, t, p, p_x, ...) = 0, \tag{1.13}$$

which is often called **integrable**.

The Lax operator L is self-adjoint i.e., L is equal to its complex conjugate. Together with the required differential operator M, they can be expressed in the form:

$$L_t = [M, L] = (ML - LM), \quad \text{or} \quad L_t - [L, M] = 0. \tag{1.14}$$

The operator M has sufficiently many freedom including unknown parameters or functions. Both operators M and L could be scalars or matrix operators.

1.4.1.2 Operator form of a Lax pair

For a given linear operator L that depends upon the function $p(x, t)$,

$$L\psi(x, t) = \lambda\psi(x, t), \tag{1.15}$$

is called a **spectral problem**, where $\lambda = \lambda(t)$ is an eigenvalue, and ψ is an eigenfunction .

The idea is to find another operator M such that

$$\psi_t(x,t) = M\psi(x,t). \tag{1.16}$$

Differentiating both sides of (1.15) with respect to t, we get

$$L_t\psi + L\psi_t = \lambda_t\psi + \lambda\psi_t. \tag{1.17}$$

Substituting (1.16) and (1.15) into (1.17) we get

$$L_t\psi_t + LM\psi = \lambda_t\psi + ML\psi, \tag{1.18}$$

so

$$(L_t + LM - ML)\psi = \lambda_t\psi. \tag{1.19}$$

Therefore (1.19) gives the following operator equation

$$L_t + [L, M] = 0, \quad \text{as long as} \quad \lambda_t = 0. \tag{1.20}$$

Thus, when each eigenvalue is constant ($\lambda_t = 0$, called an **isospectral property**), the equation (1.20) is called **a Lax representation**. Also, the Lax pairs are tools to generate conserved quantities [17].

Example 1. Let us consider the Lax pair

$$L = \frac{\partial^2}{\partial x^2} - p \tag{1.21}$$

and

$$M = -4\frac{\partial^3}{\partial x^3} + 6p\frac{\partial}{\partial x} + 3\frac{\partial p}{\partial x}, \tag{1.22}$$

and

$$L\psi(x,t) = \lambda\psi(x,t) \tag{1.23}$$

is the Sturm-Liouville problem.

Hence

$$\frac{\partial p}{\partial t} = L_t = [L, M] = -\frac{\partial^3}{\partial x^3} + 6p\frac{\partial p}{\partial x} \tag{1.24}$$

exactly gives the KdV equation

$$p_t(x,t) - 6p(x,t)p_x(x,t) + p_{xxx}(x,t) = 0. \tag{1.25}$$

1.4.2 Zero curvature representation

1.4.2.1 Introduction and concept

In 1974, Ablowitz, Kaup, Newell and Segur contructed a Lax pair formulation in a matrix form [6]. This matrix formulation uses only the first-order operation ∂_x and ∂_t instead of using higher-order Lax operators. This method was called the **AKNS scheme**. They introduced the following system:

$$\Psi_x = U\Psi, \tag{1.26}$$

$$\Psi_t = V\Psi, \tag{1.27}$$

where U represents the operator L, and V represents the operator M and Ψ represents the eigenvector function. If L is a 2nd-order operator, then $\Psi = (\psi \quad \psi_x)^T$ and U and V will be each 2×2 matrix whose components are determined by the coefficient of the operators L and M. The compatibility of the equations (1.26) and (1.27) gives

$$(\Psi_x)_t = (U\Psi)_t = (\Psi_t)_x = (V\Psi)_x, \tag{1.28}$$

and then

$$(U_t - V_x + UV - VU)\Psi = 0. \tag{1.29}$$

As the Lax pair has a nontrivial solution Ψ with $det(\Psi) \neq 0$, we obtain

$$U_t - V_x + [U, V] = 0, \tag{1.30}$$

which is equivalent to the Lax equation and called a **zero curvature equation**.

Example 2. Let's consider the following Lax matrices,

$$U = \begin{pmatrix} 0 & 1 \\ \lambda + p & 0 \end{pmatrix}, \quad V = \begin{pmatrix} -p_x & -4\lambda + 2p \\ -4\lambda^2 - 2\lambda p + 2p^2 - p_{xx} & p_x \end{pmatrix}. \tag{1.31}$$

By inserting them into (1.30) gives the NLS equation

$$p_t(x,t) - 6p(x,t)p_x(x,t) + p_{xxx}(x,t) = 0. \tag{1.32}$$

1.5 General methods of soliton hierarchies

1.5.1 Introduction and concept

We start from the spatial isospectral problem [44],[50]

$$i\psi_x(x,t) = U(u,\lambda)\psi(x,t), \tag{1.33}$$

where ψ is an eigenfunction , U is a matrix belonging to a loop algebra i.e., U is a matrix in a Lie algebra expandable in a Laurent series of the spectral parameter λ, and u is a column vector of variables x and t. Let's suppose that there exists a matrix W such that

$$W = W(u,\lambda) = \sum_{j=0}^{\infty} W_{0,j}\lambda^{-j}, \tag{1.34}$$

where $W_{0,j}$ is a matrix in the associated Lie algebra, presents a solution of the stationary zero curvature equation

$$W_x = i[U,V]. \tag{1.35}$$

Using the solution W, we define the Lax matrix

$$V^{[m]} = (\lambda^m W)_+ + \mathit{\Delta}_m, \quad \text{for} \quad m \in \{0,1,2,\ldots\}, \tag{1.36}$$

where $(\lambda^m W)_+$ will be a matrix in the loop algebra, being a polynomial in λ, and $\mathit{\Delta}_m$ are the modification terms. Choosing $\mathit{\Delta}_m \to 0$ permits to define:

the temporal spectral problems

$$i\psi_t(x,t) = V^{[m]}(u,\lambda)\psi(x,t), \tag{1.37}$$

and also leads to derive a soliton hierarchy from the zero curvature equation obtained by the compatibility condition $(\psi)_{xt} = (\psi)_{tx}$

$$U_t - V_x^{[m]} + i[U, V^{[m]}] = 0. \tag{1.38}$$

This method is an extension of the AKNS scheme.

Remark 1.5.1. The choice of the special matrix U gives rise to different sorts of hierarchies. Some well known are the AKNS hierarchy, the Kaup-Newell (KN) hierarchy, the TA hierarchy, etc.

Because those equations are integrable, they possess Hamiltonian structures [15]:

$$u_{t_m} = J\frac{\delta\mathcal{H}_m}{\delta u}, \quad \text{for} \quad m \in \{0,1,2,\dots\}, \tag{1.39}$$

where $\frac{\delta\mathcal{H}_m}{\delta u}$ is the variational derivative of the Hamiltonian with respect to u.

Example 3. (AKNS hierarchy) In $\mathfrak{sl}(2,\mathbb{R})$, the spatial isospectral problem is given by:

$$-i\psi_x = U(u,\lambda)\psi, \tag{1.40}$$

with

$$U(u,\lambda) = \begin{pmatrix} -\lambda & p \\ r & \lambda \end{pmatrix} , \tag{1.41}$$

where $u = (p,r)^T$, $\psi = (\psi_1,\psi_2)^T$, and λ is a spectral parameter.

If we set

$$W = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \tag{1.42}$$

then the stationary zero curvature equation $W_x = i[U, W]$ gives

$$\begin{cases} a_x = i(pc - br), \\ b_x = i(-2\alpha\lambda b + pd - 2ap), \\ c_x = i(2\alpha\lambda c + 2ra). \end{cases} \tag{1.43}$$

If we expand W in Laurent series, i.e.,

$$a = \sum_{j=0}^{\infty} a_j \lambda^{-j}, \quad b = \sum_{j=0}^{\infty} b_j \lambda^{-j}, \quad c = \sum_{j=0}^{\infty} c_j \lambda^{-j}, \quad \text{for} \quad j \in \{0, 1, 2, \ldots\}, \tag{1.44}$$

then we get from (1.43),

$$\begin{cases} a_{j+1,x} = i(pc_{j+1} - rb_{j+1}), \\ b_{j+1} = -\frac{1}{2}(-ib_{j,x} - 2a_j p), \\ c_{j+1} = \frac{1}{2}(ic_{j,x} - 2ra_j). \end{cases} \tag{1.45}$$

By taking the initial values

$$a_0 = -1, b_0 = 0, c_0 = 0, \tag{1.46}$$

and choosing the constant of integrations to be zero, which is equivalent to the following condition:

$$W_j|_{u=0} = 0, \quad j \in \{1, 2, \ldots\}, \tag{1.47}$$

we determine the sequence $\{a_j, b_j, c_j\}$ for $j \in \{1, 2, \ldots\}$ as follows:

$$b_1 = p, \quad c_1 = q, \quad a_1 = 0, \tag{1.48}$$

$$b_2 = i\frac{1}{2}p_x, \quad c_2 = -i\frac{1}{2}r_x, \quad a_2 = \frac{1}{2}pr, \tag{1.49}$$

$$b_3 = -\frac{1}{4}(p_{xx} + 2p^2 r), \quad c_3 = -\frac{1}{4}(r_{xx} + 2r^2 p), \quad a_3 = -\frac{1}{4}i(pr_x - p_x r), \tag{1.50}$$

$$b_4 = -\frac{1}{8}i(p_{xxx} + 6pp_x r), \quad c_4 = \frac{1}{8}i(r_{xxx} + 6rpp_x), a_4 = -\frac{1}{8}(pr_{xx} - p_x r_x + p_{xx} r + 3p^2 r^2). \tag{1.51}$$

Thus, by taking the modification terms to be zero, the temporal Lax matrix will be

$$V^{[m]} = (\lambda^m W)_{2\times 2}, \tag{1.52}$$

and it will satisfy the zero curvature equation

$$U_{t_m} - V_x^{[m]} + i[U, V^{[m]}] = 0. \tag{1.53}$$

As a result, we get the AKNS soliton hierarchy as follows

$$u_{t_m} = \begin{pmatrix} p \\ r \end{pmatrix}_{t_m} = i \begin{pmatrix} -2b^{[m+1]} \\ 2c^{[m+1]} \end{pmatrix}, \qquad m \in \{0, 1, 2, ...\}. \tag{1.54}$$

If $m = 2$, we obtain a system of integrable equations

$$\begin{cases} ip_{t_2} &= \frac{1}{2}p_{xx} - p^2 r, \\ ir_{t_2} &= -\frac{1}{2}r_{xx} + pr^2, \end{cases} \tag{1.55}$$

which is the integrable system of the standard nonlinear Schrödinger equations.

The AKNS soliton hierarchy can be written in operator form:

$$u_{t_m} = \begin{pmatrix} p \\ r \end{pmatrix}_{t_m} = i \begin{pmatrix} -2b^{[m+1]} \\ 2c^{[m+1]} \end{pmatrix} = \Phi^m \begin{pmatrix} -2p \\ 2r \end{pmatrix} = J\frac{\delta \mathcal{H}_m}{\delta u}, \qquad m \in \{0, 1, 2, ...\}. \tag{1.56}$$

Now, if $\partial^{-1} = \int$, then we derive the hereditary recursion operator

$$\Phi = \begin{pmatrix} -\frac{1}{2}\partial + p\partial^{-1} r & p\partial^{-1} p \\ -r\partial^{-1} r & \frac{1}{2}\partial - r\partial^{-1} p \end{pmatrix}, \tag{1.57}$$

and the Hamiltonian operator reads

$$J = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}. \tag{1.58}$$

Using the trace identity [50] or variational identity [18], we get the Hamiltonian functional given by

$$\mathcal{H}_m = \int \frac{2a_{m+2}}{m+1} dx. \tag{1.59}$$

If $m = 2$, then the Hamiltonian explicitly becomes

$$\mathcal{H}_2 = \int \frac{2a_4}{3} dx = -\frac{1}{12} \int (3p^2r^2 + pr_{xx} - p_x r_x + p_{xx} r) dx. \tag{1.60}$$

Chapter 2

Techniques for solving integrable equations

2.1 Inverse scattering method

2.1.1 Introduction

The inverse scattering transform (IST) is a method used to solve some non-linear partial differential equations (PDEs) by reducing nonlinear integrable PDEs to linear PDEs [1, 7, 19, 41]. It consists of recovering the time evolution of potentials from the time evolution of scattering data. The IST follows three steps and at each step we have to solve a linear problem:

1. **The direct scattering** maps the initial potential $p(x,0)$ to the initial scattering data $\{s(\lambda,0)\}$.

2. **The time evolution** of scattering data maps the stationary scattering data $\{s(\lambda,0)\}$ to the time evolution scattering data $\{s(\lambda,t)\}$ in any time.

3. **The inverse scattering** maps the scattering data $\{s(\lambda,t)\}$ to the potential $p(x,t)$ in any time.

Let's consider the Cauchy problem for a nonlinear evolution equation:

$$\begin{cases} p_t = K(p, p_x, p_{xx}, ...), \\ p(x,0) = p_0(x). \end{cases} \tag{2.1}$$

If the evolution equation can have a Lax pair

$$L\psi = \lambda\psi, \tag{2.2}$$

$$\psi_t = M\psi, \tag{2.3}$$

and $\{s(\lambda,t)\}$, the scattering data for $p(x,t)$, are:

the discrete eigenvalues $\{k_n\}$;

the norming coefficients of the eigenfunctions $\{c_n(0)\}$;

the reflection coefficient $b(k,0)$;

the transmission coefficient $a(k,0)$;

then the scheme of the inverse scattering problem is as follows:

$$p(x,0) \longrightarrow \{s(\lambda,0)\} \longrightarrow \{s(\lambda,t)\} \longrightarrow p(x,t), \tag{2.4}$$

which can be represented as:

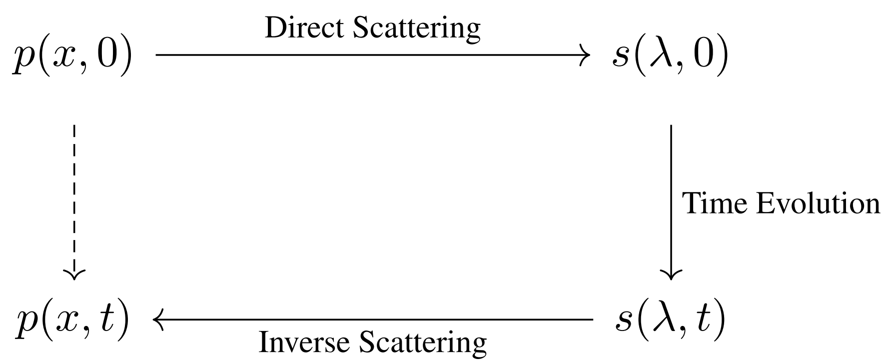

Figure 1.: Representation of the inverse scattering transform.

2.1.2 Example of the KdV solution

Let's consider the KdV equation

$$p_t(x,t) - 6p(x,t)p_x(x,t) + p_{xxx}(x,t) = 0. \tag{2.5}$$

The Lax pair reads

$$L = -\frac{\partial^2}{\partial x^2} + p, \tag{2.6}$$

$$M = -4\frac{\partial^3}{\partial x^3} + 6p\frac{\partial}{\partial x} + 3\frac{\partial p}{\partial x} + A(t), \tag{2.7}$$

and derive from (2.6) and (2.2) the Sturm–Liouville equation

$$\psi_{xx} + (\lambda - p)\psi = 0. \tag{2.8}$$

In case of the direct scattering, we need to solve this Strum–Liouville equation

$$\psi_{xx} + (\lambda - p(x,0))\psi = 0, \tag{2.9}$$

in order to derive the scattering data

$$S = \{k_n, c_n(0), a(k,0), b(k,0)\}. \tag{2.10}$$

whereas the time evolution process will derive the time dependent scattering data

$$S = \{k_n = constant,\ c_n(t) = c_0(t)e^{4k_n^3 t},\ a(k,t) = a(k,0),\ b(k,t) = b(k,0)e^{8ik^3 t}\}. \tag{2.11}$$

In order to perform the inverse scattering to recover the potential, the Gelfand–Levitan–Marchenko (GLM) integral equation found in 1950s below will be used:

$$K(x,y) + F(x+y) + \int_x^\infty K(x,z)F(y+z)dz = 0, \tag{2.12}$$

where

$$F(x,t) = \sum_{n=1}^{N} c_n^2(0)e^{8k_n^3 t - k_n x} + \frac{1}{2\pi}\int_{-\infty}^{\infty} b(k;0)e^{8ik^3 t + ikx}dk \tag{2.13}$$

satisfies the GLM equation (2.12) for $K(x,y;t)$ and allows to recover the KdV equation's potential

$$p(x,t) = -2\frac{\partial}{\partial x}K(x,x;t), \tag{2.14}$$

where

$$K(x,y;t) = \frac{-2k_1 c^2(0)e^{-k_1 x + 8k_1^2 t - k_1 y}}{2k_1 + c^2(0)e^{-2k_1 x + 8k_1^2 t}}, \tag{2.15}$$

for $N = 1$, with the discrete eigenvalue $k_n = k_1$, and the reflection coefficient $b(k,0) = 0$.

Based on this case, the potential for one soliton reads

$$p(x,t) = -2k_1^2 sech^2(k_1 x - 4k_1^3 t - x_o) \tag{2.16}$$

with the phase $x_o = \frac{1}{2}\ln\frac{c^2(0)}{2k}$.

2.1.3 Conclusion and remarks

The inverse scattering transformation is very useful in the process of solving integrable nonlinear partial differential equations, but many issues related the IST remain. It is not easy to characterize nonlinear PDEs. It is not simple as well to find a necessary, and sufficient condition that guarantees an initial value problem or the Cauchy problem for the integrable evolution equations that can be solved by inverse scattering transformation.

Another issue is that, it is not evident to find a corresponding linear ordinary equation for a given nonlinear integrable partial differential equation.

2.2 Darboux transformation

2.2.1 Introduction

In 1998, Darboux transformations [32] aim to solve the Liouville equation

$$\psi_{xx} + (\lambda - p(x,0))\psi = 0 \tag{2.17}$$

and was extended later to solve many integrable equations that are solvable by the inverse scattering transform such as the KdV, the NLS, the AKNS hierarchy, etc.

Proposition 2.2.1. *If $\psi(x,\lambda)$ and $\phi(x,t)$ are solutions of the Strum–Liouville equation*

$$\psi_{xx} + (\lambda - p(x,0))\psi = 0,$$

then $\psi[1] = \psi_x + \sigma\psi$ is a solution of the equation

$$\psi_{xx}[1] + (\lambda - p(x,0)[1])\psi[1] = 0, \tag{2.18}$$

where $\sigma = -(ln\phi)_x$ *and* $p(x,0)[1] = p(x,0) + 2\sigma_x$.

The form of $\psi[1]$ *and* $p[1]$ *represent functions of the solutions that define respectively the Darboux transformation of* ψ *and* $p(x,0)$.

In other words, $(\psi, p(x,0)) \to (\psi[1], p(x,0)[1])$.

2.2.2 Example of solutions of the KdV equation

Let's consider [51]

$$p_t(x,t) - 6p(x,t)p_x(x,t) + p_{xxx}(x,t) = 0. \tag{2.19}$$

The KdV equation has the Lax pair

$$L = -\frac{\partial^2}{\partial x^2} + p, \tag{2.20}$$

$$M = -4\frac{\partial^3}{\partial x^3} + 3p\frac{\partial}{\partial x} + 3\frac{\partial p}{\partial x} + A(t). \tag{2.21}$$

This gives the Strum–Liouville equation

$$\psi_{xx} + (\lambda - p(x,0))\psi = 0, \tag{2.22}$$

or

$$\psi_{xx} = -(\lambda - p(x,0))\psi. \tag{2.23}$$

From the Proposition 2.2.1, we have

$$\psi[1] = \psi_x + \sigma\psi, \tag{2.24}$$

where $\sigma = -(ln\psi_1)_x = -\frac{\psi_1'}{\psi_1}$ and ψ_1 satisfies

$$\psi_{xx}[1] + (\lambda - p(x,0)[1])\psi[1] = 0. \tag{2.25}$$

If we substitute (2.24) in (2.23), we get

$$\psi_{xx}[1] = -((\lambda - p(x,0)[1])\psi_x - \sigma(-p(x,0)[1] + \lambda)\psi. \tag{2.26}$$

Differentiating (2.24) twice with respect x, we get

$$\psi_{xx}[1] = \psi_{xxx} + \sigma_{xx}\psi + 2\sigma_x\psi_x + \sigma\psi_{xx}. \tag{2.27}$$

Now differentiate (2.23) we get

$$\psi_{xxx} = -(\lambda - p(x,0))\psi_x + p_x(x,0)\psi. \tag{2.28}$$

Using (2.23) and (2.27) in (2.26) gives

$$\psi_{xx}[1] = -(\lambda - p(x,0) - 2\sigma_x)\psi_x + (p_x(x,0) + \sigma_{xx} + \sigma p(x,0) - \sigma\lambda)\psi. \tag{2.29}$$

Matching (2.27) and (2.28), we deduce that

$$p(x,0)[1] = p(x,0) + 2\sigma_x. \tag{2.30}$$

If we take $p(x,0) = 0$, then the Lax pair becomes

$$\begin{aligned} \psi_{xx} &= -\lambda\psi, \\ \psi_t &= -4\psi_{xxx}, \end{aligned} \tag{2.31}$$

for $\psi_t = -4\psi_{xxx} + 6p(x,0)\psi_x + 3p_x(x,0)\psi$.

The solutions of both ODEs are given by

$$\psi_1(x,t) = e^{\frac{1}{2}(k_1 x - k_1^3 t)} + e^{-\frac{1}{2}(k_1 x - k_1^3 t)} \tag{2.32}$$

where $\lambda = \frac{k_1^2}{4}$. Thus using (2.30) with $p(x,0) = 0$ and (2.32), we obtain the one-soliton solution of the KdV equation:

$$p(x,t)[1] = -\frac{k_1^2}{2} sech^2\Big(\frac{k_1}{2}x - \frac{k_1^3}{2}t\Big). \tag{2.33}$$

Remark 2.2.2. We can recursively apply the Darboux transformation to get N-soliton solutions. For the two-soliton solution, the recursive expression gives from (2.24)

$$\psi[2] = \psi_x[1] + \sigma\psi[1] \tag{2.34}$$

where $\sigma = -(ln\psi_2[1])_x$, $\psi[1] = \psi_x + \frac{\psi_1'}{\psi}\psi$. Thus the potential for $\psi[2]$ reads

$$p(x,0)[2] = p(x,0)[1] + 2(\ln\psi_2[1])_{xx}. \tag{2.35}$$

2.2.3 Conclusion and remarks

Even though the Darboux Transformation presents explicit solutions by simple techniques, it does not solve the initial value problem. The above approach is the basic one. The general Darboux transformation could be applied with any convenient spectral problem.

2.3 Hirota's bilinear method

2.3.1 Introduction

In 1971, Hirota noticed that the best dependent variables for constructing soliton solutions are those in which the solution appears as a finite sum of exponentials. Due to this idea, he found a method to transform nonlinear evolution equations into a type of bilinear differential equations called **Hirota form** by introducing bilinear differential operators [5]. He defined the bilinear operator as follows:

$$D_t^m D_x^n(f\cdot g) = (\frac{\partial}{\partial t} - \frac{\partial}{\partial t'})^m(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'})^n f(x,t)g(x',t')|_{x'=x,t'=t}\,, \tag{2.36}$$

where m and n are positive integers, and f and g are two functions [19, 26, 27, 28].

Example 4. If $f = e^{\alpha_1}$ and $g = e^{\alpha_2}$, where $\alpha_1 = k_1x - \omega_1 t + \theta_1$ and $\alpha_2 = k_2x - \omega_2 t + \theta_2$, for x and t, then

$$D_t^m D_x^n(e^{\alpha_1}\cdot e^{\alpha_2}) = (\omega_2-\omega_1)^m(k_1-k_2)^n e^{\alpha_1+\alpha_2}. \tag{2.37}$$

Thus, if $f = g$, we get

$$D_t^m D_x^n(f\cdot f) = D_t^m D_x^n(e^{\alpha_1}\cdot e^{\alpha_1}) = 0. \tag{2.38}$$

Also if $f = 1$, we obtain

$$D_t D_x(1 \cdot g) = \frac{\partial^2 g}{\partial t \partial x}. \tag{2.39}$$

This Hirota's bilinear derivative has many properties.

If $m = 0$, then

$$D_x^n(f \cdot g) = (\frac{\partial}{\partial x} - \frac{\partial}{\partial x'})^n f(x,t) g(x',t')|_{x'=x,t'=t} \tag{2.40}$$

or

$$D_x^n(f \cdot g) = \sum_{i=0}^{n}(-1)^{n-i}. \begin{pmatrix} n \\ i \end{pmatrix} \Big(\frac{\partial^i f}{\partial x}\Big)\Big(\frac{\partial^{n-i} g}{\partial x}\Big), \quad n \in \{1,2,3,...\} \tag{2.41}$$

Example 5. If $n = 2$, then

$$D_x^2(f \cdot g) = f_{xx} g - 2 f_x g_x + f g_{xx}. \tag{2.42}$$

If n is odd, i.e., $n = 2m - 1$ and $f = g$, then

$$D_x^{2m-1}(f \cdot f) = 0, \tag{2.43}$$

because the Hirota operator is antisymmetric when it is odd.

If n is even, i.e., $n = 2m$ and $f = g$, then

$$D_x^{2m}(f \cdot f) = \sum_{i=0}^{2m}(-1)^{2m-i} \begin{pmatrix} 2m \\ i \end{pmatrix} \Big(\frac{\partial^i f}{\partial x}\Big)\Big(\frac{\partial^{2m-i} f}{\partial x}\Big), \quad m \in \{1,2,3,...\}. \tag{2.44}$$

Example 6. If $m = 2$, then

$$D_x^4(f \cdot f) = 2(f f_{xxxx} - 4 f_x f_{xxx} + 3 f_{xx}^2). \tag{2.45}$$

2.3.2 Example of solutions of the KdV equation by Hirota's method

We consider the KdV equation

$$p_t(x,t) - 6p(x,t)p_x(x,t) + p_{xxx}(x,t) = 0. \tag{2.46}$$

Let's suppose that

$$p = -2(lnf)_{xx} \tag{2.47}$$

satisfies the KdV equation (2.46), where $f(x,t)$ is the determinant of the appropriate matrix.

We deduce that if $p \to 0$, then the KdV in bilinear form is

$$(D_x D_t + D_x^4)(f \cdot f) = 0. \tag{2.48}$$

In order to solve this bilinear equation, let's substitute into (2.48) the pertubed function at the first order of

$$f = 1 + \sum_{i=0}^{\infty} \epsilon^i f_i(x,t). \tag{2.49}$$

Therefore the equation (2.48) becomes

$$(D_x D_t + D_x^4)(1 \cdot 1 + \epsilon 1 \cdot f_1 + \epsilon f_1 \cdot 1 + \epsilon^2 f_1 \cdot f_1) = 0, \tag{2.50}$$

and gives

$$(D_x D_t + D_x^4)(1 \cdot 1) + \epsilon[(D_x D_t + D_x^4)(1 \cdot f_1) + (D_x D_t + D_x^4)(f_1 \cdot 1)] + \epsilon^2 (D_x D_t + D_x^4)(f_1 \cdot f_1) = 0. \tag{2.51}$$

From (2.39),

$$(D_x D_t + D_x^4)(1 \cdot 1) = 0, \tag{2.52}$$

so to get (2.51), we need

$$\begin{cases} (D_x D_t + D_x^4)(1 \cdot f_1) + (D_x D_t + D_x^4)(f_1 \cdot 1) = 0, & \text{(2.53a)} \\ \\ (D_x D_t + D_x^4)(f_1 \cdot f_1) = 0. & \text{(2.53b)} \end{cases}$$

From (2.39), (2.53a) reads

$$\Big(\frac{\partial}{\partial t} + \frac{\partial^3}{\partial x^3}\Big) f_1 = 0. \tag{2.54}$$

From (2.37), if we assume that $f_1 = e^{kx-\omega t}$, then (2.53b) holds true. In that case, (2.54) can be rewritten as

$$(\frac{\partial}{\partial t} + \frac{\partial^3}{\partial x^3})e^{kx-\omega t} = 0, \tag{2.55}$$

which leads to the dispersion relation [29]

$$\omega = k^3. \tag{2.56}$$

Finally, we get the one-soliton solution for the KdV equation, which will be guaranteed by the choice of

$$f(x,t) = 1 + e^{kx-k^3 t}. \tag{2.57}$$

If we arbitrarily take $\epsilon = 1$, then

$$p(x,t) = -2(lnf)_{xx} = -2\big(ln(1 + e^{kx-k^3 t})\big)_{xx}. \tag{2.58}$$

Remark 2.3.1. In order to get the 2-soliton solution, one can use the perturbed second order of f

$$f = 1 + \epsilon f_1(x,t) + \epsilon^2 f_2(x,t), \tag{2.59}$$

and for N-solitons

$$f = 1 + \sum_{i=0}^{N} \epsilon^i f_i(x,t). \tag{2.60}$$

2.3.3 Conclusion and remarks

Hirota's bilinear method has turned out to be very direct and efficient in deriving soliton solutions for integrable equations, but the existence of more than three-soliton solutions requires constraints. Like the Darboux transformation, it does not solve the Cauchy problem.

2.4 Riemann-Hilbert problem approach

2.4.1 Introduction

The Riemann-Hilbert problem appears in solving inverse scattering problems, nonlinear integrable systems and other types of integral equations [52],[9],[16],[10].

This approach consists of finding a matrix of functions in the complex plane that is analytic, except at a certain contour of the plane. At that contour, the matrix of functions jumps. There are many contours that we can define such as a circle, the real line, etc. in the complex plane. In the course of this dissertation, the contour will be the real line.

2.4.2 Definition of a Riemann-Hilbert problem

Let's consider the oriented contour ($\Sigma = \mathbb{R}$). The purpose is to find a function ψ that is analytic off the Σ, that means

$$\psi^{+}(z) = \lim_{z' \in \Sigma^{+}, z' \to z} \psi(z'), \quad \text{and} \quad \psi^{-}(z) = \lim_{z' \in \Sigma^{-}, z' \to z} \psi(z') \tag{2.61}$$

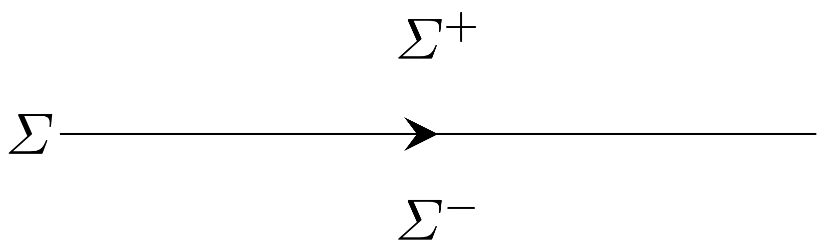

Figure 2.: Oriented contour in the complex plane λ-plane from $-\infty + 0i$ to $\infty + 0i$.

are analytic respectively in Σ^{+} and Σ^{-}, such that

$$\psi^{+}(t) = \psi^{-}(t)G(t), \quad \text{for} \quad t \in \mathbb{R}, \tag{2.62}$$

where G is the jump matrix function, and G is smooth, invertible and integrable.

The uniqueness of the solution$(\psi^{+}(z), \psi^{-}(z))$ of a RH problem requires the normalization

$$\psi(z) \to I, \text{ as } \quad z \to \infty, \quad z \in \mathbb{C} \setminus \Sigma. \tag{2.63}$$

The RP problem is solvable by the pair (Σ, G).

2.4.3 The Cauchy integral

The Riemann-Hilbert problem can be solved by the Cauchy operator defined as:

$$Cf : \Sigma \longrightarrow \mathbb{C} \setminus \Sigma \tag{2.64}$$

such that

$$(Cf)(z)) = \frac{1}{2\pi i} \int_{\Sigma} \frac{f(\tau)}{\tau - z} d\tau \tag{2.65}$$

where f has to be a smooth function.

Definition 2.4.1. (Schwartz space) The Schwartz space denoted $\mathcal{S}(\mathbb{R}^n)$ is the topological vector space of functions

$$f : \mathbb{R}^n \to \mathbb{C} \tag{2.66}$$

such that

$$f \in \mathbb{C}^{\infty}(\mathbb{R}^n), \quad and \quad x^{\alpha}\partial^{\beta} f(x) \longrightarrow 0 \quad \text{as} \quad |x| \longrightarrow \infty \tag{2.67}$$

for n, α and $\beta \in \mathbb{N}_0^n$. It is the space of all functions whose derivatives are rapidly decreasing.

Theorem 2.4.2. ***(Plemelj)*** *If $f(t) \in \mathcal{S}(\mathbb{R})$ along Σ, then the Cauchy operator*

$$(Cf)(z) = \frac{1}{2\pi i} \int_{\Sigma} \frac{f(\tau)}{\tau - z} d\tau \tag{2.68}$$

is analytic in $\mathbb{C} \setminus \Sigma$, but not on Σ, and

$$(Cf)^{+}(t) - (Cf)^{-}(t) = f(t) \tag{2.69}$$

and

$$(Cf)^{+}(t)) + (Cf)^{-}(t) = \frac{1}{\pi i} \int_{\Sigma} \frac{f(\tau)}{\tau - z} d\tau, \tag{2.70}$$

where $(Cf)^{+}(t)$ is the limit of $(Cf)^{+}(z)$ as $z \in \Sigma^{+} \longrightarrow t \in \mathbb{R}$ and $(Cf)^{-}(t)$ is the limit of $(Cf)^{-}(z)$ as $z \in \Sigma^{-} \longrightarrow t \in \mathbb{R}$.

Therefore we have

$$(Cf)^{+}(t) = \frac{1}{2\pi i} \int_{\Sigma} \frac{f(\tau)}{\tau - z} d\tau + \frac{1}{2} f(t), \tag{2.71}$$

$$(Cf)^{-}(t) = \frac{1}{2\pi i}\int_{\Sigma}\frac{f(\tau)}{\tau - z}d\tau - \frac{1}{2}f(t). \tag{2.72}$$

Corollary 2.4.3. *If $(Cf)(z) \in \mathcal{S}(\mathbb{R})$, and*

$$(Cf)^{+}(t) - (Cf)^{-}(t) = f(t) \quad as \quad t \in \mathbb{R}, \tag{2.73}$$

then

$$(Cf)(z) = \frac{1}{2\pi i}\int_{\Sigma}\frac{f(t)}{t - z}dt \tag{2.74}$$

is a particular solution of (2.73).

2.4.4 Solving the RHP problem

The factorization of the RHP problem is given by

$$\psi^{+}(t) = \psi^{-}(t)G(t), \quad for \quad t \in \mathbb{R}. \tag{2.75}$$

Subtracting both sides by $\psi^{-}(t)$ gives

$$\psi^{+}(t) - \psi^{-}(t) = \psi^{-}(t)(G(t) - I), \quad for \quad t \in \mathbb{R}. \tag{2.76}$$

Using Plemelj's theorem, we get

$$\psi^{+}(t) - I = (Cf)^{+}(t) \quad and \quad \psi^{-}(t) - I = (Cf)^{-}(t). \tag{2.77}$$

Therefore (2.76) becomes

$$(Cf)^{+}(t) - (Cf)^{-}(t) = (Cf)^{-}(t)(G(t) - I). \tag{2.78}$$

Thus, $(Cf)(z) \in \mathcal{S}(\mathbb{R})$, the solution of the RHP problem is given by

$$(Cf)(t) = \frac{1}{2\pi i}\int_{\Sigma}\frac{(Cf)^{-}(\tau)(G(\tau) - I))}{\tau - t}d(\tau). \tag{2.79}$$

Remark 2.4.4. The existence of the pair of solutions $(\psi^+(z), \psi^-(z))$ will depend on the jump function G and the contour Σ. In our theory, we assumed $f \in \mathcal{S}(\mathbb{R})$ in order to have a solution from the Plemelj's theorem. We wonder what will happen if G is in the L^2-space and in any other space .

2.4.5 Example of Riemann-Hilbert problems to inverse scattering

Let's consider the Cauchy problem for the KdV equation [10],[30]

$$\begin{cases} p_t(x,t) - 6p(x,t)p_x(x,t) + p_{xxx}(x,t) = 0, \\ p(x,0) = p_0(x). \end{cases} \tag{2.80}$$

The construction of the inverse scattering depends on the solution for solving this Sturm–Liouville equation

$$\psi_{xx} + (\lambda - p(x,0))\psi = 0. \tag{2.81}$$

In order to find a solution, the initial condition $p(x,0)$ has to belong to the Schwartz space ($p(x,0) \in \mathcal{S}(\mathbb{R})$). Thus, the boundary condition imposes

$$p(x,0) \to 0 \quad as \quad x \to \pm\infty. \tag{2.82}$$

After a tedious analysis, one derives

$$\phi^+(k) = \phi^-(k) \begin{pmatrix} 1 - |r(k)|^2 & -\overline{r}(k)e^{-i(2kx+8k^3t)} \\ r(k)e^{i(2kx+8k^3t)} & 1 \end{pmatrix}, \tag{2.83}$$

which is the RH problem in the k-plane.

$\phi^\pm$ are eigenfunctions and r is the reflection coefficient. The contour is the real line $\mathbb{R}$, and the jump function is the scattering matrix

$$G(k) = \begin{pmatrix} 1 - |r(k)|^2 & -\overline{r}(k)e^{-i(2kx+8k^3t)} \\ r(k)e^{i(2kx+8k^3t)} & 1 \end{pmatrix}, \quad as \quad k \in \mathbb{R}. \tag{2.84}$$

ϕ is analytic in $\mathbb{C} \setminus \mathbb{R}$, and the normalization condition is

$$\phi \to I \quad as \quad k \to \pm\infty. \tag{2.85}$$

As the direct scattering show a bijection from the initial potential to the scattering data, so the reflection coefficient $r(k) \in \mathcal{S}(\mathbb{R})$.

Now, if ϕ is the solution of the RH problem (2.83), and the expansion of ϕ is defined as

$$\phi(k; x, t) = I + \frac{\phi_1(x,t)}{k} + O(\frac{1}{k^2}) \quad as \quad k \to \infty, \tag{2.86}$$

then the potential solution of the KdV can be recovered as follows:

$$p(x,t) = 2i(\phi_1(x,t))_{12}. \tag{2.87}$$

2.4.6 Conclusion and remarks

The Riemann-Hilbert problem technique is very powerful in applications of solving integrable nonlinear evolution equations. It is present in the inverse scattering transform, and even in the Fokas method. In both entities, It solves the Cauchy problem with some constraints.

Chapter 3

Riemann-Hilbert problems for a nonlocal reverse-time AKNS system of fourth-order

3.1 Multi-component AKNS hierarchies

We begin with the following spatial spectral problem of $(n+1)$-order:

$$-i\psi_x = U(u,\lambda)\psi, \tag{3.1}$$

and

$$U = \begin{pmatrix} \alpha_1\lambda & p \\ r & \alpha_2\lambda I_n \end{pmatrix}, \tag{3.2}$$

where $u = (p, r^T)^T$, $p = (p_1, p_2, ..., p_n)$, $r = (r_1, r_2, ..., r_n)^T$,$\psi = (\psi_1, \psi_2, ..., \psi_n)^T$, α_1 and α_2 are real, λ is a spectral parameter, u is $2n$–dimensional potential, and I_n is $n \times n$ identity matrix [13]. Our purpose is to derive temporal Lax matrices, and the associated multi-component integrable systems.

Let's solve the stationary zero curvature equation

$$W_x = i[U, W], \tag{3.3}$$

corresponding to (3.2). If we take

$$W = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{3.4}$$

where a is a scalar, b^t and c are n-dimensional columns, d is an $n \times n$ matrix, and $\alpha = \alpha_1 - \alpha_2$, then using the zero curvature equation, we get

$$\begin{cases} a_x &= i(pc - br), \\ b_x &= i(\alpha\lambda b + pd - ap), \\ c_x &= i(-\alpha\lambda c + ra - dr), \\ d_x &= i(rb - cp). \end{cases} \tag{3.5}$$

By expanding W in Laurent series form

$$W = \sum_{j=0}^{\infty} W_{0,j}\lambda^{-j} \quad \text{with} \quad W_{0,j} = \begin{pmatrix} a^{[j]} & b^{[j]} \\ c^{[j]} & d^{[j]} \end{pmatrix}, \tag{3.6}$$

where $b^{[j]} = (b_1^{[j]}, b^{[j]}, \ldots, b_n^{[j]}), c^{[j]} = (c_1^{[j]}, c_2^{[j]}, \ldots, c_n^{[j]})^T, d^{[j]} = (d_{il}^{[j]})_{n\times n}$, and

$$a = \sum_{j=0}^{\infty} a^{[j]}\lambda^{-j}, \qquad b = \sum_{j=0}^{\infty} b^{[j]}\lambda^{-j}, \qquad c = \sum_{j=0}^{\infty} c^{[j]}\lambda^{-j}, \tag{3.7}$$

the system (3.5) will generate these recursion relations:

$$\begin{cases} b^{[0]} &= 0, \\ c^{[0]} &= 0, \\ a_x^{[0]} &= 0, \\ d_x^{[0]} &= 0, \\ b^{[j+1]} &= \frac{1}{\alpha}(-ib_x^{[j]} - pd^{[j]} + a^{[j]}p), \\ c^{[j+1]} &= \frac{1}{\alpha}(-ic_x^{[j]} + ra^{[j]} - d^{[j]}r), \\ a_x^{[j]} &= i(pc^{[j]} - b^{[j]}r), \\ d_x^{[j]} &= i(rb^{[j]} - c^{[j]}p), \quad for \quad j \geq 1. \end{cases} \tag{3.8}$$

Now, let

$$a^{[0]} = \beta_1, \quad \text{and} \quad d^{[0]} = \beta_2 I_n, \tag{3.9}$$

where β_1, β_2 are real values. If we take zero to be the constant of integration of $a_x^{[j]}$ and $d_x^{[j]}$, this requires

$$W_j|_{u=0} = 0, \quad j \geq 1, \tag{3.10}$$

then using (3.8) and (3.9), we will have the following results:

$$b_j^{[1]} = \frac{\beta}{\alpha} p_j, \quad c_j^{[1]} = \frac{\beta}{\alpha} r_j, \quad a^{[1]} = 0, \quad d_{jl}^{[1]} = 0; \tag{3.11}$$

$$b_j^{[2]} = -i\frac{\beta}{\alpha^2} p_{j,x}, \quad c_j^{[2]} = i\frac{\beta}{\alpha^2} r_{j,x}, \quad a^{[2]} = -\frac{\beta}{\alpha^2} pr, \quad d_{jl}^{[2]} = \frac{\beta}{\alpha^2} p_l r_j; \tag{3.12}$$

$$b_j^{[3]} = -\frac{\beta}{\alpha^3}(p_{j,xx} + 2prp_j), \quad c_j^{[3]} = -\frac{\beta}{\alpha^3}(q_{j,xx} + 2prr_j), \tag{3.13}$$

$$a^{[3]} = -i\frac{\beta}{\alpha^3}(pr_x - p_x r), \quad d_{jl}^{[3]} = -i\frac{\beta}{\alpha^3}(p_{l,x} r_j - p_l r_{j,x}); \tag{3.14}$$

$$b_j^{[4]} = i\frac{\beta}{\alpha^4}(p_{j,xxx} + 3prp_{j,x} + 3p_x r p_j), \tag{3.15}$$

$$c_j^{[4]} = -i\frac{\beta}{\alpha^4}(r_{j,xxx} + 3prr_{j,x} + 3pr_x r_j), \tag{3.16}$$

$$a^{[4]} = \frac{\beta}{\alpha^4}(3(pr)^2 + pr_{xx} - p_x r_x + p_{xx} r), \tag{3.17}$$

$$d_{jl}^{[4]} = -\frac{\beta}{\alpha^4}(3p_l prr_j + p_{l,xx} r_j - p_{l,x} r_{j,x} + p_l r_{j,xx}); \tag{3.18}$$

where $\beta = \beta_1 - \beta_2$, and $j \in \{1, 2, ...\},\ l \leq n$.

Hence, by taking the modification terms to be zero, the Lax temporal Lax matrices will be

$$V^{[m]} = (\lambda^m W)_{(n+1)\times(n+1)}, \tag{3.19}$$

and will satisfy the zero curvature equation

$$U_{t_m} - V_x^{[m]} + i[U, V^{[m]}] = 0. \tag{3.20}$$

As a result, we get the multi-component AKNS hierarchies of integrable equations:

$$u_{t_m} = \begin{pmatrix} p^T \\ r \end{pmatrix}_{t_m} = i \begin{pmatrix} \alpha b^{[m+1]T} \\ -\alpha c^{[m+1]} \end{pmatrix}, \quad for \quad m \in \{0, 1, 2, ...\}. \tag{3.21}$$

Remark 3.1.1. If m is odd, then the multi-component AKNS soliton hierarchy presents a modified KdV-type integrable equation, whereas if m is even, then the multi-component AKNS soliton presents an NLS-type integrable equation.

Obviously, when $n = 1$ and $m = 2$, we get the classical AKNS NLS equations given by the system of the coupled NLS equations

$$p_t(x,t) = -\frac{\beta}{\alpha^2} i(p_{xx}(x,t) + +2p(x,t)r(x,t)p(x,t)), \tag{3.22}$$

$$r_t(x,t) = \frac{\beta}{\alpha^2} i(r_{xx}(x,t) + 2r(x,t)p(x,t)r(x,t)). \tag{3.23}$$

Whereas, when $n = 1$ and $m = 3$, we get the classical AKNS mKdV equations given by the system of the coupled mKdV equations

$$p_t(x,t) = -\frac{\beta}{\alpha^2}(p_{xxx}(x,t) + 3p(x,t)r(x,t)p_x(x,t) + 3p_x(x,t)r(x,t)p(x,t)), \tag{3.24}$$

$$r_t(x,t) = -\frac{\beta}{\alpha^2}(r_{xxx}(x,t) + 3r(x,t)p(x,t)r_x(x,t) + 3r_x(x,t)p(x,t)r(x,t)). \tag{3.25}$$

3.2 Six-component AKNS hierarchy of coupled fourth-order integrable equations

Let us consider the pair of spatial and temporal spectral problems for the six-component AKNS system:

$$\psi_x = iU\psi, \tag{3.26}$$

$$\psi_t = iV^{[4]}\psi, \tag{3.27}$$

where ψ is the eigenfunction. The spectral matrix is given by

$$U(u,\lambda) = \begin{pmatrix} \alpha_1\lambda & p_1 & p_2 & p_3 \\ r_1 & \alpha_2\lambda & 0 & 0 \\ r_2 & 0 & \alpha_2\lambda & 0 \\ r_3 & 0 & 0 & \alpha_2\lambda \end{pmatrix}, \tag{3.28}$$

where λ is a spectral parameter, α_1, α_2 are real constants, $p = (p_1, p_2, p_3)$ and $r = (r_1, r_2, r_3)^T$ are vector functions of (x,t), and $u = (p, r^T)^T$ is a vector of six potentials.

The Lax matrix operator $V^{[4]}$ is given by

$$V^{[4]} = \begin{pmatrix} \begin{matrix} a^{[0]}\lambda^4 + a^{[2]}\lambda^2 \\ +a^{[3]}\lambda + a^{[4]} \end{matrix} & \begin{matrix} b_1^{[1]}\lambda^3 + b_1^{[2]}\lambda^2 \\ +b_1^{[3]}\lambda + b_1^{[4]} \end{matrix} & \begin{matrix} b_2^{[1]}\lambda^3 + b_2^{[2]}\lambda^2 \\ +b_2^{[3]}\lambda + b_2^{[4]} \end{matrix} & \begin{matrix} b_3^{[1]}\lambda^3 + b_3^{[2]}\lambda^2 \\ +b_3^{[3]}\lambda + b_3^{[4]} \end{matrix} \\ \\ \begin{matrix} c_1^{[1]}\lambda^3 + c_1^{[2]}\lambda^2 \\ +c_1^{[3]}\lambda + c_1^{[4]} \end{matrix} & \begin{matrix} d_{11}^{[0]}\lambda^4 + d_{11}^{[2]}\lambda^2 \\ +d_{11}^{[3]}\lambda + d_{11}^{[4]} \end{matrix} & \begin{matrix} d_{12}^{[2]}\lambda^2 + d_{12}^{[3]}\lambda \\ +d_{12}^{[4]} \end{matrix} & \begin{matrix} d_{13}^{[2]}\lambda^2 + d_{13}^{[3]}\lambda \\ +d_{13}^{[4]} \end{matrix} \\ \\ \begin{matrix} c_2^{[1]}\lambda^3 + c_2^{[2]}\lambda^2 \\ +c_2^{[3]}\lambda + c_2^{[4]} \end{matrix} & \begin{matrix} d_{21}^{[2]}\lambda^2 + d_{21}^{[3]}\lambda \\ +d_{21}^{[4]} \end{matrix} & \begin{matrix} d_{22}^{[0]}\lambda^4 + d_{22}^{[2]}\lambda^2 \\ +d_{22}^{[3]}\lambda + d_{22}^{[4]} \end{matrix} & \begin{matrix} d_{23}^{[2]}\lambda^2 + d_{23}^{[3]}\lambda \\ +d_{23}^{[4]} \end{matrix} \\ \\ \begin{matrix} c_3^{[1]}\lambda^3 + c_3^{[2]}\lambda^2 \\ +c_3^{[3]}\lambda + c_3^{[4]} \end{matrix} & \begin{matrix} d_{31}^{[2]}\lambda^2 + d_{31}^{[3]}\lambda \\ +d_{31}^{[4]} \end{matrix} & \begin{matrix} d_{32}^{[2]}\lambda^2 + d_{32}^{[3]}\lambda \\ +d_{32}^{[4]} \end{matrix} & \begin{matrix} d_{33}^{[0]}\lambda^4 + d_{33}^{[2]}\lambda^2 \\ +d_{33}^{[3]}\lambda + d_{33}^{[4]} \end{matrix} \end{pmatrix}, \tag{3.29}$$

where all the involved functions are defined as follows:

$$\begin{cases} a^{[0]} &= \beta_1, \\ a^{[1]} &= 0, \\ a^{[2]} &= -\frac{\beta}{\alpha^2}\sum_{i=1}^{3} p_i r_i, \\ a^{[3]} &= -i\frac{\beta}{\alpha^3}\sum_{i=1}^{3}(p_i r_{i,x} - p_{i,x} r_i), \\ a^{[4]} &= \frac{\beta}{\alpha^4}\left[3(\sum_{i=1}^{3} p_i r_i)^2 + \sum_{i=1}^{3}(p_i r_{i,xx} - p_{i,x} r_{i,x} + p_{i,xx} r_i)\right], \\ a^{[5]} &= i\frac{\beta}{\alpha^5}\left[6(\sum_{i=1}^{3} p_i r_i) - \sum_{i=1}^{3}(p_i r_{i,x} - p_{i,x} r_i) + \sum_{i=1}^{3}(p_i r_{i,xxx} - p_{i,xxx} r_i + p_{i,xx} r_{i,x} - p_{i,x} r_{i,xx})\right], \end{cases}$$

$$
\begin{cases}
b_k^{[0]} &= 0, \\
b_k^{[1]} &= \frac{\beta}{\alpha} r_k, \\
b_k^{[2]} &= -i\frac{\beta}{\alpha^2} p_{k,x}, \\
b_k^{[3]} &= -\frac{\beta}{\alpha^3}\left[p_{k,xx} + 2(\sum\limits_{i=1}^{3} p_i r_i) p_k\right], \\
b_k^{[4]} &= i\frac{\beta}{\alpha^4}\left[p_{k,xxx} + 3(\sum\limits_{i=1}^{3} p_i r_i) p_{k,x} + 3(\sum\limits_{i=1}^{3} p_{i,x} r_i) p_k\right], \\
b_k^{[5]} &= \frac{\beta}{\alpha^5}\left[p_{k,xxxx} + 4(\sum\limits_{i=1}^{3} p_i r_i) p_{k,xx} + (6\sum\limits_{i=1}^{3} p_{i,x} r_i + 2\sum\limits_{i=1}^{3} p_i r_{i,x}) p_{k,x}\right. \\
&\left. +(4\sum\limits_{i=1}^{3} p_{i,xx} r_i + 2\sum\limits_{i=1}^{3} p_{i,x} r_{i,x} + 2\sum\limits_{i=1}^{3} p_i r_{i,xx} + 6(\sum\limits_{i=1}^{3} p_i r_i)^2) p_k\right],
\end{cases}
$$

$$
\begin{cases}
c_k^{[0]} &= 0, \\
c_k^{[1]} &= \frac{\beta}{\alpha} p_k, \\
c_k^{[2]} &= i\frac{\beta}{\alpha^2} r_{k,x}, \\
c_k^{[3]} &= -\frac{\beta}{\alpha^3}\left[r_{k,xx} + 2(\sum\limits_{i=1}^{3} p_i r_i) r_k\right], \\
c_k^{[4]} &= -i\frac{\beta}{\alpha^4}\left[r_{k,xxx} + 3(\sum\limits_{i=1}^{3} p_i r_i) r_{k,x} + 3(\sum\limits_{i=1}^{3} p_{i,x} r_i) r_k\right], \\
c_k^{[5]} &= \frac{\beta}{\alpha^5}\left[r_{k,xxxx} + 4(\sum\limits_{i=1}^{3} p_i r_i) r_{k,xx} + (6\sum\limits_{i=1}^{3} p_i r_{i,x} + 2\sum\limits_{i=1}^{3} p_{i,x} r_i) r_{k,x}\right. \\
&\left. +(4\sum\limits_{i=1}^{3} p_i r_{i,xx} + 2\sum\limits_{i=1}^{3} p_{i,x} r_{i,x} + 2\sum\limits_{i=1}^{3} p_{i,xx} r_i + 6(\sum\limits_{i=1}^{3} p_i r_i)^2) r_k\right],
\end{cases}
$$

and

$$
\begin{cases}
d_{kj}^{[0]} & = \beta_2 I_3, \\
d_{kj}^{[1]} & = 0, \\
d_{kj}^{[2]} & = \frac{\beta}{\alpha^2} p_j r_k, \\
d_{kj}^{[3]} & = -i\frac{\beta}{\alpha^3}(p_{j,x} r_k - p_j r_{k,x}), \\
d_{kj}^{[4]} & = -\frac{\beta}{\alpha^4}\left[3p_j(\sum\limits_{i=1}^{3} p_i r_i) r_k + p_{j,xx} r_k - p_{j,x} r_{k,x} + p_j r_{k,xx}\right], \\
d_{kj}^{[5]} & = \frac{\beta}{\alpha^5}\left[2p_j(\sum\limits_{i=1}^{3} p_{i,x} r_i - p_i r_{i,x}) r_k + 4p_{j,x}(\sum\limits_{i=1}^{3} p_i r_i) r_k - p_j(\sum\limits_{i=1}^{3} p_i r_i) r_{k,x}\right. \\
& \left. + p_{j,xxx} r_k - p_j r_{k,xxx} + p_{j,x} r_{k,xx} - p_{j,xx} r_{k,x}\right].
\end{cases}
$$

We always assume that $a^{[i]}$ are scalars, $b^{[i]} = (b_1^{[i]}, b_2^{[i]}, b_3^{[i]})$, $c^{[i]} = (c_1^{[i]}, c_2^{[i]}, c_3^{[i]})^T$, and $d^{[i]} = (d_{kl}^{[i]})_{3\times 3}$, for $i \in \{1, 2, 3, 4, 5\}$.

The compatibility condition $\psi_{xt} = \psi_{tx}$ will lead to the zero curvature equation:

$$
U_t - V_x^{[4]} + i[U, V^{[4]}] = 0, \tag{3.30}
$$

which gives the six-component system of soliton equations

$$
u_t = \begin{pmatrix} p^T \\ r \end{pmatrix}_t = i \begin{pmatrix} \alpha b^{[5]T} \\ -\alpha c^{[5]} \end{pmatrix}, \tag{3.31}
$$

where $b^{[5]}$ and $c^{[5]}$ are defined earlier. Thus, we deduce the coupled AKNS system of fourth-order equations:

$$
\begin{cases}
p_{k,t} & = i\frac{\beta}{\alpha^4}[p_{k,xxxx} + 4(\sum\limits_{i=1}^{3} p_i r_i) p_{k,xx} + (6\sum\limits_{i=1}^{3} p_{i,x} r_i + 2\sum\limits_{i=1}^{3} p_i r_{i,x}) p_{k,x} \\
& + (4\sum\limits_{i=1}^{3} p_{i,xx} r_i + 2\sum\limits_{i=1}^{3} p_{i,x} r_{i,x} + 2\sum\limits_{i=1}^{3} p_i r_{i,xx} + 6(\sum\limits_{i=1}^{3} p_i r_i)^2) p_k], \\
r_{k,t} & = -i\frac{\beta}{\alpha^4}[r_{k,xxxx} + 4(\sum\limits_{i=1}^{3} p_i r_i) r_{k,xx} + (6\sum\limits_{i=1}^{3} p_i r_{i,x} + 2\sum\limits_{i=1}^{3} p_{i,x} r_i) r_{k,x} \\
& + (4\sum\limits_{i=1}^{3} p_i r_{i,xx} + 2\sum\limits_{i=1}^{3} p_{i,x} r_{i,x} + 2\sum\limits_{i=1}^{3} p_{i,xx} r_i + 6(\sum\limits_{i=1}^{3} p_i r_i)^2) r_k],
\end{cases} \tag{3.32}
$$

where $k \in \{1, 2, 3\}$.

3.3 Nonlocal reverse-time six-component AKNS system

Let us consider a class of specific nonlocal reverse-time reductions for the spectral matrix

$$U^T(x,-t,-\lambda) = -CU(x,t,\lambda)C^{-1}, \tag{3.33}$$

where $C = \begin{pmatrix} 1 & 0 \\ 0 & \Sigma \end{pmatrix}$ and Σ is a constant invertible symmetric 3×3 matrix, i.e., $\Sigma^T = \Sigma$ and $\det \Sigma \neq 0$.

As $U(x,t,\lambda) = \lambda \Lambda + P(x,t)$, for $P = \begin{pmatrix} 0 & p \\ r & 0 \end{pmatrix}$ and $\Lambda = \mathit{diag}(\alpha_1, \alpha_2 I_3)$, then we have

$$P^T(x,-t) = -CP(x,t)C^{-1}. \tag{3.34}$$

From the above (3.34), we get

$$p^T(x,-t) = -\Sigma r(x,t), \quad \text{i.e.,} \quad r(x,t) = -\Sigma^{-1}p^T(x,-t). \tag{3.35}$$

As $V^{[4]}(x,t,\lambda) = \lambda^4\Omega + Q(x,t,\lambda)$ and from (3.34), we prove that

$${V^{[4]}}^T(x,-t,-\lambda) = CV^{[4]}(x,t,\lambda)C^{-1} \quad \text{and} \quad Q^T(x,-t,-\lambda) = CQ(x,t,\lambda)C^{-1}, \tag{3.36}$$

where $\Omega = \mathit{diag}(\beta_1, \beta_2 I_3)$.

Importantly, the two Lax pair matrices $U^T(x,-t,-\lambda)$ and ${V^{[4]}}^T(x,-t,-\lambda)$ satisfy an equivalent zero curvature equation.

Proof: The zero curvature equation:

$$U_t - V_x^{[m]} + i[U, V^{[m]}] = 0 \tag{3.37}$$

so

$$C(U_t(x,t,\lambda) - V_x^{[m]}(x,t,\lambda) + i[U(x.t,\lambda), V^{[m]}(x,t,\lambda)])C^{-1} = 0, \tag{3.38}$$

$$C(U_t(x.t,\lambda))C^{-1} - C(V_x^{[m]}(x.t,\lambda))C^{-1} + iC[U(x.t,\lambda), V^{[m]}(x.t,\lambda)]C^{-1} = 0, \tag{3.39}$$

$$C(U_t(x.t,\lambda))C^{-1} - C(V_x^{[m]}(x.t,\lambda))C^{-1} \tag{3.40}$$

$$+ i\big(CU(x.t,\lambda)V^{[m]}(x.t,\lambda)C^{-1} - CV^{[m]}(x.t,\lambda).U(x.t,\lambda).C^{-1}\big) = 0, \tag{3.41}$$

$$(CU(x.t,\lambda)C^{-1})_t - (CV^{[m]}(x.t,\lambda)C^{-1})_x \tag{3.42}$$

$$+ i\big(CU(x.t,\lambda)C^{-1}CV^{[m]}(x.t,\lambda)C^{-1} - CV^{[m]}(x.t,\lambda)C^{-1}CU(x.t,\lambda)C^{-1}\big) = 0,$$

$$U^T(x,-t,-\lambda) - {V^{[4]}}^T(x,-t,-\lambda) \tag{3.43}$$

$$+ i(U^T(x,-t,-\lambda){V^{[4]}}^T(x,-t,-\lambda) - U^T(x,-t,-\lambda){V^{[4]}}^T(x,-t,-\lambda)) = 0, \tag{3.44}$$

$$U^T(x,-t,-\lambda) - {V^{[4]}}^T(x,-t,-\lambda) + i[U^T(x,-t,-\lambda){V^{[4]}}^T(x,-t,-\lambda)] = 0, \tag{3.45}$$

which is the zero curvature equation of the Lax pair $U^T(x,-t,-\lambda)$ and ${V^{[4]}}^T(x,-t,-\lambda)$.

From this specific nonlocal reduction, the coupled six-component fourth-order AKNS equations can be reduced to the nonlocal reverse-time six-component fourth-order equations.

As Σ is invertible and symmetric so diagonalizable, then we can take $\Sigma = diag(\rho_1^{-1}, \rho_2^{-1}, \rho_3^{-1})$, for ρ_1, ρ_2, ρ_3 non-zero real. Thus $\Sigma^{-1} = diag(\rho_1, \rho_2, \rho_3)$ leads (3.35) to

$$r_i(x,t) = -\rho_i p_i(x,-t) \quad \text{for} \quad i \in \{1,2,3\}. \tag{3.46}$$

Therefore the coupled equations (3.32) reduce to the nonlocal reverse-time fourth-order equation

$$
\begin{aligned}
p_{k,t}(x,t) = i\frac{\beta}{\alpha^4}\Bigg[& p_{k,xxxx}(x,t) - 4\Big(\sum_{i=1}^{3}\rho_i p_i(x,t)p_i(x,-t)\Big)p_{k,xx}(x,t) \\
& - \Big(6\sum_{i=1}^{3}\rho_i p_{i,x}(x,t)p_i(x,-t) + 2\sum_{i=1}^{3}\rho_i p_i(x,t)p_{i,x}(x,-t)\Big)p_{k,x}(x,t) \\
& - \Big(4\sum_{i=1}^{3}\rho_i p_{i,xx}(x,t)p_i(x,-t) + 2\sum_{i=1}^{3}\rho_i p_{i,x}(x,t)p_{i,x}(x,-t) \\
& + 2\sum_{i=1}^{3}\rho_i p_i(x,t)p_{i,xx}(x,-t) - 6\big(\sum_{i=1}^{3}\rho_i p_i(x,t)p_i(x,-t)\big)^2\Big)p_k(x,t)\Bigg]
\end{aligned} \tag{3.47}
$$

for $k \in \{1,2,3\}$.

We should notice that if Σ is negative definite, i.e., each $\rho_i < 0$ for $i \in \{1,2,3\}$, then we obtain the focusing nonlocal reverse-time six-component fourth-order equation due to the fact that the dispersive term and nonlinear terms attract [40]. If ρ_i's are not all the same sign for $i \in \{1,2,3\}$, we obtain combined focussing and defocussing cases.

3.4 Riemann-Hilbert formulation

The Lax pair of the six-component fourth-order AKNS equations can be written:

$$\psi_x = iU\psi = i(\lambda\Lambda + P)\psi, \tag{3.48}$$

$$\psi_t = iV^{[4]}\psi = i(\lambda^4\Omega + Q)\psi, \tag{3.49}$$

where $\Omega = diag(\beta_1, \beta_2, \beta_2, \beta_2)$, $\Lambda = diag(\alpha_1, \alpha_2, \alpha_2, \alpha_2)$, and

$$P = \begin{pmatrix} 0 & p_1 & p_2 & p_3 \\ r_1 & 0 & 0 & 0 \\ r_2 & 0 & 0 & 0 \\ r_3 & 0 & 0 & 0 \end{pmatrix}, \tag{3.50}$$

$$Q = \begin{pmatrix} a^{[2]}\lambda^2 + a^{[3]}\lambda & b_1^{[1]}\lambda^3 + b_1^{[2]}\lambda^2 & b_2^{[1]}\lambda^3 + b_2^{[2]}\lambda^2 & b_3^{[1]}\lambda^3 + b_3^{[2]}\lambda^2 \\ +a^{[4]} & +b_1^{[3]}\lambda + b_1^{[4]} & +b_2^{[3]}\lambda + b_2^{[4]} & +b_3^{[3]}\lambda + b_3^{[4]} \\ \\ c_1^{[1]}\lambda^3 + c_1^{[2]}\lambda^2 & d_{11}^{[2]}\lambda^2 + d_{11}^{[3]}\lambda & d_{12}^{[2]}\lambda^2 + d_{12}^{[3]}\lambda & d_{13}^{[2]}\lambda^2 + d_{13}^{[3]}\lambda \\ +c_1^{[3]}\lambda + c_1^{[4]} & +d_{11}^{[4]} & +d_{12}^{[4]} & +d_{13}^{[4]} \\ \\ c_2^{[1]}\lambda^3 + c_2^{[2]}\lambda^2 & d_{21}^{[2]}\lambda^2 + d_{21}^{[3]}\lambda & d_{22}^{[2]}\lambda^2 + d_{22}^{[3]}\lambda & d_{23}^{[2]}\lambda^2 + d_{23}^{[3]}\lambda \\ +c_2^{[3]}\lambda + c_2^{[4]} & +d_{21}^{[4]} & +d_{22}^{[4]} & +d_{23}^{[4]} \\ \\ c_3^{[1]}\lambda^3 + c_3^{[2]}\lambda^2 & d_{31}^{[2]}\lambda^2 + d_{31}^{[3]}\lambda & d_{32}^{[2]}\lambda^2 + d_{32}^{[3]}\lambda & d_{33}^{[2]}\lambda^2 + d_{33}^{[3]}\lambda \\ +c_3^{[3]}\lambda + c_3^{[4]} & +d_{31}^{[4]} & +d_{32}^{[4]} & +d_{33}^{[4]} \end{pmatrix}. \tag{3.51}$$

Our purpose is to find soliton solutions from an initial condition $(p(x,0), r^T(x,0))^T$ to $(p(x,t), r^T(x,t))^T$ at any time t. We assume that any p_i and r_i decay exponentially, i.e., $p_i \to 0$ and $r_i \to 0$ as $x,t \to \pm\infty$ for $i \in \{1,2,3\}$. Therefore from the spectral problems (3.48) and (3.49), ψ will behave asymptotically $\psi(x,t) \rightsquigarrow e^{i\lambda\Lambda x + i\lambda^4 \Omega t}$. We can then expect the solution for the spectral problems to be:

$$\psi(x,t) = \phi(x,t)e^{i\lambda\Lambda x + i\lambda^4\Omega t}. \tag{3.52}$$

For the Jost solution [9, 19], we require that

$$\phi(x,t) \to I_4, \quad \text{as} \quad x,t \to \pm\infty, \tag{3.53}$$

where I_4 is the 4×4 identity matrix. Substituting (3.52) into the Lax pair, (3.48) and (3.49), will result in the equivalent expression of the spectral problems

$$\phi_x = i\lambda[\Lambda, \phi] + iP\phi, \tag{3.54}$$

$$\phi_t = i\lambda^4[\Omega, \phi] + iQ\phi. \tag{3.55}$$

Now, we are going to work with the spatial spectral problem (3.54), assuming that the time is $t = 0$ for the direct scattering process.

Theorem 3.4.1. ***(Liouville)*** *Let $A(x)$ be a square matrix of dimension n with complex or real entries, and Y is a matrix-valued solution on an interval I. If*

$$Y_x = A(x)Y, \tag{3.56}$$

then

$$(\det Y)_x = trA \cdot \det(Y). \tag{3.57}$$

Therefore by Liouville's formula, as $tr(iP) = 0$ and $tr(iQ) = 0$, so $det(\phi)$ is a constant, and using the boundary condition (3.53), we get

$$det(\phi) = 1. \tag{3.58}$$

To construct Riemann-Hilbert problems and their solutions in the reflectionless case, we are going to use the adjoint scattering equations of the spectral problems $\psi_x = iU\psi$ and $\psi_t = iV^{[4]}\psi$. Their adjoints are

$$\tilde{\psi}_x = -i\tilde{\psi}U, \tag{3.59}$$

$$\tilde{\psi}_t = -i\tilde{\psi}V^{[4]}, \tag{3.60}$$

and the equivalent spectral adjoint equations read

$$\tilde{\phi}_x = -i\lambda[\tilde{\phi}, \Lambda] - i\tilde{\phi}P, \tag{3.61}$$

$$\tilde{\phi}_t = -i\lambda^4[\tilde{\phi}, \Omega] - i\tilde{\phi}Q. \tag{3.62}$$

As $\phi_x^{-1} = -\phi^{-1}\phi_x\phi^{-1}$, we have from (3.54),

$$\phi_x^{-1} = -i\lambda[\phi^{-1}, \Lambda] - i\phi^{-1}P. \tag{3.63}$$

Therefore, we deduce that $(\phi^{\pm})^{-1}$ satisfies the adjoint equation (3.61). Similarly, we can show that $(\phi^{\pm})^{-1}$ satisfies (3.62) as well.

Now, if the eigenfunction $\phi(x,t,\lambda)$ is a solution of the spectral problem (3.54), then $C\phi^{-1}(x,t,\lambda)$ is a solution of the spectral adjoint problem (3.61) with the same eigenvalue because $\phi_x^{-1} = -\phi^{-1}\phi_x\phi^{-1}$. Also $\phi^T(x,-t,-\lambda)C$ is a solution of the spectral adjoint problem (3.61). As both solutions have the same boundary condition as $x \to \pm\infty$ which guarantees the uniqueness of the solution, so

$$\phi^T(x,-t,-\lambda)C = C\phi^{-1}(x,t,\lambda) \quad \text{or} \quad \phi^T(x,-t,-\lambda) = C\phi^{-1}(x,t,\lambda)C^{-1}. \tag{3.64}$$

This tells us that if λ is an eigenvalue of the spectral problems, then $-\lambda$ is also an eigenvalue.

For the rest of the problem, we assume that $\alpha < 0$ and $\beta < 0$ and $Y^{\pm}$ tell us at which end of the x-axis the boundary conditions are set. We know that

$$\phi^{\pm} \to I_4 \quad \text{when} \quad x \to \pm\infty. \tag{3.65}$$

We can then write

$$\psi^{\pm} = \phi^{\pm} e^{i\lambda\Lambda x}. \tag{3.66}$$

As ψ^+ and ψ^- are two solutions of the spectral spatial differential equation of first-order (3.48), they are then linearly dependent, and so they are related by a scattering matrix $S(\lambda)$. As a result,

$$\psi^- = \psi^+ S(\lambda), \tag{3.67}$$

using (3.66), we have

$$\phi^- = \phi^+ e^{i\lambda\Lambda x} S(\lambda) e^{-i\lambda\Lambda x}, \quad \text{for} \quad \lambda \in \mathbb{R}, \tag{3.68}$$

where

$$S(\lambda) = (s_{ij})_{4\times4} = \begin{pmatrix} s_{11} & s_{12} & s_{13} & s_{14} \\ s_{21} & s_{22} & s_{23} & s_{24} \\ s_{31} & s_{32} & s_{33} & s_{34} \\ s_{41} & s_{42} & s_{43} & s_{44} \end{pmatrix}. \tag{3.69}$$

Because $det(\phi^{\pm}) = 1$, one has

$$det(S(\lambda)) = 1. \tag{3.70}$$

From (3.64) and (3.68), we have this involution relation

$$S^T(-\lambda) = CS^{-1}(\lambda)C^{-1}. \tag{3.71}$$

Proof: As

$$\phi^-(x,t,\lambda) = \phi^+(x,t,\lambda)e^{i\lambda\Lambda x}S(\lambda)e^{-i\lambda\Lambda x}, \quad \lambda \in \mathbb{R}, \tag{3.72}$$

$$\phi^-(x,-t,-\lambda) = \phi^+(x,-t,-\lambda)e^{-i\lambda\Lambda x}S(-\lambda)e^{i\lambda\Lambda x}, \tag{3.73}$$

$$(\phi^-(x,-t,-\lambda))^T = e^{i\lambda\Lambda x}S^T(-\lambda)e^{-i\lambda\Lambda x}(\phi^+(x,-t,-\lambda))^T, \tag{3.74}$$

also (3.72) gives

$$(\phi^-(x,t,\lambda))^{-1} = e^{i\lambda\Lambda x}S^{-1}(\lambda)e^{-i\lambda\Lambda x}(\phi^+(x,t,\lambda))^{-1}, \tag{3.75}$$

$$C(\phi^-(x,t,\lambda))^{-1}C^{-1} = Ce^{i\lambda\Lambda x}S^{-1}(\lambda)e^{-i\lambda\Lambda x}(\phi^+(x,t,\lambda))^{-1}C^{-1}, \tag{3.76}$$

$$C(\phi^-(x,t,\lambda))^{-1}C^{-1} = e^{i\lambda\Lambda x}CS^{-1}(\lambda)C^{-1}Ce^{-i\lambda\Lambda x}(\phi^+(x,t,\lambda))^{-1}C^{-1}, \tag{3.77}$$

$$C(\phi^-(x,t,\lambda))^{-1}C^{-1} = e^{i\lambda\Lambda x}CS^{-1}(\lambda)C^{-1}e^{-i\lambda\Lambda x}C(\phi^+(x,t,\lambda))^{-1}C^{-1}. \tag{3.78}$$

From (3.64), we have $(\phi^-(x,-t,-\lambda))^T = C(\phi^-(x,t,\lambda))^{-1}C^{-1}$ and also $(\phi^+(x,-t,-\lambda))^T = (\phi^+(x,t,\lambda))^{-1}C^{-1}$. As a result, if (3.74) and (3.78) match, we deduce then

$$S^T(-\lambda) = CS^{-1}(\lambda)C^{-1}. \tag{3.79}$$

From (3.79), we deduce that

$$\hat{s}_{11}(\lambda) = s_{11}(-\lambda), \tag{3.80}$$

where the inverse scattering data matrix $S^{-1} = (\hat{s}_{ij})_{4\times4}$ for $i,j \in \{1,2,3,4\}$.

We can see that the recovery of the potentials will depend on the information of the scattering data from the scattering matrix $S(\lambda)$. As $\phi^{\pm} \to I_4$ when $x \to \pm\infty$, we need to analyse the analyticity of the Jost matrix $\phi^{\pm}$ in order to formulate the Riemann-Hilbert problems.

One can write the solution $\phi^{\pm}$ in a uniquely manner by the Volterra integral equations using (3.48):

$$\phi^{-}(x,\lambda) = I_4 + i\int\limits_{-\infty}^{x} e^{i\lambda\Lambda(x-y)}P(y)\phi^{-}(y,\lambda)e^{i\lambda\Lambda(y-x)}dy, \tag{3.81}$$

$$\phi^{+}(x,\lambda) = I_4 - i\int\limits_{x}^{+\infty} e^{i\lambda\Lambda(x-y)}P(y)\phi^{+}(y,\lambda)e^{i\lambda\Lambda(y-x)}dy. \tag{3.82}$$

Proof: We have from (3.48)

$$\psi_x = iU\psi = i(\lambda\Lambda + P)\psi, \tag{3.83}$$

The homogeneous solutions are given by

$$\psi(x,\lambda) = e^{i\lambda\Lambda x}C, \quad C \quad \text{is a constant matrix.} \tag{3.84}$$

For the general general solution, we could take

$$\psi(x,\lambda) = e^{i\lambda\Lambda x}C(x). \tag{3.85}$$

Differentiating both sides of (3.85), and substituting into (3.83), we get

$$C^{'}(x) = e^{-i\lambda\Lambda x}(iP)e^{i\lambda\Lambda x}C(x), \tag{3.86}$$

$$\int_{-\infty}^{x} C^{'}(y)dy = \int_{-\infty}^{x} e^{-i\lambda\Lambda y}(iP)e^{i\lambda\Lambda y}C(y)dy, \tag{3.87}$$

$$C(x) = I + \int_{-\infty}^{x} e^{-i\lambda\Lambda y}(iP)e^{i\lambda\Lambda y}C(y)dy, \tag{3.88}$$

$$\int_{x}^{\infty} C^{'}(y)dy = \int_{-\infty}^{x} e^{-i\lambda\Lambda y}(iP)e^{i\lambda\Lambda y}C(y)dy, \tag{3.89}$$

$$C(x) = I - \int_{-\infty}^{x} e^{-i\lambda\Lambda y}(iP)e^{i\lambda\Lambda y}C(y)dy. \tag{3.90}$$

From (3.66) and (3.85), we have

$$\psi^{\pm}(x,\lambda) = \phi^{\pm}(x,\lambda)e^{i\lambda\Lambda x} = e^{i\lambda\Lambda x}C^{\pm}(x). \tag{3.91}$$

Thus

$$C^{\pm}(x) = e^{-i\lambda\Lambda x}\phi^{\pm}(x,\lambda)e^{i\lambda\Lambda x}, \tag{3.92}$$

and so (3.88) becomes

$$e^{-i\lambda\Lambda x}\phi^{-}(x,\lambda)e^{i\lambda\Lambda x} = I + \int_{-\infty}^{x} e^{-i\lambda\Lambda y}(iP)e^{i\lambda\Lambda y}e^{-i\lambda\Lambda y}\phi^{-}(x,\lambda)e^{i\lambda\Lambda y}dy. \tag{3.93}$$

As a result, we obtain the Volterra integrable equations

$$\phi^{-}(x,\lambda) = I + \int_{-\infty}^{x} e^{-i\lambda\Lambda(x-y)}(iP)\phi^{-}(x,\lambda)e^{i\lambda\Lambda(y-x)}dy, \tag{3.94}$$

$$\phi^{+}(x,\lambda) = I - \int_{x}^{\infty} e^{-i\lambda\Lambda(x-y)}(iP)\phi^{+}(x,\lambda)e^{i\lambda\Lambda(y-x)}dy. \tag{3.95}$$

If the matrix ϕ^{-} is

$$\phi^{-} = \begin{pmatrix} \phi^{-}_{11} & \phi^{-}_{12} & \phi^{-}_{13} & \phi^{-}_{14} \\ \phi^{-}_{21} & \phi^{-}_{22} & \phi^{-}_{23} & \phi^{-}_{24} \\ \phi^{-}_{31} & \phi^{-}_{32} & \phi^{-}_{33} & \phi^{-}_{34} \\ \phi^{-}_{41} & \phi^{-}_{42} & \phi^{-}_{43} & \phi^{-}_{44} \end{pmatrix}, \tag{3.96}$$

then the components of the first column of ϕ^{-} are

$$\phi^{-}_{11} = 1 + i\int_{-\infty}^{x} (p_1(y)\phi^{-}_{21}(y,\lambda) + p_2(y)\phi^{-}_{31}(y,\lambda) + p_3(y)\phi^{-}_{41}(y,\lambda))dy, \tag{3.97}$$

$$\phi^{-}_{21} = i\int_{-\infty}^{x} r_1(y)\phi^{-}_{11}(y,\lambda)e^{-i\lambda\alpha(x-y)}dy, \tag{3.98}$$

$$\phi^{-}_{31} = i\int_{-\infty}^{x} r_2(y)\phi^{-}_{11}(y,\lambda)e^{-i\lambda\alpha(x-y)}dy, \tag{3.99}$$

$$\phi^{-}_{41} = i\int_{-\infty}^{x} r_3(y)\phi^{-}_{11}(y,\lambda)e^{-i\lambda\alpha(x-y)}dy. \tag{3.100}$$

Similarly, the components of the second column of ϕ^- are

$$\phi_{12}^- = i\int_{-\infty}^{x}\Big(p_1(y)\phi_{22}^-(y,\lambda)+p_2(y)\phi_{32}^-(y,\lambda)+p_3(y)\phi_{42}^-(y,\lambda)\Big)e^{i\lambda\alpha(x-y)}dy, \tag{3.101}$$

$$\phi_{22}^- = 1+i\int_{-\infty}^{x} r_1(y)\phi_{12}^-(y,\lambda)dy, \tag{3.102}$$

$$\phi_{32}^- = i\int_{-\infty}^{x} r_2(y)\phi_{12}^-(y,\lambda)dy, \tag{3.103}$$

$$\phi_{42}^- = i\int_{-\infty}^{x} r_3(y)\phi_{12}^-(y,\lambda)dy, \tag{3.104}$$

and the components of the third column of ϕ^- are

$$\phi_{13}^- = i\int_{-\infty}^{x}\Big(p_1(y)\phi_{23}^-(y,\lambda)+p_2(y)\phi_{33}^-(y,\lambda)+p_3(y)\phi_{43}^-(y,\lambda)\Big)e^{i\lambda\alpha(x-y)}dy, \tag{3.105}$$

$$\phi_{23}^- = i\int_{-\infty}^{x} r_1(y)\phi_{13}^-(y,\lambda)dy, \tag{3.106}$$

$$\phi_{33}^- = 1+i\int_{-\infty}^{x} r_2(y)\phi_{13}^-(y,\lambda)dy, \tag{3.107}$$

$$\phi_{43}^- = i\int_{-\infty}^{x} r_3(y)\phi_{13}^-(y,\lambda)dy, \tag{3.108}$$

and finally the components of the fourth column of ϕ^- are

$$\phi_{14}^- = i\int_{-\infty}^{x}\Big(p_1(y)\phi_{24}^-(y,\lambda)+p_2(y)\phi_{34}^-(y,\lambda)+p_3(y)\phi_{44}^-(y,\lambda)\Big)e^{i\lambda\alpha(x-y)}dy, \tag{3.109}$$

$$\phi_{24}^- = i\int_{-\infty}^{x} r_1(y)\phi_{14}^-(y,\lambda)dy, \tag{3.110}$$

$$\phi_{34}^- = i\int_{-\infty}^{x} r_2(y)\phi_{14}^-(y,\lambda)dy, \tag{3.111}$$

$$\phi_{44}^- = 1+i\int_{-\infty}^{x} r_3(y)\phi_{14}^-(y,\lambda)dy. \tag{3.112}$$

We can see that as $\alpha=\alpha_1-\alpha_2<0$, if $Im(\lambda)>0$, then $Re(e^{-i\lambda\alpha(x-y)})$ decays exponentially when $y<x$, and so each integral of the first column of ϕ^- converges. As a result, the components of the first column of ϕ^-, i.e., $\phi_{11}^-, \phi_{21}^-, \phi_{31}^-, \phi_{41}^-$ are analytic in the upper half complex plane for $\lambda\in\mathbb{C}_+$, and continuous for $\lambda\in\mathbb{C}_+\cup\mathbb{R}$. But, if $Im(\lambda)<0$, $Re(e^{i\lambda\alpha(x-y)})$ also decays, then the components of the last three columns of ϕ^- converge, and thus they are analytic in the lower half plane for $\lambda\in\mathbb{C}_-$ and continuous for $\lambda\in\mathbb{C}_-\cup\mathbb{R}$.

In the same way for $y > x$, the components of the last three columns of ϕ^+ are analytic in the upper half plane for $\lambda \in \mathbb{C}_+$ and continuous for $\lambda \in \mathbb{C}_+ \cup \mathbb{R}$, and the components of the first column of ϕ^+ are analytic in the lower half plane for $\lambda \in \mathbb{C}_-$ and continuous for $\lambda \in \mathbb{C}_- \cup \mathbb{R}$.

Now let us construct the Riemann-Hilbert problems. Note that

$$\psi^{\pm} = \phi^{\pm} e^{i\lambda \Lambda x}, \quad \text{so} \quad \phi^{\pm} = \psi^{\pm} e^{-i\lambda \Lambda x}. \tag{3.113}$$

Let $\phi_j^{\pm}$ be the jth column of $\phi^{\pm}$ for $j \in \{1, 2, 3, 4\}$, and so the first Jost matrix solution can be taken as

$$P^{(+)}(x, \lambda) = (\phi_1^-, \phi_2^+, \phi_3^+, \phi_4^+) = \phi^- H_1 + \phi^+ H_2, \tag{3.114}$$

where $\phi_1^- = (\phi_{11}^-, \phi_{21}^-, \phi_{31}^-, \phi_{41}^-)^T$, $\phi_2^+ = (\phi_{12}^+, \phi_{22}^+, \phi_{32}^+, \phi_{42}^+)^T$, $\phi_3^+ = (\phi_{13}^+, \phi_{23}^+, \phi_{33}^+, \phi_{43}^+)^T$, $\phi_4^+ = (\phi_{14}^+, \phi_{24}^+, \phi_{34}^+, \phi_{44}^+)^T$ and $H_1 = diag(1, 0, 0, 0)$ and $H_2 = diag(0, 1, 1, 1)$.

$P^{(+)}$ is then analytic for $\lambda \in \mathbb{C}_+$ and continuous for $\lambda \in \mathbb{C}_+ \cup \mathbb{R}$.

To construct the analytic counterpart of $P^{(+)} \in \mathbb{C}_+$, it is going to be simpler to use the equivalent spectral adjoint equation (3.63). Because $\tilde{\phi}^{\pm} = (\phi^{\pm})^{-1}$ and $\psi^{\pm} = \phi^{\pm} e^{i\lambda \Lambda x}$, we have

$$(\phi^{\pm})^{-1} = e^{i\lambda \Lambda x} (\psi^{\pm})^{-1}. \tag{3.115}$$

Now, let $\tilde{\phi}_j^{\pm}$ be the jth row of $\tilde{\phi}^{\pm}$ for $j \in \{1, 2, 3, 4\}$. In the same way we proved for $P^{(+)}$ above, we can get

$$P^{(-)}(x, \lambda) = \left(\tilde{\phi}_1^-, \tilde{\phi}_2^+, \tilde{\phi}_3^+, \tilde{\phi}_4^+ \right)^T = H_1 (\phi^-)^{-1} + H_2 (\phi^+)^{-1}. \tag{3.116}$$

$P^{(-)}$ is analytic for $\lambda \in \mathbb{C}_-$ and continuous for $\lambda \in \mathbb{C}_- \cup \mathbb{R}$.

Also we have

$$P^{(-)}(x, \lambda) \to I_4 \quad \text{as} \quad \lambda \in \mathbb{C}_- \cup \mathbb{R} \to \infty. \tag{3.117}$$

From (3.114), (3.116) and (3.113) along with $\phi^T(x, -t, -\lambda) = C\phi^{-1}(x, t, \lambda) C^{-1}$, we have the nonlocal involution property

$$(P^{(+)})^T (x, -t, -\lambda) = C P^{(-)}(x, t, \lambda) C^{-1}. \tag{3.118}$$

Proof: Equation (3.114) gives

$$P^{(+)}(x,t,\lambda) = \phi^{-}(x,t,\lambda)H_1 + \phi^{+}(x,t,\lambda)H_2, \tag{3.119}$$

$$P^{(+)}(x,-t,-\lambda) = \phi^{-}(x,-t,-\lambda)H_1 + \phi^{+}(x,-t,-\lambda)H_2, \tag{3.120}$$

$$(P^{(+)})^T(x,-t,-\lambda) = H_1(\phi^{-})^T(x,-t,-\lambda) + H_2(\phi^{+})^T(x,-t,-\lambda). \tag{3.121}$$

As (3.116) gives

$$P^{(-)}(x,t,\lambda) = H_1(\phi^{-})^{-1}(x,t,\lambda) + H_2(\phi^{+})^{-1}(x,t,\lambda), \tag{3.122}$$

$$CP^{(-)}(x,t,\lambda)C^{-1} = H_1C(\phi^{-})^{-1}(x,t,\lambda)C^{-1} + CH_2(\phi^{+})^{-1}(x,t,\lambda)C^{-1}. \tag{3.123}$$

But (3.64) gives

$$\phi^T(x,-t,-\lambda) = C\phi^{-1}(x,t,\lambda)C^{-1}. \tag{3.124}$$

Therefore, matching (3.121) and (3.123), we deduce

$$(P^{(+)})^T(x,-t,-\lambda) = CP^{(-)}(x,t,\lambda)C^{-1}. \tag{3.125}$$

Through what have been done above, we have been able to construct the matrix of eigenfunctions $P^{(+)}$ and $P^{(-)}$ that are analytic in $\mathbb{C}_+$ and $\mathbb{C}_-$ respectively, and continuous in $\mathbb{C}_+ \cup \mathbb{R}$ and $\mathbb{C}_- \cup \mathbb{R}$ respectively.
From (3.121) and (3.116), we have

$$P^{(-)}(x,\lambda)P^{(+)}(x,\lambda) = e^{i\lambda\Lambda x}(H_1 + H_2S)(H_1 + S^{-1}H_2)e^{-i\lambda\Lambda x}, \text{for} \quad \lambda \in \mathbb{R}, \tag{3.126}$$

where the inverse scattering data matrix $S^{-1} = (\hat{s}_{ij})_{4\times 4}$ for $i, j \in \{1, 2, 3, 4\}$.
Using (3.68) in (3.114), we have

$$P^{(+)}(x,\lambda) = \phi^{+}(e^{i\lambda\Lambda x}Se^{-i\lambda\Lambda x}H_1 + H_2), \tag{3.127}$$

and as $\phi^{+}(x,\lambda) \to I_4$ when $x \to +\infty$, then

$$\lim_{x\to+\infty} P^{(+)} = \begin{pmatrix} s_{11}(\lambda) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{for} \quad \lambda \in \mathbb{C}_{+} \cup \mathbb{R}. \tag{3.128}$$

In the same way, we have

$$\lim_{x\to-\infty} P^{(-)} = \begin{pmatrix} \hat{s}_{11}(\lambda) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{for} \quad \lambda \in \mathbb{C}_{-} \cup \mathbb{R}. \tag{3.129}$$

Thus if we choose

$$G^{(+)}(x,\lambda) = P^{(+)}(x,\lambda) \begin{pmatrix} s_{11}^{-1}(\lambda) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.130}$$

and

$$(G^{(-)})^{-1}(x,\lambda) = \begin{pmatrix} \hat{s}_{11}^{-1}(\lambda) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} P^{(-)}(x,\lambda), \tag{3.131}$$

then on the real line, the two generalized matrices generate the matrix Riemann-Hilbert problems for the six-component AKNS system of fourth-order given by

$$G^{(+)}(x,\lambda) = G^{(-)}(x,\lambda)G_0(x,\lambda), \quad \text{for} \quad \lambda \in \mathbb{R}, \tag{3.132}$$

where the jump matrix $G_0(x,\lambda)$ can be cast as

$$G_0(x,\lambda)=e^{i\lambda\Lambda x}\begin{pmatrix}\hat{s}_{11}^{-1}(\lambda) & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}(H_1+H_2S)(H_1+S^{-1}H_2)\begin{pmatrix}s_{11}^{-1}(\lambda) & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}e^{-i\lambda\Lambda x}, \tag{3.133}$$

which can be explicitly written as

$$G_0(x,\lambda)=\begin{pmatrix} s_{11}^{-1}\hat{s}_{11}^{-1} & \hat{s}_{12}\hat{s}_{11}^{-1}e^{i\lambda x(\alpha_1-\alpha_2)} & \hat{s}_{13}\hat{s}_{11}^{-1}e^{i\lambda x(\alpha_1-\alpha_2)} & \hat{s}_{14}\hat{s}_{11}^{-1}e^{i\lambda x(\alpha_1-\alpha_2)}\\ s_{21}s_{11}^{-1}e^{-i\lambda x(\alpha_1-\alpha_2)} & 1 & 0 & 0\\ s_{31}s_{11}^{-1}e^{-i\lambda x(\alpha_1-\alpha_2)} & 0 & 1 & 0\\ s_{41}s_{11}^{-1}e^{-i\lambda x(\alpha_1-\alpha_2)} & 0 & 0 & 1\end{pmatrix}, \tag{3.134}$$

with its canonical normalization conditions being given by:

$$G^{(+)}(x,\lambda)\to I_4 \quad \text{as} \quad \lambda\in\mathbb{C}_+\cup\mathbb{R}\to\infty, \tag{3.135}$$

$$G^{(-)}(x,\lambda)\to I_4 \quad \text{as} \quad \lambda\in\mathbb{C}_-\cup\mathbb{R}\to\infty. \tag{3.136}$$

From (3.118) along with (3.80) and (3.130), we obtain

$$(G^{(+)})^T(x,-t,-\lambda)=C(G^{(-)})^{-1}(x,t,\lambda)C^{-1}. \tag{3.137}$$

Proof: From (3.130), we have

$$G^{(+)}(x,t,\lambda)=P^{(+)}(x,t,\lambda)\begin{pmatrix}s_{11}^{-1}(\lambda) & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}, \tag{3.138}$$

$$(G^{(+)})^T(x,-t,-\lambda) = \begin{pmatrix} s_{11}^{-1}(-\lambda) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} (P^{(+)})^T(x,-t,-\lambda), \tag{3.139}$$

using (3.118) and $(P^{(+)})^T(x,-t,-\lambda) = C(P^{(-)})(x,t,\lambda)C^{-1}$, we get

$$(G^{(+)})^T(x,-t,-\lambda) = \begin{pmatrix} s_{11}^{-1}(-\lambda) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} C(P^{(-)})(x,t,\lambda)C^{-1}. \tag{3.140}$$

As (3.131) is

$$(G^{(-)})^{-1}(x,\lambda) = \begin{pmatrix} \hat{s}_{11}^{-1}(\lambda) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} P^{(-)}(x,\lambda), \tag{3.141}$$

we have

$$C(G^{(-)})^{-1}(x,\lambda)C^{-1} = \begin{pmatrix} \hat{s}_{11}^{-1}(\lambda) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} CP^{(-)}(x,\lambda)C^{-1}. \tag{3.142}$$

But (3.80) is

$$\hat{s}_{11}(\lambda) = s_{11}(-\lambda), \tag{3.143}$$

so

$$\hat{s}_{11}^{-1}(\lambda) = s_{11}^{-1}(-\lambda). \tag{3.144}$$

Therefore, matching (3.140) and (3.142) we have the proof of

$$(G^{(+)})^T(x,-t,-\lambda) = C(G^{(-)})^{-1}(x,t,\lambda)C^{-1}. \tag{3.145}$$

Also, from (3.133) and (3.80), we have this involution property

$$G_0^T(x,-t,-\lambda) = CG_0(x,t,\lambda)C^{-1}. \tag{3.146}$$

Proof: As (3.133) is

$$G_0(x,\lambda) = e^{i\lambda\Lambda x}\begin{pmatrix} \hat{s}_{11}^{-1}(\lambda) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}(H_1 + H_2S(\lambda))(H_1 + S^{-1}(\lambda)H_2)\begin{pmatrix} s_{11}^{-1}(\lambda) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}e^{-i\lambda\Lambda x}, \tag{3.147}$$

so

$$CG_0(x,\lambda)C^{-1} = e^{i\lambda\Lambda x}\begin{pmatrix} \hat{s}_{11}^{-1}(\lambda) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} M \begin{pmatrix} s_{11}^{-1}(\lambda) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}e^{-i\lambda\Lambda x}, \tag{3.148}$$

$$G_0^T(x,-t,-\lambda) = e^{i\lambda\Lambda x}\begin{pmatrix} s_{11}^{-1}(-\lambda) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} N \begin{pmatrix} \hat{s}_{11}^{-1}(-\lambda) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}e^{-i\lambda\Lambda x}, \tag{3.149}$$

where

$$M = (H_1 + H_2CS(\lambda)C^{-1})(H_1 + CS^{-1}(\lambda)C^{-1}H_2), \tag{3.150}$$

$$N = (H_1 + H_2(S^{-1})^T(-t,-\lambda))(H_1 + S^T(-t,-\lambda)H_2). \tag{3.151}$$

As $S^T(-\lambda) = CS^{-1}(\lambda)C^{-1}$ and $\hat{s}_{11}^{-1}(\lambda) = s_{11}^{-1}(-\lambda)$, matching (3.148) and (3.149), we get the proof of

$$G_0^T(x,-t,-\lambda) = CG_0(x,t,\lambda)C^{-1}. \tag{3.152}$$

3.5 Time evolution of scattering data

The process of the inverse scattering transform requires the time evolution of the scattering data. Differentiating equation (3.68) with respect to time t and applying (3.55) gives

$$S_t = i\lambda^4[\Omega, S]. \tag{3.153}$$

Proof: Using (3.68), we have

$$\phi^- = \phi^+ e^{i\lambda\Lambda x} S(\lambda) e^{-i\lambda\Lambda x}, \quad \text{for} \quad \lambda \in \mathbb{R}, \tag{3.154}$$

so

$$\phi_t^- = \phi_t^+ e^{i\lambda\Lambda x} S(\lambda) e^{-i\lambda\Lambda x} + \phi^+ e^{i\lambda\Lambda x} S_t(\lambda) e^{-i\lambda\Lambda x}. \tag{3.155}$$

Using (3.55): $\phi_t = i\lambda^4[\Omega, \phi] + iQ\phi$, we get

$$\begin{aligned} i\lambda^4[\Omega, \phi^-] + iQ\phi^- &= (i\lambda^4[\Omega, \phi^+] + iQ\phi^+) e^{i\lambda\Lambda x} S(\lambda) e^{-i\lambda\Lambda x} + \phi^+ e^{i\lambda\Lambda x} S_t(\lambda) e^{-i\lambda\Lambda x} \\ &= i\lambda^4 \Omega\phi^+ e^{i\lambda\Lambda x} S(\lambda) e^{-i\lambda\Lambda x} - i\lambda^4 \phi^+ \Omega e^{i\lambda\Lambda x} S(\lambda) e^{-i\lambda\Lambda x} \\ &+ iQ\phi^+ e^{i\lambda\Lambda x} S(\lambda) e^{-i\lambda\Lambda x} + \phi^+ e^{i\lambda\Lambda x} S_t(\lambda) e^{-i\lambda\Lambda x}, \end{aligned} \tag{3.156}$$

but also

$$\begin{aligned} i\lambda^4[\Omega, \phi^-] + iQ\phi^- &= i\lambda^4[\Omega, \phi^+(e^{i\lambda\Lambda x} S(\lambda) e^{-i\lambda\Lambda x})] + iQ\phi^+(e^{i\lambda\Lambda x} S(\lambda) e^{-i\lambda\Lambda x}) \\ &= i\lambda^4 \Omega\phi^+(e^{i\lambda\Lambda x} S(\lambda) e^{-i\lambda\Lambda x}) - i\lambda^4 \phi^+(e^{i\lambda\Lambda x} S(\lambda) e^{-i\lambda\Lambda x})\Omega \\ &+ iQ\phi^+(e^{i\lambda\Lambda x} S(\lambda) e^{-i\lambda\Lambda x}). \end{aligned} \tag{3.157}$$

Matching (3.156) and (3.157), we deduce that

$$i\lambda^4 \phi^+ \Omega e^{i\lambda\Lambda x} S(\lambda) e^{-i\lambda\Lambda x} = i\lambda^4 \phi^+(e^{i\lambda\Lambda x} S(\lambda) e^{-i\lambda\Lambda x})\Omega + \phi^+ e^{i\lambda\Lambda x} S_t(\lambda) e^{-i\lambda\Lambda x}. \tag{3.158}$$

Because multiplications of diagonal matrices permute, we get

$$\phi^{+}e^{i\lambda x}(i\lambda S(\lambda)\Omega) = \phi^{+}e^{i\lambda\Lambda x}(i\lambda\Omega S(\lambda) + S_t(\lambda)), \tag{3.159}$$

$$S_t(\lambda) = i\lambda^4\Omega S(\lambda) - i\lambda^4 S(\lambda)\Omega. \tag{3.160}$$

Hence,

$$S_t(\lambda) = i\lambda^4[\Omega, S(\lambda)], \tag{3.161}$$

which is

$$S_t = \begin{pmatrix} 0 & i\beta\lambda^4 s_{12} & i\beta\lambda^4 s_{13} & i\beta\lambda^4 s_{14} \\ -i\beta\lambda^4 s_{21} & 0 & 0 & 0 \\ -i\beta\lambda^4 s_{31} & 0 & 0 & 0 \\ -i\beta\lambda^4 s_{41} & 0 & 0 & 0 \end{pmatrix}. \tag{3.162}$$

As a result, we have

$$\begin{cases} s_{12}(t,\lambda) = s_{12}(0,\lambda)e^{i\beta\lambda^4 t}, \\ s_{13}(t,\lambda) = s_{13}(0,\lambda)e^{i\beta\lambda^4 t}, \\ s_{14}(t,\lambda) = s_{14}(0,\lambda)e^{i\beta\lambda^4 t}, \\ s_{21}(t,\lambda) = s_{21}(0,\lambda)e^{-i\beta\lambda^4 t}, \\ s_{31}(t,\lambda) = s_{31}(0,\lambda)e^{-i\beta\lambda^4 t}, \\ s_{41}(t,\lambda) = s_{41}(0,\lambda)e^{-i\beta\lambda^4 t}, \end{cases} \tag{3.163}$$

and $s_{11}, s_{22}, s_{23}, s_{24}, s_{32}, s_{33}, s_{34}, s_{42}, s_{43}, s_{44}$ are constants.

Chapter 4

N-Soliton solutions

4.1 General case

In this section, we are going to compute explicitly the one-soliton and two-soliton solutions from the N-soliton solution based on the Riemann-Hilbert problems. In fact the Riemann-Hilbert problems generate a unique solution in the regular case, i.e. the $det(G^{(\pm)}) \neq 0$ when $G^{(\pm)} \to I_4$ as $\lambda \to \infty$. However, there are possible contingencies that $det(G^{(\pm)})$ could be zero for some discrete $\lambda \in \mathbb{C}_\pm$ when non-regular. In that case, it is opportune to transform the non-regular case to a regular in order to guarantee a solution.

From (3.114) and (3.116) with (3.68), as $det(\phi^{\pm}) = 1$, we prove that

$$det(P^{(+)}(x, \lambda)) = s_{11}(\lambda), \tag{4.1}$$

and

$$det(P^{(-)}(x, \lambda)) = \hat{s}_{11}(\lambda). \tag{4.2}$$

Because $det(S(\lambda)) = 1$, so $S^{-1}(\lambda) = \Big(cof(S(\lambda))\Big)^T$, thus

$$\hat{s}_{11} = \begin{vmatrix} s_{22} & s_{23} & s_{24} \\ s_{32} & s_{33} & s_{34} \\ s_{42} & s_{43} & s_{44} \end{vmatrix}. \tag{4.3}$$

In order to get soliton solutions, the solutions of $det(P^{(\pm)}(x, \lambda)) = 0$ are assumed to be simple. Let's suppose that $s_{11}(\lambda)$ has simple zeros $\lambda_k \in \mathbb{C}_+$ for $k \in \{1, 2, ..., N\}$ and $\hat{s}_{11}(\lambda)$ has simple zeros $\hat{\lambda}_k \in \mathbb{C}_-$ for $k \in \{1, 2, ..., N\}$, which are the poles of the transmission coefficients [19].

From (3.80), we know that $\hat{s}_{11}(\lambda) = s_{11}(-\lambda)$. Hence we have the involution relation

$$\hat{\lambda} = -\lambda. \tag{4.4}$$

Each $Ker(P^{(+)}(x,\lambda_k))$ contains a unique column vector v_k, and also $Ker(P^{(-)}(x,\hat{\lambda}_k))$ contains a unique row vector $\hat{v}_k$ for $k \in \{1,2,...,N\}$ such that:

$$P^{(+)}(x,\lambda_k)v_k = 0 \quad \text{for} \quad k \in \{1,2,...,N\}, \tag{4.5}$$

and

$$\hat{v}_k P^{(-)}(x,\hat{\lambda}_k) = 0 \quad \text{for} \quad k \in \{1,2,...,N\}. \tag{4.6}$$

The Riemann-Hilbert problems can be solved explicitly when $G_0 = I_4$. This will force the reflection coefficients $s_{21} = s_{31} = s_{41} = 0$ and $\hat{s}_{12} = \hat{s}_{13} = \hat{s}_{14} = 0$.
In that case, we can present the solutions to special Riemann-Hilbert problems as follows: [9, 16, 20]

$$G^{(+)}(x,\lambda) = I_4 - \sum_{k,j=1}^{N} \frac{v_k(M^{-1})_{kj}\hat{v}_j}{\lambda - \hat{\lambda}_j}, \tag{4.7}$$

and

$$(G^{(-)})^{-1}(x,\lambda) = I_4 + \sum_{k,j=1}^{N} \frac{v_k(M^{-1})_{kj}\hat{v}_j}{\lambda - \lambda_k}, \tag{4.8}$$

where $M = (m_{kj})_{N\times N}$ is a matrix defined as follows [20]

$$m_{kj} = \begin{cases} \frac{\hat{v}_k v_j}{\lambda_j - \hat{\lambda}_k} & \text{if} \quad \lambda_j \neq \hat{\lambda}_k \\ 0 & \text{if} \quad \lambda_j = \hat{\lambda}_k \end{cases}, k, j \in \{1,2,...,N\}. \tag{4.9}$$

The scattering vectors v_k and $\hat{v}_k$ are functions of (x,t), but λ_k and $\hat{\lambda}_k$ are constants, and so differentiating both sides of $P^{(+)}(x,\lambda_k)v_k = 0$ with respect to x and knowing that $P^{(+)}$ satisfies the spectral spatial equivalent equation (3.54) along with (4.5) gives

$$P^{(+)}(x,\lambda_k)\left(\frac{dv_k}{dx} - i\lambda_k \Lambda v_k\right) = 0 \quad \text{for} \quad k, j \in \{1,2,...,N\}, \tag{4.10}$$

and also differentiating it with respect to t and using the temporal equation (3.55) along with (4.5) gives

$$P^{(+)}(x,\lambda_k)\left(\frac{dv_k}{dt} - i\lambda_k^4 \Omega v_k\right) = 0 \quad \text{for} \quad k, j \in \{1,2,...,N\}. \tag{4.11}$$

In the same way by using (4.6) and the adjoint spectral equations (3.61) and (3.62), one can prove that

$$\left(\frac{d\hat{v}_k}{dx} + i\hat{\lambda}_k\hat{v}_k\Lambda\right)P^{(-)}(x,\hat{\lambda}_k) = 0, \tag{4.12}$$

and

$$\left(\frac{d\hat{v}_k}{dt} + i\hat{\lambda}_k^4\hat{v}_k\Omega\right)P^{(-)}(x,\hat{\lambda}_k) = 0. \tag{4.13}$$

As v_k is a single vector in the kernel of $P^{(+)}$, so $\frac{dv_k}{dx} - i\lambda_k\Lambda v_k$ and $\frac{dv_k}{dt} - i\lambda_k^4\Omega v_k$ are scalar multiples of v_k. This permits one to obtain

$$v_k(x,t) = e^{i\lambda_k\Lambda x + i\lambda_k^4\Omega t}w_k \quad \text{for} \quad k \in \{1,2,...,N\}. \tag{4.14}$$

In the same way, we will have for $P^{(-)}$,

$$\hat{v}_k(x,t) = \hat{w}_k e^{-i\hat{\lambda}_k\Lambda x - i\hat{\lambda}_k^4\Omega t} \quad \text{for} \quad k \in \{1,2,...,N\}, \tag{4.15}$$

where the column vector w_k and the row vector $\hat{w}_k$ are constants.

Now from (4.5) and using (3.118), we get

$$v_k^T(x,-t,-\lambda_k)(P^{(+)})^T(x,-t,-\lambda_k) = v_k^T(x,-t,-\lambda_k)CP^{(-)}(x,t,\lambda_k)C^{-1} = 0 \quad \text{for} \quad k \in \{1,2,...,N\}. \tag{4.16}$$

Because $v_k^T(x,-t,-\lambda_k)CP^{(-)}(x,t,\lambda_k)$ could be zero and using (4.6) leads to

$$v_k^T(x,-t,-\lambda_k)CP^{(-)}(x,t,\lambda_k) = \hat{v}_k(x,t,\hat{\lambda}_k)P^{(-)}(x,t,\hat{\lambda}_k) = \hat{v}_k(x,t,-\hat{\lambda}_k)P^{(-)}(x,t,-\hat{\lambda}_k) = 0. \tag{4.17}$$

As $\hat{\lambda}_k = -\lambda_k$ from (4.4), then we can take

$$\hat{v}_k(x,t,-\hat{\lambda}_k) = v_k^T(x,-t,-\lambda_k)C \quad \text{for} \quad k \in \{1,2,...,N\}. \tag{4.18}$$

These involution relations give

$$v_k(x,t) = e^{i\lambda_k \Lambda x + i\lambda_k^4 \Omega t} w_k \quad \text{for} \quad k \in \{1, 2, ..., N\}, \tag{4.19}$$

$$\hat{v}_k(x,t) = w_k^T e^{-i\hat{\lambda}_k \Lambda x - i\hat{\lambda}_k^4 \Omega t} C \quad \text{for} \quad k \in \{1, 2, ..., N\}. \tag{4.20}$$

4.2 Recovery of potentials

The jump matrix being $G = I_4$ allows to recover the potential P from the generalized matrix Jost eigenfunctions. Because $P^{(+)}$ is analytic, we can expand $G^{(+)}$ as $\lambda \to \infty$ in this form at order 2,

$$G^{(+)}(x,\lambda) = I_4 + \frac{1}{\lambda} G_1^{(+)}(x) + O(\frac{1}{\lambda^2}) \quad \text{when} \quad \lambda \to \infty. \tag{4.21}$$

Because $G^{(+)}$ satisfies the spectral problem, substituting it into (3.54) and matching the coefficients of the same powers of $\frac{1}{\lambda}$, at order $O(1)$, we get

$$P = -[\Lambda, G_1^{(+)}]. \tag{4.22}$$

If

$$G_1^{(+)} = \begin{pmatrix} (G_1^{(+)})_{11} & (G_1^{(+)})_{12} & (G_1^{(+)})_{13} & (G_1^{(+)})_{14} \\ (G_1^{(+)})_{21} & (G_1^{(+)})_{22} & (G_1^{(+)})_{23} & (G_1^{(+)})_{24} \\ (G_1^{(+)})_{31} & (G_1^{(+)})_{32} & (G_1^{(+)})_{33} & (G_1^{(+)})_{34} \\ (G_1^{(+)})_{41} & (G_1^{()})_{42} & (G_1^{(+)})_{43} & (G_1^{(+)})_{44} \end{pmatrix}, \tag{4.23}$$

then

$$P = -[\Lambda, G_1^{(+)}] = \begin{pmatrix} 0 & -\alpha(G_1^{(+)})_{12} & -\alpha(G_1^{(+)})_{13} & -\alpha(G_1^{(+)})_{14} \\ \alpha(G_1^{(+)})_{21} & 0 & 0 & 0 \\ \alpha(G_1^{(+)})_{31} & 0 & 0 & 0 \\ \alpha(G_1^{(+)})_{41} & 0 & 0 & 0 \end{pmatrix}. \tag{4.24}$$

As a result, we can now recover the potentials p_i and r_i for $i \in \{1, 2, 3\}$ as follows

$$\begin{aligned} p_1 &= -\alpha(G_1^{(+)})_{12}, \quad r_1 = \alpha(G_1^{(+)})_{21}, \\ p_2 &= -\alpha(G_1^{(+)})_{13}, \quad r_2 = \alpha(G_1^{(+)})_{31}, \\ p_3 &= -\alpha(G_1^{(+)})_{14}, \quad r_3 = \alpha(G_1^{(+)})_{41}. \end{aligned} \tag{4.25}$$

Also, from (4.21), we have

$$G_1^{(+)} = \lambda \lim_{\lambda\to\infty} (G^{(+)}(x,\lambda) - I_4), \tag{4.26}$$

and so from (4.7) we deduce that

$$G_1^{(+)} = -\sum_{k,j=1}^{N} v_k (M^{-1})_{k,j} \hat{v}_j. \tag{4.27}$$

From (3.34) and (4.22), we easily prove the nonlocal involution property

$$(G_1^{(+)})^T(x,-t) = CG_1^{(+)}(x,t)C^{-1}. \tag{4.28}$$

Proof: From (4.22), we get

$$P(x,t) = -[\Lambda, G_1^{(+)}(x,t)], \tag{4.29}$$

$$P(x,-t) = -(\Lambda G_1^{(+)}(x,-t) - G_1^{(+)}(x,-t)\Lambda), \tag{4.30}$$

$$P^T(x,-t) = -((G_1^{(+)})^T(x,-t)\Lambda - \Lambda(G_1^{(+)})^T(x,-t)), \tag{4.31}$$

$$P^T(x,-t) = [\Lambda, (G_1^{(+)})^T(x,-t)]. \tag{4.32}$$

Using (3.34) and (4.29), we get

$$P^T(x,-t) = -CP(x,t)C^{-1}, \tag{4.33}$$

$$P^T(x,-t) = C[\Lambda, G_1^{(+)}(x,t)]C^{-1}, \tag{4.34}$$

$$P^T(x,-t) = [\Lambda, CG_1^{(+)}(x,t)C^{-1}]. \tag{4.35}$$

We deduce from (5.32) and (5.35) that

$$(G_1^{(+)})^T(x,-t) = CG_1^{(+)}(x,t)C^{-1}. \tag{4.36}$$

Hence, we has this nonlocal involution property

$$(G^{(+)})^T(x,-t,-\lambda) = C(G^{(-)})^{-1}(x,-t,\lambda)C^{-1}. \tag{4.37}$$

Proof: We have from (4.7),

$$G^{(+)}(x,t,\lambda) = I_4 - \sum_{k,j=1}^{N} \frac{v_k(x,t,\lambda_k)(M^{-1})_{kj}(x,t)\hat{v}_j(x,t,\hat{\lambda}_j)}{\lambda - \hat{\lambda}_j}, \tag{4.38}$$

$$G^{(+)}(x,-t,-\lambda) = I_4 - \sum_{k,j=1}^{N} \frac{v_k(x,-t,\lambda_k)(M^{-1})_{kj}(x,-t)\hat{v}_j(x,-t,\hat{\lambda}_j)}{-\lambda - \hat{\lambda}_j}, \tag{4.39}$$

$$(G^{(+)})^T(x,-t,-\lambda) = I_4 - \sum_{k,j=1}^{N} \frac{\hat{v}_j^T(x,-t,\hat{\lambda}_j)(M^{-1})_{kj}^T(x,-t)v_k^T(x,-t,\lambda_k)}{-\lambda - \hat{\lambda}_j}. \tag{4.40}$$

We have first from (4.18),

$$\hat{v}_k(x,t,-\hat{\lambda}_k) = v_k^T(x,-t,-\lambda_k)C \quad \text{for} \quad k \in \{1,2,...,N\}, \tag{4.41}$$

$$\hat{v}_k(x,-t,\hat{\lambda}_k) = v_k^T(x,t,-\lambda_k)C, \tag{4.42}$$

as $\hat{\lambda}_k = -\lambda_k$,

$$\hat{v}_k(x,-t,\hat{\lambda}_k) = v_k^T(x,t,\hat{\lambda}_k)C, \tag{4.43}$$

$$\hat{v}_k^T(x,-t,\hat{\lambda}_k) = Cv_k(x,t,\lambda_k). \tag{4.44}$$

Secondly, using (4.9) and (4.43) and (4.44), we obtain

$$\begin{aligned}
(M^{-1})_{kj}^T(x,-t) &= (M^T)_{kj}^{-1}(x,-t) \\
&= \Big[\Big(\frac{\hat{v}_k(x,-t,\hat{\lambda}_k)v_j(x,-t,\lambda_j)}{\lambda_j - \hat{\lambda}_k}\Big)^T\Big]^{-1} = \Big(\frac{v_j^T(x,-t,\lambda_j)\hat{v}_k^T(x,-t,\hat{\lambda}_k)}{\lambda_j - \hat{\lambda}_k}\Big)^{-1} \\
&= \Big(\frac{v_j^T(x,-t,\lambda_j)CC^{-1}\hat{v}_k^T(x,-t,\hat{\lambda}_k)}{\lambda_j - \hat{\lambda}_k}\Big)^{-1} = \Big(\frac{\hat{v}_j(x,t,\hat{\lambda}_j)v_k(x,t,\lambda_k)}{\lambda_k - \hat{\lambda}_j}\Big)^{-1} \\
&= (M^{-1})_{jk}^T(x,t).
\end{aligned} \tag{4.45}$$

Finally, from (4.40),

$$(G^{(+)})^T(x,-t,-\lambda) = I_4 - \sum_{k,j=1}^{N} \frac{\hat{v}_j^T(x,-t,\hat{\lambda}_j)(M^{-1})_{kj}^T(x,-t)v_k^T(x,-t,\lambda_k)}{-\lambda - \hat{\lambda}_j}, \tag{4.46}$$

and substituting (4.44) and (4.45), into (4.46), and using the fact that $\hat{\lambda}_j = -\lambda_j$, we obtain

$$(G^{(+)})^T(x,-t,-\lambda) = I_4 - \sum_{k,j=1}^{N} \frac{Cv_j(x,t,\lambda_j)(M^{-1})_{jk}(x,t)\hat{v}_k(x,t,\lambda_k)C^{-1}}{-\lambda+\lambda_j}, \tag{4.47}$$

$$(G^{(+)})^T(x,-t,-\lambda) = C\Big(I_4 + \sum_{k,j=1}^{N} \frac{v_j(x,t,\lambda_j)(M^{-1})_{jk}(x,t)\hat{v}_k(x,t,\lambda_k)C^{-1}}{\lambda-\lambda_j}\Big)C^{-1}. \tag{4.48}$$

Interchanging k and j into (4.48), we get

$$(G^{(+)})^T(x,-t,-\lambda) = C\Big(I_4 + \sum_{k,j=1}^{N} \frac{v_k(x,t,\lambda_k)(M^{-1})_{kj}(x,t)\hat{v}_j(x,t,\lambda_j)C^{-1}}{\lambda-\lambda_k}\Big)C^{-1}. \tag{4.49}$$

As a result,

$$(G^{(+)})^T(x,-t,-\lambda) = C(G^{(-)})^{-1}C^{-1}. \tag{4.50}$$

Using the above equations along with (4.20) and (4.19) will generate the N-soliton solution to the nonlocal reverse-time six-component AKNS system of fourth-order as follows:

$$p_i = \alpha \sum_{k,j=1}^{N} v_{k1}(M^{-1})_{kj}\hat{v}_{j,i+1} \quad \text{for} \quad i \in \{1,2,3\}, \tag{4.51}$$

where $v_k = (v_{k1}, v_{k2}, v_{k3}, ..., v_{kn+1})^T,\ \hat{v}_k = (\hat{v}_{k1}, \hat{v}_{k2}, \hat{v}_{k3}, ..., \hat{v}_{kn+1})$.

4.3 Exact soliton solutions and their dynamics

4.3.1 One-soliton solution

A general explicit solution for a single-soliton in the reverse-time case when $N = 1$, $w_1 = (w_{11}, w_{12}, w_{13}, w_{14})^T$, $\lambda_1 \in \mathbb{C}$ is arbitrary, and $\hat{\lambda}_1 = -\lambda_1$ is given by

$$p_1(x,t) = \frac{2\rho_2\rho_3\lambda_1(\alpha_1-\alpha_2)w_{11}w_{12}e^{i\lambda_1(\alpha_1+\alpha_2)x+i\lambda_1^4(\beta_1-\beta_2)t}}{\rho_1\rho_2\rho_3 w_{11}^2 e^{2i\lambda_1\alpha_1 x} + (\rho_2\rho_3 w_{12}^2 + \rho_1\rho_3 w_{13}^2 + \rho_1\rho_2 w_{14}^2)e^{2i\lambda_1\alpha_2 x}}, \tag{4.52}$$

$$p_2(x,t) = \frac{2\rho_1\rho_3\lambda_1(\alpha_1-\alpha_2)w_{11}w_{13}e^{i\lambda_1(\alpha_1+\alpha_2)x+i\lambda_1^4(\beta_1-\beta_2)t}}{\rho_1\rho_2\rho_3 w_{11}^2 e^{2i\lambda_1\alpha_1 x} + (\rho_2\rho_3 w_{12}^2 + \rho_1\rho_3 w_{13}^2 + \rho_1\rho_2 w_{14}^2)e^{2i\lambda_1\alpha_2 x}}, \tag{4.53}$$

$$p_3(x,t) = \frac{2\rho_1\rho_2\lambda_1(\alpha_1-\alpha_2)w_{11}w_{14}e^{i\lambda_1(\alpha_1+\alpha_2)x+i\lambda_1^4(\beta_1-\beta_2)t}}{\rho_1\rho_2\rho_3 w_{11}^2 e^{2i\lambda_1\alpha_1 x} + (\rho_2\rho_3 w_{12}^2 + \rho_1\rho_3 w_{13}^2 + \rho_1\rho_2 w_{14}^2)e^{2i\lambda_1\alpha_2 x}}. \tag{4.54}$$

We can get the amplitude of p_1:

$$|p_1(x,t)| = 2e^{-Im(\lambda_1^4(\beta_1-\beta_2)t+\lambda_1(\alpha_1+\alpha_2)x)} \times \left| \frac{\lambda_1\rho_2\rho_3(\alpha_1-\alpha_2)w_{11}w_{12}}{\rho_1\rho_2\rho_3 w_{11}^2 e^{2i\lambda_1\alpha_1 x} + (\rho_2\rho_3 w_{12}^2 + \rho_1\rho_3 w_{13}^2 + \rho_1\rho_2 w_{14}^2)e^{2i\lambda_1\alpha_2 x}} \right|. \tag{4.55}$$

About the dynamics of the one-soliton, we can see from p_1 that there is no speed, i.e. the soliton is not a travelling wave. By choosing any arbitrary constant $x = x_0$, $\beta_1 - \beta_2 < 0$ and $\lambda_1 \notin i\mathbb{R}$ in $|p_1(x,t)|$ we see that the soliton's amplitude grows exponentially if $Im(\lambda_1^4) > 0$, while it decays exponentially if $Im(\lambda_1^4) < 0$, but when $Im(\lambda_1^4) = 0$ the amplitude is constant over the time. If we choose $x = x_0$ and $\lambda_1 \in i\mathbb{R}$ we have a constant amplitude for the soliton, indeed.

In this reverse-time case, any one-soliton does not collapse, either it strictly increases, decreases or stays constant.

From the spectral plane, let $\lambda_1 = \xi + i\eta = re^{i\theta}$, where $r > 0$ and $0 < \theta < 2\pi$ then:

$$\text{if} \begin{cases} \theta \in (0, \frac{\pi}{4}) \cup (\frac{\pi}{2}, \frac{3\pi}{4}) \cup (\pi, \frac{5\pi}{4}) \cup (\frac{3\pi}{2}, \frac{7\pi}{4}), & \text{then the amplitude of the soliton is increasing,} \\ \theta \in (\frac{\pi}{4}, \frac{\pi}{2}) \cup (\frac{3\pi}{4}, \pi) \cup (\frac{5\pi}{4}, \frac{3\pi}{2}) \cup (\frac{7\pi}{4}, 2\pi), & \text{the amplitude of the soliton is decreasing,} \\ \theta \in \{\frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}\}, & \text{the amplitude of the soliton is constant,} \\ \theta \in \{0, \pi, 2\pi\}, & \text{we obtain one breather with constant amplitude.} \end{cases} \tag{4.56}$$

This illustration is shown by the figure below.

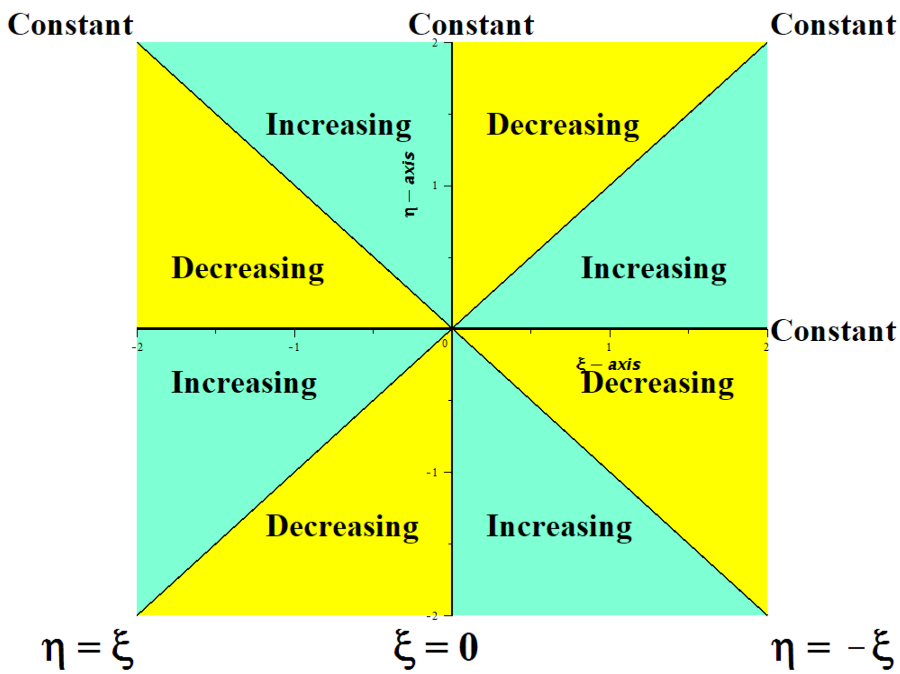

Figure 3.: Spectral plane of eigenvalues.

Let us graph the one-soliton solution. When λ_1 does not lie on the real axis ($\eta = 0$), the imaginary axis ($\xi = 0$) or the bisectors ($\eta = \pm\xi$) , the amplitude of the potential grows or decays exponentially, if $Im(\lambda_1^4) > 0$ or $Im(\lambda_1^4) < 0$ respectively. Two examples are illustrated in Figure 4 and Figure 5, where we have growing and decaying amplitudes.

When $Im(\lambda_1^4) = 0$ the amplitude does not change. In that case, λ_1 lies on the imaginary axis, the bisectors or the real axis. If λ_1 lies on the imaginary axis or on the bisectors, then we have a fundamental soliton (figure 6), whereas if $\lambda_1 \in \mathbb{R}$, then we have a periodic one-soliton with period $\frac{\pi}{\lambda_1(\alpha_1-\alpha_2)}$ which is a breather (figure 7).

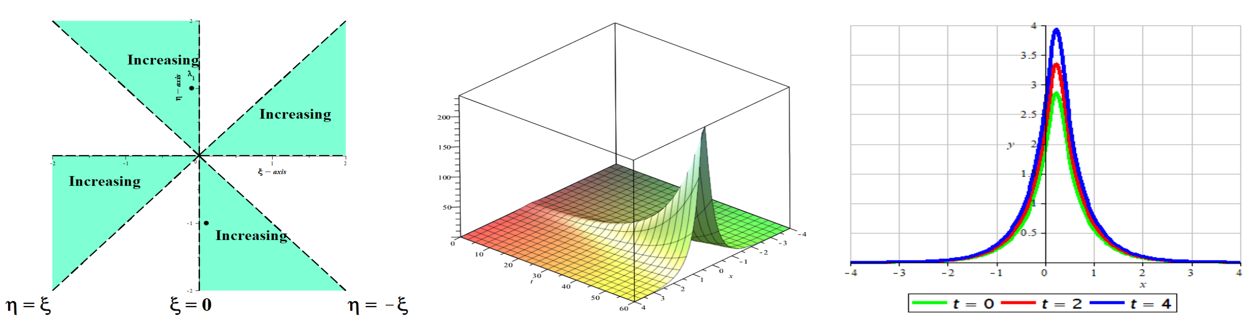

Figure 4.: Spectral plane along with 3D plot and 2D plots of $|p_1|$ of the one-soliton in the focusing case with parameter values $\rho_1 = -1$, $\rho_2 = -2$, $\rho_3 = -1$, $\lambda_1 = -0.01 + i$, $\alpha_1 = -1$, $\alpha_2 = 1$, $\beta_1 = -1$, $\beta_2 = 1$, $w_1 = (1, i, 2 + i, 1)^T$. The 2D plot is for time values, t=0, 2 and 4.

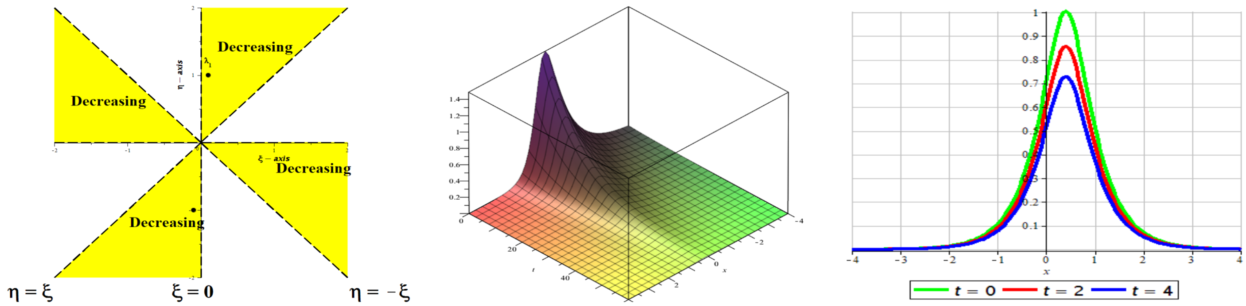

Figure 5.: Spectral plane along with 3D plot and 2D plots of $|p_1|$ of the one-soliton with parameter values $\rho_1 = 1$, $\rho_2 = 1$, $\rho_3 = 1$, $\lambda_1 = 0.01 + i$, $\alpha_1 = -1$, $\alpha_2 = 1$, $\beta_1 = -1$, $\beta_2 = 1$, $w_1 = (1, i, 2+i, 1)^T$. The 2D plot is for time values, t=0, 2 and 4.

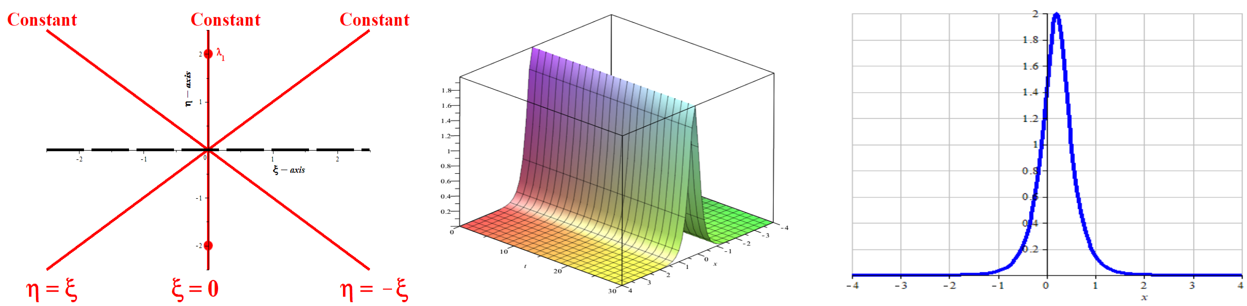

Figure 6.: Spectral plane along with 3D plot and 2D plots of $|p_1|$ of the one-soliton with parameter values $\rho_1 = 1$, $\rho_2 = 1$, $\rho_3 = 1$, $\lambda_1 = 2i$, $\alpha_1 = -1$, $\alpha_2 = 1$, $\beta_1 = -1$, $\beta_2 = 1$, $w_1 = (1, i, 2+i, 1)^T$. The 2D plot is for any time value.

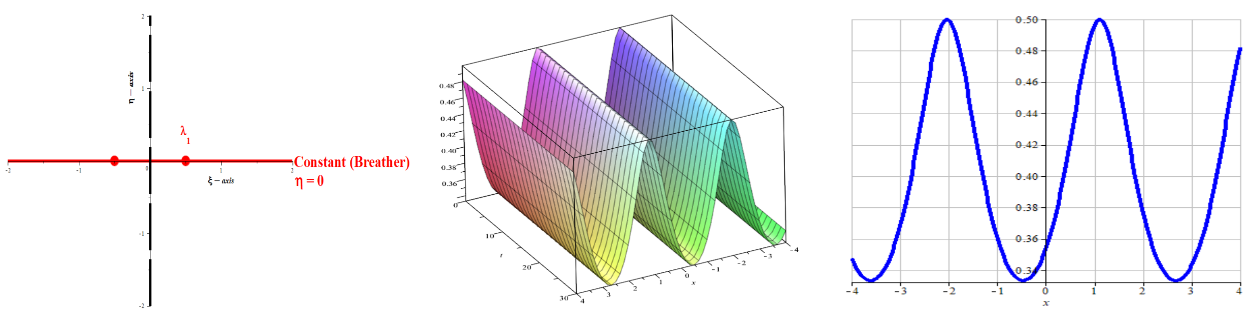

Figure 7.: Spectral plane along with 3D plot and 2D plots of $|p_1|$ of the one-soliton with parameter values $\rho_1 = 1$, $\rho_2 = 1$, $\rho_3 = 1$, $\lambda_1 = 0.5$, $\alpha_1 = -1$, $\alpha_2 = 1$, $\beta_1 = -1$, $\beta_2 = 1$, $\rho = 1$, $w_1 = (1, i, 2+i, 1)^T$. The 2D plot is for any time value.

4.3.2 Two-soliton solution

A general explicit two-soliton solution in the reverse-time case when $N = 2$, $w_1 = (w_{11}, w_{12}, w_{13}, w_{14})^T$, $w_2 = (w_{21}, w_{22}, w_{23}, w_{24})^T$, $(\lambda_1, \lambda_2) \in \mathbb{C}^2$ are arbitrary, and $\hat{\lambda}_1 = -\lambda_1$, $\hat{\lambda}_2 = -\lambda_2$, is given if $\lambda_1 \neq -\lambda_2$ by

$$p_1(x,t) = 2\rho_2\rho_3(\lambda_1+\lambda_2)(\alpha_1-\alpha_2)\frac{A(x,t)}{B(x,t)}, \tag{4.57}$$

$$p_2(x,t) = 2\rho_1\rho_3(\lambda_1+\lambda_2)(\alpha_1-\alpha_2)\frac{C(x,t)}{B(x,t)}, \tag{4.58}$$

$$p_3(x,t) = 2\rho_1\rho_2(\lambda_1+\lambda_2)(\alpha_1-\alpha_2)\frac{D(x,t)}{B(x,t)}, \tag{4.59}$$

where

$$\begin{aligned} A(x,t) = e^{i[\lambda_2^4(\beta_1-\beta_2)t+\lambda_2(\alpha_1+\alpha_2)x]}\cdot &\Bigg[\bigg(w_{22}M(\lambda_1+\lambda_2)-2w_{12}K\lambda_1\bigg)w_{21}\lambda_2 e^{2\alpha_2\lambda_1 x} \\ &- \rho_1\rho_2\rho_3(\lambda_1-\lambda_2)w_{11}^2 w_{21}w_{22}\lambda_2 e^{2\alpha_1\lambda_1 x}\Bigg] \\ +e^{i[\lambda_1^4(\beta_1-\beta_2)t+\lambda_1(\alpha_1+\alpha_2)x]}\cdot &\Bigg[\bigg(w_{12}N(\lambda_1+\lambda_2)-2w_{22}K\lambda_2\bigg)w_{11}\lambda_1 e^{2\alpha_2\lambda_2 x} \\ &+ \rho_1\rho_2\rho_3(\lambda_1-\lambda_2)w_{11}w_{12}w_{21}^2\lambda_1 e^{2\alpha_1\lambda_2 x}\Bigg], \end{aligned}$$

$$\begin{aligned} B(x,t) = &-4\rho_1\rho_2\rho_3\lambda_1\lambda_2 w_{11}w_{21}Ke^{i(\lambda_1+\lambda_2)(\alpha_1+\alpha_2)x}\cdot\Big[e^{i(\lambda_1^4-\lambda_2^4)(\beta_1-\beta_2)t}+e^{-i(\lambda_1^4-\lambda_2^4)(\beta_1-\beta_2)t}\Big] \\ &+\rho_1\rho_2\rho_3 w_{21}^2 M(\lambda_1+\lambda_2)^2 e^{i2(\alpha_1\lambda_2+\alpha_2\lambda_1)x}+\rho_1\rho_2\rho_3 w_{11}^2 N(\lambda_1+\lambda_2)^2 e^{i2(\alpha_1\lambda_1+\alpha_2\lambda_2)x} \\ &+\rho_1^2\rho_2^2\rho_3^2 w_{11}^2 w_{21}^2(\lambda_1-\lambda_2)^2 e^{i2\alpha_1(\lambda_1+\lambda_2)x}+\Big[(\lambda_1^2+\lambda_2^2)MN+(2MN-4K^2)\lambda_1\lambda_2\Big]e^{i2\alpha_2(\lambda_1+\lambda_2)x}, \end{aligned}$$

$$\begin{aligned} C(x,t) = e^{i[\lambda_2^4(\beta_1-\beta_2)t+\lambda_2(\alpha_1+\alpha_2)x]}\cdot &\Bigg[\bigg(w_{23}M(\lambda_1+\lambda_2)-2w_{13}K\lambda_1\bigg)w_{21}\lambda_2 e^{2\alpha_2\lambda_1 x} \\ &- \rho_1\rho_2\rho_3(\lambda_1-\lambda_2)w_{11}^2 w_{21}w_{23}\lambda_2 e^{2\alpha_1\lambda_1 x}\Bigg] \\ +e^{i[\lambda_1^4(\beta_1-\beta_2)t+\lambda_1(\alpha_1+\alpha_2)x]}\cdot &\Bigg[\bigg(w_{13}N(\lambda_1+\lambda_2)-2w_{23}K\lambda_2\bigg)w_{11}\lambda_1 e^{2\alpha_2\lambda_2 x} \\ &+ \rho_1\rho_2\rho_3(\lambda_1-\lambda_2)w_{11}w_{13}w_{21}^2\lambda_1 e^{2\alpha_1\lambda_2 x}\Bigg], \end{aligned}$$

$$
\begin{aligned}
D(x,t) = e^{i[\lambda_2^4(\beta_1-\beta_2)t+\lambda_2(\alpha_1+\alpha_2)x]}.&\Bigg[\Bigg(w_{24}M(\lambda_1+\lambda_2)-2w_{14}K\lambda_1\Bigg)w_{21}\lambda_2 e^{2\alpha_2\lambda_1 x} \\
&-\rho_1\rho_2\rho_3(\lambda_1-\lambda_2)w_{11}^2 w_{21}w_{24}\lambda_2 e^{2\alpha_1\lambda_1 x}\Bigg] \\
+e^{i[\lambda_1^4(\beta_1-\beta_2)t+\lambda_1(\alpha_1+\alpha_2)x]}.&\Bigg[\Bigg(w_{14}N(\lambda_1+\lambda_2)-2w_{24}K\lambda_2\Bigg)w_{11}\lambda_1 e^{2\alpha_2\lambda_2 x} \\
&+\rho_1\rho_2\rho_3(\lambda_1-\lambda_2)w_{11}w_{14}w_{21}^2\lambda_1 e^{2\alpha_1\lambda_2 x}\Bigg],
\end{aligned}
$$

and $M = \rho_2\rho_3 w_{12}^2 + \rho_1\rho_3 w_{13}^2 + \rho_1\rho_2 w_{14}^2$, $N = \rho_2\rho_3 w_{22}^2 + \rho_1\rho_3 w_{23}^2 + \rho_1\rho_2 w_{24}^2$ and $K = \rho_2\rho_3 w_{12}w_{22} + \rho_1\rho_3 w_{13}w_{23} + \rho_1\rho_2 w_{14}w_{24}$. About the dynamics of two-soliton solutions, a lot of phenomena could occur. Two solitons can travel in the same direction [46] or in opposite directions. In the present nonlocal case, either the two solitons move (repeatedly or not) in opposite directions or one moves while the other stays stationary or both are stationary.

Now, since $\lambda_1 \neq -\lambda_2$, let $\lambda_1 = \xi + i\eta$ and $\lambda_2 = \xi' + i\eta'$. If $\lambda_1 = \pm a \pm ib$, $\lambda_2 = \pm a \pm ib$ where $a \neq b$ and $a \neq 0, b \neq 0$ that means both λ_1 and λ_2 are symmetric with respect to the real axis or the imaginary axis, then the two solitons will be collapsing repeatedly or non-collapsing while moving in opposite directions. Each keeping the same amplitude before and after interaction (see Figure 8), or both keep their amplitude before interaction, but the amplitude changes after the collision to a new constant amplitude (as illustrated in Figure 9), depending on the choice of w_1,w_2.

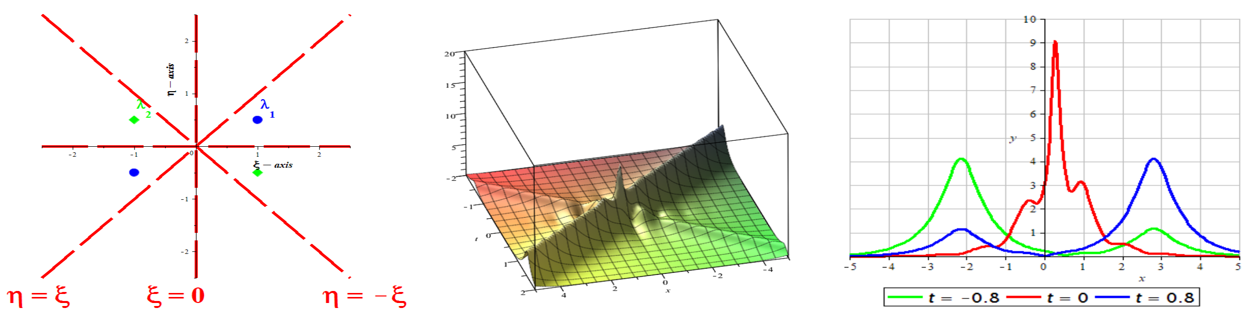

Figure 8.: Spectral plane along with 3D plot and 2D plots of $|p_1|$ of the two travelling waves in the focussing case with parameter values $\rho_1 = -1$, $\rho_2 = -2$, $\rho_3 = -3$, $\lambda_1 = 1+0.5i$, $\lambda_2 = -1+0.5i$, $\alpha_1 = -2$, $\alpha_2 = 1$, $\beta_1 = -2$, $\beta_2 = 1$, $w_1 = (1-0.5i, 1+3i, -i, 1+i)^T$ and $w_2 = (-1+2i, 1-1.5i, i, 1-i)^T$. The 2D plot is for time values $t = -0.8, 0, 0.8$.

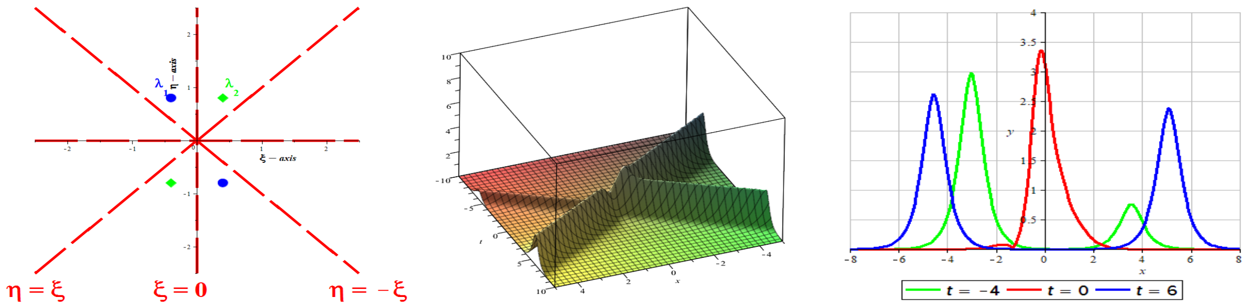

Figure 9.: Spectral plane along with 3D plot and 2D plots of $|p_1|$ of the two-soliton with parameter values $\rho_1 = -1$, $\rho_2 = 1$, $\rho_3 = -1$, $\lambda_1 = -0.4 + 0.8i$, $\lambda_2 = 0.4 + 0.8i$, $\alpha_1 = -2$, $\alpha_2 = 1$, $\beta_1 = -2$, $\beta_2 = 1$, $w_1 = (1, 1 - i, -0.1 + i, 1 + i)^T$ and $w_2 = (-1 + 2i, 1 - 0.1i, 3 + i, 0)^T$. The 2D plot is for time values $t = -4, 0, 6$.

We may have the case of two soliton waves moving in opposite directions, and after interaction they get embedded into a single wave (figure 10). Also, we can have the case where one soliton unfolds to two soliton waves [21] (figure 11). The choice of those eigenvalues may be helpful in explaining some physical phenomena [22].

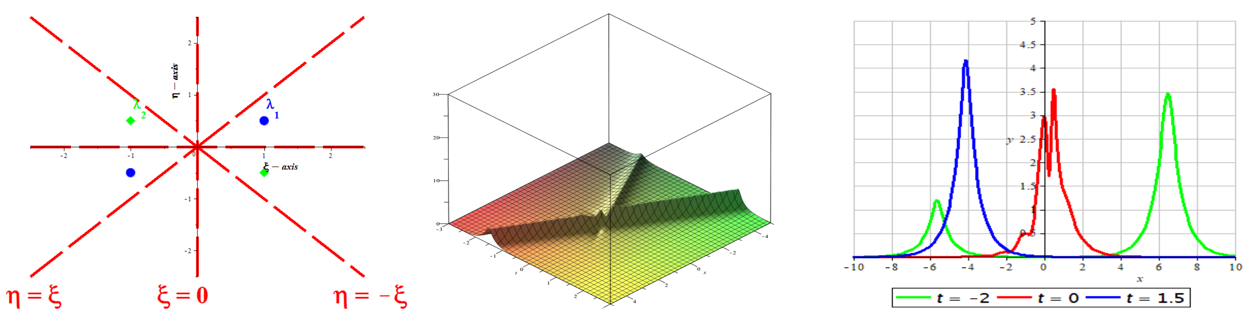

Figure 10.: Spectral plane along with 3D plot and 2D plots of $|p_1|$ of the two-soliton with parameter values $\rho_1 = -1$, $\rho_2 = -1$, $\rho_3 = -1$, $\lambda_1 = 1 + 0.5i$, $\lambda_2 = -1 + 0.5i$, $\alpha_1 = -2$, $\alpha_2 = 1$, $\beta_1 = -2$, $\beta_2 = 1$, $w_1 = (1, 0, -0.1 + i, 1 + i)^T$ and $w_2 = (-1, 1 - 2i, 3 + i, 0)^T$. The 2D plot is for time values $t = -2, 0, 1.5$.

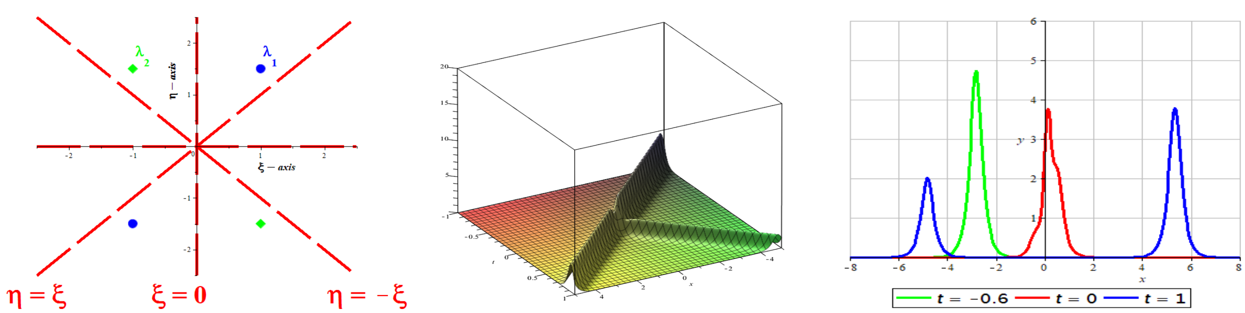

Figure 11.: Spectral plane along with 3D plot and 2D plots of $|p_1|$ of the two-soliton with parameter values $\rho_1 = 1$, $\rho_2 = 1$, $\rho_3 = 1$, $\lambda_1 = 1 + 1.5i$, $\lambda_2 = -1 + 1.5i$, $\alpha_1 = -2$, $\alpha_2 = 1$, $\beta_1 = -2$, $\beta_2 = 1$, $w_1 = (1, 0, -2 - i, 1 - i)^T$ and $w_2 = (-1, 1 - 2i, 3 - i, 0)^T$. The 2D plot is for time values $t = -0.6, 0, 1$.

Remark 4.3.1. We notice that figures (9), (10) and (11) resemble the collision of two Manakov solitons [9, 23].

If $b = a$, we have $\lambda_1 = \pm a \pm ia$, $\lambda_2 = \pm a \pm ia$, still λ_1, λ_2 are symmetric with respect to the real axis or imaginary axis and they lie on the bisectors, then the two solitons will be stationary and will have constant amplitudes. For the other choices of λ_1, λ_2, if both lie anywhere on the real axis, imaginary axis or the bisectors where both are not real, that means

$$\begin{cases} \lambda_1 = \pm a + ia, \lambda_2 = \pm b + ib, & \text{i.e., both lie on the same bisector or each on different bisector,} \\ \lambda_1 = \pm a + ia, \lambda_2 = ib, & \text{i.e., one lies on a bisector and the other one on the imaginary axis,} \\ \lambda_1 = \pm a + ia, \lambda_2 = b, & \text{i.e., one lies on a bisector and the other one on the real axis,} \\ \lambda_1 = ia, \lambda_2 = ib, & \text{i.e., both lie on the imaginary axis,} \\ \lambda_1 = a, \lambda_2 = ib, & \text{i.e., one on the real axis while the other lies on the imaginary axis,} \end{cases}$$

then the two solitons could be non-collapsing or collapsing repeatedly and/or periodically creating a standing state wave (as shown in Figure 12). Whereas if both λ_1, λ_2 are real, i.e., $\lambda_1 = a, \lambda_2 = b$, we have two breather periodic waves with period $\frac{2\pi}{(\lambda_1^4 - \lambda_2^4)(\beta_1 - \beta_2)}$ in a standing state (see Figure 13).

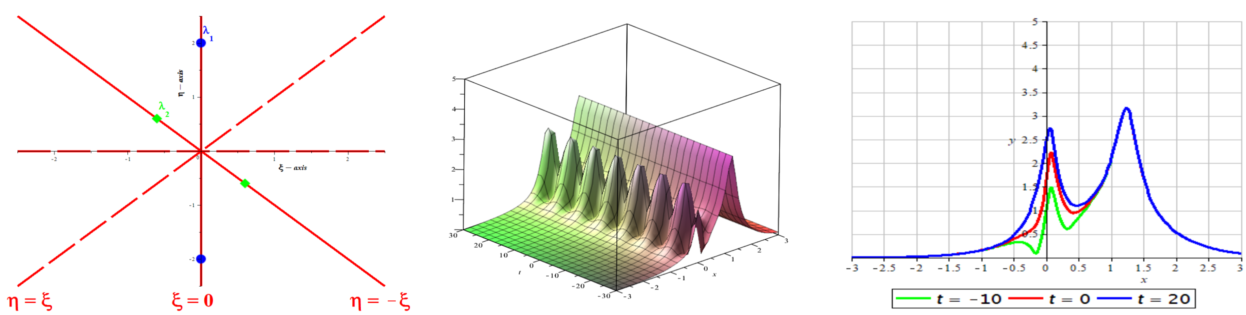

Figure 12.: Spectral plane along with 3D plot and 2D plots of $|p_1|$ of the two-soliton with parameter values $\rho_1 = -1$, $\rho_2 = 1$, $\rho_3 = -1$, $\lambda_1 = 2i$, $\lambda_2 = -0.6 + 0.6i$, $\alpha_1 = -2$, $\alpha_2 = 1$, $\beta_1 = -2$, $\beta_2 = 1$, $w_1 = (1, 0, -0.1 + i, 1 + i)^T$ and $w_2 = (-1, 1 - 2i, 3 + i, 0)^T$. The 2D plot is for time values $t = -10, 0, 20$.

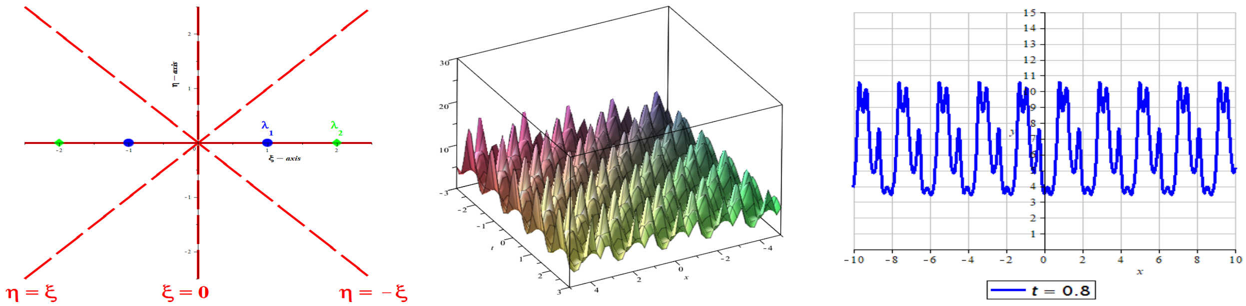

Figure 13.: Spectral plane along with 3D plot and 2D plots of $|p_1|$ of the two-soliton with parameter values $\rho_1 = -1$, $\rho_2 = -1$, $\rho_3 = -1$, $\lambda_1 = 1$, $\lambda_2 = 2$, $\alpha_1 = -2$, $\alpha_2 = 1$, $\beta_1 = -2$, $\beta_2 = 1$, $w_1 = (1, 0, 2+i, 1-i)^T$ and $w_2 = (-1, 1-2i, -i, 0)^T$. The 2D plot is for time values $t = 0.8$.

Remark 4.3.2. If $\lambda_1 \neq -\lambda_2$ and λ_1, λ_2 are symmetric about the real axis, the imaginary axis or the bisectors or also if each lies anywhere on the real axis, the imaginary axis or the bisectors, then $Im(\lambda_1^4 + \lambda_2^4) = 0$.

Remark 4.3.3. If $Im(\lambda_1^4 + \lambda_2^4) = 0$, and $|\lambda_1|^4 = |\lambda_2|^4$ then λ_1, λ_2 are symmetric about the real axis, the imaginary axis or the bisectors.
If $Im(\lambda_1^4) = 0$ and $Im(\lambda_2^4) = 0$, then this means that each of λ_1 and λ_2 lies on one of the real axis, the imaginary axis or the bisectors.

Remark 4.3.4. If λ_1, λ_2 are symmetric about the bisectors, then the dynamics of the two solitons is different from when the two eigenvalues are symmetric about the η-axis (or ξ-axis).

We can notice that any λ_1 and λ_2 satisfying any condition mentioned previously will satisfy $Im(\lambda_1^4 + \lambda_2^4) = 0$ as well. If λ_1 and λ_2 are symmetric with respect to the bisectors, i.e., $\lambda_1 = a + bi$, $\lambda_2 = b + ai$ or $\lambda_1 = a + bi$, $\lambda_2 = -b - ai$, then they still satisfy $Im(\lambda_1^4 + \lambda_2^4) = 0$, but the dynamics of the two solitons will be different from what was discussed previously.
Now, if λ_1 and λ_2 do not satisfy any of the above conditions, then the two solitons move in opposite directions and could collapse repeatedly, where they will be decreasing or increasing over the time (see Figure 14).

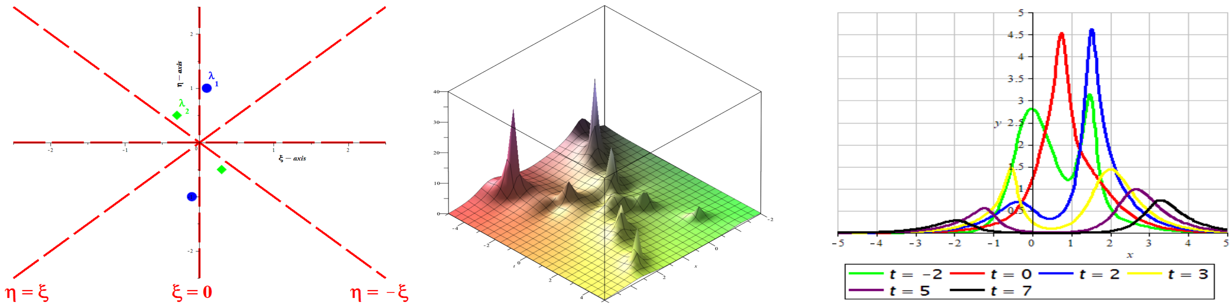

Figure 14.: Spectral plane along with 3D plot and 2D plots of $|p_1|$ of the two-soliton with parameter values $\rho_1 = -1$, $\rho_2 = -1$, $\rho_3 = -1$, $\lambda_1 = 0.1 + i$, $\lambda_2 = -0.3 + 0.5i$, $\alpha_1 = -2$, $\alpha_2 = 1$, $\beta_1 = -2$, $\beta_2 = 1$, $w_1 = (1+i, 1-3i, 2+i, 1-i)^T$ and $w_2 = (-1+i, 1+3i, 2-i, 1+i)^T$. The 2D plot is for time values $t = -2, 0, 2, 3, 5, 7$.

4.3.3 Three-soliton solution

The 3-soliton solution is given, for which $N = 3$, $w_1 = (w_{11}, w_{12}, w_{13}, w_{14})^T$, $w_2 = (w_{21}, w_{22}, w_{23}, w_{24})^T$, $w_3 = (w_{31}, w_{32}, w_{33}, w_{34})^T$, $(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{C}^3$, and $\hat{\lambda}_1 = -\lambda_1$, $\hat{\lambda}_2 = -\lambda_2$, $\hat{\lambda}_3 = -\lambda_3$, by

$$p_1 = \alpha \sum_{k,j=1}^{3} v_{k1} (M^{-1})_{kj} \hat{v}_{j,2}, \tag{4.60}$$

$$p_2 = \alpha \sum_{k,j=1}^{3} v_{k1} (M^{-1})_{kj} \hat{v}_{j,3}, \tag{4.61}$$

$$p_3 = \alpha \sum_{k,j=1}^{3} v_{k1} (M^{-1})_{kj} \hat{v}_{j,4}. \tag{4.62}$$

For the 3-solitons, if $\lambda_i = -\lambda_j$ for $i \neq j$, $i, j \in \{1, 2, 3\}$, then we have the 1-soliton dynamics. Also, if two of $\{\lambda_1, \lambda_2, \lambda_3\}$ are equal, then we have the dynamics of the 2-solitons.

Let $\lambda_1 = \xi + i\eta$, $\lambda_2 = \xi' + i\eta'$ and $\lambda_3 = \xi'' + i\eta''$. If two of the eigenvalues are symmetric about the real axis, the imaginary axis and the other eigenvalue lie on the real axis, the imaginary axis or on the bisectors then we have two solitons collapsing repeatedly or non-collapsing moving in opposite directions, while the third one stays stationary. Either each keeps the same amplitude before and after interaction (see Figure 15), or they keep their amplitudes before interaction, but their amplitudes change at the collision moment to new constant amplitudes (illustration in Figure 16), depending on the choice of w_1,w_2.

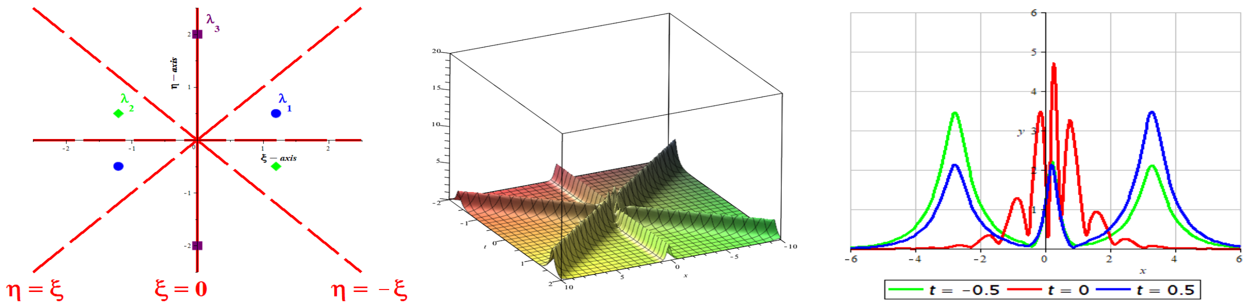

Figure 15.: Spectral plane along with 3D plot and 2D plots of $|p_1|$ for two travelling waves and a constant-amplitude stationary wave with parameter values $\rho_1 = 1$, $\rho_2 = 1$, $\rho_3 = 1$ $\lambda_1 = 1.2+0.5i$, $\lambda_2 = -1.2+0.5i$, $\lambda_3 = 2i$, $\alpha_1 = -2$, $\alpha_2 = 1$, $\beta_1 = -2$, $\beta_2 = 1$, $w_1 = (-1.5+2i, 2-3i, i, 1-i)^T$ and $w_2 = (3+2i, -1+3i, -i, 1+i)^T$, $w_3 = (1, 1, 2, 1)^T$. The 2D plot is for time values, t=-0.5,0,0.5.

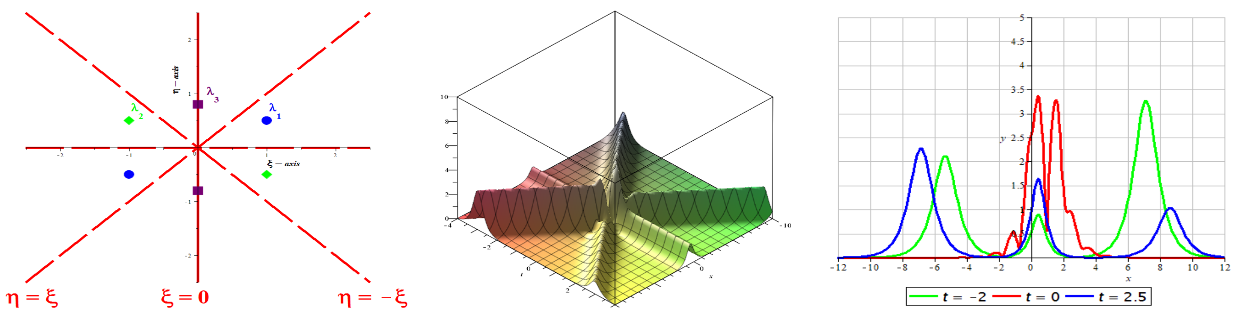

Figure 16.: Spectral plane along with 3D plot and 2D plots of $|p_1|$ of the three-soliton with parameter values $\rho_1 = -0.5$, $\rho_2 = -0.5$, $\rho_3 = -0.5$, $\lambda_1 = 1+0.5i$, $\lambda_2 = -1+0.5i$, $\lambda_3 = 0.8i$, $\alpha_1 = -2$, $\alpha_2 = 1$, $\beta_1 = -2$, $\beta_2 = 1$, $w_1 = (1, i, 3+i, 1-i)^T$, $w_2 = (-1, 1-3i, -i, 0)^T$, $w_3 = (2+i, 1+2i, 1, 2i)^T$. The 2D plot is for time values, t=-2,0,2.5.

We can also have two other different cases of interaction. The first case is where the 3-soliton after interaction are embedded into 2-soliton (figure 17). The second case happens when the two solitons after interaction unfold to 3-soliton (figure 18). As said before, those phenomena may be relevant to some nonlinear problems in applied physics.

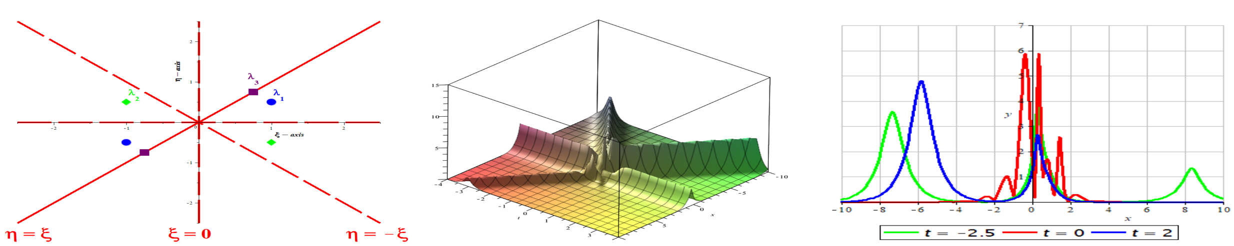

Figure 17.: Spectral plane along with 3D plot and 2D plots of $|p_1|$ of the three-soliton with parameter values $\rho_1 = 1$, $\rho_2 = 1$, $\rho_3 = 1$, $\lambda_1 = 1+0.5i$, $\lambda_2 = -1+0.5i$, $\lambda_3 = 0.75+0.75i$, $\alpha_1 = -2$, $\alpha_2 = 1$, $\beta_1 = -2$, $\beta_2 = 1$, $w_1 = (1, 0, 2+i, 1-i)^T$, $w_2 = (-1, 1-2i, -i, 0)^T$, $w_3 = (2+i, 1+2i, 1, 2i)^T$. The 2D plot is for time values, t=-2.5,0,2.

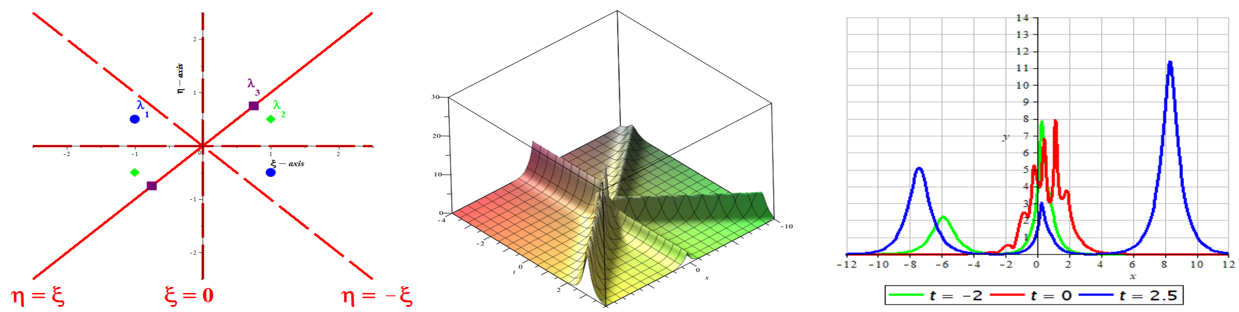

Figure 18.: Spectral plane along with 3D plot and 2D plots of $|p_1|$ of the three-soliton with parameter values $\rho_1 = 1$, $\rho_2 = 1$, $\rho_3 = 1$, $\lambda_1 = -1 + 0.5i$, $\lambda_2 = 1 + 0.5i$, $\lambda_3 = 0.75 + 0.75i$, $\alpha_1 = -2$, $\alpha_2 = 1$, $\beta_1 = -2$, $\beta_2 = 1$, $w_1 = (1, 0, 2 + i, 1 - i)^T$, $w_2 = (-1, 1 - 2i, -i, 0)^T$, $w_3 = (2 + i, 1 + 2i, 1, 2i)^T$. The 2D plot is for time values, t=-2,0,2.5.

If λ_1, λ_2, λ_3 are all real, then we have breather solitons as shown in figure 19. Otherwise, if λ_1, λ_2, λ_3 are not all real lying on the real axis, the imaginary axis or the bisectors, we will have 3 solitons collapsing repeatedly in a standing state.

If two of the λ_1, λ_2, λ_3 are symmetric (but not real) about the η-axis (or ξ-axis) and the third one lies off of the real axis, the imaginary axis, and the bisectors, then we can have three solitons interacting, with 2 solitons moving in opposite directions with constant amplitudes. After collision, their amplitudes change, but still stay constant, while the third soliton is stationary and its amplitude is either increasing or decreasing as shown in figure 20.

If λ_1, λ_2, λ_3 all lie off the real axis, the imaginary axis, and the bisectors, or one of them is real or two of them are real, then we have two solitons that could repeatedly collapse or non-collapse decreasingly or increasingly in their motion while the third one stays stationary, as in Figure 21.

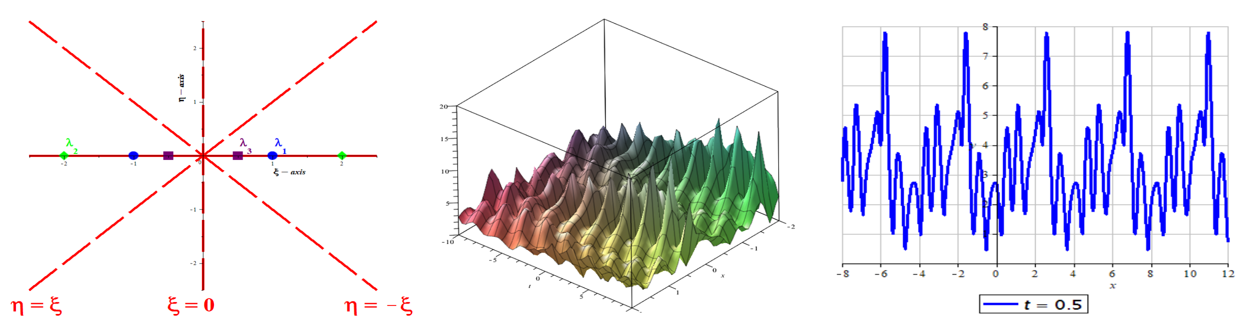

Figure 19.: Spectral plane along with 3D plot and 2D plots of $|p_1|$ of the three-soliton with parameter values $\rho_1 = 1$, $\rho_2 = 1$, $\rho_3 = 1$, $\lambda_1 = 1$, $\lambda_2 = -2$, $\lambda_3 = 0.5$, $\alpha_1 = -2$, $\alpha_2 = 1$, $\beta_1 = -2$, $\beta_2 = 1$, $w_1 = (1, 0, 2 + i, 1 - i)^T$, $w_2 = (-1, 1 - 2i, -i, 0)^T$, $w_3 = (1 + i, 1 + 2i, 0, 2i)^T$. The 2D plot is for time values, t=0.5.

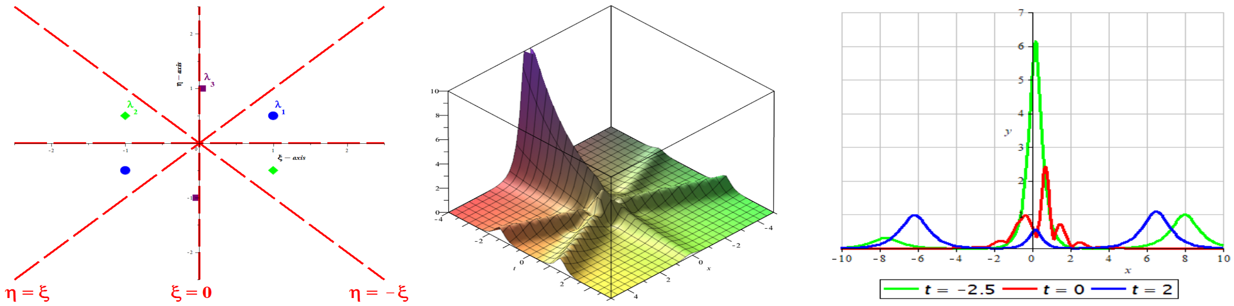

Figure 20.: Spectral plane along with 3D plot and 2D plots of $|p_1|$ of the three-soliton with parameter values $\rho_1 = 2$, $\rho_2 = 2$, $\rho_3 = 2$, $\lambda_1 = 1 + 0.5i$, $\lambda_2 = -1 + 0.5i$, $\lambda_3 = 0.04 + i$, $\alpha_1 = -2$, $\alpha_2 = 1$, $\beta_1 = -2$, $\beta_2 = 1$, $w_1 = (1 - 2i, 1 + 3i, -i, 1 + i)^T$, $w_2 = (-1 + 2i, 1 - 3i, i, 1 - i)^T$, $w_3 = (1 + i, 1 + 2i, 0, 2i)^T$. The 2D plot is for time values, t=-2.5,0,2.

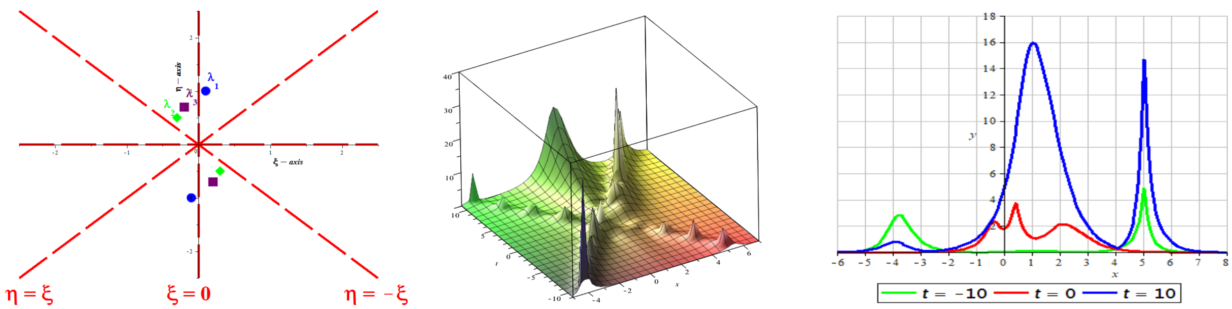

Figure 21.: Spectral plane along with 3D plot and 2D plots of $|p_1|$ of the three-soliton with parameter values $\rho_1 = -1$, $\rho_2 = -1$, $\rho_3 = -1$, $\lambda_1 = 0.1 + i$, $\lambda_2 = -0.3 + 0.5i$, $\lambda_3 = -0.2 + 0.7i$, $\alpha_1 = -2$, $\alpha_2 = 1$, $\beta_1 = -2$, $\beta_2 = 1$, $w_1 = (2, 2i, 2 + i, 1 - i)^T$, $w_2 = (-1, 1 - 2i, -i, 0)^T$, $w_3 = (-1 + i, -1 + 2i, 0, -2i)^T$. The 2D plot is for time values, t=-10,0,10.

Chapter 5

Inverse scattering of a nonlocal reverse-time nonlinear Schrödinger-type equation based on Riemann-Hilbert problems

5.1 Eight-component AKNS hierarchy of coupled second-order integrable equations

We start from the spatial spectral problem of fourth-order

$$-i\psi_x = U(u,\lambda)\psi, \tag{5.1}$$

where

$$U(u,\lambda) = \left(\begin{array}{cc|cc} \alpha_1\lambda & 0 & p_1 & p_2 \\ 0 & \alpha_1\lambda & p_3 & p_4 \\ \hline r_1 & r_2 & \alpha_2\lambda & 0 \\ r_3 & r_4 & 0 & \alpha_2\lambda \end{array}\right), \tag{5.2}$$

and $u = (p, r^T)^T$, $p = (p_1, p_2, p_3, p_4)$, $r = (r_1, r_2, r_3, r_4)^T$, where the p_i, r_i are potentials and α_1, α_2 are real, and λ is a spectral parameter, and $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)^T$. We need to find the temporal Lax matrix and the associated multi-component integrable system.

Let's solve the stationary zero curvature equation (3.3), where

$$W = \left(\begin{array}{cc|cc} a_{11} & a_{12} & b_1 & b_2 \\ a_{21} & a_{22} & b_3 & b_4 \\ \hline c_1 & c_2 & d_{11} & d_{12} \\ c_3 & c_4 & d_{21} & d_{22} \end{array}\right). \tag{5.3}$$

Solving it, we obtain

$$\begin{cases} a_{11,x} &= i(-b_1r_1 - b_2r_3 + c_1p_1 + c_3p_2), \\ a_{12,x} &= i(-b_1r_2 - b_2r_4 + c_2p_1 + c_4p_2), \\ a_{21,x} &= i(-b_3r_1 - b_4r_3 + c_1p_3 + c_3p_4), \\ a_{22,x} &= i(-b_3r_2 - b_4r_4 + c_2p_3 + c_4p_4), \end{cases} \tag{5.4}$$

$$\begin{cases} b_{1,x} = i(\lambda\alpha b_1 - a_{11}p_1 - a_{12}p_3 + d_{11}p_1 + d_{21}p_2), \\ b_{2,x} = i(\lambda\alpha b_2 - a_{11}p_2 - a_{12}p_4 + d_{12}p_1 + d_{22}p_2), \\ b_{3,x} = i(\lambda\alpha b_3 - a_{21}p_1 - a_{22}p_3 + d_{11}p_3 + d_{21}p_4), \\ b_{4,x} = i(\lambda\alpha b_4 - a_{21}p_2 - a_{22}p_4 + d_{12}p_3 + d_{22}p_4), \end{cases} \tag{5.5}$$

$$\begin{cases} c_{1,x} = i(-\lambda\alpha c_1 + a_{11}r_1 + a_{21}r_2 - d_{11}r_1 - d_{12}r_3), \\ c_{2,x} = i(-\lambda\alpha c_2 + a_{12}r_1 + a_{22}r_2 - d_{11}r_2 - d_{12}r_4), \\ c_{3,x} = i(-\lambda\alpha c_3 + a_{11}r_3 + a_{21}r_4 - d_{21}r_1 - d_{22}r_3), \\ c_{4,x} = i(-\lambda\alpha c_4 + a_{12}r_3 + a_{22}r_4 - d_{21}r_2 - d_{22}r_4), \end{cases} \tag{5.6}$$

$$\begin{cases} d_{11,x} = i(b_1r_1 + b_3r_2 - c_1p_1 - c_2p_3), \\ d_{12,x} = i(b_2r_1 + b_4r_2 - c_1p_2 - c_2p_4), \\ d_{21,x} = i(b_1r_3 + b_3r_4 - c_3p_1 - c_4p_3), \\ d_{22,x} = i(b_2r_3 + b_4r_4 - c_3p_2 - c_4p_4). \end{cases} \tag{5.7}$$

By expanding W in Laurent series form,

$$W = \sum_{j=0}^{\infty} W_m \lambda^{-m} \quad \text{with} \quad W_m = \left(\begin{array}{cc|cc} a_{11}^{[m]} & a_{12}^{[m]} & b_1^{[m]} & b_2^{[m]} \\ a_{21}^{[m]} & a_{22}^{[m]} & b_3^{[m]} & b_4^{[m]} \\ \hline c_1^{[m]} & c_2^{[m]} & d_{11}^{[m]} & d_{12}^{[m]} \\ c_3^{[m]} & c_4^{[m]} & d_{21}^{[m]} & d_{22}^{[m]} \end{array}\right), \tag{5.8}$$

the system $(5.4)-(5.7)$ is equivalent to the recursion relations:

$$\begin{cases} b_1^{[0]}=b_2^{[0]}=b_3^{[0]}=b_4^{[0]}=0, \\ c_1^{[0]}=c_2^{[0]}=c_3^{[0]}=c_4^{[0]}=0, \\ a_{11,x}^{[0]}=a_{12,x}^{[0]}=a_{21,x}^{[0]}=a_{22,x}^{[0]}=0, \\ d_{11,x}^{[0]}=d_{12,x}^{[0]}=d_{21,x}^{[0]}=d_{22,x}^{[0]}=0, \end{cases} \tag{5.9}$$

$$\begin{cases} b_1^{[m+1]}=\frac{1}{\alpha}(-ib_{1,x}^{[m]}+a_{11}^{[m]}p_1+a_{12}^{[m]}p_3-d_{11}^{[m]}p_1-d_{21}^{[m]}p_2), \\ b_2^{[m+1]}=\frac{1}{\alpha}(-ib_{2,x}^{[m]}+a_{11}^{[m]}p_2+a_{12}^{[m]}p_4-d_{12}^{[m]}p_1-d_{22}^{[m]}p_2), \\ b_3^{[m+1]}=\frac{1}{\alpha}(-ib_{3,x}^{[m]}+a_{21}^{[m]}p_1+a_{22}^{[m]}p_3-d_{11}^{[m]}p_3-d_{21}^{[m]}p_4), \\ b_4^{[m+1]}=\frac{1}{\alpha}(-ib_{4,x}^{[m]}+a_{21}^{[m]}p_2+a_{22}^{[m]}p_4-d_{12}^{[m]}p_3-d_{22}^{[m]}p_4), \end{cases} \tag{5.10}$$

$$\begin{cases} c_1^{[m+1]}=\frac{1}{\alpha}(ic_{1,x}^{[m]}+a_{11}^{[m]}r_1+a_{21}^{[m]}r_2-d_{11}^{[m]}r_1-d_{12}^{[m]}r_3), \\ c_2^{[m+1]}=\frac{1}{\alpha}(ic_{2,x}^{[m]}+a_{12}^{[m]}r_1+a_{22}^{[m]}r_2-d_{11}^{[m]}r_2-d_{12}^{[m]}r_4), \\ c_3^{[m+1]}=\frac{1}{\alpha}(ic_{3,x}^{[m]}+a_{11}^{[m]}r_3+a_{21}^{[m]}r_4-d_{21}^{[m]}r_1-d_{22}^{[m]}r_3), \\ c_4^{[m+1]}=\frac{1}{\alpha}(ic_{4,x}^{[m]}+a_{12}^{[m]}r_3+a_{22}^{[m]}r_4-d_{21}^{[m]}r_2-d_{22}^{[m]}r_4), \end{cases} \tag{5.11}$$

$$\begin{cases} a_{11,x}^{[m]} &= i(-b_1^{[m]}r_1-b_2^{[m]}r_3+c_1^{[m]}p_1+c_3^{[m]}p_2), \\ a_{12,x}^{[m]} &= i(-b_1^{[m]}r_2-b_2^{[m]}r_4+c_2^{[m]}p_1+c_4^{[m]}p_2), \\ a_{21,x}^{[m]} &= i(-b_3^{[m]}r_1-b_4^{[m]}r_3+c_1^{[m]}p_3+c_3^{[m]}p_4), \\ a_{22,x}^{[m]} &= i(-b_3^{[m]}r_2-b_4^{[m]}r_4+c_2^{[m]}p_3+c_4^{[m]}p_4), \quad \text{for} \quad m\in\{1,2,3\}, \end{cases} \tag{5.12}$$

$$
\begin{cases}
d_{11,x}^{[m]} = i(b_1^{[m]} r_1 + b_3^{[m]} r_2 - c_1^{[m]} p_1 - c_2^{[m]} p_3), \\
d_{12,x}^{[m]} = i(b_2^{[m]} r_1 + b_4^{[m]} r_2 - c_1^{[m]} p_2 - c_2^{[m]} p_4), \\
d_{21,x}^{[m]} = i(b_1^{[m]} r_3 + b_3^{[m]} r_4 - c_3^{[m]} p_1 - c_4^{[m]} p_3), \\
d_{22,x}^{[m]} = i(b_2^{[m]} r_3 + b_4^{[m]} r_4 - c_3^{[m]} p_2 - c_4^{[m]} p_4), \quad \text{for} \quad m \in \{1,2,3\}.
\end{cases} \tag{5.13}
$$

Now by fixing the following initial values:

$$
\begin{cases}
a_{11}^{[0]} = \beta_1, a_{12}^{[0]} = 0, a_{21}^{[0]} = 0, a_{22}^{[0]} = \beta_1, \\
d_{11}^{[0]} = \beta_2, d_{12}^{[0]} = 0, d_{21}^{[0]} = 0, d_{22}^{[0]} = \beta_2,
\end{cases} \tag{5.14}
$$

where β_1, β_2 are real arbitrary constants, and taking the constant of integration in (5.12) and (5.13) to be zero, it requires that

$$
W_m|_{u=0} = 0, \quad m \in \{1,2,3\}. \tag{5.15}
$$

Upon using (5.9)–(5.13), this allows to generate

$$
\begin{cases}
b_1^{[1]} = \frac{\beta}{\alpha} p_1, \quad b_2^{[1]} = \frac{\beta}{\alpha} p_2, \quad b_3^{[1]} = \frac{\beta}{\alpha} p_3, \quad b_4^{[1]} = \frac{\beta}{\alpha} p_4, \\
c_1^{[1]} = \frac{\beta}{\alpha} r_1, \quad c_2^{[1]} = \frac{\beta}{\alpha} r_2, \quad c_3^{[1]} = \frac{\beta}{\alpha} r_3, \quad c_4^{[1]} = \frac{\beta}{\alpha} r_4, \\
a_{11}^{[1]} = 0, \quad a_{12}^{[1]} = 0, \quad a_{21}^{[1]} = 0, \quad a_{22}^{[1]} = 0, \\
d_{11}^{[1]} = 0, \quad d_{12}^{[1]} = 0, \quad d_{21}^{[1]} = 0, \quad d_{22}^{[1]} = 0,
\end{cases} \tag{5.16}
$$

$$
\begin{cases}
b_1^{[2]} = -i\frac{\beta}{\alpha^2} p_{1,x}, \quad b_2^{[2]} = -i\frac{\beta}{\alpha^2} p_{2,x}, \quad b_3^{[2]} = -i\frac{\beta}{\alpha^2} p_{3,x}, \quad b_4^{[2]} = -i\frac{\beta}{\alpha^2} p_{4,x}, \\
c_1^{[2]} = i\frac{\beta}{\alpha^2} r_{1,x}, \quad c_2^{[2]} = i\frac{\beta}{\alpha^2} r_{2,x}, \quad c_3^{[2]} = i\frac{\beta}{\alpha^2} r_{3,x}, \quad c_4^{[2]} = i\frac{\beta}{\alpha^2} r_{4,x}, \\
a_{11}^{[2]} = -\frac{\beta}{\alpha^2}(p_1 r_1 + p_2 r_3), \quad a_{12}^{[2]} = -\frac{\beta}{\alpha^2}(p_1 r_2 + p_2 r_4), \\
a_{21}^{[2]} = -\frac{\beta}{\alpha^2}(p_3 r_1 + p_4 r_3), \quad a_{22}^{[2]} = -\frac{\beta}{\alpha^2}(p_3 r_2 + p_4 r_4), \\
d_{11}^{[2]} = \frac{\beta}{\alpha^2}(p_1 r_1 + p_3 r_2), \quad d_{12}^{[2]} = \frac{\beta}{\alpha^2}(p_2 r_1 + p_4 r_2), \\
d_{21}^{[2]} = \frac{\beta}{\alpha^2}(p_1 r_3 + p_3 r_4), \quad d_{22}^{[2]} = \frac{\beta}{\alpha^2}(p_4 r_4 + p_2 r_3),
\end{cases} \tag{5.17}
$$

$$
\begin{cases}
b_1^{[3]} = -\frac{\beta}{\alpha^3} p_{1,xx} + 2(p_1^2 r_1 + p_1 p_2 r_3 + p_1 p_3 r_2 + p_2 p_3 r_4), \\
b_2^{[3]} = -\frac{\beta}{\alpha^3} p_{2,xx} + 2(p_2^2 r_3 + p_1 p_2 r_1 + p_2 p_4 r_4 + p_1 p_4 r_2), \\
b_3^{[3]} = -\frac{\beta}{\alpha^3} p_{3,xx} + 2(p_3^2 r_2 + p_1 p_3 r_1 + p_3 p_4 r_4 + p_1 p_4 r_3), \\
b_4^{[3]} = -\frac{\beta}{\alpha^3} p_{4,xx} + 2(p_4^2 r_4 + p_2 p_4 r_3 + p_3 p_4 r_2 + p_2 p_3 r_1), \\
c_1^{[3]} = -\frac{\beta}{\alpha^3} r_{1,xx} + 2(r_1^2 p_1 + r_1 r_2 p_3 + r_1 r_3 p_2 + r_2 r_3 p_4), \\
c_2^{[3]} = -\frac{\beta}{\alpha^3} r_{2,xx} + 2(r_2^2 p_3 + r_1 r_2 p_2 + r_1 r_4 p_2 + r_2 r_4 p_4), \\
c_3^{[3]} = -\frac{\beta}{\alpha^3} r_{3,xx} + 2(r_3^2 p_2 + r_1 r_3 p_1 + r_3 r_4 p_4 + r_1 r_4 p_3), \\
c_4^{[3]} = -\frac{\beta}{\alpha^3} r_{4,xx} + 2(r_4^2 p_4 + r_2 r_3 p_1 + r_2 r_4 p_3 + r_3 r_4 p_2),
\end{cases} \tag{5.18}
$$

$$
\begin{cases}
a_{11}^{[3]} = -i\frac{\beta}{\alpha^3}(p_1 r_{1,x} - p_{1,x} r_1 + p_2 r_{3,x} - p_{2,x} r_3), \\
a_{12}^{[3]} = -i\frac{\beta}{\alpha^3}(p_1 r_{2,x} - p_{1,x} r_2 + p_2 r_{4,x} - p_{2,x} r_4), \\
a_{21}^{[3]} = -i\frac{\beta}{\alpha^3}(p_3 r_{1,x} - p_{3,x} r_1 + p_4 r_{3,x} - p_{4,x} r_3), \\
a_{22}^{[3]} = -i\frac{\beta}{\alpha^3}(p_3 r_{2,x} - p_{3,x} r_2 + p_4 r_{4,x} - p_{4,x} r_4),
\end{cases} \tag{5.19}
$$

$$
\begin{cases}
d_{11}^{[3]} = -i\frac{\beta}{\alpha^3}(r_1 p_{1,x} - r_{1,x} p_1 + r_2 p_{3,x} - r_{2,x} p_3), \\
d_{12}^{[3]} = -i\frac{\beta}{\alpha^3}(r_1 p_{2,x} - r_{1,x} p_2 + r_2 p_{4,x} - r_{2,x} p_4), \\
d_{21}^{[3]} = -i\frac{\beta}{\alpha^3}(r_3 p_{1,x} - r_{3,x} p_1 + r_4 p_{3,x} - r_{4,x} p_3), \\
d_{22}^{[3]} = -i\frac{\beta}{\alpha^3}(r_3 p_{2,x} - r_{3,x} p_2 + r_4 p_{4,x} - r_{4,x} p_4),
\end{cases} \tag{5.20}
$$

where $\beta = \beta_1 - \beta_2$ and $\alpha = \alpha_1 - \alpha_2$.

By taking the modification term to be zero, the Lax matrix will be

$$
V^{[2]} = (\lambda^2 W)_{4\times 4}, \tag{5.21}
$$

and will satisfy the zero curvature equation

$$U_t - V_x^{[2]} + i[U, V^{[2]}] = 0. \tag{5.22}$$

As a result, we get the multicomponent AKNS hierarchies of integrable equations:

$$u_t = \begin{pmatrix} p^T \\ r \end{pmatrix}_t = i \begin{pmatrix} \alpha b^{[3]T} \\ -\alpha c^{[3]} \end{pmatrix}, \tag{5.23}$$

where $b^{[3]} = (b_1^{[3]}, b_2^{[3]}, b_3^{[3]}, b_4^{[3]})$ and $c^{[3]} = (c_1^{[3]}, c_2^{[3]}, c_3^{[3]}, c_4^{[3]})^T$. Thus, we derive the nonlinear system in the corresponding soliton hierarchy:

$$p_{1,t} = -i\frac{\beta}{\alpha^2}\big[p_{1,xx} + 2(p_1^2 r_1 + p_1 p_2 r_3 + p_1 p_3 r_2 + p_2 p_3 r_4)\big], \tag{5.24}$$
$$p_{2,t} = -i\frac{\beta}{\alpha^2}\big[p_{2,xx} + 2(p_2^2 r_3 + p_1 p_2 r_1 + p_2 p_4 r_4 + p_1 p_4 r_2)\big], \tag{5.25}$$
$$p_{3,t} = -i\frac{\beta}{\alpha^2}\big[p_{3,xx} + 2(p_3^2 r_2 + p_1 p_3 r_1 + p_3 p_4 r_4 + p_1 p_4 r_3)\big], \tag{5.26}$$
$$p_{4,t} = -i\frac{\beta}{\alpha^2}\big[p_{4,xx} + 2(p_4^2 r_4 + p_2 p_4 r_3 + p_3 p_4 r_2 + p_2 p_3 r_1)\big], \tag{5.27}$$

$$r_{1,t} = i\frac{\beta}{\alpha^2}\big[r_{1,xx} + 2(r_1^2 p_1 + r_1 r_2 p_3 + r_1 r_3 p_2 + r_2 r_3 p_4)\big], \tag{5.28}$$
$$r_{2,t} = i\frac{\beta}{\alpha^2}\big[r_{2,xx} + 2(r_2^2 p_3 + r_1 r_2 p_2 + r_1 r_4 p_2 + r_2 r_4 p_4)\big], \tag{5.29}$$
$$r_{3,t} = i\frac{\beta}{\alpha^2}\big[r_{3,xx} + 2(r_3^2 p_2 + r_1 r_3 p_1 + r_3 r_4 p_4 + r_1 r_4 p_3)\big], \tag{5.30}$$
$$r_{4,t} = i\frac{\beta}{\alpha^2}\big[r_{4,xx} + 2(r_4^2 p_4 + r_2 r_3 p_1 + r_2 r_4 p_3 + r_3 r_4 p_2)\big]. \tag{5.31}$$

Now one can get the Lax pair of the eight-component AKNS equations of nonlinear Schrödinger type as follows:

$$\psi_x = iU\psi = i(\lambda \Lambda + P)\psi, \tag{5.32}$$

$$\psi_t = iV^{[2]}\psi = i(\lambda^2 \Omega + Q)\psi, \tag{5.33}$$

where $\Lambda = diag(\alpha_1, \alpha_1, \alpha_2, \alpha_2)$, $\Omega = diag(\beta_1, \beta_1, \beta_2, \beta_2)$, and

$$P = \left(\begin{array}{cc|cc} 0 & 0 & p_1 & p_2 \\ 0 & 0 & p_3 & p_4 \\ \hline r_1 & r_2 & 0 & 0 \\ r_3 & r_4 & 0 & 0 \end{array}\right), \tag{5.34}$$

$$Q = \begin{pmatrix} a_{11}^{[1]}\lambda + a_{11}^{[2]} & \begin{matrix} a_{12}^{[0]}\lambda^2 + a_{12}^{[1]}\lambda \\ +a_{12}^{[2]} \end{matrix} & \begin{matrix} b_1^{[0]}\lambda^2 + b_1^{[1]}\lambda \\ +b_1^{[2]} \end{matrix} & \begin{matrix} b_2^{[0]}\lambda^2 + b_2^{[1]}\lambda \\ +b_2^{[2]} \end{matrix} \\ \begin{matrix} a_{21}^{[0]}\lambda^2 + a_{21}^{[1]}\lambda \\ +a_{21}^{[2]} \end{matrix} & a_{22}^{[1]}\lambda + a_{22}^{[2]} & \begin{matrix} b_3^{[0]}\lambda^2 + b_3^{[1]}\lambda \\ +b_3^{[2]} \end{matrix} & \begin{matrix} b_4^{[0]}\lambda^2 + b_4^{[1]}\lambda \\ +b_4^{[2]} \end{matrix} \\ \begin{matrix} c_1^{[0]}\lambda^2 + c_1^{[1]}\lambda \\ +c_1^{[2]} \end{matrix} & \begin{matrix} c_2^{[0]}\lambda^2 + c_2^{[1]}\lambda \\ +c_2^{[2]} \end{matrix} & d_{11}^{[1]}\lambda + d_{11}^{[2]} & \begin{matrix} d_{12}^{[0]}\lambda^2 + d_{12}^{[1]}\lambda \\ +d_{12}^{[2]} \end{matrix} \\ \begin{matrix} c_3^{[0]}\lambda^2 + c_3^{[1]}\lambda \\ +c_3^{[2]} \end{matrix} & \begin{matrix} c_4^{[0]}\lambda^2 + c_4^{[1]}\lambda \\ +c_4^{[2]} \end{matrix} & \begin{matrix} d_{21}^{[0]}\lambda^2 + d_{21}^{[1]}\lambda \\ +d_{21}^{[2]} \end{matrix} & d_{22}^{[1]}\lambda + d_{22}^{[2]} \end{pmatrix}, \tag{5.35}$$

$$V^{[2]} = \begin{pmatrix} \begin{matrix} a_{11}^{[0]}\lambda^2 + a_{11}^{[1]}\lambda \\ +a_{11}^{[2]} \end{matrix} & \begin{matrix} a_{12}^{[0]}\lambda^2 + a_{12}^{[1]}\lambda \\ +a_{12}^{[2]} \end{matrix} & \begin{matrix} b_1^{[0]}\lambda^2 + b_1^{[1]}\lambda \\ +b_1^{[2]} \end{matrix} & \begin{matrix} b_2^{[0]}\lambda^2 + b_2^{[1]}\lambda \\ +b_2^{[2]} \end{matrix} \\ \begin{matrix} a_{21}^{[0]}\lambda^2 + a_{21}^{[1]}\lambda \\ +a_{21}^{[2]} \end{matrix} & \begin{matrix} a_{22}^{[0]}\lambda^2 + a_{22}^{[1]}\lambda \\ +a_{22}^{[2]} \end{matrix} & \begin{matrix} b_3^{[0]}\lambda^2 + b_3^{[1]}\lambda \\ +b_3^{[2]} \end{matrix} & \begin{matrix} b_4^{[0]}\lambda^2 + b_4^{[1]}\lambda \\ +b_4^{[2]} \end{matrix} \\ \begin{matrix} c_1^{[0]}\lambda^2 + c_1^{[1]}\lambda \\ +c_1^{[2]} \end{matrix} & \begin{matrix} c_2^{[0]}\lambda^2 + c_2^{[1]}\lambda \\ +c_2^{[2]} \end{matrix} & \begin{matrix} d_{11}^{[0]}\lambda^2 + d_{11}^{[1]}\lambda \\ +d_{11}^{[2]} \end{matrix} & \begin{matrix} d_{12}^{[0]}\lambda^2 + d_{12}^{[1]}\lambda \\ +d_{12}^{[2]} \end{matrix} \\ \begin{matrix} c_3^{[0]}\lambda^2 + c_3^{[1]}\lambda \\ +c_3^{[2]} \end{matrix} & \begin{matrix} c_4^{[0]}\lambda^2 + c_4^{[1]}\lambda \\ +c_4^{[2]} \end{matrix} & \begin{matrix} d_{21}^{[0]}\lambda^2 + d_{21}^{[1]}\lambda \\ +d_{21}^{[2]} \end{matrix} & \begin{matrix} d_{22}^{[0]}\lambda^2 + d_{22}^{[1]}\lambda \\ +d_{22}^{[2]} \end{matrix} \end{pmatrix}. \tag{5.36}$$

5.2 A specific reduction for a nonlocal reverse-time AKNS system

Consider a class of specific nonlocal reverse-time reductions for the spatial spectral matrix

$$U^T(x,-t,-\lambda) = -CU(x,t,\lambda)C^{-1}, \tag{5.37}$$

where $C = \begin{pmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{pmatrix}$, where Σ_1, Σ_2 are constant invertible symmetric 2×2 matrices, i.e., $\Sigma_i^T = \Sigma_i$ and $\det \Sigma_i \neq 0$ for $i \in \{1,2\}$.

From (5.37), one gets

$$P^T(x,-t) = -CP(x,t)C^{-1}. \tag{5.38}$$

Thus from (5.38), we deduce

$$r(x,t) = -\Sigma_2^{-1} p^T(x,-t)\Sigma_1, \tag{5.39}$$

where p, r are vector potentials.

Also one can prove that $Q^T(x,-t,-\lambda) = CQ(x,-t,\lambda)C^{-1}$ and

${V^{[2]}}^T(x,-t,-\lambda) = CV^{[2]}(x,t,\lambda)C^{-1}$. Remarkably the two non-local Lax pair matrices $U^T(x,-t,-\lambda)$ and $V^{[2]}(x,-t,-\lambda)$ satisfy an equivalent zero curvature equation.

As Σ_1 and Σ_2 are invertible and symmetric, so they are diagonalizable, then we can take $\Sigma_1 = diag(\rho_1,\rho_2)$, $\Sigma_2 = diag(\rho_3^{-1},\rho_4^{-1})$, for $\rho_1,\rho_2,\rho_3,\rho_4$ non-zero real. Thus this leads to

$$r_1(x,t,\lambda) = -\rho_1\rho_3 p_1(x,-t,-\lambda), \tag{5.40}$$

$$r_2(x,t,\lambda) = -\rho_2\rho_3 p_3(x,-t,-\lambda), \tag{5.41}$$

$$r_3(x,t,\lambda) = -\rho_1\rho_4 p_2(x,-t,-\lambda), \tag{5.42}$$

$$r_4(x,t,\lambda) = -\rho_2\rho_4 p_4(x,-t,-\lambda). \tag{5.43}$$

From this specific reduction, we can reduce these integrable equations (5.24)–(5.31) to a nonlocal reverse-time nonlinear Schrödinger type equations.

$$p_{1,t}(x,t) = -i\frac{\beta}{\alpha^2}\big[p_{1,xx}(x,t) - 2\rho_1\rho_3 p_1^2(x,t)p_1(x,-t) - 2\rho_1\rho_4 p_1(x,t)p_2(x,t)p_2(x,-t)$$
$$- 2\rho_2\rho_3 p_1(x,t)p_3(x,t)p_3(x,-t) - 2\rho_2\rho_4 p_2(x,t)p_3(x,t)p_4(x,-t)\big], \tag{5.44}$$
$$p_{2,t}(x,t) = -i\frac{\beta}{\alpha^2}\big[p_{2,xx}(x,t) - 2\rho_1\rho_4 p_2^2(x,t)p_2(x,-t) - 2\rho_1\rho_3 p_2(x,t)p_(x,t)p_1(x,-t)$$
$$- 2\rho_2\rho_4 p_2(x,t)p_4(x,t)p_4(x,-t) - 2\rho_2\rho_3 p_1(x,t)p_4(x,t)p_3(x,-t)\big], \tag{5.45}$$
$$p_{3,t}(x,t) = -i\frac{\beta}{\alpha^2}\big[p_{3,xx}(x,t) - 2\rho_2\rho_3 p_3^2(x,t)p_3(x,-t) - 2\rho_1\rho_3 p_3(x,t)p_1(x,t)p_1(x,-t)$$
$$- 2\rho_2\rho_4 p_3(x,t)p_4(x,t)p_4(x,-t) - 2\rho_1\rho_4 p_1(x,t)p_4(x,t)p_2(x,-t)\big], \tag{5.46}$$
$$p_{4,t}(x,t) = -i\frac{\beta}{\alpha^2}\big[p_{4,xx}(x,t) - 2\rho_2\rho_4 p_4^2(x,t)p_4(x,-t) - 2\rho_1\rho_4 p_4(x,t)p_2(x,t)p_2(x,-t)$$
$$- 2\rho_2\rho_3 p_4(x,t)p_3(x,t)p_3(x,-t) - 2\rho_1\rho_3 p_2(x,t)p_3(x,t)p_1(x,-t)\big]. \tag{5.47}$$

Remark 5.2.1. $p(-x,t)$ and $p^*(x,-t)$ are both solutions of the PT symmetric nonlocal Schrödinger type equations (5.44)–(5.47).

Remark 5.2.2. If Σ_1, Σ_2 are both positive definite such that $\rho_1, \rho_2 > 0$, $\rho_3, \rho_4 < 0$ or $\rho_1, \rho_2 < 0$, $\rho_3, \rho_4 > 0$, then we have a focussing nonlocal reverse-time eight-component NLS equations. On the other hand, if $\rho_1, \rho_2, \rho_3, \rho_4 > 0$ or $\rho_1, \rho_2, \rho_3, \rho_4 < 0$, then we obtained the defocussing case. Otherwise, we could have a combined focussing and defocussing case.

5.3 Direct scattering

Our objective is to find soliton solutions from an initial condition $(p(x,0), r^T(x,0))^T$ to $(p(x,t), r^T(x,t))^T$ at any time t [31]. We assume that any p_i and r_i decay exponentially, i.e., $p_i \to 0$ and $r_i \to 0$ as $x, t \to \pm\infty$ for $i \in \{1,2,3\}$. With an infinite number of bound states, this requires [41, 44]

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |x|^m |t|^n \big(\sum_{i=1}^{4}(|p_i| + |q_i|)\big) dx dt < \infty, \quad m, n \geq 0. \tag{5.48}$$

Therefore from the spectral problems (5.32), (5.33), we derive asymptotically $\psi(x,t) \rightsquigarrow e^{i\lambda\Lambda x + i\lambda^2\Omega t}$.

We can then expect the solution for the spectral problems to be:

$$\psi(x,t) = \phi(x,t)e^{i\lambda \Lambda x + i\lambda^2 \Omega t}. \tag{5.49}$$

For the Jost solution, we require that

$$\phi(x,t) \to I_4, \quad \text{as} \quad x,t \to \pm\infty, \tag{5.50}$$

where I_4 is the 4×4 identity matrix. Substituting (5.49) into the Lax pair, (5.32) and (5.33), will result in the equivalent expression of the spectral problems

$$\phi_x = i\lambda[\Lambda, \phi] + iP\phi, \tag{5.51}$$

$$\phi_t = i\lambda^2[\Omega, \phi] + iQ\phi. \tag{5.52}$$

Now, we are going to work with the spatial spectral problem (5.51), assuming that the time is $t = 0$ for the direct scattering process.

From Liouville's theorem (3.4.1), as $tr(iP) = 0$ and $tr(iQ) = 0$, so $(det(\phi))_x = 0$, thus $det(\phi)$ is a constant, and using the boundary condition (5.50), we get

$$det(\phi) = 1. \tag{5.53}$$

To construct the Riemann–Hilbert problems and their solutions in the reflectionless case, we are going to use the adjoint scattering equations of the spectral problems (5.32) and (5.33).

Therefore it follows that the adjoints are

$$\tilde{\psi}_x = -i\tilde{\psi}U, \tag{5.54}$$

$$\tilde{\psi}_t = -i\tilde{\psi}V^{[2]}, \tag{5.55}$$

and the equivalent spectral adjoint equations read

$$\tilde{\phi}_x = -i\lambda[\tilde{\phi}, \varLambda] - i\tilde{\phi}P, \tag{5.56}$$

$$\tilde{\phi}_t = -i\lambda^2[\tilde{\phi}, \varOmega] - i\tilde{\phi}Q. \tag{5.57}$$

As $\phi_x^{-1} = -\phi^{-1}\phi_x\phi^{-1}$, we have from (5.51),

$$\phi_x^{-1} = -i\lambda[\phi^{-1}, \varLambda] - i\phi^{-1}P. \tag{5.58}$$

Therefore, we deduce that $(\phi^{\pm})^{-1}$ satisfies both adjoint equations (5.56) and (5.57).

Now, if the eigenfunction $\phi(x,t,\lambda)$ is a solution of the spectral problem (3.54), then $C\phi^{-1}(x,t,\lambda)$ is a solution of the spectral adjoint problem (5.56) with the same eigenvalue because $\phi_x^{-1} = -\phi^{-1}\phi_x\phi^{-1}$. Also $\phi^T(x,-t,-\lambda)C$ is a solution of the spectral adjoint problem (5.56). As both solutions have the same boundary condition as $x \to \pm\infty$ which guarantees the uniqueness of the solution, so

$$\phi^T(x,-t,-\lambda) = C\phi^{-1}(x,t,\lambda)C^{-1}. \tag{5.59}$$

This tells us that if λ is an eigenvalue of the spectral problems, then $-\lambda$ is also an eigenvalue.

We suppose for the rest of the problem that, $\alpha < 0$ and $\beta < 0$ and as before $Y^{\pm}$ denotes which end of the x-axis the boundary conditions are set. Knowing that

$$\phi^{\pm} \to I_4 \quad \text{when} \quad x \to \pm\infty, \tag{5.60}$$

we can then write

$$\psi^{\pm} = \phi^{\pm}e^{i\lambda\varLambda x}. \tag{5.61}$$

As ψ^+ and ψ^- are two solutions of the spectral spatial differential equation of first-order (5.32) and hence they are linearly dependent, and so they are related by a scattering matrix $S(\lambda)$. As a result,

$$\psi^- = \psi^+S(\lambda), \tag{5.62}$$

using (5.61) and we have

$$\phi^- = \phi^+ e^{i\lambda\Lambda x} S(\lambda) e^{-i\lambda\Lambda x}, \quad \text{for} \quad \lambda \in \mathbb{R}, \tag{5.63}$$

where

$$S(\lambda) = (s_{ij})_{4\times 4} = \begin{pmatrix} s_{11} & s_{12} & s_{13} & s_{14} \\ s_{21} & s_{22} & s_{23} & s_{24} \\ s_{31} & s_{32} & s_{33} & s_{34} \\ s_{41} & s_{42} & s_{43} & s_{44} \end{pmatrix}. \tag{5.64}$$

Because $det(\phi^\pm) = 1$, one has

$$det(S(\lambda)) = 1. \tag{5.65}$$

From (5.59) and (5.63), we have this involution relation

$$S^T(-\lambda) = CS^{-1}(\lambda)C^{-1}. \tag{5.66}$$

From (5.66), we deduce that

$$s_{11}(-\lambda) = \hat{s}_{11}(\lambda), \quad s_{22}(-\lambda) = \hat{s}_{22}(\lambda), \tag{5.67}$$

$$s_{21}(-\lambda) = \rho_1\rho_2^{-1}\hat{s}_{12}(\lambda), \quad s_{12}(-\lambda) = \rho_2\rho_1^{-1}\hat{s}_{21}(\lambda), \tag{5.68}$$

where the inverse scattering data matrix $S^{-1} = (\hat{s}_{ij})_{4\times 4}$ for $i, j \in \{1, 2, 3, 4\}$.

We can see that the recovery of the potentials will depend on the information of the scattering data from the scattering matrix $S(\lambda)$. As $\phi^\pm \to I_4$ when $x \to \pm\infty$, we need to analyse the analyticity of the Jost matrix $\phi^\pm$ in order to formulate the Riemann-Hilbert problems.

One can write the solution $\phi^\pm$ in a uniquely manner by the Volterra integral equations using (5.32):

$$\phi^-(x,\lambda) = I_4 + i \int_{-\infty}^{x} e^{i\lambda\Lambda(x-y)} P(y)\phi^-(y,\lambda) e^{i\lambda\Lambda(y-x)} dy, \tag{5.69}$$

$$\phi^+(x,\lambda) = I_4 - i \int_{x}^{+\infty} e^{i\lambda\Lambda(x-y)} P(y)\phi^+(y,\lambda) e^{i\lambda\Lambda(y-x)} dy. \tag{5.70}$$

By assumption $\alpha = \alpha_1 - \alpha_2 < 0$. So, if $Im(\lambda) > 0$, then $Re(e^{-i\lambda\alpha(x-y)})$ decays exponentially when $y < x$, and so each integral of the first column and second column of ϕ^- converges, so they are analytic in the upper-half plane, i.e. where $\lambda \in \mathbb{C}_+$ and continuous for $\lambda \in \mathbb{C}_+ \cup \mathbb{R}$.

If $Im(\lambda) < 0$, $Re(e^{i\lambda\alpha(x-y)})$ also decays, then the components of the third and fourth columns of ϕ^- converge, and thus they are analytic in the lower-half plane for $\lambda \in \mathbb{C}_-$ and continuous for $\lambda \in \mathbb{C}_- \cup \mathbb{R}$.

Similarly, for $y > x$, the components of the third and fourth columns of ϕ^+ are analytic in the upper-half plane for $\lambda \in \mathbb{C}_+$ and continuous for $\lambda \in \mathbb{C}_+ \cup \mathbb{R}$, and the components of the first and second columns of ϕ^+ are analytic in the lower-half plane for $\lambda \in \mathbb{C}_-$ and continuous for $\lambda \in \mathbb{C}_- \cup \mathbb{R}$.

Now let us construct the Riemann-Hilbert problems. Note that

$$\phi^{\pm} = \psi^{\pm} e^{-i\lambda \Lambda x}. \tag{5.71}$$

Let $\phi_j^{\pm}$ be the jth column of $\phi^{\pm}$ for $j \in \{1, 2, 3, 4\}$, and so the first Jost matrix solution can be taken as

$$P^{(+)}(x, \lambda) = (\phi_1^-, \phi_2^-, \phi_3^+, \phi_4^+) = \phi^- H_1 + \phi^+ H_2, \tag{5.72}$$

where $H_1 = diag(1, 1, 0, 0)$ and $H_2 = diag(0, 0, 1, 1)$.

$P^{(+)}$ is then analytic for $\lambda \in \mathbb{C}_+$ and continuous for $\lambda \in \mathbb{C}_+ \cup \mathbb{R}$.

To construct the analytic counterpart of $P^{(+)} \in \mathbb{C}_+$, it is going to be simpler to use the equivalent spectral adjoint equation (5.58). Because $\tilde{\phi}^{\pm} = (\phi^{\pm})^{-1}$ and $\psi^{\pm} = \phi^{\pm} e^{i\lambda \Lambda x}$, we have

$$(\phi^{\pm})^{-1} = e^{i\lambda \Lambda x} (\psi^{\pm})^{-1}. \tag{5.73}$$

Now, let $\tilde{\phi}_j^{\pm}$ be the jth row of $\tilde{\phi}^{\pm}$ for $j \in \{1, 2, 3, 4\}$. In the same way we proved for $P^{(+)}$ above, we can get

$$P^{(-)}(x, \lambda) = \left(\tilde{\phi}_1^-, \tilde{\phi}_2^-, \tilde{\phi}_3^+, \tilde{\phi}_4^+ \right)^T = H_1 (\phi^-)^{-1} + H_2 (\phi^+)^{-1}. \tag{5.74}$$

$P^{(-)}$ is analytic for $\lambda \in \mathbb{C}_-$ and continuous for $\lambda \in \mathbb{C}_- \cup \mathbb{R}$.

Also we have

$$P^{(-)}(x, \lambda) \to I_4 \quad \text{as} \quad \lambda \in \mathbb{C}_- \cup \mathbb{R} \to \infty. \tag{5.75}$$

From (5.72), (5.74) and (5.71) along with $\phi^T(x,-t,-\lambda) = C\phi^{-1}(x,t,\lambda)C^{-1}$, we have the nonlocal involution property

$$(P^{(+)})^T(x,-t,-\lambda) = CP^{(-)}(x,t,\lambda)C^{-1}. \tag{5.76}$$

We know that the eigenfunctions $P^{(+)}$ and $P^{(-)}$ that are analytic in $\mathbb{C}_+$ and $\mathbb{C}_-$ and continuous in $\mathbb{C}_+ \cup \mathbb{R}$ and $\mathbb{C}_- \cup \mathbb{R}$, respectively.

From (5.72) and (5.74), we have

$$P^{(-)}(x,\lambda)P^{(+)}(x,\lambda) = e^{i\lambda\Lambda x}(H_1 + H_2 S)(H_1 + S^{-1}H_2)e^{-i\lambda\Lambda x}, \text{for} \quad \lambda \in \mathbb{R}, \tag{5.77}$$

where the inverse scattering data matrix $S^{-1} = (\hat{s}_{ij})_{4\times 4}$ for $i,j \in \{1,2,3,4\}$.

Using (5.63) in (5.72), we have

$$P^{(+)}(x,\lambda) = \phi^+(e^{i\lambda\Lambda x}Se^{-i\lambda\Lambda x}H_1 + H_2), \tag{5.78}$$

and as $\phi^+(x,\lambda) \to I_4$ when $x \to +\infty$, then

$$\lim_{x\to+\infty} P^{(+)} = \left(\begin{array}{cc|cc} s_{11}(\lambda) & s_{12}(\lambda) & 0 & 0 \\ s_{21}(\lambda) & s_{22}(\lambda) & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \quad \text{for} \quad \lambda \in \mathbb{C}_+ \cup \mathbb{R}. \tag{5.79}$$

In the same way, we have

$$\lim_{x\to-\infty} P^{(-)} = \left(\begin{array}{cc|cc} \hat{s}_{11}(\lambda) & \hat{s}_{12}(\lambda) & 0 & 0 \\ \hat{s}_{21}(\lambda) & \hat{s}_{22}(\lambda) & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \quad \text{for} \quad \lambda \in \mathbb{C}_- \cup \mathbb{R}. \tag{5.80}$$

Thus if we choose

$$G^{(+)}(x,\lambda) = P^{(+)}(x,\lambda)\left(\begin{array}{cc|cc} \frac{s_{22}(\lambda)}{z_1(\lambda)} & -\frac{s_{12}(\lambda)}{z_1(\lambda)} & 0 & 0 \\ -\frac{s_{21}(\lambda)}{z_1(\lambda)} & \frac{s_{11}(\lambda)}{z_1(\lambda)} & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right), \tag{5.81}$$

where $z_1(\lambda) = s_{11}(\lambda)s_{22}(\lambda) - s_{12}(\lambda)s_{21}(\lambda)$, and

$$(G^{(-)})^{-1}(x,\lambda) = \left(\begin{array}{cc|cc} \frac{\hat{s}_{22}(\lambda)}{z_2(\lambda)} & -\frac{\hat{s}_{12}(\lambda)}{z_2(\lambda)} & 0 & 0 \\ -\frac{\hat{s}_{21}(\lambda)}{z_2(\lambda)} & \frac{\hat{s}_{11}(\lambda)}{z_2(\lambda)} & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) P^{(-)}(x,\lambda) \tag{5.82}$$

where $z_2(\lambda) = \hat{s}_{11}(\lambda)\hat{s}_{22}(\lambda) - \hat{s}_{12}(\lambda)\hat{s}_{21}(\lambda)$.

So on the real line, the two generalized matrices generate the matrix Riemann-Hilbert problems for the eight-component AKNS system of second-order given by

$$G^{(+)}(x,\lambda) = G^{(-)}(x,\lambda)G_0(x,\lambda), \quad \text{for} \quad \lambda \in \mathbb{R}, \tag{5.83}$$

where the jump matrix $G_0(x,\lambda)$ can be cast as

$$G_0(x,\lambda) = e^{i\lambda\Lambda x}\left(\begin{array}{cc|cc} \frac{\hat{s}_{22}(\lambda)}{z_2(\lambda)} & -\frac{\hat{s}_{12}(\lambda)}{z_2(\lambda)} & 0 & 0 \\ -\frac{\hat{s}_{21}(\lambda)}{z_2(\lambda)} & \frac{\hat{s}_{11}(\lambda)}{z_2(\lambda)} & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) A \left(\begin{array}{cc|cc} \frac{s_{22}(\lambda)}{z_1(\lambda)} & -\frac{s_{12}(\lambda)}{z_1(\lambda)} & 0 & 0 \\ -\frac{s_{21}(\lambda)}{z_1(\lambda)} & \frac{s_{11}(\lambda)}{z_1(\lambda)} & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) e^{-i\lambda\Lambda x}, \tag{5.84}$$

where $A = (H_1 + H_2 S)(H_1 + S^{-1}H_2)$. Moreover, $G_0(x,\lambda)$ can be explicitly written as

$$G_0(x,\lambda)=\begin{pmatrix} \frac{s_{21}\hat{s}_{12}+s_{22}\hat{s}_{22}}{z_1(\lambda)z_2(\lambda)} & -\frac{s_{11}\hat{s}_{12}+s_{12}\hat{s}_{22}}{z_1(\lambda)z_2(\lambda)} & -e^{i\lambda\alpha x}\frac{\hat{s}_{12}\hat{s}_{23}-\hat{s}_{13}\hat{s}_{22}}{z_2(\lambda)} & -e^{i\lambda\alpha x}\frac{\hat{s}_{12}\hat{s}_{24}-\hat{s}_{14}\hat{s}_{22}}{z_2(\lambda)} \\ -\frac{s_{21}\hat{s}_{11}+s_{22}\hat{s}_{21}}{z_1(\lambda)z_2(\lambda)} & \frac{s_{11}\hat{s}_{11}+s_{12}\hat{s}_{21}}{z_1(\lambda)z_2(\lambda)} & e^{i\lambda\alpha x}\frac{\hat{s}_{11}\hat{s}_{23}-\hat{s}_{13}\hat{s}_{21}}{z_2(\lambda)} & e^{i\lambda\alpha x}\frac{\hat{s}_{11}\hat{s}_{24}-\hat{s}_{14}\hat{s}_{21}}{z_2(\lambda)} \\ -e^{-i\lambda\alpha x}\frac{s_{21}s_{32}-s_{22}s_{31}}{z_1(\lambda)} & e^{-i\lambda\alpha x}\frac{s_{11}s_{32}-s_{12}s_{31}}{z_1(\lambda)} & 1 & 0 \\ -e^{-i\lambda\alpha x}\frac{s_{21}s_{42}-s_{22}s_{41}}{z_1(\lambda)} & e^{-i\lambda\alpha x}\frac{s_{11}s_{42}-s_{12}s_{41}}{z_1(\lambda)} & 0 & 1 \end{pmatrix}, \tag{5.85}$$

with its canonical normalization conditions given by:

$$G^{(+)}(x,\lambda)\to I_4 \quad \text{as} \quad \lambda\in\mathbb{C}_+\cup\mathbb{R}\to\infty, \tag{5.86}$$

$$G^{(-)}(x,\lambda)\to I_4 \quad \text{as} \quad \lambda\in\mathbb{C}_-\cup\mathbb{R}\to\infty. \tag{5.87}$$

From (5.76) along with (5.67)–(5.68) and (5.81), we obtain

$$(G^{(+)})^T(x,-t,-\lambda)=C(G^{(-)})^{-1}(x,t,\lambda)C^{-1}. \tag{5.88}$$

5.4 Time evolution of scattering data

The process of the inverse scattering transform requires the time evolution of the scattering data. Differentiating equation (5.63) with respect to time t and applying (5.52) gives

$$S_t=i\lambda^2[\Omega,S], \tag{5.89}$$

which implies

$$S_t=\begin{pmatrix} 0 & 0 & i\beta\lambda^2 s_{13} & i\beta\lambda^2 s_{14} \\ 0 & 0 & i\beta\lambda^2 s_{23} & i\beta\lambda^4 s_{24} \\ -i\beta\lambda^2 s_{31} & -i\beta\lambda^2 s_{32} & 0 & 0 \\ -i\beta\lambda^2 s_{41} & -i\beta\lambda^2 s_{42} & 0 & 0 \end{pmatrix}. \tag{5.90}$$

As a result, we have

$$\begin{cases} s_{13}(t,\lambda) = s_{13}(0,\lambda)e^{i\beta\lambda^2 t}, \quad s_{14}(t,\lambda) = s_{14}(0,\lambda)e^{i\beta\lambda^2 t}, \\ s_{23}(t,\lambda) = s_{23}(0,\lambda)e^{i\beta\lambda^2 t}, \quad s_{24}(t,\lambda) = s_{24}(0,\lambda)e^{i\beta\lambda^2 t}, \\ s_{31}(t,\lambda) = s_{31}(0,\lambda)e^{-i\beta\lambda^2 t}, \quad s_{32}(t,\lambda) = s_{32}(0,\lambda)e^{-i\beta\lambda^2 t}, \\ s_{41}(t,\lambda) = s_{41}(0,\lambda)e^{-i\beta\lambda^2 t}, \quad s_{42}(t,\lambda) = s_{42}(0,\lambda)e^{-i\beta\lambda^2 t}. \end{cases} \tag{5.91}$$

and $s_{11}, s_{12}, s_{21}, s_{22}, s_{33}, s_{34}, s_{43}, s_{44}$ are constants.

5.5 Inverse scattering

5.5.1 Soliton solutions: General case

In this section, we are going to compute explicitly the one-soliton solution from the N-soliton solution based on the Riemann-Hilbert problems. In fact the Riemann-Hilbert problems generate a unique solution in the regular case, i.e., the $det(G^{(\pm)}) \neq 0$ when $G^{(\pm)} \to I_4$ as $\lambda \to \infty$.

However, there are possible contingencies that $det(G^{(\pm)})$ could be zero for some discrete $\lambda \in \mathbb{C}_\pm$ when non-regular. In that case, it is opportune to transform the non-regular case to a regular in order to guarantee a solution.

From (5.72) and (5.74) with (5.63), as $det(\phi^{\pm}) = 1$, we prove that

$$det(P^{(+)}(x,\lambda)) = s_{11}(\lambda)s_{22}(\lambda) - s_{12}(\lambda)s_{21}(\lambda) = z_1(\lambda), \tag{5.92}$$

and

$$det(P^{(-)}(x,\lambda)) = \hat{s}_{11}(\lambda)\hat{s}_{22}(\lambda) - \hat{s}_{12}(\lambda)\hat{s}_{21}(\lambda) = z_2(\lambda). \tag{5.93}$$

To get soliton solutions, the solutions of $det(P^{(\pm)}(x,\lambda)) = 0$ are assumed to be simple. Let's suppose that $z_1(\lambda)$ has simple zeros $\lambda_k \in \mathbb{C}_+$ for $k \in \{1, 2, ..., N\}$ and $z_2(\lambda)$ has simple zeros $\hat{\lambda}_k \in \mathbb{C}_-$ for $k \in \{1, 2, ..., N\}$, which are the poles of the transmission coefficients.

From (5.92), (5.93) and using (5.67)–(5.68), we have the involution relation

$$\hat{\lambda} = -\lambda. \tag{5.94}$$

Each $Ker(P^{(+)}(x,\lambda_k))$ contains a single column eigenvector v_k, and also $Ker(P^{(-)}(x,\hat{\lambda}_k))$ contains a single row eigenvector $\hat{v}_k$ for $k \in \{1,2,...,N\}$ such that:

$$P^{(+)}(x,\lambda_k)v_k = 0 \quad \text{for} \quad k \in \{1,2,...,N\}, \tag{5.95}$$

and

$$\hat{v}_k P^{(-)}(x,\hat{\lambda}_k) = 0 \quad \text{for} \quad k \in \{1,2,...,N\}. \tag{5.96}$$

The Riemann-Hilbert problems can be solved explicitly when $G_0 = I_4$. This will force the following reflection coefficients

$$\hat{s}_{12}\hat{s}_{23} - \hat{s}_{13}\hat{s}_{22} = 0, \quad \hat{s}_{12}\hat{s}_{24} - \hat{s}_{14}\hat{s}_{22} = 0, \quad \hat{s}_{11}\hat{s}_{23} - \hat{s}_{13}\hat{s}_{21} = 0, \quad \hat{s}_{11}\hat{s}_{24} - \hat{s}_{14}\hat{s}_{21} = 0,$$

$$s_{21}s_{32} - s_{22}s_{31} = 0, \quad s_{11}s_{32} - s_{12}s_{31} = 0, \quad s_{21}s_{42} - s_{22}s_{41} = 0, \quad s_{11}s_{42} - s_{12}s_{41} = 0.$$

In that case, we can present the solutions to special Riemann-Hilbert problems as follows:

$$G^{(+)}(x,\lambda) = I_4 - \sum_{k,j=1}^{N} \frac{v_k (M^{-1})_{kj}\hat{v}_j}{\lambda - \hat{\lambda}_j}, \tag{5.97}$$

and

$$(G^{(-)})^{-1}(x,\lambda) = I_4 + \sum_{k,j=1}^{N} \frac{v_k (M^{-1})_{kj}\hat{v}_j}{\lambda - \lambda_k}, \tag{5.98}$$

where $M = (m_{kj})_{N\times N}$ is a matrix defined as follows

$$m_{kj} = \begin{cases} \frac{\hat{v}_k v_j}{\lambda_j - \hat{\lambda}_k} & \text{if} \quad \lambda_j \neq \hat{\lambda}_k \\ 0 & \text{if} \quad \lambda_j = \hat{\lambda}_k \end{cases}, k,j \in \{1,2,...,N\}. \tag{5.99}$$

The scattering vectors v_k and $\hat{v}_k$ are functions of (x,t), but λ_k and $\hat{\lambda}_k$ are constants, and so differentiating both sides of $P^{(+)}(x,\lambda_k)v_k = 0$ with respect to x and knowing that $P^{(+)}$ satisfies the spectral spatial equivalent equation (5.51) along with (5.95) gives

$$P^{(+)}(x,\lambda_k)\left(\frac{dv_k}{dx} - i\lambda_k \Lambda v_k\right) = 0 \quad \text{for} \quad k,j \in \{1,2,...,N\}, \tag{5.100}$$

and also differentiating it with respect to t and using the temporal equation (5.52) along with (5.95) gives

$$P^{(+)}(x,\lambda_k)\left(\frac{dv_k}{dt} - i\lambda_k^2 \Omega v_k\right) = 0 \quad \text{for} \quad k,j \in \{1,2,...,N\}. \tag{5.101}$$

In the same way by using (5.96) and the adjoint spectral equations (5.56) and (5.57), one can prove that

$$\left(\frac{d\hat{v}_k}{dx} + i\hat{\lambda}_k \hat{v}_k \Lambda\right) P^{(-)}(x,\hat{\lambda}_k) = 0, \tag{5.102}$$

and

$$\left(\frac{d\hat{v}_k}{dt} + i\hat{\lambda}_k^2 \hat{v}_k \Omega\right) P^{(-)}(x,\hat{\lambda}_k) = 0. \tag{5.103}$$

As v_k is a single vector in the kernel of $P^{(+)}$, so $\frac{dv_k}{dx} - i\lambda_k \Lambda v_k$ and $\frac{dv_k}{dt} - i\lambda_k^2 \Omega v_k$ are scalar multiples of v_k.

This permits to obtain

$$v_k(x,t) = e^{i\lambda_k \Lambda x + i\lambda_k^2 \Omega t} w_k, \quad \text{for} \quad k \in \{1,2,...,N\}. \tag{5.104}$$

In the same way, we will have for $P^{(-)}$,

$$\hat{v}_k(x,t) = \hat{w}_k e^{-i\hat{\lambda}_k \Lambda x - i\hat{\lambda}_k^2 \Omega t}, \quad \text{for} \quad k \in \{1,2,...,N\} \tag{5.105}$$

where the column vector w_k and the row vector $\hat{w}_k$ are constants.

Now from (5.95) and using (5.76) we get

$$v_k^T(x,-t,-\lambda_k)(P^{(+)})^T(x,-t,-\lambda_k) = v_k^T(x,-t,-\lambda_k) C P^{(-)}(x,t,\lambda_k) C^{-1} = 0 \quad \text{for} \quad k \in \{1,2,...,N\}. \tag{5.106}$$

Because $v_k^T(x,-t,-\lambda_k) C P^{(-)}(x,t,\lambda_k)$ could be zero and using (5.96) leads to

$$v_k^T(x,-t,-\lambda_k) C P^{(-)}(x,t,\lambda_k) = \hat{v}_k(x,t,\hat{\lambda}_k) P^{(-)}(x,t,\hat{\lambda}_k) = \hat{v}_k(x,t,-\hat{\lambda}_k) P^{(-)}(x,t,-\hat{\lambda}_k) = 0. \tag{5.107}$$

As $\hat{\lambda}_k = -\lambda_k$ from (5.94), then we can take

$$\hat{v}_k(x,t,-\hat{\lambda}_k) = v_k^T(x,-t,-\lambda_k) C \quad \text{for} \quad k \in \{1,2,...,N\}. \tag{5.108}$$

These involution relations will then give

$$v_k(x,t) = e^{i\lambda_k \Lambda x + i\lambda_k^2 \Omega t} w_k, \quad \text{for} \quad k \in \{1, 2, ..., N\}, \tag{5.109}$$

$$\hat{v}_k(x,t) = w_k^T e^{-i\hat{\lambda}_k \Lambda x - i\hat{\lambda}_k^2 \Omega t} C, \quad \text{for} \quad k \in \{1, 2, ..., N\}. \tag{5.110}$$

5.5.2 Recovery of potentials

The jump matrix being $G = I_4$ allows to recover the potential P from the generalized matrix Jost eigenfunctions. Because $P^{(+)}$ is analytic, we can expand $G^{(+)}$ as $\lambda \to \infty$ in this form at order 2,

$$G^{(+)}(x,\lambda) = I_4 + \frac{1}{\lambda} G_1^{(+)}(x) + O(\frac{1}{\lambda^2}) \quad \text{when} \quad \lambda \to \infty. \tag{5.111}$$

Because $G^{(+)}$ satisfies the spectral problem, substituting it into (5.51) and matching the coefficients of the same powers of $\frac{1}{\lambda}$, at order $O(1)$, we get

$$P = -[\Lambda, G_1^{(+)}]. \tag{5.112}$$

If

$$G_1^{(+)} = \begin{pmatrix} (G_1^{(+)})_{11} & (G_1^{(+)})_{12} & (G_1^{(+)})_{13} & (G_1^{(+)})_{14} \\ (G_1^{(+)})_{21} & (G_1^{(+)})_{22} & (G_1^{(+)})_{23} & (G_1^{(+)})_{24} \\ (G_1^{(+)})_{31} & (G_1^{(+)})_{32} & (G_1^{(+)})_{33} & (G_1^{(+)})_{34} \\ (G_1^{(+)})_{41} & (G_1^{(+)})_{42} & (G_1^{(+)})_{43} & (G_1^{(+)})_{44} \end{pmatrix}, \tag{5.113}$$

then

$$P = -[\Lambda, G_1^{(+)}] = \begin{pmatrix} 0 & 0 & -\alpha(G_1^{(+)})_{13} & -\alpha(G_1^{(+)})_{14} \\ 0 & 0 & -\alpha(G_1^{(+)})_{23} & -\alpha(G_1^{(+)})_{24} \\ \alpha(G_1^{(+)})_{31} & \alpha(G_1^{(+)})_{32} & 0 & 0 \\ \alpha(G_1^{(+)})_{41} & \alpha(G_1^{(+)})_{42} & 0 & 0 \end{pmatrix}. \tag{5.114}$$

As a result, we can now recover the potentials p_i and r_i for $i \in \{1, 2, 3, 4\}$ as follows

$$
\begin{aligned}
p_1 &= -\alpha(G_1^{(+)})_{13}, \quad r_1 = \alpha(G_1^{(+)})_{31}, \\
p_2 &= -\alpha(G_1^{(+)})_{14}, \quad r_2 = \alpha(G_1^{(+)})_{32}, \\
p_3 &= -\alpha(G_1^{(+)})_{23}, \quad r_3 = \alpha(G_1^{(+)})_{41}. \\
p_4 &= -\alpha(G_1^{(+)})_{24}, \quad r_4 = \alpha(G_1^{(+)})_{42}.
\end{aligned} \tag{5.115}
$$

Also, from (5.111), we have

$$
G_1^{(+)} = \lambda \lim_{\lambda \to \infty} (G^{(+)}(x,\lambda) - I_4), \tag{5.116}
$$

and so from (5.97), we deduce that

$$
G_1^{(+)} = -\sum_{k,j=1}^{N} v_k (M^{-1})_{k,j} \hat{v}_j. \tag{5.117}
$$

From (5.38) and (5.112), we easily prove the nonlocal involution property

$$
(G_1^{(+)})^T(x,-t) = C G_1^{(+)}(x,t) C^{-1}. \tag{5.118}
$$

Using the above equations along with (5.110) and (5.109) will generate the N-soliton solutions to the nonlocal reverse-time eight-component AKNS system of second-order as follows:

$$
p_1 = \alpha \sum_{k,j=1}^{N} v_{k1} (M^{-1})_{kj} \hat{v}_{j,3}, \tag{5.119}
$$

$$
p_2 = \alpha \sum_{k,j=1}^{N} v_{k1} (M^{-1})_{kj} \hat{v}_{j,4}, \tag{5.120}
$$

$$
p_3 = \alpha \sum_{k,j=1}^{N} v_{k2} (M^{-1})_{kj} \hat{v}_{j,3}, \tag{5.121}
$$

$$
p_4 = \alpha \sum_{k,j=1}^{N} v_{k2} (M^{-1})_{kj} \hat{v}_{j,4}, \tag{5.122}
$$

where $v_k = (v_{k1}, v_{k2}, v_{k3}, v_{k4})^T, \hat{v}_k = (\hat{v}_{k1}, \hat{v}_{k2}, \hat{v}_{k3}, \hat{v}_4), \quad k \in \{1, ..., N\}$.

5.6 Exact one-soliton solution

A general explicit solution for a single-soliton in the reverse-time case when $N = 1$, $w_1 = (w_{11}, w_{12}, w_{13}, w_{14})^T$, $\lambda_1 \in \mathbb{C}$ is arbitrary, and $\hat{\lambda}_1 = -\lambda_1$ is given by

$$p_1(x,t) = \frac{2\alpha\rho_4 w_{11} w_{13} \lambda_1 e^{i\alpha\lambda_1 x + i\beta\lambda_1^2 t}}{\rho_3\rho_4(\rho_1 w_{11}^2 + \rho_2 w_{12}^2)e^{i2\alpha\lambda_1 x} + (\rho_4 w_{13}^2 + \rho_3 w_{14}^2)}, \tag{5.123}$$

$$p_2(x,t) = \frac{2\alpha\rho_3 w_{11} w_{14} \lambda_1 e^{i\alpha\lambda_1 x + i\beta\lambda_1^2 t}}{\rho_3\rho_4(\rho_1 w_{11}^2 + \rho_2 w_{12}^2)e^{i2\alpha\lambda_1 x} + (\rho_4 w_{13}^2 + \rho_3 w_{14}^2)}, \tag{5.124}$$

$$p_3(x,t) = \frac{2\alpha\rho_4 w_{12} w_{13} \lambda_1 e^{i\alpha\lambda_1 x + i\beta\lambda_1^2 t}}{\rho_3\rho_4(\rho_1 w_{11}^2 + \rho_2 w_{12}^2)e^{i2\alpha\lambda_1 x} + (\rho_4 w_{13}^2 + \rho_3 w_{14}^2)}, \tag{5.125}$$

$$p_4(x,t) = \frac{2\alpha\rho_3 w_{12} w_{14} \lambda_1 e^{i\alpha\lambda_1 x + i\beta\lambda_1^2 t}}{\rho_3\rho_4(\rho_1 w_{11}^2 + \rho_2 w_{12}^2)e^{i2\alpha\lambda_1 x} + (\rho_4 w_{13}^2 + \rho_3 w_{14}^2)}, \tag{5.126}$$

where $\alpha = \alpha_1 - \alpha_2$ and $\beta = \beta_1 - \beta_2$.

The amplitude of p_1 is:

$$|p_1(x,t)| = 2Ae^{-Im(\lambda_1^2\beta t + \lambda_1(\alpha_1+\alpha_2)x)}, \tag{5.127}$$

where

$$A = \left| \frac{\lambda_1\rho_4(\alpha_1 - \alpha_2) w_{11} w_{13}}{\rho_3\rho_4(\rho_1 w_{11}^2 + \rho_2 w_{12}^2)e^{i2\alpha\lambda_1 x} + (\rho_4 w_{13}^2 + \rho_3 w_{14}^2)} \right|. \tag{5.128}$$

We can see that the soliton is not a travelling wave. It moves over the time, but not over the space.

If $Im(\lambda_1^2) > 0$, its amplitude $|p_1|$ grows exponentially, while it decays exponentially when $Im(\lambda_1^2) < 0$. Now when $Im(\lambda_1^2) = 0$, the soliton conserves its amplitude over the time.

In the first and the third quadrant of the spectral plane (ξ, η), the amplitude of the one soliton is increasing. While in the second and the fourth, it is decreasing and constant on the imaginary axis $\xi = 0$ and a breather with a constant amplitude on the real axis $\eta = 0$, as shown in figure 22.

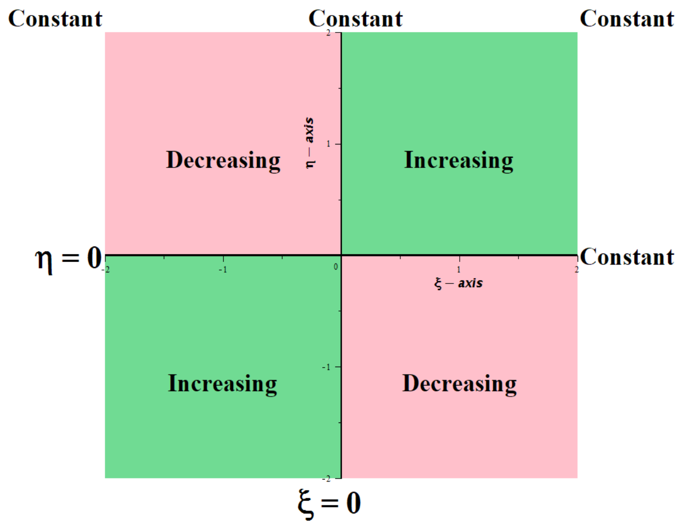

Figure 22.: Spectral plane of eigenvalues.

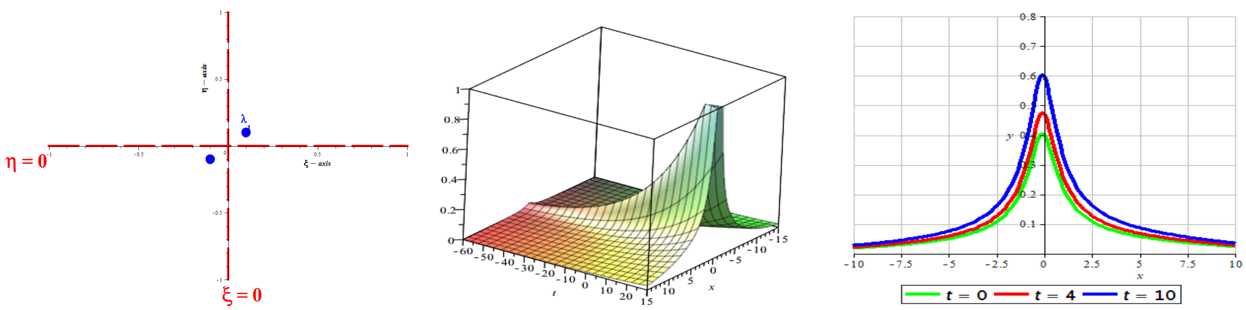

Figure 23.: Spectral plane along with 3D plot and 2D plots of $|p_1|$ of the one-soliton with parameter values $\rho_1 = -1$, $\rho_2 = -1$, $\rho_3 = 1$,$\rho_4 = 1$, $\lambda_1 = 0.1 + 0.1i$, $\alpha_1 = -1$, $\alpha_2 = 1$, $\beta_1 = -1$, $\beta_2 = 1$, $w_1 = (1 + i, 2, 1, 2 + i)^T$. The 2D plot is at $t = 0, 4, 10$.

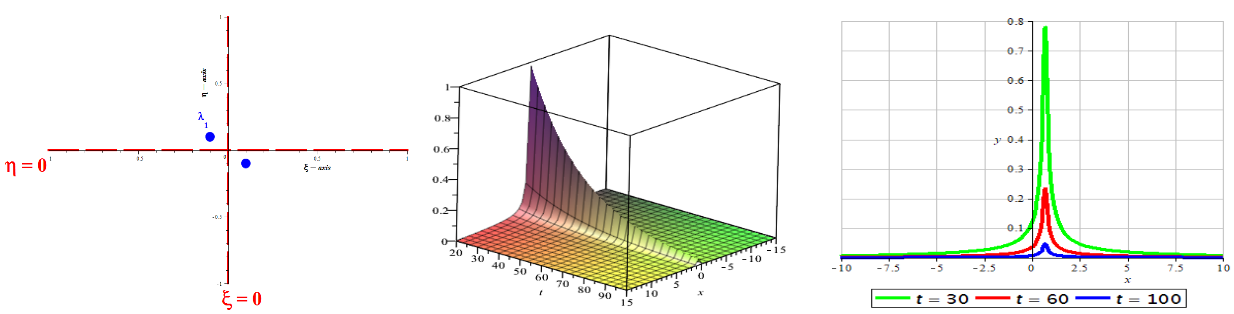

Figure 24.: Spectral plane along with 3D plot and 2D plots of $|p_1|$ of the one-soliton with parameter values $\rho_1 = -1$, $\rho_2 = -1$, $\rho_3 = 1$,$\rho_4 = 1$, $\lambda_1 = -0.1 + 0.1i$, $\alpha_1 = -1$, $\alpha_2 = 1$, $\beta_1 = -1$, $\beta_2 = 1$, $w_1 = (1 + i, 2, 1, 2 + i)^T$. The 2D plot is at $t = 30, 60, 100$.

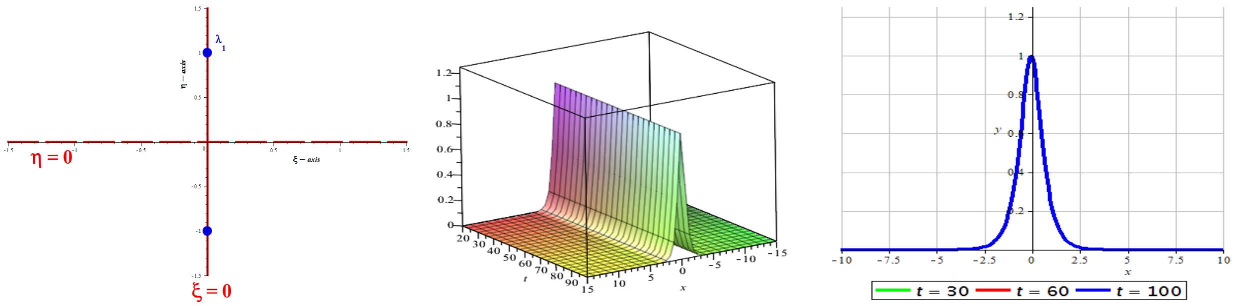

Figure 25.: Spectral plane along with 3D plot and 2D plots of $|p_1|$ of the one-soliton with parameter values $\rho_1 = -1$, $\rho_2 = -1$, $\rho_3 = 1$,$\rho_4 = 1$, $\lambda_1 = i$, $\alpha_1 = -1$, $\alpha_2 = 1$, $\beta_1 = -1$, $\beta_2 = 1$, $w_1 = (-1+3i, 2, 1, 2+i)^T$. The 2D plot is at any time value of t.

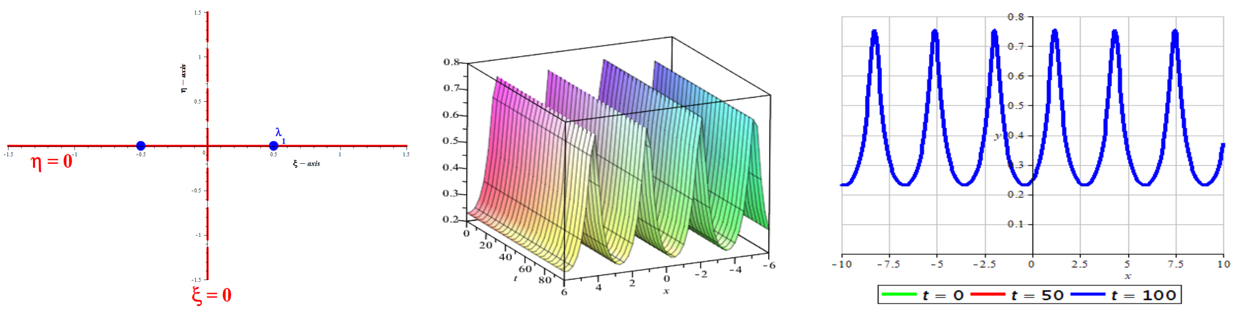

Figure 26.: Spectral plane along with 3D plot and 2D plots of $|p_1|$ of the one-soliton with parameter values $\rho_1 = 1$, $\rho_2 = 1$, $\rho_3 = 1$,$\rho_4 = 1$, $\lambda_1 = 0.5$, $\alpha_1 = -1$, $\alpha_2 = 1$, $\beta_1 = -1$, $\beta_2 = 1$, $w_1 = (i, 2, 1, 2+i)^T$. The 2D plot is at any time value of t.

Chapter 6

Conclusion

In this dissertation, by using the Riemann-Hilbert technique, we have obtained the N-soliton solution of a nonlocal nonlinear six-component fourth-order AKNS system and a nonlocal nonlinear eight-component second-order AKNS system under a reverse-time reduction, and particularly, the one- and two-soliton solutions have been presented explicitly. What is interesting in this reverse-time case is that the symmetry involution guarantees a pair of eigenvalues, one being in the upper half complex plane and its symmetric partner being in the lower half plane. Therefore, the Riemann-Hilbert problem becomes easier to solve than in the reverse-space or in the reverse-space-time cases [24]. Also, we have noticed that in comparison to classical solitons which keep their shapes and amplitudes over the time, in the reverse-time case, the amplitude of the soliton potential changes and sometimes the soliton itself collapses while moving. Such solutions show that they have singularities at a finite time. Moreover, at least two nonlocal solitons do not always collide elastically in a nonlinear superposition manner like classical solitons. Besides the Riemann-Hilbert approach, one could investigate the solvability of those nonlocal integrable equations by Hirota's bilinear method or the Darboux transformation.

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