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THE RADIUS OF CONVEXITY OF CERTAIN ANALYTIC FUNCTIONS II

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(Received August 7, 1979)

<u>ABSTRACT</u>. In [2], MacGregor found the radius of convexity of the functions $f(z) = z + a_2 z^2 + a_3 z^3 + \ldots$, analytic and univalent such that |f'(z) - 1| < 1. This paper generalized MacGregor's theorem, by considering another univalent function $g(z) = z + b_2 z^2 + b_3 z^3 + \ldots$ such that $\left| \frac{f'(z)}{g'(z)} - 1 \right| < 1$ for |z| < 1. Several theorems are proved with sharp results for the radius of convexity of the subfamilies of functions associated with the cases: g(z) is starlike for |z| < 1, g(z) is convex for |z| < 1, $Re\{g'(z)\} > \alpha$ ($\alpha=0$, 1/2).

KEY WORDS AND PHRASES. Univalent, analytic, starlike, convex, radius of starlikeness and radius of convexity.

1980 MATHEMATICS SUBJECT CLASSIFICATION CODES. 30A32, 30A36, 30A42.

1. INTROUDCTION.

Throughout we suppose that $f(z) = z + a_2 z^2 + ...$ is analytic for |z| < and $g(z) = z + b_2 z^2 + ...$ is analytic and univalent for |z| < 1. In [4] the author solved the following problem: what is the radius of convexity of the family of

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functions f(z) which satisfy $\operatorname{Re}(\frac{f'(z)}{g'(z)}) > 0$ for |z| < 1? The problem was solved also for each of the subfamilies associated with the cases: g(z) is starlike for |z| < 1, $\operatorname{Re}\{g'(z)\} > \alpha$ ($\alpha = 0$, $\frac{1}{2}$) for |z| < 1, g(z) is convex of order $\alpha(0 \le \alpha < 1)$ for |z| < 1.

In this paper we consider functions f(z) which satisfy $\left|\frac{f'(z)}{g'(z)} - 1\right| < 1$ for |z| < 1. The radius of convexity of this family of functions is determined. Also, we find the radius of convexity for the subfamilies associated with each of the cases: g(z) is starlike for |z| < 1, g(z) is convex for |z| < 1, $Re\{g'(z)\} > \alpha$ ($\alpha = 0, \frac{1}{2}$). The case g(z) = z has already been proved by MacGregor $\left[2\right]$,

2. The following lemmas will be used in the proofs of our theorems.

LEMMA 1. $\begin{bmatrix} 4 \end{bmatrix}$ The function h(z) is analytic for |z| < 1 and satisfies h(0) = 1 and Re{h(z)} > α (0 $\leq \alpha < 1$) for |z| < 1 if and only if h(z) = $\frac{1 + (2\alpha - 1)z\phi(z)}{1 + z\phi(z)}$, where $\phi(z)$ is analytic and satisfies $|\phi(z)| \leq 1$ for |z| < 1.

LEMMA 2. If $\phi(z)$ is analytic for $\left|z\right|<1$ and $\left|\phi(z)\right|$ ≤ 1 for $\left|z\right|<1$, then

(i)
$$| \phi'(z) | \leq \frac{1 - |\phi(z)|^2}{1 - |z|^2}$$

(ii)
$$\left| \frac{z\phi'(z) + \phi(z)}{1 + z\phi(z)} \right| \leq \frac{1}{1 - |z|}$$

Part (i) of Lemma 2 is well-known $\begin{bmatrix} 3 \end{bmatrix}$, and part (ii) follows easily, by first applying triangular inequalities and then using part (i).

LEMMA 3. If $h(z) = 1 + c_1 z + \dots$ is analytic for |z| < 1 and $Re\{h(z)\} > 0$ for |z| < 1, then $Re\{h(z)\} \ge \frac{1 - |z|}{1 + |z|} .$

This is a well-known result due to C. Caratheodory.

3. THEOREM 1. Suppose $f(z) = z + a_2 z^2 + ...$ is analytic for |z| < 1 and $g(z) = z + b_2 z^2 + ...$ is analytic and univalent for |z| < 1.

If $\left|\frac{f'(z)}{g'(z)} - 1\right| < 1$ for |z| < 1, then f(z) maps |z| < 1/5 onto a convex domain. The result is sharp.

PROOF. Let $h(z) = \frac{f'(z)}{g'(z)} - 1$. The function g(z) is univalent for |z| < 1, therefore $g'(z) \neq 0$ for |z| < 1. The function h(z) is analytic for |z| < 1, h(0) = 0 and |h(z)| < 1 for |z| < 1. Thus by Schwarz's lemma we have $h(z) = z\phi(z)$ with $|\phi(z)| < 1$.

Therefore

$$f'(z) = g'(z)(1 + z\phi(z)).$$

Taking the logarithmic derivative we obtain

$$\frac{f''(z)}{f'(z)} = \frac{g''(z)}{g'(z)} + \frac{z\phi'(z) + \phi(z)}{1 + z\phi(z)}.$$

Using lemma 2(ii) we get

$$Re \left\{ \frac{zf''(z)}{f'(z)} + 1 \right\} \ge Re \left\{ \frac{zg''(z)}{g'(z)} + 1 \right\} - \frac{|z|}{1 - |z|} . \tag{3.1}$$

Since g(z) is univalent, we have
$$\begin{bmatrix}1\end{bmatrix}$$

$$\text{Re } \{\frac{zg''(z)}{g'(z)}+1\} \geq 1+\frac{\left|z\right|(2\left|z\right|-4)}{1-\left|z\right|^2}.$$

Using this estimate in (3.1) we obtain

Re
$$\frac{zf''(z)}{f'(z)} + 1$$
 $\geq \frac{1-5|z|}{1+|z|^2}$.

This last expression is positive if |z| < 1/5. Since the condition $\text{Re}\left\{\frac{zf''(z)}{f^{\dagger}(z)} + 1\right\}$ |z| < r is necessary and sufficient for f(z) to map |z| < r onto a convex domain, we conclude that f(z) maps |z| < 1/5 onto a convex domain. To show that the estimate obtained in the theorem is sharp, we consider the function f(z) such that $f'(z) = \frac{(1+z)^2}{(1-z)^3}$ with $g(z) = \frac{z}{(1-z)^2}$.

This function f(z) satisfies the hypotheses of the theorem and a short computation shows that $\frac{zf''(z)}{f'(z)} + 1 = \frac{1+5z}{1-z^2}$. This expression vanishes at z = -1/5.

THEOREM 2. Let $f(z) = z + a_2 z^2 + ...$ be analytic for |z| < 1 and

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 $g(z) = z + b_2 z^2 + \dots$ be analytic and starlike for |z| < 1. If $\left| \frac{f'(z)}{g'(z)} - 1 \right| < 1$ for |z| < 1, then f(z) maps |z| < 1/5 onto a convex domain. The result is sharp.

PROOF: Since g(z) is starlike for |z| < 1 implies g(z) is univalent there, the proof of this theorem follows from that of theorem 1.

THEOREM 3. Suppose $f(z) = z + a_2 z^2 + \ldots$ is analytic for |z| < 1 and $g(z) = z + b_2 z^2 + \ldots$ is analytic and convex for |z| < 1. If $\left| \frac{f'(z)}{g'(z)} - 1 \right| < 1$ for |z| < 1, then f(z) maps |z| < 1/3 onto a convex domain. The result is sharp.

PROOF. Since g(z) is convex for |z| < 1 it is univalent there. Therefore $g'(z) \neq 0$ for |z| < 1 and $Re\{\frac{zg''(z)}{g'(z)} + 1\} > 0$ for |z| < 1.

The function $\frac{zg''(z)}{g'(z)} + 1 = 1 + c_1 z + ...$ is regular for |z| < 1 and has positive real part, therefore by lemma 3,

Re
$$\{\frac{zg''(z)}{g'(z)}+1\} \ge \frac{1-|z|}{1+|z|}$$
.

Using this estimate in (3.1) we get

$$\operatorname{Re}\left\{\frac{zf''(z)}{f'(z)} + 1\right\} \ge \frac{1 - |z|}{1 + |z|} - \frac{|z|}{1 - |z|} = \frac{1 - 3|z|}{1 - |z|^2}.$$

This last expression is positive for |z| < 1/3. Thus f(z) maps |z| < 1/3 onto a convex domain. To see that the estimate obtained is sharp, we consider f(z) such that $f'(z) = \frac{1+z}{(1-z)^2}$, with $g(z) = \frac{z}{1-z}$. Thus f(z) satisfies the

hypotheses of the theorem. However $\frac{zf''(z)}{f'(z)} + 1 = \frac{1+3z}{1-z^2}$, which vanishes at z = -1/3.

THEOREM 4. Suppose $f(z) = z + a_2 z^2 + \ldots$ is analytic for |z| < 1 and $g(z) = z + b_2 z^2 + \ldots$ is analytic and Re g'(z) > 0 for |z| < 1. If $\left|\frac{f'(z)}{g'(z)} - 1\right| < 1$ for |z| < 1, then f(z) maps $|z| < (\sqrt{17} - 3)/4$ onto a convex domain. The result is sharp.

PROOF. Since Re g'(z) > 0 for |z| < 1, it follows from lemma 1, with $\alpha = 0$

that

$$g'(z) = \frac{1 - z\phi(z)}{1 + z\phi(z)}$$
 where $|\phi(z)| \le 1$.

Taking the logarithmic derivative of this expression we get

$$\frac{g''(z)}{g'(z)} = \frac{-2(z\phi'(z) + \phi(z))}{1 - z^2\phi^2(z)}$$

Using lemma 2 (ii) and simplifying we get

$$\left|\frac{g''(z)}{g'(z)}\right| \leq \frac{2[|z|+|\phi(z)|]}{(1-|z|^2)(1+|z||\phi(z)|)} \leq \frac{2}{1-|z|^2}.$$

Thus

$$Re\{\frac{zg''(z)}{g'(z)} + 1\} \ge 1 - \frac{2|z|}{1 - |z|^2}$$

Using this estimate in (3.1) we get

$$\operatorname{Re}\left\{\frac{zf''(z)}{f'(z)} + 1\right\} \ge 1 - \frac{2|z|}{1 - |z|^2} - \frac{|z|}{1 - |z|} = \frac{1 - 3|z| - 2|z|^2}{1 - |z|^2}$$

This last expression is positive for $|z| < (\sqrt{17} - 3)/4$. To show that the estimate obtained is sharp we consider f(z) such that $f'(z) = \frac{(1+z)^2}{1-z}$ with

 $g(z) = -z - 2 \log(1 - z)$. This f(z) satisfies the hypotheses of the theorem.

However

$$\frac{zf''(z)}{f'(z)} + 1 = \frac{1 + 3z - 2z^2}{1 - z^2}$$

This last expression vanishes at $z = (3 - \sqrt{17})/4$.

THEOREM 5. Suppose $f(z) = z + a_2 z^2 + \ldots$ is analytic for |z| < 1 and $g(z) = z + b_2 z^2 + \ldots$ is analytic for |z| < 1 and $Re\{g'(z)\} > 1/2$ for |z| < 1. If $\left|\frac{f'(z)}{g'(z)} - 1\right| < 1$ for |z| < 1, then f(z) maps $|z| < r_0$ onto a convex domain, where r_0 is the smallest positive root of $4-4r - 13r^2 - 2r^3 - r^4 = 0$. The result is sharp.

PROOF. Since Re{g'(z)} > 1/2 for |z| < 1, we have by lemma 1 with $\alpha = 1/2$,

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$$g'(z) = \frac{1}{1 + z\phi(z)}$$
. Thus
$$\frac{g''(z)}{g'(z)} = \frac{z\phi'(z) + \phi(z)}{1 + z\phi(z)}$$

From (3.1) we get

$$Re\{\frac{zf''(z)}{f'(z)} + 1\} \ge Re\{\frac{zg''(z)}{g'(z)} + 1 - \frac{|z|}{1 - |z|}\}$$

$$= Re\{\frac{zg''(z)}{g'(z)} + \frac{1 - 2|z|}{1 - |z|}\}$$

$$= Re\{\frac{-z^2\phi'(z) - z\phi(z)}{1 + z\phi(z)} + \frac{1 - 2|z|}{1 - |z|}\}.$$

Therefore, $\operatorname{Re}\left\{\frac{zf''(z)}{f'(z)}+1\right\}$ is positive if

$$Re\{\frac{1-3|z|+|z|(1-z\phi(z))-(1-|z|)z^{2}\phi'(z)}{(1-|z|)(1+z\phi(z))}\} > 0.$$

This will be true if

$$\text{Re}\{ \left[\left| z \right| (1 - z\phi(z)) - \left\{ (3 \left| z \right| - 1) + (1 - \left| z \right|) z^2 \phi '(z) \right\} \right] \quad \left[1 + z\phi(z) \right]^* \} > 0,$$
 (asterisks denote the conjugate of a complex number)

$$\text{Re}\{ \left| z \right| \left| (1 - \left| z \right|^2 \middle| \phi(z) \right|^2) - \left[(3 \middle| z \middle| - 1) + (1 - \middle| z \middle|) z^2 \phi^{\dagger} \left(z \right) \right] \left[1 + z \phi(z) \right]^{\frac{1}{4}} \} > 0,$$

$$\text{Re}\{ \left[(3 \middle| z \middle| - 1) + (1 - \middle| z \middle|) z^2 \phi^{\dagger} \left(z \right) \right] \left[1 + z \phi(z) \right]^{\frac{1}{4}} \} < \left| z \middle| (1 - \middle| z \middle|^2 \middle| \phi(z) \middle|^2 \right).$$

By lemma 2, it is easily seen that this last inequality will be true if

$$3|z| - 1 + (1 - |z|)|z|^2 (\frac{1 - |\phi(z)|^2}{1 - |z|^2}) < |z|(1 - |z||\phi(z)|).$$

This inequality is equivalent to showing

$$r + 3r^2 + r^2(1 + r)x - r^2x^2 < 1$$

where |z| = r (0 < r < 1) and $|\phi(z)| = x (0 \le x \le 1)$.

Let
$$p(x) = r + 3r^2 + r^2(1 + r)x - r^2x^2$$
.

We see that p(x) attains its maximum value q(r) at $x = \frac{1+r}{2}$, consequently

$$q(r) = r + 3r^2 + \frac{r^2}{4} (1 + r)^2$$
.

Since $r + 3r^2 + \frac{r^2}{4}(1+r)^2 < 1$ holds for all $r < r_0$, where r_0 is the smallest positive root of the equation $r + 3r^2 + \frac{r^2}{4}(1+r)^2 = 1$.

This simplifies to

$$4 - 4r - 13r^2 - 2r^3 - r^4 = 0 (3.2)$$

To show that the estimate obtained above is sharp, we let

 $g'(z)=\frac{1}{1+z\phi(z)}$, where $\phi(z)=\frac{z+b}{1+bz}$, $b=\frac{1}{2+r_0}$ where r_0 is the smallest positive root of (3.2); and we select f(z) so that f'(z)=(1-z)g'(z). Since $|\phi(z)|<1$ for |z|<1, we have Re g'(z)>1/2 for |z|<1. Thus f(z) satisfies the hypotheses of the theorem, and

$$\frac{zf''(z)}{f'(z)} + 1 = \frac{1 + (2b - 2)z + (2b^2 - 5b - 1)z^2 - 4b^2z^3 - bz^4}{(1 - z)(1 + bx)(1 + 2bz + z^2)}.$$

Setting $z = r_0$ and $b = \frac{1}{2 + r_0}$, we see that the numerator of the above expression is

$$(1 + r_0)$$
 $(4 - 4r_0 - 13r_0^2 - 2r_0^3 - r_0^4)$

which vanishes.

Theorems 4 and 5 give the radius of convexity for the class of functions f(z) associated with g(z) such that Re $g'(z) > \alpha$ when $\alpha = 0$ and 1/2. For $\alpha \neq 0$, 1/2 our method seems to give only estimates for r_c the radius of convexity and determination of r_c is still open.

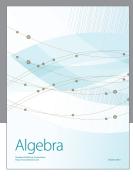
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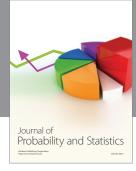
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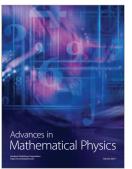






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