Numeracy
Advancing Education in Quantitative Literacy

Volume 4 | Issue 1 Article 9

2011

# Parts of the Whole : Cognition, Schemas, and Quantitative Reasoning

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## Abstract
Based loosely on ideas of Jean Piaget and Richard Skemp, this Parts of the Whole column considers the construction of knowledge in mathematics and quantitative reasoning. Examples are chosen that illustrate an important cognitive difference between quantitative numeracy and classical mathematics, and which illuminate the particular choices instructors must make in order to teach either or both of these.

## Keywords
cognition, schema, numeracy, quantitative reasoning



## Cover Page Footnote
Dorothy Wallace is a professor of mathematics at Dartmouth. She was 2000 New Hampshire CASE Professor of the Year, and the lead PI of the seminal NSF project, Mathematics Across the Curriculum. She recently finished a text in mathematical biology for first-year students, "Situated Complexity." She was a charter board member of the National Numeracy Network and is now co-editor of this journal.

# Parts of the Whole

A Column by D. Wallace

The problem of how best to improve the numeracy of a society is a thorny one, embracing the learning process of a single student but rising in scale to include the management and alteration of an entire system of education. With the issue of quantitative literacy always in mind, this column considers various aspects of the systemic workings of education, the forces acting on classrooms, teachers and students, and mechanisms of both stasis and change.

## Cognition, Schemas, and Quantitative Reasoning

Any serious attempt to improve the extent to which students learn critical material has to be based on an understanding of how people learn. This understanding, in turn, is linked to theories of cognition in general. That is, what does it mean to know something? How does the acquisition of knowledge work? It is appropriate, at this point, to return briefly to the issues raised in the first column in this series. Why are some kinds of knowledge easy to acquire on one's own, whereas other kinds require a guide? Anybody who plans to meddle with the educational system as a whole had best have a firm idea of how cognition is built. This column will outline some of the basic principles gleaned from the writings of Jean Piaget[1] and Richard Skemp,[2] and give examples of how they work. Everything presented is further colored by the author's personal teaching experiences and reflection upon these.

A brief caveat is in order before we proceed. The overview presented here is based on theories that are on substantially less firm ground than the statistics discussed in the last column in this series, for example. Nonetheless, these theories explain many aspects of cognition convincingly and give us more to work with than just our best guesses. Enough work has been done with these ideas to show that they are good guideposts, even though the exact ways in which the mind manages to do these things are still quite opaque.

Consider a simple example: the concept of "circle." Very small children know what a circle is, and every parent who pays attention to the process knows that they learn it by example. The moon, the bicycle tire and the hula-hoop are all circles. The concept is built by abstraction, categorizing numerous familiar

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[1] J. Piaget, *The Psychology of Intelligence*, London: Routledge and Paul, 1950, among other writings.

[2] R. Skemp, *The Psychology of Learning Mathematics*, Hillsdale NJ: Lawrence Erlbaum Associates, 1987.

objects under a single heading that captures a particular common property. The process, although simple, requires guidance from someone that suggests the category and names it, and offers the first few examples. Even into adulthood, the process of looking at unrelated things, abstracting a new, still nameless property that is common among them, and categorizing them accordingly, remains difficult for most people to accomplish without guidance.

As the child grows older, his or her concept of "circle" grows larger. "Circle" becomes more than just a category of objects. Related abstract ideas like "diameter" and "radius" are connected intimately to the concept of circle. The circle in all its abstraction becomes an object of categorization itself, as in "Circles and squares are both geometric things." In classical geometry the circle becomes an object of study, leading to concepts such as "All circles are similar figures" or "A circle is the set of points equidistant from a given one." The idea of "circle" becomes very rich by being connected in the mind with many other concepts such as "square," "point," "equidistant," and "similar." The multitude of concepts, along with the connections among them, is called a "cognitive map."

The cognitive map, along with the individual's ability to utilize it, is called a "schema." A schema includes processes, such as "I can draw a circle," or "I can prove theorems about all circles." The complexity of the processes available to the individual is part of the growth of cognition also. Enlarging the schema that children have when they say "I can draw a circle" to the one that a young adult has when saying "I can prove theorems about circles" represents a major shift in cognition for the student. In a sense, the schema for the older person is a completely different structure from that of the child and must be built entirely through the artifice of education, even though it certainly includes the earlier schema intact within it.

Another necessary form of cognitive growth does not require enlarging the schema of the individual, but merely extending the cognitive map. For example, children who know what a circle is may very well observe on their own that round objects are capable of rolling. The idea of rolling and the idea of circle will be forever connected in that child's mind, yet the learning may have been entirely spontaneous with no guidance whatsoever. The child now has a richer concept of "circle," but may obtain this richness without altering the basic schema for the concept. The schema is extended and reinforced in the process. The vast majority of learning that we do in school falls into this category and may be pursued in a relatively independent way by a student.

The mechanism by which a schema is substantially enlarged or changed to enable the student to think in a new way about an old concept is called "cognitive dissonance." Now, this may not be the only mechanism by which such growth can happen, but it is the one that has been best understood and articulated. The way it works is by confronting the student with numerous examples or problems or questions that cannot be approached with their present schema, and then offering the student a way of looking at these questions that resolves the dilemma without completely erasing the previous schema. Many of the experiments done by Piaget are of this nature. It can be an uncomfortable process for the student, because it requires the realization that one does not understand something previously supposed mastered. People do not usually change an intact schema easily or willingly. Moments in history where an individual arrived at a new schema for a concept on his or her own have been true breakthroughs. Cognitive dissonance almost always requires a teacher, although making productive use of it in the classroom requires a real understanding of the subject material and an overt recognition of how it is to be progressively understood.

Again, these issues are closely linked to the work of Piaget on developmental stages, each of which is characterized by particular schema. Piaget was looking for stages characterized roughly by the age of the child, at which certain concepts become teachable. For older students the question of what is teachable resides more in the current structure of their schema and the experiences and tutelage we offer based on that knowledge. Recent research on late stage growth spurts of the brain may also help our understanding of how to teach material more effectively. If, indeed, the brain experiences a growth spurt during late adolescence, it must surely result in new learning possibilities for the individual at that point. Such physiological changes must necessarily be a major source of variation in learning among students during the time frame when they happen, although until maturity this variation is a temporary phenomenon that could be addressed by a flexible educational structure.

Given the two kinds of schema construction we have described, it might be useful to visualize the cognitive map as a sort of pyramid. As the schema for a given concept grows, there are points at which it must be enlarged and substantially changed in order to accommodate new information. This is the vertical direction in the pyramid. At every stage, however, it is necessary to reinforce the existing schema by connecting it broadly with as many contexts as possible. A subject that can be learned independently will have little growth in the vertical direction and much growth horizontally. A subject that requires a lot of vertical growth, such as mathematics, will need substantial instruction and leading. In order to grow vertically, the subject will also require horizontal growth, because it is difficult to enlarge a schema that is not a fully functioning part of the cognitive structure in the first place. The range of possibilities is pictured in Figure 1.

The existence of multiple schemas for a concept, arranged from simplest to most complex or abstract (in the case of mathematics), is a major source of variation in student achievement that has an impact across the entire educational system. It is possible, for example, to teach algebra in such a way that a low-level schema is consistently reinforced even though the problems being solved by the student get harder and harder. Another teacher of algebra may demand that the students shift their schema to one of the more abstract ones in order to solve a new kind of problem. The result will be a wide variation in the achievement of students in those two classes. A sketch of the corresponding pyramids would look like Figures 2 and 3.

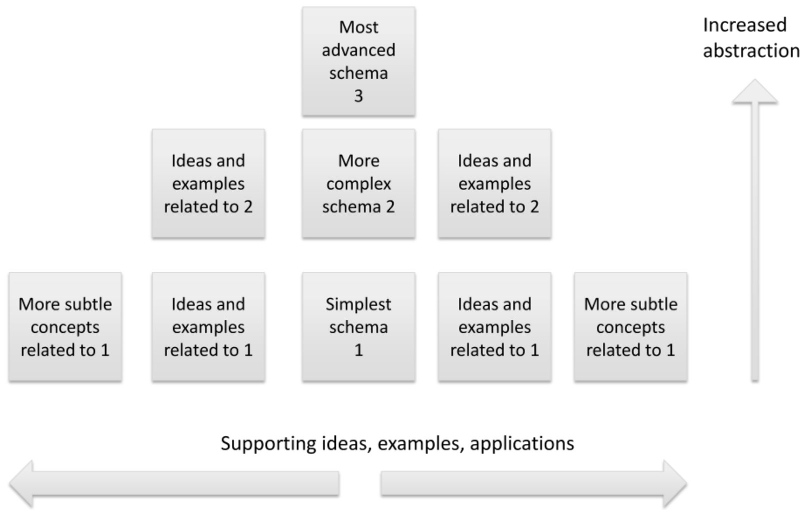


**Figure 1:** A metaphor for how cognition is built.

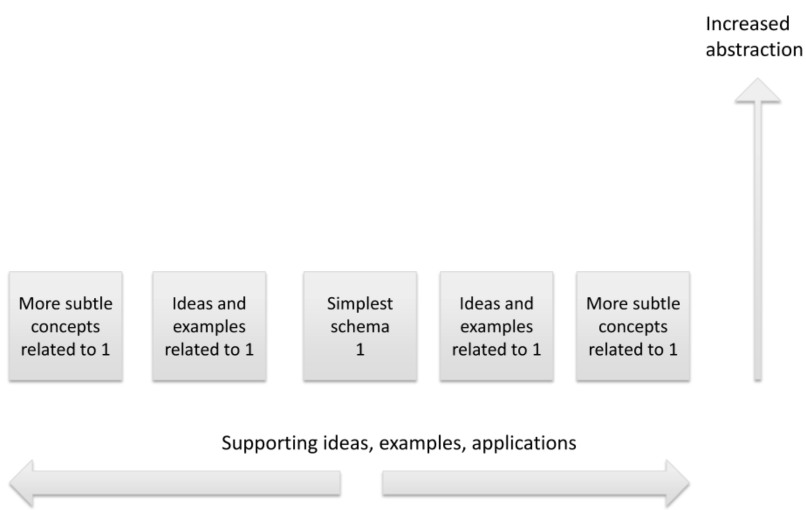


**Figure 2:** An intact, useful and well-supported schema.


DOI: http://dx.doi.org/10.5038/1936-4660.4.1.9

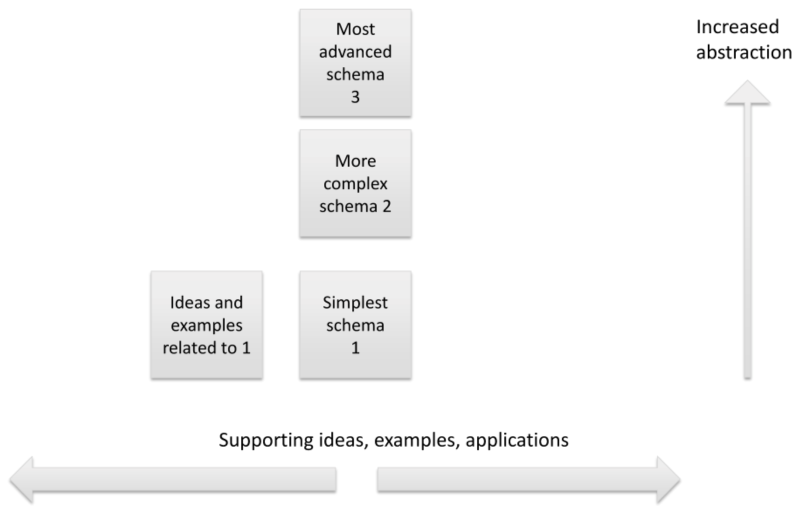


**Figure 3:** A path to abstraction with few supporting connections.

The same amount of learning has taken place, but completely different kinds of learning predominate in each class. If these students are mixed together in the subsequent course, what should the next teacher do? It is completely unclear. It is possible to complete either pyramid to look like Figure 1, yet the process for teaching would be completely different for each of the two starting structures. None of this is the fault of any teacher or student, but rather the fault of an educational system that hasn't figured out its own internal process or what to aim for at each stage of the process, and textbooks that amplify the confusion.

## Improving quantitative reasoning

If numeracy, as many authors have suggested, resides in using mathematics to understand the real worlds of data and science and money, then it seems clear that teaching quantitative reasoning specifically means broadening the pyramid in Figure 1 at each level as mathematics concepts are learned. Both mathematics and quantitative reasoning textbooks really try to do this by providing applications of core concepts in both the text and exercises for students. And yet we see that the resulting cognitive structure is far from secure. Students do not recognize the math out of context and it does not occur to them to use it even when encountering similar applications. It is worth considering various reasons why the system might be failing the student.

## The core concept

It is of little use to know that an income tax rate is a proportion of the salary if the concept of proportion is not firmly in place. It is of no use to include problems on interest rates in the section of the text describing exponential growth if the concept for exponential growth is "something that grows kind of like the exponential function." Interest rates only make sense in the context of exponential growth if the concept in play is "growth that is in proportion to the current amount." In this example, the student could have a very firm grasp of the first concept for exponential growth yet be fairly unaware of the second one. In this case there is a robust intact core concept, but it's not the one needed. The student has to move vertically up the scale pictured in Figure 1 at least a little bit in order to tie the example of interest rates to the math concept of exponential growth.

This kind of breakdown in the system is probably the most frequently mentioned source of difficulty in teaching quantitative reasoning. "They don't know the basic math, so how can they apply it?" But as the example of exponential growth shows, there can be multiple schemas associated to a concept, and reinforcing the wrong one may leave the student still unable to make necessary connections to a given application. Rarely are schema made explicit in textbooks or in the classroom, and instructors may be unaware of which schema are necessary to support particular kinds of quantitative reasoning.

Of course, some of the time the students really do not understand the basic concepts, and this has to be fixed before they can apply them. Two things are important to remember in this case. First, it is hard to tell this situation from the last one, in which a schema is mastered that is inadequate to the task. Second, because it really is difficult to apply a piece of mathematics you don't understand, the push to introduce applications requiring quantitative reasoning can backfire. By pushing the application before the relevant schema is mastered, the system wastes the student's valuable time and energy. Worse, it reinforces the idea that mathematics makes no sense and the belief that the student is not good at math and has no hope of mastering it.

## *How many examples?*

The images in Figures 1–3 look like stacked bricks. It is worth considering what kind of effort might be needed to put one of these bricks in place. In the case of constructing a simple schema for "circle," it takes exposure to a lot of examples of circles, as well as quite a few examples of things that are not circles, to solidify the criteria the mind uses to identify something as being a circle or not. Piaget and Skemp both argue forcefully that the mind constructs schema from examples, and requires quite a few of them before it can recognize an incoming situation as belonging to that schema or not.

Yet we do not teach as if this were the case. Mathematics texts in particular tend to work from the general to the particular. First a definition or concept is presented, and then afterwards examples of the concept are offered. The actual concept appears to come from abstract considerations (which is, granted, sometimes the case). This makes it difficult to see that the examples, which embody the quantitative reasoning we wish students to do, are not only the instantiations of the concept but often the actual source of it.

A second source of difficulty lies in the number and kind of examples offered. This author remembers sitting through half an abstract algebra class where the only examples of groups offered were the symmetric group and the integers mod N. She spent way too much time wondering why anybody would bother to generalize the definition of group just to take care of these two (families of) cases. She now believes (without, alas, benefit of research) that it takes about five well-chosen examples and non-examples of a concept or definition to get it secure in a student's head. One hundred identical problems do not do the job, nor does one brilliantly constructed example. And yet, instructors rarely allow themselves the luxury of letting their class work through five deeply illustrative examples of anything.

## *Mortar*

It is completely possible for a student to have a good grasp of exponential growth and a good grasp of how interest rates work, and yet not make the connection between these things. This is a weakness in the cognitive structure because it deprives a mathematical concept (exponential growth) from drawing on one of its most useful examples, and prevents the example from generalizing. One could say that the bricks in Figure 1 lacked mortar between them. The ability to transfer knowledge from one context to another depends on the connections made from one cognitive structure to another. It is worth asking how one might address the question of connection directly.

The most obvious connections are through language. It is possible to explain exponential growth without using the words "interest" or "principal" and equally possible to explain bank accounts without using the word "exponential" or the phrase "relative rate." Not only is it possible, it is common. The duplication of language can be initially confusing, because there are multiple terms that mean the same thing and some words that seem peripheral to the immediate question. But seeing words redundantly across a multiple situations creates a useful cognitive link that allows the student to access information stored (metaphorically of course) under a different heading.

Logic can also provide a deep connection between concepts. Although one can explain what happens to money invested at a certain interest rate with just percents and a calculator, it is much more powerful to set it in the context of an exponential growth problem. Part of that power lies in the direct link between potentially separate concepts. It is always a temptation to give a student the simplest possible explanation of a phenomenon and leave it at that. This may be a quick and efficient route to teaching a student to solve a particular kind of problem, but it doesn't necessarily build the best understanding in the long run.

Calculation can provide a strong link between concepts. When the calculator used to explore principal and interest is recruited to do the same calculations for population growth, a concrete physical action now links the two ideas. If the equations governing exponential growth are then used to solve problems of money and biology, the link becomes even stronger. Textbooks attempt to do this, but usually the relationship is one directional, from math to example. Quantitative reasoning demands that the student seek the math in the situation. This is a behavior that can certainly be reinforced in the classroom.

Finally, it is worth mentioning the potential emotional links that might be available to cement ideas together. Many concrete problems that require a quantitative reasoning approach are also very motivating. Questions in medical and social contexts have a natural ability to engage people. Yet instructors often teach those very topics in the least emotional way possible. It is natural to want to avoid upsetting students. But distress creates memory. If the math predicts that something really unpleasant or really fantastic is going to happen in a medical context (for example), the associated emotion becomes a strong link between the context and the math.

## *Pity the poor math instructors*

A serious education in mathematics is a steep climb up the vertical direction of the cognitive structure in Figure 1. The whole point of a "real" math course is to give the student a new, larger schema for concepts lower on the structure. Linear algebra, for example, abstracts systems of equations into matrix algebra. Then it abstracts the matrix algebra into coordinate-free linear operators. A course in linear algebra that failed to create these cognitive structures in students would be useless preparation for mathematics. Creating new, more powerful schema for mathematical objects is what mathematicians do. It is a useful and beautiful vocation.

It is a habit of mind that poorly prepares one to educate students in the horizontal directions pictured by Figure 1. Even when a successful layer in that diagram is laid, and all the connections made, it feels like no mathematics has been gained. And even when the instructor values such growth, few assessment instruments are available that can distinguish between a student whose command of material resembles Figure 2 versus one whose understanding is more like Figure 3. How then is the instructor supposed to know what, let alone how, to teach? Clearly there is room here for much analysis of specific concepts and methods for evaluating the mastery of these that would be a guide for those of us wishing to teach quantitative reasoning in our courses.