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Symbolic Computation of Lump Solutions to a Combined (2+1)-dimensional Nonlinear Evolution Equation

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by

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Table of Contents

Abstract

This thesis aims to consider a (2+1)-dimensional nonlinear evolution equation and its lump solutions. By using symbolic computation, two classes of lump solutions are presented. And for two specific chosen examples, we will show three-dimensional plots and density plots to exhibit dynamical features of the lump solution, which are made by Maple plot tools.

Chapter 1

Introduction

Solvable partial differential equations always involve constant coefficients, and they are linear. However, it's difficult to solve partial differential equations with variable coefficients or nonlinear terms via analytic ways. Nevertheless, soliton theory provides some methods to solve nonlinear partial differential equations ([1],[15]). The Hirota bilinear method, historically developed for integrable equations, is an essential method to get soliton solutions and lump solutions because of its simplicity and directness ([2], [4]).

Soliton solutions are analytic and exponentially localized in all directions in time and space, and lump solutions are a class of rational function solutions that are localized in all directions in space [7], which can be derived from taking long wave limits of soliton equations ([14]). Also, lump solutions are originated from solving integrable equations in (2+1)-dimensions (see, e.g, [12], [13], [17]). Long wave limits of N-soliton solutions can produce special lumps as envelope solutions [16]. Many existing studies on (2+1)-dimensional integrable equations show abundance of lump solutions (see, e.g., [12], [13]), which include BKP equation [3], the (2+1)-dimensional Ito equation [18], the Davey-Stewartson equation II [16], the Ishimori-I equation [5], the KP equation with a self-consistent source [19], and the second KP equation [10]. In order to compute lump solutions, a significant step is to find a positive quadratic function solution to Hirota bilinear equation [11]. Then from the solutions of such positive quadratic function, we are able to compute lump solutions to nonlinear partial differential equations through the logarithmic transformations.

In Hirota bilinear formation, we have a (2+1)-dimensional partial differential equation with a variable u, which connects with a Hirota bilinear differential equation

$$P(D_x, D_y, D_t)f \cdot f = 0,$$

where P is a polynomial and D_x, D_y, D_t are Hirota's bilinear derivatives. The dependent variable u is often defined by one of the logarithmic transforms

$$u = 2(ln\ f)_x, \quad u = 2(ln\ f)_{xx}.$$

Then soliton solutions can be formulated as

$$f = \sum_{\mu=0,1} exp(\sum_{i=1}^{N} \mu_i \xi_i + \sum_{i<j} \mu_i \mu_j a_{ij}),$$

where

$$\xi_i = k_i x + l_i y - w_i t + \xi_{i,0},\ 1 \leq i \leq N,$$

and k_i, l_i, w_i, $1 \leq i \leq N$, satisfy the corresponding dispersion relations. Also, $\xi_{i,0}$, $1 \leq i \leq N$, are arbitrary phase shifts.

In this paper, we propose a combined fourth-order nonlinear evolution equation in (2+1)-dimensions and determine its lump solutions. Our combined nonlinear equation contains all second-order linear terms to ensure that our equation processes lump solutions. Also, the nonlinear equation contains three types of nonlinear terms that include two types of the '2+2'-type terms and one '3+1'-type term, which never appear in recent research. Symbolic computation is conducted in Maple to compute and simplify the lump solution expression. We analyze its coefficients for two specific examples of the nonlinear equation and illustrate the corresponding specific lump solutions using 3-dimensional plots and density plots to show the structures of our solutions.

Chapter 2

A Combined Nonlinear PDE and Its Lump Solutions

2.1 Combined Fourth-order Nonlinear Model

We would like to consider a general combined fourth-order nonlinear partial differential equation in $(2+1)$-dimensions as:

$$\begin{aligned}P(u) &= \alpha_1[4u_tu_{xt}+u_xu_{tt}+u_{xx}k+u_{xxtt}]+\alpha_2[3(u_xu_y)_x+u_{xxxy}] \\ &+\alpha_3[4u_yu_{xy}+u_xu_{yy}+u_{xx}v+u_{xxyy}]+\delta_1u_{yt}+\delta_2u_{xx}+\delta_3u_{xt}+\delta_4u_{xy}+\delta_5u_{yy}+\delta_6u_{tt}=0,\end{aligned} \tag{2.1}$$

where $k_x=u_{tt}$, $v_x=u_{yy}$, and α_i and δ_j are arbitrary constant with $1\le i\le 3$, $1\le j\le 6$. The coefficients α_i, $1\le i\le 3$ correspond to three combinations of fourth-order derivative terms, and δ_j, $1\le j\le 6$ correspond to all linear second-order derivative terms.

When $\alpha_1=0$, $\alpha_2=1$, $\alpha_3=0$, $\delta_3=\delta_5=1$ and the other δ_j's are all zero, the equation (2.1) obtains an equation that process lump solutions named the generalized Calogero-Bogoyavlenskii-Schiff equation:

$$3(u_xu_y)_x+u_{xxxy}+u_{xt}+u_{yy}=0, \tag{2.2}$$

which possesses a Hirota bilinear form under $u=2(ln\ f)_x$:

$$(D_x^3D_y+D_xD_t+D_y^2)f\cdot f=0. \tag{2.3}$$

Generally, under the logarithmic transformation

$$u=2(ln\ f)_x=2\frac{f_x}{f},\quad v_x=2(ln\ f)_{yy}=\frac{2(f_{yy}f-f_y^2)}{f^2},\quad k_x=2(ln\ f)_{tt}=\frac{2(f_{tt}f-f_t^2)}{f^2}, \tag{2.4}$$

we can transform the equation (2.1) into the Hirota bilinear form

$$\begin{aligned}B(f) &= (\alpha_1D_x^2D_t^2+\alpha_2D_x^3D_y+\alpha_3D_x^2D_y^2 \\ &+\delta_1D_yD_t+\delta_2D_x^2+\delta_3D_xD_t+\delta_4D_xD_y+\delta_5D_y^2+\delta_6D_t^2)f\cdot f=0.\end{aligned} \tag{2.5}$$

Precisely, the actual relation between the bilinear equation and the combined nonlinear equation states $P(u) = (\frac{B(f)}{f^2})_x$, where u, v, k satisfy (2.4). Therefore, if f solves the bilinear equation (2.5), then $u = 2(ln\, f)_x$ solves our combined nonlinear equation (2.1).

This combined fourth-order nonlinear equation has three types of fourth-order derivative term and all linear second-order derivative terms. If $\alpha_1 \neq 0$ and $\alpha_3 \neq 0$, the equation contains two '2+2'-type fourth-order term (like $D_x^2 D_y^2$) which is barely mentioned in the past.

2.2 Lump Solutions

In this section, we are going to compute lump solutions to the nonlinear partial differential equation (2.1) through symbolic computation.

A crucial step in finding lump solutions is to determine the positive quadratic solutions, and so we start with

$$f = (a_1 x + a_2 y + a_3 t + a_4)^2 + (a_5 x + a_6 y + a_7 t + a_8)^2 + a_9, \tag{2.6}$$

where a_i, $1 \leq i \leq 9$, are constant parameters to be determined, which generate lump solutions to our combined fourth-order nonlinear equation (2.1).

2.2.1 the Case of $\delta_6 = 0$

Firstly, we consider the case of $\delta_6 = 0$ for our combined nonlinear equation (2.1). Through symbolic computation, it directly gives us a solution to the parameters

$$\begin{aligned} a_3 &= -\frac{b_1}{(a_1\delta_3 + a_2\delta_1)^2 + (a_5\delta_3 + a_6\delta_1)^2}, \\ a_7 &= -\frac{b_2}{(a_1\delta_3 + a_2\delta_1)^2 + (a_5\delta_3 + a_6\delta_1)^2}, \\ a_9 &= -\frac{b_3\alpha_1 + b_4\alpha_2 + b_5\alpha_3}{(a_1a_6 - a_2a_5)^2(\delta_1^2\delta_2 - \delta_1\delta_2\delta_4 + \delta_3^2\delta_5)((a_1\delta_3 + a_2\delta_1)^2 + (a_5\delta_3 + a_6\delta_1)^2)} \end{aligned} \tag{2.7}$$

and all other $a_i's$ are arbitrary. The above constants $b_i, 1 \leq i \leq 5$, are defined as belows:

$$
\begin{aligned}
b_1 &= [((a_1^2 - a_5^2)a_2 + 2a_1a_5a_6)\delta_2 + a_1(a_2^2 + a_6^2)\delta_4 + a_2(a_2^2 + a_6^2)\delta_5]\delta_1 \\
&\quad + [a_1(a_1^2 + a_5^2)\delta_2 + a_2(a_1^2 + a_5^2)\delta_4 + ((a_2^2 - a_6^2)a_1 + 2a_2a_5a_6)\delta_5]\delta_3, \\
b_2 &= [(-a_1^2a_6 + 2a_1a_2a_5 + a_5^2a_6)\delta_2 + (a_2^2a_5 + a_5a_6^2)\delta_4 + (a_2^2a_6 + a_6^3)\delta_5]\delta_1 \\
&\quad + [(a_1^2a_5 + a_5^3)\delta_3\delta_2 + (a_1^2a_6 + a_5^2a_6)\delta_4 + (2a_1a_2a_6 - a_2^2a_5 + a_5a_6^2)\delta_5]\delta_3, \\
b_3 &= [(a_1^2 + a_5^2)^2p_1\delta_2^2 + (6(a_2^2 + a_6^2)(a_1^2 + a_5^2)p_2\delta_4 + 2(a_2^2 + a_6^2)(a_1^2 + a_5^2)p_3\delta_5)\delta_2 \\
&\quad + 3(a_2^2 + a_6^2)^2(a_1^2 + a_5^2)^2\delta_4^2 + 6(a_2^2 + a_6^2)^2p_2\delta_4\delta_5 + (a_2^2 + a_6^2)^2p_1\delta_5^2]\delta_1^2 \\
&\quad + [6(a_1^2 + a_5^2)^2p_2\delta_3\delta_2^2 + (4(a_1^2 + a_5^2)^2p_1\delta_4 + 4p_2p_3)\delta_5)\delta_2\delta_3 \\
&\quad + (6(a_2^2 + a_6^2)(a_1^2 + a_5^2)p_2\delta_4^2 + 4(a_2^2 + a_6^2)(a_1^2 + a_5^2)p_3\delta_4\delta_5 \\
&\quad + 2(a_2^2 + a_6^2)(a_1a_2 + a_5a_6)p_3\delta_5^2)\delta_3]\delta_1 \\
&\quad + 3(a_1^2 + a_5^2)^4\delta_2^2\delta_3^2 + (6(a_1^2 + a_5^2)^2p_2\delta_4 + 6(a_1^2 + a_5^2)^2p_4\delta_5)\delta_2\delta_3^2 \\
&\quad + ((a_1^2 + a_5^2)^2p_1\delta_4^2 + 2p_2p_3\delta_4\delta_5 + p_5\delta_5^2)\delta_3^2, \\
b_4 &= 3p_2((a_1\delta_3 + a_2\delta_1)^2 + (a_5\delta_3 + a_6\delta_1)^2)^2, \\
b_5 &= p_1((a_2^2 + a_6^2)\delta_1^2 + 2(a_1a_2 + a_5a_6)\delta_1\delta_3 + (a_1^2 + a_5^2)\delta_3^2).
\end{aligned}
\tag{2.8}
$$

where

$$
\begin{aligned}
p_1 &= 3a_1^2a_2^2 + a_1^2a_6^2 + 4a_1a_2a_5a_6 + a_2^2a_5^2 + 3a_5^2a_6^2, \\
p_2 &= (a_1^2 + a_5^2)(a_1a_2 + a_5a_6), \\
p_3 &= 3a_1^2a_2^2 - a_1^2a_6^2 + 8a_1a_2a_5a_6 - a_2^2a_5^2 + 3a_5^2a_6^2, \\
p_4 &= (a_1a_2 - a_1a_6 + a_2a_5 + a_5a_6)(a_1a_2 + a_1a_6 - a_2a_5 + a_5a_6), \\
p_5 &= 3a_1^4a_2^4 - 2a_1^4a_2^2a_6^2 + 3a_1^4a_6^4 + 16a_1^3a_2^3a_5a_6 - 16a_1^3a_2a_5a_6^3 - 2a_1^2a_2^4a_5^2 + 44a_1^2a_2^2a_5^2a_6^2 \\
&\quad - 2a_1^2a_5^2a_6^4 - 16a_1a_2^3a_5^3a_6 + 16a_1a_2a_5^3a_6^3 + 3a_2^4a_5^4 - 2a_2^2a_5^4a_6^2 + 3a_5^4a_6^4.
\end{aligned}
\tag{2.9}
$$

The simplification of presenting the formulas in (2.7)-(2.9) has been conducted with Maple. Based on the above solution formulas, we need to ensure that the above solutions present a lump, and so we require

$$(\delta_1 + \delta_3)^2(\delta_1^2\delta_2 - \delta_1\delta_2\delta_4 + \delta_3^2\delta_5) \neq 0 \tag{2.10}$$

to generate lump solutions to the nonlinear equation (2.1).

2.2.2 The Case of $\delta_5 = 0$

Secondly, we will consider the case of $\delta_5 = 0$ for the equation (2.1). The same direct symbolic computation provides us with a set of solutions for the parameters, which are

$$\begin{aligned} a_2 &= -\frac{c_1}{(a_1\delta_4 + a_3\delta_1)^2 + (a_5\delta_4 + a_7\delta_1)^2}, \\ a_6 &= -\frac{c_2}{(a_1\delta_4 + a_3\delta_1)^2 + (a_5\delta_4 + a_7\delta_1)^2}, \\ a_9 &= \frac{-c_3\alpha_1 + 3c_4\alpha_2 - c_5\alpha_3}{(a_1a_7 - a_3a_5)^2(\delta_1^2\delta_2 - \delta_1\delta_3\delta_4 + \delta_4^2\delta_6)((a_3\delta_1 + a_1\delta_4)^2 + (a_7\delta_1 + a_5\delta_4)^2)}, \end{aligned} \tag{2.11}$$

and all other $a_i's$ are arbitrary. The involved constant $c_i's$, $1 \le i \le 5$, are defined as follows:

$$\begin{aligned} c_1 =& [a_1(a_3^2 + a_7^2)\delta_3 + (a_1^2a_3 + 2a_1a_5a_7 - a_3a_5^2)\delta_2 + a_3(a_3^2 + a_7^2)\delta_6]\delta_1 \\ & + [a_3(a_1^2 + a_5^2)\delta_3 + (a_1a_3^2 - a_1a_7^2 + 2a_3a_5a_7)\delta_6 + a_1(a_1^2 + a_5^2)\delta_2]\delta_4, \\ c_2 =& [a_5(a_3^2 + a_7^2)\delta_3 + (-a_1^2a_7 + 2a_1a_3a_5 + a_7a_5^2)\delta_2 + a_7(a_3^2 + a_7^2)\delta_6]\delta_1 \\ & + [a_7(a_1^2 + a_5^2)\delta_3 + (2a_1a_3a_7 - a_3^2a_5 + a_5a_7^2)\delta_6 + a_5(a_1^2 + a_5^2)\delta_2]\delta_4 \\ c_3 =& (a_3^2 + a_7^2)^2q_3\delta_1^4 + 4(a_3^2 + a_7^2)(a_1a_3 + a_5a_7)q_3\delta_1^3\delta_4 + 2q_3\delta_1^2\delta_4^2 \\ & +4q_2q_3\delta_1\delta_4^3 + (a_1^2 + a_5^2)^2q_3\delta_4^4, \\ c_4 =& [(a_3^2 + a_7^2)(a_1^2 + a_5^2)q_2\delta_2 + (a_3^2 + a_7^2)^2(a_1^2 + a_5^2)^2\delta_3 + (a_3^2 + a_7^2)^2q_2\delta_6]\delta_1^3 \\ & +[(a_1^2 + a_5^2)^2q_3\delta_2\delta_4 + 3(a_3^2 + a_7^2)(a_1^2 + a_5^2)q_2\delta_3\delta_4 + (a_3^2 + a_7^2)(a_1^2 + a_5^2)q_4\delta_4\delta_6]\delta_1^2 \\ & +[3(a_1^2 + a_5^2)^2q_2\delta_2\delta_4^2 + (a_1^2 + a_5^2)^2q_3\delta_3\delta_4^2 + q_2q_4\delta_4^2\delta_6]\delta_1 \\ & +(a_1^2 + a_5^2)^4\delta_2\delta_4^3 + (a_1^2 + a_5^2)^2q_2\delta_3\delta_4^3 + (a_1^2 + a_5^2)^2q_1\delta_4^3\delta_6, \\ c_5 =& [(a_1^2 + a_5^2)^2q_3\delta_2^2 + (6(a_3^2 + a_7^2)(a_1^2 + a_5^2)q_2\delta_3 + 2(a_3^2 + a_7^2)(a_1^2 + a_5^2)q_4\delta_6)\delta_2 \\ & +3(a_3^2 + a_7^2)^2(a_1^2 + a_5^2)^2\delta_3^2 + 6(a_3^2 + a_7^2)^2q_2\delta_3\delta_6 + (a_3^2 + a_7^2)^2q_3\delta_6^2]\delta_1^2 \\ & +[6(a_1^2 + a_5^2)^2q_2\delta_2^2\delta_4 + (4(a_1^2 + a_5^2)^2q_3\delta_3\delta_4 + 4q_2q_4\delta_6\delta_4)\delta_2 \\ & +6(a_3^2 + a_7^2)(a_1^2 + a_5^2)q_2\delta_4\delta_3^2 + 4(a_3^2 + a_7^2)(a_1^2 + a_5^2)q_4\delta_3\delta_4\delta_6 \\ & +2(a_3^2 + a_7^2)(a_1a_3 + a_5a_7)q_4\delta_4\delta_6^2]\delta_1 \\ & +3(a_1^2 + a_5^2)^4\delta_2^2\delta_4^2 + (6(a_1^2 + a_5^2)^2q_2\delta_3\delta_4^2 + 6(a_1^2 + a_5^2)^2q_1\delta_4^2\delta_6)\delta_2 \\ & +(a_1^2 + a_5^2)^2q_3\delta_3^2\delta_4^2 + 2q_2q_4\delta_3\delta_4^2\delta_6 + q_5\delta_4^2\delta_6^2, \end{aligned} \tag{2.12}$$

where

$$
\begin{aligned}
q_1 &= (a_1a_3 - a_1a_7 + a_3a_5 + a_5a_7)(a_1a_3 + a_1a_7 - a_3a_5 + a_5a_7),\\
q_2 &= (a_1^2 + a_5^2)(a_1a_3 + a_5a_7),\\
q_3 &= 3a_1^2a_3^2 + a_1^2a_7^2 + 4a_1a_3a_5a_7 + a_3^2a_5^2 + 3a_5^2a_7^2,\\
q_4 &= 3a_1^2a_3^2 - a_1^2a_7^2 + 8a_1a_3a_5a_7 - a_3^2a_5^2 + 3a_5^2a_7^2,\\
q_5 &= 3a_1^4a_3^4 - 2a_1^4a_3^2a_7^2 + 3a_1^4a_7^4 + 16a_1^3a_3^3a_5a_7 - 16a_1^3a_3a_5a_7^3 - 2a_1^2a_3^4a_5^2\\
&\quad + 44a_1^2a_3^2a_5^2a_7^2 - 2a_1^2a_5^2a_7^4 - 16a_1a_3^3a_5^3a_7 + 16a_1a_3a_5^3a_7^3 + 3a_3^4a_5^4 - 2a_3^2a_5^4a_7^2 + 3a_5^4a_7^4.
\end{aligned}
\tag{2.13}
$$

All above formulas from (2.11) to (2.13) have been simplified directly through Maple process. For $\delta_5 = 0$ we need to check when the solutions are lumps, and so we need the basic condition

$$
(\delta_1 + \delta_4)^2(\delta_1^2\delta_2 - \delta_1\delta_3\delta_4 + \delta_4^2\delta_6) \neq 0. \tag{2.14}
$$

Also, for this case, we need to check when the set of the proceed parameters gives lumps, and thus we compute

$$
\begin{aligned}
&a_1a_6 - a_2a_5\\
&= \frac{(a_1a_7 - a_3a_5)((a_1^2\delta_2 + \delta_2a_5^2 - \delta_6(a_3^2 + a_7^2))\delta_1 - \delta_4(a_1^2\delta_3 + 2a_1a_3\delta_6 + a_5^2\delta_3 + 2a_5a_7\delta_6))}{(a_1\delta_4 + a_3\delta_1)^2 + (a_5\delta_4 + a_7\delta_1)^2}.
\end{aligned}
\tag{2.15}
$$

It follows that $a_1a_6 - a_2a_5 \neq 0$ if and only if

$$
\begin{cases}
a_1a_7 - a_3a_5 \neq 0, \quad \delta_1^2 + \delta_4^2 \neq 0,\\
(a_1^2\delta_2 + \delta_2a_5^2 - \delta_6(a_3^2 + a_7^2))\delta_1 - \delta_4(a_1^2\delta_3 + 2a_1a_3\delta_6 + a_5^2\delta_3 + 2a_5a_7\delta_6) \neq 0.
\end{cases}
\tag{2.16}
$$

Both $a_1a_6 - a_2a_5 \neq 0$ and $a_9 > 0$ assure that the corresponding set of parameters will present lump solutions.

2.2.3 Equivalence Between Two Cases of Solutions

When $\delta_5 = \delta_6 = 0$, we have two sets of parameters that process lumps, determined by section 2.2.1 and section 2.2.2. Also, we show the equivalence between these two cases of corresponding lump solutions.

When taking $\delta_5 = \delta_6 = 0$, the combined equation (2.1) becomes

$$P(u) = \alpha_1[4u_t u_{xt} + u_x u_{tt} + u_{xx}k + u_{xxtt}] + \alpha_2[3(u_x u_y)_x + u_{xxxy}] \\ + \alpha_3[4u_y u_{xy} + u_x u_{yy} + u_{xx}v + u_{xxyy}] + \delta_1 u_{yt} + \delta_2 u_{xx} + \delta_3 u_{xt} + \delta_4 u_{xy} = 0. \tag{2.17}$$

So the related Hirota bilinear form becomes

$$B(f) = (\alpha_1 D_x^2 D_t^2 + \alpha_2 D_x^3 D_y + \alpha_3 D_x^2 D_y^2 \\ + \delta_1 D_y D_t + \delta_2 D_x^2 + \delta_3 D_x D_t + \delta_4 D_x D_y) f \cdot f = 0. \tag{2.18}$$

For the first class of the lump solutions defined by (2.7) with (2.8) (2.9), we have

$$a_3 = -\frac{[((a_1^2 - a_5^2)a_2 + 2a_1a_5a_6))\delta_2 + a_1(a_2^2 + a_6^2)\delta_4]\delta_1 + [a_1(a_1^2 + a_5^2)\delta_2 + a_2(a_1^2 + a_5^2)\delta_4]\delta_3}{(a_1\delta_3 + a_2\delta_1)^2 + (a_5\delta_3 + a_6\delta_1)^2},$$

$$a_7 = -\frac{[(-a_1^2a_6 + 2a_1a_2a_5 + a_5^2a_6)\delta_2 + (a_2^2a_5 + a_5a_6^2)\delta_4]\delta_1 + [(a_1^2a_5 + a_5^3)\delta_3\delta_2 + (a_1^2a_6 + a_5^2a_6)\delta_4]\delta_3}{(a_1\delta_3 + a_2\delta_1)^2 + (a_5\delta_3 + a_6\delta_1)^2}, \tag{2.19}$$

$$a_9 = -\frac{d_3\alpha_1 + d_4\alpha_2 + d_5\alpha_3}{(a_1a_6 - a_2a_5)^2(\delta_1^2\delta_2 - \delta_1\delta_2\delta_4)((a_1\delta_3 + a_2\delta_1)^2 + (a_5\delta_3 + a_6\delta_1)^2)}$$

where

$$b_3 = [(a_1^2 + a_5^2)^2 p_1\delta_2^2 + (6(a_2^2 + a_6^2)(a_1^2 + a_5^2)p_2\delta_4)\delta_2 + 3(a_2^2 + a_6^2)^2(a_1^2 + a_5^2)^2\delta_4^2]\delta_1^2 \\ + [6(a_1^2 + a_5^2)^2 p_2\delta_2^2\delta_3 + 6(a_2^2 + a_6^2)(a_1^2 + a_5^2)p_2\delta_3\delta_4^2]\delta_1 \\ + 3(a_1^2 + a_5^2)^4\delta_2^2\delta_3^2 + 6(a_1^2 + a_5^2)^2 p_2\delta_2\delta_3^2\delta_4 + (a_1^2 + a_5^2)^2 p_1\delta_3^2\delta_4^2, \tag{2.20}$$

$$b_4 = 3p_2((a_1\delta_3 + a_2\delta_1)^2 + (a_5\delta_3 + a_6\delta_1)^2)^2,$$

$$b_5 = p_1((a_2^2 + a_6^2)\delta_1^2 + 2(a_1a_2 + a_5a_6)\delta_1\delta_3 + (a_1^2 + a_5^2)\delta_3^2)$$

with p_i, $1 \le i \le 5$, given by (2.9).

Furthermore, the set of parameters by (2.11) with (2.12) (2.13) are

$$a_2 = -\frac{[a_1(a_3^2 + a_7^2)\delta_3 + (a_1^2a_3 + 2a_1a_5a_7 - a_3a_5^2)\delta_2]\delta_1 + [a_3(a_1^2 + a_5^2)\delta_3 + a_1(a_1^2 + a_5^2)\delta_2]\delta_4}{(a_1\delta_4 + a_3\delta_1)^2 + (a_5\delta_4 + a_7\delta_1)^2},$$

$$a_6 = -\frac{[a_5(a_3^2 + a_7^2)\delta_3 + (-a_1^2a_7 + 2a_1a_3a_5 + a_7a_5^2)\delta_2]\delta_1 + [a_7(a_1^2 + a_5^2)\delta_3 + a_5(a_1^2 + a_5^2)\delta_2]\delta_4}{(a_1\delta_4 + a_3\delta_1)^2 + (a_5\delta_4 + a_7\delta_1)^2}, \tag{2.21}$$

$$a_9 = \frac{-e_3\alpha_1 + 3e_4\alpha_2 - e_5\alpha_3}{(a_1a_7 - a_3a_5)^2(\delta_1^2\delta_2 - \delta_1\delta_3\delta_4)((a_3\delta_1 + a_1\delta_4)^2 + (a_7\delta_1 + a_5\delta_4)^2)},$$

where

$$
\begin{aligned}
e_3 &= (a_3^2+a_7^2)^2 q_3\delta_1^4 + 4(a_3^2+a_7^2)(a_1a_3+a_5a_7)q_3\delta_1^3\delta_4 + 2q_3\delta_1^2\delta_4^2 + 4q_2q_3\delta_1\delta_4^3 + (a_1^2+a_5^2)^2 q_3\delta_4^4,\\
e_4 &= [(a_3^2+a_7^2)(a_1^2+a_5^2)q_2\delta_2 + (a_3^2+a_7^2)^2(a_1^2+a_5^2)^2\delta_3]\delta_1^3\\
&\quad + [(a_1^2+a_5^2)^2 q_3\delta_2\delta_4 + 3(a_3^2+a_7^2)(a_1^2+a_5^2)q_2\delta_3\delta_4]\delta_1^2\\
&\quad + [3(a_1^2+a_5^2)^2 q_2\delta_2\delta_4^2 + (a_1^2+a_5^2)^2 q_3\delta_3\delta_4^2]\delta_1 + (a_1^2+a_5^2)^4\delta_2\delta_4^3 + (a_1^2+a_5^2)^2 q_2\delta_3\delta_4^3, \qquad (2.22)\\
e_5 &= [(a_1^2+a_5^2)^2 q_3\delta_2^2 + 6(a_3^2+a_7^2)(a_1^2+a_5^2)q_2\delta_2\delta_3 + 3(a_3^2+a_7^2)^2(a_1^2+a_5^2)^2\delta_3^2]\delta_1^2\\
&\quad + [6(a_1^2+a_5^2)^2 q_2\delta_2^2\delta_4 + (4(a_1^2+a_5^2)^2 q_3\delta_3\delta_4)\delta_2 + 6(a_3^2+a_7^2)(a_1^2+a_5^2)q_2\delta_4\delta_3^2]\delta_1\\
&\quad + 3(a_1^2+a_5^2)^4\delta_2^2\delta_4^2 + (6(a_1^2+a_5^2)^2 q_2\delta_3\delta_4^2)\delta_2 + (a_1^2+a_5^2)^2 q_3\delta_3^2\delta_4^2,
\end{aligned}
$$

with q_i, $1 \le i \le 5$, given by (2.13).

A straightforward symbolic computation can show that the above two classes of lump solutions ((2.19-(2.20)) and ((2.21)-(2.22)) are equivalent to each other. Furthermore, we have

$$a_1a_6 - a_2a_5 = \frac{(a_1^2+a_5^2)(a_1a_7-a_3a_5)(\delta_1\delta_2-\delta_3\delta_4)}{(a_1\delta_4+a_3\delta_1)^2+(a_5\delta_4+a_7\delta_1)^2} \qquad (2.23)$$

and

$$a_1a_7 - a_3a_5 = \frac{(a_1^2+a_5^2)(a_1a_6-a_2a_5)(\delta_1\delta_2-\delta_3\delta_4)}{(a_1\delta_4+a_3\delta_1)^2+(a_5\delta_4+a_7\delta_1)^2}. \qquad (2.24)$$

Therefore, when

$$\delta_1^2\delta_2 - \delta_1\delta_2\delta_4 \neq 0, \qquad (2.25)$$

the two sets determine the absolutely same values for all the parameters and thus they process the same lump solutions.

Chapter 3

Examples in Two Cases

3.1 Example in The Case $\delta_6 = 0$

We firstly consider the case of $\delta_6 = 0$, and choose

$$\alpha_1 = 1, \quad \alpha_2 = 1, \quad \alpha_3 = 2, \quad \delta_1 = 1, \quad \delta_2 = 2, \quad \delta_3 = 0, \quad \delta_4 = 2, \quad \delta_5 = 3. \tag{3.1}$$

Then we get our specific combined nonlinear equation

$$\begin{aligned} P(u) &= 4u_t u_{xt} + u_x u_{tt} + u_{xx} k + u_{xxtt} \\ &+ 3(u_x u_y)_x + u_{xxxy} \\ &+ 2[4u_y u_{xy} + u_x u_{yy} + u_{xx} v + u_{xxyy}] \\ &+ u_{yt} + 2u_{xx} + 2u_{xy} + 3u_{yy} = 0, \end{aligned} \tag{3.2}$$

where $k_x = u_{tt}$ and $v_x = u_{yy}$. Therefore, under the logarithmic transformation in (2.4), the Hirota bilinear form becomes

$$(D_x^2 D_t^2 + D_x^3 D_y + 2D_x^2 D_y^2 + D_y D_t + 2D_x^2 + 2D_x D_y + 3D_y^2) f \cdot f = 0. \tag{3.3}$$

Associated with

$$a_1 = 4, \quad a_2 = -2, \quad a_4 = 5, \quad a_5 = 2, \quad a_6 = 4, \quad a_8 = 6, \tag{3.4}$$

the transformations in (2.4) with (2.6) provide a specific lump solution to our first specific combined nonlinear equation (3.2):

$$
\begin{aligned}
u_1 &= \frac{2(-80t + 40x + 64)}{(-6t + 4x - 2y + 5)^2 + (-8t + 2x + 4y + 6)^2 - 150}, \\
v_1 &= \frac{80}{(-6t + 4x - 2y + 5)^2 + (-8t + 2x + 4y + 6)^2 - 150} \\
&\quad - \frac{2(-40t + 40y + 28)^2}{[(-6t + 4x - 2y + 5)^2 + (-8t + 2x + 4y + 6)^2 - 150]^2}, \\
k_1 &= \frac{400}{(-6t + 4x - 2y + 5)^2 + (-8t + 2x + 4y + 6)^2 - 150} \\
&\quad - \frac{2(200t - 80x - 40y - 156)^2}{[(-6t + 4x - 2y + 5)^2 + (-8t + 2x + 4y + 6)^2 - 150]^2}.
\end{aligned}
\tag{3.5}
$$

Figure 1 presents three 3-dimensional plots and density plots of the above lump solutions u_1 at three different times.

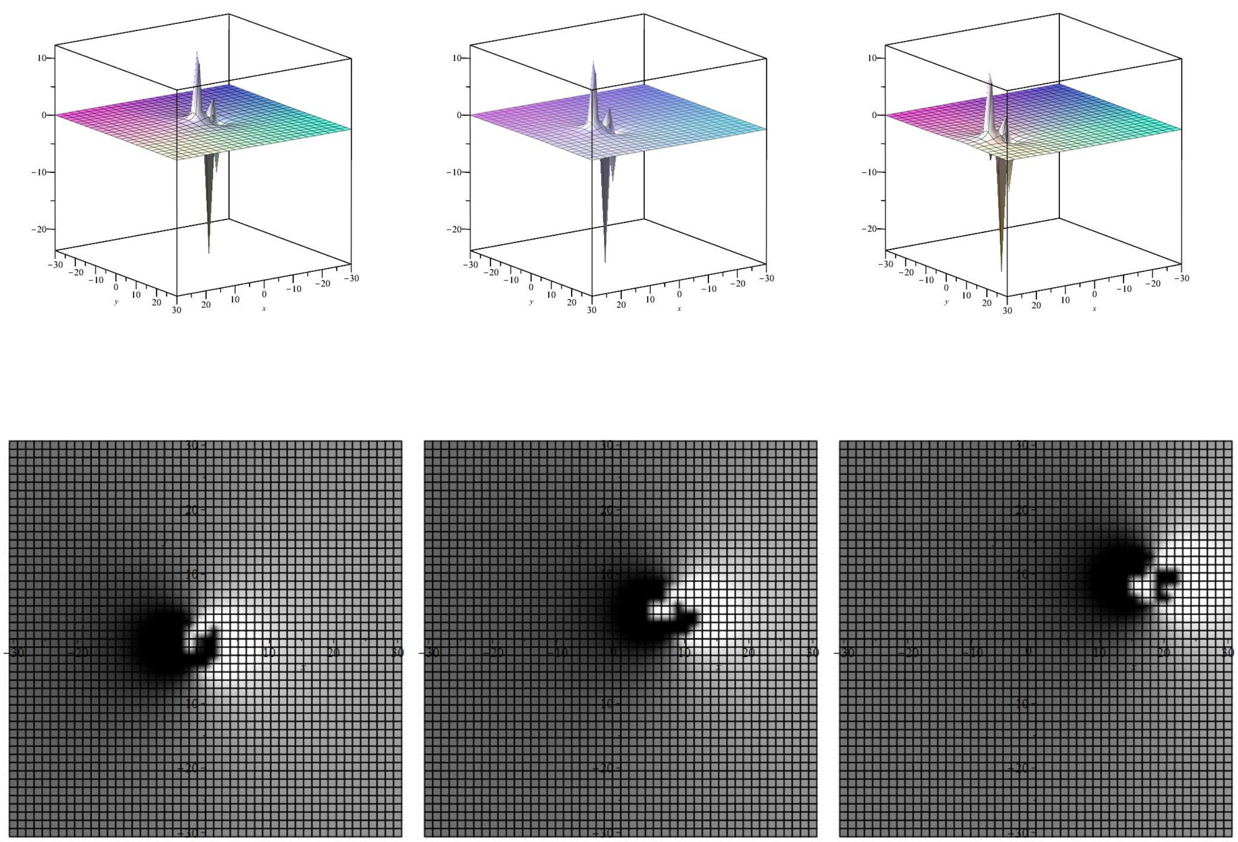

Figure 1.: profiles of u_1 when t = 0, 5, 10: 3d plots (top) and density plots (bottom)

3.2 Example in The case $\delta_5 = 0$

For the case of $\delta_5 = 0$, we take

$$\alpha_1 = 1, \quad \alpha_2 = 1, \quad \alpha_3 = 2, \quad \delta_1 = 1, \quad \delta_2 = 2, \quad \delta_3 = 0, \quad \delta_4 = 2, \quad \delta_6 = 3, \tag{3.6}$$

which leads to another particular combined nonlinear equation

$$\begin{aligned} P(u) = {} & 4u_t u_{xt} + u_x u_{tt} + u_{xx} k + u_{xxtt} \\ & + 3(u_x u_y)_x + u_{xxxy} \\ & + 2[4u_y u_{xy} + u_x u_{yy} + u_{xx} v + u_{xxyy}] \\ & + u_{yt} + 2u_{xx} + 2u_{xy} + 3u_{tt} = 0, \end{aligned} \tag{3.7}$$

where $k_x = u_{tt}$ and $v_x = u_{yy}$. Under the logarithmic transformations in (2.4), the above nonlinear equation also has a Hirota bilinear form

$$(D_x^2 D_t^2 + D_x^3 D_y + 2D_x^2 D_y^2 + D_y D_t + 2D_x^2 + 2D_x D_y + 3D_t^2) f \cdot f = 0. \tag{3.8}$$

Furthermore, associated with

$$a_1 = 4, \quad a_3 = -2, \quad a_4 = 5, \quad a_5 = 2, \quad a_7 = 4, \quad a_8 = 2, \tag{3.9}$$

the transformations in (2.4) with (2.6) presents a specific lump solution to our second specific combined nonlinear equation (3.7):

$$\begin{aligned} u_2 &= \frac{2(40x + 16y + 48)}{(-2t + 4x + 2y + 5)^2 + (4t + 2x + 2)^2 - \frac{162}{7}}, \\ v_2 &= \frac{16}{(-2t + 4x + 2y + 5)^2 + (4t + 2x + 2)^2 - \frac{162}{7}} \\ &\quad - \frac{2(-8t + 16x + 8y + 20)^2}{[(-2t + 4x + 2y + 5)^2 + (4t + 2x + 2)^2 - \frac{162}{7}]^2}, \\ k_2 &= \frac{80}{(-2t + 4x + 2y + 5)^2 + (4t + 2x + 2)^2 - \frac{162}{7}} \\ &\quad - \frac{2(40t - 8y - 4)^2}{[(-2t + 4x + 2y + 5)^2 + (4t + 2x + 2)^2 - \frac{162}{7}]^2}. \end{aligned} \tag{3.10}$$

Similarly, we use figure 2 and figure 3 to show three 3-dimensional plots and density plots of the lump functions v_2 and k_2 respectively at three different times.

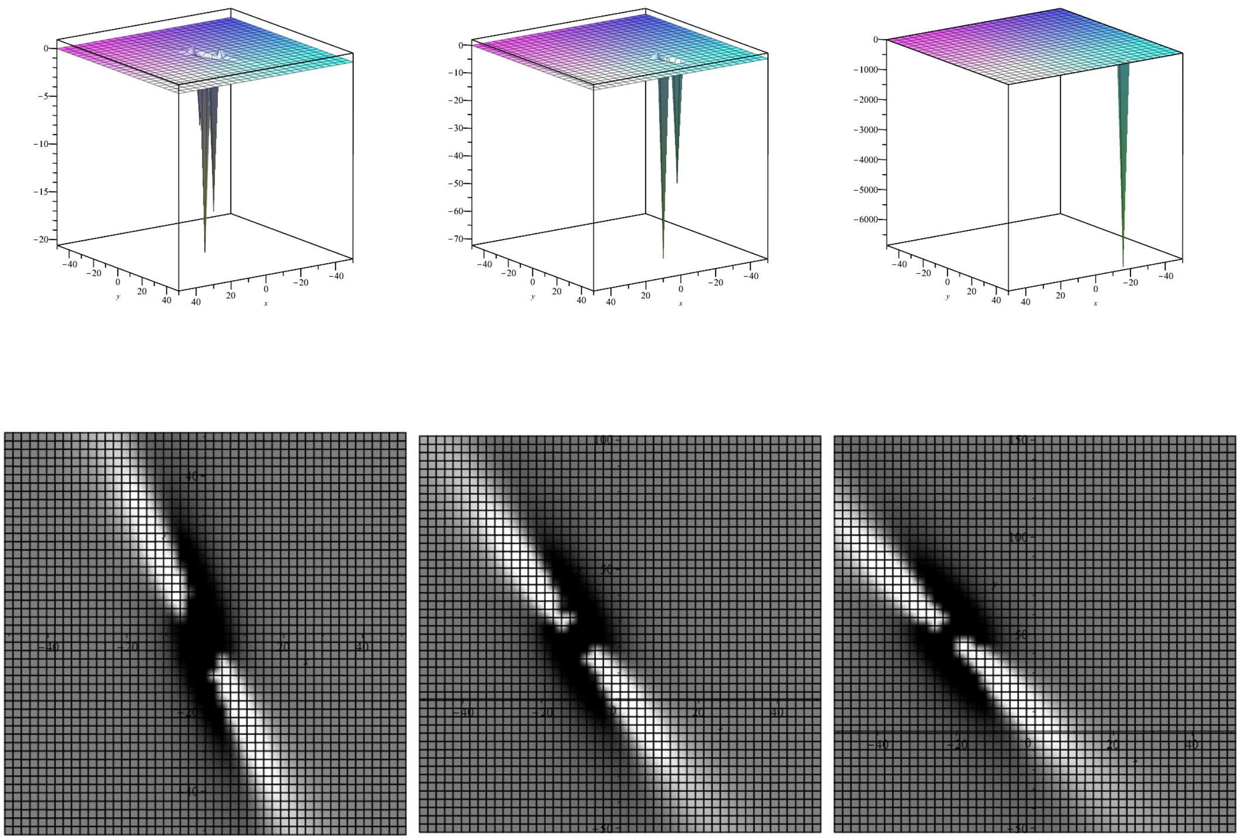

Figure 2.: profiles of v_2 when t = 0, 5, 10: 3d plots (top) and density plots (bottom)

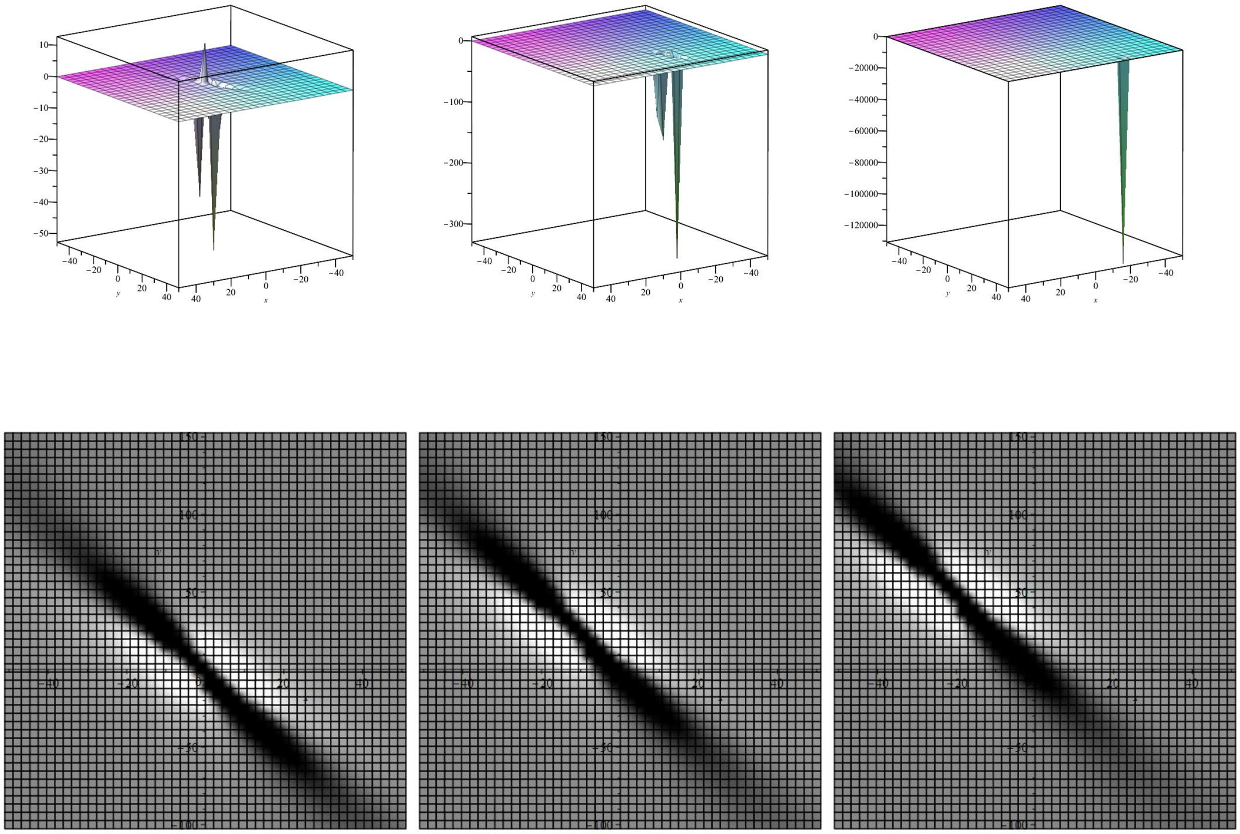

Figure 3.: profiles of k_2 when t = 0, 5, 10: 3d plots (top) and density plots (bottom)

Chapter 4
Conclusion

By using symbolic computation and the Hirota bilinear method, we have presented two classes of lump solutions to a combined fourth-order nonlinear equation in (2.1), including two types of nonlinear terms in (2+1)-dimensions. The results of our study present a new example of (2+1)-dimensional combined nonlinear equations that process lump solutions. Also, we use 3-dimensional plots of our two specific lump solutions in the two cases of the combined nonlinear equation to show the structure of the lump solutions. The simplification process, the plots, and the solutions are directly made by using Maple. It is important to remark that the three nonlinear terms can be merged into the considered nonlinear equation (2.1). The terms $D_x^2 D_y^2$ and $D_x^2 D_t^2$ reflect a new and more complex structure of the solutions. However, these types of terms ('2+2'-type terms) have been rarely mentioned in the past. In this paper, we combined two '2+2'-type terms and one '3+1'-type term, which never happened in the past research.

This study shows the richness and variation of nonlinear partial differential equations that possess lump solutions. It is a commonly known fact that many nonlinear waves can be described by interaction solutions between lump solutions and soliton solutions [9]. Furthermore, there are lots of studies showing the existence of interaction solutions between lump solutions and lump-kind solutions and other kinds of exact solutions to nonlinear equations [8] and nonlinear integrable equations ([6], [20]). Since the interaction properties involve much more complicated mathematical computations, further researches for interaction and lump solutions for partial differential equations are meaningful.

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