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*Research Article*

# Mixed Rational-Exponential Solutions to the Kadomtsev-Petviashvili-II Equation with a Self-Consistent Source

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Explicit rational-exponential solutions for the Kadomtsev-Petviashvili-II equation with a self-consistent source (KPIIESCS) are studied by the Hirota bilinear method. One typical feature for this hybrid type of solutions is that they contain two arbitrary functions of time variable $t$ which affect the amplitudes and propagation trajectories. The dynamics of solutions are demonstrated by the three-dimensional figures. The method used here is quite general and can be applied to other equations with self-content sources.

## 1. Introduction

The Kadomtsev-Petviashvili (KP) equation with self-consistent sources arose in the pioneering work of Mel'nikov for describing the interaction of waves on the $x$, $y$ plane [1]. After that, the study of the KP equation with self-consistent sources has become a subject of intense investigation [2–11]. For example, the $N$-soliton solution was obtained by the Wronskian technique [12] and the generalized binary Darboux transformation method [13]. The source generation procedure was applied to construct and solve a hybrid type of KP equations with self-consistent sources [14]. The general high-order rogue waves and lump-type solutions were derived via the Hirota bilinear method [15–17].

Ablowitz and Satsuma obtained rational solutions of certain nonlinear evolution equations by choosing the phase constants appropriately and taking the long-way limit [18]. However, the procedure of choosing the phase constants as definite singular functions of physical parameters is unknown and, in fact, is not solvable, even for three- and four-soliton solutions. Then Johnson and Thompson employed the method of separation of variables to solve the appropriate scalar Gelfand–Levitan equation and introduced a new rational-exponential solution (afterwards referred as RE solutions) for the KP equation [19]. And Pöppe obtained new types of RE solutions, corresponding to multipe poles in the scattering data for the hyperbolic sine-Gordon (sG) and Korteweg-de Vries (KdV) equations using the Fredholm determinant method [20]. Later, Bezmaternih and Borisov presented a new approach to the construction of RE solutions for nonlinear partial differential equations based on the formal perturbation theory in Hirota's bilinear form with another choice of starting solution [21]. These solutions are the rational functions of polynomials multiplied by exponents. The proposed procedure was applied to the elliptic sine-Gordon, the Korteweg-de Vries (KdV), the Kadomtsev-Petviashvili (KP), and the Landau–Lifshitz (L–L) equations [22].

In our present work, we will construct RE solutions of the Kadomtsev-Petviashvili-II equation with one source (KPIIESCS)

$$\begin{aligned}&[u_t + 6uu_x + u_{xxx} + 8(\Phi\Psi)_x]_x + 3u_{yy} = 0,\\ &\Phi_y = \Phi_{xx} + u\Phi,\\ &\Psi_y = -\Psi_{xx} - u\Psi,\end{aligned} \tag{1}$$

based on its Hirota bilinear forms and the method suggested in [21]. Equation (1) is a member of the KP hierarchy with self-consistent sources and admits some interesting solutions [13]. The paper is arranged as follows. We first present its bilinear forms in Section 2. Then the representation of RE solutions which contains two arbitrary functions of one independent variable is obtained. Furthermore, we will investigate the dynamic behaviors of the RE solutions. At last, a few concluding remarks will be given in the final section.

## 2. RE Solutions to KPIIESCS

In the following, we shall construct RE solutions of the KPIIESCS by virtue of the Hirota method.

With the help of the dependent variable transformations

$$\begin{aligned}&u = 2(\ln F)_{xx},\\ &\Phi = \tfrac{G}{F},\\ &\Psi = \tfrac{H}{F},\end{aligned} \tag{2}$$

the KPIIESCS can be transformed into the bilinear forms

$$\begin{aligned}&\left(D_xD_t + D_x^4 + 3D_y^2\right)F\cdot F + 8GH = 0,\\ &\left(D_y - D_x^2\right)G\cdot F = 0,\\ &\left(D_y + D_x^2\right)H\cdot F = 0,\end{aligned} \tag{3}$$

where $D$ is the well-known operator defined as in [23]

$$\begin{aligned}D_t^m D_x^n G\cdot F = {}&\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^m \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^n\\ &\cdot G(x,t)F(x',t')\big|_{x'=x,t'=t}.\end{aligned} \tag{4}$$

Consequently the soliton solutions of KPIIESCS can be derived through the standard Hirota's approach by expanding $F, G, H$ as the series

$$\begin{aligned}F &= 1 + F^{(2)}\epsilon^2 + F^{(4)}\epsilon^4 + \dots,\\ G &= G^{(1)}\epsilon + G^{(3)}\epsilon^3 + \dots,\\ H &= H^{(1)}\epsilon + H^{(3)}\epsilon^3 + \dots,\end{aligned} \tag{5}$$

and finding each coefficient successively and truncating the expansion at an appropriate finite order.

For example, assuming that $G^{(1)}, H^{(1)}$ take the form

$$\begin{aligned}&G^{(1)} = \tfrac{1}{2}\sqrt{2(k_1+q_1)\dot{\beta}_1(t)}e^{\xi_1-\beta_1(t)}, \quad \xi_1 = k_1x + k_1^2y - 4k_1^3t + \xi_1^{(0)},\\ &H^{(1)} = \tfrac{1}{2}\sqrt{2(k_1+q_1)\dot{\beta}_1(t)}e^{\eta_1-\beta_1(t)}, \quad \eta_1 = q_1x - q_1^2y - 4q_1^3t + \eta_1^{(0)},\end{aligned} \tag{6}$$

which leads to

$$\begin{aligned}&F^{(2)} = e^{\xi_1+\eta_1-2\beta_1(t)}, \quad F^{(j)} = 0,\ j = 4,6,\dots,\\ &G^{(i)} = 0, \quad H^{(i)} = 0, \quad i = 3,5,\dots,\end{aligned} \tag{7}$$

where $k_1(\neq \pm q_1), q_1$ are arbitrary constants, and $\beta_1(t)$ is an arbitrary differentiable function, we can make the infinite expansion truncate with a finite number of terms and get the exact one-soliton solution

$$\begin{aligned}&u = 2\left[\ln\left(1 + e^{\xi_1+\eta_1-2\beta_1(t)}\right)\right]_{xx},\\ &\Phi = \frac{1}{2}\frac{\sqrt{2(k_1+q_1)\dot{\beta}_1(t)}e^{\xi_1-\beta_1(t)}}{1 + e^{\xi_1+\eta_1-2\beta_1(t)}},\\ &\Psi = \frac{1}{2}\frac{\sqrt{2(k_1+q_1)\dot{\beta}_1(t)}e^{\eta_1-\beta_1(t)}}{1 + e^{\xi_1+\eta_1-2\beta_1(t)}}.\end{aligned} \tag{8}$$

Next, let us discuss the two-soliton solution by choosing

$$\begin{aligned}G^{(1)} &= \frac{1}{2}\sqrt{2(k_1+q_1)\dot{\beta}_1(t)}e^{\xi_1-\beta_1(t)},\\ G^{(3)} &= \frac{1}{2}\sqrt{2(k_1+q_1)\dot{\beta}_1(t)}\frac{(k_1-k_2)}{(k_1+q_2)}e^{\xi_1-\beta_1(t)+\xi_2+\eta_2},\\ H^{(1)} &= \frac{1}{2}\sqrt{2(k_1+q_1)\dot{\beta}_1(t)}e^{\eta_1-\beta_1(t)},\\ H^{(3)} &= \frac{1}{2}\sqrt{2(k_1+q_1)\dot{\beta}_1(t)}\frac{(q_1-q_2)}{(q_1+k_2)}e^{\eta_1-\beta_1(t)+\xi_2+\eta_2},\\ G^{(i)} &= 0, \quad H^{(i)} = 0, i = 5,7,\dots\\ F^{(2)} &= e^{\xi_1+\eta_1-2\beta_1(t)} + e^{\xi_2+\eta_2},\\ F^{(4)} &= A_{12}e^{\xi_1+\eta_1+\xi_2+\eta_2-2\beta_1(t)}, \quad A_{12} = \frac{(k_1-k_2)(q_1-q_2)}{(k_1+q_2)(q_1+k_2)},\\ F^{(j)} &= 0, \quad j = 6,8,\dots,\end{aligned} \tag{9}$$

where

$$\begin{aligned}\xi_2 &= k_2x + k_2^2y - 4k_2^3t + \xi_2^{(0)},\\ \eta_2 &= q_2x - q_2^2y - 4q_2^3t + \eta_2^{(0)},\end{aligned} \tag{10}$$

and so the two-soliton solution is

$$\begin{aligned}u &= 2\left[\ln\left(1 + e^{\xi_1+\eta_1-2\beta_1(t)} + e^{\xi_2+\eta_2} + A_{12}e^{\xi_1+\eta_1+\xi_2+\eta_2-2\beta_1(t)}\right)\right]_{xx},\\ \Phi &= \frac{1}{2}\sqrt{2(k_1+q_1)\dot{\beta}_1(t)}\frac{e^{\xi_1-\beta_1(t)}\left[1 + (k_1-k_2)/(k_1+q_2)e^{\xi_2+\eta_2}\right]}{1 + e^{\xi_1+\eta_1-2\beta_1(t)} + e^{\xi_2+\eta_2} + A_{12}e^{\xi_1+\eta_1+\xi_2+\eta_2-2\beta_1(t)}},\\ \Psi &= \frac{1}{2}\sqrt{2(k_1+q_1)\dot{\beta}_1(t)}\frac{e^{\eta_1-\beta_1(t)}\left[1 + (q_1-q_2)/(q_1+k_2)e^{\xi_2+\eta_2}\right]}{1 + e^{\xi_1+\eta_1-2\beta_1(t)} + e^{\xi_2+\eta_2} + A_{12}e^{\xi_1+\eta_1+\xi_2+\eta_2-2\beta_1(t)}}.\end{aligned} \tag{11}$$

It appears that by choosing the phase constants as definite singular functions of physical parameters and performing an appropriate limiting procedure, the two-soliton solution reduces to the simplest RE solution [18]. Here, based on the RE solutions of the KP equation [21] and the two-soliton solution obtained above, we construct a definite type of RE solutions for the KPIIESCS as follows

$$\begin{aligned}F &= 1 + Q(x,y,t)e^{\xi+\eta} + R(t)e^{2(\xi+\eta)},\\ G &= G_1(x,y,t)e^{\xi} + G_2(t)e^{2\xi+\eta},\\ H &= H_1(x,y,t)e^{\eta} + H_2(t)e^{\xi+2\eta},\end{aligned} \tag{12}$$

with

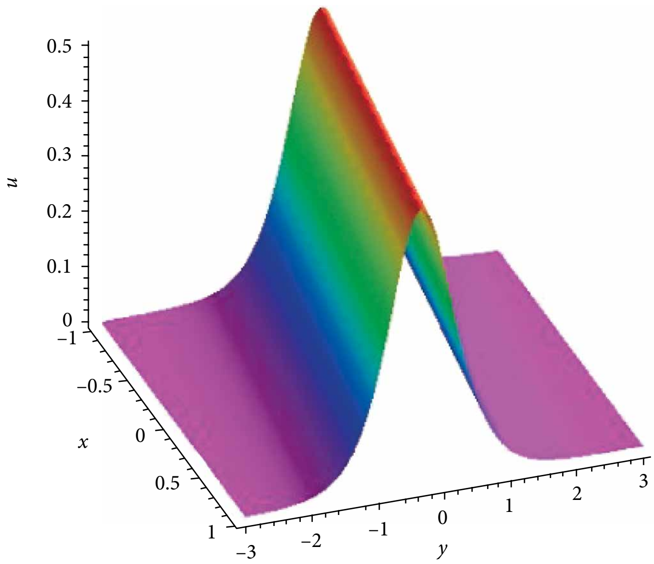


FIGURE 1: Line-soliton solution of the KPIIESCS for $t = 0$ and $F_1(t) = 0, \beta(t) = F_2(t) = t, k = 2, q = -1, \xi_0 = \zeta_0 = R_0 = 0$.

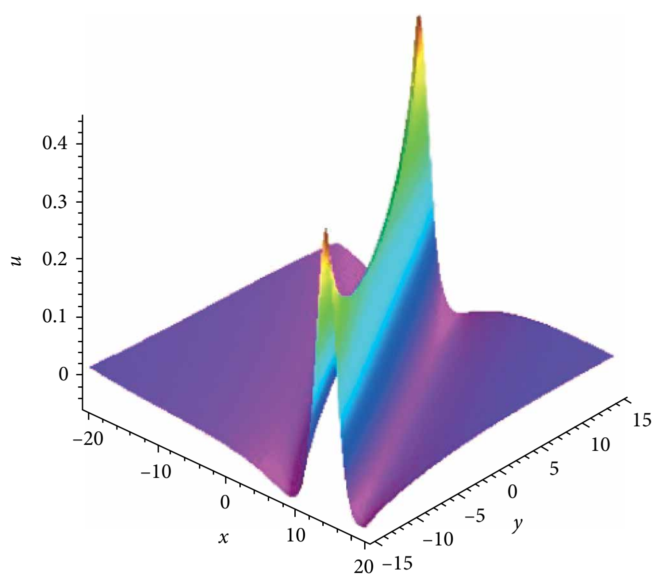


FIGURE 2: Mixed RE solution of the KPIIESCS for $t = 0$ and $F_1(t) = F_2(t) = 1$, $\beta(t) = t, k = 1/2$, $q = -1/4$, $\xi_0 = \zeta_0 = R_0 = 0$.

$$\begin{aligned}\xi &= kx + k^2 y - 4k^3 t - \beta(t) + \xi^{(0)},\\ \eta &= qx - q^2 y - 4q^3 t - \beta(t) + \eta^{(0)}.\end{aligned} \tag{13}$$

After careful calculations, the substitution of Equation (13) into Equation (3) yields

$$\begin{aligned}G_1(x,y,t) &= \left(x + 2ky - 12k^2 t + C_3\right)F_1(t),\\ H_1(x,y,t) &= \left(x - 2qy - 12q^2 t + C_2\right)F_2(t),\\ G_2(t) &= C_1 e^{2\beta(t)} F_1(t), \quad H_2(t) = C_1 e^{2\beta(t)} F_2(t),\\ R(t) &= \frac{e^{4\beta(t)}}{(k+q)^2}\left[R_0 e^{-4\beta_0}(k+q)^2 - 4C_1 \int_0^t F_1(s) F_2(s) e^{-2\beta(s)} ds\right],\\ Q(x,y,t) &= C_1 (k+q) e^{2\beta(t)} + \frac{2}{C_1 (k+q) e^{2\beta(t)}} + \frac{\Delta}{C_1 e^{2\beta(t)}},\end{aligned} \tag{14}$$

where

$$\begin{aligned}\Delta = {} & (k+q)x^2 - 4qk(k+q)y^2 + 144k^2q^2(k+q)t^2\\ & + 2\left(k^2 - q^2\right)xy - 12\left(k^2 + q^2\right)(k+q)xt\\ & + 24qk\left(k^2 - q^2\right)yt + (k+q)\Big[\left(C_2 + C_3\right)x\\ & + \left(kC_2 - qC_3\right)y + \left(k^2 C_2 + q^2 C_3\right)t + C_2 C_3\Big]\\ & - 2x + (q-k)y - \left(k^2 + q^2\right)t - 2C_2.\end{aligned} \tag{15}$$

Here $C_1, C_2, C_3$ are arbitrary constants, and $F_1(t), F_2(t)$ are two arbitrary functions provided that all formulas are well defined and the analyticity of the solutions is guaranteed. This generates a class of general RE solutions to the KPIIESCS equation in Equation (1) through the transformation of Equation (2).

Furthermore, this family of solutions contains two arbitrary functions of time variable $t$ and there are a variety of shapes. If we further set

$$\begin{aligned} C_1 &= \frac{e^{-\zeta_0}}{k+q}, \quad C_2 = C_3 = 0, \\ F_1(t) &= \frac{1}{2}\sqrt{2(k+q)\dot{\beta}(t)}e^{-\xi_0}, \\ F_2(t) &= \frac{1}{2}\sqrt{2(k+q)\dot{\beta}(t)}e^{-\zeta_0}, \end{aligned} \tag{16}$$

we can recreate the mixture of exponential and rational solutions of the KPIIESCS presented in [13].

Moreover, without loss of generality, we can normalize $C_1 = 1, C_2 = C_3 = 0$ due to the translation and scaling invariance. For illustration, the dynamical features of some RE solutions are shown via three-dimensional figures.

In Figure 1, we take $F_1(t) = 0, \beta(t) = F_2(t) = t, k = 2, q = -1, \xi_0 = \zeta_0 = R_0 = 0$, which leads to

$$\begin{aligned} F &= 1 + e^{x+3y-28t}, \\ G &= 0, \\ H &= \left(xt + 2yt - 12t^2\right)e^{-x-y+3t} + te^{2y-25t}. \end{aligned} \tag{17}$$

Under this case, the RE solution reduces to a usual line one-soliton of the KP equation without source.

Whereas, in Figure 2, we take $F_1(t) = F_2(t) = 1, \beta(t) = t, k = 1/2, q = -1/4, \xi_0 = \zeta_0 = R_0 = 0$ which results in

$$\begin{aligned} F &= 1 + \Big(12xy - 30xt - 18yt + (1/4)e^{2t} + 8x^2 - 64x \\ &\quad -48y + 120t + 18t^2 + 4y^2 + 256\Big)e^{x/4+3y/16-39t/16} \\ &\quad + 32e^{x/2+3y/8-23t/8}, \\ G &= (x + y - 3t)e^{x/2+y/4-3t/2} + e^{3x/4+7y/16-31t/16}, \\ H &= (x + y/2 - 3t/4)e^{-x/4-y/16-15t/16} + e^{y/8-11t/8}. \end{aligned} \tag{18}$$

In this case, the solution describes a soliton which exhibits both exponential and rational properties. The shape and motion of the RE solution presents a time-dependent effect. Indeed, the insertion of a source may cause the variation of the velocity of a solution, the amplitudes and trajectories vary with time, and this time dependence is an effect of the source.

## 3. Results and Discussion

In this paper, we studied RE solutions to the KPIIESC equation. Several constraint conditions for the existence of such RE solutions were given. The proposed method here permits one to obtain RE solutions directly in an explicit form and an entirely analogous technique can be used to obtain more complicated RE solutions.

## Data Availability

The data used to support the findings of this study are included within the article.

## Conflicts of Interest

The authors declare that there are no conflict of interest regarding the publication of this paper.

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