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Research Article

Lump Solutions of a Nonlinear PDE Combining with a New Fourth-Order Term $D_x^2 D_t^2$ *

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A nonlinear PDE combining with a new fourth-order term $D_x^2 D_t^2$ is studied. Adding three new fourth-order derivative terms and some second-order derivative terms, we formulate a combined fourth-order nonlinear partial differential equation, which possesses a Hirota's bilinear form. The class of lump solutions is constructed explicitly through Hirota's bilinear method. Their dynamical behaviors are analyzed through plots.

1. Introduction

In the theory of differential equations, one of the fundamental problems is the Cauchy problem to determine a solution of a differential equation satisfying what are known as initial data. The classical approaches, such as Laplace's method and Fourier transform method, were developed for solving Cauchy problems for linear ordinary differential equations and for linear partial differential equations. In modern soliton theory, the isomonodromic transform method and the inverse scattering transform method have been created for handling Cauchy problems for nonlinear ordinary and partial differential equations, respectively [1, 2]. One of the most exciting and extremely active areas of research investigation arises on the study of exact solution and the related issue of

the construction of solutions to a wide class of nonlinear equations.

Exact solutions of partial differential equations describe significant mathematical and physical phenomena. A soliton solution is an exact solution determined by exponentially localized functions, which localized in all directions both in time and in space. A lump solution is also a kind of exact solutions of partial differential equations, obtained from soliton theory by taking long wave limits [1]. Nevertheless, a lump solution is localized in all directions just in space. In addition, it is well known that interaction solutions between lump solutions and soliton solutions allow to describe more nonlinear phenomena [3]. However, the interaction properties are rarely discussed because the involved mathematical computation is much more complicated.

Generally, through a depended variable transformation, a partial differential equation can be mapped into a Hirota's bilinear form.

$$P(D_x, D_y, D_t)f \cdot f = 0, \quad (1)$$

where P is a polynomial, and D_x, D_y, D_t are the Hirota bilinear derivatives [4], defined by

$$\begin{aligned} & D_x^l, D_y^n, D_t^m f(x, y, t) \cdot g(x, y, t) \\ &= \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^l \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'} \right)^n \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^m f(x, y, t) \\ & \quad \cdot g(x', y', t') \Big|_{x'=x, y'=y, t'=t}. \end{aligned} \quad (2)$$

When f solves (2), it presents the N -soliton solution in $(2+1)$ -dimensions to the corresponding PDE under the transformation $u = 2(\ln f)_{xx}$:

$$f = \sum_{\mu=0,1} \exp \left(\sum_{i=1}^N \mu_i \xi_i + \sum_{i < j} \mu_i \mu_j a_{ij} \right), \quad (3)$$

where $\sum_{\mu=0,1}$ denotes the sum over all possibilities for μ_1, \dots, μ_N in 0, 1, and

$$\begin{cases} \xi_i = k_i x + l_i y - \omega_i t + \xi_{i,0}, & 1 \leq i \leq N, \\ e^{a_{ij}} = -\frac{P(k_i - k_j, l_i - l_j, \omega_j - \omega_i)}{P(k_i + k_j, l_i + l_j, \omega_j + \omega_i)}, & 1 \leq i < j \leq N, \end{cases} \quad (4)$$

with k_i, l_i, ω_i satisfying the dispersion relation, and $\xi_{i,0}$ being arbitrary shifts.

As is well known, the KPI equation possesses lump solutions [5]: $u = 2(\ln f)_{xx}$, where

$$\begin{aligned} f &= \left(a_1 x + a_2 y + \frac{a_1 a_2^2 - a_1 a_6^2 + 2a_2 a_5 a_6}{a_1^2 + a_5^2} t + a_4 \right)^2 \\ &+ \left(a_5 x + a_6 y + \frac{2a_1 a_2 a_6 - a_2^2 a_5 + a_5 a_6^2}{a_1^2 + a_5^2} t + a_8 \right)^2 \\ &+ \frac{3(a_1^2 - a_5^2)^3}{(a_1 a_6 - a_2 a_5)^2}. \end{aligned} \quad (5)$$

The condition $a_1 a_6 - a_2 a_5 \neq 0$ guarantees the rational localisation in all directions in the (x, y) -plane.

In the past few decades, many researchers have studied soliton solutions, lump solutions, and other classes of solutions to integrable equations, such as the Ishimori-I equation [6], the Davey-Stewartson equation II [1], the BKP equation [7, 8], the three-dimensional three-wave resonant interaction [9], the KP equation with a self-consistent source [10], and so on [11–14]. Some nonintegrable equations have

lump solutions as well, such as the generalized KP, Sawada-Kotera equations [15–17]. Moreover, various studies show the existence of interaction solutions between lumps and other kinds of exact solutions to nonlinear integrable equation [2, 18–23].

This paper is concerned with a nonlinear PDE combining with a new fourth-order term $D_x^2 D_t^2$. Adding three new fourth-order derivative terms and some second-order derivative terms, we formulate a combined fourth-order nonlinear partial differential equation, which possesses a Hirota's bilinear form. Based on a bilinear transformation, lump solutions are obtained through symbolic computation with Maple. This new term $D_x^2 D_t^2$ is the second derivative with respect to time, which really makes the calculation more difficult. It is also reflected that the structure of the solution is more complex, and we pay more attention to deal with and analyze the dispersion relation of the solution. We also exhibit three-dimensional plot and contour plot profiles of these solutions and study their dynamic behaviors. Some concluding remarks are given in the final section.

2. Bilinear Form

We would like to consider a general combined fourth-order nonlinear equation as follows:

$$\begin{aligned} P(u) &= \alpha_1 [3(u_x u_t)_x + u_{xxx t}] + \alpha_2 [3(u_x u_y)_x + u_{xxx y}] \\ &+ \alpha_3 [(v u_x)_x + 4u_t u_{xt} + u_{xxt t}] + \delta_1 u_{yt} + \delta_2 u_{xy} \\ &+ \delta_3 u_{xx} + \delta_4 u_{tt} + \delta_5 u_{yy} + \delta_6 u_{xt} = 0, \end{aligned} \quad (6)$$

where v is a function satisfying $v_x = u_{tt}$, the constants α_i , $i = 1, 2, 3$ all are not zero, and the constants δ_i , $i = 1, 2, 3, \dots, 6$ are all arbitrary. We substitute the following logarithmic transformation into (6):

$$\begin{aligned} u &= 2(\ln f)_x, \\ v &= 2(\ln f)_{tt}, \end{aligned} \quad (7)$$

and then the equation has the following Hirota's bilinear form:

$$\begin{aligned} B(f) &= \left(\alpha_1 D_x^3 D_t + \alpha_2 D_x^3 D_y + \alpha_3 D_x^2 D_t^2 + \delta_1 D_y D_t + \delta_2 D_x D_y \right. \\ & \quad \left. + \delta_3 D_x^2 + \delta_4 D_t^2 + \delta_5 D_y^2 + \delta_6 D_x D_t \right) f \cdot f = 0. \end{aligned} \quad (8)$$

Therefore, if f solves the bilinear Equation (8), then $u = 2(\ln f)_x$, $v = 2(\ln f)_{tt}$ will solve Equation (6), where we have the relation $P(u) = ((B(f))/f^2)_x$.

When $\alpha_1 = 1$, $\alpha_2 = \alpha_3 = 0$, $\delta_1 = \delta_3 = 1$, and the other δ_i , $1 \leq i \leq 6$ are zero, we obtain an integrable extension of the Hirota-Satsuma equation [4]:

$$3(u_x u_t)_x + u_{xxx} + u_{yt} + u_{xx} = 0, \tag{9}$$

which has a bilinear form under the logarithmic transformation $u = 2(\ln f)_x$ as follows:

$$(D_x^3 D_t + D_y D_t + D_x^2) f \cdot f = 0, \tag{10}$$

that is called the bilinear HSI equation.

When $\alpha_2 = 1, \alpha_1 = \alpha_3 = 0, \delta_5 = \delta_6 = 1$, and the other $\delta_i, 1 \leq i \leq 6$ are zero, we obtain a generalized Calogero-Bogoyavlenskii-Schiff equation [16]:

$$3(u_x u_y)_x + u_{xxx} + u_{xt} + u_{yy} = 0, \tag{11}$$

which has a bilinear form under the logarithmic transformation $u = 2(\ln f)_x$ as follows:

$$(D_x^3 D_y + D_x D_t + D_y^2) f \cdot f = 0, \tag{12}$$

that has lump solutions [16].

3. Lump Solutions

A search for positive quadratic solutions to the bilinear Equation (8) generates a class of lump solutions to Equation (6):

$$f = (a_1 x + a_2 y + a_3 t + a_4)^2 + (a_5 x + a_6 y + a_7 t + a_8)^2 + a_9, \tag{13}$$

where $a_i, i = 1, 2, 3, \dots, 9$ are arbitrary real constants to be determined. We insert (13) into (8) and then solve the resulting algebraic system by Maple symbolic computations.

We first consider the case of $\delta_2 = \delta_4 = \delta_5 = 0$ for the combined nonlinear Equation (6). We obtain the following parameters:

$$\begin{cases} a_3 = -\frac{a_1(a_1^2 + a_5^2)\delta_3\delta_6 + (a_1^2 a_2 + 2a_1 a_5 a_6 - a_2 a_5^2)\delta_1\delta_3}{(a_1\delta_6 + a_2\delta_1)^2 + (a_5\delta_6 + a_6\delta_1)^2}, \\ a_7 = -\frac{a_5(a_1^2 + a_5^2)\delta_3\delta_6 - (a_1^2 a_6 + 2a_1 a_2 a_5 - a_5^2 a_6)\delta_1\delta_3}{(a_1\delta_6 + a_2\delta_1)^2 + (a_5\delta_6 + a_6\delta_1)^2}, \\ a_9 = -\frac{b_1\alpha_1 + b_2\alpha_2 + b_3\alpha_3}{(a_1 a_6 - a_2 a_5)^2 [(a_1\delta_6 + a_2\delta_1)^2 + (a_5\delta_6 + a_6\delta_1)^2] \delta_1^2 \delta_3^2}, \end{cases} \tag{14}$$

and all other a_i are arbitrary. The above involved three constants $b_i, i = 1, 2, 3$, are defined as follows:

$$\begin{cases} b_1 = -3(a_1^2 + a_5^2)^2 [(a_1^2 + a_5^2)\delta_6 + (a_1 a_2 + a_5 a_6)\delta_1] [(a_1\delta_6 + a_2\delta_1)^2 + (a_5\delta_6 + a_6\delta_1)^2], \\ b_2 = 3(a_1^2 + a_5^2)(a_1 a_2 + a_5 a_6) [(a_1\delta_6 + a_2\delta_1)^2 + (a_5\delta_6 + a_6\delta_1)^2]^2, \\ b_3 = 3(a_1^2 + a_5^2)^2 [3(a_1 a_2 + a_5 a_6)^2 + (a_1 a_6 - a_2 a_5)^2] \delta_1^2 \delta_3^2 + 3(a_1^2 + a_5^2)^4 \delta_3^2 \delta_6^2 + 6(a_1^2 + a_5^2)^3 (a_1 a_2 + a_5 a_6)\delta_1 \delta_6 \delta_3^2. \end{cases} \tag{15}$$

Therefore, the condition guaranteeing the nonsingularity of the lump solution is $(a_1 a_6 - a_2 a_5)^2 [(a_1\delta_6 + a_2\delta_1)^2 + (a_5\delta_6 + a_6\delta_1)^2] \delta_1^2 \delta_3 \neq 0$, and they should satisfy the following constraint conditions:

$$-\frac{b_1\alpha_1 + b_2\alpha_2 + b_3\alpha_3}{\delta_3} < 0. \tag{16}$$

Now, the transformations in (7) generate a large class of lump solutions to (6), determined by $u = 2(\ln f)_x, v = 2(\ln f)_{tt}$:

$$\begin{cases} u = \frac{4a_1(a_1 x + a_2 y + a_3 t + a_4) + 4a_5(a_5 x + a_6 y + a_7 t + a_8)}{f}, \\ v = \frac{4(a_3^2 + a_7^2)f - 4[a_1(a_1 x + a_2 y + a_3 t + a_4) + 4a_5(a_5 x + a_6 y + a_7 t + a_8)]^2}{f^2}. \end{cases} \tag{17}$$

We take a special choice for the parameters: $\delta_1 = 1, \delta_3 = 1, \delta_6 = 1, \alpha_1 = \alpha_2 = \alpha_3 = 1, a_1 = 2, a_2 = -1, a_3 = 224/785, a_4 = 5, a_5 = -2, a_6 = -30, a_7 = -8/785, a_8 = 6, a_9 = 227007188/660185$, under which by f (13), (14), and (15) will present the lump solution to (6).

$$f = \left(2x - y - \frac{224}{785}t + 5\right)^2 + \left(-2x - 30y - \frac{8}{785}t + 6\right)^2 + \frac{227007188}{660185}. \tag{18}$$

We plot the graphs of this solution in Figures 1 and 2.

By calculating, we find that the lump solution reaches the peak when $t = 0$. As time goes on, the waves moving forward reach the same amplitude at other points. The results provide new insights into the generation of the large amplitude waves and thus are useful in the application or prevention of the large amplitude waves in the propagation of many important

nonlinear physical situations. Next, we will consider the application of these results to physical theory and experiments and its relation with the initial boundary value problem is also considered.

We secondly consider the case of $\delta_1 = \delta_5 = \delta_6 = 0$ for the combined nonlinear Equation (6). We obtain the following parameters:

$$\begin{cases} a_7 = b, \\ a_3 = \frac{2a_1a_5\delta_3 + (a_1a_6 + a_2a_5)\delta_2}{2b\delta_4}, \\ a_9 = -\frac{(a_1^2 + a_5^2)(c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3)}{(2a_1a_5\delta_3 + \delta_2(a_1a_6 + a_2a_5))^2 + 4b^2a_1a_2\delta_2\delta_4 + 4b^2a_1^2\delta_3\delta_4}, \end{cases} \tag{19}$$

and all other a_i are arbitrary. The above involved three constants $b_i, i = 1, 2, 3$ are defined as follows:

$$\begin{cases} c_1 = 3[-a_1(a_1a_6 + a_2a_5)\delta_2 - 2a_1^2a_5\delta_3 + 2b^2a_5\delta_4]b\delta_4, \\ c_2 = 6(a_1a_2 + a_5a_6)b^2\delta_4^2, \\ c_3 = 2[(a_1^2 - 3a_5^2)b^2\delta_3\delta_4 + (a_1a_2 - 3a_5a_6)b^2\delta_2\delta_3 + (2a_1a_5\delta_3 + (a_1a_6 + a_2a_5)\delta_2)^2]. \end{cases} \tag{20}$$

Therefore, the condition guaranteeing the nonsingularity of the lump solution is as follows:

$$(2a_1a_5\delta_3 + \delta_2(a_1a_6 + a_2a_5))^2 + 4b^2a_1a_2\delta_2\delta_4 + 4b^2a_1^2\delta_3\delta_4 \neq 0, \tag{21}$$

and they should satisfy the following constraint conditions: $a_9 > 0$.

We take a special choice for the parameters: $\delta_2 = 1, \delta_3 = 1, \delta_4 = 1, \alpha_1 = \alpha_2 = \alpha_3 = 1, a_1 = -1, a_2 = 11, a_3 = 10(1/(\sqrt{-5 + 5\sqrt{5}})), a_4 = 1, a_5 = -2, a_6 = 2, a_7 = 1/(\sqrt{-5 + 5\sqrt{5}})$,

$a_8 = 1, a_9 = -1/2((6\sqrt{-5 + 5\sqrt{5}}\sqrt{5} + 55\sqrt{5} - 135)/(\sqrt{5} - 3))$, and

$$f = \left(10\frac{t}{\sqrt{-5 + 5\sqrt{5}}} - x + 11y + 1\right)^2 + \left(\frac{t}{\sqrt{-5 + 5\sqrt{5}}} - 2x + 2y + 1\right)^2 - \frac{6\sqrt{-5 + 5\sqrt{5}}\sqrt{5} + 55\sqrt{5} - 135}{2\sqrt{5} - 6}, \tag{22}$$

under which f by (13), (19), and (20) will present the lump solution to (6), determined by $u = 2(\ln f)_x, v = 2(\ln f)_{tt}$.

$$\begin{cases} u = \frac{-4\left(10\left(\frac{t}{\sqrt{-5 + 5\sqrt{5}}}\right) - x + 11y + 1\right) - 8\left(\left(\frac{t}{\sqrt{-5 + 5\sqrt{5}}}\right) - 2x + 2y + 1\right)}{f}, \\ v = \frac{2f_{tt}f - 2[f_t]^2}{f^2}. \end{cases} \tag{23}$$

We plot the graphs of this solution in Figures 3 and 4.

We have presented lump solutions to a combined fourth-order nonlinear partial differential equation by means of the

Hirota bilinear formulation. The solutions have been depicted for special values of the parameters and three different values of t . Figure 3 shows lump wave. By calculating,

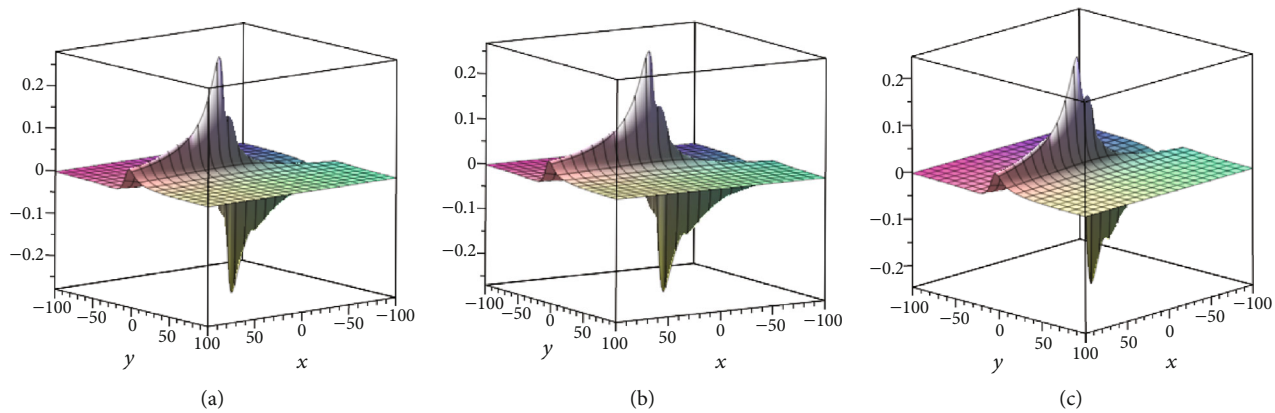


FIGURE 1: (Color online) soliton solution u with $t = 0, 15, 30$: 3D plots.

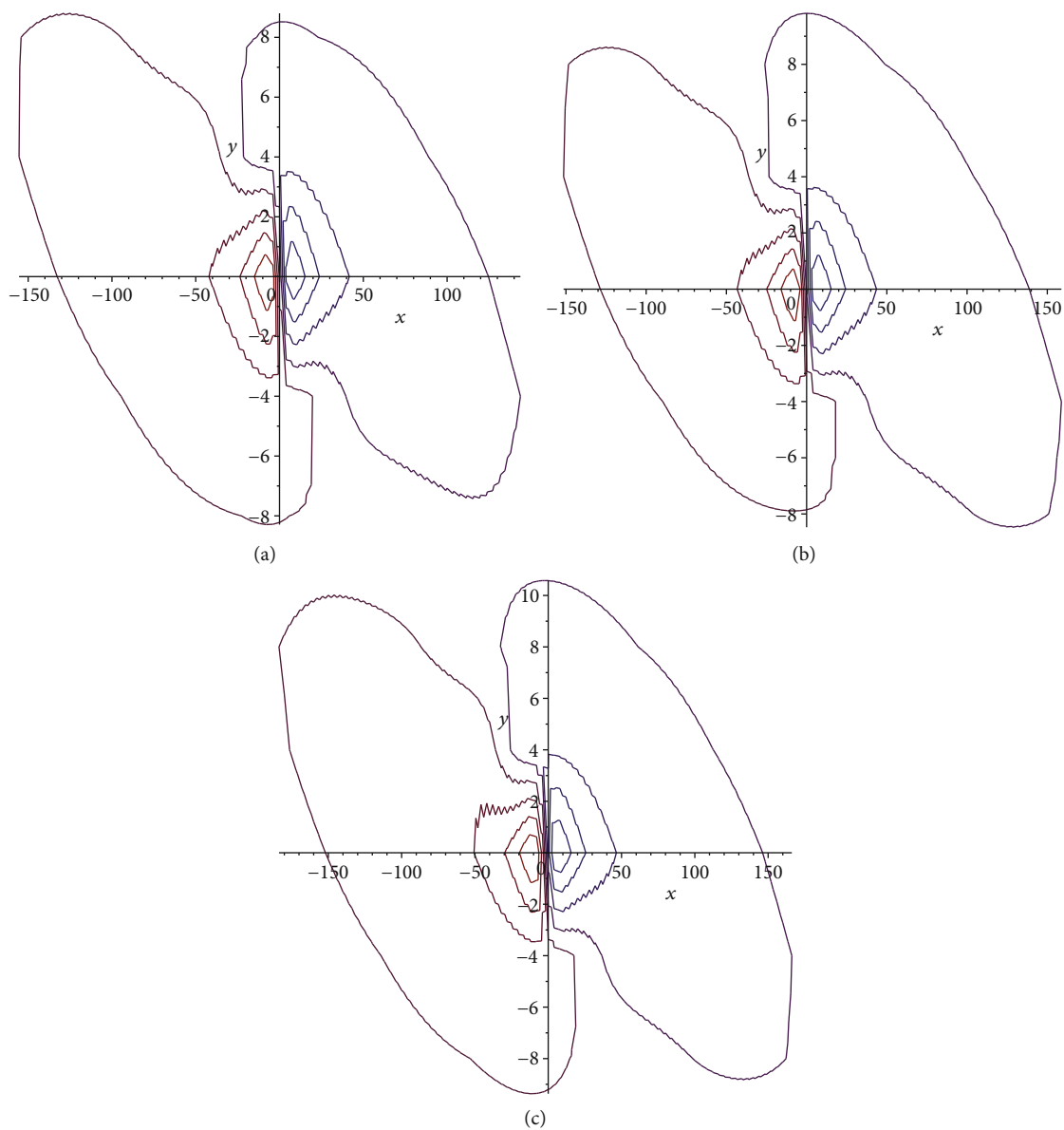


FIGURE 2: (Color online) soliton solution u with $t = 0, 10, 30$: contour plots.

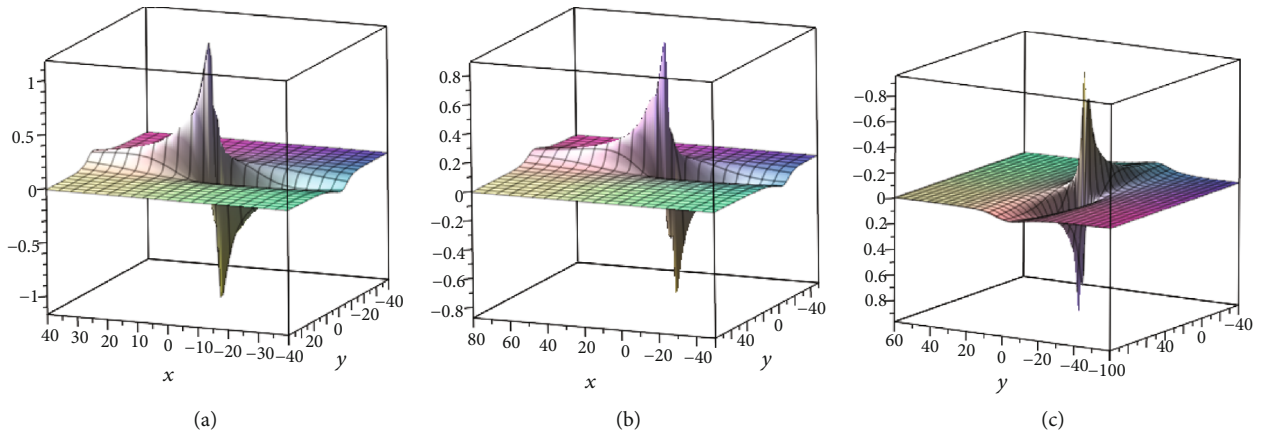


FIGURE 3: (Color online) soliton solution u with $t = 0, 15, 30$: 3D plots.

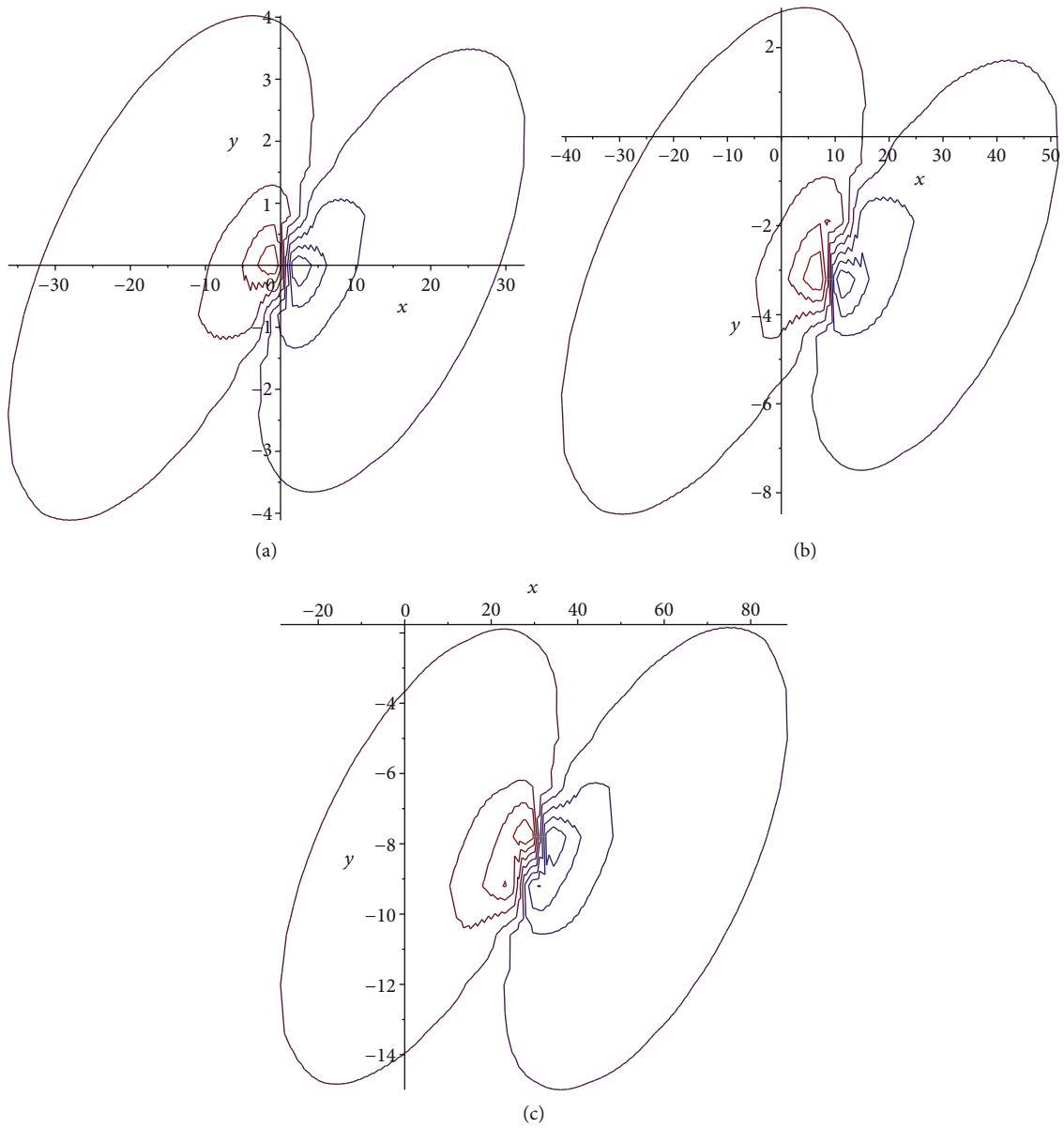


FIGURE 4: (Color online) soliton solution u with $t = 0, 10, 30$: contour plots.

find that the lump solution has a peak when $t = 0$. As time goes on, the waves moving forward reach the same amplitude at other points.

4. Conclusion and Remarks

In this paper, we have considered a combined fourth-order nonlinear partial differential equation. We have worked out a class of lump solutions by Hirota's bilinear method. It is important to remark that the three nonlinear terms can be merged together into the considered nonlinear model. It is worth pointing out that the term $D_x^2 D_t^2$ makes the calculation more difficult. It is also reflected that the structure of the solution is more complex. No one has considered it over before. All the above results offer us abundant new exact solutions, which enrich the existing theories of solutions [24–29] to equations, and add valuable insights into soliton solutions and dromion-type solutions, developed through various powerful solution techniques including the Hirota perturbation approach, the Riemann-Hilbert approach, the Wronskian technique, symmetry reductions, and symmetry constraints. It is also interesting to research for lump and interaction solutions to other generalized bilinear differential equations [30–34]. The research to establish a fundamental theory of lump and interaction solutions for PDEs deserve our further effort.

Data Availability

The data used to support the findings of this study are available from the corresponding authors upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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