April 2022

Advances and Applications of Optimal Polynomial Approximants

Raymond Centner
University of South Florida

Advances and Applications of Optimal Polynomial Approximants

by

Raymond Centner

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
Department of Mathematics & Statistics
College of Arts & Sciences
University of South Florida

Major Professor: Catherine Bénéteau, Ph.D.
Dmitry Khavinson, Ph.D.
Myrto Manolaki, Ph.D.
Evguenii Rakhmanov, Ph.D.
Boris Shekhtman, Ph.D.

Date of Approval:
March 25, 2022

Keywords: L^p space, Hardy space, digital filter, Shanks' Conjecture, Moore-Penrose inverse

Acknowledgments

I would like to thank Dr. Catherine Bénéteau for leading me down the path that I've always wanted to take. The conversations that we've had over the years have been crucial for my mathematical and pedagogical development. I would also like to thank Dr. Dmitry Khavinson for introducing me to many different areas of analysis. He has been a major inspiration to me since I started at USF. I would like to thank Dr. Myrto Manolaki, Dr. Evguenii Rakhmanov, and Dr. Boris Shekhtman for serving on my Doctoral Committee. I greatly appreciate their valuable feedback throughout my candidacy. I would also like to thank Dr. Howard Cohl for providing helpful feedback on this dissertation. In addition, I would like to thank Dr. Scott Rimbey and Dr. Dmytro Savchuk for giving me the opportunity to develop a vast teaching experience. Lastly, I would like to thank Dr. Brian Curtin and former classmate Robert Freeman for the early advice on transitioning into graduate life.

Table of Contents

List of Figures

Abstract

The history of optimal polynomial approximants (OPAs) dates back to the engineering literature of the 1970s. Here, these polynomials were studied in the context of the Hardy space $H^2(X)$, where X denotes the open unit disk $\mathbb{D}$ or the bidisk $\mathbb{D}^2$. Under certain conditions, it was thought that these polynomials had all of their zeros outside the closure of X. Hence, it was suggested that these polynomials could be used to design a stable digital filter. In recent mathematics literature, OPAs have been studied in many different function spaces; in these settings, numerous papers have been devoted to studying the properties of their zeros. In this dissertation, we begin by introducing the notion of optimal polynomial approximant in the space $L^p(\mathbb{T})$, where $\mathbb{T}$ denotes the unit circle and $1 \leq p \leq \infty$. Here, we shed light on an orthogonality condition that allows us to study OPAs in $L^p(\mathbb{T})$ with the additional tools from the $L^2(\mathbb{T})$ setting. We later use this orthogonality condition to compute the coefficients of some OPAs in $L^p(\mathbb{T})$; this will give us insight into the location of their zeros. We continue the dissertation by discussing the connection of OPAs to 1D digital filter design; a majority of these discussions will be devoted to surveying the design process of Chui and Chan [13]. Toward the end of this dissertation, we extend the operator theoretic approach of Izumino [22] to the $L^2(\mathbb{T})$ setting; in light of the orthogonality condition, this provides us additional tools to study the zeros of OPAs in $L^p(\mathbb{T})$.

Chapter 1: Introduction[1]

The methods of least-squares approximation have been used in many areas of engineering since the latter half of the 20th century. In 1963, Robinson [27] studied these methods in the context of digital signal processing and geophysical studies. His goal was to obtain a finite-length wavelet whose convolution with a given finite-length wavelet best approximates the unit spike. In other words, given a finite sequence of real numbers $b := (b_0, b_1, \ldots, b_m)$ and a particular value of $n \in \mathbb{N}$, he was trying to find another finite sequence of real numbers $a := (a_0, a_1, \ldots, a_n)$ such that the difference between $a * b$ and the unit spike $(1, 0, \ldots, 0)$ has the smallest l^2-norm. This sequence a was referred to as a least-squares approximate inverse.

In the 1970s, least-squares approximation appeared in connection with 2D recursive digital filter design. Applications of 2D digital filtering include the processing of medical pictures, satellite photographs, seismic data mappings, gravity waves data, magnetic recordings, and radar and sonar maps [20]. In 1972, Justice, Shanks, and Treitel [23] applied the methods of least-squares approximation in an effort to ensure 2D filter stability. Generally speaking, a filter is called stable if it corresponds to a system in which bounded input yields bounded output. In their studies, they considered the following problem: given a polynomial $f(z, w)$ of two complex variables, find another polynomial $q(z, w)$ such that the difference between qf and 1 has the smallest $H^2(\mathbb{D}^2)$-norm. The polynomial q was referred to as a planar least-squares inverse (PLSI) polynomial. It is one of the earliest examples of an *optimal polynomial approximant*.

In 1980, Chui [12] formulated the problem of Robinson in the $H^2(\mathbb{D})$ setting. More specifically, he considered the following problem: given $n \in \mathbb{N}$ and a polynomial $f(z) := \sum_{k=0}^{m} b_k z^k$ with $f(0) \neq 0$, find an nth degree polynomial $q(z) := \sum_{k=0}^{n} a_k z^k$ such that the

[1]This chapter has been modified from [8].

difference between qf and 1 has the smallest $H^2(\mathbb{D})$-norm. The polynomial q was referred to as a least-squares inverse polynomial. For brevity, we'll refer to these as LSI polynomials. Chui's formulation of the problem ties in well with signal processing since digital filters are often represented by quotients of polynomials. Now, in many contexts of 1D recursive digital filter design, it is important to ensure that the poles of the filter are outside of $\overline{\mathbb{D}}$. If this is the case, then one could easily modify the filter to be stable. In 1982, Chui and Chan [13] proposed a design method that guaranteed filter stability. In the build-up to their method, they reformulated the problem of Robinson to include arbitrary $f \in H^2(\mathbb{D})$ with $f(0) \neq 0$. They proceeded to show that for this general class of functions, the LSI polynomials are zero-free in $\overline{\mathbb{D}}$.

Since the 1980s, least-squares inverse polynomials have been studied in many different function spaces. In the mathematics community, these polynomials became known as optimal polynomial approximants (OPAs). In 2015, OPAs were introduced in the Dirichlet-type spaces $\mathcal{D}_\alpha$, $\alpha \in \mathbb{R}$ (see [3] for details). Recall that $\mathcal{D}_\alpha$ denotes the space of analytic functions $f(z) = \sum_{k=0}^{\infty} c_k z^k$ in the open unit disk $\mathbb{D}$ such that

$$\sum_{k=0}^{\infty}(k+1)^\alpha |c_k|^2 < \infty.$$

It can be shown that $\mathcal{D}_\alpha$ is a Hilbert space with inner product

$$\langle f, g\rangle_\alpha := \sum_{k=0}^{\infty}(k+1)^\alpha c_k \overline{d_k},$$

where $f(z) = \sum_{k=0}^{\infty} c_k z^k$ and $g(z) = \sum_{k=0}^{\infty} d_k z^k$. For any $f \in \mathcal{D}_\alpha \setminus \{0\}$ and $n \in \mathbb{N}$, the nth OPA of $1/f$ in $\mathcal{D}_\alpha$ is defined as the unique polynomial that minimizes the norm $\|qf - 1\|_\alpha$, where q varies over $\mathcal{P}_n$. Here, $\mathcal{P}_n$ denotes the space of analytic polynomials of degree at most n, $\|\cdot\|_\alpha$ denotes the norm induced by the inner product, and $\mathbb{N} := \{0, 1, 2, \dots\}$. Since 2015, several papers have been devoted to investigating the properties of these polynomials (see, e.g., [5, 6, 7]). Such properties include boundary behavior, connections to reproducing kernel functions, and location of zeros. In [5], it was shown that if $\alpha \geq 0$, $f \in \mathcal{D}_\alpha$, and $f(0) \neq 0$, then the zeros of the associated OPA lie outside of $\overline{\mathbb{D}}$. In more recent literature, optimal polynomial approximants and their zeros have been studied in the context of ℓ^p_A (see [11] for details).

The primary focus of this dissertation is to investigate OPAs in the context of $H^p := H^p(\mathbb{D})$, $1 \leq p \leq \infty$. In particular, we would like to know if the nontrivial OPAs are zero-free in the closed disk $\overline{\mathbb{D}}$, which is the case in the Hardy space H^2 and the Dirichlet-type spaces $\mathcal{D}_\alpha$, $\alpha \geq 0$. Recall that for $1 \leq p < \infty$, H^p denotes the space of analytic functions f in the open unit disk $\mathbb{D}$ such that

$$\sup_{0 \leq r < 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{it})|^p dt < \infty.$$

It can be shown that H^p is a Banach space with norm

$$\|f\|_p := \left\{ \sup_{0 \leq r < 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{it})|^p dt \right\}^{\frac{1}{p}}.$$

Recall that H^∞ denotes the space of bounded analytic functions in $\mathbb{D}$. This is a Banach space with norm

$$\|f\|_\infty := \sup_{z \in \mathbb{D}} |f(z)|.$$

To facilitate our study in the H^p setting, we introduce the notion of optimal polynomial approximant in the space $L^p := L^P(\mathbb{T})$. For $1 \leq p < \infty$, L^p denotes the space of Lebesgue measurable $\mathbb{C}$-valued functions f on the unit circle $\mathbb{T}$ such that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(e^{it})|^p dt < \infty.$$

It can be shown that L^p is a Banach space with norm

$$\|f\|_p := \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(e^{it})|^p dt \right\}^{\frac{1}{p}}.$$

Recall that L^∞ denotes the space of essentially bounded Lebesgue measurable $\mathbb{C}$-valued functions f on $\mathbb{T}$. It can be shown that L^∞ is a Banach space with norm

$$\|f\|_\infty := \operatorname*{ess\,sup}_{t\in[-\pi,\pi]} |f(e^{it})|.$$

It is well known that for any $f \in H^p$, $1 \le p \le \infty$, the nontangential limits $F(e^{it})$ exist almost everywhere on $\mathbb{T}$ and $F \in L^p$ (see, e.g., [15, 17, 21, 28]). Futhermore, $\|F\|_p = \|f\|_p$. This shows that H^p is isometrically imbedded in L^p. Therefore, we regard any $f \in H^p$ as a function in L^p with the understanding that we are identifying f with its nontangential limit function F. Accordingly, any result we obtain for functions in L^p will automatically hold for functions in H^p.

After developing the notion of optimal polynomial approximant in the space L^p, we focus our attention to the special case of L^2. As we will see in Section 2.3, many OPAs in L^p are equivalent to an OPA in L^2. In Chapter 3, we focus on applications in the H^2 setting. In particular, we survey the filter design process of Chui and Chan [13]. We organize their process in three stages: in the first stage, we approximate an ideal digital filter by the magnitude of a non-vanishing function in H^2; in the second stage, we use OPAs to approximate the magnitude of this non-vanishing function with the magnitude of a rational function; in the third stage, we make modifications to the rational function to ensure stability.

We later introduce additional tools to the L^2 setting by formulating the theory of OPAs using Moore-Penrose inverses; this idea was presented in 1985 by Izumino [22] in the context of H^2. As one of the fundamental results, we show that an OPA in L^2 is contained in the image of a particular Moore-Penrose inverse; this allows one to study the properties of OPAs by studying the respective operator. As a benefit, one may find that the operator theoretic framework provides a convenient way to deduce results. At any rate, due to the relationship between OPAs in L^p and OPAs in L^2, there seems to be a lot of potential for the operator theoretic approach.

Chapter 2: Optimal Polynomial Approximants in L^p[1]

The context of this chapter will be the space L^p, $1 \leq p \leq \infty$. We would like to have an adequate structure to support the theory, so it seems natural to consider only the values of p for which L^p is a Banach space. We begin by discussing existence and uniqueness of optimal polynomial approximants in L^p, $1 \leq p \leq \infty$. For the extreme cases of $p = 1$ and $p = \infty$, there seems to be a lot of fertile ground for research. We then introduce OPAs in the Hilbert space L^2. This is a generalization of the familiar theory in H^2 (see [2] for a survey of OPAs in $\mathcal{D}_\alpha$). In Section 2.3, we use an orthogonality condition to characterize OPAs in L^p, $1 < p < \infty$. We later use this condition to compute some OPAs for several values of p. Throughout this chapter, we discuss what we know and pose several open questions about the zeros of optimal polynomial approximants in L^p.

2.1 Existence and Uniqueness

The notion of optimal polynomial approximant has been studied extensively in the context of the Dirichlet-type spaces $\mathcal{D}_\alpha$, $\alpha \in \mathbb{R}$. In this case, it is easy to see that such a polynomial must exist and that it is unique: since $\mathcal{D}_\alpha$ is a Hilbert space, the orthogonal projection of 1 onto the subspace $f\mathcal{P}_n := \{fq : q \in \mathcal{P}_n\}$ gives the unique function fQ that minimizes the distance from 1; since f is not identically zero, the polynomial Q must also be unique. In less structured normed spaces, existence and uniqueness of best approximations become more of a delicate issue. However, consider the case where Y is a finite dimensional subspace of a normed space X. Then for each $x \in X$, there exists a best approximation to x out of Y (see, e.g., [24, Theorem 6.1-1]). This allows us to define the notion of optimal polynomial approximant in L^p, $1 \leq p \leq \infty$.

[1]This chapter has been taken from [8]. Slight modifications have been made for clarity.

Proposition 2.1.1 (Existence)**.** *Let $1 \leq p \leq \infty$ and $n \in \mathbb{N}$. For $f, g \in L^p$, $f \not\equiv 0$, there exists a polynomial $Q \in \mathcal{P}_n$ such that*

$$\inf_{q \in \mathcal{P}_n} \|qf - g\|_p = \|Qf - g\|_p.$$

Proof. Since $f\mathcal{P}_n$ is a finite dimensional subspace of L^p, the result is immediate. □

Proposition 2.1.1 states that there is a best approximation to g out of the space $f\mathcal{P}_n$. If Qf is such a best approximation, then the polynomial $Q \in \mathcal{P}_n$ is referred to as an *optimal polynomial approximant in L^p*. Now, it's important to note that best approximations in the Banach space setting are not always unique. However, suppose that $(X, \|\cdot\|)$ is a strictly convex normed space. This means that $\|u + v\| < 2$ whenever u and v are distinct unit vectors. Then there is at most one best approximation to any element $x \in X$ out of a given subspace Y (see, e.g., [24, Theorem 6.2-3]). This fact guarantees that optimal polynomial approximants in L^p, $1 < p < \infty$, are unique. (See [29] for a similar discussion of uniqueness of OPAs in ℓ^p_A.)

Proposition 2.1.2 (Uniqueness)**.** *Let $1 < p < \infty$ and $n \in \mathbb{N}$. For $f, g \in L^p$, $f \not\equiv 0$, there exists a unique polynomial $Q \in \mathcal{P}_n$ such that*

$$\inf_{q \in \mathcal{P}_n} \|qf - g\|_p = \|Qf - g\|_p.$$

Proof. It is well-known that L^p is strictly convex for $1 < p < \infty$. Since g has a best approximation out of $f\mathcal{P}_n$, it must be unique. □

In a normed space X, the set of best approximations to a given point $x \in X$ out of a subspace Y is convex (see, e.g., [24, Lemma 6.2-1]). In the case of L^1, Proposition 2.1.1 implies that the set of best approximations to a given function g out of $f\mathcal{P}_n$ is nonempty. Therefore, there is either a unique best approximation or an infinite number of best approximations. The following proposition shows that the former does not need to hold.

Proposition 2.1.3. *Optimal polynomial approximants in L^1 are not necessarily unique.*

Proof. Let $f \equiv 1$, and consider the characteristic function $g(e^{it}) := \chi_{[-\pi,0)}(t)$. We show that

$$\inf_{q \in \mathcal{P}_0} \|qf - g\|_1 = \|af - g\|_1$$

for all $a \in [0,1]$. Let $\Lambda := \{h \in L^\infty \colon \|h\|_\infty \leq 1 \text{ and } \frac{1}{2\pi}\int_{-\pi}^{\pi} h dt = 0\}$. By Hahn-Banach duality, it follows that

$$\inf_{q \in \mathcal{P}_0} \|qf - g\|_1 = \sup_{h \in \Lambda} \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} ghdt \right|.$$

If we let $h_0(e^{it}) := \chi_{[-\pi,0)}(t) - \chi_{[0,\pi)}(t)$, then we see that $h_0 \in \Lambda$. Consequently,

$$\inf_{q \in \mathcal{P}_0} \|qf - g\|_1 \geq \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} gh_0 dt \right| = \frac{1}{2}. \tag{2.1}$$

Now, suppose that $a \in [0,1]$. Then

$$\begin{aligned} \|af - g\|_1 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |a - g| dt \\ &= \frac{1}{2\pi} \int_{-\pi}^{0} (1-a) dt + \frac{1}{2\pi} \int_{0}^{\pi} a dt \\ &= \frac{1}{2}. \end{aligned} \tag{2.2}$$

From (2.1) and (2.2), we conclude that

$$\inf_{q \in \mathcal{P}_0} \|qf - g\|_1 = \|af - g\|_1$$

whenever $a \in [0,1]$. □

On the other hand, consider the case of L^∞. From Proposition 2.1.1 and our previous discussion, the set of best approximations to a given function g out of $f\mathcal{P}_n$ consists of either a single element or an infinite number of elements. As in the case of L^1, the former does not need to hold.

Proposition 2.1.4. *Optimal polynomial approximants in L^∞ are not necessarily unique.*

Proof. Let $f(z) = 1 - z$. We show that

$$\inf_{q \in \mathcal{P}_0} \|qf - 1\|_\infty = \|af - 1\|_\infty$$

for all $a \in [0, 1]$. For any $a \in \mathbb{C}$,

$$\|af - 1\|_\infty = \|(a-1) - az\|_\infty \leq |a-1| + |a|. \tag{2.3}$$

If $a = 0$, then equality in (2.3) clearly holds. If $a \neq 0$, let

$$w := -\frac{|a|}{a} e^{i \arg(a-1)}.$$

Note that

$$\begin{aligned} |af(w) - 1| &= |(a-1) - aw| \\ &= \left| |a-1| e^{i \arg(a-1)} + |a| e^{i \arg(a-1)} \right| \\ &= |a-1| + |a|. \end{aligned}$$

Thus, equality in (2.3) holds, and it follows that

$$\|af - 1\|_\infty = |a+1| + |a| \geq (1 - |a|) + |a| = 1$$

for all $a \in \mathbb{C}$. Consequently,

$$\inf_{q \in \mathcal{P}_0} \|qf - 1\|_\infty = 1, \tag{2.4}$$

where the infimum is attained for all $a \in \mathbb{C}$ such that $|a-1| + |a| = 1$.

If $a \in \mathbb{C}$ satisfies $|a-1| + |a| = 1$, then

$$|1-a| + |a| = |(1-a) + a|.$$

Hence, $a = \alpha(1-a)$ for some positive real α. It follows that a is a real number in $[0,1]$. Conversely, if $a \in [0,1]$, then

$$|a-1| + |a| = (1-a) + a = 1.$$

Therefore, we conclude that $|a-1| + |a| = 1$ if and only if $a \in [0,1]$. It follows that

$$\|af - 1\|_\infty = 1 \quad \text{for all} \quad a \in [0,1]. \tag{2.5}$$

By combining (2.4) and (2.5), the result follows. $\square$

Although optimal polynomial approximants in L^p are not necessarily unique for $p = 1$ or $p = \infty$, there are still interesting questions that one can ask about best approximations in these spaces. For example, when there are infinitely many best approximations to g out of $f\mathcal{P}_n$, is there an optimal polynomial approximant that has norm strictly smaller than all the others? In the case of Proposition 2.1.3 (or Proposition 2.1.4), this optimal polynomial approximant is clearly the constant 0. As another example, is there a way to characterize the functions f such that the constant function 0 is a best approximation to 1 out of $f\mathcal{P}_n$? In Proposition 2.4.2, we will see an analogous characterization in the context of H^p, $1 < p < \infty$. Nevertheless, a best approximation to g out of $f\mathcal{P}_n$ is unique whenever $1 < p < \infty$. This leads us to the following definition.

Definition 2.1.5 (nth OPA)**.** Let $1 < p < \infty$ and $n \in \mathbb{N}$. For $f, g \in L^p$, $f \not\equiv 0$, the unique polynomial that minimizes the norm $\|qf - g\|_p$, where q varies over $\mathcal{P}_n$, is denoted by $q_{n,p}[f,g]$ and is called the nth OPA of g/f in L^p.

In the next section, we develop the theory of OPAs in L^2; the inner product structure gives us an efficient way to compute the coefficients of OPAs; it also gives us an easy way to verify whether or not the OPA $q_{1,2}[f,g]$ is zero-free in $\overline{\mathbb{D}}$. As we will see in Section 2.3, it is possible to relate certain OPAs in L^p, $p \geq 2$, to OPAs in L^2.

2.2 OPAs in L^2

In 1980, Chui [12] presented the idea of least-squares inverse polynomials in the context of H^2. The LSI polynomials being studied can be viewed as OPAs of the form $q_{n,2}[f,1]$, where f is a polynomial. In 1982, Chui and Chan [13] presented a filter design method that extended this study to include arbitrary functions $f \in H^2$, $f(0) \neq 0$. The reason for considering more general f was to approximate certain outer functions that were defined in conjunction with an ideal digital filter (see Chapter 3). One of the selling points of this method was that the coefficients of the LSI polynomial could be expressed as the solution to a system of linear equations. Moreover, the associated matrix is positive definite and Toeplitz. Therefore, its inverse can be computed quickly with any of the available algorithms (see [1] for details).

In this section, we will see that some of the most important OPA properties in H^2 generalize naturally to the L^2 setting. Before we begin, let's note that the following proposition follows immediately from the notion of orthogonal projection.

Proposition 2.2.1 (Linearity)**.** *Let $f, g, h \in L^2$, $f \not\equiv 0$, $n \in \mathbb{N}$, and $\alpha, \beta \in \mathbb{C}$. Then*

$$q_{n,2}[f, \alpha g + \beta h] = \alpha q_{n,2}[f, g] + \beta q_{n,2}[f, h].$$

Proof. By definition, the OPA $q_{n,2}[f,g]$ minimizes the norm $\|qf - g\|_2$, where q varies over $\mathcal{P}_n$. Since L^2 is a Hilbert space and $f\mathcal{P}_n$ is a closed subspace, it follows that $q_{n,2}[f,g]f$ is the orthogonal projection of g onto $f\mathcal{P}_n$. The proposition follows by linearity of the projection. □

In the following result, we characterize OPAs of the form $q_{n,2}[f,g]$, where $f, g \in L^2$.

Proposition 2.2.2. *Let $f, g \in L^2$, $f \not\equiv 0$, $n \in \mathbb{N}$, and $Q \in \mathcal{P}_n$. Then*

$$\langle Qf - g, z^k f \rangle = 0$$

for $k = 0, \ldots, n$ if and only if $Q = q_{n,2}[f,g]$.

Proof. Let $q_{n,2} := q_{n,2}[f,g]$. If $\langle Qf - g, z^k f\rangle = 0$ for $k = 0,\dots,n$, then $(Qf - g) \perp f\mathcal{P}_n$. Since $g = Qf + (g - Qf)$, it follows that Qf is the projection of g onto $f\mathcal{P}_n$, i.e., $Q = q_{n,2}$. Conversely, since $q_{n,2}f$ is the projection of g onto $f\mathcal{P}_n$, and since $g = q_{n,2}f + (g - q_{n,2}f)$, it follows that $(q_{n,2}f - g) \in L^2 \ominus f\mathcal{P}_n$. In particular, $\langle q_{n,2}f - g, z^k f\rangle = 0$ for $k = 0,\dots,n$. $\square$

A valuable aspect of Proposition 2.2.2 is that it allows us to compute the coefficients of $q_{n,2}[f,g]$ in an efficient way; as we will see, the coefficients are expressed as a solution to a system of linear equations. In fact, the associated matrix has the same properties as the one that Chui and Chan were considering in [13]. (For a recent discussion of the computation of coefficients in the context of analytic function spaces, see [4, 7].)

Theorem 2.2.3 (Coefficients in L^2)**.** *Let $f,g \in L^2$, $f \not\equiv 0$, and $n \in \mathbb{N}$. Set $q_{n,2}[f,g](z) = \sum_{j=0}^n a_j z^j$. Then the coefficients of $q_{n,2}[f,g]$ satisfy*

$$\begin{bmatrix} \langle f,f\rangle & \langle zf,f\rangle & \dots & \langle z^n f,f\rangle \\ \langle f,zf\rangle & \langle zf,zf\rangle & \dots & \langle z^n f,zf\rangle \\ \vdots & \vdots & & \vdots \\ \langle f,z^n f\rangle & \langle zf,z^n f\rangle & \dots & \langle z^n f,z^n f\rangle \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} \langle g,f\rangle \\ \langle g,zf\rangle \\ \vdots \\ \langle g,z^n f\rangle \end{bmatrix}.$$

Proof. From Proposition 2.2.2, it follows that

$$\sum_{j=0}^{n} \langle z^j f, z^k f\rangle a_j = \langle g, z^k f\rangle \quad \text{for} \quad k = 0,\dots,n. \tag{2.6}$$

Let a denote the vector with components a_j, $j = 0,\dots,n$, and y denote the vector with components $\langle g, z^k f\rangle$, $k = 0,\dots,n$. If B is the $(n+1)\times(n+1)$ matrix with entries

$$b_{kj} = \langle z^j f, z^k f\rangle, \quad 0 \le k,j \le n,$$

it is easy to see that (2.6) is equivalent to $Ba = y$. $\square$

Example 2.2.4. Let $f(z) = 1 - z$. Then $q_{1,2}[f,1](z) = \frac{1}{3}z + \frac{2}{3}$.

Proof. Set $q_{1,2}[f,1](z) = a_1 z + a_0$. By Theorem 2.2.3, the coefficients of $q_{1,2}[f,1](z)$ satisfy

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Therefore, it easily follows that $q_{1,2}[f,1](z) = \frac{1}{3}z + \frac{2}{3}$. □

As previously suggested, the design method of Chui and Chan was motivated by the fact that all nontrivial LSI polynomials are zero-free in $\overline{\mathbb{D}}$. Since we are extending these polynomials to the L^2 setting, it's natural to question if OPAs of the form $q_{n,2}[f,g]$, where $f, g \in L^2$, are zero-free in $\overline{\mathbb{D}}$. It turns out that this is not the case; the following theorem shows that the function g plays a prominent role.

Theorem 2.2.5. *Let $f, g \in L^2$ and $f \not\equiv 0$. The OPA $q_{1,2}[f,g]$ is zero-free in $\overline{\mathbb{D}}$ if and only if*

$$|\langle g, f\rangle| > |\langle g, zf\rangle|.$$

Proof. Set $q_{1,2}[f,g](z) = a_1 z + a_0$. From Proposition 2.2.2, we have that

$$a_0\langle f, z^k f\rangle + a_1\langle zf, z^k f\rangle = \langle g, z^k f\rangle \quad \text{for} \quad k = 0, 1. \tag{2.7}$$

This is equivalent to

$$\begin{bmatrix} \langle f, f\rangle & \langle zf, f\rangle \\ \langle f, zf\rangle & \langle zf, zf\rangle \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} \langle g, f\rangle \\ \langle g, zf\rangle \end{bmatrix}. \tag{2.8}$$

Let A denote the 2×2 matrix in (2.8). Since $|\langle f, f\rangle| > |\langle f, zf\rangle|$, it follows that $\det A \neq 0$. Therefore

$$a_0 = \frac{\langle zf, zf\rangle\langle g, f\rangle - \langle zf, f\rangle\langle g, zf\rangle}{\det A}$$

and

$$a_1 = \frac{\langle f, f\rangle\langle g, zf\rangle - \langle f, zf\rangle\langle g, f\rangle}{\det A}.$$

Case 1 ($a_1 = 0$): If $q_{1,2}[f,g]$ is zero-free in $\overline{\mathbb{D}}$, then $a_0 \neq 0$. It follows from (2.7) that

$$|\langle g, f\rangle| = |\langle a_0 f, f\rangle| > |\langle a_0 f, zf\rangle| = |\langle g, zf\rangle|.$$

Conversely, suppose that $|\langle g, f\rangle| > |\langle g, zf\rangle|$. Then $\langle g, f\rangle$ and $\langle g, zf\rangle$ cannot simultaneously be zero. From (2.8), it follows that $a_0 \neq 0$, i.e., $q_{1,2}[f,g]$ is zero-free in $\overline{\mathbb{D}}$.

Case 2 ($a_1 \neq 0$): Let $q_{1,2}[f,g](z_0) = 0$. Then

$$z_0 = -\frac{a_0}{a_1} = \frac{\langle zf, f\rangle\langle g, zf\rangle - \langle zf, zf\rangle\langle g, f\rangle}{\langle f, f\rangle\langle g, zf\rangle - \langle f, zf\rangle\langle g, f\rangle}. \tag{2.9}$$

If $\langle g, zf\rangle = 0$, then it follows from (2.8) that $\langle g, f\rangle \neq 0$. From (2.9), we see that

$$|z_0| = \frac{|\langle zf, zf\rangle|}{|\langle f, zf\rangle|} > 1.$$

Hence, both implications in the theorem are trivially true. We thus assume that $\langle g, zf\rangle \neq 0$. Let $w := \frac{\langle g,f\rangle}{\langle g,zf\rangle}$ and $\alpha := \frac{\langle zf,f\rangle}{\langle zf,zf\rangle}$. From (2.9), it follows that

$$|z_0| = \left|\frac{w-\alpha}{1-\overline{\alpha}w}\right|.$$

Since $\alpha \in \mathbb{D}$, $z_0 \in \overline{\mathbb{D}}$ if and only if $w \in \overline{\mathbb{D}}$. Equivalently, $q_{1,2}[f,g]$ is zero-free in $\overline{\mathbb{D}}$ if and only if $|\langle g, f\rangle| > |\langle g, zf\rangle|$. □

The following example demonstrates the relative ease of applying Theorem 2.2.5.

Example 2.2.6 (Zero in $\mathbb{D}$)**.** Let $f(z) = z^2 + 1$ and $g(z) = 2z - 1$. Then $q_{1,2}[f,g]$ has its zero in $\overline{\mathbb{D}}$.

Proof. Note that

$$|\langle g, f\rangle| = 1 < 2 = |\langle g, zf\rangle|.$$

By Theorem 2.2.5, the result is immediate. □

Given any functions $f, g \in L^2$, $f \not\equiv 0$, Theorem 2.2.5 makes it easy to determine whether or not the OPA $q_{1,2}[f, g]$ is zero-free in $\overline{\mathbb{D}}$; as we will see in the upcoming sections, this will give us insight into the zeros of certain OPAs in L^p. In the case where n is arbitrary, the following is still unknown.

Open Question 1. Let $n \in \mathbb{N}$. For which functions $f, g \in L^2$ can we guarantee that $q_{n,2}[f, g]$ is zero-free in $\overline{\mathbb{D}}$?

2.3 Orthogonality Condition

Our main motivation for this dissertation is to investigate the zeros of optimal polynomial approximants in the context of H^p. For the Hilbert space H^2, we already know that OPAs of the form $q_{n,2}[f, 1]$, where $f \in H^2$ and $f(0) \neq 0$, are zero-free in $\overline{\mathbb{D}}$; one proof of this follows from a special case of Proposition 2.2.2. Now, in an effort to understand the zeros of OPAs in H^p, we generalize Proposition 2.2.2 to functions in L^p. As a result, this allows us to relate OPAs in L^p to OPAs in the more structured space L^2.

Recall that in a normed space $(X, \|\cdot\|)$, an element $x \in X$ is said to be *orthogonal* to a subspace Y if $\|x\| \leq \|x + y\|$ for all $y \in Y$. In the case where Y is a subspace of L^p, $1 < p < \infty$, Shapiro [30, Theorem 4.2.1] characterized the functions $f \in L^p$ that are orthogonal to Y. Now, it follows from the definition of OPA that $Qf - g$ is orthogonal to $f\mathcal{P}_n$ if and only if $Q = q_{n,p}[f, g]$. This observation, along with Shapiro's result, allows us to generalize Proposition 2.2.2. We refer to this generalization as the *orthogonality condition in L^p*.

Proposition 2.3.1 (Orthogonality Condition)**.** *Let $1 < p < \infty$, $f, g \in L^p$, $f \not\equiv 0$, $n \in \mathbb{N}$, and $Q \in \mathcal{P}_n$. Then*

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |Qf - g|^{p-1} \operatorname{sgn}(Qf - g) e^{-ikt} \overline{f} dt = 0$$

for $k = 0, \ldots, n$ if and only if $Q = q_{n,p}[f, g]$.

Proof. Let $F := Qf - g$. From [30, Theorem 4.2.1],

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |F|^{p-1}\overline{\operatorname{sgn}(F)} qf\, dt = 0 \tag{2.10}$$

for all $q \in \mathcal{P}_n$ if and only if F is orthogonal to $f\mathcal{P}_n$. From our previous discussion, it follows that (2.10) is true for all $q \in \mathcal{P}_n$ if and only if $Q = q_{n,p}[f, g]$. This is equivalent to our assertion. $\square$

One of the primary reasons Proposition 2.2.2 is useful is that it allows us to easily compute the coefficients of OPAs in L^2; since Proposition 2.3.1 is a generalization of Proposition 2.2.2, it's natural to wonder if Proposition 2.3.1 can be used to readily compute the coefficients of OPAs in L^p. By imposing certain conditions on f and p, we will see in Section 2.5 that this is the case.

On a different note, Proposition 2.3.1 can give us insight into OPAs of the form $q_{n,p}[f, 1]$, where $2 \leq p < \infty$. Before we look into the details, consider the following remark.

Remark 2.3.2. Let $2 \leq p < \infty$, $f \in L^p \setminus \{0\}$, and $n \in \mathbb{N}$. Define the function $g := \{q_{n,p}[f, 1]f - 1\}^{\frac{p-2}{2}}$. If $g \equiv 0$, then $q_{n,p}[f, 1]$ is zero-free on $\mathbb{T}$. Moreover, if $f \in H^p$ and $g \equiv 0$, then $q_{n,p}[f, 1]$ is zero-free in $\overline{\mathbb{D}}$.

We will reference this remark when we study the location of zeros in the next section. Nevertheless, perhaps the most valuable insight into the zeros of OPAs can be obtained by the following theorem.

Theorem 2.3.3. *Let* $2 \leq p < \infty$, $f \in L^p \setminus \{0\}$, $n \in \mathbb{N}$, *and* $g := \{q_{n,p}[f, 1]f - 1\}^{\frac{p-2}{2}}$. *If* $g \not\equiv 0$, *then*

$$q_{n,p}[f, 1] = q_{n,2}[fg, g].$$

Proof. Let $Q := q_{n,p}[f, 1]$. By Proposition 2.3.1, we have that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |Qf - 1|^{p-1} \operatorname{sgn}(Qf - 1) e^{-ikt}\overline{f} dt = 0 \quad \text{for} \quad k = 0, \ldots, n.$$

Now, we can write

$$
\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi}|Qf-1|^{p-1}\operatorname{sgn}(Qf-1)e^{-ikt}\overline{f}dt &= \frac{1}{2\pi}\int_{-\pi}^{\pi}|Qf-1|^{p-2}(Qf-1)e^{-ikt}\overline{f}dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}(Qf-1)^{\frac{p-2}{2}}(\overline{Qf}-1)^{\frac{p-2}{2}}(Qf-1)e^{-ikt}\overline{f}dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}g(Qf-1)e^{-ikt}\overline{fg}dt \\
&= \langle Qfg-g, z^k fg\rangle.
\end{aligned}
$$

Thus, $\langle Qfg - g, z^k fg\rangle = 0$ for $k = 0, \ldots, n$. By Proposition 2.2.2, we conclude that $Q = q_{n,2}[fg, g]$. $\square$

Theorem 2.3.3 suggests that we can gain a better understanding of OPAs in L^p by studying OPAs of the form $q_{n,2}[fg, g]$, where f and g are arbitrary functions in L^2 such that $fg \in L^2 \setminus \{0\}$; this amounts to studying the unique polynomial that minimizes the norm $\|\{qf-1\}g\|_2$, where q varies over $\mathcal{P}_n$. As a particular point of interest, we would like to know the necessary conditions to impose on g in order for $q_{n,2}[fg, g]$ to be zero-free in $\overline{\mathbb{D}}$. The following example shows that g cannot be arbitrary.

Example 2.3.4 (Zero in $\mathbb{D}$)**.** Let $f(z) = 2z+1$ and $g(z) = 1-z$. Then $q_{1,2}[fg, g] = -\frac{6}{35}z - \frac{1}{35}$.

To better align with our main motivation, let's consider the case where $f \in H^p$. If we assume that $2 \leq p < \infty$, then it's easy to see that the function g in Theorem 2.3.3 belongs to L^2. However, it's not at all clear if g belongs to H^2. Nevertheless, the following proposition shows that the OPA $q_{n,2}[fg, g]$ is equal to an OPA obtained by replacing g with a particular function in H^2.

Theorem 2.3.5. *Let $2 \leq p < \infty$, $f \in H^p \setminus \{0\}$, $n \in \mathbb{N}$, and $g := \{q_{n,p}[f,1]f - 1\}^{\frac{p-2}{2}}$. If $g \not\equiv 0$, then there exists a function $h \in H^2$, which is zero-free in $\mathbb{D}$, such that*

$$q_{n,p}[f, 1] = q_{n,2}[fh, h].$$

Proof. Define the function $g := \{q_{n,p}[f,1]f - 1\}^{\frac{p-2}{2}}$. Note that we can write $q_{n,p}[f,1]f - 1 = u\Phi$, where u is an inner function and Φ is an outer function (see Section 4.3 for a discussion of inner and outer functions). Let $h := \Phi^{\frac{p-2}{2}}$. Since Φ is zero-free in $\mathbb{D}$, it follows that $h \in H^2$. Furthermore, h is zero-free in $\mathbb{D}$. Now, since $g = u^{\frac{p-2}{2}} h$, we see that

$$\begin{aligned}
\|q_{n,2}[fh,h]fg - g\|_2 &= \|q_{n,2}[fh,h]fh - h\|_2 \\
&= \inf_{q \in \mathcal{P}_n} \|qfh - h\|_2 \\
&= \inf_{q \in \mathcal{P}_n} \|qfg - g\|_2 \\
&= \|q_{n,2}[fg,g]fg - g\|_2.
\end{aligned}$$

By uniqueness of OPAs and Theorem 2.3.3, the result follows.

□

Theorem 2.3.5 indicates that it's worth studying OPAs of the form $q_{n,2}[fg,g]$, where f and g are arbitrary functions in H^2 such that $fg \in H^2 \setminus \{0\}$. Accordingly, we would like to address a modified version of Open Question 1 .

Open Question 2. Let $n \in \mathbb{N}$. For which functions $f, g \in H^2$ can we guarantee that $q_{n,2}[fg,g]$ is zero-free in $\overline{\mathbb{D}}$?

2.4 Location of Zeros

Our motivation to study the zeros of optimal polynomial approximants comes from the early application of PLSI polynomials to digital filter design. One of the most important aspects to designing a recursive digital filter is ensuring filter stability; recall that a filter is stable if it corresponds to a system in which bounded input yields bounded output. In 2D signal processing, a stable filter can be obtained by a rational function of two complex variables such that the denominator is zero-free in $\overline{\mathbb{D}^2}$. In 1972, Justice, Shanks, and Treitel [23] proposed that such a rational function can be created with the use of planar least-squares inverse polynomials. Accordingly, they made the following conjecture: for any polynomial $f(z, w) \not\equiv 0$, the associated PLSI polynomial is zero-free in $\overline{\mathbb{D}^2}$. This conjecture became known in the engineering community as *Shanks' Conjecture*.

In 1975, Genin and Kamp [18] gave a counterexample to Shanks' Conjecture. In particular, they discovered a polynomial with zeros in $\mathbb{D}^2$ such that the associated PLSI polynomial has zeros in $\mathbb{D}^2$. A few years later, they developed a method [19] to construct polynomials with which the associated PLSI polynomial is guaranteed to have zeros in $\mathbb{D}^2$. At this point, it was unclear whether it was possible to impose conditions on the polynomial $f(z, w)$ to guarantee that the associated PLSI polynomial is zero-free in $\overline{\mathbb{D}^2}$. In 1980, Delsarte, Genin, and Kamp [14] made the following conjecture: for any polynomial $f(z, w)$ that doesn't vanish in $\overline{\mathbb{D}^2}$, the associated PLSI polynomial is zero-free in $\overline{\mathbb{D}^2}$. Although there have been several attempts to prove this conjecture, it is still unresolved. This conjecture is referred to as *Weak Shanks' Conjecture*.

Since our primary motivation for this dissertation is to understand the zeros of OPAs in the context of H^p, it's natural for us to wonder if "Shanks-type" results hold in this setting. In fact, we already know that such a result holds in H^2. More specifically, we know that the OPA $q_{n,2}[f, 1]$ is zero-free in $\overline{\mathbb{D}}$ whenever $f \in H^2$ and $f(0) \neq 0$. With this in mind, we can start by asking if the OPA $q_{n,p}[f, 1]$ is zero-free in $\overline{\mathbb{D}}$ whenever $f \in H^p$ and $f(0) \neq 0$. Moreover, we can ask if it's necessary to impose the condition $f(0) \neq 0$. On a different note, if there are values of n for which the OPAs have a zero in $\mathbb{D}$, then we should ask if there's a limit to how far in $\mathbb{D}$ the zeros can lie. To begin our investigation, we establish the following proposition.

Proposition 2.4.1. *Let $1 < p < \infty$, $f \in H^p \setminus \{0\}$, and $n \in \mathbb{N}$. If there exists some $z_0 \in \mathbb{D}$ such that $q_{n,p}[f, 1](z_0) = 0$, then*

$$\sqrt{1 - \|q_{n,p}[f, 1]f - 1\|_p^p} \leq |z_0|.$$

Proof. Let $Q := q_{n,p}[f,1]$. Suppose there exists some $z_0 \in \mathbb{D}$ such that $Q(z_0) = 0$. Define the analytic functions

$$\varphi(z) := \frac{z_0 - z}{1 - \overline{z_0} z} \quad \text{and} \quad \psi(z) := \left\{ \frac{1 - |z_0|^2}{(1 - \overline{z_0} z)^2} \right\}^{\frac{1}{p}}.$$

Let $\psi C_\varphi : H^p \to H^p$ denote the weighted composition operator $h(z) \mapsto \psi(z) h\big(\varphi(z)\big)$. Since $\big|\psi C_\varphi(Qf - 1)\big|^p$ is subharmonic in $\mathbb{D}$, it follows that

$$\begin{aligned} 1 - |z_0|^2 &= \big|\psi C_\varphi(Qf - 1)(0)\big|^p \\ &\leq \frac{1}{2\pi i} \int_{\mathbb{T}} \left| Q\Big(\frac{z_0 - z}{1 - \overline{z_0} z}\Big) f\Big(\frac{z_0 - z}{1 - \overline{z_0} z}\Big) - 1 \right|^p \frac{1 - |z_0|^2}{|1 - \overline{z_0} z|^2} \frac{dz}{z} \\ &= \frac{1}{2\pi i} \int_{\mathbb{T}} |Q(w) f(w) - 1|^p \frac{dw}{w} \\ &= \|Qf - 1\|_p^p. \end{aligned}$$

Equivalently, we have that $\sqrt{1 - \|Qf - 1\|_p^p} \leq |z_0|$. $\square$

Proposition 2.4.1 shows that if the OPA has a zero in $\mathbb{D}$ and the quantity $\|q_{n,p}[f,1]f - 1\|_p$ is small, then we can expect the zero to be close to $\mathbb{T}$. Now, it's still interesting to know which functions $f \in H^p$ correspond to a quantity $\|q_{n,p}[f,1]f - 1\|_p$ that's close to 1. Of course, this quantity is equal to 1 whenever $q_{n,p}[f,1] \equiv 0$. In the case where $p = 2$, it is well-known that $q_{n,p}[f,1] \equiv 0$ if and only if $f(0) = 0$ (see, e.g., [5]). Interestingly enough, the following proposition shows that this is true for all $1 < p < \infty$.

Proposition 2.4.2. *Let $1 < p < \infty$, $f \in H^p \setminus \{0\}$, and $n \in \mathbb{N}$. The following are equivalent:*

(i) $f(0) = 0$

(ii) $q_{n,p}[f,1] \equiv 0$

(iii) $q_{n,p}[f,1](0) = 0$

Proof. Let $q_{n,p} := q_{n,p}[f,1]$. We first suppose that $f(0) = 0$. Since $|q_{n,p} f - 1|^p$ is subharmonic in $\mathbb{D}$, it follows that

$$\begin{aligned}1 &= |q_{n,p}(0)f(0) - 1|^p \\ &\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |q_{n,p}f - 1|^p dt \\ &= \|q_{n,p}f - 1\|_p^p.\end{aligned}$$

Since $\|q_{n,p}f - 1\|_p \leq 1$, we have that $\|q_{n,p}f - 1\|_p = 1$. By uniqueness of OPAs, it follows that $q_{n,p} \equiv 0$. This shows that (i) implies (ii).

To prove that (ii) implies (iii) is trivial. Now, suppose that $q_{n,p}(0) = 0$. In a similar argument as above, it follows that $q_{n,p} \equiv 0$. Therefore, Proposition 2.3.1 implies that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{f(e^{it})}dt = 0. \tag{2.11}$$

By conjugating both sides of (2.11), we see that

$$f(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{it})dt = 0.$$

This shows that (iii) implies (i). □

As a result of the previous two propositions, we are able to address the question about how far in $\mathbb{D}$ the zeros of OPAs can lie.

Theorem 2.4.3. *Let $1 < p < \infty$, $f \in H^p$, $f(0) \neq 0$, and $n \in \mathbb{N}$. Then $q_{n,p}[f,1](z)$ is zero-free for $|z| < \sqrt{1 - \|q_{0,p}[f,1]f - 1\|_p^p}$.*

Proof. Note that since $f(0) \neq 0$, it follows from Proposition 2.4.2 that $\|q_{0,p}[f,1]f - 1\|_p < 1$, i.e., $\left\{z \in \mathbb{C} \colon |z| < \sqrt{1 - \|q_{0,p}[f,1]f - 1\|_p^p}\right\}$ is nonempty. Now, suppose $q_{n,p}[f,1](z_0) = 0$ for some $z_0 \in \mathbb{D}$. As a result of Proposition 2.4.1 and the fact that $\left\{\|q_{k,p}[f,1]f - 1\|_p\right\}_{k=0}^{\infty}$ is a decreasing sequence of nonnegative numbers, we conclude that

$$\begin{aligned}\sqrt{1-\|q_{0,p}[f,1]f-1\|_p^p} &= \sqrt{1-\sup_{k\in\mathbb{N}}\|q_{k,p}[f,1]f-1\|_p^p}\\ &= \inf_{k\in\mathbb{N}}\sqrt{1-\|q_{k,p}[f,1]f-1\|_p^p}\\ &\leq \sqrt{1-\|q_{n,p}[f,1]f-1\|_p^p}\\ &\leq |z_0|.\end{aligned}$$

Therefore, $q_{n,p}[f,1](z)$ is zero-free for $|z| < \sqrt{1-\|q_{0,p}[f,1]f-1\|_p^p}$. $\square$

For a function $f \in H^p$ with $f(0) \neq 0$, Theorem 2.4.3 gives us considerable insight into the zeros of $q_{n,p}[f,1]$. In fact, it provides us with the radius of a disk in which $q_{n,p}[f,1]$ is zero-free for every $n \in \mathbb{N}$. Now, the existence of this disk is partially due to Proposition 2.4.2: Since $f(0) \neq 0$, we have that $q_{0,p}[f,1] \not\equiv 0$ and hence $\sqrt{1-\|q_{0,p}[f,1]f-1\|_p^p} > 0$. Nonetheless, the following result (which was noted in [5] for a particular class of weighted Hilbert spaces of analytic functions) demonstrates another interesting consequence of Proposition 2.4.2.

Proposition 2.4.4. *Let $1 < p < \infty$. For each $n \in \mathbb{N} \setminus \{0\}$, let*

$$M_n := \inf\{|z| \colon q_{n,p}[f,1](z) = 0 \textit{ for at least one } f \in H^p \textit{ with } f(0) \neq 0\}.$$

Then $M_1 \leq M_n$.

Proof. For each $n \in \mathbb{N} \setminus \{0\}$, define the set

$$E_n := \{|z| \colon q_{n,p}[f,1](z) = 0 \text{ for at least one } f \in H^p \text{ with } f(0) \neq 0\}.$$

If $|z_0| \in E_n$, then there exists some $f \in H^p$, with $f(0) \neq 0$, such that $q_{n,p}[f,1](z_0) = 0$. Define the function

$$h(z) := \frac{q_{n,p}[f,1](z)f(z)}{z-z_0},$$

and note that

$$\|q_{n,p}[f,1]f-1\|_p = \|(z-z_0)h-1\|_p. \tag{2.12}$$

Now, suppose there exists some $q \in \mathcal{P}_1$ such that

$$\|(z-z_0)h-1\|_p > \|qh-1\|_p.$$

From (2.12), it follows that

$$\|q_{n,p}[f,1]f-1\|_p > \|qh-1\|_p = \left\| q\frac{q_{n,p}[f,1]}{z-z_0}f-1\right\|_p,$$

a contradiction since $q\frac{q_{n,p}[f,1]}{z-z_0}$ is a polynomial of degree at most n. We conclude that $q_{1,p}[h,1](z) = z-z_0$. Now, since $f(0) \neq 0$, it follows from Proposition 2.4.2 that $q_{n,p}[f,1](0) \neq 0$. This implies that $h(0) \neq 0$. We then see that $|z_0| \in E_1$, and hence $E_n \subset E_1$. We conclude that $M_1 \leq M_n$. □

In an effort to find a lower bound for the zeros of $q_{n,p}[f,1]$, where $f \in H^p$ and $f(0) \neq 0$, Proposition 2.4.4 suggests that we should focus our attention to the case where $n = 1$. Now, by making the additional assumption that $2 \leq p < \infty$, we can gain insight into the zeros of all the OPAs by implementing the test from Theorem 2.2.5. With this in mind, it becomes our great interest to address the following question.

Open Question 3. Let $2 \leq p < \infty$, $f \in H^p$, $f(0) \neq 0$, $g := \{q_{1,p}[f,1]f-1\}^{\frac{p-2}{2}}$, and $g \not\equiv 0$. Is it true that $|\langle g, fg\rangle| > |\langle g, zfg\rangle|$?

Note that if $g \equiv 0$, then it follows from Remark 2.3.2 that $q_{1,p}[f,1]$ is zero-free in $\overline{\mathbb{D}}$. Otherwise, it follows from Theorems 2.2.5 and 2.3.3 that the OPA $q_{1,p}[f,1]$ is zero-free in $\overline{\mathbb{D}}$ if and only if $|\langle g, fg\rangle| > |\langle g, zfg\rangle|$. Therefore, if the answer to Open Question 3 is "Yes", then we could deduce from Propositions 2.4.2 and 2.4.4 that $q_{n,p}[f,1]$ is zero-free in $\overline{\mathbb{D}}$ for every $n \in \mathbb{N}$ and every $f \in H^p$ with $f(0) \neq 0$.

Although many of our discussions have been focusing on functions in H^p, there are other functions in L^p that are worthy of our attention, e.g., the modulus of any nontrivial function in H^p. In the following proposition, we use Theorem 2.2.5 to gain insight into the zeros of OPAs associated with some of these functions.

Proposition 2.4.5. *Let $2 \leq p < \infty$. If f is a real-valued nonnegative function in $L^p \setminus \{0\}$, then $q_{1,p}[f, 1]$ is zero-free in $\overline{\mathbb{D}}$.*

Proof. Let $g := \{q_{1,p}[f, 1]f - 1\}^{\frac{p-2}{2}}$. If $g \equiv 0$, then $q_{1,p}[f, 1]$ must be a non-zero constant, i.e., $q_{1,p}[f, 1]$ is zero-free in $\overline{\mathbb{D}}$. If $g \not\equiv 0$, then it follows from Theorem 2.3.3 that $q_{1,p}[f, 1] = q_{1,2}[fg, g]$. Since

$$\begin{aligned}
|\langle g, zfg\rangle| &= \left|\left\langle g, zf^{\frac{1}{2}} f^{\frac{1}{2}} g\right\rangle\right| \\
&= \left|\left\langle gf^{\frac{1}{2}}, zf^{\frac{1}{2}} g\right\rangle\right| \\
&< \left\|gf^{\frac{1}{2}}\right\|_2^2 \\
&= \left|\left\langle 1, |g|^2 f\right\rangle\right| \\
&= |\langle g, fg\rangle|,
\end{aligned}$$

we conclude from Theorem 2.2.5 that $q_{1,p}[f, 1]$ is zero-free in $\overline{\mathbb{D}}$. □

Example 2.4.6. Let $f(z) = |1 - z|^2$. From Proposition 2.4.5, it follows that $q_{1,2}[f, 1]$ is zero-free in $\overline{\mathbb{D}}$. More specifically, since $f(z) = -z + 2 - z^{-1}$ on $\mathbb{T}$, we can easily show that $q_{1,2}[f, 1](z) = \frac{1}{10}z + \frac{2}{5}$.

As of this point in the chapter, our computations of OPAs have only pertained to the case where $p = 2$. In the following section, we develop a method that allows us to compute the coefficients of OPAs for many different values of p. This method is based on the orthogonality condition that was presented in Section 2.3.

As we saw in Section 2.2, the coefficients of OPAs in L^2 can be computed by using Proposition 2.2.2. In fact, these coefficients can be expressed as the solution to a system of linear equations for which the associated matrix can easily be inverted. Now, recall that Proposition 2.3.1 is a generalization of Proposition 2.2.2 to the L^p setting. Therefore, one would hope that this result could be used to compute the coefficients of OPAs in L^p; after making restrictions on the functions and considering specific values of p, we find that we are able to solve for the coefficients numerically; in light of Proposition 2.4.4, we will focus most of our computations on the case where $n = 1$.

Before we begin the numerical computations, let's consider an example in which $n = 0$.

Example 2.5.1. Let $f(z) = 1 - z$. Then $q_{0,p}[f, 1] \equiv \frac{1}{2}$ for all integers $p > 1$.

Proof. Let $Q := \frac{1}{2}$. Note that

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\pi}^{\pi} |Qf-1|^{p-1}\operatorname{sgn}(Qf-1)e^{-ikt}\overline{f}dt \\
&\quad= \frac{1}{2\pi}\int_{-\pi}^{\pi} |Qf-1|^{p-2}(Qf-1)\overline{f}dt \\
&\quad= \frac{1}{2\pi i}\int_{\mathbb{T}} \left|-\frac{1}{2}z-\frac{1}{2}\right|^{p-2}\left(-\frac{1}{2}z-\frac{1}{2}\right)(1-\overline{z})\frac{dz}{z} \\
&\quad= \frac{1}{2\pi i}\int_{\mathbb{T}} \left(-\frac{1}{2}z-\frac{1}{2}\right)^{\frac{p-2}{2}}\left(-\frac{1}{2}\overline{z}-\frac{1}{2}\right)^{\frac{p-2}{2}}\left(-\frac{1}{2}z-\frac{1}{2}\right)(1-\overline{z})\frac{dz}{z} \\
&\quad= \frac{1}{2\pi i}\int_{\mathbb{T}} \left(-\frac{1}{2}z-\frac{1}{2}\right)^{p-1}\frac{z-1}{z^{\frac{p}{2}}}\frac{dz}{z} \\
&\quad= \left(-\frac{1}{2}\right)^{p-1}\frac{1}{2\pi i}\int_{\mathbb{T}}\left[\sum_{j=0}^{p-1}\binom{p-1}{j}z^j\right]\frac{z-1}{z^{\frac{p}{2}}}\frac{dz}{z} \\
&\quad= \left(-\frac{1}{2}\right)^{p-1}\frac{1}{2\pi i}\sum_{j=0}^{p-1}\binom{p-1}{j}\left[\int_{\mathbb{T}} z^{j-(\frac{p-2}{2})}\frac{dz}{z} - \int_{\mathbb{T}} z^{j-\frac{p}{2}}\frac{dz}{z}\right].
\end{aligned} \tag{2.13}$$

If p is odd, then each integral in (2.13) is zero for every j. In this case,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |Qf-1|^{p-1}\operatorname{sgn}(Qf-1)e^{-ikt}\overline{f}dt = 0.$$

If p is even, then the first integral in (2.13) is nonzero only when $j = \frac{p-2}{2}$. The second integral is nonzero only when $j = \frac{p}{2}$. Since

$$\binom{p-1}{\frac{p-2}{2}} = \frac{(p-1)!}{\left(\frac{p}{2}\right)!\left(\frac{p-2}{2}\right)!} = \binom{p-1}{\frac{p}{2}},$$

it follows that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |Qf-1|^{p-1}\operatorname{sgn}(Qf-1)e^{-ikt}\overline{f}dt = 0.$$

In either case, we conclude from Proposition 2.3.1 that

$$Q = q_{0,p}[f,1] \equiv \frac{1}{2}.$$

□

Let us now begin to formulate a general setting to carry out our computations; for simplicity, we will only consider OPAs of the form $q_{n,p}[f,1]$. We will see that it works to our advantage to make certain assumptions on the function f. One of our assumptions will be based on the observation that conjugating a function's Fourier coefficients does not change its norm. More specifically, if f belongs to L^p and has Fourier series $\sum_{k=-\infty}^{\infty} c_k z^k$, then the function $\tilde{f}$ with Fourier series $\sum_{k=-\infty}^{\infty} \overline{c}_k z^k$ belongs to L^p and satisfies $\|\tilde{f}\|_p = \|f\|_p$. This allows us to deduce the following result.

Proposition 2.5.2. *Let $1 < p < \infty$, $f \in L^p \setminus \{0\}$, and $n \in \mathbb{N}$. If the Fourier coefficients of f are all real, then the coefficients of $q_{n,p}[f,1]$ are all real.*

Proof. Let $q_{n,p} := q_{n,p}[f,1]$. Then we see that

$$\inf_{q\in\mathcal{P}_n} \|qf-1\|_p = \|q_{n,p}f-1\|_p = \|(q_{n,p}f-1)^{\sim}\|_p = \|\tilde{q}_{n,p}f-1\|_p.$$

By uniqueness of OPAs, we have that $q_{n,p} = \tilde{q}_{n,p}$. $\square$

In the computations that later follow, we will make the assumption that the Fourier coefficients of the function f are all real. Under this assumption, Proposition 2.5.2 will drastically simplify the equations involved in solving for the OPA coefficients. Nonetheless, the following theorem gives us a method for computing the coefficients of OPAs whenever f is a polynomial and p is even.

Theorem 2.5.3. *Let $n, N \in \mathbb{N}$ and $Q \in \mathcal{P}_n$. Suppose that f is a polynomial of degree at most N and $p \geq 2$ is even. Define the polynomials $G(z) := Q(z)f(z)-1$, $R(z) := z^{n+N}\overline{G\left(\frac{1}{\bar{z}}\right)}$, and $P(z) := z^N G(z)^{\frac{p}{2}} R(z)^{\frac{p-2}{2}} \overline{f\left(\frac{1}{\bar{z}}\right)}$. Then*

$$\frac{d^k}{dz^k}P(z)\bigg|_{z=0} = 0$$

for $\frac{p}{2}(n+N) - n \leq k \leq \frac{p}{2}(n+N)$ if and only if $Q = q_{n,p}[f,1]$.

Proof. Note that

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\pi}^{\pi} |Qf-1|^{p-1}\operatorname{sgn}(Qf-1)e^{-ikt}\overline{f}dt \\
&\qquad= \frac{1}{2\pi}\int_{-\pi}^{\pi} |Qf-1|^{p-2}(Qf-1)e^{-ikt}\overline{f}dt \\
&\qquad= \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{(Qf-1)^{\frac{p}{2}}(\overline{Qf}-1)^{\frac{p-2}{2}}\overline{f}}{z^k}\frac{dz}{z} \\
&\qquad= \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{z^N G(z)^{\frac{p}{2}} R(z)^{\frac{p-2}{2}}\overline{f\left(\frac{1}{\bar{z}}\right)}}{z^{k+\frac{p}{2}(n+N)-n}}\frac{dz}{z} \\
&\qquad= \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{P(z)}{z^{k+\frac{p}{2}(n+N)-n}}\frac{dz}{z}.
\end{aligned}$$

Therefore, it follows from Proposition 2.3.1 that

$$\frac{1}{2\pi i}\int_{\mathbb{T}} \frac{P(z)}{z^{k+\frac{p}{2}(n+N)-n}}\frac{dz}{z} = 0 \quad \text{for} \quad k = 0, \ldots, n \tag{2.14}$$

if and only if $Q = q_{n,p}[f,1]$. Since P is a polynomial, we can easily see that (2.14) is equivalent to

$$\left.\frac{d^k}{dz^k}P(z)\right|_{z=0} = 0 \quad \text{for} \quad \frac{p}{2}(n+N) - n \leq k \leq \frac{p}{2}(n+N). \tag{2.15}$$

Hence, (2.15) is true if and only if $Q = q_{n,p}[f,1]$. $\square$

Let's have a look at an example to demonstrate this method.

Example 2.5.4. Let $f(z) = 1 - z$; let z_0 be the zero of $q_{1,p}[f,1]$.

(i) $q_{1,2}[f,1](z) \approx 0.3333333333z + 0.6666666667, \quad z_0 \approx -2.00000$

(ii) $q_{1,4}[f,1](z) \approx 0.3734388420z + 0.6265611579, \quad z_0 \approx -1.67781$

(iii) $q_{1,6}[f,1](z) \approx 0.3964823122z + 0.6035176878, \quad z_0 \approx -1.52218$

(iv) $q_{1,8}[f,1](z) \approx 0.4117075962z + 0.5882924038, \quad z_0 \approx -1.42891$

(v) $q_{1,10}[f,1](z) \approx 0.4226290585z + 0.5773709415, \quad z_0 \approx -1.36614$

(vi) $q_{1,20}[f,1](z) \approx 0.4336619002z + 0.5313377953, \quad z_0 \approx -1.22524$

(vii) $q_{1,30}[f,1](z) \approx 0.2117474393z + 0.2701449218, \quad z_0 \approx -1.27578$

Proof. We only show the work for (ii). However, note the similarity between the OPA in (i) and the OPA in Example 2.2.4. Let $Q := q_{1,4}[f,1]$. Define the polynomials $G(z)$, $R(z)$, and $P(z)$ as in Theorem 2.5.3. By setting $q_{1,4}[f,1] = a_1 z + a_0$, it is easy to see that

$$G(z) = -a_1 z^2 + (a_1 - a_0)z + (a_0 - 1).$$

Moreover, we have that

$$\overline{zf\left(\frac{1}{\overline{z}}\right)} = z - 1.$$

Now, it follows from Proposition 2.5.2 that a_0 and a_1 are real. Therefore,

$$R(z) = z^2\overline{G\left(\frac{1}{\overline{z}}\right)} = z^2 G\left(\frac{1}{z}\right) = (a_0 - 1)z^2 + (a_1 - a_0)z - a_1.$$

We thus find that

$$P(z) = \big\{ - a_1 z^2 + (a_1 - a_0)z + (a_0 - 1)\big\}^2 \big\{(a_0 - 1)z^2 + (a_1 - a_0)z - a_1\big\}(z - 1).$$

To solve for a_0 and a_1, we set

$$\frac{d^k}{dz^k}P(z)\bigg|_{z=0} = 0$$

for $k = 3$ and $k = 4$. This gives us the nonlinear equations

$$\begin{aligned} &-24a_1^3 + 78a_1^2a_0 - 72a_1a_0^2 + 36a_0^3 \\ &\qquad - 36a_1^2 + 60a_1a_0 - 54a_0^2 - 12a_1 + 30a_0 - 6 = 0 \end{aligned}$$

and

$$\begin{aligned} &144a_1^3 - 288a_1^2a_0 + 312a_1a_0^2 - 96a_0^3 \\ &\qquad + 72a_1^2 - 288a_1a_0 + 120a_0^2 + 96a_1 - 48a_0 = 0. \end{aligned}$$

Through the use of numerical methods, we find that

$$\begin{cases} a_1 \approx 0.3734388420 \\ a_0 \approx 0.6265611579. \end{cases}$$

$\square$

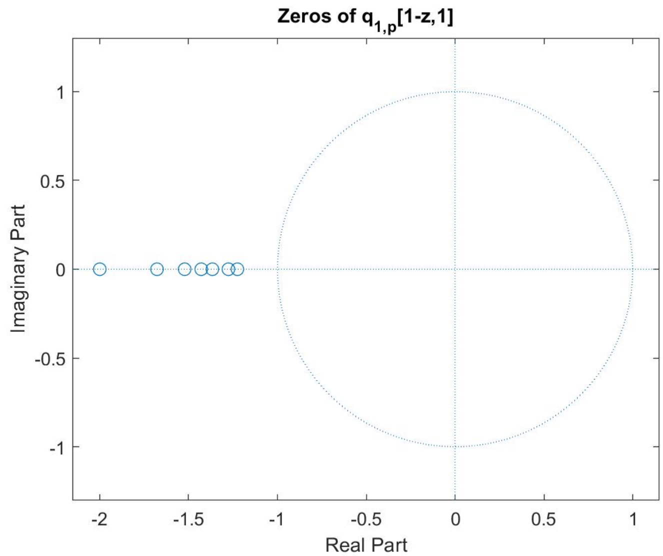

Figure 2.1: Zeros of $q_{1,p}[1-z,1]$

In the above example, it seems plausible that the zeros of $q_{1,p}[f,1]$ remain outside of $\overline{\mathbb{D}}$ for all p. Moreover, the zeros seem to converge as $p \to \infty$. As a general question, for a polynomial f with $f(0) \neq 0$, do the zeros of $q_{1,p}[f,1]$ converge as $p \to \infty$? Under these conditions, it would be interesting to determine the smallest disk that contains all the zeros. Perhaps this disk is disjoint from the closed unit disk $\overline{\mathbb{D}}$.

Chapter 3: Digital Filter Design[1]

Several problems in engineering ultimately depend on a system's response to an input. In the case of a digital system, the input is given as a sampling sequence $\{x(n)\}_{n=-\infty}^{\infty}$, and the output $\{y(n)\}_{n=-\infty}^{\infty}$ can often be described by a difference equation

$$y(n) = \sum_{k=0}^{M} b_k x(n-k) - \sum_{j=1}^{N} a_j y(n-j), \tag{3.1}$$

where the coefficients a_j and b_k are real numbers.

If the coefficients remain constant over time, the system is known as *linear time-invariant*, or LTI. If the a_j's are not all zero, then the system is referred to as *recursive*. This means that one or more of the system's output is used as an input. Now, if we consider an input sequence $\{x(n)\}_{n=-\infty}^{\infty}$ that is bounded, it seems problematic in practice for $|y(n)|$ to increase without bound as $n \to \infty$. Therefore, it is of interest to seek for properties of a system that preserves boundedness. A system in which a bounded input yields a bounded output is called *BIBO stable*.

In order to facilitate our discussion of filters, we will assume that our input sequences $\{x(n)\}_{n=-\infty}^{\infty}$ have the property that $x(n) = 0$ for $n < 0$. A sequence with this property is known as *causal*. Moreover, we will assume that the input sequences are *exponentially bounded*. That is, we assume that

$$|x(n)| \le K^n, \quad n \ge n_0$$

for some constant K and some integer n_0. Now, to better understand the relationship between the input and output sequences, we make use of the following operator. For any

[1] This chapter has been reproduced from [2] with permission from Springer Nature. Slight modifications have been made for clarity.

causal sequence $\{a(n)\}_{n=-\infty}^{\infty}$ that's exponentially bounded, consider the mapping

$$\{a(n)\}_{n=-\infty}^{\infty} \mapsto \sum_{n=0}^{\infty} a(n)z^{-n}.$$

This mapping is known as the *z-transform*. It is a linear operator from the space of exponentially bounded causal sequences onto the space of functions analytic at ∞. The z-transform of a sequence $\{a(n)\}_{n=-\infty}^{\infty}$ has a region of convergence (ROC) given by $z \in \mathbb{C} \cup \{\infty\}$ such that

$$\limsup_{n\to\infty} \sqrt[n]{|a(n)|} < |z|. \tag{3.2}$$

The sum and product of two transformed sequences are defined to be in the intersection of both ROCs, and the product is given by the expression

$$\left(\sum_{n=0}^{\infty} a(n)z^{-n}\right)\left(\sum_{n=0}^{\infty} b(n)z^{-n}\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a(k)b(n-k)\right)z^{-n}.$$

The z-transform has many properties that make it useful in the analysis of digital systems. In particular, if $A(z)$ is the z-transform of the sequence $\{a(n)\}_{n=-\infty}^{\infty}$, then the z-transform of the sequence $\{a(n-N)\}_{n=-\infty}^{\infty}$ is $z^{-N}A(z)$ for any $N \in \mathbb{N}$. Therefore, by applying the z-transform to both sides of (3.1), we see that the z-transforms of the input and output are related by the equation

$$Y(z) = H(z)X(z),$$

where $H(z)$ is given by the rational function

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{j=1}^{N} a_j z^{-j}}. \tag{3.3}$$

We use $X(z)$ and $Y(z)$ to represent the the z-transforms of the input and output sequences, respectively. The rational function H is known as the transfer function, or *filter*, of the system. In the case of a recursive system, the transfer function is commonly called an *infinite impulse response filter*, or IIR filter.

For the purpose of our discussion, we will only be considering recursive LTI systems. For simplicity, we refer to a BIBO stable system as *stable*. Likewise, we refer to the filter of a BIBO stable system as stable. The goal of this chapter is to demonstrate how optimal polynomial approximants are used in designing a stable filter. It's worth noting that although we will be considering filters of a single variable, several applications are concerned with filters of multiple variables. As mentioned in [20], the processing of medical pictures, satellite photographs, seismic data mappings, gravity waves data, magnetic recordings, and radar and sonar maps are examples in which 2D signal processing is needed. Here, one is concerned with designing filters $H(z,w)$ of two complex variables that correspond to BIBO stable systems of doubly indexed input and output sequences.

3.1 Stability

An important part of designing a digital filter is ensuring that the filter is stable. The stability of the filter will prevent the magnitude of the output from increasing without bound, which could be damaging to the physical system. Therefore, we seek for properties of the expression in (3.3) that guarantee stability. If we define the polynomials $A(z) = 1 + \sum_{j=1}^{N} a_j z^j$ and $B(z) = \sum_{k=0}^{M} b_k z^k$, we see that

$$H(z) = \frac{z^{N-M} B^*(z)}{A^*(z)}, \tag{3.4}$$

where $A^*(z)$ and $B^*(z)$ are the reverse polynomials of $A(z)$ and $B(z)$, respectively. Recall that if q is a polynomial of degree n, then the *reverse polynomial* of q is defined as $q^*(z) := z^n \overline{q\left(\frac{1}{\bar{z}}\right)}$. From (3.4), it is easy to check that $H(z)$ is a rational function analytic at ∞. Therefore, $H(z)$ is the z-transform of some causal sequence $\{h(n)\}_{n=-\infty}^{\infty}$, and we write

$$H(z) = \sum_{n=0}^{\infty} h(n) z^{-n} \tag{3.5}$$

for some specified ROC.

Now if we assume that the poles of $H(z)$ are contained in the disk $\mathbb{D}$, the expression in (3.5) must be vaild on $\mathbb{T}$. Since the series defined by $\sum_{n=0}^{\infty} |h(n)|z^{-n}$ has the same ROC, it follows that

$$\sum_{n=0}^{\infty} |h(n)| < \infty. \tag{3.6}$$

This observation leads us to the following theorem (see [26, Section 3.7.5] for details about stability and causality).

Theorem 3.1.1. *A filter $H(z)$ is stable if its poles are contained in the disk $\mathbb{D}$.*

Proof. Let $\{x(n)\}_{n=-\infty}^{\infty}$ be an input sequence with $|x(n)| \leq M$ for all n. If the poles of $H(z)$ are contained in $\mathbb{D}$, then (3.6) holds. Consequently, the output sequence $\{y(n)\}_{n=-\infty}^{\infty}$ is bounded with

$$\begin{aligned} |y(n)| &= \left| \sum_{k=0}^{n} h(k)x(n-k) \right| \\ &\leq M \sum_{n=0}^{\infty} |h(n)| \end{aligned}$$

for all n. By definition, $H(z)$ must be stable. $\square$

This theorem gives a sufficient condition for a filter to be stable. However, it's important to note that a filter with a pole outside of $\mathbb{D}$ need not be stable. As an example, consider the function

$$H(z) = \frac{1}{z-1}.$$

For all $|z| > 1$, this function is represented by the series

$$H(z) = \sum_{n=1}^{\infty} z^{-n}.$$

If we consider the input defined by $x(n) = 1$ for $n \geq 0$, the magnitude of the output is given by

$$|y(n)| = \left| \sum_{k=0}^{n} h(k)x(n-k) \right| = n,$$

which clearly increases without bound as $n \to \infty$.

3.2 Frequency Response

Many problems are concerned with how a system responds to a sinusoidal input. This is particularly evident in audio equalizing, where the input function represents a superposition of multiple sound waves. Under the assumption that the system is linear, it is therefore advantageous to study the response of a system to the input $x(n) = e^{ins}$, where $s \in \mathbb{R}$ is a particular frequency. In this case, the output is known as the *frequency response* of the system. If we let $H(z)$ be the corresponding filter, then the frequency response is expressed as

$$\begin{aligned} y(n) = \sum_{k=0}^{n} h(k)e^{i(n-k)s} &= \sum_{k=0}^{\infty} h(k)e^{i(n-k)s} \\ &= \left[\sum_{k=0}^{\infty} h(k)e^{-iks} \right] e^{ins} \\ &= H(e^{is})e^{ins}. \end{aligned}$$

We then see that the frequency response is bounded, with

$$|y(n)| = |H(e^{is})|, \;\; n \geq 0.$$

The quantity $|H(e^{is})|$ is referred to as the *magnitude of the frequency response.* To get an idea of what this function looks like, consider the filter

$$H(z) = \frac{0.3(z^2 + 2z + 1)}{1.3z^2 + 1}.$$

The poles and zeros of $H(z)$ are displayed in the following diagram:

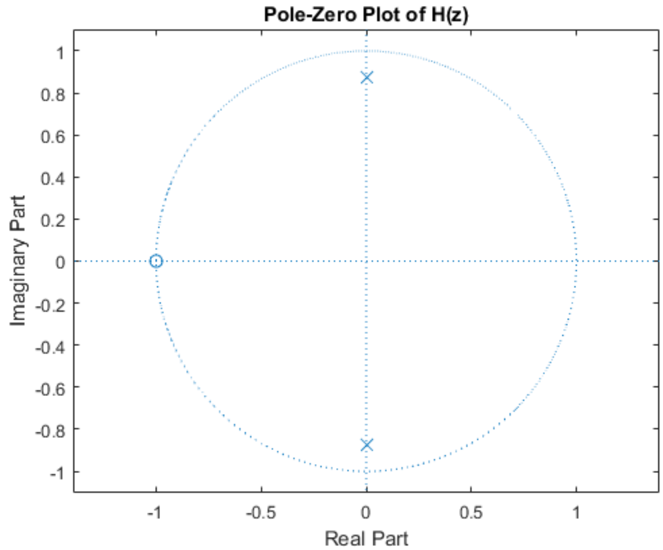

Figure 3.1: Pole-Zero Plot of $H(z)$

The poles of $H(z)$ are marked with a cross and the zero of $H(z)$ is marked with a circle. On the interval $[0, \pi]$, we therefore expect $|H(e^{is})|$ to have a maximum around 1.5 radians and a minimum around 3.1 radians. This can be seen in the following graph:

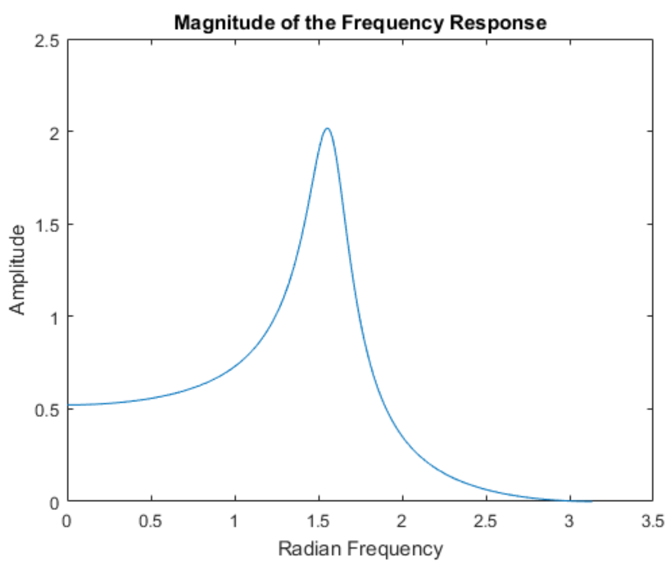

Figure 3.2: Magnitude of the Frequency Response

Often in the design of a digital filter, the goal is to develop a rational function $H(z)$ of which the modulus satisfies a set of specifications on the boundary $\mathbb{T}$. The effect of this would control the response of the system to the input $x(n) = e^{ins}$. For the purpose of our discussion, we will assume that the specifications are given in the form of a non-negative even step function on $[-\pi, \pi]$. Such a step function is known as an *ideal digital filter*. The problem of digital filter design can then be stated as follows:

Problem 3.2.1. For a given ideal digital filter $\chi(e^{is})$, find a rational function $H(z)$ with poles inside of $\mathbb{D}$ such that $|H(e^{is})|$ approximates (in some sense) the filter $\chi(e^{is})$.

Methods of approximation that guarantee stability of the filter is an interesting topic of research. We present (slightly differently) the method that Chui and Chan presented in [13]. This method creates a rational function $p(z)/q(z)$, with poles in $\mathbb{D}$, such that $|p(e^{is})|/|q(e^{is})|$ approximates $\chi(e^{is})$ in the *least-squares sense*.

Definition 3.2.2 (Least-squares sense)**.** Let $f \in L^2$ and $\eta > 0$. The quotient g/h of two functions in L^2 approximates f within η in the least-squares sense if

$$\|hf - g\|_2 < \eta.$$

In this case, we call g/h an (LS)-approximant of f and write $f \approx_{LS} g/h$.

We will present the method of approximation in three stages. In the first stage, we will approximate the ideal filter by the magnitude of a non-vanishing function in H^2. In the second stage, we will use optimal polynomial approximants to approximate the magnitude of this non-vanishing function with the magnitude of a rational function. In the third stage, we will alter the numerator and denominator of the rational function in order to ensure stability.

3.3 First Stage of Approximation

We start the first stage by defining a continuous function $\chi_\varepsilon(e^{is})$ in the following way. Let $\mathcal{S} = \{s_j\}_{j=1}^N$ denote the points of discontinuity of $\chi(e^{is})$. For each $s_j \in \mathcal{S}$, let $I_j := (s_j - \varepsilon/2, s_j + \varepsilon/2)$. Here, ε is a positive number chosen so that the intervals do not overlap and such that ε is smaller than the minimum of the step values. If $s \notin \cup_{j=1}^N I_j$, set

$$\chi_\varepsilon(e^{is}) := \begin{cases} \chi(e^{is}) & \text{if } \chi(e^{is}) > 0 \\ \varepsilon & \text{if } \chi(e^{is}) = 0. \end{cases}$$

This creates a positive step function on $[-\pi, \pi] \setminus \cup_{j=1}^N I_j$. Then connect each successive step with a straight line segment. For each $s \in \cup_{j=1}^N I_j$, set $\chi_\varepsilon(e^{is})$ to coincide with these segments. This creates a non-vanishing continuous function on $[-\pi, \pi]$.

We then create an analytic function on $\mathbb{D}$ by using $\chi_\varepsilon(e^{is})$. For any $z \in \mathbb{D}$, define the function

$$f_\varepsilon(z) := \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{is}+z}{e^{is}-z} \log \chi_\varepsilon(e^{is}) ds\right).$$

Note that $f_\varepsilon(z)$ is analytic in $\mathbb{D}$, non-vanishing in $\overline{\mathbb{D}}$, and has the property that

$$\log|f_\varepsilon(z)| = \frac{1}{2\pi}\int_{-\pi}^{\pi} \operatorname{Re}\left(\frac{e^{is}+z}{e^{is}-z}\right) \log \chi_\varepsilon(e^{is}) ds.$$

i.e., $\log|f_\varepsilon(z)|$ solves the Dirichlet problem in $\mathbb{D}$ with boundary values defined by $\log \chi_\varepsilon(e^{is})$. Therefore, the analytic function satisfies

$$|f_\varepsilon(e^{is})| = \chi_\varepsilon(e^{is}) \tag{3.7}$$

for all $s \in [-\pi, \pi]$. This leads us to the following theorem.

Theorem 3.3.1. *Any ideal digital filter χ can be approximated in the least-squares sense by the magnitude of a non-vanishing function in H^2.*

Proof. Let $\chi(e^{is})$ be an ideal filter with a collection of discontinuities $\mathcal{S} = \{s_j\}_{j=1}^N$. Given any $\eta > 0$, choose ε to satisfy

$$0 < \varepsilon < \min\left\{\frac{\eta^2\pi}{N(\|\chi\|_\infty^2 + 1)}, \frac{\eta}{\sqrt{2}}\right\}.$$

Let $\{E_k\}_k$ denote the collection of intervals for which $\chi_\varepsilon(e^{is}) = \varepsilon$. From (3.7), it follows that

$$\begin{aligned}
\|\chi - |f_\varepsilon|\|_2^2 &= \|\chi - \chi_\varepsilon\|_2^2 \\
&= \sum_{j=1}^{N} \frac{1}{2\pi}\int_{I_j} |\chi(e^{is}) - \chi_\varepsilon(e^{is})|^2 ds \\
&+ \sum_k \frac{1}{2\pi}\int_{E_k} |\chi(e^{is}) - \chi_\varepsilon(e^{is})|^2 ds \\
&\le \varepsilon\frac{N}{2\pi}\|\chi\|_\infty^2 + \varepsilon^2 \sum_k \frac{1}{2\pi}\int_{E_k} ds \\
&\le \varepsilon\frac{N}{2\pi}\|\chi\|_\infty^2 + \varepsilon^2 \\
&< \eta^2.
\end{aligned}$$

□

This theorem states that $|f_\varepsilon(z)|$ is an (LS)-approximant of $\chi(e^{is})$. Now, since $f_\varepsilon(z)$ is a function in H^2 that doesn't vanish at the origin, the nth OPA of $1/f_\varepsilon$ in L^2 is non-vanishing on $\overline{\mathbb{D}}$. This suggests that $q_{n,2}[f_\varepsilon, 1]$ (more specifically, the reverse polynomial of $q_{n,2}[f_\varepsilon, 1]$) should be a part of our rational function $H(z)$. This observation leads to the second stage of the approximation.

3.4 Second Stage of Approximation

Given an ideal filter $\chi(e^{is})$, the first stage of the approximation involved determining the function $f_\varepsilon(z)$. It then followed that $\chi(e^{is}) \approx_{LS} |f_\varepsilon(e^{is})|$. In the next stage, we approximate $|f_\varepsilon|$ with the magnitude of a rational function.

Since f_ε is an outer function in H^2 (see Section 4.3), it follows that $\|q_n f_\varepsilon - 1\|_2 \to 0$ as $n \to \infty$, where $q_n := q_{n,2}[f_\varepsilon, 1]$. Therefore, given any $\eta > 0$, we can choose N so that $\|q_N f_\varepsilon - 1\|_2 < \eta$. Moreover for any $M \geq 0$, we see that

$$\|q_N f_\varepsilon - p_M\|_2 = \inf_{p \in \mathcal{P}_M} \|q_N f_\varepsilon - p\|_2 < \eta,$$

where p_M denotes the orthogonal projection of $q_N f_\varepsilon$ onto $\mathcal{P}_M$. Hence, we have that p_M/q_N is an (LS)-approximant of f_ε. Consequently, we have that $|f_\varepsilon(e^{is})| \approx_{LS} |p_M(e^{is})|/|q_N(e^{is})|$.

It's important to note that it's computationally efficient to determine the polynomials p_M and q_N. We have already seen that the coefficients of q_N can be expressed as the solution of a system of $N+1$ linear equations, each of which are dependent only on the function f_ε. The entries of the associated matrix B can be expressed as the Fourier coefficients of the L^2 function $|f_\varepsilon|^2$, i.e.,

$$B_{jk} = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f_\varepsilon(e^{is})|^2 e^{i(k-j)s} ds.$$

Hence, they can be computed efficiently through any available FFT algorithm. Furthermore, B is a Gram matrix generated by the vectors $\{z^k f_\varepsilon\}_{k=0}^N$. Since these vectors are linearly independent, it follows that B is invertible. Moreover, since Gram matrices are positive definite, and since B is Hermitian and Toeplitz, we can use any of the fast algorithms to compute its inverse. We then see that the coefficients of q_N, say $a_0, \ldots, a_N$, are given by the expression

$$\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_N \end{bmatrix} = B^{-1} \begin{bmatrix} \overline{f_\varepsilon(0)} \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Hence, the coefficients are determined by the first column of B^{-1} scaled by $\overline{f_\varepsilon(0)}$. On the other hand, the coefficients of p_M are given by the first $M+1$ Fourier coefficients of $q_N f_\varepsilon$.

3.5 Third Stage of Approximation

In the first two stages of approximation, we were able to approximate an ideal filter $\chi(e^{is})$ with the magnitude of a rational function. More specifically,

$$\chi(e^{is}) \approx_{LS} |f_\varepsilon(e^{is})| \approx_{LS} \frac{|p_M(e^{is})|}{|q_N(e^{is})|}.$$

Since the ideal filter is assumed to be an even function on $[-\pi, \pi]$, we have that

$$\begin{aligned}\chi(e^{is}) &= \chi(e^{-is}) \\ &\approx_{LS} |f_\varepsilon(e^{-is})| \\ &\approx_{LS} \left|\frac{p_M(e^{-is})}{q_N(e^{-is})}\right| \\ &= \left|\frac{p_M^*(e^{is})}{q_N^*(e^{is})}\right|,\end{aligned}$$

where p_M^* and q_N^* are the reverse polynomials of p_M and q_N, respectively. Now, since M is arbitrary, choose $M \leq N$ and define the rational function

$$H(z) = \frac{p_M^*(z)}{q_N^*(z)}. \tag{3.8}$$

Then $H(z)$ is a rational function that's analytic at ∞. It has the property that $|H(e^{is})|$ is an approximation of $\chi(e^{is})$. Furthermore, since the zeros of q_N are outside of $\overline{\mathbb{D}}$, and since $H(z)$ is analytic at ∞, it follows that the poles of $H(z)$ are contained in the disk $\mathbb{D}$. Therefore, the expression in (3.8) gives us our stable filter.

This three stage method of approximation summarizes the main ideas presented by Chui and Chan [13] in 1982. The use of OPAs in filter design doesn't seem to have gone much further in the one variable case, although some papers later in the 1980s and 1990s discuss related IIR filter designs (see, e.g., [9, 10, 16]). It would be interesting to know if OPAs might have some further applications in signal processing research.

Chapter 4: An Operator Theoretic Approach

In 1980, Chui [12] introduced least-squares inverse polynomials as a way to approximate a given polynomial f with another polynomial P that's zero-free in $\overline{\mathbb{D}}$ [12]. As a viable candidate, he used a polynomial of the form $P_{n,m} := q_{n,2}[Q, 1]$, where $Q := q_{m,2}[f, 1]$ for some $m \in \mathbb{N}$. Such a polynomial $P_{n,m}$ was referred to as a double least-squares inverse. For simplicity, we'll refer to these as DLSI polynomials. In his paper, Chui made the conjecture that if f is a polynomial that's zero-free in $\mathbb{D}$ and has N zeros (counting multiplicities) on $\mathbb{T}$, then for $n = 0, \ldots, N-1$, $\|P_{n,m}\|_2 \to 0$ as $m \to \infty$. In 1985, Izumino [22] proved a generalized version of this conjecture. In doing so, he expressed the LSI and DLSI polynomials as the image of 1 under an appropriate operator. It was then possible to approach various problems relating to least-squares inverses by using the powerful tools of operator theory.

In this chapter, we extend the ideas of Izumino to the general L^2 setting. Accordingly, we begin by introducing the idea of Moore-Penrose inverse. To maintain generality, we present this notion in the context of arbitrary Hilbert spaces. Afterward, we use this idea to study optimal polynomial approximants in L^2. Since many OPAs in L^2 can be expressed as an equivalent OPA in L^p, the theory we present can be used to study the OPAs $q_{n,p}[f, 1]$. In the last section, we will see how this operator theoretic framework ties in nicely with functions in H^∞.

4.1 The Moore-Penrose Inverse[1]

Many problems in physics and engineering rely on solving an inverse problem. These problems typically involve some Hilbert spaces H and K and a bounded linear operator $A \in \mathcal{L}(H, K)$. For a given $b \in K$, the problem might be to find some $x_0 \in H$ such that

[1] The material in this section is well known and can be found in many introductory textbooks (see, e.g., [25]).

$Ax_0 = b$. If A is bijective, then such an x_0 exists and is uniquely given by $x_0 = A^{-1}b$. However, it's often the case that b is an element outside the range of A. In this case, it might suffice to find some $x_0 \in H$ that minimizes the norm $\|Ax - b\|$, where x varies over H. Such an x_0 is called a least-squares solution of $Ax = b$. Now, if A is not injective, then there may be more than one least-squares solution, if not an infinite number of them. In this case, it's often desirable to seek a least-squares solution with smallest norm.

In the discussion that follows, we develop the tools needed to find a least-squares solution of minimal norm. We begin by constructing a bounded linear operator from K to H. This operator will subsequently be called the Moore-Penrose inverse (or pseudoinverse) of A.

Proposition 4.1.1 (Moore-Penrose inverse)**.** *Let H and K be Hilbert spaces and $A \in \mathcal{L}(H, K)$ have a closed range. Define the mapping $T : K \to H$ by*

$$T(A(x + z) + y) = x,$$

where $x \in (\ker A)^{\perp}$, $z \in \ker A$, and $y \in$ (Range $A)^{\perp}$. Then $T \in \mathcal{L}(K, H)$.

Proof. It is easy to see that T is a well-defined linear operator from K to H. To see that T is bounded, suppose that $\{A(x_n + z_n) + y_n\} \to y$ as $n \to \infty$, where $\{x_n\}_{n=1}^{\infty} \subset (\ker A)^{\perp}$, $\{z_n\}_{n=1}^{\infty} \subset \ker A$, and $\{y_n\}_{n=1}^{\infty} \subset$ (Range $A)^{\perp}$. Moreover, suppose that $x_n \to x$ for some $x \in H$. Since $(\ker A)^{\perp}$ is closed, we have that $x \in (\ker)^{\perp}$. Now, let $y = y' + y''$, where $y' \in$ Range A and $y'' \in$ (Range $A)^{\perp}$. Note that

$$\begin{aligned} 0 \le \|y_n - y''\|^2 &\le \|A(x_n + z_n) - y'\|^2 + \|y_n - y''\|^2 \\ &= \|(A(x_n + z_n) - y') + (y_n - y'')\|^2 \\ &= \|(A(x_n + z_n) + y_n) - y\|^2. \end{aligned} \tag{4.1}$$

Since the right-hand side of (4.1) tends to zero as $n \to \infty$, we have that $y_n \to y''$ as $n \to \infty$.

By the boundedness of A, we see that

$$
\begin{aligned}
y &= \lim_{n\to\infty} (A(x_n + z_n) + y_n) \\
&= \lim_{n\to\infty} (Ax_n + y_n) \\
&= Ax + y''.
\end{aligned}
$$

Hence, $Ty = x$. By the Closed Graph Theorem, it follows that T is bounded. Therefore, $T \in \mathcal{L}(K, H)$. □

By how the operator T is defined, we will see that it has a special relationship with the operator A. More specifically, we will see that the operators A and T satisfy some of the same identities that an invertible operator and its inverse satisfy. We summarize these "Penrose identities" in the following proposition.

Proposition 4.1.2 (Penrose identities)**.** *Let H and K be Hilbert spaces, $A \in \mathcal{L}(H, K)$ have a closed range, and $T \in \mathcal{L}(K, H)$ be the operator from Proposition 4.1.1. Then*

(i) TA is Hermitian.

(ii) AT is Hermitian.

(iii) $TAT = T$.

(iv) $ATA = A$.

Proof. (i) For $x_1, x_2 \in (\ker A)^\perp$ and $z_1, z_2 \in \ker A$, we have

$$
\begin{aligned}
\langle TA(x_1 + z_1), x_2 + z_2\rangle &= \langle x_1, x_2 + z_2\rangle \\
&= \langle x_1, x_2\rangle \\
&= \langle x_1 + z_1, x_2\rangle \\
&= \langle x_1 + z_1, TA(x_2 + z_2)\rangle.
\end{aligned}
$$

(ii) For $x_1, x_2 \in (\ker A)^\perp$, $z_1, z_2 \in \ker A$, and $y_1, y_2 \in (\text{Range } A)^\perp$, we have

$$\begin{aligned}
\langle AT(A(x_1+z_1)+y_1), A(x_2+z_2)+y_2\rangle &= \langle Ax_1, Ax_2+y_2\rangle \\
&= \langle Ax_1, Ax_2\rangle \\
&= \langle A^*Ax_1, x_2\rangle \\
&= \langle A^*Ax_1, T(A(x_2+z_2)+y_2)\rangle \\
&= \langle Ax_1, AT(A(x_2+z_2)+y_2)\rangle \\
&= \langle A(x_1+z_1), AT(A(x_2+z_2)+y_2)\rangle \\
&= \langle A(x_1+z_1)+y_1, AT(A(x_2+z_2)+y_2)\rangle.
\end{aligned}$$

Hence, $(AT)^* = AT$.

(iii) For $x \in (\ker A)^\perp$, $z \in \ker A$, and $y \in (\text{Range } A)^\perp$, we have

$$\begin{aligned}
TAT(A(x+z)+y) &= TAx \\
&= TA(x+z) \\
&= T(A(x+z)+y).
\end{aligned}$$

Therefore, $TAT = T$.

(iv) For $x \in (\ker A)^\perp$ and $z \in \ker A$, we have

$$ATA(x+z) = Ax = A(x+z).$$

Thus, $ATA = A$.

$\square$

Proposition 4.1.2 shows that the operator T acts in a way that resembles an inverse of A. Therefore, it's reasonable to question if T could be used to find a least-squares solution of

$Ax = b$. Of course, the same could be asked about any operator $B \in \mathcal{L}(K, H)$ that satisfies these Penrose identities. However, in the case when such an operator B exists, the following proposition shows that it must be equal to T.

Proposition 4.1.3 (Uniqueness)**.** *Let H and K be Hilbert spaces and $A \in \mathcal{L}(H, K)$ have a closed range. There is only one operator $B \in \mathcal{L}(K, H)$ that satisfies the Penrose identities from Proposition 4.1.2.*

Proof. Let $B_1, B_2 \in \mathcal{L}(K, H)$ satisfy the Moore-Penrose identities from Proposition 4.1.2. Then

$$\begin{aligned}
B_1 &= B_1AB_1 \\
&= B_1(AB_1)^* \\
&= B_1B_1^*A^* \\
&= B_1B_1^*(AB_2A)^* \\
&= B_1B_1^*A^*(AB_2)^* \\
&= B_1(AB_1)^*(AB_2)^* \\
&= B_1AB_1AB_2 \\
&= B_1AB_2 \\
&= (B_1A)^*B_2AB_2 \\
&= A^*B_1^*A^*B_2^*B_2 \\
&= A^*B_2^*B_2 \\
&= (B_2A)^*B_2 \\
&= B_2AB_2 \\
&= B_2.
\end{aligned}$$

□

Remark 4.1.4. Let H and K be Hilbert spaces, $A \in \mathcal{L}(H, K)$ have a closed range, and T be the operator from Proposition 4.1.1. If A is bijective, then A^{-1} satisfies the Penrose identities from Proposition 4.1.2. It then follows from Proposition 4.1.3 that $A^{-1} = T$.

The operator T from Proposition 4.1.1 is an example of a "generalized inverse". Examples of generalized inverses include the Bott-Duffin inverse, the Drazin inverse, and the Moore-Penrose inverse (see [25] for details). In our case, T is an example of a Moore-Penrose inverse.

Definition 4.1.5 (Moore-Penrose inverse)**.** Let H and K be Hilbert spaces and $A \in \mathcal{L}(H, K)$ have a closed range. The unique operator $T \in \mathcal{L}(K, H)$ that satisfies the Penrose identities from Proposition 4.1.2 is called the Moore-Penrose inverse of A and is denoted by $A^\dagger$.

In the case where A is bijective, we see from Remark 4.1.4 that $A^\dagger A$ and $AA^\dagger$ represent the identity operators on H and K, respectively. On the other hand, if A is not bijective, then it turns out that these operators still have desirable properties. More specifically, $A^\dagger A : H \to H$ represents the orthogonal projection of H onto Range $A^\dagger$ and $AA^\dagger : K \to K$ represents the orthogonal projection of K onto Range A. We prove this for $AA^\dagger$ in the following proposition. The proof for $A^\dagger A$ is similar.

Proposition 4.1.6. *Let H and K be Hilbert spaces and $A \in \mathcal{L}(H, K)$ have a closed range. Then the operator $AA^\dagger : K \to K$ is the orthogonal projection onto Range A.*

Proof. Let $P := AA^\dagger$. Note that

$$\begin{aligned} P^2 &= (AA^\dagger)(AA^\dagger) \\ &= A(A^\dagger AA^\dagger) \\ &= AA^\dagger \\ &= P. \end{aligned}$$

Furthermore, let $w_1 := A(x_1 + z_1) + y_1$ and $w_2 := A(x_2 + z_2) + y_2$, where $x_1, x_2 \in (\ker A)^\perp$,

$z_1, z_2 \in \ker A$, and $y_1, y_2 \in (\text{Range } A)^\perp$. Then,

$$\begin{aligned}\langle Pw_1, w_2\rangle &= \langle Ax_1, Ax_2 + y_2\rangle \\ &= \langle A(x_1 + z_1), Ax_2\rangle \\ &= \langle A(x_1 + z_1) + y_1, AA^\dagger(A(x_2 + z_2) + y_2)\rangle \\ &= \langle w_1, Pw_2\rangle,\end{aligned}$$

i.e., $P^* = P$. Since $P^2 = P = P^*$, it follows that P is the orthogonal projection of K onto Range $AA^\dagger$.

Now, it's clear that Range $(AA^\dagger) \subset$ Range A. However, it's true that equality holds. To see that this is the case, if $w \in$ Range A, then $w = Ax$ for some $x \in (\ker A)^\perp$. Consequently,

$$AA^\dagger w = AA^\dagger Ax = Ax = w.$$

Therefore, Range $AA^\dagger =$ Range A, so the result follows. □

If our goal is to find a least-squares solution of $Ax = b$ with minimal norm, we might be tempted to guess that $A^\dagger b$ will suffice. In the case when A is bijective, we see from Remark 4.1.4 that this is certainly true. To our satisfaction, the following Theorem shows that this is also the case for any closed range operator $A \in \mathcal{L}(H, K)$.

Theorem 4.1.7. *Let H and K be Hilbert spaces and $A \in \mathcal{L}(H, K)$ have a closed range. For any $b \in K$, $A^\dagger b$ is a least-squares solution of $Ax = b$. Moreover, if $x_0 \in H$ is a different least-squares solution of $Ax = b$, then $\|x_0\| > \|A^\dagger b\|$.*

Proof. For any $x \in H$, we have

$$\begin{aligned}\langle Ax - AA^\dagger b, AA^\dagger b - b\rangle &= \langle Ax, AA^\dagger b\rangle - \langle Ax, b\rangle + \langle AA^\dagger b, b\rangle - \langle AA^\dagger b, AA^\dagger b\rangle \\ &= \langle AA^\dagger Ax, b\rangle - \langle Ax, b\rangle + \langle AA^\dagger b, b\rangle - \langle AA^\dagger AA^\dagger b, b\rangle \\ &= \langle Ax, b\rangle - \langle Ax, b\rangle + \langle AA^\dagger b, b\rangle - \langle AA^\dagger b, b\rangle = 0.\end{aligned}$$

Therefore,

$$\|Ax - b\|^2 = \|Ax - AA^\dagger b + AA^\dagger b - b\|^2 = \|Ax - AA^\dagger b\|^2 + \|AA^\dagger b - b\|^2$$
$$\geq \|AA^\dagger b - b\|^2.$$

Consequently,

$$\|AA^\dagger b - b\| = \inf_{x \in H} \|Ax - b\|,$$

i.e., $A^\dagger b$ is a least-squares solution of $Ax = b$.

Since A has a closed range, there exists a unique $y \in$ Range A such that

$$\inf_{x \in H} \|Ax - b\| = \|y - b\|. \tag{4.2}$$

Now, suppose that x_0 is a least-squares solution of $Ax = b$ different from $A^\dagger b$. From (4.2), we see that

$$Ax_0 = y = AA^\dagger b.$$

This means that $A(x_0 - A^\dagger b) = 0$. Therefore, $x_0 = z + A^\dagger b$ for some $z \in \ker A$. Since

$$\langle z, A^\dagger b\rangle = \langle z, A^\dagger AA^\dagger b\rangle = \langle A^\dagger Az, A^\dagger b\rangle = 0,$$

it follows that

$$\|x_0\|^2 = \|z\|^2 + \|A^\dagger b\|^2 > \|A^\dagger b\|^2.$$

□

Theorem 4.1.7 shows that the Moore-Penrose inverse can be used to solve certain extremal problems involving a closed range operator A. In the following section, we will see how the extremal problem of determining an OPA can be formulated to fit this operator theoretic framework.

One of the most useful aspects of the Moore-Penrose inverse is that it gives us an explicit way to represent a least-squares solution. As a result, one could gain insight into the solution by studying the properties of this operator. Now, since the problem of determining the OPA $q_{n,2}[f,g]$ involves minimizing the distance from g to some finite-dimensional space, one might ask if it's possible to express the OPA as the least-squares solution of some operator equation $Ax = g$. It turns out that this is possible when we impose some mild conditions on the function f. This was demonstrated in 1985 by Izumino ([22]) in reference to functions $f \in H^\infty$ and $g \in H^2$. In this section, we extend those ideas to functions $f \in L^\infty$ and $g \in L^2$.

If we assume that $f \in L^\infty$, then its easy to see that

$$\inf_{q \in \mathcal{P}_n} \|qf - g\|_2 = \inf_{h \in H^2} \|T_f E_n h - g\|_2,$$

where $T_f : L^2 \to L^2$ denotes the Toeplitz operator $h \mapsto fh$ and $E_n : L^2 \to \mathcal{P}_n$ denotes the orthogonal projection of L^2 onto $\mathcal{P}_n$. Therefore, we are tempted to define the operator A to be the product $T_f E_n$ and express the OPA in terms of $A^\dagger$. However, before we can continue in this direction, we must convince ourselves that the operator $T_f E_n$ has a closed range.

Proposition 4.2.1. *For any $f \in L^\infty$ and $n \in \mathbb{N}$, the operator $T_f E_n$ has finite rank.*

Proof. Suppose f has Fourier series $\sum_{k=-\infty}^{\infty} c_k z^k$. Let $M = [M_{ij}]$ denote the matrix of $T_f E_n$ with respect to the basis $\{z^k\}_{k=-\infty}^{\infty}$ of L^2. Then

$$M_{ij} = \left\langle T_f E_n(z^j), z^i \right\rangle$$

for $i, j \in \mathbb{Z}$. In particular, $M_{ij} = 0$ whenever $|j| > n$. On the other hand, for $|j| \le n$,

$$M_{ij} = \left\langle \sum_{k=-\infty}^{\infty} c_k z^{k+j}, z^i \right\rangle$$

$$= \sum_{k=-\infty}^{\infty} c_k \langle z^{k+j}, z^i \rangle$$

$$= c_{i-j}.$$

Thus, we see that

$$M = \begin{bmatrix} & & \vdots & & & \vdots & & & \vdots & & \\ \dots & 0 & 0 & c_{n-1} & \dots & c_0 & c_{-1} & c_{-2} & \dots & c_{-n-1} & 0 & 0 & \dots \\ \dots & 0 & 0 & c_n & \dots & c_1 & \boxed{c_0} & c_{-1} & \dots & c_{-n} & 0 & 0 & \dots \\ \dots & 0 & 0 & c_{n+1} & \dots & c_2 & c_1 & c_0 & \dots & c_{-n+1} & 0 & 0 & \dots \\ & & \vdots & & & \vdots & & & \vdots & & \end{bmatrix}.$$

Therefore, $T_f E_n$ has finite rank. □

Since the operator $T_f E_n$ has finite rank, its range must be closed. Therefore, it immediately follows that $(T_f E_n)^\dagger(g)$ is a least-squares solution of $T_f E_n h = g$. In the following theorem, we will see how this solution is related to the OPA $q_{n,2}[f, g]$.

Theorem 4.2.2. *Let $n \in \mathbb{N}$ and $f \in L^\infty \setminus \{0\}$. Then Range$(T_f E_n)^\dagger \subset \mathcal{P}_n$. Furthermore, for any $g \in L^2$,*

$$q_{n,2}[f, g] = (T_f E_n)^\dagger(g).$$

Proof. Let $Q := q_{n,2}[f, g]$ and $k := (T_f E_n)^\dagger(g)$. Note that

$$\begin{aligned} \|Qf - g\|_2 &= \inf_{q \in \mathcal{P}_n} \|qf - g\|_2 \\ &= \inf_{h \in L^2} \|T_f E_n h - g\|_2 \qquad (4.3) \\ &= \|T_f E_n k - g\|_2 \\ &= \|f E_n k - g\|_2. \end{aligned}$$

By uniqueness of OPAs, we see that $Q = E_n k$, and hence

$$\|Q\|_2 \leq \|k\|_2. \tag{4.4}$$

Now, it follows from (4.3) that

$$\|T_f E_n Q - g\|_2 = \|Qf - g\|_2 = \inf_{h \in L^2} \|T_f E_n h - g\|_2,$$

i.e., Q is a least-squares solution of $T_f E_n h = g$. By (4.4) and Theorem 4.1.7, we see that $Q = (T_f E_n)^\dagger(g)$. $\square$

Theorem 4.2.2 allows us to gain insight into the OPA $q_{n,2}[f, g]$ by studying the operator $(T_f E_n)^\dagger$. As it turns out, several properties of OPAs can be easily deduced in this way. In the following proposition, we use the fact that $AA^\dagger$ is an orthogonal projection to gain insight into the norm of $q_{n,2}[f, g]$.

Proposition 4.2.3. *Let $n \in \mathbb{N}$, $f \in L^\infty \setminus \{0\}$, and $g \in L^2$. Then*

$$\|f q_{n,2}[f, g]\|_2 \leq \|g\|_2.$$

Proof. It is easy to see that $f q_{n,2}[f, g] = (T_f E_n)(T_f E_n)^\dagger g$. Since $(T_f E_n)(T_f E_n)^\dagger$ is the orthogonal projection of L^2 onto $f\mathcal{P}_n$, it follows that

$$\|f q_{n,2}[f, g]\|_2 = \|(T_f E_n)(T_f E_n)^\dagger g\|_2 \leq \|g\|_2.$$

$\square$

Despite the fact that Proposition 4.2.3 could have been easily be deduced from Proposition 2.2.2, there seems to be a lot of utility to the Moore-Penrose framework. In the following section, we will see some examples in the context of the Hardy space H^∞.

In 1980, Chui [12] formulated the problem of Robinson in the H^2 setting in an effort to approximate the denominator of a digital filter. However, since digital filters are represented by rational functions, his approximation method was only in reference to polynomials. As we mentioned, the denominator h was approximated by a DLSI polynomial of the form $q_{n,2}[Q,1]$, where $Q = q_{m,2}[h,1]$. A few years later, Chui and Chan [13] reformulated the problem to include arbitrary functions in H^2. The reason was to approximate the analytic function f_ε that was defined using the continuous function χ_ε (see Section 3.3 for details). Now, recall that these functions had a representation that gave them desirable properties, such as $|f_\varepsilon(e^{it})| = \chi_\varepsilon(e^{it})$ for all $t \in [-\pi, \pi]$. The representation was an example of the more generalized notion of outer function. An *outer function* is a function f that can be written in the form

$$f(z) = c \exp\left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} \log \varphi(e^{it}) dt \right\},$$

where φ is a positive Lebesgue measurable function on $\mathbb{T}$ such that $\log \varphi \in L^1$ and c is a constant with $|c| = 1$. One of the most useful properties of outer functions is that they are cyclic. Recall that a function $f \in H^2$ is called *cyclic* if $f\mathcal{P}$ is dense in H^2. Here, $\mathcal{P}$ denotes the space of all polynomials. In the following proposition, we will see how the cyclicity of f contributes to the convergence of $\{(T_f E_n)^\dagger\}_{n=0}^\infty$.

Proposition 4.3.1 (Izumino [22]). *Let f be an outer function in H^∞. Then $T_f(T_f E_n)^\dagger : H^2 \to H^2$ converges strongly to the identity $I : H^2 \to H^2$ as $n \to \infty$.*

Proof. Since $f \in H^\infty$ is an outer function, it follows that there exists a sequence of polynomials $\{\varphi_k\}_{k=0}^\infty$ such that $\deg \varphi_k = k$ and $\{\varphi_k f\}_{k=0}^\infty$ constitutes an orthonormal basis of H^2. Thus, given any $h \in H^2$ and $\varepsilon > 0$, there exists some N such that

$$\left\| \sum_{k=0}^{m} \langle h, \varphi_k f \rangle \varphi_k f - h \right\|_2 < \varepsilon$$

whenever $n \geq N$. Consequently,

$$\begin{aligned}
\|(T_f(T_fE_n)^\dagger - I)h\|_2 &= \|T_f(T_fE_n)^\dagger h - h\|_2 \\
&= \|T_fE_n(T_fE_n)^\dagger(h) - h\|_2 \\
&\leq \left\| T_fE_n\left(\sum_{k=0}^{n}\langle h, \varphi_k f\rangle \varphi_k\right) - h\right\|_2 \\
&= \left\| \sum_{k=0}^{n}\langle h, \varphi_k f\rangle \varphi f - h\right\|_2 \\
&< \varepsilon
\end{aligned}$$

whenever $n \geq N$. Therefore, $T_f(T_fE_n)^\dagger$ converges strongly to I as $n \to \infty$. $\square$

There are several properties of $q_{n,2}[f, g]$ that are desirable under the assumption that the function $f \in H^\infty$ is outer. For the functions f that are not outer, it turns out that some of these properties are still preserved. The reason for this is because any function in H^∞ is the product of an outer function with an inner function. An *inner function* is a function $f \in H^\infty$ such that $|f(z)| = 1$ a.e. on $\mathbb{T}$. The fact that every nontrivial function $f \in H^\infty$ admits and inner-outer factorization is well-known and can be found in many introductory textbooks. In fact, this result holds for all functions $f \in H^p \setminus \{0\}$, $1 \leq p \leq \infty$ (see, e.g., [15, 17, 21, 28]).

As we see from Proposition 4.3.1, outer functions $F \in H^\infty$ yield desirable convergence properties for the operators $\{(T_FE_n)^\dagger\}_{n=0}^\infty$. In fact, the result that $\|q_{n,2}[F, g]F - g\|_2 \to 0$ as $n \to \infty$ is easily deduced. For the functions $f \in H^\infty$ that are not outer, the inner-outer factorization gives us a way to relate f with such an outer function. Therefore, one might question if this can be used to gain insight into the convergence of $\{(T_fE_n)^\dagger\}_{n=0}^\infty$. Or more generally, one might question if it's possible to relate the OPAs $q_{n,2}[F, 1]$ and $q_{n,2}[f, 1]$, where F is the outer function corresponding to the inner-outer factorization of f. In the next two lemmas, we present some results that allow us to address this general question.

Lemma 4.3.2. *Let $f \in H^\infty \setminus \{0\}$ have factorization $f = uF$, where u is inner and F is outer. Then*

$$(T_f E_n)^\dagger = (T_F E_n)^\dagger T_u^*.$$

Proof. For any $h \in H^2$, note that

$$\|T_u h\|_2 = \|uh\|_2 = \|h\|_2.$$

Consequently, $T_u^* T_u = I$. Now, let $A := T_f E_n$ and $B := (T_F E_n)^\dagger T_u^*$. Then

$$\begin{aligned}
(AB)^* &= \{T_f E_n (T_F E_n)^\dagger T_u^*\}^* \\
&= \{T_u T_F E_n (T_F E_n)^\dagger T_u^*\}^* \\
&= T_u \{(T_F E_n)(T_F E_n)^\dagger\}^* T_u^* \\
&= T_u (T_F E_n)(T_F E_n)^\dagger T_u^* \\
&= T_f E_n (T_F E_n)^\dagger T_u^* \\
&= AB. \qquad (4.5)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
(BA)^* &= \{(T_F E_n)^\dagger T_u^* T_f E_n\}^* \\
&= \{(T_F E_n)^\dagger T_u * T_u T_F E_n\}^* \\
&= \{(T_F E_n)^\dagger T_F E_n\}^* \\
&= (T_F E_n)^\dagger T_F E_n \\
&= (T_F E_n)^\dagger T_u^* T_u T_F E_n \\
&= (T_F E_n)^\dagger T_u^* T_f E_n \\
&= BA. \qquad (4.6)
\end{aligned}$$

Furthermore, note that

$$
\begin{aligned}
ABA &= T_f E_n (T_F E_n)^\dagger T_u^* T_f E_n \\
&= T_f E_n (T_F E_n)^\dagger T_u^* T_u T_F E_n \\
&= T_f E_n (T_F E_n)^\dagger T_F E_n \\
&= T_f E_n \\
&= A. \qquad (4.7)
\end{aligned}
$$

Lastly,

$$
\begin{aligned}
BAB &= (T_F E_n)^\dagger T_u^* T_f E_n (T_F E_n)^\dagger T_u^* \\
&= (T_F E_n)^\dagger T_u^* T_u T_F E_n (T_F E_n)^\dagger T_u^* \\
&= (T_F E_n)^\dagger T_F E_n (T_F E_n)^\dagger T_u^* \\
&= (T_F E_n)^\dagger T_u^* \\
&= B. \qquad (4.8)
\end{aligned}
$$

By (4.5), (4.6), (4.7), and (4.8), we see that the operators A and B satisfy the Moore-Penrose properties of Proposition 4.1.2. Therefore, $A^\dagger = B$. □

Since $(T_h E_n)^\dagger(1) = q_{n,2}[h,1]$ for any $h \in H^\infty \setminus \{0\}$, Lemma 4.3.2 suggests that the OPA $q_{n,2}[F,1]$ might be a factor of $q_{n,1}[f,1]$. Before we confirm this, let's get a better grasp on the operator T_u^*.

Lemma 4.3.3. *Let $f \in H^\infty$. Then*

$$T_f^* = P_+ M_{\overline{f}}, \qquad (4.9)$$

where $M : H^2 \to L^2$ is the multiplication operator $h \mapsto \overline{f}h$ and $P_+ : L^2 \to H^2$ is the orthogonal projection of L^2 onto H^2.

Proof. Let $g, h \in H^2$. Then

$$\langle g, P_+ M_{\overline{f}} h \rangle = \langle g, P_+(\overline{f} h) \rangle = \langle g, \overline{f} h \rangle = \langle f g, h \rangle = \langle T_f g, h \rangle.$$

Hence, $T_f^* = P_+ M_{\overline{f}}$. □

As a result of Lemma 4.3.2 and Lemma 4.3.3, we can easily show that the inner-outer factorization $f = uF$ yields a convenient relationship between $q_{n,2}[f, 1]$ and $q_{n,2}[F, 1]$.

Theorem 4.3.4 (Izumino [22])**.** *Let $n \in \mathbb{N}$ and $f \in H^\infty \setminus \{0\}$. Let f have a factorization $f = uF$, where u is inner and F is outer. Then*

$$q_{n,2}[f, 1] = \overline{u(0)} q_{n,2}[F, 1].$$

Proof. From Theorem 4.2.2, Lemma 4.3.2 and Lemma 4.3.3, it follows that

$$\begin{aligned} q_{n,2}[f, 1] &= (T_f E_n)^\dagger(1) \\ &= (T_F E_n)^\dagger T_u^*(1) \\ &= (T_F E_n)^\dagger (P_+ M_{\overline{u}}(1)) \\ &= (T_F E_n)^\dagger (P_+ \overline{u}) \\ &= (T_F E_n)^\dagger (\overline{u(0)}) \\ &= \overline{u(0)} (T_F E_n)^\dagger(1) \\ &= \overline{u(0)} q_{n,2}[F, 1]. \end{aligned}$$

□

Since any function $f \in H^2 \setminus \{0\}$ has an inner-outer factorization $f = uF$, it turns out that Theorem 4.3.4 holds for these functions as well (see [4] for details). At any rate, we conclude that the zero-set of $q_{n,2}[f, 1]$ is the same as the zero-set of $q_{n,2}[F, 1]$. Now, we already know that the zero-set of $q_{n,2}[f, 1]$ is disjoint from $\overline{\mathbb{D}}$ [13]. In this case, we do not need to rely on the properties of outer functions. However, perhaps the inner-outer factorization of f will help us gain insight into the zero-sets of the more general OPAs $q_{n,2}[f, g]$. Accordingly, we pose the following problem.

Problem 4.3.5. Let $f \in H^\infty \setminus \{0\}$, $g \in L^2$, and $n \in \mathbb{N}$. Let f have a factorization $f = uF$, where u is inner and F is outer. Find a relationship between $q_{n,2}[f, g]$ and $q_{n,2}[F, g]$.

4.4 Concluding Remarks

In this dissertation, I have focused many of the discussions on the zeros of OPAs in L^p; my motivation for this was to try and extend "Shanks-type" results to a larger collection of function spaces. In the process, I have shown that OPAs of the form $q_{n,p}[f, 1]$ can often be written as an OPA of the form $q_{n,2}[fg, g]$. On the other hand, I have shown that OPAs of the form $q_{n,2}[fg, g]$ can be written as $(T_{fg}E_n)^\dagger(g)$. From the developed theory and the resulting computations, I am led to conclude that a Shanks-type result holds for functions in H^p. Accordingly, I make the following conjecture.

Conjecture 4.4.1. Let $1 < p < \infty$, $f \in H^p$, and $n \in \mathbb{N}$. If $f(0) \neq 0$, then $q_{n,p}[f, 1]$ is zero-free in $\overline{\mathbb{D}}$.

If this conjecture is true, then the corresponding OPAs could be used to design a BIBO stable filter. Now, one would need an efficient algorithm for computing these OPAs in order for this to be marketable in engineering practice; although I demonstrated some computations in this dissertation, there seems to be a lot of room for future development. Moreover, one would need a solid grasp on the location of their zeros; the location has an effect on the attenuating and amplifying properties of the filter. Nonetheless, I am very interested in understanding the relationship between p and the zeros of optimal polynomial approximants; it is my hope that this dissertation inspires other researchers to follow me in this pursuit.

References

[1] Ammar, G., Classical foundations of algorithms for solving positive definite Toeplitz equations, *Calcolo* **33** (1996), no. 1–2, 99–113.

[2] Bénéteau, C. and Centner, R., A survey of optimal polynomial approximants, applications to digital filter design, and related open problems, *Complex Anal. Synerg.* **7** (2021), no. 2, Paper No. 16.

[3] Bénéteau, C., Condori, A., Liaw, C., Seco, D., and Sola, A., Cyclicity in Dirichlet-type spaces and extremal polynomials, *J. Anal. Math.* **126** (2015), 259–286.

[4] Bénéteau, C., Ivrii, O., Manolaki, M., and Seco, D., Simultaneous zero-free approximation and universal optimal polynomial approximants, *J. Approx. Theory* **256** (2020).

[5] Bénéteau, Khavinson, D., Liaw, C., Seco, D., and Simanek, B., Optimal polynomial approximants: Jacobi matrices and Jentzsch-type theorems, *Rev. Mat. Iberoam.* **35** (2019), 607–642.

[6] Bénéteau, C., Khavinson, D., Liaw, C., Seco, D., and Sola, A., Orthogonal polynomials, reproducing kernels, and zeros of optimal approximants, *J. Lond. Math. Soc. (2)* **94** (2016), no. 3, 726–746.

[7] Bénéteau, C., Manolaki, M., and Seco, D., Boundary behavior of optimal polynomial approximants, *Constr. Approx.* **54** (2021), no. 1, 157–183.

[8] Centner, R., Optimal polynomial approximants in L^p, arXiv:2112.14002.

[9] Chen, B., Chiou, B., and Peng, S., Minimum sensitivity IIR filter design using principal component approach, *IEE Proceedings G (Circuits, Devices and Systems)* **138** (1991), no. 4, 474–482.

[10] Chen, B., Peng, S., and Chiou, B., IIR filter design via optimal Hankel-norm approximation, *IEE Proceedings G (Circuits, Devices and Systems)* **139** (1992), no. 5, 586–590.

[11] Cheng, R., Ross, W., and Seco, D., Zeros of optimal polynomial approximants in ℓ^p_A, arXiv:2104.08014.

[12] Chui, C., Approximation by double least-squares inverses, *J. Math. Anal. Appl.* **75** (1980), no. 1, 149–163.

[13] Chui, C. and Chan, A., Application of approximation theory methods to recursive digital filter design, *IEEE Transactions on Acoustics, Speech, and Signal Processing* **ASSP-30** (1982), no. 1, 18–24.

[14] Delsarte, P., Genin, Y., and Kamp, Y., Planar least-squares inverse polynomials: part II - asymptotic behavior, *SIAM J. Alg. Disc. Meth.* **1** (1980), no. 3, 336–344.

[15] Duren, P., "Theory of H^p Spaces", Academic Press, New York, 1970.

[16] Durrani, T., and Chapman, R., Optimal all-pole filter design based on discrete prolate spheroidal sequences, *IEEE transactions on acoustics, speech, and signal processing* **32** (1984), no. 4, 716–721.

[17] Garnett, J., "Bounded Analytic Functions", Academic Press, New York, 1981.

[18] Genin, Y. and Kamp, Y., Counterexample in the least-squares inverse stabilisation of 2D recursive filters, *Elec. Letters* **11** (1975), 330–331.

[19] Genin, Y. and Kamp, Y., Two-dimensional stability and orthogonal polynomials of the hypercircle, *Proceedings of the IEEE* **65** (1977), no. 6, 873–881.

[20] Gonos, I., Mastorakis, N., Swamy, M., and Virirakis, L., Evolutionary design of 2-dimensional recursive filters via the computer language GENETICA, *IEEE Transactions on Circuits and Systems II: Express Briefs* **53** (2006), no. 4, 254–258.

[21] Hoffman, K., "Banach Spaces of Analytic Functions", Prentice-Hall, New Jersey, 1962.

[22] Izumino, S., Generalized inverses of Toeplitz operators and inverse approximation in H^2, *Tohoku Math. J.* (2) **37** (1985), no. 1, 95–99.

[23] Justice, J., Shanks, J., and Treitel, S., Stability and synthesis of two-dimensional recursive filters, *IEEE Transactions on Audio and Electroacoustics* **AU-20** (1972), no. 2, 115–128.

[24] Kreyszig, E., "Introductory Functional Analysis with Applications", Wiley, New York, 1978.

[25] Qiao, S., Wang, G., and Wei, Y., "Generalized Inverses: Theory and Computations", Springer, Singapore, 2018.

[26] Rao, K. and Swamy, M., "Digital Signal Processing: Theory and Practice", Springer, Singapore, 2018.

[27] Robinson, E., Structural properties of stationary stochastic processes with applications, in "Proceedings of the Symposium on Time Series Analysis", Wiley, New York, 1963.

[28] Rudin, W., "Real and Complex Analysis", McGraw-Hill, New York, 1987.

[29] Seco, D. and Téllez, R., Polynomial approach to cyclicity for weighted ℓ^p_A, *Banach J. Math. Anal.* **15** (2021), no.1, Paper No.1.

[30] Shapiro, H., "Topics in approximation theory", Springer, New York, 1971.