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Global and Stochastic Dynamics of Diffusive Hindmarsh-Rose Equations in Neurodynamics

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by

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Dedication

In memory of my grandparents.

To my parents and sister, with eternal appreciation and love.

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Abstract

This dissertation consisting of three parts is the study of the open problems of global dynamics of the diffusive Hindmarsh-Rose equations, random dynamics of the stochastic Hindmarsh-Rose equations with multiplicative noise and additive noise respectively, and synchronization of the boundary coupled Hindmarsh-Rose neuron networks.

In Part I (Chapters 2, 3 and 4) of this dissertation, we study the global dynamics for single neuron models of diffusive and partly diffusive Hindmarsh-Rose equations on a three-dimensional bounded domain. The existence of global attractors as well as its regularity and structure is established by showing the absorbing properties and the asymptotically compact characteristics, especially for the partly diffusive Hindmarsh-Rose equations by means of the Kolmogorov-Riesz theorem. A new general theorem on the squeezing property for reaction-diffusion equations is proved, through which we proved the existence of an exponential attractor and the finite fractal dimensionality of the global attractor. For nonautonomous diffusive Hindmarsh-Rose equations with translation bounded input, the existence of pullback exponential attractor is shown with the leverage of proving the L^2 to H^1 smoothing Lipschitz continuity in a long run of the nonautonomous solution process.

In Part II (Chapters 5 and 6), we investigate the pullback long-term behavior of the random dynamical system or called cocycle of the stochastic Hindmarsh-Rose equations driven by multiplicative white noise on a 3D domain and by additive noise on a 2D domain, respectively. The existence of a random attractor for both problems is proved respectively through the exponential transformation and the additive transformation by means of the Ornstein-Uhlenbeck process. Through the sharp uniform estimates, we proved the pullback absorbing property and the pullback asymptotically compactness of these two dynamical system in the L^2 Hilbert space.

Lastly in Part III (Chapters 7 and 8), the two new mathematical models of partly coupled neurons and of a boundary coupled neuron network are proposed in terms of the systems of partly diffusive Hindmarsh-Rose equations and with the coupling boundary conditions for the network. Through the absorbing and asymptotic analysis for the differencing Hindmarsh-Rose equations of the neuron network, the main result in Chapter 8 shows that the neuronal network is asymptotically synchronized at a uniform exponential rate provided that the combined boundary coupling strength and the stimulating signals exceed a quantified threshold explicitly in terms of the parameters.

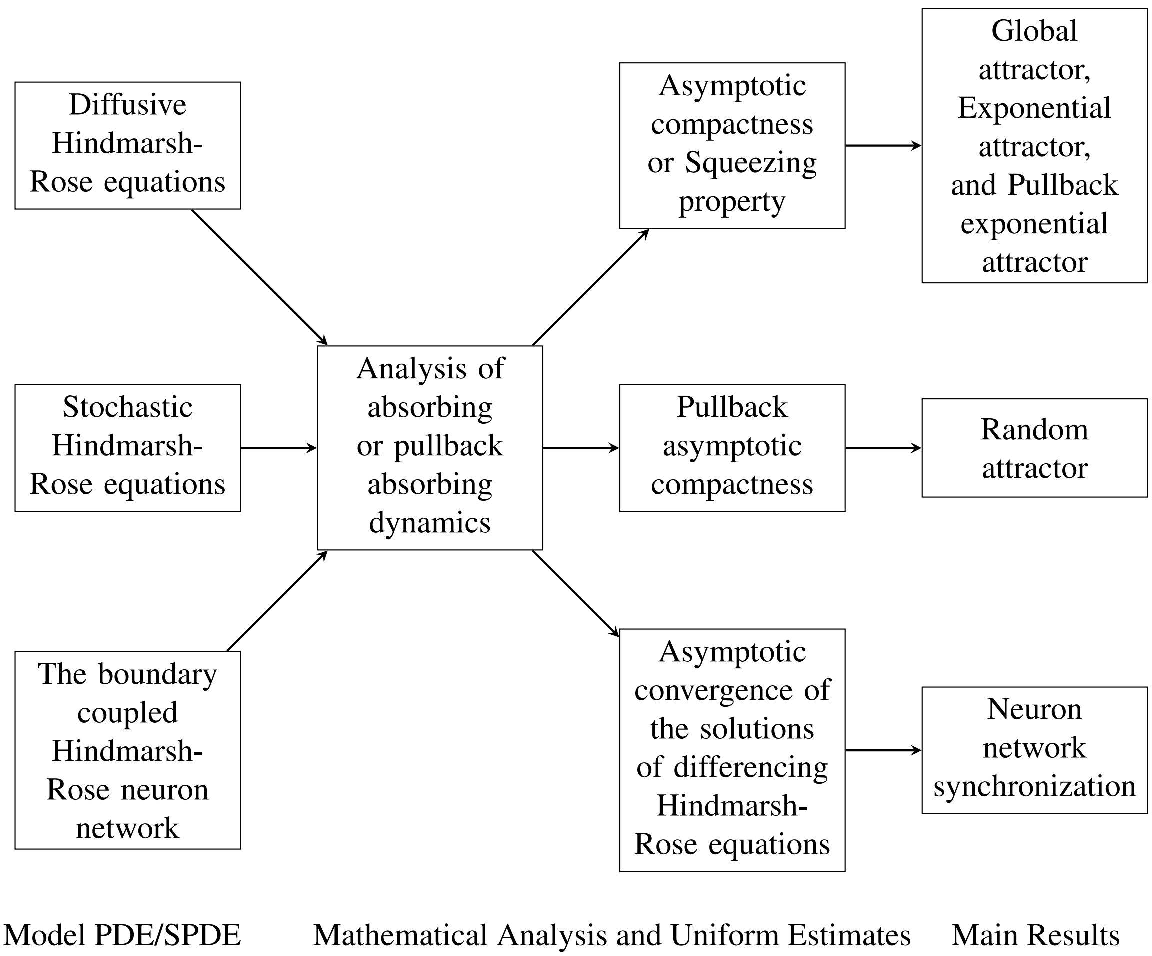

Figure 1.: Route Map of the Dissertation Research

Chapter 1
Introduction

1.1 Overview of the Topics and Achieved Results

The mathematical model of Hindmarsh-Rose equations for describing biological neuron spiking-bursting of the intracellular membrane potential observed in experiments was originally proposed by Hindmarsh and Rose in 1982–1984 [35, 36]. This model composed of three ordinary differential equations has been studied through numerical simulations and mathematical analysis in recent years, cf. [35, 36, 38, 50, 67, 83] and the references therein. It exhibits rich and interesting temporal bursting patterns, especially the three-dimensional complex bifurcations leading to chaotic bursting and dynamics.

This dissertation aims to study the global and longtime dynamics of the diffusive and partly diffusive Hindmarsh-Rose equations as well as the random dynamics of the stochastic Hindmarsh-Rose equations with the white noise on a bounded domain. As a significant extension of these topics on single neuron dynamics, this dissertation put forward two new and meaningful mathematical models respectively for two coupled neurons and for a boundary coupled Hindmarsh-Rose neuronal network. In comparison with all the existing results on several ODE models and the FitzHugh-Nagumo neuron models, the synchronization of this boundary coupled Hindmarsh-Rose neuronal network with the explicitly expressed threshold condition achieved in this work is a breakthrough in neuroscience.

In Chapter 2, the existence of global attractor together with its regularity and structure for the diffusive and partly diffusive Hindmarsh-Rose equations is presented. A new approach through the use of Kolmogorov-Riesz Theorem is taken to prove the challenging asymptotic compactness of the solution semiflow for the partly diffusive system of coupled PDE and ODE.

In Chapter 3, a new theorem with the weakened and accessible conditions for the squeezing property of solutions of the general reaction-diffusion equations is proved and applied to establish the existence of an exponential attractor, which consequently shows that the global attractor located in the infinite-dimensional function space actually has a finite fractal dimension.

In Chapter 4, the existence of a pullback exponential attractor for the nonautonomous diffusive Hindmarsh-Rose equations with time-varying external input is proved by conducting the sharp and uniform estimates of solutions to show the smoothing Lipschitz condition, especially using the fractional Sobolev space interpolation with the exquisite Gagliardo-Nirenberg inequalities.

In Chapter 5, the random dynamics of stochastic diffusive Hindmarsh-Rose equations driven by a multiplicative noise on a 3D bounded domain is presented. A transformation of exponential multiplication is used to covert the stochastic PDE driven by the multiplicative noise terms to a system of PDE with random coefficients and random initial data. Then the two-step pullback estimates of the pathwise solutions demonstrate the pullback random absorbing sets in both L^2 and H^1 spaces and the existence of a random attractor.

In Chapter 6, the existence of a random attractor for the random dynamical system generated by the stochastic diffusive Hindmarsh-Rose equations with additive noise on a two-dimensional bounded domain (due to mathematical technicality) is proved through the approach of the additive transformation by means of the noise driven Ornstein-Uhlenbeck processes and the sharp uniform estimates.

In Chapter 7, a new model of the coupled neurons represented by the partly diffusive Hindmarsh-Rose equations with linear coupling terms us proposed. It is shown that the solution semiflow exhibits globally absorbing characteristics. We defined the asynchronous degree of the semiflow and proved the asymptotic synchronization for the coupled Hindmarsh-Rose neuron system provided that the coupling strength coefficient exceeds the threshold quantitatively expressed by the parameters, which can be generalized to ensemble coupling neurons.

In Chapter 8, a new mathematical model of neuronal networks is presented by the system of partly diffusive Hindmarsh-Rose equations with the non-overlapping, pairwise coupling, Robin boundary condition. The global absorbing property of the solution semiflow is proved. Through the solution estimates of the differencing Hindmarsh-Rose equations and by the generalized Poincaré inequality, the asymptotic synchronization of this neuron network at a uniform exponential rate is proved, provided that the multiplied boundary coupling strength and the stimulating signals exceed a threshold explicitly expressed by the parameters. This model is biologically more realistic and better than the other models for the neuron network dynamics, because the neuron coupling and signal transmission usually take place on the boundary of the cell domain through bio-electric potential stimulations which are related to the first component equation for the membrane potential.

Finally, the dissertation concludes with the future research directions in Chapter 9.

1.2 Motivation and Biological Neurons and Neuronal Network Dynamics

In 1982–1984, J.L. Hindmarsh and R.M. Rose developed the mathematical model to describe neuronal activity and dynamics:

$$\begin{aligned} \frac{du}{dt} &= au^2 - bu^3 + v - w + J, \\ \frac{dv}{dt} &= \alpha - \beta u^2 - v, \\ \frac{dw}{dt} &= q(u - u_R) - rw. \end{aligned} \tag{1.1}$$

This neuron model was motivated by the discovery of neuronal cells in the pond snail *Lymnaea* which generated a burst after being depolarized by a short current pulse. This model charactrerizes the phenomena of synaptic bursting and especially chaotic bursting in a three-dimensional (u, v, w) space, which incorporates the third variable representing a slow channel of ions that hyperpolarizes the neuronal cell.

In the above system, the variable $u(t)$ refers to the membrane electric potential of a neuronal cell, the variable $v(t)$ represents the transport rate of the ions of sodium and potassium through the fast ion channels and is called the spiking variable, while the variables $w(t)$ represents the transport rate across the neuronal cell membrane through slow channels of calcium and other ions correlated to the bursting phenomenon and is called the bursting variable.

Neurons are the nerve cells which form the major pathways of communication and create biological networks capable of processing and coordinating biochemical and bio-electrical information. Human brain consists of approximately 10^{11} (100 billion) neurons. The soma of neuron cell predominantly processes and integrates synaptic inputs and determine whether the neuron becomes activated and/or transmits signals to others.

Neuronal signals are short electrical pulses called spike or action potential. The synaptic pulse inputs received by a neuron from its dendrite branches modify the intracellular membrane potentials and can be excitatory or inhibitory, which gives rise to temporal and spatial summation and may cause the busting in alternating phases of rapid firing spikes and then refractory quiescence. Neuron signals often triggered at the axon hillock can propagate along the axon or diffuse to the neighbors.

Bursting constitutes a mechanism to modulate and set the pace for brain functionalities and to communicate with the neighbor neurons. Bursting behavior and patterns occur in a variety of bio-systems such as pituitary melanotropic gland, thalamic neurons, respiratory pacemaker neurons, and insulin-secreting pancreatic β-cells, cf. [6, 12, 16, 36].

Synaptic coupling in neuron activities has to reach certain threshold for release of quantal vesicles and achieving an synchronization [21, 57, 63]. The fast threshold modulation for neurons synchronization for certain ODE models was initiated and analyzed in [66, 85].

The mathematical analysis mainly using bifurcations together with numerical simulations of the above Hindmarsh-Rose equations and several other ODE models on bursting dynamics of single neurons has been studied by a number of authors, cf. [5, 27, 39, 50, 55, 67, 69, 74, 83]. The more interesting study is on the behavior of neurons coupling and synchronization [27, 58, 64, 69]. It was rigorously proved in [67, 83] that chaotic bursting solutions can be quickly synchronized and regularized when the coupling strength is large enough to topologically change the bifurcation diagram based on this Hindmarsh-Rose model in ODE.

It is well known that the Hodgkin-Huxley equations [37] (1952) provided a four-dimensional model describing the dynamics of neuron membrane potential taking into account of the sodium, potassium as well as leak ions current. It is a highly nonlinear system if without simplification assumptions. The FitzHugh-Nagumo equations [29] (1961–1962) as a two-dimensional model for an excitable neuron with two variables of the membrane potential and the combined ion current admits exquisite phase plane analysis showing excitation and sustained periodic bursting, but the 2D nature of FitzHugh-Nagumo equations exclude any chaotic solutions so that no chaotic bursting can be generated. Another drawback of FitzHugh-Nagumo model with only few parameters is difficult to adapt to characterizing some specific properties of neuron dynamics.

The Hindmarsh-Rose equations (1.1) contributed a three-dimensional model with cubic nonlinearity is capable to generate significant mechanisms for rapid firing and regular or chaotic busting in the research of neurodynamics. This ODE model shows that geometric deformation of the three nullclines with varying parameters can demonstrate coexistence of more than one steady states and a limit cycle, which yields a variety of complex bifurcations and lasting chaotic dynamics.

It has been indicated by the research on this model (1.1) that adding the third equation of $w(t)$ in this Hindmarsh-Rose model causes lower down the neuron firing threshold and that $w(t)$ will return to zero when the membrane potential $u(t)$ has reached its rest state value $c = u_R$. Moreover the Hindmarsh-Rose model allows for varying interspike interval. This 3D model (1.1) is a suitable choice for investigation of neurodynamics and attracts more research interests exposed to a wide range of applications in neuroscience, including the self-synchronization and self-regulation of neuronal ensembles and networks.

Understanding of the mechanisms in biological brain through mathematical models and analysis is in some sense critical for advancing the researches of medicine and artificial intelligence. In particular, synchronization and desynchronization of neuronal firing and bursting are very important both for improvement to and degeneration of the brain's functionality and performance. Increased synchronization may lead to enhanced information processing or to neurological disorders such as epilepsy, Alzheimer's disease, and Parkinson's disease [22, 28, 32, 41, 52, 53, 59].

It is desirable to research under what conditions (in terms of neurons connectivity, coupling strength, configuration, signals and noise) synchronization can arise in neural networks and the possibility of controlling its prevalence. Synchronization for neuron ensembles and for complex neuron networks as well as artificial neural networks is one of the central and significant topics in neuroscience and in the theory of artificial intelligence.

1.3 Preliminary Mathematical Concepts

Here we introduce some mathematical concepts and basics commonly used in this dissertation. Let $\Omega \subset \mathbb{R}^n$ ($n \leq 3$) be a bounded domain with locally Lipschitz continuous boundary. Define the real Hilbert space $H = [L^2(\Omega)]^3 = L^2(\Omega, \mathbb{R}^3)$ and the Sobolev space $E = [H^1(\Omega)]^3 = H^1(\Omega, \mathbb{R}^3)$. The norm and inner-product of H or $L^2(\Omega)$ will be denoted by $\| \cdot \|$ and $\langle \cdot, \cdot \rangle$, respectively. The norm and inner-product of E will be denoted by $\| \cdot \|_E$ and $\langle \cdot, \cdot \rangle_E$, respectively. The norm of $L^p(\Omega)$ or $L^p(\Omega, \mathbb{R}^3)$ will be dented by $\| \cdot \|_{L^p}$ if $p \neq 2$. We use $| \cdot |$ to denote vector norm or set measure in a Euclidean space.

Consider an initial value problem of a nonlinear evolutionary equation in the Hilbert space H:

$$\begin{aligned} \frac{\partial g}{\partial t} &= Ag + f(g), \ \ t > 0, \\ g(0) &= g_0 \in H. \end{aligned} \tag{1.2}$$

Here $A : D(A)(\subset H) \to H$ is a densely defined, closed, linear operator, usually a second-order differential operator with respect to the spatial variables, and $f(g) : E \to H$ is a nonlinear Nemytskii operator which is locally Lipschitz continuous.

Definition 1.3.1. A function $g(t, x), (t, x) \in [0, \tau] \times \Omega$, is called a *weak solution* to the initial value problem (1.2), if the following conditions are satisfied:

(i) $\frac{d}{dt} \langle g, \zeta \rangle_E = (Ag, \zeta)_E + (f(g), \zeta)_E$ is satisfied for a.e. $t \in [0, \tau]$ and any $\zeta \in E$;

(ii) $g(t, \cdot) \in C([0, \tau]; H) \cap L^2([0, \tau]; E)$ such that $g(0) = g_0$.

Lemma 1.3.2. *Under the aforementioned conditions, for any given initial state $g_0 \in H$, there exists a unique weak solution $g(t, g_0)$, $t \in [0, T_{max})$, for some $T_{max} > 0$, of the initial value problem* (1.2)*, which satisfies*

$$g \in C([0, T_{max}); H) \cap C^1((0, T_{max}); H) \cap L^2_{loc}([0, T_{max}); E), \tag{1.3}$$

where $I_{max} = [0, T_{max})$ is the maximal interval of existence. The weak solutions continuously depends on the initial data.

Proof. The existence and uniqueness of a local weak solution in time can be proved by the Galerkin approximation method based on the *a priori* estimates exemplarily similar to what we shall present in Section 2.2 and followed by the Lions-Magenes type of weak and weak* compactness argument [13, Section XV.3] and [62]. □

Definition 1.3.3. A time-parametrized family of bounded linear operators $\{S(t)\}_{t \geq 0} \subset \mathcal{L}(X)$ is called a semiflow on a Banach space X, if the following conditions are satisfied:

(i) $S(0) = I_X$, the identity operator on X.

(ii) $S(t)S(\tau) = S(t + \tau)$, for any $t, \tau \geq 0$.

(iii) The mapping $(t, x) \longmapsto S(t)x$ is continuous on $[0, \infty) \times X$.

Definition 1.3.4. Let $\{S(t)\}_{t\geq 0}$ be a semiflow on a Banach space $\mathscr{X}$. A bounded set B_0 of $\mathscr{X}$ is called an absorbing set for this semiflow, if for any given bounded subset $B \subset \mathscr{X}$ (including B_0 itself) there is a finite time $T_B \geq 0$ such that $S(t)B \subset B_0$ for all $t \geq T_B$. If there exists an absorbing set, then the dynamical system represented by the semiflow is called dissipative.

Definition 1.3.5. A semiflow $\{S(t)\}_{t\geq 0}$ on a Banach space $\mathscr{X}$ is called asymptotically compact if for any bounded sequence $\{w_n\}$ in $\mathscr{X}$ and any monotone increasing sequences $0 < t_n \to \infty$, there exist subsequences $\{w_{n_k}\}$ of $\{w_n\}$ and $\{t_{n_k}\}$ of $\{t_n\}$ such that $\lim_{k\to\infty} S(t_{n_k})w_{n_k}$ exists in $\mathscr{X}$.

Definition 1.3.6 (Global Attractor)**.** A set $\mathscr{A}$ in a Banach space $\mathscr{X}$ is called a global attractor for a semiflow $\{S(t)\}_{t\geq 0}$ on $\mathscr{X}$, if the following two properties are satisfied:

(i) $\mathscr{A}$ is a nonempty, compact, and invariant set meaning $S(t)\mathscr{A} = \mathscr{A}$ for $t \geq 0$, in the space $\mathscr{X}$.

(ii) $\mathscr{A}$ attracts any given bounded set $B \subset \mathscr{X}$ with respect to the semi-Hausdorff distance,

$$\text{dist}_{\mathscr{X}}(S(t)B, \mathscr{A}) = \sup_{x\in B} \inf_{y\in\mathscr{A}} \|S(t)x - y\|_{\mathscr{X}} \to 0, \quad \text{as} \quad t \to \infty.$$

As specified in [13,62,68], global attractor is a depository (usually of finite fractal dimension) of all the permanent regimes including the steady states, periodic orbits, homoclinic and heteroclinic orbits, and invariant sets with the associate unstable manifolds for an infinite-dimensional dynamical system. Roughly speaking, global attractor (if exists) qualitatively characterizes the global, longtime and asymptotic dynamics of the solutions to nonlinear PDEs. Its counterpart for random dynamical systems is random attractor.

Below is the main existing result on the existence of a global attractor, cf. [13, 56, 62, 68, 76].

Proposition 1.3.7. *Let $\{S(t)\}_{t\geq 0}$ be a semiflow on a Banach space $\mathscr{X}$. If the following two conditions are satisfied*:

(i) *there exists a bounded absorbing set $B_0 \subset \mathscr{X}$ for $\{S(t)\}_{t\geq 0}$, and*

(ii) *the semiflow $\{S(t)\}_{t\geq 0}$ is asymptotically compact on $\mathscr{X}$,*

then there exists a global attractor $\mathscr{A}$ in $\mathscr{X}$ for the semiflow $\{S(t)\}_{t\geq 0}$ and $\mathscr{A}$ is the ω-limit set of *the absorbing set B_0,*

$$\mathscr{A} = \bigcap_{\tau \geq 0} \overline{\bigcup_{t \geq \tau} (S(t)B_0)}. \tag{1.4}$$

Definition 1.3.8. Let $\{S(t)\}_{t\geq 0}$ be a semiflow on a Banach space X and let Y be a Banach space which is compactly embedded in X. Then a set $\mathscr{A} \subset Y$ is called a bispace (X, Y)-global attractor for this semiflow if the following two conditions are satisfied:

(i) $\mathscr{A}$ is a nonempty, compact, and invariant set in Y, and

(ii) $\mathscr{A}$ attracts any bounded set B of X with respect to the Y-norm.

We refer to [8, 9, 11, 19, 20, 25, 43, 46, 49, 86] for the concepts and some existing results in the theory of nonautonomous dynamical systems, especially on the topics of pullback attractors and pullback exponential attractors. Recall that these concepts are rooted in the theory of global attractors and other invariant sets for the autonomous infinite-dimensional dynamical systems [13, 51, 56, 62, 68, 76–78] and the theory of exponential attractors [23, 24, 48, 51, 75].

Let X be a Banach space and suppose that a nonautonomous partial differential equation with certain initial-boundary condition, which usually involves a time-dependent external input term, has global solutions in space-time. Then the solution operator

$$\{S(t, \tau) : X \to X\}_{t \geq \tau \in \mathbb{R}}$$

is called a *nonautonomous process* [11, 13], if it satisfies the three conditions:

1) $S(\tau, \tau) = I_X$ (the identity) for any $\tau \in \mathbb{R}$.

2) The cocycle property is satisfied:

$$S(t, s)S(s, \tau) = S(t, \tau) \quad \text{for any} \quad -\infty < \tau \leq s \leq t < \infty.$$

3) The mapping $(t, \tau, g) \to S(t, \tau)g \in X$ is continuous with respect to $(t, \tau, g) \in \mathfrak{T} \times X$, where $\mathfrak{T} = \{(t, \tau) \in \mathbb{R}^2 : t \geq \tau \in \mathbb{R}\}$.

Definition 1.3.9 (Nonautonomous semiflow)**.** A mapping $\Phi(t, \tau, g) : \mathbb{R}^+ \times \mathbb{R} \times X \to X$ is called a *nonautonomous semiflow* (or called nonautonomous dynamical system) on a Banach space X over $\mathbb{R}$, if the following conditions are satisfied:

1) $\Phi(0, \tau, \cdot)$ is the identity on X, for any $\tau \in \mathbb{R}$.

2) $\Phi(t+s, \tau, \cdot) = \Phi(t, \tau+s, \Phi(s, \tau, \cdot))$, for any $t, s \geq 0$ and $\tau \in \mathbb{R}$.

3) $\Phi(t, \tau, g) : \mathfrak{T} \times X \to X$ is continuous.

If $\{S(t,\tau) : X \to X\}_{(t,\tau)\in\mathfrak{T}}$ is a continuous evolution process on X, then it generates a nonautonomous semiflow defined by

$$\Phi(t, \tau, g) = S(t+\tau, \tau, g), \quad (t, \tau, g) \in \mathfrak{T} \times X. \tag{1.5}$$

This relation in the pullback sense is the following important identity

$$\Phi(t, \tau - t, g) = S(\tau, \tau - t)g, \quad (t, \tau, g) \in \mathbb{R}^+ \times \mathbb{R} \times X. \tag{1.6}$$

Definition 1.3.10 (Pullback Attractor)**.** A time-parametrized set $\mathcal{A} = \{\mathcal{A}(\tau)\}_{\tau\in\mathbb{R}}$ in a Banach space X is called a pullback attractor for the nonautonomous semiflow $\{\Phi(t, \tau, \cdot)\}_{(t,\tau)\in\mathfrak{T}}$ generated by a continuous evolution process $\{S(t,\tau) : X \to X\}_{(t,\tau)\in\mathfrak{T}}$, if the following conditions are satisfied:

1) $\mathcal{A}$ is compact in the sense that for each $\tau \in \mathbb{R}$ the set $\mathcal{A}(\tau)$ is compact in X.

2) $\mathcal{A}$ is invariant,

$$S(t,\tau)\,\mathcal{A}(\tau) = \mathcal{A}(t+\tau), \quad t \geq 0,\ \tau \in \mathbb{R}.$$

it is equivalent to $\Phi(t, \tau, \mathcal{A}(\tau)) = \mathcal{A}(t+\tau)$ for $t \geq \tau$.

3) $\mathcal{A}$ pullback attracts every bounded set $B \subset X$ with respect to the semi-Hausdorff distance,

$$\lim_{t\to\infty} dist_X(\Phi(t,\, \tau - t, B),\, \mathcal{A}(\tau)) = \lim_{t\to\infty} dist_X(S(\tau,\, \tau - t)B,\, \mathcal{A}(\tau)) = 0.$$

Definition 1.3.11 (Pullback Exponential Attractor)**.** A time-parametrized set $\mathscr{M} = \{\mathscr{M}(t)\}_{t\in\mathbb{R}} \subset X$, where X is a Banach space, is called a pullback exponential attractor of a continuous evolution process $\{S(t,\tau)\}_{t\geq\tau\in\mathbb{R}}$ on X, if the following conditions are satisfied:

1) For any $t \in \mathbb{R}$, the set $\mathscr{M}(t)$ is a compact and positively invariant sel in X with respect to this process,

$$S(t,\tau)\mathscr{M}(\tau) \subset \mathscr{M}(t) \quad \text{for any } \infty < \tau \leq t < \infty.$$

2) The fractal dimension $\dim_F \mathscr{M}(t)$ for all $t \in \mathbb{R}$ is finite and

$$\sup_{t\in\mathbb{R}} \dim_F \mathscr{M}(t) < \infty.$$

3) $\mathscr{M} = \{\mathscr{M}(t)\}_{t\in\mathbb{R}}$ exponentially pullback attracts every bounded set $B \subset X$ in the sense that there exists a constant $\sigma > 0$, such that for every bounded set B in X and any $\tau \in \mathbb{R}$,

$$\lim_{t\to\infty} e^{\sigma t} \mathrm{dist}_X(S(\tau, \tau - t)B, \mathscr{M}(\tau)) = 0.$$

To study the dynamics generated by the pathwise solutions of stochastic partial differential equations in the asymptotically long run, we first recall the preliminary concepts for random dynamical systems, or called stochastic cocycles, cf. [2, 4, 14, 17, 18, 26, 30, 54, 60].

Let $(\mathfrak{Q}, \mathcal{F}, P)$ be a probability space and let X be a real Banach space.

Definition 1.3.12. $(\mathfrak{Q}, \mathcal{F}, P, \{\theta_t\}_{t\in\mathbb{R}})$ is called a *metric dynamical system* (MDS), if $(\mathfrak{Q}, \mathcal{F}, P)$ is a probability space and θ_t is a time-shifting mapping with the following conditions satisfied:

(i) the mapping $\theta : \mathbb{R} \times \mathfrak{Q} \to \mathfrak{Q}$ is $(\mathscr{B}(\mathbb{R}) \otimes \mathcal{F}, \mathcal{F})$ - measurable,

(ii) θ_0 is the identity on $\mathfrak{Q}$,

(iii) $\theta_{t+s} = \theta_t \circ \theta_s$ for all $t, s \in \mathbb{R}$, and

(iv) θ_t is probability invariant, meaning $\theta_t P = P$ for all $t \in \mathbb{R}$.

Here $\mathscr{B}(X)$ stands for the σ-algebra of Borel sets in a Banach space X and $(\theta_t P)(S) = P(\theta_t S)$ for any $S \in \mathcal{F}$.

Definition 1.3.13. A continuous *random dynamical system* (RDS) briefly called a cocycle on X over an MDS $(\mathfrak{Q}, \mathcal{F}, P, \{\theta_t\}_{t\in\mathbb{R}})$ is a mapping

$$\varphi(t, \omega, x) : [0, \infty) \times \mathfrak{Q} \times X \to X,$$

which is $(\mathscr{B}(\mathbb{R}^+) \otimes \mathcal{F} \otimes \mathscr{B}(X), \mathscr{B}(X))$–measurable and satisfies the following conditions for every ω in $\mathfrak{Q}$:

(i) $\varphi(0, \omega, \cdot)$ is the identity operator on X.

(ii) The cocycle property holds:

$$\varphi(t+s,\omega,\cdot)=\varphi(t,\theta_s\omega,\varphi(s,\omega,\cdot)),\quad \text{for all } t,s\geq 0.$$

(iii) The mapping $\varphi(\cdot,\omega,\cdot):[0,\infty)\times X\to X$ is strongly continuous.

Definition 1.3.14. A set-valued function $B:\mathfrak{Q}\to 2^X$ is called a random set in X if its graph $\{(\omega,x): x\in B(\omega)\}\subset\mathfrak{Q}\times X$ is an element of the product σ-algebra $\mathcal{F}\otimes\mathscr{B}(X)$. A *bounded* random set $B(\omega)\subset X$ means that there is a random variable $r(\omega)\in[0,\infty),\omega\in\mathfrak{Q}$, such that $|||B(\omega)|||:=\sup_{x\in B(\omega)}\|x\|\leq r(\omega)$ for all $\omega\in\mathfrak{Q}$. A bounded random set $B(\omega)$ is called *tempered* with respect to $\{\theta_t\}_{t\in\mathbb{R}}$ on $(\mathfrak{Q},\mathcal{F},P)$, if for any $\omega\in\mathfrak{Q}$ and for any constant $\beta>0$,

$$\lim_{t\to\infty}e^{-\beta t}\,|||\,B(\theta_{-t}\omega)|||=0.$$

A random set $S(\omega)\subset X$ is called *compact* (reps. *precompact*) for every $\omega\in\mathfrak{Q}$ the set $S(\omega)$ is a compact (reps. precompact) set in X.

Definition 1.3.15. A *tempered* random variable $R:(\mathfrak{Q},\mathcal{F},P)\to(0,\infty)$ respect to a metric dynamical system $\{\theta_t\}_{t\in\mathbb{R}}$ on $(\mathfrak{Q},\mathcal{F},P)$ means that for any $\omega\in\mathfrak{Q}$,

$$\lim_{t\to-\infty}\frac{1}{t}\log R(\theta_t\omega)=0.$$

We shall let $\mathscr{D}_X$ denote an inclusion-closed family of random sets in X, meaning that if $D=\{D(\omega)\}_{\omega\in\mathfrak{Q}}\in\mathscr{D}_X$ and $\hat{D}=\{\hat{D}(\omega)\}_{\omega\in\mathfrak{Q}}$ with $\hat{D}(\omega)\subset D(\omega)$ for all $\omega\in\mathfrak{Q}$, then $\hat{D}\in\mathscr{D}_X$. Such a family of random sets in X is called a *universe*. In this work, we define $\mathscr{D}_H$ to be the universe of all the tempered random sets in the Hilbert space $H=L^2(\Omega,\mathbb{R}^3)$.

Definition 1.3.16. For a given universe $\mathscr{D}_X$ of random sets in a Banach space X, a random set $K\in\mathscr{D}_X$ is called a *pullback absorbing set* with respect to an RDS (cocycle) φ over the MDS $(\mathfrak{Q},\mathcal{F},P,\{\theta_t\}_{t\in\mathbb{R}})$, if for any bounded random set $B\in\mathscr{D}_X$ and any $\omega\in\mathfrak{Q}$ there exists a finite time $T_B(\omega)>0$ such that

$$\varphi(t,\theta_{-t}\omega,B(\theta_{-t}\omega))\subset K(\omega),\quad \text{for all }\ t\geq T_B(\omega).$$

Definition 1.3.17. Let a universe $\mathscr{D}_X$ of random sets in a Banach space X be given, A random dynamical system (cocycle) φ is *pullback asymptotically compact* with respect to $\mathscr{D}_X$, if for any $\omega \in \mathfrak{Q}$, the sequence

$$\{\varphi(t_m, \theta_{-t_m}\omega, x_m)\}_{m=1}^{\infty} \text{ has a convergent subsequence in } X,$$

whenever $t_m \to \infty$ and $x_m \in B(\theta_{-t_m}\omega)$ for any given $B \in \mathscr{D}_X$.

Definition 1.3.18. Let a universe $\mathscr{D}_X$ of tempered random sets in a Banach space X be given. A random set $\mathcal{A} \in \mathscr{D}_X$ is called a *random attractor* for a given random dynamical system (cocycle) φ over the metric dynamical system $(\mathfrak{Q}, \mathcal{F}, P, \{\theta_t\}_{t\in\mathbb{R}})$, if the following conditions are satisfied:

(i) $\mathcal{A}$ is a compact random set in the space X.

(ii) $\mathcal{A}$ is invariant in the sense that

$$\varphi(t, \omega, \mathcal{A}(\omega)) = \mathcal{A}(\theta_t\omega), \quad \text{for all } \ t \geq 0, \ \omega \in \mathfrak{Q}.$$

(iii) $\mathcal{A}$ attracts every $B \in \mathscr{D}_X$ in the pullback sense that

$$\lim_{t\to\infty} dist_X(\varphi(t, \theta_{-t}\omega, B(\theta_{-t}\omega)), \mathcal{A}(\omega)) = 0, \quad \omega \in \mathfrak{Q},$$

where $dist_X(\cdot, \cdot)$ is the semi-Hausdorff distance with respect to the X-norm. Then $\mathscr{D}_X$ is called the *basin* of attraction for $\mathcal{A}$.

We list the following useful inequalities to study global dynamics, random dynamics, and synchronization of neuronal networks:

(1) The Gagliardo-Nirenberg inequality [62, Appendtx B] of Sobolev space interpolation is instrumental in estimates of solutions of deterministic and stochastic partial differential equations:

$$\|y\|_{W^{k,p}(\Omega)} \leq C\|y\|_{W^{m,q}(\Omega)}^{\theta}\, \|y\|_{L^r(\Omega)}^{1-\theta}, \quad \text{for all } \ y \in W^{m,q}(\Omega), \tag{1.7}$$

where $C > 0$ is a constant, provided that $p, q, r \geq 1, 0 < \theta < 1$, and

$$k - \frac{n}{p} \leq \theta\left(m - \frac{n}{q}\right) - (1-\theta)\,\frac{n}{r}, \quad n = \dim{(\Omega)}.$$

(2) The Young's inequality in the general form for any nonnegative x, y is

$$xy \leq \varepsilon x^p + C(\varepsilon, p) y^q, \qquad \frac{1}{p} + \frac{1}{q} = 1, \ \ (p, q \geq 1), \qquad C(\varepsilon, p) = \varepsilon^{-q/p}. \tag{1.8}$$

where constant $\varepsilon > 0$ can be arbitrarily small.

(3) By the fact that $H^1(\Omega) \hookrightarrow L^6(\Omega)$ is a continuous imbedding for space dimension $n \leq 3$ and by the Young's inequality, there is a constant $C_0 > 0$ such that for $\varphi(u) = au^2 - bu^3$ and $\psi(u) = \alpha - \beta u^2$ in the diffusive and partly diffusive Hindmarsh-Rose equations,

$$\|\varphi(u)\| \leq C_0(1 + \|u\|^3_{L^6}) \quad \text{and} \quad \|\psi(u)\| \leq C_0(1 + \|u\|^2_{L^4}) \quad \text{for } u \in H^1(\Omega).$$

(4) Gronwall inequality [62, Lemma D.3].

Let y and h be nonnegative functions in $L^1_{\text{loc}}[0, T; \mathbb{R})$, where $0 < T \leq \infty$. Assume that y is absolutely continuous on $[0, T)$ and that

$$\frac{dy}{dt} \leq ay(t) + h(t), \quad \text{almost everywhere on } (0, T).$$

Then $y \in L^\infty_{\text{loc}}(0, T; \mathbb{R})$ and one has

$$y(t) \leq y_0 \, \mathrm{e}^{-at} + \frac{h_0}{a}, \quad a \neq 0$$

(5) Uniform Gronwall inequality [62, Appendix D].

Let y, g and h be nonnegative functions in $L^1_{\text{loc}}[0, T; \mathbb{R})$, where $0 < T \leq \infty$. Assume that y is absolutely continuous on $(0, T)$ such that

$$\frac{dy}{dt} \leq g(t)y(t) + h(t), \quad \text{almost everywhere on } (0, T)$$

and

$$\int_t^{t+\tau} g(s)ds \leq a_1, \quad \int_t^{t+\tau} h(s)ds \leq a_2, \quad \int_t^{t+\tau} y(s)ds \leq a_3,$$

where τ, a_1, a_2 and a_3 are positive constants. Then

$$y(t+\tau) \leq \left(\frac{a_3}{\tau} + a_2\right) \mathrm{e}^{a_1}.$$

(6) Generalized Poincaré Inquality

Let Ω be an open, bounded, and connected subset of $\mathbb{R}^d$ for some $d \geq 1$ and let $g \in H^1(\Omega)$. Then there exist positive constants C_1 and C_2 such that

$$\int_\Omega g^2(x)dx \leq C_1 \int_\Omega |\nabla g(x)|^2 dx + C_2 \left[\int_\Omega |g(x)|\, dx\right]^2.$$

Remark. In all the chapters of this dissertation, we do not assume nor require the unknown variables $u(t,x), v(t,x), w(t,x)$ in the deterministic scenario or $u(t,x,\omega), v(t,x,\omega), w(t,x,\omega)$ in the stochastic environment to be nonnegative, since the action potential of neuron membrane as a voltage and the transportation rates of ions are not sign-definite.

Chapter 2

Global Attractors for Diffusive Hindmarsh-Rose Equations in Neurodynamics

In this chapter, we shall study the global dynamics in terms of the existence of a global attractor for the diffusive Hindmarsh-Rose equations, which is a new PDE model in neurodynamics:

$$\frac{\partial u}{\partial t} = d_1 \Delta u + \varphi(u) + v - w + J, \tag{2.1}$$

$$\frac{\partial v}{\partial t} = d_2 \Delta v + \psi(u) - v, \tag{2.2}$$

$$\frac{\partial w}{\partial t} = d_3 \Delta w + q(u - c) - rw, \tag{2.3}$$

for $t > 0,\ x \in \Omega \subset \mathbb{R}^n$ $(n \leq 3)$, where Ω is a bounded domain with locally Lipschitz continuous boundary. The nonlinear terms

$$\varphi(u) = au^2 - bu^3, \quad \text{and} \quad \psi(u) = \alpha - \beta u^2. \tag{2.4}$$

In this system (2.1)–(2.3), the variable $u(t,x)$ refers to the membrane electric potential of a neuron cell, the variable $v(t,x)$ represents the transport rate of the ions of sodium and potassium through the fast ion channels and is called the spiking variable, while the variables $w(t,x)$ represents the transport rate across the neuronal cell membrane through slow channels of calcium and other ions correlated to the bursting phenomenon and is called the bursting variable.

All the involved parameters $d_1, d_2, d_3, a, b, \alpha, \beta, q, r$, and the inject current J are positive constants except $c\,(= u_R) \in \mathbb{R}$, which is a reference value of the membrane potential of neuron cell. In the original model of ODE [83], a set of the typical parameters are

$$J = 3.281, \quad r = 0.0021, \quad S = 4.0, \quad q = rS, \quad c = -1.6,$$

$$\varphi(s) = 3.0s^2 - s^3, \quad \psi(s) = 1.0 - 5.0s^2.$$

We impose the Neumann boundary conditions for the three components,

$$\frac{\partial u}{\partial \nu}(t,x) = 0, \;\; \frac{\partial v}{\partial \nu}(t,x) = 0, \;\; \frac{\partial w}{\partial \nu}(t,x) = 0, \quad t > 0, \; x \in \partial \Omega, \tag{2.5}$$

and the initial conditions are denoted by

$$u(0,x) = u_0(x), \; v(0,x) = v_0(x), \; w(0,x) = w_0(x), \quad x \in \Omega. \tag{2.6}$$

We shall also consider the following two models of partly diffusive Hindmarsh-Rose equations

$$\begin{aligned} \frac{\partial u}{\partial t} &= d_1 \Delta u + \varphi(u) + v - w + J, \\ \frac{\partial v}{\partial t} &= \psi(u) - v, \\ \frac{\partial w}{\partial t} &= q(u - c) - rw \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} \frac{\partial u}{\partial t} &= d_1 \Delta u + \varphi(u) + v - w + J, \\ \frac{\partial v}{\partial t} &= d_2 \Delta v + \psi(u) - v, \\ \frac{\partial w}{\partial t} &= q(u - c) - rw. \end{aligned} \tag{2.8}$$

In neuronal dynamics, the partly diffusive models (2.7) or (2.8) are more commonly interesting, since the ions currents may or may not diffuse quickly.

In this chapter, the following Section 2.1 is the formulation of the system (2.1)–(2.3) with preliminary concepts. In Section 2.2 we shall conduct uniform estimates to show the absorbing properties of the Hindmarsh-Rose semiflow. In Section 2.3 and Section 2.4, the main result on the existence of global attractor for the diffusive Hindmarsh-Rose system is proved together with its regularity and structure. And in Section 2.5 we shall prove the asymptotic compactness and the existence of global attractors of the partly diffusive systems (2.7) and (2.8) by means of the Kolmogorov-Riesz theorem.

2.1 Formulation, Uniform Estimates and Absorbing Properties

2.1.1 Formulation

The initial-boundary value problem (2.1)–(2.6) is formulated as an initial value problem of the evolutionary equation:

$$\begin{aligned} \frac{\partial g}{\partial t} &= Ag + f(g), \quad t > 0, \\ g\,(0) &= g_0 = (u_0, v_0, w_0) \in H. \end{aligned} \tag{2.9}$$

Here the nonpositive self-adjoint operator

$$A = \begin{pmatrix} d_1\Delta & 0 & 0 \\ 0 & d_2\Delta & 0 \\ 0 & 0 & d_3\Delta \end{pmatrix} : D(A) \to H, \tag{2.10}$$

where $D(A) = \{g \in H^2(\Omega, \mathbb{R}^3) : \partial g/\partial \nu = 0\}$ is the generator of an analytic C_0-semigroup $\{e^{At}\}_{t\geq 0}$ on the Hilbert space H due to the Lumer-Phillips theorem [62]. By the fact that $H^1(\Omega) \hookrightarrow L^6(\Omega)$ is a continuous imbedding for space dimension $n \leq 3$ and by the Young's inequality, there is a constant $C_0 > 0$ such that

$$\|\varphi(u)\| \leq C_0(1 + \|u\|_{L^6}^3) \quad \text{and} \quad \|\psi(u)\| \leq C_0(1 + \|u\|_{L^4}^2) \quad \text{for } u \in H^1(\Omega).$$

Therefore, the nonlinear mapping

$$f(u, v, w) = \begin{pmatrix} \varphi(u) + v - w + J \\ \psi(u) - v, \\ q(u - c) - rw \end{pmatrix} : E \longrightarrow H \tag{2.11}$$

is a locally Lipschitz continuous mapping. We can simply write the column vector $g(t)$ as $(u(t, \cdot), v(t, \cdot), w(t, \cdot))$ and write $g_0 = (u_0, v_0, w_0)$.

The existence and uniqueness of the weak solutions local in time has been shown in Lemma

1.3.2 with Definition 1.3.1 in Section 1.3.

2.1.2 Global Existence and Dissipative Property in H

In this section, first we shall prove the global existence of all the weak solutions of the problem (2.9) in time and the existence of an absorbing set in H for the solution semiflow.

Theorem 2.1.1. *For any given initial state $g_0 = (u_0, v_0, w_0) \in H$, there exists a unique global weak solution in time, $g(t) = (u(t), v(t), w(t))$, $t \in [0, \infty)$, of the initial value problem* (2.9) *for the diffusive Hindmarsh-Rose equations* (2.1)–(2.3). *The weak solution turns out to be a strong solution on the interval* $(0, \infty)$.

Proof. Taking the L^2 inner-product $\langle(2.1), C_1 u(t)\rangle$ with an adjustable constant $C_1 > 0$, we get

$$\frac{C_1}{2}\frac{d}{dt}\|u\|^2 + C_1 d_1 \|\nabla u\|^2 = \int_\Omega C_1(au^3 - bu^4 + uv - uw + Ju)\,dx. \tag{2.12}$$

Taking the L^2 inner-products $\langle(2.2), v(t)\rangle$ and $\langle(2.3), w(t)\rangle$, by Young's inequality, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|v\|^2 + d_2\|\nabla v\|^2 \\
&= \int_\Omega (\psi(u)v - v^2)\,dx = \int_\Omega (\alpha v - \beta u^2 v - v^2)\,dx \\
&\le \int_\Omega \left(\alpha v + \frac{1}{2}(\beta^2 u^4 + v^2) - v^2\right) dx = \int_\Omega \left(\alpha v + \frac{1}{2}\beta^2 u^4 - \frac{1}{2}v^2\right) dx \\
&\le \int_\Omega \left(2\alpha^2 + \frac{1}{8}v^2 + \frac{1}{2}\beta^2 u^4 - \frac{1}{2}v^2\right) dx = \int_\Omega \left(2\alpha^2 + \frac{1}{2}\beta^2 u^4 - \frac{3}{8}v^2\right) dx
\end{aligned} \tag{2.13}$$

and

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|w\|^2 + d_3\|\nabla w\|^2 = \int_\Omega (q(u-c)w - rw^2)\,dx \\
&\le \int_\Omega \left(\frac{q^2}{2r}(u-c)^2 + \frac{1}{2}rw^2 - rw^2\right) dx \le \int_\Omega \left(\frac{q^2}{r}(u^2 + c^2) - \frac{1}{2}rw^2\right) dx.
\end{aligned} \tag{2.14}$$

Choose the scaling constant in (2.12) to be $C_1 = \frac{1}{b}(\beta^2 + 4)$ so that

$$\int_\Omega (-C_1 b u^4)\,dx + \int_\Omega (\beta^2 u^4)\,dx \le \int_\Omega (-4u^4)\,dx.$$

Then we estimate all the mixed product terms on the right-hand side of the above inequalities by using Young's inequality in an appropriate way as follows. In (2.12),

$$\int_\Omega C_1 a u^3\, dx \le \frac{3}{4}\int_\Omega u^4\, dx + \frac{1}{4}\int_\Omega (C_1 a)^4\, dx \le \int_\Omega u^4\, dx + (C_1 a)^4|\Omega|,$$
$$\int_\Omega C_1(uv - uw + Ju)\, dx \le \int_\Omega \left(2(C_1 u)^2 + \frac{1}{8}v^2 + \frac{(C_1 u)^2}{r} + \frac{1}{4}rw^2 + C_1 u^2 + C_1 J^2\right) dx,$$

where on the right-hand side of the second inequality we further treat the three terms involving u^2 as

$$\int_\Omega \left(2(C_1 u)^2 + \frac{(C_1 u)^2}{r} + C_1 u^2\right) dx \le \int_\Omega u^4\, dx + \left[C_1^2\left(2 + \frac{1}{r}\right) + C_1\right]^2 |\Omega|.$$

Then in (2.14),

$$\int_\Omega \frac{1}{r} q^2 u^2\, dx \le \int_\Omega \left(\frac{u^4}{2} + \frac{q^4}{2r^2}\right) dx \le \int_\Omega u^4\, dx + \frac{q^4}{r^2}|\Omega|.$$

Substitute the above term estimates into (2.12) and (2.14) and then sum up the three inequalities (2.12)–(2.14) to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(C_1\|u\|^2 + \|v\|^2 + \|w\|^2) + (C_1 d_1\|\nabla u\|^2 + d_2\|\nabla v\|^2 + d_3\|\nabla w\|^2) \\
&\le \int_\Omega C_1(au^3 - bu^4 + uv - uw + Ju)\, dx \\
&\quad + \int_\Omega \left(2\alpha^2 + \frac{1}{2}\beta^2 u^4 - \frac{3}{8}v^2\right) dx + \int_\Omega \left(\frac{q^2}{r}(u^2 + c^2) - \frac{1}{2}rw^2\right) dx \\
&\le \int_\Omega (3-4)u^4\, dx + \int_\Omega \left(\frac{1}{8} - \frac{3}{8}\right) v^2\, dx + \int_\Omega \left(\frac{1}{4} - \frac{1}{2}\right) rw^2\, dx \\
&\quad + |\Omega| \left((C_1 a)^4 + C_1 J^2 + \left[C_1^2\left(2 + \frac{1}{r}\right) + C_1\right]^2 + 2\alpha^2 + \frac{q^2c^2}{r} + \frac{q^4}{r^2}\right) \\
&= - \int_\Omega \left(u^4(t,x) + \frac{1}{4}v^2(t,x) + \frac{1}{4}rw^2(t,x)\right) dx + C_2|\Omega|
\end{aligned} \tag{2.15}$$

where $C_2 > 0$ is the constant given by

$$C_2 = (C_1 a)^4 + C_1 J^2 + \left[C_1^2\left(2 + \frac{1}{r}\right) + C_1\right]^2 + 2\alpha^2 + \frac{q^2c^2}{r} + \frac{q^4}{r^2}.$$

Set

$$d = 2\min\{d_1, d_2, d_3\}.$$

Then (2.15) yields the following uniform group estimate for all solutions,

$$\begin{aligned} &\frac{d}{dt}(C_1\|u(t)\|^2 + \|v(t)\|^2 + \|w(t)\|^2) + d(C_1\|\nabla u\|^2 + \|\nabla v\|^2 + \|\nabla w\|^2) \\ &+ \int_\Omega \left(2u^4(t,x) + \frac{1}{2}v^2(t,x) + \frac{1}{2}rw^2(t,x)\right) dx \le 2C_2|\Omega|, \end{aligned} \tag{2.16}$$

where $t \in I_{max} = [0, T_{max})$, the maximal time interval of solution existence. Since

$$2u^4 \ge \frac{1}{2}\left(C_1u^2 - \frac{C_1^2}{16}\right),$$

it follows from (2.16) that

$$\begin{aligned} &\frac{d}{dt}(C_1\|u(t)\|^2 + \|v(t)\|^2 + \|w(t)\|^2) + d(C_1\|\nabla u\|^2 + \|\nabla v\|^2 + \|\nabla w\|^2) \\ &+ \int_\Omega \frac{1}{2}\left(C_1u^2(t,x) + v^2(t,x) + rw^2(t,x)\right) dx \le \left(2C_2 + \frac{C_1^2}{32}\right)|\Omega|. \end{aligned}$$

Set $r_1 = \frac{1}{2}\min\{1, r\}$. Then we have

$$\begin{aligned} &\frac{d}{dt}(C_1\|u(t)\|^2 + \|v(t)\|^2 + \|w(t)\|^2) + d(C_1\|\nabla u\|^2 + \|\nabla v\|^2 + \|\nabla w\|^2) \\ &\quad + r_1(C_1\|u\|^2 + \|v\|^2 + \|w\|^2) \le \left(2C_2 + \frac{C_1^2}{32}\right)|\Omega|, \quad t \in [0, T_{max}). \end{aligned} \tag{2.17}$$

Apply the Gronwall inequality to

$$\frac{d}{dt}(C_1\|u(t)\|^2 + \|v(t)\|^2 + \|w(t)\|^2) + r_1(C_1\|u\|^2 + \|v\|^2 + \|w\|^2) \le \left(2C_2 + \frac{C_1^2}{32}\right)|\Omega|$$

and we obtain

$$C_1\|u(t)\|^2 + \|v(t)\|^2 + \|w(t)\|^2 \le e^{-r_1t}(C_1\|u_0\|^2 + \|v_0\|^2 + \|w_0\|^2) + M|\Omega| \tag{2.18}$$

for any $t \in [0, T_{max})$, where

$$M = \frac{1}{r_1}\left(2C_2 + \frac{C_1^2}{32}\right).$$

The estimate (2.18) shows that the weak solutions will never blow up at any finite time because it is uniformly bounded,

$$C_1\|u(t)\|^2 + \|v(t)\|^2 + \|w(t)\|^2 \le C_1\|u_0\|^2 + \|v_0\|^2 + \|w_0|^2 + M|\Omega|.$$

Therefore the weak solution of the initial value problem (2.9) exists globally in time for any initial data. The time interval of maximal existence is always $[0, \infty)$. □

The global existence and uniqueness of the weak solutions and their continuous dependence on the initial data enable us to define the solution semiflow of the diffusive Hindmarsh-Rose equations (2.1)–(2.3) on the space H as follows:

$$S(t) : g_0 \longmapsto g(t, g_0) = (u(t,\cdot), v(t,\cdot), w(t,\cdot)), \quad g_0 = (u_0, v_0, w_0) \in H, \quad t \ge 0,$$

where $g(t, g_0)$ is the weak solution with $g(0) = g_0$. We call this semiflow $\{S(t)\}_{t\ge 0}$ the Hindmarsh-Rose semiflow associated with (2.9).

Theorem 2.1.2. *There exists an absorbing set in the space H for the Hindmarsh-Rose semiflow $\{S(t)\}_{t\ge 0}$, which is the bounded ball*

$$B_H = \{g \in H : \|g\|^2 \le K_1\} \tag{2.19}$$

where $K_1 = \frac{M|\Omega|}{\min\{C_1, 1\}} + 1$.

Proof. From the uniform estimate (2.18) in Theorem 2.1.1 we see that

$$\limsup_{t\to\infty} \left(\|u(t)\|^2 + \|v(t)\|^2 + \|w(t)\|^2\right) < K_1 = \frac{M|\Omega|}{\min\{C_1, 1\}} + 1 \tag{2.20}$$

for all weak solutions of (2.9) with any initial state $g_0 \in H$. Moreover, for any given bounded set

$B = \{g \in H : \|g\| \leq R\}$ in H, there exists a finite time

$$T_0(B) = \frac{1}{r_1} \log^+(R^2 \max\{C_1, 1\}) \tag{2.21}$$

such that $\|u(t)\|^2 + \|v(t)\|^2 + \|w(t)\|^2 < K_1$ for all $t \geq T_0(B)$ and for all $g_0 \in B$. Thus, by Definition 1.3.4, the bounded ball B_H is an absorbing set for the Hindmarsh-Rose semiflow in the phase space H. □

Corollary 2.1.3. *There exists an absorbing set in the space H for the semiflow generated by the partly diffusive Hindmarsh-Rose equations* (2.7) *and* (2.8)*, respectively.*

Proof. The proof of Lemma 2.1.1 is still valid without the terms of $\|\nabla v\|^2$ and $\|\nabla w\|^2$. Instead of (8.16) we have the inequality

$$\begin{aligned} &\frac{d}{dt}(C_1\|u(t)\|^2 + \|v(t)\|^2 + \|w(t)\|^2) + d_1 C_1 \|\nabla u\|^2 \\ &+ r_1(C_1\|u\|^2 + \|v\|^2 + \|w\|^2) \leq \left(2C_2 + \frac{C_1^2}{32}\right)|\Omega|, \;\; t \in [0, T_{max}). \end{aligned} \tag{2.22}$$

It leads to the same differential inequality (2.18). Thus the same result as in Theorem 2.1.2 holds for the semiflow generated by (2.7) and by (2.8), respectively. □

2.1.3 Absorbing Property of the Hindmarsh-Rose Semiflow in Spaces L^4 and L^6

The following theorem plays a key role to show the asymptotic compactness of the solution semiflow generated by the partly diffusive Hindmarsh-Rose equations (2.7) and (2.8) in Section 2.5 and the H^2– regularity of the global attractor in Section 2.4.

Theorem 2.1.4. *For $p = 2$ and 3, there exists a constant $K_p > 0$ such that the absorbing property with respect to the space $L^{2p}(\Omega, \mathbb{R}^3)$,*

$$\limsup_{t\to\infty} \left(\|u(t)\|^{2p}_{L^{2p}} + \|v(t)\|^{2p}_{L^{2p}} + \|w(t)\|^{2p}_{L^{2p}}\right) < K_p \tag{2.23}$$

is satisfied by every weak solution $S(t)g_0 = g(t, g_0) = (u(t), v(t), w(t))$ of the diffusive Hindmarsh-Rose equations (2.9) *for any initial state $g_0 \in H$.*

Proof. By Lemma 1.3.2 and Theorem 2.1.1, for any given weak solution $(u(t), v(t), w(t))$ of the Hindmarsh-Rose evolutionary equation (2.9), there exists a time $t_0 \in (0, 1)$ such that

$$S(t_0)g_0 = g(t_0, g_0) \in E = H^1(\Omega, \mathbb{R}^3) \subset L^6(\Omega, \mathbb{R}^3) \subset L^4(\Omega, \mathbb{R}^3).$$

According to Lemma 1.3.2 the weak solution $S(t)g_0$ becomes a strong solution on $[t_0, \infty)$ and satisfies

$$S(\cdot)g_0 \in C([t_0, \infty), E) \subset C([t_0, \infty), L^6(\Omega, \mathbb{R}^3)) \subset C([t_0, \infty), L^4(\Omega, \mathbb{R}^3)).$$

This parabolic regularity with a reset t_0 as the initial time point enables us to simply assume that the initial state $g_0 \in E \subset L^6(\Omega, \mathbb{R}^3) \subset L^4(\Omega, \mathbb{R}^3))$ and, by the bootstrap argument for the resulting strong solutions, we can even assume that $g_0 \in H^2(\Omega, \mathbb{R}^3) \subset L^8(\Omega, \mathbb{R}^3)$ in proving the longtime dynamical property (2.23).

Step 1. Common estimates.

For $p = 2$ and $p = 3$, we take the L^2 inner-product $\langle(2.1), u^{2p-1}\rangle$ to get

$$\begin{aligned} &\frac{1}{2p}\frac{d}{dt}\|u(t)\|^{2p}_{L^{2p}} + d_1(2p-1)\|u^{p-1}\nabla u\|^2_{L^2} \\ &= \int_\Omega [au^{2p+1} - bu^{2p+2} + u^{2p-1}(v - w - J)]\, dx. \end{aligned} \tag{2.24}$$

On the right-hand side of the inequality (2.24), by Young's inequality and

$$\frac{2p+2}{3} \le 2p \quad \text{for} \quad p = 2,\ 3,$$

we have

$$\begin{aligned} &au^{2p+1} + u^{2p-1}(v - w + J) \\ &\le \frac{b}{4}u^{2p+2} + C_b\left(a^{2p+2} + (|v(t)|^{\frac{2p+2}{3}} + |w(t)|^{\frac{2p+2}{3}} + J^{\frac{2p+2}{3}}\right) \\ &\le \frac{1}{4}bu^{2p+2} + C_b\left(a^{2p+2} + C_{p,r} + J^{\frac{2p+2}{3}}\right) + \frac{1}{4}|v(t)|^{2p} + \frac{r}{4}|w(t)|^{2p} \end{aligned} \tag{2.25}$$

where $C_b > 0$ is a constant depending on the parameter b and $C_{p,r} > 0$ is the constant depending on p and the parameter r, which is generated from the inequality

$$|v(t,x)|^{\frac{2p+2}{3}} + |w(t,x)|^{\frac{2p+2}{3}} \le C_{p,r} + \frac{1}{4}|v(t,x)|^{2p} + \frac{r}{4}|w(t,x)|^{2p}.$$

From (2.24) and (2.25) it follows that

$$\begin{aligned} &\frac{1}{2p}\frac{d}{dt}\|u(t)\|_{L^{2p}}^{2p} + d_1(2p-1)\|u^{p-1}\nabla u\|_{L^2}^2 \\ =& \int_\Omega [au^{2p+1} - bu^{2p+2} + u^{2p-1}(v-w-J)]\, dx \\ \le& C_b\left(a^{2p+2} + C_{p,r} + J^{\frac{2p+2}{3}}\right)|\Omega| - \frac{3}{4}\int_\Omega bu^{2p+2}\, dx + \frac{1}{4}\int_\Omega (v^{2p} + rw^{2p})\, dx. \end{aligned} \tag{2.26}$$

The main controlling term on the right-hand side is

$$-\frac{3}{4}\int_\Omega bu^{2p+2}\, dx$$

. Then taking the L^2 inner-product $\langle (2.2), v^{2p-1}\rangle$, we obtain

$$\begin{aligned} &\frac{1}{2p}\frac{d}{dt}\|v(t)\|_{L^{2p}}^{2p} + d_2(2p-1)\|v^{p-1}\nabla v\|_{L^2}^2 \\ =&\frac{1}{2p}\frac{d}{dt}\|v(t)\|_{L^{2p}}^{2p} + \frac{1}{p^2}d_2(2p-1)\|\nabla(v^p)\|_{L^2}^2 \\ =& \int_\Omega [\psi(u)v^{2p-1} - v^{2p}]\, dx = \int_\Omega [\alpha v^{2p-1} - \beta u^2 v^{2p-1} - v^{2p}]\, dx. \end{aligned} \tag{2.27}$$

It is challenging to control the middle term on the right-hand side,

$$-\int_\Omega \beta u^2 v^{2p-1}\, dx. \tag{2.28}$$

In the next two steps, we shall exploit the Gagliardo–Nirenberg inequality (1.7) and the absorbing result in Theorem 2.1.2 to handle this issue for $p = 2$ and $p = 3$. First use the Young's inequality (1.8) and get

$$\begin{aligned} -\int_\Omega \beta u^2 v^{2p-1}\, dx &\le \frac{1}{8}\int_\Omega bu^{2p+2}\, dx + C_{b,\beta}\int_\Omega v^{(2p-1)(1+\frac{1}{p})}\, dx \\ &= \frac{1}{8}\int_\Omega bu^{2p+2}\, dx + C_{b,\beta}\int_\Omega v^{2p+1-\frac{1}{p}}\, dx \end{aligned} \tag{2.29}$$

Step 2. Prove (2.23) for $p = 2$.

For $p = 2$, the second term on the right-hand side of (2.29) is

$$C_{b,\beta} \int_\Omega v^{5-\frac{1}{2}}\, dx \leq \varepsilon_1 \int_\Omega v^5\, dx + C_{b,\beta,\varepsilon_1} |\Omega|, \tag{2.30}$$

where $\varepsilon_1 > 0$ is a small constant to be chosen and the constant $C_{b,\beta,\varepsilon_1}$ depends on $C_{b,\beta}$ and ε_1. Apply the Gagliardo–Nirenberg inequality (1.7) to the spaces

$$L^1(\Omega) \hookrightarrow L^{2.5}(\Omega) \hookrightarrow H^1(\Omega) \tag{2.31}$$

combined with the Poincaré inequality that there are constants $c_1, c_2 > 0$ only depending on the space domain and its dimension such that

$$\|h\|_{H^1} \leq c_1 \|\nabla h\| + c_2 \|h\|_{L^1}, \quad \text{for any } h \in H^1(\Omega).$$

We see that there exists a constant $C > 0$ and then constants $\eta_0, \eta_1 > 0$ such that

$$\begin{aligned} \int_\Omega v^5\, dx &= \|v^2\|_{L^{2.5}}^{2.5} \leq C \left(\|v^2\|_{H^1}^{\theta} \, \|v^2\|_{L^1}^{1-\theta} \right)^{2.5} \\ &\leq C \left[(c_1 \|\nabla (v^2)\| + c_2 \|v^2\|_{L^1})^{\theta} \, \|v^2\|_{L^1}^{1-\theta} \right]^{2.5} \\ &\leq \eta_1 \left[\|\nabla (v^2)\|^{\theta} \|v^2\|_{L^1}^{1-\theta} \right]^{2.5} + \eta_0 \, \|v^2\|_{L^1}^{2.5} \end{aligned} \tag{2.32}$$

where the fractional exponent θ in (1.7) is calculated by

$$-\frac{3}{2.5} = \theta \left(1 - \frac{3}{2} \right) + (1-\theta)(-3) \quad \text{so that} \quad \theta = \frac{18}{25}, \;\; 1 - \theta = \frac{7}{25}.$$

Then for any weak solution $g(t) = (u(t), v(t), w(t))$ of the Hindmarsh-Rose evolutionary equation (2.9) with $g_0 \in H$, there is a finite time $T(g_0)$ such that for $t \geq T(g_0)$,

$$\begin{aligned} &\int_\Omega v^5\, dx \leq \eta_1 \|\nabla (v^2)\|^{18/10} \, \|v^2\|_{L^1}^{7/10} + \eta_0 \, \|v^2\|_{L^1}^{2.5} \\ &\leq \eta_1 \|\nabla (v^2)\|^2 + \eta_1 \|v^2\|_{L^1}^{7} + \eta_0 \, \|v^2\|_{L^1}^{2.5} \leq \eta_1 \|\nabla (v^2)\|^2 + \eta_1 K_1^7 + \eta_0 K_1^{2.5}, \end{aligned} \tag{2.33}$$

where the constant K_1 is given in (2.19) and the inequality (2.33) follows from Theorem 2.1.2.

Substitute (2.33) into (2.30) and then into (2.29). We obtain

$$
\begin{aligned}
&-\int_\Omega \beta u^2 v^{2p-1}\, dx \\
&\le \frac{1}{8}\int_\Omega b u^{2p+2}\, dx + \varepsilon_1 \int_\Omega v^5\, dx + C_{b,\beta,\,\varepsilon_1}|\Omega| \\
&\le \frac{1}{8}\int_\Omega b u^{2p+2}\, dx + \varepsilon_1 \left(\eta_1 \|\nabla (v^2)\|^2 + \eta_1 K_1^7 + \eta_0 K_1^{2.5}\right) + C_{b,\beta,\varepsilon_1}|\Omega|.
\end{aligned} \tag{2.34}
$$

Now we choose

$$
\varepsilon_1 = \frac{1}{2\eta_1\, p^2}\, d_2(2p-1) = \frac{3\, d_2}{8\,\eta_1}, \quad \text{for } p = 2,
$$

so that

$$
\varepsilon_1 \eta_1 \|\nabla(v^2)\|^2 \le \frac{1}{2p^2} d_2 (2p-1) \|\nabla (v^p)\|^2_{L^2}. \tag{2.35}
$$

Put together (2.27) for $p = 2$ with the estimates (2.34) and (8.2.1). We obtain

$$
\begin{aligned}
&\frac{1}{2p}\frac{d}{dt}\|v(t)\|^{2p}_{L^{2p}} + \frac{1}{2} d_2 (2p-1) \|v^{p-1}\nabla v\|^2_{L^2} \\
\le &\int_\Omega [\alpha v^{2p-1} - v^{2p}]\, dx + \frac{1}{8}\int_\Omega b u^{2p+2}\, dx + \varepsilon_1(\eta_1 K_1^7 + \eta_0 K_1^{2.5}) + C_{b,\beta,\varepsilon_1}|\Omega| \\
\le &\frac{1}{8}\int_\Omega b u^{2p+2}\, dx - \frac{1}{2}\int_\Omega v^{2p}\, dx + \varepsilon_1(\eta_1 K_1^7 + \eta_0 K_1^{2.5}) + (C_2(\alpha)\alpha^{2p} + C_{b,\beta,\varepsilon_1})|\Omega|
\end{aligned} \tag{2.36}
$$

for $t \ge T(g_0)$, where $\alpha v^{2p-1} \le \frac{1}{2} v^{2p} + C_2(\alpha)\alpha^{2p}$ for $p = 2$ and $C_2(\alpha) > 0$ is a constant.
Next take the L^2 inner-product $\langle (2.3), w^{2p-1}\rangle$ to obtain

$$
\begin{aligned}
&\frac{1}{2p}\frac{d}{dt}\|w(t)\|^{2p}_{L^{2p}} + d_3(2p-1)\|w^{p-1}\nabla w\|^2_{L^2} \\
= &\int_\Omega [q(u-c)w^{2p-1} - r w^{2p}]\, dx \le \int_\Omega \left(C|q(u-c)|^{2p} + \frac{1}{4} r w^{2p} - r w^{2p}\right) dx \\
\le &\int_\Omega \left(C(u^{2p} + c^{2p}) - \frac{3}{4} r w^{2p}\right) dx \le \int_\Omega \left(C_3 + \frac{1}{8} b u^{2p+2} + C_3\, c^{2p} - \frac{3}{4} r w^{2p}\right) dx
\end{aligned} \tag{2.37}
$$

where $C > 0$ is a constant from the expansion of $|q(u-c)|^{2p}$ and $C_3 > 0$ is from raising the power $u^{2p} = u^4$ to $u^{2p+2} = u^6$ in the last step.
Then we sum up the estimates (2.26), (2.36) and (2.37) for $p = 2$:

$$
\begin{aligned}
&\frac{1}{2p}\frac{d}{dt}(\|u(t)\|^{2p}_{L^{2p}} + \|v(t)\|^{2p}_{L^{2p}} + \|w(t)\|^{2p}_{L^{2p}}) \\
&\quad + \frac{1}{2}(2p-1)(d_1\|u^{p-1}\nabla u\|^2_{L^2} + d_2\|v^{p-1}\nabla v\|^2_{L^2} + d_3\|w^{p-1}\nabla w\|^2_{L^2}) \\
&\le C_b\left(a^{2p+2} + C_{p,r} + J^{\frac{2p+2}{3}}\right)|\Omega| - \frac{3}{4}\int_\Omega bu^{2p+2}\,dx + \frac{1}{4}\int_\Omega (v^{2p} + rw^{2p})\,dx \\
&\quad + \frac{1}{8}\int_\Omega bu^{2p+2}\,dx - \frac{1}{2}\int_\Omega v^{2p}\,dx + \varepsilon_1(\eta_1 K_1^7 + \eta_0 K_1^{2.5}) + (C_2(\alpha)\alpha^{2p} + C_{b,\beta,\varepsilon_1})|\Omega| \\
&\quad + \int_\Omega \left(C_3 + \frac{1}{8}bu^{2p+2} + C_3\,c^{2p} - \frac{3}{4}rw^{2p}\right)dx \\
&\le C_b\left(a^{2p+2} + C_{p,r} + J^{\frac{2p+2}{3}}\right)|\Omega| + (C_2(\alpha)\alpha^{2p} + C_{b,\beta,\varepsilon_1} + C_3(1+c^{2p}))|\Omega| \\
&\quad + \varepsilon_1(\eta_1 K_1^7 + \eta_0 K_1^{2.5}) - \left(\int_\Omega \frac{1}{2}bu^{2p+2}\,dx + \frac{1}{4}\int_\Omega v^{2p}\,dx + \frac{1}{2}\int_\Omega rw^{2p}dx\right), \quad t \ge T(g_0).
\end{aligned}
$$

It follows that, for $p = 2$,

$$
\begin{aligned}
&\frac{1}{2p}\frac{d}{dt}(\|u(t)\|^{2p}_{L^{2p}} + \|v(t)\|^{2p}_{L^{2p}} + \|w(t)\|^{2p}_{L^{2p}}) \\
&\le C_4|\Omega| + \varepsilon_1(\eta_1 K_1^7 + \eta_0 K_1^{2.5}) - \left[\int_\Omega \frac{1}{2}bu^{2p+2}\,dx + \int_\Omega \frac{1}{4}v^{2p}\,dx + \int_\Omega \frac{1}{2}rw^{2p}\,dx\right] \\
&\le C_4|\Omega| + \varepsilon_1(\eta_1 K_1^7 + \eta_0 K_1^{2.5}) - \frac{1}{4}\left[\int_\Omega bu^{2p}\,dx - \frac{b}{p+1}|\Omega| + \int_\Omega (v^{2p} + rw^{2p})\,dx\right] \\
&\le \left(C_4 + \frac{b}{4(p+1)}\right)|\Omega| + \varepsilon_1(\eta_1 K_1^7 + \eta_0 K_1^{2.5}) - \frac{1}{4}\int_\Omega (bu^{2p} + v^{2p} + rw^{2p})\,dx,
\end{aligned}
\tag{2.38}
$$

for $t \ge T(g_0),\ g_0 \in H$, where

$$
C_4 = C_b\left(a^{2p+2} + C_{p,r} + J^{\frac{2p+2}{3}}\right) + C_2(\alpha)\alpha^{2p} + C_{b,\beta,\varepsilon_1} + C_3(1+c^{2p}).
$$

Then we can apply the Gronwall inequality to the following differential inequality reduced from (2.38) by moving the negative integral term to the left-hand side,

$$
\begin{aligned}
&\frac{d}{dt}(\|u(t)\|^4_{L^4} + \|v(t)\|^4_{L^4} + \|w(t)\|^4_{L^4}) \\
&+ \min\{b, r, 1\}(\|u(t)\|^4_{L^4} + \|v(t)\|^4_{L^4} + \|w(t)\|^4_{L^4}) \\
&\le 4\left(C_4 + \frac{b}{12}\right)|\Omega| + 4\varepsilon_1(\eta_1 K_1^7 + \eta_0 K_1^{2.5}), \quad \text{for } t \ge T(g_0).
\end{aligned}
\tag{2.39}
$$

Hence we obtain the bounded estimate that, for any $g_0 \in H$ and any $t \geq T(g_0)$,

$$\begin{aligned} \|(u(t), v(t), w(t))\|_{L^4}^4 \leq e^{-\lambda(t-T(g_0))} \|g(T(g_0), g_0)\|_{L^4}^4 \\ + M_1|\Omega| + 4\lambda^{-1}\varepsilon_1 \left(\eta_1 K_1^7 + \eta_0 K^{2.5}\right), \end{aligned} \tag{2.40}$$

where K_1 is shown in (2.19) and

$$\lambda = \min\{b, r, 1\} \quad \text{and} \quad M_1 = \frac{1}{\lambda}\left(4C_4 + \frac{b}{3}\right).$$

Let $t \to \infty$ in (2.40). Then the absorbing property (2.23) is proved for $p = 2$,

$$\limsup_{t\to\infty} \left(\|u(t)\|_{L^4}^4 + \|v(t)\|_{L^4}^4 + \|w(t)\|_{L^4}^4\right) < K_2 \tag{2.41}$$

and

$$K_2 = M_1|\Omega| + 4\lambda^{-1}\varepsilon_1(\eta_1 K_1^7 + \eta_0 K^{2.5}) + 1. \tag{2.42}$$

Step 3. Prove (2.23) *for* $p = 3$ *by means of bootstrap argument.*

For $p = 3$, the second term on the right-hand side of (2.29) is

$$C_{b,\beta} \int_\Omega v^{7-\frac{1}{3}}\, dx \leq \varepsilon_2 \int_\Omega v^7\, dx + C_{b,\beta,\varepsilon_2}|\Omega|, \tag{2.43}$$

where $\varepsilon_2 > 0$ is a small constant to be chosen and the constant $C_{b,\beta,\varepsilon_2}$ depends on $C_{b,\beta}$ and ε_2. Similar to (2.32), we use the Gagliardo–Nirenberg inequality (1.7) for the interpolation of spaces

$$L^1(\Omega) \hookrightarrow L^{7/3}(\Omega) \hookrightarrow H^1(\Omega) \tag{2.44}$$

and the Poincaré inequality to claim that there exists constants $\eta_2 > 0$ and $\tilde{\eta}_0 > 0$ such that

$$\begin{aligned} \int_\Omega v^7\, dx = \|v^3\|_{L^{7/3}}^{7/3} &\leq C\left(\|v^3\|_{H^1}^\theta \|v^3\|_{L^1}^{1-\theta}\right)^{7/3} \\ &\leq C\left[(c_1\|\nabla(v^3)\| + c_2\|v^3\|_{L^1})^\theta \|v^3\|_{L^1}^{1-\theta}\right]^{7/3} \\ &\leq \eta_2\left[\|\nabla(v^3)\|^\theta \|v^3\|_{L^1}^{1-\theta}\right]^{7/3} + \tilde{\eta}_0\, \|v^3\|_{L^1}^{7/3} \end{aligned} \tag{2.45}$$

where the fractional exponent θ is calculated by

$$-\frac{3}{7/3} = \theta\left(1 - \frac{3}{2}\right) + (1-\theta)(-3) \quad \text{so that} \quad \theta = \frac{24}{35}, \ \ 1-\theta = \frac{11}{35}.$$

Note that by Hölder inequality,

$$\|v^3\|_{L^1} = \int_\Omega v^3\, dx \le \|v^3\|_{L^{4/3}} |\Omega|^{1/4} = \|v\|_{L^4}^3 |\Omega|^{1/4}.$$

By (2.41) we just proved in Step 2, for any $g_0 \in H$, there is $\tilde{T}(g_0) > 0$ such that

$$\|v(t)\|_{L^4}^3 |\Omega|^{1/4} \le K_2^{3/4} |\Omega|^{1/4}, \qquad \text{for} \quad t \ge \tilde{T}(g_0).$$

Then we have

$$\begin{aligned} \int_\Omega v^7\, dx &\le \eta_2 \|\nabla(v^3)\|^{8/5} \, \|v^3\|_{L^1}^{11/15} + \tilde{\eta}_0 \, \|v^3\|_{L^1}^{7/3} \\ &\le \eta_2 \|\nabla(v^3)\|^2 + \eta_2 \|v^3\|_{L^1}^{11/3} + \tilde{\eta}_0 \, \|v^3\|_{L^1}^{7/3} \\ &\le \eta_2 \|\nabla(v^3)\|^2 + \eta_2 K_2^{11/4} |\Omega|^{11/12} + \tilde{\eta}_0 \, K_2^{7/4} |\Omega|^{7/12}, \quad \text{for } \ t \ge \tilde{T}(g_0). \end{aligned} \tag{2.46}$$

Substitute (2.46) into (2.43) and then into (2.29) for $p = 3$. We obtain

$$\begin{aligned} & -\int_\Omega \beta u^2 v^{2p-1}\, dx \\ \le\ & \frac{1}{8}\int_\Omega b u^{2p+2}\, dx + \varepsilon_2 \int_\Omega v^7\, dx + C_{b,\beta,\varepsilon_2} |\Omega| \\ \le\ & \frac{1}{8}\int_\Omega b u^{2p+2}\, dx + \varepsilon_2 (\eta_2 \|\nabla(v^3)\|^2 + \eta_2 K_2^{11/4} |\Omega|^{11/12} + \tilde{\eta}_0 \, K_2^{7/4} |\Omega|^{7/12}) + C_{b,\beta,\varepsilon_2} |\Omega|. \end{aligned} \tag{2.47}$$

Now we choose

$$\varepsilon_2 = \frac{1}{2\eta_2\, p^2} d_2(2p-1) = \frac{5\, d_2}{18\, \eta_2} \quad \text{for } \ p = 3,$$

so that

$$\varepsilon_2 \eta_2 \|\nabla(v^3)\|^2 \le \frac{1}{2p^2} d_2(2p-1) \|\nabla(v^p)\|_{L^2}^2. \tag{2.48}$$

Put together (2.27) for $p = 3$ with the estimates (2.47) and (2.48). We obtain

$$\begin{aligned}
&\frac{1}{2p}\frac{d}{dt}\|v(t)\|_{L^{2p}}^{2p} + \frac{1}{2}d_2(2p-1)\|v^{p-1}\nabla v\|_{L^2}^2 \\
\le &\int_\Omega [\alpha v^{2p-1} - v^{2p}]\, dx + \frac{1}{8}\int_\Omega bu^{2p+2}\, dx + C_{b,\beta,\varepsilon_2}|\Omega| \\
&+ \varepsilon_2(\eta_2 K_2^{11/4}|\Omega|^{11/12} + \tilde{\eta}_0\, K_2^{7/4}|\Omega|^{7/12}) \\
\le &\frac{1}{8}\int_\Omega bu^{2p+2}\, dx - \frac{1}{2}\int_\Omega v^{2p}\, dx + (C_5(\alpha)\alpha^{2p} + C_{b,\beta,\varepsilon_2})|\Omega| \\
&+ \varepsilon_2(\eta_2 K_2^{11/4}|\Omega|^{11/12} + \tilde{\eta}_0\, K_2^{7/4}|\Omega|^{7/12}), \quad t \ge \tilde{T}(g_0),
\end{aligned} \tag{2.49}$$

where we used $\alpha v^{2p-1} \le \frac{1}{2}v^{2p} + C_5(\alpha)\alpha^{2p}$ for $p = 3$ and $C_5(\alpha) > 0$ is a constant.

Sum up the same (2.26), (2.37) and the new estimate (2.49) for the second component $v(t,x)$ with $p = 3$. We obtain

$$\begin{aligned}
&\frac{1}{2p}\frac{d}{dt}(\|u(t)\|_{L^{2p}}^{2p} + \|v(t)\|_{L^{2p}}^{2p} + \|w(t)\|_{L^{2p}}^{2p}) \\
\le &C_b\left(a^{2p+2} + C_{p,r} + J^{\frac{2p+2}{3}}\right)|\Omega| - \frac{3}{4}\int_\Omega bu^{2p+2}\, dx + \frac{1}{4}\int_\Omega (v^{2p} + rw^{2p})\, dx \\
&+ \frac{1}{8}\int_\Omega bu^{2p+2}\, dx - \frac{1}{2}\int_\Omega v^{2p}\, dx + (C_5(\alpha)\alpha^{2p} + C_{b,\beta,\varepsilon_2})|\Omega| \\
&+ \varepsilon_2(\eta_2 K_2^{11/4}|\Omega|^{11/12} + \tilde{\eta}_0\, K_2^{7/4}|\Omega|^{7/12}) + \int_\Omega \left(C_3 + \frac{1}{8}\, bu^{2p+2} + C_3 c^{2p} - \frac{3}{4}\, rw^{2p}\right) dx \\
\le &C_b\left(a^{2p+2} + C_{p,r} + J^{\frac{2p+2}{3}}\right)|\Omega| + (C_5(\alpha)\alpha^{2p} + C_{b,\beta,\varepsilon_1} + C_3(1 + c^{2p}))|\Omega| \\
&+ \varepsilon_2(\eta_2 K_2^{11/4}|\Omega|^{11/12} + \tilde{\eta}_0\, K_2^{7/4}|\Omega|^{7/12}) - \int_\Omega \left(\frac{1}{2}bu^{2p+2} + \frac{1}{4}v^{2p} + \frac{1}{2}rw^{2p}\right) dx
\end{aligned}$$

for $t \ge \widetilde{T}(g_0)$. It follows that

$$\begin{aligned}
&\frac{1}{2p}\frac{d}{dt}(\|u(t)\|_{L^{2p}}^{2p} + \|v(t)\|_{L^{2p}}^{2p} + \|w(t)\|_{L^{2p}}^{2p}) \\
\le &C_6|\Omega| + \varepsilon_2(\eta_2 K_2^{11/4}|\Omega|^{11/12} + \tilde{\eta}_0\, K_2^{7/4}|\Omega|^{7/12}) - \int_\Omega \left(\frac{1}{2}bu^{2p+2} + \frac{1}{4}v^{2p} + \frac{1}{2}rw^{2p}\right) dx \\
\le &C_6|\Omega| + \varepsilon_2(\eta_2 K_2^{11/4}|\Omega|^{11/12} + \tilde{\eta}_0\, K_2^{7/4}|\Omega|^{7/12}) - \frac{1}{4}\left[\int_\Omega (bu^{2p} + v^{2p} + rw^{2p})\, dx - \frac{b}{p+1}|\Omega|\right] \\
\le &\left(C_6 + \frac{b}{16}\right)|\Omega| + \varepsilon_2(\eta_2 K_2^{11/4}|\Omega|^{11/12} + \tilde{\eta}_0\, K_2^{7/4}|\Omega|^{7/12}) - \frac{1}{4}\int_\Omega (bu^{2p} + v^{2p} + rw^{2p})\, dx,
\end{aligned}$$

for $t \geq \widetilde{T}(g_0),\ g_0 \in H$, where

$$C_6 = C_b \left(a^{2p+2} + C_{p,r} + J^{\frac{2p+2}{3}} \right) + C_5(\alpha)\alpha^{2p} + C_{b,\beta,\varepsilon_1} + C_3(1 + c^{2p}).$$

Moving the negative integral term in the above inequality to the left-hand side, we end up with the following differential inequality for $p = 3$,

$$\begin{aligned}
&\frac{d}{dt}(\|u(t)\|_{L^{2p}}^{2p} + \|v(t)\|_{L^{2p}}^{2p} + \|w(t)\|_{L^{2p}}^{2p}) \\
&+ \frac{p}{2}\min\{b, r, 1\}(\|u(t)\|_{L^{2p}}^{2p} + \|v(t)\|_{L^{2p}}^{2p} + \|w(t)\|_{L^{2p}}^{2p}) \\
&\leq \left(6C_6 + \frac{3b}{8}\right)|\Omega| + 6\varepsilon_2(\eta_2 K_2^{11/4}|\Omega|^{11/12} + \tilde{\eta}_0\, K_2^{7/4}|\Omega|^{7/12}), \quad \text{for } \ t \geq \widetilde{T}(g_0).
\end{aligned} \tag{2.50}$$

Apply the Gronwall inequality to (2.50). It holds that for $p = 3$,

$$\begin{aligned}
\|(u(t), v(t), w(t))\|_{L^6}^6 \leq e^{-\lambda(t - T(g_0))}\|g(\tilde{T}(g_0), g_0)\|_{L^6}^6 + M_2\,|\Omega| \\
+ 6\lambda^{-1}\varepsilon_2(\eta_2 K_2^{11/4}|\Omega|^{11/12} + \tilde{\eta}_0\, K_2^{7/4}|\Omega|^{7/12})
\end{aligned} \tag{2.51}$$

for $t \geq \widetilde{T}(g_0),\ g_0 \in H$. Here

$$\lambda = \frac{3}{2}\min\{b, r, 1\} \quad \text{and} \quad M_2 = \frac{1}{\lambda}\left(6\, C_6 + \frac{3b}{8}\right).$$

Let $t \to \infty$. Then the absorbing property (2.23) is proved for the case $p = 3$, namely,

$$\limsup_{t\to\infty}\,(\|u(t)\|_{L^6}^6 + \|v(t)\|_{L^6}^6 + \|w(t)\|_{L^6}^6) < K_3$$

and

$$K_3 = M_2|\Omega| + 6\lambda^{-1}\varepsilon_2(\eta_2 K_2^{11/4}|\Omega|^{11/12} + \tilde{\eta}_0\, K_2^{7/4}|\Omega|^{7/12}) + 1. \tag{2.52}$$

The proof is completed. □

2.2 Asymptotic Compactness and the Existence of Global Attractor

In this section, we show that the Hindmarsh-Rose semiflow $\{S(t)\}_{t\geq 0}$ is asymptotically compact and then reach the main result on the existence of a global attractor for this dynamical system generated by the diffusive Hindmarsh-Rose equations.

Theorem 2.2.1. *For any given bounded set $B \in H$, there exists a finite time $T_1(B) > 0$ such that for any initial state $g_0 = (u_0, v_0, w_0) \in B$, the weak solution $g(t) = S(t)g_0 = (u(t), v(t), w(t))$ of the initial value problem* (2.9) *satisfies*

$$\|(u(t), v(t), w(t))\|_E^2 \leq Q_1, \quad \textit{for} \;\; t \geq T_1(B) \tag{2.53}$$

where $Q_1 > 0$ is a constant depending only on K_1 in (2.19).

Proof. Take the L^2 inner-product $\langle$(2.1), $-\Delta u(t)\rangle$ to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2 + d_1\|\Delta u\|^2 = \int_\Omega (-au^2\Delta u - 3bu^2|\nabla u|^2 - v\Delta u + w\Delta u - J\Delta u)\, dx \\
&\leq \int_\Omega \left(\frac{2v^2}{d_1} + \frac{d_1}{8}|\Delta u|^2 + \frac{2w^2}{d_1} + \frac{d_1}{8}|\Delta u|^2 + \frac{2J^2}{d_1} + \frac{d_1}{8}|\Delta u|^2 + \frac{2a^2u^4}{d_1} + \frac{d_1}{8}|\Delta u|^2\right) dx \\
&- \int_\Omega 3bu^2|\nabla u|^2\, dx.
\end{aligned}$$

It follows that

$$\frac{d}{dt}\|\nabla u\|^2 + d_1\|\Delta u\|^2 + 6b\|u\nabla u\|^2 \leq \frac{4}{d_1}\|v\|^2 + \frac{4}{d_1}\|w\|^2 + \frac{4J^2}{d_1}|\Omega| + \frac{4a^2}{d_1}\|u\|_{L^4}^4. \tag{2.54}$$

Next take the L^2 inner-product $\langle$(2.2), $-\Delta v(t)\rangle$ to get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla v\|^2 + d_2\|\Delta v\|^2 = \int_\Omega (-\alpha\Delta v + \beta u^2\Delta v - |\nabla v|^2)\, dx \\
&\leq \int_\Omega \left(\frac{\alpha^2}{d_2} + \frac{d_2}{4}|\Delta v|^2 + \frac{\beta^2u^4}{d_2} + \frac{d_2}{4}|\Delta v|^2\right) dx - \|\nabla v\|^2.
\end{aligned}$$

It follows that

$$\frac{d}{dt}\|\nabla v\|^2 + d_2\|\Delta v\|^2 + 2\|\nabla v\|^2 \leq \frac{2\alpha^2}{d_2}|\Omega| + \frac{2\beta^2}{d_2}\|u\|_{L_4}^4. \tag{2.55}$$

Then taking the L_2 inner-product $\langle(2.3), -\Delta w(t)\rangle$, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla w\|^2 + d_3\|\Delta w\|^2 &= \int_\Omega (qc\Delta w - qu\Delta w - r|\nabla w|^2)\, dx \\ \le \int_\Omega \left(\frac{q^2c^2}{d_3} + \frac{d_3}{4}|\Delta w|^2 + \frac{q^2u^2}{d_3} + \frac{d_3}{4}|\Delta w|^2\right)\, dx - r\|\nabla w\|^2.\end{aligned}$$

It follows that

$$\frac{d}{dt}\|\nabla w\|^2 + d_3\|\Delta w\|^2 + 2r\|\nabla w\|^2 \le \frac{2q^2c^2}{d_3}|\Omega| + \frac{2q^2}{d_3}\|u\|^2_{L_2}. \tag{2.56}$$

Sum up the above estimates (2.54), (2.55) and (2.56) to obtain

$$\begin{aligned}&\frac{d}{dt}(\|\nabla u\|^2 + \|\nabla v\|^2 + \|\nabla w\|^2) + d_1\|\Delta u\|^2 + d_2\|\Delta v\|^2 + d_3\|\Delta w\|^2 \\ &+ 6b\|u\nabla u\|^2 + 2\|\nabla v\|^2 + 2r\|\nabla w\|^2 \\ \le\ &\frac{4}{d_1}\|v\|^2 + \frac{4}{d_1}\|w\|^2 + \frac{2q^2}{d_3}\|u\|^2 + \left(\frac{4a^2}{d_1} + \frac{2\beta^2}{d_2}\right)\|u\|^4_{L^4} \\ &+ \left(\frac{4J^2}{d_1} + \frac{2\alpha^2}{d_2} + \frac{2q^2c^2}{d_3}\right)|\Omega|.\end{aligned} \tag{2.57}$$

Since $H^1(\Omega) \hookrightarrow L^4(\Omega)$ is a continuous embedding, there is a positive constant $\eta > 0$ such that

$$\|u\|_{L^4} \le \eta\|u\|_{H^1} \le \eta\sqrt{\|u\|^2 + \|\nabla u\|^2}.$$

Then we have

$$\|u\|^4_{L^4} \le \eta^4(\|u\|^2 + \|\nabla u\|^2)^2 \le 2\eta^4\|u\|^4 + 2\eta^4\|\nabla u\|^4.$$

According to Theorem 2.1.2 and (2.21), there is a finite time $T_0(B) > 0$ such that the solution $g(t) = (u(t), v(t), w(t))$ with any initial state $g_0 \in B$ will permanently enter the absorbing ball B_0 shown in (2.19). It implies that the sum of the L^2–norms of all three components of the solution satisfies

$$\|u(t)\|^2 + \|v(t)\|^2 + \|w(t)\|^2 \le K_1, \quad \text{for any } t > T_0(B),\ g_0 \in B. \tag{2.58}$$

Then (2.57) yields the following differential inequality

$$
\begin{aligned}
&\frac{d}{dt}(\|\nabla u\|^2 + \|\nabla v\|^2 + \|\nabla w\|^2) + d_1\|\Delta u\|^2 + d_2\|\Delta v\|^2 + d_3\|\Delta w\|^2 \\
&+ 6b\|u\nabla u\|^2 + 2\|\nabla v\|^2 + 2r\|\nabla w\|^2 \\
\leq\ & \max\left\{\frac{4}{d_1}, \frac{2q^2}{d_3}\right\} K_1 + \left(\frac{8a^2}{d_1} + \frac{4\beta^2}{d_2}\right)\eta^4\|\nabla u\|^4 \\
&+ \eta^4 K_1^2\left(\frac{8a^2}{d_1} + \frac{4\beta^2}{d_2}\right) + \left(\frac{4J^2}{d_1} + \frac{2\alpha^2}{d_2} + \frac{2q^2c^2}{d_3}\right)|\Omega|, \quad t > T_0(B),\ g_0 \in B.
\end{aligned}
\tag{2.59}
$$

The inequality (2.59) implies that for any initial data $g_0 \in B$ it holds that

$$
\begin{aligned}
&\frac{d}{dt}\|(\nabla u, \nabla v, \nabla w)\|^2 \\
\leq\ & \eta^4\left(\frac{8a^2}{d_1} + \frac{4\beta^2}{d_2}\right)\|(\nabla u, \nabla v, \nabla w)\|^2\ \|(\nabla u, \nabla v, \nabla w)\|^2 \\
&+ \max\left\{\frac{4}{d_1}, \frac{2q^2}{d_3}\right\} K_1 + \eta^4 K_1^2\left(\frac{8a^2}{d_1} + \frac{4\beta^2}{d_2}\right) + \left(\frac{4J^2}{d_1} + \frac{2\alpha^2}{d_2} + \frac{2q^2c^2}{d_3}\right)|\Omega|
\end{aligned}
\tag{2.60}
$$

for all $t > T_0(B)$.

Now we can apply the uniform Gronwall inequality [62] to the differential inequality (2.60), which is written as

$$
\frac{d}{dt}\sigma(t) \leq \rho(t)\,\sigma(t) + h(t), \quad \text{for } t > T_0(B),\ g_0 \in B, \tag{2.61}
$$

where

$$
\sigma(t) = \|(\nabla u(t), \nabla v(t), \nabla w(t))\|^2,
$$

$$
\rho(t) = \eta^4\left(\frac{8a^2}{d_1} + \frac{4\beta^2}{d_2}\right)\|(\nabla u(t), \nabla v(t), \nabla w(t))\|^2,
$$

and $h(t)$ is a constant

$$
h(t) = \max\left\{\frac{4}{d_1}, \frac{2q^2}{d_3}\right\} K_1 + \eta^4 K_1^2\left(\frac{8a^2}{d_1} + \frac{4\beta^2}{d_2}\right) + \left(\frac{4J^2}{d_1} + \frac{2\alpha^2}{d_2} + \frac{2q^2c^2}{d_3}\right)|\Omega|.
$$

For any $t > T_0(B)$, integration of (8.16) implies that

$$\int_t^{t+1} \min\{d_1, d_2, d_3\}(C_1\|\nabla u(s)\|^2 + \|\nabla v(s)\|^2 + \|\nabla w(s)\|^2)\, ds$$
$$\le C_1\|u(t)\|^2 + \|v(t)\|^2 + \|w(t)\|^2 + r_1 M|\Omega| \le \max\{1, C_1\}K_1 + r_1 M|\Omega|, \quad t > T_0(B).$$

Here the constant $M > 0$ is shown in (2.18). Thus we get

$$\int_t^{t+1} \sigma(s)\, ds \le \frac{r_1 M|\Omega| + \max\{1, C_1\}K_1}{\min\{d_1, d_2, d_3\}\min\{1, C_1\}} \quad \text{for } t > T_0(B),\ g_0 \in B. \tag{2.62}$$

Hence we also have

$$\int_t^{t+1} \rho(s)\, ds \le \eta^4 \left(\frac{8a^2}{d_1} + \frac{4\beta^2}{d_2}\right)\left(\frac{r_1 M|\Omega| + \max\{1, C_1\}K_1}{\min\{d_1, d_2, d_3\}\min\{1, C_1\}}\right). \tag{2.63}$$

Denote by

$$N = \eta^4 \left(\frac{8a^2}{d_1} + \frac{4\beta^2}{d_2}\right)\left(\frac{r_1 M|\Omega| + \max\{1, C_1\}K_1}{\min\{d_1, d_2, d_3\}\min\{1, C_1\}}\right).$$

The uniform Gronwall inequality applied to (2.61) yields

$$\|(\nabla u(t), \nabla v(t), \nabla w(t))\|^2 \le C_7\, e^N, \quad \text{for any. } t \ge T_0(B) + 1,\ g_0 \in B, \tag{2.64}$$

where

$$\begin{aligned} C_7 \; &= \frac{r_1 M|\Omega| + \max\{1, C_1\}K_1}{\min\{d_1, d_2, d_3\}\min\{1, C_1\}} + \max\left\{\frac{4}{d_1}, \frac{2q^2}{d_3}\right\} K_1 \\ &+ \eta^4 K_1^2 \left(\frac{8a^2}{d_1} + \frac{4\beta^2}{d_2}\right) + \left(\frac{4J^2}{d_1} + \frac{2\alpha^2}{d_2} + \frac{2q^2c^2}{d_3}\right)|\Omega|. \end{aligned}$$

Finally, we complete the proof of (2.53):

$$\|(u(t), v(t), w(t))\|_E^2 = \|(u, v, w)\|^2 + \|\nabla(u, v, w)\|^2 \le Q_1 = K_1 + C_7\, e^N$$

for $t \ge T_1(B) = T_0(B) + 1$. The proof is completed. $\square$

We now prove the first main result on the existence of a global attractor.

Theorem 2.2.2. *For any positive parameters $d_1, d_2, d_3, a, b, \alpha, \beta, q, r, J$ and $c \in \mathbb{R}$, there exists a global attractor $\mathscr{A}$ in the space $H = L^2(\Omega, \mathbb{R}^3)$ for the Hindmarsh-Rose semiflow $\{S(t)\}_{t\geq 0}$ generated by the weak solutions of the diffusive Hindmarsh-Rose equations* (2.9)*. Moreover, the global attractor $\mathscr{A}$ is an (H, E)–global attractor.*

Proof. In Theorem 2.1.2, it is proved that there is an absorbing set $B_H \in H$ for the Hindmarsh-Rose semiflow $\{S(t)\}_{t\geq 0}$. In Theorem 2.2.1, it is shown that for any given bounded set $B \subset H$,

$$\|S(t)g_0\|_E^2 \leq Q_1, \quad \text{for } t \geq T_1(B) \text{ and all } g_0 \in B.$$

This implies that $\bigcup_{t\geq T_1(B)} S(t)B$ is a bounded set in E. Hence it is a precompact set in H due to the compact embedding $E \hookrightarrow H$. Therefore, the Hindmarsh-Rose semiflow $\{S(t)\}_{t\geq 0}$ is asymptotically compact in H. Since the two conditions in Proposition 1.3.7 are satisfied, we conclude that there exists a global attractor $\mathscr{A}$ in the space H for this Hindmarsh-Rose semiflow $\{S(t)\}_{t\geq 0}$ and

$$\mathscr{A} = \bigcap_{\tau\geq 0} \overline{\bigcup_{t\geq\tau} (S(t)B_H)}. \tag{2.65}$$

The proof of the fact that the global attractor $\mathscr{A}$ is a bi-space (H, E)–global attractor is similar to the corresponding proof in [77]. Here it is omitted. $\square$

2.3 Regularity and Structure of the Global Attractor

In this section, we shall prove the regularity properties of the global attractor $\mathscr{A}$ in the spaces $L^\infty(\Omega, \mathbb{R}^3)$ and $H^2(\Omega, \mathbb{R}^3)$. And we shall show that the Hindmarsh-Rose semiflow is a gradient system so that the global attractor $\mathscr{A}$ is structurally the union of the unstable manifolds of all the steady states.

Theorem 2.3.1. *The global attractor $\mathscr{A}$ for the Hindmarsh-Rose semiflow $\{S(t)\}_{t\geq 0}$ in the space H is a bounded set in $L^\infty(\Omega, \mathbb{R}^3)$. There is a constant $C_\infty > 0$ such that*

$$\sup_{g\in\mathscr{A}} \|g\|_{L^\infty} \leq C_\infty. \tag{2.66}$$

Proof. The analytic C_0-semigroup $\{e^{At}\}_{t\geq 0}$ has the regularity property [62] that $e^{At} : L^p(\Omega) \to L^\infty(\Omega)$ for $p \geq 1, t > 0$, and there is a constant $c(p) > 0$ such that

$$\|e^{At}\|_{\mathcal{L}(L^p, L^\infty)} \leq c(p)\, t^{-\frac{n}{2p}}, \quad \text{where } n = \dim\, \Omega. \tag{2.67}$$

Since any weak solution of (2.9) as defined is a mild solution [62] and the global attractor $\mathscr{A}$ is an invariant set, for any $g \in \mathscr{A} \subset E$, we have

$$\begin{aligned}\|S(t)g\|_{L^\infty} &\leq \|e^{At}\|_{\mathcal{L}(L^2, L^\infty)} \|g\| + \int_0^t \|e^{A(t-\sigma)}\|_{(L^2, L^\infty)} \|f(S(\sigma)g) - f(S(\sigma)0)\|\, d\sigma \\ &\leq c(2) t^{-\frac{3}{4}} \|g\| + \int_0^t c(2)(t-\sigma)^{-\frac{3}{4}} L(Q_1)(\|S(\sigma)g\|_E + \|S(\sigma)0\|_E)\, d\sigma, \quad t > 0,\end{aligned} \tag{2.68}$$

where $L(Q_1)$ is the Lipschitz constant of the nonlinear map f restricted on the closed, bounded ball $B_E = \{g \in E : \|g\|_E^2 \leq Q_1\}$ in E. Since the global attractor $\mathscr{A}$ is an invariant set, by Theorem 2.2.1, we see that

$$\{S(t)\mathscr{A} : t \geq 0\} \subset B_H\, (\subset H) \cap B_E\, (\subset E).$$

Then from (2.68) we obtain

$$\begin{aligned}\|S(t)g\|_{L^\infty} &\leq c(2)\sqrt{K_1} t^{-\frac{3}{4}} + \int_0^t c(2) L(Q_1) \left(\sqrt{Q_1} + \sqrt{Q_2}\right) (t-\sigma)^{-\frac{3}{4}}\, d\sigma \\ &= c(2) \left[\sqrt{K_1} t^{-\frac{3}{4}} + 4L(Q_1) \left(\sqrt{Q_1} + \sqrt{Q_2}\right) t^{\frac{1}{4}}\right], \quad \text{for } 0 < t \leq 1,\end{aligned} \tag{2.69}$$

where

$$Q_2 = \sup_{0 \leq \sigma \leq t \leq 1} \|S(\sigma)0\|_E^2\,.$$

Take $t = 1$ in (2.69) and get

$$\|S(1)g\|_{L^\infty} \leq c(2) \left[\sqrt{K_1} + 4L(Q_1) \left(\sqrt{Q_1} + \sqrt{Q_2}\right)\right], \quad \text{for any } g \in \mathscr{A}.$$

The invariance of $\mathscr{A}$ implies that $S(1)\mathscr{A} = \mathscr{A}$. Therefore, the global attractor $\mathscr{A}$ is a bounded subset in $L^\infty(\Omega)$. $\square$

Theorem 2.3.2. *The global attractor $\mathscr{A}$ in the space H for the Hindmarsh-Rose semiflow $\{S(t)\}_{t\geq 0}$ is a bounded set in $H^2(\Omega, \mathbb{R}^3)$.*

Proof. Consider the solution trajectories inside the global attractor $\mathscr{A}$.

Step 1. Take the L^2 inner-product $\langle(2.1), u_t\rangle$ to obtain

$$
\begin{aligned}
&\|u_t\|^2 + \frac{d_1}{2}\frac{d}{dt}\|\nabla u\|^2 = \int_\Omega (au^2 - bu^2 + v - w + J) u_t \, dx \\
\le &\int_\Omega \left(a\, C_\infty^2 + b\, C_\infty^3 + 2C_\infty + J\right) |u_t| \, dx \\
= &\frac{1}{2}\left(a\, C_\infty^2 + b\, C_\infty^3 + 2C_\infty + J\right)^2 |\Omega| + \frac{1}{2}\|u_t\|^2,
\end{aligned}
$$

where C_∞ is from (2.3.1), for the first component $u(t,x)$ of all the solution trajectories in $\mathscr{A}$. Also take the L^2 inner-product $\langle(2.2), v_t\rangle$ to obtain

$$
\begin{aligned}
&\|v_t\|^2 + \frac{d_2}{2}\frac{d}{dt}\|\nabla v\|^2 = \int_\Omega (\alpha - \beta u^2 - v) v_t \, dx \\
\le &\int_\Omega (\alpha + \beta {C_\infty}^2 + C_\infty)|v_t| \, dx \\
= &\frac{1}{2}(\alpha + \beta {C_\infty}^2 + C_\infty)^2 |\Omega| + \frac{1}{2}\|v_t\|^2
\end{aligned}
$$

for the second component $v(t,x)$ of all the trajectories in $\mathscr{A}$. Then take the L^2 inner-product $\langle(2.3), w_t\rangle$ to acquire

$$
\begin{aligned}
&\|w_t\|^2 + \frac{d_3}{2}\frac{d}{dt}\|\nabla w\|^2 = \int_\Omega (qu - qc - rw) w_t \, dx \\
\le &\int_\Omega (qC_\infty + q|c| + rC_\infty)|w_t| \, dx \\
= &\frac{1}{2}(qC_\infty + q|c| + rC_\infty)^2 |\Omega| + \frac{1}{2}\|w_t\|^2
\end{aligned}
$$

for the third component $w(t,x)$ of all the trajectories in $\mathscr{A}$.

Summing up the above three estimates we get

$$
\begin{aligned}
&\|u_t\|^2 + \|v_t\|^2 + \|w_t\|^2 + \frac{d}{dt}\left\{d_1\|\nabla u\|^2 + d_2\|\nabla v\|^2 + d_3\|\nabla w\|^2\right\} \\
\le &\left((a{C_\infty}^2 + b{C_\infty}^3 + 2C_\infty + J)^2 + (\alpha + \beta{C_\infty}^2 + C_\infty)^2 + (qC_\infty + q|c| + rC_\infty)^2\right)|\Omega|.
\end{aligned}
\tag{2.70}
$$

Integrating the inequality (2.70) over the time interval $[0,1]$, we obtain

$$\begin{aligned}
&\int_0^1 (\|u_t(s)\|^2 + \|v_t(s)\|^2 + \|w_t(s)\|^2)\, ds \\
&\le d_1\|\nabla u(0)\|^2 + d_2\|\nabla v(0)\|^2 + d_3\|\nabla w(0)\|^2 + (a{C_\infty}^2 + b{C_\infty}^3 + 2C_\infty + J)^2|\Omega| \\
&\quad + (\alpha + \beta {C_\infty}^2 + C_\infty)^2|\Omega| + (qC_\infty + q|c| + rC_\infty)^2|\Omega| \\
&\le (d_1 + d_2 + d_3)Q_1 + (a{C_\infty}^2 + b{C_\infty}^3 + 2C_\infty + J)^2|\Omega| \\
&\quad + (\alpha + \beta {C_\infty}^2 + C_\infty)^2|\Omega| + (qC_\infty + q|c| + rC_\infty)^2|\Omega|.
\end{aligned} \tag{2.71}$$

Step 2. For the diffusive Hindmarsh-Rose equations confined in the set of the global attractor $\mathscr{A}$, we can differentiate the equations (2.1), (2.2), and (2.3) to get

$$\begin{aligned}
u_{tt} &= d_1\Delta u_t + 2auu_t - 3bu^2u_t + v_t - w_t, \\
v_{tt} &= d_2\Delta v_t - 2\beta uu_t - v_t, \\
w_{tt} &= d_3\Delta w_t + qu_t - rw_t.
\end{aligned} \tag{2.72}$$

Take the inner products $\langle (2.1), t^2u_t\rangle, \langle (2.2), t^2v_t\rangle, \langle (2.3), t^2w_t\rangle$ and then sum them up,

$$\begin{aligned}
&- t\|u_t\|^2 - t\|v_t\|^2 - t\|w_t\|^2 + \frac{1}{2}\frac{d}{dt}(\|tu_t\|^2 + \|tv_t\|^2 + \|tw_t\|^2) \\
&\qquad + t^2(d_1\|\nabla u_t\|^2 + d_2\|\nabla v_t\|^2 + d_3\|\nabla w_t\|^2) \\
&= \int_\Omega t^2(2auu_t^2 - 3bu^2u_t^2 + v_tu_t - w_tu_t - 2\beta uu_tv_t - v_t^2 + qu_tw_t - rw_t^2)\, dx \\
&\le \int_\Omega t^2\left[2aC_\infty u_t^2 + \frac{1}{2}(v_t^2 + u_t^2) + \frac{1}{2}(w_t^2 + u_t^2) + \beta C_\infty(u_t^2 + v_t^2) + \frac{q}{2}(u_t^2 + w_t^2)\right] dx \\
&= t^2\left(2aC_\infty + 1 + \beta C_\infty + \frac{q}{2}\right)\|u_t\|^2 + t^2\left(\frac{1}{2} + \beta C_\infty\right)\|v_t\|^2 + t^2\left(\frac{1}{2} + \frac{q}{2}\right)\|w_t\|^2,
\end{aligned} \tag{2.73}$$

where the u-component portion is deduced by

$$\begin{aligned}
&- t\|u_t\|^2 + \frac{1}{2}\frac{d}{dt}\|tu_t\|^2 = -t\|u_t\|^2 + \frac{1}{2}\frac{d}{dt}\langle tu_t, tu_t\rangle \\
&= -t\|u_t\|^2 + \frac{1}{2}\left(\left\langle \frac{d}{dt}(tu_t), tu_t\right\rangle + \left\langle tu_t, \frac{d}{dt}(tu_t)\right\rangle\right) \\
&= -t\|u_t\|^2 + \left\langle \frac{d}{dt}(tu_t), tu_t\right\rangle = -t\|u_t\|^2 + \langle u_t, tu_t\rangle + \langle tu_{tt}, tu_t\rangle \\
&= -t\|u_t\|^2 + t\|u_t\|^2 + \langle u_{tt}, t^2u_t\rangle = \langle u_{tt}, t^2u_t\rangle.
\end{aligned} \tag{2.74}$$

Similar derivation goes to the v-component and w-component portion as well.

Now we integrate the differential inequality (2.73) on $[0, t]$ to obtain

$$
\begin{aligned}
&\frac{1}{2}(\|tu_t\|^2 + \|tv_t\|^2 + \|tw_t\|^2) \\
\le &\int_0^t s^2 \left(2aC_\infty + 1 + \beta C_\infty + \frac{q}{2}\right) \|u_t(s)\|^2 \, ds + \int_0^t s^2 \left(\frac{1}{2} + \beta C_\infty\right) \|v_t(s)\|^2 \, ds \\
&+ \int_0^t s^2 \left(\frac{1}{2} + \frac{q}{2}\right) \|w_t(s)\|^2 \, ds + \int_0^t s(\|u_t(s)\|^2 + \|v_t(s)\|^2 + \|w_t(s)\|^2) \, ds.
\end{aligned} \tag{2.75}
$$

In the above inequality we can take $t = 1$ and get

$$
\begin{aligned}
&\|u_t(1)\|^2 + \|v_t(1)\|^2 + \|w_t(1)\|^2 \\
\le &\, 2 \int_0^1 \left(2aC_\infty + 1 + \beta C_\infty + \frac{q}{2}\right) \|u_t(s)\|^2 \, ds \\
&+ 2 \int_0^1 \left(\frac{1}{2} + \beta C_\infty\right) \|v_t(s)\|^2 \, ds + 2 \int_0^1 \left(\frac{1}{2} + \frac{q}{2}\right) \|w_t(s)\|^2 \, ds \\
&+ 2 \int_0^1 (\|u_t(s)\|^2 + \|v_t(s)\|^2 + \|w_t(s)\|^2) \, ds \\
\le &\, 2\,(2aC_\infty + 5 + 2\beta C_\infty + q) \int_0^1 (\|u_t(s)\|^2 + \|v_t(s)\|^2 + \|w_t(s)\|^2) \, ds \le D
\end{aligned} \tag{2.76}
$$

where, by the inequality in (2.71) from the Step 1,

$$
\begin{aligned}
D = &\, 2\,(2aC_\infty + 5 + 2\beta C_\infty + q) \left\{ (d_1 + d_2 + d_3) Q_1 \right. \\
&+ ({aC_\infty}^2 + {bC_\infty}^3 + 2C_\infty + J)^2 |\Omega| \\
&\left. + (\alpha + {\beta C_\infty}^2 + C_\infty)^2 |\Omega| + (qC_\infty + q|c| + rC_\infty)^2 |\Omega| \right\}.
\end{aligned}
$$

where Q_1 is given in (2.53).

Step 3. Since the global attractor $\mathscr{A}$ is an invariant set, for any trajectory $g(t) = (u(t), v(t), w(t)) \in \mathscr{A}$, one has $\tilde{g}(t) = g(t-1) \in \mathscr{A}$ such that $g(t) = S(1)\tilde{g}(t)$. Then the inequality (2.76) together with the equations (2.1), (2.2) and (2.3) implies that

$$
\begin{aligned}
&d_1\|\Delta u(t)\| + d_2\|\Delta v(t)\| + d_3\|\Delta w(t)\| \\
&\le \|u_t(t)\| + \|v_t(t)\| + \|w_t(t)\| + a\|u^2(t)\| + b\|u^3(t)\| + \|v(t)\| + \|w(t)\| \\
&\quad + \beta\|u^2(t)| + \|v(t)\| + q\|u(t)\| + r\|w(t)\| + (J+\alpha+q|c|)|\Omega|^{\frac{1}{2}} \\
&= \|\tilde{u}_t(t+1)\| + \|\tilde{v}_t(t+1)\| + \|\tilde{w}_t(t+1)\| + q\|u(t)\| + 2\|v(t)\| \\
&\quad + (1+r)\|w(t)\| + (a+\beta)\|u(t)\|_{L^4}^2 + b\|u(t)\|_{L^6}^3 + (J+\alpha+q|c|)|\Omega|^{\frac{1}{2}} \\
&\le D^{\frac{1}{2}} + (q+3+r)K_1^{\frac{1}{2}} + (a+\beta)K_2^{\frac{1}{2}} + bK_3^{\frac{1}{2}} + (J+\alpha+q|c|)|\Omega|^{\frac{1}{2}},
\end{aligned}
\tag{2.77}
$$

where the positive constants K_1, K_2, K_3 are defined in (2.19) of Theorem 2.1.2 and (2.23) of Theorem 2.1.4.

Since the Laplacian operator $A_0 = \Delta$ with the Neumann boundary condition (2.5) is self-adjoint and negative definite modulo constant functions, the Sobolev space norm of any $\varphi \in H^2(\Omega, \mathbb{R}^3)$ is equivalent to $\|\varphi\|^2 + \|\nabla\varphi\|^2 + \|\Delta\varphi\|^2$. Therefore, the inequality (2.77) together with Theorem 2.1.2, Theorem 2.2.1, and Theorem 2.2.2 shows that the global attractor $\mathscr{A}$ is a bounded set in $H^2(\Omega, \mathbb{R}^3)$. $\square$

Theorem 2.3.3. *The dynamical system $\{S(t)\}_{t\ge 0}$ generated by the diffusive Hindmarsh-Rose equations* (2.9) *is a gradient system and its global attractor $\mathscr{A}$ in $H \cap E$ is structurally given by*

$$
\mathscr{A} = \bigcup_{g\in G} W^u(g) \tag{2.78}
$$

where G is the set of all the steady states with respect to $\{S(t)\}_{t\ge 0}$ and $W^u(g)$ stands for the unstable manifold associated with the steady state g.

Proof. According to [56, Definition 10.11], it suffices to show that there is a continuous Lyapunov functional Γ on a positively invariant set $\mathfrak{S}$ with respect to this semiflow $\{S(t)\}_{t\ge 0}$, which contains the global attractor $\mathscr{A}$ in H, such that $\frac{d}{dt}\Gamma(S(t)g) \le 0$ along any solution trajectory in $\mathfrak{S}$ of the evolutionary equation (2.9) and that if $\Gamma(S(\tau)g) = \Gamma(g)$ for some $\tau > 0$, then g is a steady state.

For the system (2.9), we can construct the following Lyapunov functional on the global attractor:

$$
\Gamma(g(t)) = -\left(\frac{1}{2}\|\nabla g(t)\|^2 + \int_\Omega F(g(t,x))\,dx\right) \tag{2.79}
$$

where

$$F(g(t,x)) = \int_0^t f(g(s,x)) \cdot dg, \quad \gamma(g) \subset \mathscr{A},$$

which is the line integral along the trajectory $\gamma(g) \subset \mathbb{R}^3$ over a time interval $[0,t]$. By the H^2–regularity of the global attractor $\mathscr{A}$ shown in Theorem 2.3.2, for all solution trajectories $g(t) = (u(t), v(t), w(t))$ of the equation (2.9) in $\mathscr{A}$, we have

$$\frac{d}{dt}\Gamma(g(t)) = -\left\langle Ag(t), \frac{dg}{dt}\right\rangle - \left\langle f(g(t)), \frac{dg}{dt}\right\rangle = -\|g_t\|^2 \le 0, \quad t \ge 0.$$

If $\Gamma(S(\tau)g) = \Gamma(g)$ for some $\tau > 0$, then $\frac{dg}{dt} = 0$ for almost all $t \in [0,\tau]$, which implies that $g(t) \equiv g(0)$ so that g must be a steady state.

Moreover, we can prove the functional $\Gamma : \mathscr{A}(\subset E) \to \mathbb{R}$ is continuous. Therefore, by Theorem 10.13 in [56], $\Gamma(g)$ is a continuous Lyapunov functional and the Hindmarsh-Rose semiflow a gradient system. Consequently (2.78) is proved. $\square$

2.4 Global Attractors for Partly Diffusive Hindmarsh-Rose Equations

In this section we shall prove the existence of a global attractor for the partly diffusive Hindmarsh-Rose equations (2.7) and (2.8), respectively. Note that the partly diffusive system (2.7) can be formulated into the evolutionary equation:

$$\begin{aligned} \frac{\partial g}{\partial t} &= \hat{A}g + \hat{f}(g), \quad t > 0, \\ g\,(0) &= g_0 = (u_0, v_0, w_0) \in H. \end{aligned} \tag{2.80}$$

Here the nonnegative self-adjoint operator

$$\hat{A} = \begin{pmatrix} d_1\Delta & 0 & 0 \\ 0 & -I & 0 \\ 0 & 0 & -rI \end{pmatrix} : D(\hat{A}) \to H, \tag{2.81}$$

where $D(\hat{A}) = \{g \in H^2(\Omega) \times L^2(\Omega, \mathbb{R}^2) : \partial g/\partial \nu = 0\}$, and

$$\hat{f}(u,v,w) = \begin{pmatrix} \varphi(u) + v - w + J \\ \psi(u), \\ q(u-c) \end{pmatrix} : H^1(\Omega) \times L^2(\Omega, \mathbb{R}^2) \longrightarrow H. \tag{2.82}$$

Another partly diffusive system (2.8) can be formulated into the evolutionary equation:

$$\begin{aligned} &\frac{\partial g}{\partial t} = \widetilde{A} g + \widetilde{f}(g), \quad t > 0, \\ &g(0) = g_0 = (u_0, v_0, w_0) \in H. \end{aligned} \tag{2.83}$$

Here the nonnegative self-adjoint operator

$$\widetilde{A} = \begin{pmatrix} d_1 \Delta & 0 & 0 \\ 0 & d_2 \Delta & 0 \\ 0 & 0 & -rI \end{pmatrix} : D(\widetilde{A}) \to H, \tag{2.84}$$

where $D(\widetilde{A}) = \{g \in H^2(\Omega, \mathbb{R}^2) \times L^2(\Omega) : \partial g/\partial \nu = 0\}$, and

$$\widetilde{f}(u,v,w) = \begin{pmatrix} \varphi(u) + v - w + J \\ \psi(u) - v, \\ q(u-c) \end{pmatrix} : H^1(\Omega, \mathbb{R}^2) \times L^2(\Omega) \longrightarrow H. \tag{2.85}$$

Below is the Kolmogorov-Riesz compactness Theorem shown in [33, Theorem 5].

Lemma 2.4.1. *Let $1 \le p < \infty$ and Ω be a bounded domain with locally Lipschitz boundary in $\mathbb{R}^n$. A subset $\mathcal{F}$ in $L^p(\Omega)$ is precompact if and only if the following two conditions are satisfied*:

1) *$\mathcal{F}$ is a bounded set in $L^p(\Omega)$.*

2) *For every $\varepsilon > 0$, there is some $\delta > 0$ such that, for all $f \in \mathcal{F}$ and $y \in \mathbb{R}^n$ with $|y| < \delta$,*

$$\int_\Omega |f(x+y) - f(x)|^p \, dx < \varepsilon^p.$$

It is a convention that $f(x) = 0$ for $x \in \mathbb{R}^n \backslash \Omega$.

Theorem 2.4.2. *There exists a global attractor $\mathscr{A}_1$ in the space $H = L^2(\Omega, \mathbb{R}^3)$ for the semiflow generated by the solutions of the partly diffusive Hindmarsh-Rose equations* (2.7).

Proof. Since Corollary 2.1.3 has proved that there exists an absorbing set for each of the partly diffusive Hindmarsh-Rose system (2.7) and (2.8), it suffices to show that the semiflow generated by this system (2.7) is asymptotically compact via an approach different from Theorem 2.2.1, but by means of Theorem 2.1.4 and Lemma 2.4.1.

The Laplacian operator $d_1\Delta$ with the Neumann boundary condition generates a parabolic semigroup $e^{d_1\Delta t}, t \geq 0$. The u-component of the solutions to (2.7) and to (2.8) is expressed by

$$u(t) = e^{d_1\Delta t}u_0 + \int_0^t e^{d_1\Delta(t-s)}(\varphi(u) + v - w + J)\, ds, \quad t \geq 0.$$

For $1 \leq p < q$, the $L^p \to L^q$ regularity of parabolic semigroup [62, Theorem 38.10] indicates that, for space dimension $n \leq 3$,

$$\|e^{d_1\Delta t}u_0\|_{L^q} \leq c(p,q)\, t^{-\frac{3}{2}(\frac{1}{p} - \frac{1}{q})}\|u_0\|_{L^p}, \quad t > 0. \tag{2.86}$$

Step 1. Let $p = 2$ and $q = 4$ in (2.86). We have

$$\|e^{d_1\Delta t}u_0\|_{L^4} \leq \widetilde{c}\, t^{-\frac{3}{8}}\|u_0\|_{L^2}, \quad t > 0.$$

where $\widetilde{c}$ is a constant. Then we see

$$\|u(t)\|_{L^4} = \frac{\widetilde{c}}{t^{3/8}}\|u_0\| + \int_0^t \|e^{d_1\Delta(T-s)}(\varphi(u(s)) + v(s) - w(s) + J)\|_{L^4}\, ds, \tag{2.87}$$

where $\varphi(u) = au^2 - bu^3$. From (8.16) adapted to the partly diffusive Hindmarsh-Rose equations (2.7), we have

$$\begin{aligned}&\frac{d}{dt}(C_1\|u(t)\|^2 + \|v(t)\|^2 + \|w(t)\|^2) + d_1C_1\|\nabla u(t)\|^2 \\ &+ (C_1\|u\|^2 + \|v\|^2 + r\|w\|^2) \leq (2C_2 + C_1^2)|\Omega|, \quad t \geq 0.\end{aligned}$$

Set $d_* = \min\{d_1, 1\}$. Then it holds that

$$
\begin{aligned}
&\frac{d}{dt}(C_1\|u(t)\|^2 + \|v(t)\|^2 + \|w(t)\|^2) + d_* C_1(\|\nabla u(t)\|^2 + \|u\|^2) \\
&\qquad + (\|v\|^2 + r\|w\|^2) \leq (2C_2 + C_1^2)|\Omega|, \quad t \geq 0.
\end{aligned} \tag{2.88}
$$

Integrate (2.88) over the time interval $[0,t]$ to get

$$
\begin{aligned}
&\int_0^t (d_* C_1 \|u\|_{H^1(\Omega)}^2 + \|v\|^2 + r\|w\|^2)\, ds \\
&\leq \max\{C_1, 1\}\|g_0\|^2 + t(2C_2 + C_1^2)|\Omega|,\ t \geq 0.
\end{aligned} \tag{2.89}
$$

Since the C_0-semigroup $e^{d_1 \Delta t}$ is a contraction semigroup both on $L^2(\Omega)$ and $H^1(\Omega)$ so that $\|e^{d_1\Delta(t-s)}\|_{\mathcal{L}(L^2)} \leq 1$ and $\|e^{d_1\Delta(t-s)}\|_{\mathcal{L}(H^1)} \leq 1$, from (2.87) and (2.89) it follows that

$$
\begin{aligned}
\|u(t)\|_{L^4} &\leq \frac{\widetilde{C}}{t^{3/8}}\|u_0\| + \int_0^t \|e^{d_1\Delta(t-s)}\|_{\mathcal{L}(L^2)} \|au^2 - bu^3 + v - w + J\|\, ds \\
&\leq \widetilde{C}\left(\frac{1}{t^{3/8}}\|u_0\| + \int_0^t (\|au^2 - bu^3 + v - w\|^2 + 1)\, ds + Jt|\Omega|^{1/2}\right) \\
&= \widetilde{C}\left(\frac{1}{t^{3/8}}\|u_0\| + \int_0^t (d_* C_1\|u\|_{H^1(\Omega)}^2 + \|v\|^2 + r\|w\|^2)\, ds + t + Jt|\Omega|^{1/2}\right) \\
&\leq \widetilde{C}\left(\frac{1}{t^{3/8}}\|g_0\| + \max\{C_1, 1\}\|g_0\|^2 + t(2C_2 + C_1^2)|\Omega| + t + Jt|\Omega|^{1/2}\right),\ t > 0,
\end{aligned} \tag{2.90}
$$

where $\widetilde{C} > 0$ is a constant. Take $t = 1$ in (2.90) and we can confirm that for any given bounded set $B \subset H$ and any initial state $g_0 \in B$, it holds that

$$
\|u(1)\|_{L^4} \leq \widetilde{C}\left(|||B||| + \max\{C_1, 1\}\ |||\ B\ |||^2 + (2C_2 + C_1^2)|\Omega| + 1 + J|\Omega|^{1/2}\right) \tag{2.91}
$$

where $|||B||| = \sup_{g_0 \in B}\|g_0\|$.

The uniform boundedness (2.91) allows us to use and adapt the trajectory-wise estimates in Theorem 2.1.4. From (2.24), (2.25) and the fact that $\frac{1}{3}(2p+2) = 2$ for $p = 2$, we can improve the inequality (2.26) for this system (2.7) and get

$$
\begin{aligned}
&\frac{1}{4}\frac{d}{dt}\|u(t)\|_{L^4}^4 + 3d_1\|u\nabla u\|^2 = \int_\Omega [au^5 - bu^6 + u^3(v - w + J)]\,dx \\
&\le C_b(a^6 + C_r + J^2)|\Omega| - \frac{3}{4}\int_\Omega bu^6\,dx + \frac{1}{4}\int_\Omega (v^2(t,x) + rw^2(t,x))\,dx \\
&\le C_b(a^6 + C_r + J^2)|\Omega| + \|v(t)\|^2 + r\|w(t)\|^2 - \frac{3}{4}\int_\Omega bu^6\,dx \\
&\le C_b(a^6 + C_r + J^2)|\Omega| + K_1 - \frac{3}{4}\int_\Omega bu^6\,dx, \quad t \ge T_B + 1,
\end{aligned}
\tag{2.92}
$$

because Corollary 2.1.3 shows that for any bounded set $B \subset H$ there is a time $T_B > 0$ such that for $t \ge T_B$, $\|v(t)\|^2 + \|w(t)\|^2$ is uniformly bounded by K_1 given in (2.20).

Note that

$$
\frac{1}{4}\int_\Omega bu^6 dx \ge \frac{1}{4}\int_\Omega u^4 dx - \frac{1}{4b}|\Omega|.
$$

Then from (2.92) we obtain

$$
\frac{d}{dt}\|u(t)\|_{L^4}^4 + \|u(t)\|_{L^4}^4 + 2b\int_\Omega u^6\,dx \le K, \quad t \ge T_B + 1,
\tag{2.93}
$$

where

$$
K = \left(4C_b(a^6 + C_r + J^2) + \frac{1}{b}\right)|\Omega| + 4K_1.
$$

Apply Gronwall inequality to (2.93) without $2b\int_\Omega u^6\,dx$ and use (2.91) to get

$$
\begin{aligned}
&\|u(t)\|_{L^4}^4 \le e^{-(t-1)}\|u(1)\|_{L^4}^4 + K \\
&\le \widetilde{C}^4 \left(|||B\,||| + \max\{C_1, 1\}\,|||\,B\,|||^2 + (2C_2 + C_1^2)|\Omega| + 1 + J|\Omega|^{1/2}\right)^4 + K
\end{aligned}
\tag{2.94}
$$

for $t \ge T_B + 1$ and any $g_0 \in B$. Moreover, integrating (2.93) yields the following important bound to be used a little later: For $t \ge T_B + 1$,

$$
\begin{aligned}
&\int_{T_B+1}^t e^{-(t-s)}\int_\Omega u^6(s,x)\,dx\,ds \le \frac{1}{2b}\left(\|u(T_B+1)\|_{L^4}^4 + K\right) \\
&\le \frac{1}{2b}\left(\widetilde{C}^4(|||B\,||| + \max\{C_1, 1\}\,|||\,B\,|||^2 + (2C_2 + C_1^2)|\Omega| + 1 + J|\Omega|^{1/2})^4 + 2K\right).
\end{aligned}
\tag{2.95}
$$

The inequality (2.94) shows that, for any given bounded set $B \subset H$,

$$\bigcup_{t \geq T_B+1} \left(\bigcup_{g_0 \in B} u(t, \cdot) \right) \quad \text{is a bounded set in } L^4(\Omega)$$

so that, by the compact embedding $L^4(\Omega) \hookrightarrow L^2(\Omega)$ and $L^4(\Omega) \hookrightarrow L^3(\Omega)$,

$$\bigcup_{t \geq T_B+1} \left(\bigcup_{g_0 \in B} u(t, \cdot) \right) \quad \text{is a precompact set in } L^2(\Omega) \text{ and in } L^3(\Omega). \tag{2.96}$$

Step 2. It remains to prove the precompactness of the two component functions $v(t,x)$ and $w(t,x)$, which satisfy the ordinary differential equations in (2.7). By the variation-of-constant formula for the solutions of ODE, we have

$$\begin{aligned} v(t,x) &= e^{-t} v_0(x) + \int_0^t e^{-(t-s)} (\alpha - \beta u^2)\, ds \\ &\leq \alpha + e^{-t} v_0 - \beta \int_0^t e^{-(t-s)} u^2(s,x)\, ds, \\ w(t,x) &= e^{-rt} w_0(x) + \int_0^t e^{-r(t-s)} q(u-c)\, ds \\ &\leq \frac{q|c|}{r} + e^{-rt} w_0 + q \int_0^t e^{-r(t-s)} u(s,x)\, ds. \end{aligned} \tag{2.97}$$

By Lemma 2.4.1 and (2.96), for any $\varepsilon > 0$, there is some $\delta > 0$ such that, for any given bounded set $B \subset H$ and all $g_0 \in B$, and for $y \in \mathbb{R}^3$ with $|y| < \delta$, it holds that

$$\int_\Omega |u(t, x+y) - u(t,x)|^3\, dx < \varepsilon^3, \quad \text{for all } t \geq T_B + 1. \tag{2.98}$$

Consider (2.97) on the time interval $[T_B + 1, \infty)$. Using the Hölder inequality we can infer that, for any $t \geq T_B + 1$ and any $g_0 \in B$,

$$\begin{aligned} &\int_\Omega |v(t,x+y) - v(t,x)|^2 dx = e^{-(t-T_B-1)} \int_\Omega |v(T_B+1, x+y) - v(T_B+1, x)|^2\, dx \\ &+ \beta \int_{T_B+1}^t e^{-(t-s)} \int_\Omega |u^2(s,x+y) - u^2(s,x)|^2\, dx\, ds \leq 2\, e^{-(t-T_B-1)} \|v(T_B+1)\|^2 \\ &+ \beta \int_{T_B+1}^t e^{-(t-s)} \int_\Omega |u(s,x+y) - u(s,x)|^2 |u(s,x+y) + u(s,x)|^2\, dx\, ds \end{aligned}$$

$$
\begin{aligned}
&\le 2\, e^{-(t-T_B-1)} \|v(T_B+1)\|^2 \\
&\quad + \beta \int_{T_B+1}^{t} e^{-(t-s)} \|(u(s,x+y)-u(s,x))^2\|_{L^{3/2}} \|(u(s,x+y)+u(s,x))^2\|_{L^3}\, ds \\
&\le 2\, e^{-(t-T_B-1)} \|v(T_B+1)\|^2 \\
&\quad + 24\beta \int_{T_B+1}^{t} e^{-(t-s)} \|u(s,x+y)-u(s,x)\|_{L^3}^2 \left(\|u(s,x+y)\|_{L^6}^2 + \|u(s,x)\|_{L^6}^2\right)\, ds \\
&\le 2\, e^{-(t-T_B-1)} \|v(T_B+1)\|^2 \\
&\quad + 48\beta \int_{T_B+1}^{t} e^{-(t-s)} \|u(s,x+y)-u(t,x)\|_{L^3}^2 \|u(s,x)\|_{L^6}^2\, ds \\
&\le 2\, e^{-(t-T_B-1)} \|v(T_B+1)\|^2 \\
&\quad + 48\beta \int_{T_B+1}^{t} e^{-(t-s)} \|u(s,x+y)-u(t,x)\|_{L^3}^2 (\|u(s,x)\|_{L^6}^6 + 1)\, ds
\end{aligned}
\tag{2.99}
$$

where in the last step of the above inequality, we used Young's inequality

$$
\|u(s,\cdot)\|_{L^6}^2 \le \frac{1}{3}\|u(s,\cdot)\|_{L^6}^6 + \frac{2}{3} \le \|u(s,\cdot)\|_{L^6}^6 + 1.
$$

By (2.95), (2.98) and (2.99), for any $\varepsilon > 0$, there is some $\delta > 0$ such that, for any given bounded set $B \subset H$ and all $g_0 \in B$, and for $y \in \mathbb{R}^3$ with $|y| < \delta$, we have

$$
\begin{aligned}
&\int_\Omega |v(t,x+y) - v(t,x)|^2 dx \\
&\le 2\, e^{-(t-T_B-1)} \|v(T_B+1)\|^2 + 48\,\beta \int_{T_B+1}^{t} e^{-(t-s)}\, \varepsilon^2 \left(\int_\Omega u(s,s)^6\, dx + 1\right)\, ds \\
&= 2\, e^{-(t-T_B-1)} \|v(T_B+1)\|^2 + 48\,\beta\, \varepsilon^2 \left(K^* + 1\right), \quad t \ge T_B+1,\ g_0 \in B,
\end{aligned}
\tag{2.100}
$$

where the constant K^* is given by the right-hand side of (2.95),

$$
K^* = \frac{1}{2b}\left(\widetilde{C}^4(\|\!| B \|\!| + \max\{C_1, 1\} \|\!| B \|\!|^2 + (2C_2 + C_1^2)|\Omega| + 1 + J|\Omega|^{1/2})^4 + 2K\right).
$$

Moreover, there exists a time

$$
T^*(B) = T_B + 1 + \log_e\left(\frac{\varepsilon^2}{4K_1}\right)
$$

such that

$$2\, e^{-(t-T_B-1)} \|v(T_B+1)\|^2 < \varepsilon^2, \quad \text{for } t \geq T^*(B), \tag{2.101}$$

where K_1 is given in (2.20). It follows from (2.100) and (2.101) that

$$\int_\Omega |v(t,x+y) - v(t,x)|^2 dx < [1 + 48\,\beta(K^*+1)]\,\varepsilon^2, \quad t \geq T^*(B),\, g_0 \in B. \tag{2.102}$$

Since $\varepsilon > 0$ is arbitrary, by Lemma 2.4.1, (2.112) demonstrates that

$$\bigcup_{t \geq T^*(B)} \left(\bigcup_{g_0 \in B} v(t,\cdot) \right) \text{ is precompact in } L^2(\Omega). \tag{2.103}$$

Similarly, by Lemma 2.4.1 and (2.96), for any $\varepsilon > 0$, there is some $\delta > 0$ such that, for any given bounded set $B \subset H$ and all $g_0 \in B$, and for $y \in \mathbb{R}^3$ with $|y| < \delta$, it holds that

$$\int_\Omega |u(t,x+y) - u(t,x)|^2\, dx < \varepsilon^2, \quad \text{for all } t \geq T_B + 1,$$

and we can show that, for any $g_0 \in B$,

$$\begin{aligned}
&\int_\Omega |w(t,x+y) - w(t,x)|^2 dx \\
&\leq 2\, e^{-r(t-T_B-1)} \|w(T_B+1)\|^2 + q \int_{T_B+1}^t e^{-r(t-s)} \int_\Omega |u(s,x+y) - u(s,x)|^2 dx\, ds \\
&< \left(1 + \frac{q}{r}\right) \varepsilon^2, \quad t \geq \widetilde{T}(B),
\end{aligned} \tag{2.104}$$

where

$$\widetilde{T}(B) = T_B + 1 + \frac{1}{r} \log_e \left(\frac{\varepsilon^2}{4K_1} \right).$$

(2.104) shows that

$$\bigcup_{t \geq \widetilde{T}(B)} \left(\bigcup_{g_0 \in B} w(t,\cdot) \right) \text{ is precompact in } L^2(\Omega). \tag{2.105}$$

Finally, put together (2.96), (2.103) and (2.105). Then we see

$$\bigcup_{t \geq \max\{T^*(B), \widetilde{T}(B)\}} \left(\bigcup_{g_0 \in B} g(t,\cdot) \right) \text{ is precompact in } H. \tag{2.106}$$

Therefore, the solution semiflow generated by the system (2.7) is asymptotically compact. By Proposition 1.3.7, there exists a global attractor $\mathscr{A}_1$ in the space $H = L^2(\Omega, \mathbb{R}^3)$ for the partly diffusive Hindmarsh-Rose equations (2.7). $\square$

Theorem 2.4.3. *There exists a global attractor $\mathscr{A}_2$ in the space $H = L^2(\Omega, \mathbb{R}^3)$ for the semiflow generated by the solutions of the partly diffusive Hindmarsh-Rose equations* (2.8).

Proof. Corollary 2.1.3 with $d_2\|\nabla v(t)\|^2$ added to the right-hand side of (2.22) shows that there exists an absorbing set in H for this system. The proof of the asymptotic compactness of the u-component functions and the w-component functions for this system (2.8) is the same as in the proof of Theorem 2.4.2.

Thus it suffices to show the asymptotic compactness of the v-component functions for this system. Since Theorem 2.1.4 and (2.41) already proved the v-absorbing property for each individual solution trajectory,

$$\limsup_{t\to\infty} \|v(t)\|^4_{L^4} < K_2, \quad \text{for any } \; g_0 \in H, \tag{2.107}$$

we need only to show that for any given bounded set $B \subset H$ and all the initial states $g_0 \in B$, the bunch of v-component functions of all these solutions $g(t, g_0)$ admits a uniform bound in the space $L^4(\Omega)$ at the unified time point $t = 1$. This will be the counterpart to (2.91) in the proof of Theorem 2.4.2.

According to (2.86), here we have

$$\|e^{d_2\Delta t} v_0\|_{L^4} \le \widetilde{c}\, t^{-\frac{3}{8}} \|v_0\|_{L^2}, \quad t > 0,$$

where $\widetilde{c}$ is a constant, and

$$\|v(t)\|_{L^4} = \frac{\widetilde{c}}{t^{3/8}}\|v_0\| + \int_0^t \|e^{d_2\Delta(T-s)}(\alpha - \beta u^2(s))\|_{L^4}\, ds, \quad t > 0. \tag{2.108}$$

Set $d_* = \min\{d_1, d_2, 1\}$. Adapt (2.22) to the following inequality

$$\begin{aligned} &\frac{d}{dt}(C_1\|u(t)\|^2 + \|v(t)\|^2 + \|w(t)\|^2) + d_* C_1(\|\nabla u\|^2 + \|u\|^2) \\ &+ d_*(\|\nabla v\|^2 + \|v\|^2) + r\|w\|^2 \le (2C_2 + C_1^2)|\Omega|, \quad t \ge 0. \end{aligned} \tag{2.109}$$

Integrate (2.109) over the time interval $[0, t]$ to yield

$$\int_0^t (d_*(C_1\|u\|^2_{H^1(\Omega)} + \|v\|^2_{H^1(\Omega)}) + r\|w\|^2)\, ds \le \max\{C_1, 1\} \,|||\, B \,|||^2 + t\,(2C_2 + C_1^2)|\Omega| \quad (2.110)$$

By the fact that $L^4(\Omega)$ and $L^6(\Omega)$ are continuously embedded in $H^1(\Omega)$, it follows from (2.108) and (2.110) that, for $t > 0$,

$$\begin{aligned}
\|v(t)\|_{L^4} &\le \frac{\widetilde{c}}{t^{3/8}}\,\|v_0\| + \alpha t\,|\Omega|^{1/2} + \beta\int_0^t \|e^{d_2\Delta(t-s)}u^2(s)\|_{L^4}\, ds \\
&\le \frac{\widetilde{c}}{t^{3/8}}\|v_0\| + \alpha t\,|\Omega|^{1/2} + \beta\int_0^t \|e^{d_2\Delta(t-s)}\|_{\mathcal{L}(L^2,L^4)}\,\|u^2(s)\|_{L^2}\, ds \\
&= \frac{\widetilde{c}}{t^{3/8}}\|v_0\| + \alpha t\,|\Omega|^{1/2} + \beta\int_0^t \|e^{d_2\Delta(t-s)}\|_{\mathcal{L}(L^2,L^4)}\,\|u(s)\|^2_{L^4}\, ds \qquad (2.111)\\
&\le \frac{\widetilde{c}}{t^{3/8}}\|v_0\| + \alpha t\,|\Omega|^{1/2} + \beta\kappa\int_0^t \|e^{d_2\Delta(t-s)}\|_{\mathcal{L}(H^1)}\,\|u(s)\|^2_{H^1}\, ds \\
&\le \frac{\widetilde{c}}{t^{3/8}}\,|||\, B \,|||\, + \alpha t\,|\Omega|^{1/2} + \frac{\beta\kappa}{d_*C_1}\left(\max\{C_1, 1\}\,|||\, B \,|||^2 + t(2C_2 + C_1^2)|\Omega|\right),
\end{aligned}$$

where $\kappa > 0$ is the $H^1 \hookrightarrow L^4$ embedding constant and $e^{d_2\Delta t}$ is a contraction semigroup on $H^1(\Omega)$. Take $t = 1$ in (2.111) and we reach a uniform bound

$$\sup_{g_0\in B}\|v(1)\|_{L^4} \le \left(\widetilde{c} + \frac{\beta\kappa}{d_*C_1}\left(\max\{C_1, 1\}\right)\right)|||\, B \,|||\, + \alpha|\Omega|^{1/2} + \frac{\beta\kappa}{d_*C_1}(2C_2 + C_1^2)|\Omega|.$$

for any given bounded set $B \subset H$ and all $g_0 = (u_0, v_0, w_0) \in B$. Now the bunch of v-component functions at time $t = 1$ of all these solutions are uniformly bounded in the space $L^4(\Omega)$. Then the subsequent proof for Theorem 2.1.4 remains valid here and shows that there exists an bounded absorbing set in $L^4(\Omega)$ for the v-component functions of all these solutions started from B in H, which in turn shows that by the compact Sobolev embedding,

$$\bigcup_{t\ge T^*(B)}\left(\bigcup_{g_0\in B} v(t,\cdot)\right) \text{ is precompact in } L^2(\Omega). \quad (2.112)$$

The proof is completed. □

Chapter 3

Exponential Attractor for Hindmarsh-Rose Equations in Neurodynamics

Note to Reader

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In this chapter, we shall explore the existence of an exponential attractor for the dynamical systems generated by the diffusive Hindmarsh-Rose equations (2.1)–(2.6) with the same assumptions as in Chapter 2.

3.1 Formulation and Preliminaries

The formulation of the initial-boundary value problem (2.1)–(2.6) is the same initial value problem (2.9) with (2.10) and (2.11). The existence of a global attractor for the solution semiflow of (2.9) has been proved in the previous chapter.

Note that global attractor for an infinite-dimensional dynamical systems generated by evolutionary PDE may exhibit slow rates and complicated behavior in attraction of solution trajectories. Here we investigate the existence of exponential attractor, the notion introduced in [23].

Theorem 3.1.1. *For any given bounded set $B \subset H$, there exists a finite time $T_B > 0$ such that for any initial state $g_0 = (u_0, v_0, w_0) \in B$, the weak solution $g(t) = S(t)g_0 = (u(t), v(t), w(t))$ of the initial value problem* (2.9) *uniquely exists for $t \in [0, \infty)$ and satisfies*

$$\|(u(t), v(t), w(t))\|_E \leq Q, \quad \text{for } \; t \geq T_B, \tag{3.1}$$

where $Q > 0$ is a constant independent of any bounded set B in H, and the finite $T_B > 0$ only depends on the bounded set B.

Definition 3.1.2 (Exponential Attractor)**.** Suppose that X is a Banach space and $\{S(t)\}_{t\geq 0}$ is a semiflow on X. A subset $\mathscr{E} \subset X$ is called an exponential attractor for this semiflow if the following three conditions are satisfied:

1) $\mathscr{E}$ is a compact in X with a finite fractal dimension.

2) $\mathscr{E}$ is positively invariant with respect to the semiflow $\{S(t)\}_{t\geq 0}$ in the sense

$$S(t)\mathscr{E} \subset \mathscr{E} \quad \text{for all} \;\; t \geq 0.$$

3) $\mathscr{E}$ attracts all the solution trajectories exponentially with a uniform rate $\sigma > 0$ in the sense that for any given bounded set $B \subset X$ there is a constant $C(B) > 0$ and

$$\text{dist}_X(S(t)B, \mathscr{E}) \leq C(B)e^{-\sigma t}, \quad t \geq 0.$$

If there exists an exponential attractor $\mathscr{E}$ (may not be unique) as well as a global attractor $\mathscr{A}$ for a semiflow in a Banach space X, then it is always true that

$$\mathscr{A} \subset \mathscr{E}.$$

Consequently, the global attractor must have a finite fractal dimension as a subset of the exponential attractor.

There are two approaches in terms of sufficient conditions for construction of an exponential attractor. The first approach is the squeezing property which was introduced in the book [23] and expounded in [51]. The second approach is the compact smoothing property introduced by Efendiev-Miranville-Zelik [24, 25]. Conceptually, the two properties are essentially equivalent when the phase space is a Hilbert space. From the application viewpoint, the squeezing property fits more to the semilinear reaction-diffusion equations. The second approach has been exploited in proving the existence of exponential attractors for quasilinear reaction-diffusion systems [75].

The following definition of squeezing property [44, 51] for a mapping means that either the mapping (which can be s snapshot of a semiflow at any time t^*) is a contraction or that higher modes are dominated by lower modes.

Definition 3.1.3 (Squeezing Property)**.** Let H be a Hilbert space and $\{S(t)\}_{t\geq 0}$ be a semiflow on

H whose norm is $\|\cdot\|$. Let $S = S(t^*)$ for some fixed $t^* \in (0, \infty)$. If there is a positively invariant set $Z \subset H$ with respect to this semiflow and there is a constant $0 < \delta < 1$ and an orthogonal projection P from H onto a finite-dimensional subspace of $PH \subset H$, such that either

$$\|Su - Sv\| \leq \delta \|u - v\|, \quad \text{for any } u, v \in Z,$$

or

$$\|(I - P)(Su - Sv)\| \leq \|P(Su - Sv)\|, \quad \text{for any } u, v \in Z,$$

then we say that the mapping S has the squeezing property and the affiliated semiflow $\{S(t)\}_{t \geq 0}$ has the squeezing property on the set Z.

Definition 3.1.4 (Fractal Dimension)**.** The fractional dimension of a bounded subset $\mathscr{M}$ in a Banach space is defined by

$$\dim_f \mathscr{M} = \limsup_{\varepsilon \to 0^+} \frac{\log N_\varepsilon[\mathscr{M}]}{\log(1/\varepsilon)}$$

where $N_\varepsilon[\mathscr{M}]$ is the infimum number of open balls with the radius ε for a covering of the set $\mathscr{M}$.

The following theorem states sufficient conditions for the existence of an exponential attractor with respect to a semiflow in a Hilbert space. Its proof is seen in [51, Theorems 4.4 and 4.5].

Theorem 3.1.5. *Let $\{S(t)\}_{t \geq 0}$ be a semiflow on a Hilbert space H with the following conditions satisfied*:

1) *The squeezing property is satisfied for $S = S(t^*)$ at some $t^* > 0$ on a nonempty, compact, positively invariant, and absorbing set $M \subset H$.*
2) *For all $t \in [0, t^*]$, the mapping $S(t) : M \to M$ is Lipschitz continuous and the Lipschitz constant $K(t) : [0, t^*] \to (0, \infty)$ is a bounded function.*
3) *For any $g \in M$, the mapping $S(\cdot)g : [0, t^*] \to M$ is Lipschitz continuous and the Lipschitz constant $L(g) : M \to (0, \infty)$ is a bounded function.*

Then there exists an exponential attractor $\mathscr{E}$ in the space H for this semiflow. Moreover, for any $\theta \in (0, 1)$, the fractal dimension of the exponential attractor $\mathscr{E}$ has the estimate

$$\dim_F(\mathscr{E}) \leq N \max\left\{1, \frac{\log(2\sqrt{2}L/\theta + 1)}{-\log\theta}\right\} \tag{3.2}$$

where N is the rank of the spectral projection associated with the squeezing property of the mapping $S(t^*)$ and L is the Lipschitz constant of the mapping $S(t^*)$ on the positively invariant absorbing set M.

3.2 Squeezing Property for Reaction-Diffusion Systems

The approach to proving the squeezing property for an evolutionary PDE is to study the difference of two solutions, $w(t) = g(t) - h(t)$, and conduct estimates to bound the time derivatives of the lower and higher modes, $d\|Pw\|/dt$ and $d\|Qw\|/dt$.

Consider a general system of reaction-diffusion equations in the form of an evolutionary equation on a real Hilbert space $H = L^2(\Omega, \mathbb{R}^d)$, where the higher dimensional $\Omega \subset \mathbb{R}^d$ $(d \geq 3)$, is a bounded Lipschitz domain,

$$\frac{dg}{dt} + \mathcal{A}g = f(g) \tag{3.3}$$

where $f \in C^1(\mathbb{R}^d, \mathbb{R}^d)$ is a nonlinear vector function and the differential operator $\mathcal{A} : \mathscr{D}(\mathcal{A}) \to H$ is a densely defined, nonnegative self-adjoint operator with compact resolvent so that its spectrum consists of a nonnegative sequence of the eigenvalues $\{\lambda_m\}$ with finite multiplicities for $\lambda_m > 0$ and $\lambda_m \to \infty$ as $m \to \infty$.

Assume that the weak solution $g(t)$ of the evolutionary equation (3.3) exists on the time interval $[0, \infty)$ for any initial data $g_0 \in H$, such that

$$g \in C([0, \infty), H) \cap L^2_{loc}([0, \infty), E) \tag{3.4}$$

where $E = H^1(\Omega, \mathbb{R}^d)$ whose norm is defined by $\|u\|_E^2 = \|\nabla u\|^2 + \|u\|^2$. Suppose that there exists a positively invariant, closed and bounded set $M \subset E$ for the solution semiflow such that

$$\|f(g) - f(\tilde{g})\|_H \leq C\|g - \tilde{g}\|_E, \quad \text{for any } \; g, \tilde{g} \in M, \tag{3.5}$$

where the positive Lipschitz constant $C = C(M) > 0$, and

$$\langle f(g) - f(\widetilde{g}),\, g - \widetilde{g}\rangle_H \leq C^*\|g - \widetilde{g}\|_H^2, \quad \text{for any } \; g, \tilde{g} \in M, \tag{3.6}$$

where $C^* > 0$ is a constant independent of M.

Let the complete set of the orthonormal eigenvectors of $\mathcal{A} : \mathscr{D}(\mathcal{A}) \to H$ associated with the eigenvalues $\{\lambda_i\}$ (each repeated to the respective multiplicity) be $\{e_i\}$, $\mathcal{A}e_i = \lambda_i e_i$ and $\lambda_i \leq \lambda_{i+1} \to \infty$. Let $P_m : H \to \text{Span } \{e_1, ..., e_m\}$ and $Q_m = I - P_m$ be the orthogonal spectral projections. Then

$$\|p\|_E = \left(\sum_{k=1}^{m} |\langle p, e_k \rangle|^2 \lambda_k \right)^{\frac{1}{2}} \leq \left(\lambda_m^{\frac{1}{2}} + 1 \right) \|p\|_H, \qquad p \in PH,$$
$$\|q\|_E = \left(\sum_{k=m+1}^{\infty} |\langle q, e_k \rangle|^2 \lambda_k \right)^{\frac{1}{2}} \geq \left(\lambda_{m+1}^{\frac{1}{2}} + 1 \right) \|q\|_H, \quad q \in QH,$$

where we briefly write $P = P_m$ and $Q = Q_m = I - P_m$.

We now prove a theorem on the squeezing property for the abstract reaction-diffusion equation (3.3) on a higher dimensional bounded domain.

Theorem 3.2.1. *Under the assumptions* (3.4)*,* (3.5) *and* (3.6)*, there exists an integer $m \geq 1$ sufficiently large such that the squeezing property is satisfied on the compact, positively invariant and bounded set $M \subset H$ with respect to the projection mapping $P = P_m$ for the solution semiflow of the reaction-diffusion system* (3.3)*.*

Proof. For two solutions $g(t)$ and $h(t)$ of (3.3) in the positively invariant set M, the difference $\xi(t) = g(t) - h(t)$ satisfies the equation

$$\frac{d\xi}{dt} + \mathcal{A}\xi = f(g) - f(h), \quad t \geq 0. \tag{3.7}$$

Write $p(t) = P\xi(t)$ and $q(t) = Q\xi(t)$ so that $\xi(t) = p(t) + q(t)$ is an orthogonal decomposition of $\xi(t)$. Note that the closed and bounded set $M \subset E$ in the assumptions (3.5) and (3.6) must be a compact set in the space H.

Step 1. Take L^2 inner-product $\langle (3.7), p(t) \rangle$ and note that $\mathcal{A}P = P\mathcal{A}$ on $\mathscr{D}(\mathcal{A})$ and $P^2 = P$. We have

$$\frac{1}{2}\frac{d}{dt}\|p(t)\|^2 + \|\nabla p\|^2 = \langle f(g) - f(h), p \rangle \geq -C\|\xi\|_E \|p\| \geq -C(\lambda_m^{\frac{1}{2}} + 1)\|\xi\|\|p\|$$

due to the Lipschitz condition (3.5). Then

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|p(t)\|^2 &\geq -\lambda_m\|p\|^2 - C(\lambda_m^{\frac{1}{2}}+1)(\|p\|+\|q\|)\|p\| \\ &= -(\lambda_m + C(\lambda_m^{\frac{1}{2}}+1))\|p\|^2 - C(\lambda_m^{\frac{1}{2}}+1)\|p\|\|q\|.\end{aligned} \tag{3.8}$$

On the other side, we take the inner product $\langle(3.7), q(t)\rangle$ and obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|q(t)\|^2 &\leq -\lambda_{m+1}\|q\|^2 + C(\lambda_m^{\frac{1}{2}}+1)(\|p\|+\|q\|)\|q\| \\ &\leq -(\lambda_m - C(\lambda_m^{\frac{1}{2}}+1))\|q\|^2 + C(\lambda_m^{\frac{1}{2}}+1)\|p\|\|q\|.\end{aligned} \tag{3.9}$$

We choose m sufficiently large such that

$$\lambda_m - C(\lambda_m^{\frac{1}{2}}+1) > 2C(\lambda_m^{\frac{1}{2}}+1). \tag{3.10}$$

Let $S = S(1)$ for $t^* = 1$. Then either

$$\|(I-P)(Sg-Sh)\| \leq \|P(Sg-Sh)\|, \quad \text{i.e.} \quad \|q(1)\| \leq \|p(1)\|,$$

or otherwise

$$\|(I-P)\xi(1)\| = \|Q\xi(1)\| > \|P\xi(1)\|, \quad \text{i.e.} \quad \|q(1)\| > \|p(1)\|. \tag{3.11}$$

Below we consider the case that (3.11) occurs. By the choice (3.10), we have

$$(\lambda_m - C(\lambda_m^{\frac{1}{2}}+1))\|Q\xi(1)\| > 2C(\lambda_m^{\frac{1}{2}}+1)\|P\xi(1)\|.$$

Namely,

$$(\lambda_m - C(\lambda_m^{\frac{1}{2}}+1))\|q(1)\| > 2C(\lambda_m^{\frac{1}{2}}+1)\|p(1)\|. \tag{3.12}$$

The continuity of $\xi(t)$ in H implies that the strict inequality as above holds for t in a small neighborhood of $t^* = 1$. There are two possibilities to be considered.

Step 2. The first possibility is that

$$(\lambda_m - C(\lambda_m^{\frac{1}{2}} + 1))\|q(t)\| > 2C(\lambda_m^{\frac{1}{2}} + 1)\|p(t)\| \tag{3.13}$$

holds for all $t \in \left[\frac{1}{2}, 1\right]$. Then

$$\begin{aligned}&(\lambda_m - C(\lambda_m^{\frac{1}{2}} + 1))\|q(t)\| - C(\lambda_m^{\frac{1}{2}} + 1)\|p(t)\| \\ &> \frac{1}{2}(\lambda_m - C(\lambda_m^{\frac{1}{2}} + 1))\|q(t)\| > \frac{\lambda_m}{3}\|q(t)\|, \quad \text{for } \ t \in [1/2, 1],\end{aligned} \tag{3.14}$$

where we used (3.12) in the first inequality and (3.10) in the second inequality of (3.14). Then (3.9) becomes

$$\frac{d}{dt}\|q\|^2 \leq -\frac{2}{3}\,\lambda_m\|q\|^2, \quad t \in [1/2, 1].$$

Integrating this inequality over the time interval $[\frac{1}{2}, 1]$, we obtain

$$\|q(1)\|^2 \leq e^{-\lambda_m/3}\,\|q(1/2)\|^2.$$

Since $\|\xi(1)\|^2 = \|p(1)\|^2 + \|q(1)|^2 \leq 2\|q(1)\|^2$ due to (3.11), it infers that

$$\|\xi(1)\| \leq \sqrt{2}\,\|q(1)\| \leq \sqrt{2}\,e^{-\lambda_m/6}\|q(1/2)\| \leq \sqrt{2}e^{-\lambda_m/3}\|\xi(1/2)\|. \tag{3.15}$$

On the other hand, taking the inner product $\langle(3.7), \xi(t)\rangle$ and using the monotone property (3.6), we can get

$$\frac{1}{2}\frac{d}{dt}\|\xi\|^2 \leq \frac{d}{dt}\|\xi\|^2 + \|\nabla\xi\|^2 \leq \langle f(g) - f(h), g - h\rangle \leq C^*\|g - h\|^2 = C^*\|\xi\|^2.$$

Integrate the above inequality over the time interval $[0, t]$, we get

$$\|g(t) - h(t)\| \leq e^{C^* t}\|g_0 - h_0\|, \quad \text{for any } \ t \geq 0. \tag{3.16}$$

It yields, in particular,

$$\|\xi(1/2)\| \leq e^{C^*/2}\|\xi(0)\|, \tag{3.17}$$

Then (3.15) and (3.17) give rise to the inequality

$$\|Sg_0 - Sh_0\| = \|S(1)g_0 - S(1)h_0\| = \|\xi(1)\| \le \delta\|\xi(0)\| = \delta\|g_0 - h_0\| \tag{3.18}$$

with

$$0 < \delta = \sqrt{2}\, e^{-\lambda_m/6}\, e^{C^*/2} < 1 \tag{3.19}$$

provided that m is large enough so that λ_m is large enough. Thus it is proved that for this first possibility the squeezing property is satisfied by the mapping S and by the solution semiflow $\{S(t)\}_{t\ge 0}$ of the reaction-diffusion system (3.3).

Step 3. The second possibility is that (3.13) does not hold for all $t \in [1/2, 1]$. Then there is a time $\frac{1}{2} < t_0 < 1$ such that (3.13) is valid for $t \in (t_0, 1]$ and

$$(\lambda_m - C(\lambda_m^{\frac{1}{2}} + 1))\|q(t_0)\| = 2C(\lambda_m^{\frac{1}{2}} + 1)\|p(t_0)\|. \tag{3.20}$$

Define a function

$$\Phi(t) = (\|p(t)\| + \|q(t)\|) \exp\left(\frac{\lambda_m\|q(t)\|}{C_m(\|p(t)\| + \|q(t)\|)}\right) \tag{3.21}$$

where $C_m = C(\lambda_m^{\frac{1}{2}} + 1)$. From (3.8) and (3.9), since $\frac{1}{2}\frac{d}{dt}\|p(t)\|^2 = \|p(t)\|\frac{d}{dt}\|p(t)\|$ and similarly for $\|q(t)\|$, we have

$$\begin{aligned}
&\frac{d}{dt}\|p\| \ge -(\lambda_m + C(\lambda_m^{\frac{1}{2}} + 1))\|p\| - C(\lambda_m^{\frac{1}{2}} + 1)\|q\|, \\
&\frac{d}{dt}\|q\| \le -(\lambda_m - C(\lambda_m^{\frac{1}{2}} + 1))\|q\| + C(\lambda_m^{\frac{1}{2}} + 1)\|p\|.
\end{aligned}$$

Then

$$\frac{d}{dt}\Phi(t) = \exp\left[\frac{\lambda_m\|q\|}{C_m(\|p\| + \|q\|)}\right]\left[\frac{d}{dt}(\|p\| + \|q\|) + (\|p\| + \|q\|)\frac{d}{dt}\left(\frac{\lambda_m\|q\|}{C_m(\|p\| + \|q\|)}\right)\right]. \tag{3.22}$$

Since the exponential factor is positive, in order to know the sign of the derivative $\frac{d}{dt}\Phi(t)$, we only need to estimate the second factor on the right side of (3.22):

$$
\begin{aligned}
&\frac{d}{dt}(\|p\|+\|q\|)+(\|p\|+\|q\|)\,\frac{d}{dt}\left(\frac{\lambda_m\|q\|}{C_m(\|p\|+\|q\|)}\right)\\
=&\frac{d}{dt}\|p\|+\frac{d}{dt}\|q\|+\frac{\lambda_m}{C_m}\frac{d}{dt}\|q\|-\frac{\lambda_m\|q\|}{C_m(\|p\|+\|q\|)}\left(\frac{d}{dt}\|p\|+\frac{d}{dt}\|q\|\right)\\
=&\frac{d}{dt}\|q\|\left[1+\frac{\lambda_m}{C_m}-\frac{\lambda_m\|q\|}{C_m(\|p\|+\|q\|)}\right]-\frac{d}{dt}\|p\|\left[\frac{\lambda_m\|q\|}{C_m(\|p\|+\|q\|)}-1\right]\\
=&\frac{d}{dt}\|q\|\left[1+\frac{\lambda_m\|p\|}{C_m(\|p\|+\|q\|)}\right]-\frac{d}{dt}\|p\|\left[\frac{\lambda_m\|q\|}{C_m(\|p\|+\|q\|)}-1\right]\\
\le&(-(\lambda_m-C_m)\|q\|+C_m\|p\|)\left[1+\frac{\lambda_m\|p\|}{C_m(\|p\|+\|q\|)}\right]\\
&+((\lambda_m+C_m)\|p\|+C_m\|q\|)\left[\frac{\lambda_m\|q\|}{C_m(\|p\|+\|q\|)}-1\right]\\
=&-(\lambda_m-C_m)\|q\|-\frac{\lambda_m(\lambda_m-C_m)\|p\|\|q\|}{C_m(\|p\|+\|q\|)}+C_m\|p\|+\frac{\lambda_m\|p\|^2}{\|p\|+\|q\|}\\
&+\frac{\lambda_m(\lambda_m+C_m)\|p\|\|q\|}{C_m(\|p\|+\|q\|)}-(\lambda_m+C_m)\|p\|+\frac{\lambda_m\|q\|^2}{\|p\|+\|q\|}-C_m\|q\|\\
=&-\lambda_m\|q\|-\lambda_m\|p\|+\frac{2\lambda_m\|p\|\|q\|}{\|p\|+\|q\|}+\frac{\lambda_m\|p\|^2}{\|p\|+\|q\|}+\frac{\lambda_m\|q\|^2}{\|p\|+\|q\|}\\
=&-\lambda_m\|q\|-\lambda_m\|p\|+\frac{\lambda_m(\|p\|^2+\|q\|^2+2\|p\|\|q\|)}{\|p\|+\|q\|}=0.
\end{aligned}
\tag{3.23}
$$

Hence we obtain

$$\frac{d}{dt}\,\Phi(t)\le 0,\quad \text{for } t\in[t_0,1].$$

It follows that

$$\Phi(1)\le\Phi(t_0). \tag{3.24}$$

At $t=1$, $\|q(1)\|=\|Q\xi(1)\|>\|P\xi(1)\|=\|p(1)\|$ by (3.11). Then from (3.21) we see that

$$\Phi(1)\ge\|q(1)\|\exp\left(\frac{\lambda_m\|q(1)\|}{2C_m\|q(1)\|}\right)=\|q(1)\|e^{\lambda_m/(2C_m)}. \tag{3.25}$$

At $t=t_0$, (3.20) indicates that

$$(\lambda_m-C_m)\|q(t_0)\|=2C_m\|p(t_0)\|$$

and then

$$2C_m(\|p(t_0)\| + \|q(t_0)\|) = (\lambda_m + C_m)\|q(t_0)\|.$$

Thus,

$$\Phi(t_0) = \frac{\lambda_m + C_m}{2C_m} \|q(t_0)\| \exp\left(\frac{2\lambda_m}{\lambda_m + C_m}\right). \tag{3.26}$$

Note that $t_0 \in (1/2, 1]$. Put together (3.24), (3.25) and (3.26). We use the Lipschitz continuous dependence on initial data to obtain

$$\begin{aligned}
\|q(1)\| &\le \exp\left(-\frac{\lambda_m}{2C_m}\right)\Phi(1) \le \exp\left(-\frac{\lambda_m}{2C_m}\right)\Phi(t_0) \\
&\le \exp\left(-\frac{\lambda_m}{2C_m}\right)\frac{\lambda_m + C_m}{2C_m}\exp\left(\frac{2\lambda_m}{\lambda_m + C_m}\right)\|q(t_0)\|. \\
&\le \exp\left(-\frac{\lambda_m}{2C_m}\right)\frac{\lambda_m + C_m}{2C_m}\, e^2\, \|q(t_0)\| \\
&\le \exp\left(-\frac{\lambda_m}{2C_m}\right)\frac{\lambda_m + C_m}{2C_m}\, e^2\, \|\xi(t_0)\|.
\end{aligned} \tag{3.27}$$

According to the solution expression of the evolutionary equation (3.7),

$$\xi(t) = e^{-\mathcal{A}t}\xi(0) + \int_0^t e^{-\mathcal{A}(t-s)}(f(g(s)) - f(h(s)))\,ds \quad t \ge 0,$$

By using the Lipschitz condition (3.5) and the fact that $e^{-\mathcal{A}t}$ is a contraction semigroup, we can deduce that

$$\begin{aligned}
\|\xi(t)\| &\le \|e^{-\mathcal{A}t}\|_{\mathcal{L}(H)}\|\xi(0)\| + \int_0^t \|e^{-\mathcal{A}(t-s)}\|_{\mathcal{L}(H)}\|f(g(s)) - f(h(s))\|\,ds \\
\le \|\xi(0)\| &+ \int_0^t C\|g(s) - h(s)\|_E\,ds \le \|\xi(0)\| + \int_0^t C\|\xi(s)\|_E\,ds, \quad t \ge 0.
\end{aligned} \tag{3.28}$$

Then the Gronwall inequality applied to (3.28) shows that

$$\|\xi(t)\| \le \|\xi(0)\|\, e^{Ct}, \quad t \ge 0.$$

Substitute this inequality at t_0 into (3.27) to obtain

$$\|q(1)\| \le \exp\left(-\frac{\lambda_m}{2C_m}\right)\frac{\lambda_m + C_m}{2C_m}\; e^2\,\|\xi(t_0)\| \le \exp\left(-\frac{\lambda_m}{2C_m}\right)\frac{\lambda_m + C_m}{2C_m}\; e^{2+C}\,\|\xi(0)\|.$$

Since $\|p(1)\| < \|q(1)\|$, we end up with

$$\|\xi(1)\| \le \sqrt{2}\,\exp\left(-\frac{\lambda_m}{2C_m}\right)\frac{\lambda_m + C_m}{2C_m}\; e^{2+C}\|\xi(0)\|.$$

For m sufficiently large, we can assert

$$\begin{aligned} 0 < \delta &= \sqrt{2}\,\exp\left(-\frac{\lambda_m}{2C_m}\right)\frac{\lambda_m + C_m}{2C_m}\; e^{2+2C} \\ &= \sqrt{2}\left(\frac{\lambda_m}{2C(\lambda_m^{\frac{1}{2}}+1)} + \frac{1}{2}\right)\exp\left(-\frac{\lambda_m}{2C(\lambda_m^{\frac{1}{2}}+1)}\right)e^{2+2C} < 1. \end{aligned}$$

We have proved that

$$\|Sg_0 - Sh_0\| = \|\xi(1)\| \le \delta\|\xi(0)\| = \delta\|g_0 - h_0\|, \quad \text{for any } \; g_0, h_0 \in M. \tag{3.29}$$

Finally (3.18) and (3.29) show that, in any case as in Step 2 and Step 3, if the spectral number m of the finite-rank orthogonal projection P_m on the space H is chosen to be large enough, then the squeezing property holds for the Hindmarsh-Rose semiflow $\{S(t)\}_{t\ge 0}$ generated by the equation (3.3) on this compact, positively invariant and bounded set $M \subset H$. The proof is completed. □

3.3 The Existence of Exponential Attractor

In this section, we prove the main result on the existence of exponential attractor for the solution semiflow of the diffusive Hindmarsh-Rose equations. We start with the squeezing property stated Theorem 3.2.1 and check its two conditions (3.5) and (3.6) are satisfied by the Hindmarsh-Rose semiflow.

Lemma 3.3.1. *Under the same assumptions as in Theorem* 2.2.2*, the Nemytskii operator f defined by* (2.11) *satisfies the E to H Lipschitz condition*

$$\|f(g) - f(\tilde{g})\|_H \le C_E(M)\|g - \tilde{g}\|_E, \quad \textit{for any} \;\; g, \tilde{g} \in M, \tag{3.30}$$

on any given positively invariant and bounded set $M \subset E$, where $C_E(M) > 0$ is a constant only depending on M. Moreover, f satisfies the monotone property that there exists a constant $C^* > 0$ independent of M and

$$\langle f(g) - f(\tilde{g}), g - \tilde{g} \rangle \leq C^* \|g - \tilde{g}\|^2, \quad \textit{for any} \;\; g,\, \tilde{g} \in M. \tag{3.31}$$

Proof. First we prove the claim (3.31). For any $g = (u, v, w)$ and $\tilde{g} = (\tilde{u}, \tilde{v}, \tilde{w})$ in the set M and denote the three components of f by f_1, f_2, f_3. For the first component f_1, we have

$$\begin{aligned}
&\langle f_1(g) - f_1(\tilde{g}), u - \tilde{u} \rangle \\
={}& \langle f_1(u, v, w) - f_1(\tilde{u}, \tilde{v}, \tilde{w}), u - \tilde{u} \rangle \\
\leq{}& \langle \varphi(u) - \varphi(\tilde{u}), u - \tilde{u} \rangle + \langle v - \tilde{v}, u - \tilde{u} \rangle + \langle w - \tilde{w}, u - \tilde{u} \rangle \\
\leq{}& a\langle u^2 - \tilde{u}^2, u - \tilde{u} \rangle - b\langle u^3 - \tilde{u}^3, u - \tilde{u} \rangle + \|v - \tilde{v}\| \|u - \tilde{u}\| + \|w - \tilde{w}\| \|u - \tilde{u}\| \\
\leq{}& a \int_\Omega |u(x) - \tilde{u}(x)|^2 (u(x) + \tilde{u}(x))\, dx + \|u - \tilde{u}\|^2 + \|v - \tilde{v}\|^2 + \|w - \tilde{w}\|^2 \\
&- b \int_\Omega |u(x) - \tilde{u}(x)|^2 (u^2(x) + u(x)\tilde{u}(x) + \tilde{u}^2(x)) dx \\
\leq{}& \frac{2a^2}{b} \int_\Omega |u(x) - \tilde{u}(x)|^2\, dx + \|u - \tilde{u}\|^2 + \|v - \tilde{v}\|^2 + \|w - \tilde{w}\|^2 \\
&- \frac{b}{4} \int_\Omega |u(x) - \tilde{u}(x)|^2 (u^2(x) + \tilde{u}^2(x))\, dx,
\end{aligned} \tag{3.32}$$

where we used

$$a(u(x) + v(x)) \leq \frac{b}{4}(u^2(x) + \tilde{u}^2(x)) + \frac{2a^2}{b}.$$

For the second component f_2, we do the estimate

$$\begin{aligned}
&\langle f_2(g) - f_2(\tilde{g}), v - \tilde{v} \rangle = \langle f_2(u, v, w) - f_2(\tilde{u}, \tilde{v}, \tilde{w}), v - \tilde{v} \rangle \\
\leq{}& \langle \psi(u) - \psi(\tilde{u}), v - \tilde{v} \rangle + \|v - \tilde{v}\|^2 \\
={}& \beta \int_\Omega (u(x) - \tilde{u}(x))(u(x) + \tilde{u}(x))(v(x) - \tilde{v}(x))\, dx + \|v - \tilde{v}\|^2 \\
\leq{}& \frac{b}{8} \int_\Omega |u(x) - \tilde{u}(x)|^2 |u(x) + \tilde{u}(x)|^2\, dx + \frac{2\beta^2}{b} \|v - \tilde{v}\|^2 + \|v - \tilde{v}\|^2 \\
\leq{}& \frac{b}{4} \int_\Omega |u(x) - \tilde{u}(x)|^2 (u^2(x) + \tilde{u}^2(x))\, dx + \left(1 + \frac{2\beta^2}{b}\right) \|v - \tilde{v}\|^2.
\end{aligned} \tag{3.33}$$

For the third component f_3, we have

$$\begin{aligned}
&\langle f_3(g) - f_3(\tilde{g}), w - \tilde{w}\rangle = \langle f_3(u,v,w) - f_3(\tilde{u},\tilde{v},\tilde{w}), w - \tilde{w}\rangle \\
&\le q\|u-\tilde{u}\|\|w-\tilde{w}\| + r\|w-\tilde{w}\|^2 \le q\|u-\tilde{u}\|^2 + (q+r)\|w-\tilde{w}\|^2.
\end{aligned} \tag{3.34}$$

Summing up (3.32), (3.33) and (3.34), and two integral terms with plus and minus signs respectively on the right-hand side of (3.32) and (3.33) being cancelled out, we obtain

$$\begin{aligned}
&\langle f(g) - f(\tilde{g}), g - \tilde{g}\rangle = \langle f_1(g) - f_1(\tilde{g}), u - \tilde{u}\rangle \\
&+\langle f_2(g) - f_2(\tilde{g}), v - \tilde{v}\rangle + \langle f_3(g) - f_3(\tilde{g}), w - \tilde{w}\rangle \\
&\le \left(1 + \frac{2a^2}{b}\right)\|u-\tilde{u}\|^2 + \|v-\tilde{v}\|^2 + \|w-\tilde{w}\|^2 \\
&+ \left(1 + \frac{2\beta^2}{b}\right)\|v-\tilde{v}\|^2 + q\|u-\tilde{u}\|^2 + (q+r)\|w-\tilde{w}\|^2 \\
&= \left(1 + q + \frac{2a^2}{b}\right)\|u-\tilde{u}\|^2 + \left(2 + \frac{2\beta^2}{b}\right)\|v-\tilde{v}\|^2 + (1+q+r)\|w-\tilde{w}\|^2 \\
&\le C^*\left(\|u-\tilde{u}\|^2 + \|v-\tilde{v}\|^2 + \|w-\tilde{w}\|^2\right) = C^*\|g-\tilde{g}\|^2.
\end{aligned} \tag{3.35}$$

Thus the inequality (3.31) is satisfied by f on the set M with a uniform coefficient

$$C^* = \max\left\{1 + q + \frac{2a^2}{b},\ 2 + \frac{2\beta^2}{b},\ 1 + q + r\right\}. \tag{3.36}$$

Next we prove the E to H Lipschitz condition (3.30) of the Nemytskii operator f. Due to the Sobolev embedding $E = H^1(\Omega, \mathbb{R}^3) \hookrightarrow L^6(\Omega, \mathbb{R}^3) \hookrightarrow L^4(\Omega, \mathbb{R}^3)$, there are positive constants δ_1 and δ_2 such that

$$\|\cdot\|_{L^4(\Omega)} \le \delta_1 \|\cdot\|_{H^1(\Omega)} \quad \text{and} \quad \|\cdot\|_{L^6(\Omega)} \le \delta_2 \|\cdot\|_{H^1(\Omega)}.$$

Since M is an invariant and bounded set in E, we define

$$N_1 = \max_{g\in M} \|u\|_{L^4}, \quad N_2 = \max_{g\in M} \|u\|_{L^6}.$$

Then we obtain

$$
\begin{aligned}
\|f(g)- f(\tilde g)\|_H^2 &= \|f_1(g) - f_1(\tilde g)\|^2 + \|f_2(g) - f_2(\tilde g)\|^2 + \|f_3(g) - f_3(\tilde g)\|^2 \\
&\le (a\|u^2 - \tilde u^2\| + b\|u^3 - \tilde u^3\| + \|v - \tilde v\| + \|w - \tilde w\|)^2 \\
&\quad + (\beta\|u^2 - \tilde u^2\| + \|v - \tilde v\|)^2 + (q\|u - \tilde u\| + r\|w - \tilde w\|)^2 \\
&\le 4(a^2\|u^2 - \tilde u^2\|^2 + b^2\|u^3 - \tilde u^3\|^2 + \|v - \tilde v\|^2 + \|w - \tilde w\|^2) \\
&\quad + 2(\beta^2\|u^2 - \tilde u^2\|^2 + \|v - \tilde v\|^2) + 2(q^2\|u - \tilde u\|^2 + r^2\|w - \tilde w\|^2) \\
&= (4a^2 + 2\beta^2)\|u^2 - \tilde u^2\|^2 + 4b^2\|u^3 - \tilde u^3\|^2 + 2q^2\|u - \tilde u\|^2 \\
&\quad + 6\|v - \tilde v\|^2 + (4 + 2r^2)\|w - \tilde w\|^2.
\end{aligned}
\tag{3.37}
$$

Note that Hölder inequality implies

$$
\begin{aligned}
\|u^2 - \tilde u^2\|^2 = \|(u - \tilde u)(u + \tilde u)\|^2 &= \int_\Omega |u(x) - \tilde u(x)|^2|u(x) + \tilde u(x)|^2\, dx \\
&\le \|u - \tilde u\|_{L^4}^2\|u + \tilde u\|_{L^4}^2 \le 4\,\delta_1^2 N_1^2\|u - \tilde u\|_{H^1(\Omega)}^2
\end{aligned}
$$

and

$$
\begin{aligned}
&\|u^3 - \tilde u^3\|^2 = \|(u - \tilde u)(u^2 + u\tilde u + \tilde u)^2\|^2 \\
&= \int_\Omega |u(x) - \tilde u(x)|^2|u^2(x) + u(x)\tilde u(x) + \tilde u^2(x)|^2\, dx \\
&\le \left(\int_\Omega |u(x) - \tilde u(x)|^6\, dx\right)^{1/3}\left(\int_\Omega |u^2(x) + u(x)\tilde u(x) + \tilde u^2(x)|^3\, dx\right)^{2/3} \\
&= \|u - \tilde u\|_{L^6}^2\|u^2 + u\tilde u + \tilde u^2\|_{L^6}^4 \le 4\delta_2^2\|u - \tilde u\|_{H^1}^2\|2u^2 + 2\tilde u^2\|_{L^3}^2 \\
&\le 4\delta_2^2\|u - \tilde u\|_{H^1}^2 \cdot (4\|u\|_{L^6}^4 + 4\|\tilde u\|_{L^6}^4) \le 32\,\delta_2^2\, N_2^4\|u - \tilde u\|_{H^1(\Omega)}^2.
\end{aligned}
$$

Substitute the above two inequalities into (3.37). We obtain

$$
\begin{aligned}
\|f(g) - f(\tilde g)\|_H^2 \le &\left(4\,\delta_1^2N_1^2(4a^2 + 2\beta^2) + 128\, b^2\delta_2^2\, N_2^4 + 2q^2\right)\|u - \tilde u\|_{H^1(\Omega)}^2 \\
&+ 6\|v - \tilde v\|^2 + (4 + 2r^2)\|w - \tilde w\|^2,
\end{aligned}
\tag{3.38}
$$

which shows that (3.30) is valid with the constant $C_E(M) > 0$ given by

$$C_E(M) = \sqrt{\max\{4\,\delta_1^2 N_1^2(4a^2 + 2\beta^2) + 128\,b^2\delta_2^2\,N_2^4 + 2q^2,\ 6,\ 4 + 2r^2\}}.$$

The proof is completed. $\square$

Finally we prove the main result on the existence of an exponential attractor for the Hindmarsh-Rose semiflow $\{S(t)\}_{t\geq 0}$ generated by the Hindmarsh-Rose evolutionary equation (2.9)

Theorem 3.3.2. *Under the same assumptions as in Theorem* 2.2.2, *there exists an exponential attractor $\mathscr{E}$ in the space $H = L^2(\Omega, \mathbb{R}^3)$ for the Hindmarsh-Rose semiflow $\{S(t)\}_{t\geq 0}$ generated by the weak solutions of the diffusive Hindmarsh-Rose equations* (2.9).

Proof. The following steps will check all the three conditions stated in Theorem 3.1.5 are satisfied for the diffusive Hindmarsh-Rose equations (2.9).

Step 1. First we show that there exists a compact, positively invariant and absorbing set $M \subset H$ for the Hindmarsh-Rose semiflow $\{S(t)\}_{t\geq 0}$ such that (3.5) and (3.6) are satisfied. Then according to Theorem 3.2.1 the squeezing property is satisfied for the mapping $S(t^*)$ at $t^* = 1$ on this set M.

Theorem 3.1.1 has shown that the closed and bounded ball $B_E(Q)$ centered at the origin with radius $Q > 0$ in the space $E = H^1(\Omega, \mathbb{R}^3)$ is an absorbing set for this semiflow. We can easily verify that the set

$$M = \overline{\bigcup_{0\leq t\leq T^*} S(t)B_E(Q)} \tag{3.39}$$

is a compact, positively invariant and absorbing set in the space H for this semiflow, where $T^* = T^*(B_E(Q))$ is the permanently entering time for the solution trajectories starting from the ball $B_E(Q)$ into itself, as indicated in (3.1). By the boundedness of M in the space E and the compact embedding $E \hookrightarrow H$ so that the cylinder $[0, T^*] \times B_E(Q)$ is a compact set in $\mathbb{R} \times H$ and by the fact that the function

$$\gamma(t, g) = S(t)g \ \text{ is continuous on } \ [0, T^*] \times B_E(Q). \tag{3.40}$$

These two facts infer that the set M is a compact set in H.

In Lemma 3.3.1 it has been shown that the nonlinear mapping $f(g)$ given in (2.11) satisfies the Lipschitz continuous condition (3.5) and the monotone condition (3.6) on this set M given in

(3.39). Moreover by the continuity of the functions $\gamma(t, g)$ in (3.40), we see that

$$G = \max\{\|\gamma(t, g)\|_E : (t, g) \in [0, T^*] \times B_E(Q)\} < \infty. \tag{3.41}$$

Thus we can apply Theorem 3.2.1 with its proof to confirm that the squeezing property is satisfied by the mapping $S(t^*)$ at $t^* = 1$ so that the squeezing property is satisfied by the Hindmarsh-Rose semiflow $\{S(t)\}_{t\geq 0}$ on this set M in H. Therefore, the first condition in Theorem 3.1.5 is satisfied by the Hindmarsh-Rose semiflow.

Step 2. Next we show that, for the Hindmarsh-Rose semiflow and for any $t \in [0, t^*] = [0, 1]$, the mapping $S(t) : M \to M$ is Lipschitz continuous in H and the associated Lipschitz constant $K(t) : [0, 1] \to (0, \infty)$ is a bounded function.

For this purpose, consider any two $g_0 = (u_0, v_0, w_0), \tilde{g}_0 = (\tilde{u}_0, \tilde{v}_0, \tilde{w}_0) \in M$ and the solutions $g(t) = S(t)g_0$ and $\tilde{g}(t) = S(t)\tilde{g}_0$ for $t \in [0, 1]$. Then $h(t) = g(t) - \tilde{g}(t)$ satisfies the equation and the following initial condition,

$$\begin{aligned} \frac{dh}{dt} &= Ah + f(g) - f(\tilde{g}), \quad t > 0, \\ h(0) &= h_0 = g_0 - \tilde{g}_0. \end{aligned} \tag{3.42}$$

The three component functions of $h(t) = (U(t), V(t), W(t))$ can be estimated as follows. First,

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|U\|^2 + d_1\|\nabla U\|^2 = \langle f_1(g) - f_1(\tilde{g}), u - \tilde{u}\rangle \\ &= \langle (\varphi(u) - \varphi(\tilde{u})) + (v - \tilde{v}) - (w - \tilde{w}), u - \tilde{u}\rangle \\ &= \int_\Omega \big(a(u^2 - \tilde{u}^2) - b(u^3 - \tilde{u}^3) + (v - \tilde{v}) - (w - \tilde{w})\big)(u - \tilde{u})\, dx \\ &= \int_\Omega \big(a(u - \tilde{u})^2(u + \tilde{u}) - b(u - \tilde{u})^2(u^2\tilde{u} + u\tilde{u} + \tilde{u}^2)\big)\, dx \\ &\quad + \int_\Omega ((v - \tilde{v})(u - \tilde{u}) - (w - \tilde{w})(u - \tilde{u}))\, dx \\ &\leq \int_\Omega (u - \tilde{u})^2\left[a(u + \tilde{u}) - b(u^2 + u\tilde{u} + \tilde{u}^2)\right] dx \\ &\quad + \|u - \tilde{u}\|(\|v - \tilde{v}\| + \|w - \tilde{w}\|) \\ &\leq \int_\Omega (u - \tilde{u})^2\left[a(u + \tilde{u}) - b(u^2 + u\tilde{u} + \tilde{u}^2)\right] dx + 2\|g - \tilde{g}\|^2 \end{aligned} \tag{3.43}$$

and by Young's inequality we have

$$
\begin{gathered}
a(u+\tilde{u}) - b(u^2 + u\tilde{u} + \tilde{u}^2) = [a(u+\tilde{u}) - bu\tilde{u}] - b(u^2+\tilde{u}^2) \\
\le \left(\frac{b}{4}u^2 + \frac{a^2}{b}\right) + \left(\frac{b}{4}\tilde{u}^2 + \frac{a^2}{b}\right) + \frac{b}{2}(u^2+\tilde{u}^2) - b(u^2+\tilde{u}^2) \le -\frac{b}{4}(u^2+\tilde{u}^2) + \frac{2a^2}{b}.
\end{gathered}
$$

It follows that

$$
\begin{aligned}
&\frac{d}{dt}\|U\|^2 \le \frac{d}{dt}\|U\|^2 + 2d_1\|\nabla U\|^2 \\
&\le 2\int_\Omega (u-\tilde{u})^2 \left(-\frac{b}{4}(u^2+\tilde{u}^2) + \frac{2a^2}{b}\right) dx + 4\|g-\tilde{g}\|^2 \\
&\le \int_\Omega (u-\tilde{u})^2 \left(-\frac{b}{2}(u^2+\tilde{u}^2)\right) dx + \frac{4a^2}{b}\|u-\tilde{u}\|^2 + 4\|g-\tilde{g}\|^2 \\
&\le -\frac{b}{2}\int_\Omega (u-\tilde{u})^2(u^2+\tilde{u}^2)\, dx + 4\left(1+\frac{a^2}{b}\right)\|h\|^2.
\end{aligned}
\tag{3.44}
$$

Similarly, for the second and third components of $h(t) = g(t) - \tilde{g}(t) = (U(t), V(t), W(t))$, we get

$$
\begin{aligned}
&\frac{d}{dt}\|V\|^2 \le \frac{d}{dt}\|V\|^2 + 2d_2\|\nabla V\|^2 \le 2\langle \psi(u) - \psi(\tilde{u}) - (v-\tilde{v}), v-\tilde{v}\rangle \\
&= 2\int_\Omega \left(-\beta(u^2-\tilde{u}^2) - (v-\tilde{v})\right)(v-\tilde{v})\, dx \\
&\le 2\int_\Omega \left(-\beta(u-\tilde{u})u(v-\tilde{v}) - \beta(u-\tilde{u})\tilde{u}(v-\tilde{v})\right) dx \\
&\le \int_\Omega \left(\frac{bu^2}{2}(u-\tilde{u})^2 + \frac{b\tilde{u}^2}{2}(u-\tilde{u})^2\right) dx + \frac{4\beta}{b}\|v-\tilde{v}\|^2 \\
&\le \frac{b}{2}\int_\Omega (u^2+\tilde{u}^2)(u-\tilde{u})^2\, dx + \frac{4\beta}{b}\|h\|^2
\end{aligned}
\tag{3.45}
$$

and

$$
\begin{aligned}
&\frac{d}{dt}\|W\|^2 \le \frac{d}{dt}\|W\|^2 + 2d_3\|\nabla W\|^2 \le 2\langle q(u-\tilde{u}) - r(w-\tilde{w}), w-\tilde{w}\rangle \\
&= 2\int_\Omega \left(q(u-\tilde{u}) - r(w-\tilde{w})\right)(w-\tilde{w})\, dx \\
&\le q\|u-\tilde{u}\|^2 + (q+2r)\|w-\tilde{w}\|^2 \le 2(q+r)\|h\|^2.
\end{aligned}
\tag{3.46}
$$

Add up the inequalities (3.44), (3.45) and (3.46) with a cancellation of the first terms on the rightmost side of (3.44) and (3.45). Then we obtain

$$\frac{d}{dt}\|h\|^2 = \frac{d}{dt}\left(\|U\|^2 + \|V\|^2 + \|W\|^2\right) \le C_* \|h\|^2, \quad t > 0, \tag{3.47}$$

where C_* is a positive constant given by

$$C_* = 4\left(1 + \frac{\beta}{b} + \frac{a^2}{b}\right) + 2(q + r).$$

Solve the differential inequality (3.47) to get

$$\|g(t) - \tilde{g}(t)\| = \|h(t)\| \le e^{C_* t/2}\|h(0)\| = K(t)\|g_0 - \tilde{g}_0\|, \quad t \ge 0, \tag{3.48}$$

where $K(t) = e^{C_* t/2} \in [1, e^{C_*/2}]$ is a bounded function on the time interval $t \in [0, t^*]$, $t^* = 1$. The claim at the beginning of this step is proved.

Step 3. Finally we show that for any given $g \in M$ the mapping $S(\cdot)g_0 : [0, t^*] = [0, 1] \to M$ is Lipschitz continuous and the associated Lipschitz constant $L(g_0) : M \to (0, \infty)$ is a bounded function.

For any given $g_0 \in M$, since the weak solution $S(t)g_0, t \ge 0$, is a mild solution for the evolutionary equation (2.9), we have

$$S(t)g_0 = e^{At}g_0 + \int_0^t e^{A(t-s)} f(g(s, g_0))\, dt, \quad t \ge 0, \tag{3.49}$$

where the operator A and the nonlinear mapping f are defined in (2.10) and (2.11), respectively. Note that the parabolic semigroup $\{e^{At}\}_{t \ge 0}$ is a self-adjoint contraction semigroup so that $\max_{t \ge 0} \|e^{At}\|_{\mathcal{L}(H)} = 1$. A fundamental theorem on sectorial operators [62, Theorem 37.5] shows that the operator function $e^{At} : [0, \infty) \to \mathcal{L}(H)$ is uniformly Lipschitz continuous. Actually, the spectral expansion of e^{At} shows

$$(e^{At}g_0)(x) = \sum_{k=1}^{\infty} e^{-\lambda_k t}\langle g_0, e_k\rangle e_k(x), \quad g \in H, \; t \ge 0,$$

where $\{-\lambda_k\}_{k=1}^{\infty}$, with $0 \le \lambda_k \to \infty$ as $k \to \infty$, is the set of all the eigenvalues (repeated to the respective multiplicities) of $A : D(A) \to H$, and $\{e_k\}_{k=1}^{\infty}$ with $Ae_k = -\lambda_k e_k$ is the complete set of the orthonormal eigenvectors of A. Then we can derive the Lipschitz continuity of e^{At} as follow. For any $g_0 \in M$ and any $0 \le \tau < t$,

$$\begin{aligned}
&\|e^{At}g_0 - e^{A\tau}g_0\|^2 = \sum_{k=1}^{\infty} |e^{-\lambda_k(t-\tau)}|^2 |\langle g_0, e_k\rangle|^2 \\
=& \sum_{k=1}^{\infty} |e^{-\zeta_k}|^2 \lambda_k |t-\tau| |\langle g_0, e_k\rangle|^2 \ \ (\text{where } 0 \le \lambda_k \tau \le \zeta_k \le \lambda_k t) \\
\le& |t-\tau| \|\nabla g_0\|^2 \le |t-\tau| \|g_0\|_E^2 \le G^2 |t-\tau|.
\end{aligned} \tag{3.50}$$

Therefore, we can deduce that, for any $0 \le \tau < t$,

$$\begin{aligned}
&\|S(t)g_0 - S(\tau)g_0\|_H \le \|e^{At}g_0 - e^{A\tau}g_0\| + \int_\tau^t \|e^{A(t-s)} f(g(s,g_0))\| \, dt \\
\le& G^2|t-\tau| + \int_\tau^t \|e^{A(t-s)}\|_{\mathcal{L}(H)} \|f(g(s,g_0))\|_H \, dt \\
\le& G^2|t-\tau| + \int_\tau^t \|f(g(s,g_0)) - f(0)\|_H \, dt + \int_\tau^t \|f(0)\|_H \, dt \\
\le& G^2|t-\tau| + \int_\tau^t C_E(M) \|g(s,g_0)\|_E \, dt + (J + \alpha + q|c|)|t-\tau| \\
\le& G^2|t-\tau|\|| + C_E(M)G^2|t-\tau| + (J+\alpha+q|c|)|t-\tau| \\
\le& L(M)|t-\tau|,
\end{aligned} \tag{3.51}$$

where the Lipschitz constant $C_E(M)$ is given in (3.30) and

$$L(M) = (1 + C_E(M))G^2 + (J + \alpha + q|c|).$$

Then clearly the claim in Step 3 is proved.

Since we have proved that all the three conditions in Theorem 3.1.5 are satisfied by the Hindmarsh-Rose semiflow, there exists an exponential attractor $\mathscr{E}$ in the space H for this Hindmarsh-Rose semiflow. The proof is completed. $\square$

The existence of an exponential attractor as well as the squeezing property have the following meaningful corollaries on the finite fractal dimensionality of the global attractor shown in Chapter

2 and on the determining modes.

Corollary 3.3.3. *The global attractor $\mathscr{A}$ of the Hindmarsh-Rose semiflow has a finite fractal dimension*

$$\dim_F(\mathscr{A}) \le N \max\left\{1, \frac{\log(2\sqrt{2}K/\theta + 1)}{-\log\theta}\right\}, \quad \theta \in (0,1), \tag{3.52}$$

where N is the rank of the spectral projection associated with the squeezing property of the mapping $S(1)$ and K is the Lipschitz constant of the mapping $S(1)$ on the compact, positively invariant, absorbing set M.

Proof. This result is simply implied by the inclusion of the global attractor $\mathscr{A}$ in the exponential attractor $\mathscr{E}$,

$$\mathscr{A} \subset \mathscr{E}$$

because $\lim_{t\to\infty} dist_H(S(t)\mathscr{A}, \mathscr{E}) = dist_H(\mathscr{A}, \mathscr{E}) = 0$, and that by definition the exponential attractor $\mathscr{E}$ has a finite fractal dimension. The estimate (3.52) follows from Theorem 3.1.5. □

Corollary 3.3.4. *Under the same assumptions as in Theorem* 2.2.2*, the orthogonal projection of the trajectories in the global attractor $\mathscr{A}$ on the finite dimensional subspace PH of the low modes is determining in the sense that, for two trajectories $g(t)$ and $\tilde{g}(t)$ in $\mathscr{A}$, if*

$$\|Pg(t) - P\tilde{g}(t)\|_H \to 0, \quad \textit{as} \;\; t \to \infty,$$

then

$$\|g(t) - \tilde{g}(t)\|_H \to 0, \quad \textit{as} \;\; t \to \infty.$$

Here the finite-rank orthogonal projection P is affiliated with the corresponding squeezing property of the Hindmarsh-Rose semiflow.

This Corollary 3.3.4 is a consequence of the squeezing property of the Hindmarsh-Rose semiflow shown in Theorem 3.1.2 above and Theorem 14.3 in [56].

Chapter 4

Global Dynamics of Nonautonomous Diffusive Hindmarsh-Rose Equations

Note to Reader

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In this chapter, we shall study the global dynamics for the nonautonomous diffusive Hindmarsh-Rose equations with time-dependent external inputs:

$$\frac{\partial u}{\partial t} = d_1 \Delta u + \varphi(u) + v - w + J + p_1(t,x) \tag{4.1}$$

$$\frac{\partial v}{\partial t} = d_2 \Delta v + \psi(u) - v + p_2(t,x), \tag{4.2}$$

$$\frac{\partial w}{\partial t} = d_3 \Delta w + q(u - c) - rw + p_3(t,x), \tag{4.3}$$

for $t > \tau \in \mathbb{R},\ x \in \Omega \subset \mathbb{R}^n$ ($n \leq 3$), where

$$\varphi(u) = au^2 - bu^3, \quad \text{and} \quad \psi(u) = \alpha - \beta u^2. \tag{4.4}$$

Assume that the external input terms $p_i \in L^2_{loc}(\mathbb{R}, L^2(\Omega)), i = 1, 2, 3$, satisfy the condition of translation boundedness [13],

$$\|p_i\|^2_{L^2_b} = \sup_{t \in \mathbb{R}} \int_t^{t+1} \int_\Omega |p_i(t,x)|^2 \, dx \, ds < \infty, \quad i = 1, 2, 3. \tag{4.5}$$

We impose the homogeneous Neumann boundary conditions

$$\frac{\partial u}{\partial \nu}(t,x) = 0, \ \ \frac{\partial v}{\partial \nu}(t,x) = 0, \ \ \frac{\partial w}{\partial \nu}(t,x) = 0, \quad t > \tau \in \mathbb{R},\ x \in \partial\Omega, \tag{4.6}$$

and the initial conditions to be specified are denoted by

$$u(\tau, x) = u_\tau(x),\ v(\tau, x) = v_\tau(x),\ w(\tau, x) = w_\tau(x), \quad x \in \Omega. \tag{4.7}$$

The nonautonomous system (4.1)–(4.3) with the initial-boundary conditions (4.6) and (4.7) can be written in the vector form

$$\begin{aligned} \frac{\partial g}{\partial t} &= Ag + f(g) + p(t, x), \quad t > \tau \in \mathbb{R}, \\ g(\tau) &= g_\tau, \end{aligned} \tag{4.8}$$

where

$$g(t) = \text{col}\,(u(t, \cdot), v(t, \cdot), w(t, \cdot)), \quad g_\tau = \text{col}\,(u_\tau, v_\tau, w_\tau),$$

and $p(t, x) = \text{col}\,(p_1(t, x), p_2(t, x), p(t, x))$, the nonpositive self-adjoint operator

$$A = \begin{pmatrix} d_1\Delta & 0 & 0 \\ 0 & d_2\Delta & 0 \\ 0 & 0 & d_3\Delta \end{pmatrix} : D(A) \to H, \tag{4.9}$$

where

$$D(A) = \{g \in H^2(\Omega, \mathbb{R}^3) : \partial g/\partial \nu = 0\},$$

is the generator of an analytic C_0-semigroup $\{e^{At}\}_{t \geq 0}$ on the Hilbert space H [62]. By the fact that $H^1(\Omega) \hookrightarrow L^6(\Omega)$ is a continuous imbedding for space dimension $n \leq 3$ and by the Hölder inequality, there is a constant $C_0 > 0$ such that

$$\|\varphi(u)\| \leq C_0 \left(1 + \|u\|_{L^6}^3\right) \quad \text{and}$$

$$\|\psi(u)\| \leq C_0 \left(1 + \|u\|_{L^4}^2\right) \quad \text{for } u \in L^6(\Omega).$$

Therefore, the nonlinear mapping

$$f(u,v,w) = \begin{pmatrix} \varphi(u) + v - w + J \\ \\ \psi(u) - v, \\ \\ q(u-c) - rw \end{pmatrix} : E \longrightarrow H \tag{4.10}$$

is a locally Lipschitz continuous mapping. The existence and uniqueness of the weak solutions local in time is defined by Definition 1.3.1 with simple time-varying adaption and similarly proved as in Lemma 1.3.2.

4.1 Pullback Attractor and Pullback Exponential Attractor

Below we present two existing results on the sufficient conditions for the existence of pullback attractor and for the existence of pullback exponential attractor, respectively.

Proposition 4.1.1. [8, 9, 11, 43] *A nonautonomous process $\{S(t,\tau)\}_{t\geq\tau\in\mathbb{R}}$ on a Banach space X has a unique pullback attractor $\mathcal{A} = \{\mathcal{A}(\tau)\}_{\tau\in\mathbb{R}}$, if the following two conditions are satisfied*:

(i) *There is a pullback absorbing set M in X, which means that for any given bounded set $B \subset X$, there is a finite time $T_B > 0$ such that*

$$S(\tau, \tau - t)B \subset M, \quad \text{for all } \; t > T_B. \tag{4.11}$$

(ii) *The nonautonomous process $S(t,\tau)$ is pullback asymptotically compact in the sense that for any sequences $t_k \to \infty$ and $\{x_k\} \subset B$, where B is any given bounded set in X, the sequence $\{S(\tau, \tau - t_k)x_k)\}_{k=1}^{\infty}$ has a convergent subsequence.*

Moreover, the pullback attractor is given by

$$\mathcal{A}(\tau) = \bigcap_{s\geq 0} \overline{\bigcup_{t\geq s} S(\tau, \tau - t)M}. \tag{4.12}$$

Proposition 4.1.2. [19, 20] *Let X and Y be Banach spaces and Y compactly embedded in X. Assume that $\{S(t,\tau) \in \mathcal{L}(X) \cap \mathcal{L}(Y) : t \geq \tau \in \mathbb{R}\}$ be a nonautonomous process such that the following three conditions are satisfied*:

1) *There exists a bounded pullback absorbing set $M^* \subset Y$ uniformly in time in the sense that,*

for any bounded set $B \subset X$, there is a finite time $T_B > 0$ such that

$$\bigcup_{\tau \in \mathbb{R}} S(\tau, \tau - t)B \subset M^*, \quad \text{for all } \ t > T_B. \tag{4.13}$$

2) *The smoothing Lipschitz continuity is satisfied*: *There is a constant $\kappa > 0$ such that for the aforementioned bounded pullback absorbing set $M^* \subset Y$,*

$$\sup_{\tau \in \mathbb{R}} \|S(\tau, \tau - T_{M^*})g_1 - S(\tau, \tau - T_{M^*})g_2\|_Y \leq \kappa \|g_1 - g_2\|_X, \ \textit{for any} \ \ g_1, g_2 \in M^*. \tag{4.14}$$

3) *The Hölder/Lipschitz continuity in time is satisfied*: *There exist two exponents $\gamma_1, \gamma_2 \in (0, 1]$ such that for the aforementioned set $M^* \subset Y$,*

$$\sup_{\tau \in \mathbb{R}} \|S(\tau, \tau - T_{M^*})g - S(\tau, \tau - T_{M^*} - t)g\|_X \leq c_1 |t|^{\gamma_1}, \ \ t \in [0, T_{M^*}], \ g \in M^*, \tag{4.15}$$

$$\sup_{\tau \in \mathbb{R}} \|S(\tau, \tau - t_1)g - S(\tau, \tau - t_2)g\|_X \leq c_2 |t_1 - t_2|^{\gamma_2}, \ \ t_1, t_2 \in [T_{M^*}, 2T_{M^*}], \ g \in M^*. \tag{4.16}$$

In (4.14)–(4.16)*, $T_{M^*} > 0$ is the time when all the pullback trajectories starting from M^* permanently enter the absorbing set M^* itself, and $c_1 = c_1(M^*), c_2 = c_2(M^*)$ are two positive constants. Then there exists a pullback exponential attractor $\mathscr{M} = \{\mathscr{M}(\tau)\}_{\tau \in \mathbb{R}}$ in X for this process.*

Remark 4.1.3. The pullback absorbing set can be a time-parametrized set $M(\tau)$ in X or in Y. Here the pullback absorbing sets specified in the above Proposition (4.1.1) and Proposition (4.1.2) are time-invariant, which is what we only need.

Remark 4.1.4. Another concept to describe the asymptotic global dynamics of a nonautonomous PDE is a skew-product dynamical systems [62]. It is to embed a nonautonomous semiflow into an augmented autonomous semiflow. The corresponding topic is uniform attractor [13, Chapter IV].

Although a uniform attractor is not a time-parametrized set, the major drawback is that the fractal dimension and Hausdorff dimension of a uniform attractor are in general infinite. The finite dimensionality reduction is lost. Moreover, it is usually difficult to estimate the oftentimes slow rate of attraction for a uniform attractor in terms of physical parameters in the mathematical model. Therefore, pullback attractor and pullback exponential attractor are favorable pursuit of the asymptotic behavior of nonautonomous dynamical systems generated by PDEs.

4.2 Pullback Attractor for Nonautonomous Diffusive Hindmarsh-Rose Process

In this section, we shall first prove the global existence in time of the weak solutions to the system (4.8) and then show the pullback absorbing property of the nonautonomous Hindmarsh-Rose process in the space H and also in the space E, which leads to the existence of a pullback attractor for this nonautonomous semiflow.

Lemma 4.2.1. *The weak solution of the nonautonomous system* (4.8) *for any initial time* $\tau \in \mathbb{R}$ *and any initial data* $g_\tau \in H$ *exists globally for* $t \in [\tau, \infty)$ *and it generates a continuous evolution process* $\{S(t,\tau) \in \mathcal{L}(H) \cap \mathcal{L}(E) : t \geq \tau \in \mathbb{R}\}$,

$$S(t,\tau)g_\tau = g(t,\tau,g_\tau) = \text{col}\,(u,v,w)(t,\tau,g_\tau) \tag{4.17}$$

which is called the nonautonomous Hindmaersh-Rose process. Moreover, there exists a time-invariant pullback absorbing set in the space H,

$$M_H^* = \{g \in H : \|g\|^2 \leq K_1\} \tag{4.18}$$

where K_1 *is a positive constant independent of* τ *and* t *such that for any given bounded set* $B \subset H$,

$$S(\tau, \tau - t)B \subset M_H^*, \quad \text{for } \ t \geq T_B, \tag{4.19}$$

where the constant $T_B > 0$ *depend only on* $\|B\| = \sup_{g \in B} \|g\|$.

Proof. Take the H inner-product $\langle (4.8), (c_1 u, v, w)\rangle$ with constant $c_1 > 0$ to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(c_1\|u\|^2 + \|v\|^2 + \|w\|^2\right) + \left(c_1 d_1\|\nabla u\|^2 + d_2\|\nabla v\|^2 + d_3\|\nabla w\|^2\right)\\
&= \int_\Omega c_1(au^3 - bu^4 + uv - uw + Ju + up_1(t,x))\,dx\\
&\quad + \int_\Omega (\alpha v - \beta u^2 v - v^2 + vp_2(t,x) + q(u-c)w - rw^2 + wp_3(t,x))\,dx\\
&\leq \int_\Omega c_1(au^3 - bu^4 + uv - uw + Ju + up_1(t,x))\,dx
\end{aligned} \tag{4.20}$$

$$+\int_\Omega \left[\left(2\alpha^2+\frac{1}{2}\beta^2u^4-\frac{3}{8}v^2\right)+\left(\frac{q^2}{r}(u^2+c^2)-\frac{1}{2}rw^2\right)+vp_2+wp_3\right]dx$$
$$\leq \int_\Omega c_1(au^3-bu^4+uv-uw+Ju+up_1(t,x))\,dx$$
$$+\int_\Omega \left[\left(2\alpha^2+\frac{1}{2}\beta^2u^4-\frac{3}{8}v^2\right)+\left(\frac{q^2}{r}(u^2+c^2)-\frac{1}{2}rw^2\right)\right]dx$$
$$+\int_\Omega \left[\frac{1}{8}v^2+2|p_2(t,x)|^2+\frac{1}{8}rw^2+\frac{2}{r}|p_3(t,x)|^2\right]dx.$$

Choose the positive constant in (4.20) to be $c_1=\frac{1}{b}(\beta^2+3)$ so that

$$-c_1\int_\Omega bu^4\,dx+\int_\Omega \beta^2u^4\,dx\leq -3\int_\Omega u^4\,dx.$$

Note that

$$\int_\Omega c_1au^3\,dx\leq \frac{3}{4}\int_\Omega u^4\,dx+\frac{1}{4}(c_1a)^4|\Omega|\leq \int_\Omega u^4\,dx+(c_1a)^4|\Omega|,$$

and

$$\int_\Omega c_1(uv-uw+Ju+up_1(t,x))\,dx\leq \int_\Omega \Bigg[2(c_1u)^2+\frac{1}{8}v^2+\frac{(c_1u)^2}{r}+\frac{1}{4}rw^2$$
$$+\frac{1}{2}\left((c_1u)^2+J^2+(c_1u)^2+|p_1(t,x)|^2\right)\Bigg]dx.$$

The collection of all integral terms of u^2 in the above inequality and in (4.20) satisfies

$$\int_\Omega \left(2(c_1u)^2+\frac{(c_1u)^2}{r}+(c_1u)^2+\frac{q^2}{r}u^2\right)dx\leq \int_\Omega u^4\,dx+\left[c_1^2\left(3+\frac{1}{r}\right)+\frac{q^2}{r}\right]^2|\Omega|.$$

Substitute these inequalities into (4.20). Then we get

$$\frac{1}{2}\frac{d}{dt}\left(c_1\|u\|^2+\|v\|^2+\|w\|^2\right)+\left(c_1d_1\|\nabla u\|^2+d_2\|\nabla v\|^2+d_3\|\nabla w\|^2\right)$$
$$\leq \int_\Omega c_1(au^3-bu^4+uv-uw+Ju+up_1(t,x))\,dx$$
$$+\int_\Omega \left[\left(2\alpha^2+\frac{1}{2}\beta^2u^4-\frac{3}{8}v^2\right)+\left(\frac{q^2}{r}(u^2+c^2)-\frac{1}{2}rw^2\right)\right]dx$$

$$+\int_\Omega \left[\frac{1}{8}v^2 + 2|p_2(t,x)|^2 + \frac{1}{8}rw^2 + \frac{2}{r}|p_3(t,x)|^2\right] dx$$

$$\begin{aligned}
&\le \int_\Omega (2-3)u^4\,dx + \int_\Omega \left(\frac{1}{8} - \frac{3}{8} + \frac{1}{8}\right) v^2\,dx + \int_\Omega \left(\frac{1}{4} - \frac{1}{2} + \frac{1}{8}\right) rw^2\,dx \\
&\quad + \int_\Omega \left[\frac{1}{2}|p_1(t,x)|^2 + 2|p_2(t,x)|^2 + \frac{2}{r}|p_3(t,x)|^2\right] dx \\
&\quad + \left((c_1a)^4 + J^2 + \left[c_1^2\left(3+\frac{1}{r}\right) + \frac{q^2}{r}\right]^2 + 2\alpha^2 + \frac{q^2c^2}{r}\right)|\Omega| \\
&\le -\int_\Omega \left(u^4(t,x) + \frac{1}{8}v^2(t,x) + \frac{1}{8}rw^2(t,x)\right) dx + \left(2+\frac{2}{r}\right)\|p(t)\|^2 + c_2|\Omega|,
\end{aligned} \tag{4.21}$$

where

$$c_2 = (c_1a)^4 + J^2 + \left[c_1^2\left(3+\frac{1}{r}\right) + \frac{q^2}{r}\right]^2 + 2\alpha^2 + \frac{q^2c^2}{r}.$$

It follows that

$$\begin{aligned}
&\frac{d}{dt}(c_1\|u(t)\|^2 + \|v(t)\|^2 + \|w(t)\|^2) + 2d(c_1\|\nabla u\|^2 + \|\nabla v\|^2 + \|\nabla w\|^2) \\
&+ \int_\Omega \left(2u^4(t,x) + \frac{1}{4}v^2(t,x) + \frac{1}{4}rw^2(t,x)\right) dx \le 4\left(1+\frac{1}{r}\right)\|p(t)\|^2 + 2c_2|\Omega|,
\end{aligned}$$

where $d = \min\{d_1, d_2, d_3\}$ and we used

$$2u^4 \ge \frac{1}{4}\left(c_1u^2 - \frac{c_1^2}{32}\right).$$

Therefore,

$$\begin{aligned}
&\frac{d}{dt}(c_1\|u(t)\|^2 + \|v(t)\|^2 + \|w(t)\|^2) + 2d(c_1\|\nabla u\|^2 + \|\nabla v\|^2 + \|\nabla w\|^2) \\
&+\frac{1}{4}\left(c_1\|u(t)\|^2 + \|v(t)\|^2 + r\|w(t)\|^2\right) \le 4\left(1+\frac{1}{r}\right)\|p(t)\|^2 + \left(\frac{c_1^2}{128} + 2c_2\right)|\Omega|
\end{aligned} \tag{4.22}$$

for $t \in [\tau, T_{max})$, the maximum time interval of existence. Set

$$\delta = \frac{1}{4}\min\{1, r\}.$$

Then the Gronwall inequality applied to the inequality reduced from (4.22),

$$
\begin{aligned}
&\frac{d}{dt}(c_1\|u(t)\|^2 + \|v(t)\|^2 + \|w(t)\|^2) + \delta(c_1\|u(t)\|^2 + \|v(t)\|^2 + \|w(t)\|^2) \\
\le\, &4\left(1 + \frac{1}{r}\right)\|p(t)\|^2 + \left(\frac{c_1^2}{128} + 2c_2\right)|\Omega|,
\end{aligned}
$$

shows that

$$
\begin{aligned}
&c_1\|u(t)\|^2 + \|v(t)\|^2 + \|w(t)\|^2 \\
\le\, &e^{-\delta t}(c_1\|u_\tau\|^2 + \|v_\tau\|^2 + \|w_\tau\|^2) \\
&+ 4\left(1 + \frac{1}{r}\right)\int_\tau^t e^{-\delta(t-s)}\|p(s)\|^2\,ds + \frac{1}{\delta}\left(\frac{c_1^2}{128} + 2c_2\right)|\Omega|, \quad t \in [\tau, T_{max}).
\end{aligned}
\tag{4.23}
$$

By the assumption (4.5) on the translation boundedness of the external input terms and the upper bound estimate (4.23), the weak solutions will never blow up at any finite time so that $T_{max} = +\infty$ for all $\tau \in \mathbb{R}$ and any initial data $g_\tau \in H$. Thus, the global existence in time of the weak solutions in the space H is proved. Together with the uniqueness and the continuous dependence of (t, τ, g_τ) which can be shown, the statement of the continuous evolution process $S(t, \tau)$ in (4.17) is proved.

In order to prove the claimed existence of a pullback absorbing set, we can exploit the bounded translation property (4.5) of the time-dependent forcing terms to treat the integral in (4.23) on the time interval $[\tau, t+\tau]$, or equivalently the time interval $[\tau - t, \tau]$, for $t > 0$, as follows:

$$
\begin{aligned}
&c_1\|u(t+\tau)\|^2 + \|v(t+\tau)\|^2 + \|w(t+\tau)\|^2 \\
\le\, &e^{-\delta t}(c_1\|u(\tau)\|^2 + \|v(\tau)\|^2 + \|w(\tau)\|^2) + \frac{1}{\delta}\left(\frac{c_1^2}{128} + 2c_2\right)|\Omega| \\
&+ 4\left(1 + \frac{1}{r}\right)\int_\tau^{t+\tau} e^{-\delta(t+\tau-s)}\|p(s)\|^2\,ds \\
\le\, &e^{-\delta t}(c_1\|u(\tau)\|^2 + \|v(\tau)\|^2 + \|w(\tau)\|^2) + \frac{1}{\delta}\left(\frac{c_1^2}{128} + 2c_2\right)|\Omega| \\
&+ 4\left(1 + \frac{1}{r}\right)\sum_{k=0}^{\infty}\int_{t+\tau-k-1}^{t+\tau-k} e^{-\delta(t+\tau-s)}\|p(s)\|^2\,ds
\end{aligned}
\tag{4.24}
$$

$$\le e^{-\delta t}(c_1\|u(\tau)\|^2 + \|v(\tau)\|^2 + \|w(\tau)\|^2) + \frac{1}{\delta}\left(\frac{c_1^2}{128} + 2c_2\right)|\Omega|$$

$$+ 4\left(1 + \frac{1}{r}\right)\sum_{k=0}^{\infty} e^{-k\delta}\left(\|p_1\|_{L_b^2}^2 + \|p_2\|_{L_b^2}^2 + \|p_3\|_{L_b^2}^2\right)$$

$$= e^{-\delta t}(c_1\|u_\tau\|^2 + \|v_\tau\|^2 + \|w_\tau\|^2) + \frac{1}{\delta}\left(\frac{c_1^2}{128} + 2c_2\right)|\Omega|$$

$$+ 4\left(1 + \frac{1}{r}\right)\frac{1}{1 - e^{-\delta}}\left(\|p_1\|_{L_b^2}^2 + \|p_2\|_{L_b^2}^2 + \|p_3\|_{L_b^2}^2\right).$$

It implies that the global weak solutions of the nonautonomous diffusive Hindmarsh-Rose system (4.8) admit the estimate that, for any $t \ge \tau \in \mathbb{R}$,

$$\|g(t)\|^2 \le \frac{\max\{1, c_1\}}{\min\{1, c_1\}} e^{-\delta(t-\tau)}\|g(\tau)\|^2 + \frac{1}{\delta}\left(\frac{c_1^2}{128} + 2c_2\right)|\Omega| + 4\left(1 + \frac{1}{r}\right)\frac{\|p\|_{L_b^2}^2}{1 - e^{-\delta}}. \tag{4.25}$$

Hence, for any $\tau - t \le \tau \in \mathbb{R}$ with $t > 0$, it holds that

$$\|g(\tau)\|^2 \le \frac{\max\{1, c_1\}}{\min\{1, c_1\}} e^{-\delta t}\|g(\tau - t)\|^2 + \frac{1}{\delta}\left(\frac{c_1^2}{128} + 2c_2\right)|\Omega| + 4\left(1 + \frac{1}{r}\right)\frac{\|p\|_{L_b^2}^2}{1 - e^{-\delta}}. \tag{4.26}$$

Since

$$\lim_{t\to\infty} e^{-\delta(t-\tau)}\|g(\tau)\|^2 = 0$$

uniformly for $g(\tau) = g_\tau$ in any given bounded set $B \subset H$ in regard to (4.25), and

$$\lim_{t\to\infty} e^{-\delta t}\|g(\tau - t)\|^2 = 0$$

uniformly for $g(\tau - t)$ in any given bounded set $B \subset H$ in regard to (4.26), there exists a pullback absorbing set as claimed in (4.18) with the constant

$$K_1 = 1 + \frac{1}{\delta}\left(\frac{c_1^2}{128} + 2c_2\right)|\Omega| + 4\left(1 + \frac{1}{r}\right)\frac{\|p\|_{L_b^2}^2}{1 - e^{-\delta}}, \tag{4.27}$$

which is independent of initial time and initial state in H. Therefore, the pullback absorbing property (4.19) for any given bounded set $B \subset H$ is proved:

$$S(\tau, \tau - t)B \subset M_H^*, \quad \text{for all } \ t \geq T_B,$$

and

$$T_B = \frac{1}{\delta} \log^+ \left(\frac{\max\{1, c_1\}}{\min\{1, c_1\}} \|B\|^2 \right) > 0 \tag{4.28}$$

depends only on $\|B\|$. The proof is completed. □

Lemma 4.2.2. *For the nonautonomous diffusive Hindmarsh-Rose system* (4.8)*, there also exists a time-invariant pullback absorbing set in the space E,*

$$M_E^* = \{g \in E : \|g\|_E^2 \leq K_2\}, \tag{4.29}$$

where K_2 is a positive constant, such that for any given bounded set $B \subset H$,

$$S(\tau, \tau - t)B \subset M_E^* \quad \text{for all } \quad t \geq T_B + 1, \tag{4.30}$$

for any $\tau \in \mathbb{R}$, where the constant T_B is given in (4.28)*.*

Proof. Take the H inner-product $\langle$(4.8), $-\Delta g(t)\rangle$ to obtain

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \left(\|\nabla u\|^2 + \|\nabla v\|^2 + \|\nabla w\|^2 \right) + d_1 \|\Delta u\|^2 + d_2 \|\Delta v\|^2 + d_3 \|\Delta w\|^2 \\
&= \int_\Omega \left(-au^2 \Delta u - 3bu^2 |\nabla u\|^2 - v\Delta u + w\Delta u - J\Delta u - p_1(t,x)\Delta u \right) \, dx \\
&+ \int_\Omega \left(-\alpha \Delta v + \beta u^2 \Delta v - |\nabla v|^2 \right) \, dx + \int_\Omega \left(qc\Delta w - qu\Delta w - r|\nabla w|^2 \right) \, dx \\
&- \int_\Omega \left(p_2(t,x)\Delta v + p_3(t,x)\Delta w \right) \, dx.
\end{aligned}$$

By using Young's inequality appropriately to treat the integral terms on the right-hand side of the above inequality, we can get

$$\begin{aligned}
&\frac{d}{dt} \left(\|\nabla u\|^2 + \|\nabla v\|^2 + \|\nabla w\|^2 \right) + d_1 \|\Delta u\|^2 + d_2 \|\Delta v\|^2 + d_3 \|\Delta w\|^2 \\
&+ 6b\|u\nabla u\|^2 + 2\|\nabla v\|^2 + 2r\|\nabla w\|^2
\end{aligned} \tag{4.31}$$

$$
\begin{aligned}
&\le \frac{4}{d_1}\|v\|^2 + \frac{4}{d_1}\|w\|^2 + \frac{4a^2}{d_1}\|u\|_{L^4}^4 + \frac{8J^2}{d_1}|\Omega| + \frac{8}{d_1}\|p_1(t)\|^2 \\
&\quad + \frac{2\beta^2}{d_2}\|u\|_{L^4}^4 + \frac{4\alpha^2}{d_2}|\Omega| + \frac{4}{d_2}\|p_2(t)\|^2 + \frac{2q^2}{d_3}\|u\|^2 + \frac{4q^2c^2}{d_3}|\Omega| + \frac{4}{d_3}\|p_3(t)\|^2 \\
&= \frac{4}{d_1}\|v\|^2 + \frac{4}{d_1}\|w\|^2 + \frac{2q^2}{d_3}\|u\|^2 + \left(\frac{4a^2}{d_1} + \frac{2\beta^2}{d_2}\right)\|u\|_{L^4}^4 \\
&\quad + \left(\frac{8J^2}{d_1} + \frac{4\alpha^2}{d_2} + \frac{4q^2c^2}{d_3}\right)|\Omega| + \frac{8}{d_1}\|p_1(t)\|^2 + \frac{4}{d_2}\|p_2(t)\|^2 + \frac{4}{d_3}\|p_3(t)\|^2.
\end{aligned}
$$

The Sobolev imbedding $H^1(\Omega) \hookrightarrow L^4(\Omega)$ tells us that there is a positive constant $\rho > 0$ such that

$$
\|u\|_{L^4}^4 \le \rho(\|u\|^2 + \|\nabla u\|^2)^2 \le 2\rho(\|u\|^4 + \|\nabla u\|^4). \tag{4.32}
$$

According to Lemma 4.2.1, for any given bounded set $B \subset H$, we have

$$
\|u(t)\|^2 + \|v(t))\|^2 + \|w(t)\|^2 \le K_1, \quad \text{for any } \ t \ge T_B, \ g_\tau \in B. \tag{4.33}
$$

Then (4.31) yields the following inequality that for any $t \ge T_B$ and $g_\tau \in B$,

$$
\begin{aligned}
&\frac{d}{dt}\left(\|\nabla u\|^2 + \|\nabla v\|^2 + \|\nabla w\|^2\right) + d_1\|\Delta u\|^2 + d_2\|\Delta v\|^2 + d_3\|\Delta w\|^2 \\
&+ 6b\|u\nabla u\|^2 + 2\|\nabla v\|^2 + 2r\|\nabla w\|^2 \\
&\le \max\left\{\frac{4}{d_1}, \frac{4q^2c^2}{d_3}\right\} K_1 + \left(\frac{8a^2}{d_1} + \frac{4\beta^2}{d_2}\right)\rho K_1^2 + \left(\frac{8a^2}{d_1} + \frac{4\beta^2}{d_2}\right)\rho\|\nabla u\|^4 \\
&+ \left(\frac{8J^2}{d_1} + \frac{4\alpha^2}{d_2} + \frac{4q^2c^2}{d_3}\right)|\Omega| + \frac{8}{d_1}\|p_1(t)\|^2 + \frac{4}{d_2}\|p_2(t)\|^2 + \frac{4}{d_3}\|p_3(t)\|^2.
\end{aligned} \tag{4.34}
$$

Hence we can apply the uniform Gronwall inequality [62, Lemma D.3] to the following inequality reduced from (4.34) on $\nabla g(t) = \text{col}\,(\nabla u(t), \nabla v(t), \nabla w(t))$,

$$
\begin{aligned}
&\frac{d}{dt}\|\nabla g(t)\|^2 \le \rho\left(\frac{8a^2}{d_1} + \frac{4\beta^2}{d_2}\right)\|\nabla g\|^2\|\nabla g\|^2 \\
&\quad + \max\left\{\frac{4}{d_1}, \frac{4q^2c^2}{d_3}\right\} K_1 + \left(\frac{8a^2}{d_1} + \frac{4\beta^2}{d_2}\right)\rho K_1^2 \\
&\quad + \left(\frac{8J^2}{d_1} + \frac{4\alpha^2}{d_2} + \frac{4q^2c^2}{d_3}\right)|\Omega| + \frac{8}{d_1}\|p_1(t)\|^2 + \frac{4}{d_2}\|p_2(t)\|^2 + \frac{4}{d_3}\|p_3(t)\|^2
\end{aligned} \tag{4.35}
$$

which is written in the form

$$\frac{d\sigma}{dt} \le \xi\,\sigma + h, \quad \text{for } \ t \ge T_B,\ g_\tau \in B, \tag{4.36}$$

where

$$\sigma(t) = \|\nabla g(t)\|^2, \quad \xi(t) = \rho\left(\frac{8a^2}{d_1} + \frac{4\beta^2}{d_2}\right)\|\nabla g\|^2,$$

and

$$h(t) = \max\left\{\frac{4}{d_1}, \frac{4q^2c^2}{d_3}\right\}K_1 + \left(\frac{8a^2}{d_1} + \frac{4\beta^2}{d_2}\right)\rho K_1^2$$
$$+\left(\frac{8J^2}{d_1} + \frac{4\alpha^2}{d_2} + \frac{4q^2c^2}{d_3}\right)|\Omega| + \frac{8}{d_1}\|p_1(t)\|^2 + \frac{4}{d_2}\|p_2(t)\|^2 + \frac{4}{d_3}\|p_3(t)\|^2.$$

For $t \ge T_B$, by integration of the inequality (4.22) we can deduce that

$$\begin{aligned}
&\int_t^{t+1} 2d(c_1\|\nabla u(s)\|^2 + \|\nabla v(s)\|^2 + \|\nabla w(s)\|^2)\,ds \\
\le\, & c_1\|u(t)\|^2 + \|v(t)\|^2 + \|w(t)\|^2 \\
&+ 4\left(1 + \frac{1}{r}\right)\int_t^{t+1}\|p(s)\|^2 ds + \left(\frac{c_1^2}{128} + 2c_2\right)|\Omega| \\
\le\, & \max\{1, c_1\}K_1 + 4\left(1 + \frac{1}{r}\right)\|p\|^2_{L_b^2} + \left(\frac{c_1^2}{128} + 2c_2\right)|\Omega|,
\end{aligned}$$

where $\|p\|^2_{L_b^2} = \sum_{i=1}^3 \|p_i\|^2_{L_b^2}$. Denote by

$$N_1 = \frac{1}{2d\min\{1, c_1\}}\left[\max\{1, c_1\}K_1 + 4\left(1 + \frac{1}{r}\right)\|p\|^2_{L_b^2} + \left(\frac{c_1^2}{128} + 2c_2\right)|\Omega|\right]$$

and

$$N_2 = \max\left\{\frac{4}{d_1}, \frac{4q^2c^2}{d_3}\right\}K_1 + \left(\frac{8a^2}{d_1} + \frac{4\beta^2}{d_2}\right)\rho K_1^2 + \left(\frac{8J^2}{d_1} + \frac{4\alpha^2}{d_2} + \frac{4q^2c^2}{d_3}\right)|\Omega|.$$

Then we have

$$\begin{aligned}
&\int_t^{t+1} \sigma(s)\,ds \le N_1, \\
&\int_t^{t+1} \xi(s)\,ds \le \rho\left(\frac{8a^2}{d_1} + \frac{4\beta^2}{d_2}\right) N_1, \\
&\int_t^{t+1} h(s)\,ds \le N_2 + \max\left\{\frac{8}{d_1}, \frac{4}{d_2}, \frac{4}{d_3}\right\} \|p\|^2_{L^2_b}.
\end{aligned} \tag{4.37}$$

Thus the uniform Gronwall inequality applied to (4.36) shows that

$$\|\nabla g(t)\|^2 \le \left(N_1 + N_2 + \max\left\{\frac{8}{d_1}, \frac{4}{d_2}, \frac{4}{d_3}\right\} \|p\|^2_{L^2_b}\right) \exp\left\{\rho\left(\frac{8a^2}{d_1} + \frac{4\beta^2}{d_2}\right) N_1\right\}, \tag{4.38}$$

for all $t \ge T_B + 1$ and all $g_\tau \in B$. Therefore, the claim (4.29) of a pullback absorbing ball M_E^* in the space E is proved and the constant K_2 is given by

$$K_2 = K_1 + \left(N_1 + N_2 + \max\left\{\frac{8}{d_1}, \frac{4}{d_2}, \frac{4}{d_3}\right\} \|p\|^2_{L^2_b}\right) \exp\left\{\rho\left(\frac{8a^2}{d_1} + \frac{4\beta^2}{d_2}\right) N_1\right\}.$$

Indeed, for any given bounded set $B \subset H$, we have

$$S(\tau, \tau - t)B \subset M_E^* \quad \text{for all } \ t \ge T_B + 1.$$

The proof is completed. $\square$

Now we prove the first main result of this paper.

Theorem 4.2.3. *Under the assumption* (4.5)*, for any positive parameters and* $c \in \mathbb{R}$ *in the Hindmarsh-Rose equations* (4.1)–(4.3)*, there exists a pullback attractor* $\mathcal{A} = \{\mathcal{A}(\tau)\}_{\tau\in\mathbb{R}}$ *in* H *for the nonautonomous Hindmarsh-Rose process* $\{S(t,\tau)\}_{t\ge\tau\in\mathbb{R}}$.

Proof. By Lemma 4.2.1, there exists a pullback absorbing set M_H^* in H for the solution process $\{S(t,\tau) : t \ge \tau \in \mathbb{R}\}$ of the nonautononous Hindmarsh-Rose system (4.8) so that the first condition in Proposition 4.1.1 is satisfied.

By Lemma 4.2.2 and the compact embedding $E \hookrightarrow H$, the existence of a pullback absorbing set M_E^* in E for this nonautonomous process shows that any sequence $\{S(\tau, \tau - t_k)g_k\}_{k=1}^\infty$, where

$t_k \to \infty$ and $\{g_k\}$ in any given bounded set of H, has a convergent subsequence. Thus the second condition of the pullback asymptotic compactness in Proposition 4.1.1 is also satisfied.

Then by Proposition 4.1.1, there exists a pullback attractor $\mathcal{A} = \{\mathcal{A}(\tau)\}_{\tau \in \mathbb{R}}$,

$$\mathcal{A}(\tau) = \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} S(\tau, \tau - t) M_H^*},$$

for this nonautonomous Hindmarsh-Rose process. $\square$

4.3 The Existence of Pullback Exponential Attractor

In this section, we shall prove the existence of a pullback exponential attractor for the nonautonomous Hindmarsh-Rose process based on Proposition 4.1.2. The key leverage as well as the crucial challenge here is to prove the H^1 smoothing Lipschitz continuity of this nonautonomous process with respect to the initial data in the space L^2.

Theorem 4.3.1 (Smoothing Lipschitz Continuity)**.** *For the nonautonomous Hindmarsh-Rose process* $\{S(t,\tau)\}_{t \geq \tau \in \mathbb{R}}$ *in* (4.17) *generated by the weak solutions of the nonautonomous diffusive Hinsmarsh-Rose system* (4.8)*, there exists a constant* $\kappa > 0$ *such that*

$$\sup_{\tau \in \mathbb{R}} \|S(\tau, \tau - T_{M_E^*}) g_\tau - S(\tau, \tau - T_{M_E^*}) \tilde{g}_\tau\|_E \leq \kappa \|g_\tau - \tilde{g}_\tau\|, \quad \textit{for } g_\tau, \tilde{g}_\tau \in M_E^*, \tag{4.39}$$

where $T_{M_E^*} > 0$ *is the time when all the pullback solution trajectories of* (4.8) *starting from the set* M_E^* *in* (4.29) *permanently enter the set* M_E^* *itself shown in Lemma* 4.2.2.

Proof. It is equivalent to prove that

$$\sup_{\tau \in \mathbb{R}} \|S(\tau + T_{M_E^*}, \tau) g_\tau - S(\tau + T_{M_E^*}, \tau) \tilde{g}_\tau\|_E \leq \kappa \|g_\tau - \tilde{g}_\tau\|, \quad g_\tau, \tilde{g}_\tau \in M_E^*. \tag{4.40}$$

Denote two weak solutions of the nonautonomous diffusive Hindmarsh-Rose equations with any given initial data g_τ and $\tilde{g}_\tau$ by $g(t) = (u(t), v(t), w(t))$ and $\tilde{g}(t) = (\tilde{u}(t), \tilde{v}(t), \tilde{w}(t))$, respectively. Denote the difference by $\Pi(t) = g(t) - \tilde{g}(t) = (U(t), V(t), W(t))$. Then $\Pi(t)$ is the solution of the following intial value problem

$$\begin{aligned}\frac{d\Pi}{dt} &= A\Pi + f(g) - f(\tilde{g}), \quad t \geq \tau \in \mathbb{R}, \\ \Pi(\tau) &= g_\tau - \tilde{g}_\tau.\end{aligned} \tag{4.41}$$

Step 1. Take the inner-product $\langle (4.41), \Pi(t)\rangle$ through three component equations (4.1)–(4.3). For the first component equation of $\Pi(t) = g(t) - \tilde{g}(t)$, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|U(t)\|^2 + d_1\|\nabla U(t)\|^2 = \langle f_1(g) - f_1(\tilde{g}), u - \tilde{u}\rangle \\ &= \int_\Omega \big(a(u-\tilde{u})^2(u+\tilde{u}) - b(u-\tilde{u})^2(u^2\tilde{u} + u\tilde{u} + \tilde{u}^2)\big)\, dx \\ &\quad + \int_\Omega ((v-\tilde{v})(u-\tilde{u}) - (w-\tilde{w})(u-\tilde{u}))\, dx \\ &\leq \int_\Omega (u-\tilde{u})^2 \left[a(u+\tilde{u}) - b(u^2 + u\tilde{u} + \tilde{u}^2)\right] dx + \|u-\tilde{u}\|(\|v-\tilde{v}\| + \|w-\tilde{w}\|) \\ &\leq \int_\Omega (u-\tilde{u})^2 \left[a(u+\tilde{u}) - b(u^2 + u\tilde{u} + \tilde{u}^2)\right] dx + 2\|g-\tilde{g}\|^2\end{aligned} \tag{4.42}$$

and by Young's inequality we have

$$\begin{aligned}&a(u+\tilde{u}) - b(u^2 + u\tilde{u} + \tilde{u}^2) = [a(u+\tilde{u}) - bu\tilde{u}] - b(u^2+\tilde{u}^2) \\ &\leq \left(\frac{b}{4}u^2 + \frac{a^2}{b}\right) + \left(\frac{b}{4}\tilde{u}^2 + \frac{a^2}{b}\right) + \frac{b}{2}(u^2+\tilde{u}^2) - b(u^2+\tilde{u}^2) \leq -\frac{b}{4}(u^2+\tilde{u}^2) + \frac{2a^2}{b}.\end{aligned}$$

It follows that

$$\begin{aligned}&\frac{d}{dt}\|U(t)\|^2 \leq \frac{d}{dt}\|U(t)\|^2 + 2d_1\|\nabla U(t)\|^2 \\ &\leq 2\int_\Omega (u-\tilde{u})^2 \left(-\frac{b}{4}(u^2+\tilde{u}^2) + \frac{2a^2}{b}\right) dx + 4\|g-\tilde{g}\|^2 \\ &\leq \int_\Omega (u-\tilde{u})^2 \left(-\frac{b}{2}(u^2+\tilde{u}^2)\right) dx + \frac{4a^2}{b}\|u-\tilde{u}\|^2 + 4\|g-\tilde{g}\|^2 \\ &\leq -\frac{b}{2}\int_\Omega (u-\tilde{u})^2(u^2+\tilde{u}^2)\, dx + 4\left(1 + \frac{a^2}{b}\right)\|\Pi(t)\|^2.\end{aligned} \tag{4.43}$$

Similarly, for the second and third components of $\Pi(t) = g(t) - \tilde{g}(t))$, we get

$$
\begin{aligned}
&\frac{d}{dt}\|V(t)\|^2 \le \frac{d}{dt}\|V(t)\|^2 + 2d_2\|\nabla V(t)\|^2 \le 2\langle \psi(u) - \psi(\tilde{u}) - (v - \tilde{v}), v - \tilde{v}\rangle \\
&= 2\int_\Omega \big(-\beta(u^2 - \tilde{u}^2) - (v - \tilde{v})\big)(v - \tilde{v})\, dx \\
&\le 2\int_\Omega (-\beta(u - \tilde{u})u(v - \tilde{v}) - \beta(u - \tilde{u})\tilde{u}(v - \tilde{v}))\, dx \\
&\le \int_\Omega \left(\frac{bu^2}{2}(u - \tilde{u})^2 + \frac{b\tilde{u}^2}{2}(u - \tilde{u})^2\right) dx + \frac{4\beta}{b}\|v - \tilde{v}\|^2 \\
&\le \frac{b}{2}\int_\Omega (u^2 + \tilde{u}^2)(u - \tilde{u})^2\, dx + \frac{4\beta}{b}\|\Pi(t)\|^2
\end{aligned}
\tag{4.44}
$$

and

$$
\begin{aligned}
&\frac{d}{dt}\|W(t)\|^2 \le \frac{d}{dt}\|W(t)\|^2 + 2d_3\|\nabla W(t)\|^2 \le 2\langle q(u - \tilde{u}) - r(w - \tilde{w}), w - \tilde{w}\rangle \\
&= 2\int_\Omega (q(u - \tilde{u}) - r(w - \tilde{w}))(w - \tilde{w})\, dx \\
&\le q\|u - \tilde{u}\|^2 + (q + 2r)\|w - \tilde{w}\|^2 \le 2(q + r)\|\Pi(t)\|^2.
\end{aligned}
\tag{4.45}
$$

Sum up the inequalities (4.43), (4.44) and (4.45) with a cancellation of the first terms on the rightmost side of (4.43) and (4.44). Then we obtain

$$
\begin{aligned}
&\frac{d}{dt}\|\Pi\|^2 + 2(d_1\|\nabla U\|^2 + d_2\|\nabla V\|^2 + d_3\|\nabla W\|^2) = 2\langle f(g) - f(\tilde{g}), \Pi\rangle \\
&\le \left(4\left[1 + \frac{1}{b}(a^2 + \beta)\right] + 2(q + r)\right)\|\Pi\|^2.
\end{aligned}
\tag{4.46}
$$

It follows that, for any $g_\tau, \tilde{g}_\tau \in M_E^*$ and indeed for any $g_\tau, \tilde{g}_\tau \in H$,

$$
\frac{d}{dt}\|\Pi\|^2 \le C_*\|\Pi\|^2 \tag{4.47}
$$

where the constant $C_* = 4\left(1 + \frac{1}{b}(a^2 + \beta)\right) + 2(q + r)$. Consequently,

$$
\begin{aligned}
&\|S(t + \tau, \tau)g_\tau - S(t + \tau, \tau)\tilde{g}_\tau\|^2 = \|\Pi(t + \tau)\|^2 \\
&\le e^{C_* t}\|\Pi(\tau)\|^2 = e^{C_* t}\|g_\tau - \tilde{g}_\tau\|^2, \quad t \ge 0, \ \tau \in \mathbb{R}.
\end{aligned}
\tag{4.48}
$$

Step 2. In oder to prove (4.40), we express the weak solution of (4.41) by using the mild solution formula,

$$\Pi(t+\tau) = e^{At}\,\Pi(\tau) + \int_\tau^{t+\tau} e^{A(t+\tau-s)}(f(g(s)) - f(\tilde{g}(s))\,ds, \quad t \geq 0, \tag{4.49}$$

where the C_0-semigroup $\{e^{At}\}_{t\geq 0}$ is generated by the operator A defined in (4.9). By the regularity property of the analytic C_0-semigroup $\{e^{At}\}_{t\geq 0}$ [56, 62], it holds that $e^{At} : H \to E$ for $t > 0$ and there is a constant $C_1 > 0$ such that

$$\|e^{At}\|_{\mathcal{L}(H,\,E)} \leq C_1\, t^{-1/2}, \quad t > 0. \tag{4.50}$$

Thus we have

$$\begin{aligned}\|\Pi(t+\tau)\|_E &\leq \|e^{At}\|_{\mathcal{L}(H,\,E)}\|\Pi(\tau)\| + \int_\tau^{t+\tau} \|e^{A(t+\tau-s)}\|_{\mathcal{L}(H,E)}\|(f(g(s)) - f(\tilde{g}(s))\|\,ds \\ &\leq \frac{C_1}{\sqrt{t}}\,\|g_\tau - \tilde{g}_\tau\| + \int_\tau^{t+\tau} \frac{C_1}{\sqrt{t+\tau-s}}\,\|(f(g(s)) - f(\tilde{g}(s))\|\,ds, \quad t > 0.\end{aligned} \tag{4.51}$$

Here we estimate the norm of the difference in the last integral of (4.51),

$$\begin{aligned}&\|f(g) - f(\tilde{g})\|^2 = \|\varphi(u) - \varphi(\tilde{u}) + (v-\tilde{v}) - (w-\tilde{w})\|^2 \\ &\quad + \|\psi(u) - \psi(\tilde{u}) - (v-\tilde{v})\|^2 + \|q(u-\tilde{u}) - r(w-\tilde{w})\|^2 \\ &\leq 3\|\varphi(u) - \varphi(\tilde{u})\|^2 + 2\|\psi(u) - \psi(\tilde{u})\|^2 + 2q\|u-\tilde{u}\|^2 + 5\|v-\tilde{v}\|^2 + (3+2r)\|w-\tilde{w}\|^2 \\ &= (3a^2 + 2\beta^2)\|u^2 - \tilde{u}^2\|^2 + 3b^2\|u^3 - \tilde{u}^3\|^2 + 2q\|u-\tilde{u}\|^2 + 5\|v-\tilde{v}\|^2 + (3+2r)\|w-\tilde{w}\|^2 \\ &\leq (6a^2 + 4\beta^2)(\|u\|^2 + \|\tilde{u}\|^2)\|u-\tilde{u}\|^2 + 3b^2\|u^2 + u\tilde{u} + \tilde{u}^2\|^2\|u-\tilde{u}\|^2 \\ &\quad + 2q\|u-\tilde{u}\|^2 + 5\|v-\tilde{v}\|^2 + (3+2r)\|w-\tilde{w}\|^2 \\ &\leq (6a^2 + 4\beta^2)(\|u\|^2 + \|\tilde{u}\|^2)\|u-\tilde{u}\|^2 + 3b^2\|u^2 + u\tilde{u} + \tilde{u}^2\|^2\|u-\tilde{u}\|^2 \\ &\quad + 2q\|u-\tilde{u}\|^2 + 5\|v-\tilde{v}\|^2 + (3+2r)\|w-\tilde{w}\|^2,\end{aligned}$$

where in the term $3b^2\|u^2+u\tilde{u}+\tilde{u}^2\|^2\|u-\tilde{u}\|^2$, we deduce that

$$
\begin{aligned}
&\|u^2+u\tilde{u}+\tilde{u}^2\|^2 = \int_\Omega (u^2+u\tilde{u}+u^2)^2\,dx = \int_\Omega (u^4+3u^2\tilde{u}^2+\tilde{u}^4+2u\tilde{u}(u^2+\tilde{u}^2))\,dx \\
&\le \left(u^4+\tilde{u}^4+\frac{3}{2}(u^4+\tilde{u}^4)+u^2\tilde{u}^2+(u^2+\tilde{u}^2)^2\right)dx \\
&\le 5\int_\Omega (u^4+\tilde{u}^4)\,dx = 5\left(\|u\|_{L^4}^4+\|\tilde{u}\|_{L^4}^4\right).
\end{aligned}
$$

Substitute the above inequalities into the integral term in (4.51) and use the embedding inequality (4.32) to obtain

$$
\begin{aligned}
&\|\Pi(t+\tau)\|_E \le \frac{C_1}{\sqrt{t}}\|g_\tau-\tilde{g}_\tau\| + \int_\tau^{t+\tau}\frac{C_1}{\sqrt{t+\tau-s}}\|(f(g(s))-f(\tilde{g}(s))\|\,ds \\
&\le \frac{C_1}{\sqrt{t}}\|g_\tau-\tilde{g}_\tau\| + \int_\tau^{t+\tau}\frac{C_1}{\sqrt{t+\tau-s}}(6a^2+4\beta^2)(\|u\|^2+\|\tilde{u}\|^2)\|u-\tilde{u}\|^2\,ds \\
&\quad + \int_\tau^{t+\tau}\frac{C_1}{\sqrt{t+\tau-s}}30b^2\rho\left(\|u\|_{H^1}^4+\|\tilde{u}\|_{H^1}^4\right)\|u-\tilde{u}\|^2\,ds \\
&\quad + \int_\tau^{t+\tau}\frac{C_1}{\sqrt{t+\tau-s}}(2q\|u-\tilde{u}\|^2+5\|v-\tilde{v}\|^2+(3+2r)\|w-\tilde{w}\|^2)\,ds,\ t\ge 0,
\end{aligned}
\tag{4.52}
$$

for any $\tau\in\mathbb{R}$.

Note that from (4.25) and (4.29), since both g_τ and $\tilde{g}_\tau$ are in M_E^*, we have

$$
\|u(t+\tau)\|^2 \le \|g(t+\tau)\|^2 \le G_1 = \frac{\max\{1,c_1\}}{\min\{1,c_1\}}K_2+K_1, \quad \text{for } t\ge 0, \tau\in\mathbb{R}, \tag{4.53}
$$

where the positive constants K_1 and K_2 are given in (4.27) and (4.29) respectively, and independent of t and τ.

Step 3. We want to improve the inequality (4.31):

$$
\begin{aligned}
&\frac{d}{dt}\left(\|\nabla u\|^2+\|\nabla v\|^2+\|\nabla w\|^2\right)+d_1\|\Delta u\|^2+d_2\|\Delta v\|^2+d_3\|\Delta w\|^2 \\
&+6b\|u\nabla u\|^2+2\|\nabla v\|^2+2r\|\nabla w\|^2 \\
&\le \frac{4}{d_1}\|v\|^2+\frac{4}{d_1}\|w\|^2+\frac{2q^2}{d_3}\|u\|^2+\left(\frac{4a^2}{d_1}+\frac{2\beta^2}{d_2}\right)\|u\|_{L^4}^4 \\
&+\left(\frac{8J^2}{d_1}+\frac{4\alpha^2}{d_2}+\frac{4q^2c^2}{d_3}\right)|\Omega|+\frac{8}{d_1}\|p_1(t)\|^2+\frac{4}{d_2}\|p_2(t)\|^2+\frac{4}{d_3}\|p_3(t)\|^2.
\end{aligned}
$$

Specifically we need to further treat the following term on the right-hand side of the above (4.31),

$$\left(\frac{4a^2}{d_1}+\frac{2\beta^2}{d_2}\right)\|u\|_{L^4}^4$$

by using the Gagliardo-Nirenberg inequality (1.7) for the interpolation spaces

$$L^1(\Omega)\hookrightarrow L^2(\Omega)\hookrightarrow H^1(\Omega).$$

By the Poincaré inequality that there are constants $\eta_1,\eta_2>0$ such that $\|h\|_{H^1}\le\eta_1\|\nabla h\|+\eta_2\|h\|_{L^1}$ for any $h\in H^1(\Omega)$, it implies that there is a constant $C>0$ and

$$\|u^2\|^2\le C\left(\|\nabla(u^2)\|^{6/5}+\|u^2\|_{L^1}^{6/5}\right)\|u^2\|_{L^1}^{4/5}. \tag{4.54}$$

because $-\frac{3}{2}=\theta(1-\frac{3}{2})-3(1-\theta)$ with $\theta=3/5$ and $1-\theta=2/5$. Therefore, the inequality (4.54) and the Young's inequality imply that there exists a constant $0<\varepsilon<b$ such that

$$\begin{aligned}
&\left(\frac{4a^2}{d_1}+\frac{2\beta^2}{d_2}\right)\|u\|_{L^4}^4=\left(\frac{4a^2}{d_1}+\frac{2\beta^2}{d_2}\right)\|u^2\|^2\\
\le{}&C\left(\frac{4a^2}{d_1}+\frac{2\beta^2}{d_2}\right)\left(\|\nabla(u^2)\|^{6/5}+\|u^2\|_{L^1}^{6/5}\right)\|u^2\|_{L^1}^{4/5}\le\varepsilon\|\nabla u^2\|^2+C_\varepsilon\|u^2\|_{L^1}^2\\
={}&4\varepsilon\,\|u\nabla u\|^2+C_\varepsilon\|u\|^4\le 4b\,\|u\nabla u\|^2+C_\varepsilon\|u\|^4,
\end{aligned}\tag{4.55}$$

where $C_\varepsilon>0$ is a constant only depending on ε. Substitute (4.55) into the above inequality (4.31) to obtain

$$\begin{aligned}
&\frac{d}{dt}\left(\|\nabla u\|^2+\|\nabla v\|^2+\|\nabla w\|^2\right)+d\left(\|\Delta u\|^2+\|\Delta v\|^2+\|\Delta w\|^2\right)\\
&\quad+2b\,\|u\nabla u\|^2+2\|\nabla v\|^2+2r\|\nabla w\|^2
\end{aligned}\tag{4.56}$$

$$\begin{aligned}
&\le G_2(\|u\|^2+\|v\|^2+\|w\|^2)+G_3|\Omega|\\
&\quad+\frac{8}{d}\left(\|p_1(t)\|^2+\|p_2(t)\|^2+\|p_3(t)\|^2\right)+C_\varepsilon\|u\|^4,\quad t\ge\tau\in\mathbb{R},
\end{aligned}$$

where $d=\min\{d_1,d_2,d_3\}$,

$$G_2=\frac{1}{d}\max\{4,2q^2\}\quad\text{and}\quad G_3=\frac{1}{d}\left(8J^2+4\alpha^2+4q^2c^2\right).$$

Then the inequalities (4.56) with (4.53) infer that

$$\begin{aligned} \frac{d}{dt}\|\nabla(u,v,w)\|^2 &\le G_2\|(u,v,w)\|^2 + G_3|\Omega| + \frac{8}{d}\,\|p(t)\|^2 + C_\varepsilon\|u\|^4 \\ &\le G_1G_2 + G_3|\Omega| + \frac{8}{d}\,\|p(t)\|^2 + C_\varepsilon G_1^2 \end{aligned} \tag{4.57}$$

for any $t \ge \tau \in \mathbb{R}$. It follows that for any $0 \le t \le T_{M_E^*}$, we have

$$\begin{aligned} &\|u(t+\tau)\|^2_{H^1(\Omega)} = \|u(t+\tau)\|^2 + \|\nabla u(t+\tau)\|^2 \\ &\le \|u(t+\tau)\|^2 + \|\nabla u(\tau)\|^2 + \int_\tau^{\tau+t}\left(G_1G_2 + G_3|\Omega| + \frac{8}{d}\,\|p(s)\|^2 + C_\varepsilon G_1^2\right) ds \\ &\le G_1 + \|\nabla u(\tau)\|^2 + t(G_1G_2 + G_3|\Omega| + C_\varepsilon G_1^2) + \frac{8}{d}\int_\tau^{t+\tau}\|p(s)\|^2\,ds \\ &\le G_1 + K_2 + T_{M_E^*}(G_1G_2 + G_3|\Omega| + C_\varepsilon G_1^2) + \frac{8}{d}\left(T_{M_E^*}+1\right)\|p\|^2_{L_b^2}. \end{aligned} \tag{4.58}$$

Step 4. Finally we substitute (4.53) and (4.58) into the inequality (4.52) for any two solutions $g(t)$ and $\tilde{g}(t)$ of the nonautonomous system (4.8) with initial states in M_E^*. Then for any $t > 0$ and $\tau \in \mathbb{R}$, it holds that

$$\begin{aligned} &\|\Pi(t+\tau)\|_E \le \frac{C_1}{\sqrt{t}}\,\|g_\tau - \tilde{g}_\tau\| + \int_\tau^{t+\tau}\frac{C_1}{\sqrt{t+\tau-s}}\,\|(f(g(s)) - f(\tilde{g}(s))\|\,ds \\ &\le \frac{C_1}{\sqrt{t}}\,\|g_\tau - \tilde{g}_\tau\| + \int_\tau^{t+\tau}\frac{C_1}{\sqrt{t+\tau-s}}\,G_1(12a^2+8\beta^2)\|u-\tilde{u}\|^2\,ds \\ &+ \int_\tau^{t+\tau}\frac{C_1}{\sqrt{t+\tau-s}}\,30b^2\rho\left(\|u\|^4_{H^1} + \|\tilde{u}\|^4_{H^1}\right)\|u-\tilde{u}\|^2\,ds \\ &+ \int_\tau^{t+\tau}\frac{C_1}{\sqrt{t+\tau-s}}\max\{2q,5,3+2r\}(\|u-\tilde{u}\|^2 + \|v-\tilde{v}\|^2 + \|w-\tilde{w}\|^2)\,ds \\ &\le \frac{C_1}{\sqrt{t}}\,\|g_\tau - \tilde{g}_\tau\| + \int_\tau^{t+\tau}\frac{C_1}{\sqrt{t+\tau-s}}\,G_p\,\|g(s)-\tilde{g}(s)\|^2\,ds \\ &\le \frac{C_1}{\sqrt{t}}\,\|g_\tau - \tilde{g}_\tau\| + \int_\tau^{t+\tau}\frac{C_1}{\sqrt{t+\tau-s}}\,G_p\,e^{C_*(s-\tau)}\,\|g_\tau - \tilde{g}_\tau\|^2\,ds \\ &\le \frac{C_1}{\sqrt{t}}\,\|g_\tau - \tilde{g}_\tau\| + \int_\tau^{t+\tau}\frac{C_1}{\sqrt{t+\tau-s}}\,G_p\,e^{C_*(s-\tau)}2\sqrt{G_1}\|g_\tau - \tilde{g}_\tau\|\,ds, \end{aligned} \tag{4.59}$$

where we used (4.48) and (4.53) in the last two steps, and the positive constant G_p is given by

$$\begin{aligned} G_p = & G_1(12a^2 + 8\beta^2) + \max\{2q, 5, 3 + 2r\} \\ & + 60b^2\rho \left[G_1 + K_2 + T_{M_E^*}(G_1G_2 + G_3|\Omega| + C_\varepsilon G_1^2) + \frac{8}{d}(T_{M_E^*} + 1)\|p\|^2_{L_b^2} \right]^2 \end{aligned} \tag{4.60}$$

which depends on the nonautonomous terms $p_i(t, x), i = 1, 2, 3$, and the permanently entering time $T_{M_E^*}$. Calculating the integral in the inequality (4.59) on the time interval $[\tau, \tau + T_{M_E^*}]$, here without of generality $T_{M_E^*} > 0$, we then obtain the result that

$$\begin{aligned} & \|S(\tau + T_{M_E^*}, \tau)g_\tau - S(\tau + T_{M_E^*}, \tau)\tilde{g}_\tau\|_E = \|\Pi(\tau + T_{M_E^*})\|_E \\ \le & C_1 \left(\frac{1}{\sqrt{T_{M_E^*}}} + 4\sqrt{G_1 T_{M_E^*}} \exp\left\{C_* T_{M_E^*}\right\} G_p \right) \|g_\tau - \tilde{g}_\tau\| \end{aligned} \tag{4.61}$$

for any $g_\tau, \tilde{g}_\tau \in M_E^*$ and any $\tau \in \mathbb{R}$. Therefore, (4.40) and then (4.39) are proved with the uniform Lipschitz constant

$$\kappa = C_1 \left(\frac{1}{\sqrt{T_{M_E^*}}} + 4\sqrt{G_1 T_{M_E^*}} \exp\left\{C_* T_{M_E^*}\right\} G_p \right).$$

The proof of this theorem is completed. □

After the challenging Theorem 4.3.1 has been proved, now we can prove the second main result of this paper.

Theorem 4.3.2. *For the nonautonomous Hindmarsh-Rose process* $\{S(t, \tau)_{t \ge \tau \in \mathbb{R}}$ *generated by the nonautonomous Hindmarsh-Rose equations* (4.1)–(4.3)*, there exists a pullback exponential attractor* $\mathscr{M} = \{\mathscr{M}(\tau)\}_{\tau \in \mathbb{R}}$ *in the space H.*

Proof. We can apply Proposition 4.1.2 to prove this theorem. Indeed Lemma 4.2.2 and Theorem 4.3.1 have shown that the first two conditions in that Proposition 4.1.2 are satisfied with the pullback absorbing set $M^* = M_E^*$ in (4.13) by the nonautonomous Hindmarsh-Rose process $S(t, \tau)_{t \ge \tau \in \mathbb{R}}$. Thus it suffices to show that the third condition of (4.15) and (4.16) in Proposition 4.1.2 is satisfied.

Recall that the Hindmarsh-Rose process $S(t, \tau)$ is defined by (4.17) and let $g(t, \tau, g_\tau)$ be the weak solution to the initial value problem of the nonautonomous Hindmarsh-Rose evolutionary equation (4.8). For any $t_1 < t_2$ with $|t_1 - t_2| \le I$, where I is any given positive constant, we can estimate the H-norm of the difference of two pullback solution trajectories

$$g^1(t) = S(t, \tau - t_1)g_0 \quad \text{and} \quad g^2(t) = S(t, \tau - t_2)g_0, \quad 0 \le t_1 \le t_2, \; g_0 \in H,$$

as follows.

Using the notation in (4.41) but here $\Pi(t) = g^1(t) - g^2(t)$. Then $\Pi(t)$ is the solution of the initial value problem

$$\begin{aligned} \frac{d\Pi}{dt} &= A\Pi + f(g^1) - f(g^2), \quad t \ge \tau - t_1 \in \mathbb{R}, \\ \Pi(\tau - t_1) &= g_0 - S(\tau - t_1, \tau - t_2)g_0. \end{aligned} \tag{4.62}$$

By (4.47), we have

$$\frac{d\|\Pi\|^2}{dt} \le C_* \|\Pi\|^2, \quad t \ge \tau, \tag{4.63}$$

where C_* is the same constant as in (4.47).

The Lipschitz and Hölder continuity associated with the regularity property of the parabolic C_0-semigroup of contraction $\{e^{At}\}_{t \ge 0}$, cf. [62], gives rise to

$$\|e^{A(t_0+h)}g_0 - e^{At_0}g_0\| \le \|e^{At_0}\| \|e^{Ah}g_0 - g_0\| \le C_0 |h| \|g_0\|, \quad \text{for all } \; t_0 \ge 0, \tag{4.64}$$

where $C_0 > 0$ is a constant depending only on the contraction operator semigroup e^{At}. Then, it follows from (4.63) and (4.64) that

$$\begin{aligned} &\|S(t, \tau - t_1)g_0 - S(t, \tau - t_2)g_0\| = \|g^1(t, \tau - t_1, g_0) - g^2(t, \tau - t_2, g_0)\| \\ &= \Bigg\| e^{A(t-(\tau-t_1))}g_0 + \int_{\tau - t_1}^t e^{A(t-s)}[f(g^1(s, \tau - t_1, g_0)) + p(s,x)]\, ds \\ &\quad - e^{A(t-(\tau-t_2))}g_0 - \int_{\tau-t_2}^t e^{A(t-s)}[f(g^2(s, \tau - t_2, g_0)) + p(s,x)]\, ds \Bigg\| \\ &\le \|\Pi(t, \tau - t_1, \Pi(\tau - t_1)\| \le e^{\frac{1}{2}C_*|t-(\tau-t_1)|} \|\Pi(\tau - t_1, \tau - t_2, g_0)\| \end{aligned} \tag{4.65}$$

$$\le e^{\frac{1}{2}C_*|t-(\tau-t_1)|}\,\|e^{A(t_2-t_1)}g_0 - g_0)\|$$

$$+\, e^{\frac{1}{2}C_*|t-(\tau-t_1)|}\int_{\tau-t_2}^{\tau-t_1}\|e^{A(\tau-t_1-s)}[f(g^2(s,\tau-t_2,g_0)) + p(s,x)]\|\,ds$$

$$\le e^{\frac{1}{2}C_*|t-(\tau-t_1)|}\,C_0\,|t_1-t_2|\|g_0\|$$

$$+\, e^{\frac{1}{2}C_*|t-(\tau-t_1)|}\int_{\tau-t_2}^{\tau-t_1}\|e^{A(t-s)}[f(g^2(s,\tau-t_2,g_0)) + p(s,x)]\|\,ds.$$

Denote by $T^* = T_{M_E^*} > 0$, which is the finite time when all the pullback solution trajectories started from the pullback absorbing set M_E^* in Lemma 4.2.2 permanently enter into itself. Define the following set, where the closure is taken in the space E,

$$\Gamma = \overline{\bigcup_{0\le t\le T^*} S(\tau,\tau-t)M_E^*} \tag{4.66}$$

Lemma 4.2.2 demonstrated that M_E^* and T^* are independent of $\tau \in \mathbb{R}$ and $t \ge 0$. Denote by $D_\Gamma = \max_{g\in\Gamma}\|f(g)\|_H$, since the Nemytskii operator $f : E \to H$ is bounded on the bounded set Γ in E. Here $\|e^{At}\|_{\mathcal{L}(H)} \le 1$ and by Hölder inequality,

$$\begin{aligned}
&\int_{\tau-t_2}^{\tau-t_1}\|e^{A(t-s)}\|_{\mathcal{L}(H)}(\|f(g^2(s,\tau-t_2,g_0))\| + \|p(s,x)\|)\,ds\\
&\le (D_\Gamma + K_2)|t_1-t_2| + \int_{\tau-t_2}^{\tau-t_1}\|p(s,\cdot)\|\,ds\\
&\le (D_\Gamma + K_2)|t_1-t_2| + |t_1-t_2|^{1/2}\sqrt{\int_{\tau-t_2}^{\tau-t_1}\|p(s,\cdot)\|^2\,ds}\\
&\le (D_\Gamma + K_2)|t_1-t_2| + |t_1-t_2|^{1/2}\sqrt{(|t_1-t_2|+1)\,\Sigma_{i=1}^3\|p_i\|^2_{L_b^2}}\\
&\le (D_\Gamma + K_2)|t_1-t_2| + |t_1-t_2|^{1/2}(|t_1-t_2|^{1/2}+1)\|p\|_{L_b^2}\\
&\le (|t_1-t_2| + |t_1-t_2|^{1/2})(D_\Gamma + K_2 + \|p\|_{L_b^2}),
\end{aligned} \tag{4.67}$$

for any $t_1 \ge T^*$ and $g_0 \in M_E^*$, where K_2 is given in (4.29).

Substituting (4.67) into (4.65) we obtain

$$
\begin{aligned}
&\|S(t,\tau-t_1)g_0 - S(t,\tau-t_2)g_0\| = \|g^1(t,\tau-t_1,g_0) - g^2(t,\tau-t_2,g_0)\| \\
\le\, & e^{\frac{1}{2}C_*|t-(\tau-t_1)|}C_0\,|t_1-t_2|\|g_0\| \\
+\, & e^{\frac{1}{2}C_*|t-(\tau-t_1)|}(|t_1-t_2| + |t_1-t_2|^{1/2})(D_\Gamma + K_2 + \|p\|_{L_b^2}) \\
\le\, & \lambda(M_E^*)\, e^{\frac{1}{2}C_*|t-(\tau-t_1)|}|t_1-t_2|^\gamma, \quad \text{for } t \ge \tau - t_1,\ t_1 \ge T^*,\ g_0 \in M_E^*.
\end{aligned}
\tag{4.68}
$$

where

$$
\lambda(M_E^*) = C_0K_2 + 2(D_\Gamma + K_2 + \|p\|_{L_b^2})
$$

and

$$
\gamma = \begin{cases} \frac{1}{2}, & \text{if } |t_1-t_2| < 1; \\ 1, & \text{if } |t_1-t_2| \ge 1. \end{cases}
$$

For any given $\tau \in \mathbb{R}$, in the above inequality (4.68), take

$$
t = \tau, \quad t_1 = T_{M_E^*}, \quad \text{and separately,} \quad t_2 = T_{M_E^*} + t \quad \text{for } t \in [0,\, T^*_{M_E^*}].
$$

Then we obtain

$$
\sup_{\tau\in\mathbb{R}} \|S(\tau,\tau-T_{M_E^*})g_0 - S(\tau,\tau-T_{M_E^*}-t)g_0\| \le \lambda(M_E^*)\, \exp\left\{\frac{C_*}{2}T_{M_E^*}\right\}|t|^\gamma \tag{4.69}
$$

for $t \in [0, T_{M_E^*}]$, $g_0 \in M_E^*$. It shows that the Lipschitz condition (4.15) with $M^* = M_E^*$ in Proposition 4.1.2 is satisfied. Moreover, for any given $\tau \in \mathbb{R}$, take $t = \tau$ and $t_1, t_2 \in [T_{M_E^*}, 2T_{M_E^*}]$ in (4.68), we see that

$$
\|S(\tau,\tau-t_1)g_0 - S(\tau,\tau-t_2)g_0\| \le \lambda(M_E^*)\, \exp\left\{\frac{C_*}{2}T_{M_E^*}\right\}|t_1-t_2|^\gamma \tag{4.70}
$$

for any $g_0 \in M_E^*$. It shows that the Lipschitz condition (4.16) with $M^* = M_E^*$ is also satisfied by the nonautonomouss Hindmarsh-Rose process. According to Proposition 4.1.2, there exists a pullback exponential attractor $\mathscr{M} = \{\mathscr{M}(\tau)\}_{\tau\in\mathbb{R}}$ in the space H. The proof of this theorem is completed. □

Chapter 5

Random Attractor for Stochastic Hindmarsh-Rose Equations with Multiplicative Noise

Note to Reader

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In this work, we shall study the longtime random dynamics in terms of the existence of a random attractor for the stochastic diffusive Hindmarsh-Rose equations driven by a multiplicative white noise:

$$\frac{\partial u}{\partial t} = d_1 \Delta u + \varphi(u) + v - z + J + \varepsilon u \circ \frac{dW}{dt}, \tag{5.1}$$

$$\frac{\partial v}{\partial t} = d_2 \Delta v + \psi(u) - v + \varepsilon v \circ \frac{dW}{dt}, \tag{5.2}$$

$$\frac{\partial z}{\partial t} = d_3 \Delta z + q(u - c) - rz + \varepsilon z \circ \frac{dW}{dt}, \tag{5.3}$$

for $t > 0,\ x \in \Omega \subset \mathbb{R}^n$ ($n \leq 3$), where Ω is a bounded domain with locally Lipschitz continuous boundary, and the nonlinear terms

$$\varphi(u) = au^2 - bu^3, \quad \text{and} \quad \psi(u) = \alpha - \beta u^2. \tag{5.4}$$

with the Neumann boundary condition

$$\frac{\partial u}{\partial \nu}(t,x) = 0, \ \ \frac{\partial v}{\partial \nu}(t,x) = 0, \ \ \frac{\partial z}{\partial \nu}(t,x) = 0, \quad t > 0,\ x \in \partial\Omega, \tag{5.5}$$

and an initial condition

$$u(0,x) = u_0(x),\ v(0,x) = v_0(x),\ z(0,x) = z_0(x), \quad x \in \Omega. \tag{5.6}$$

Here $W(t), t \in \mathbb{R}$, is a one-dimensional standard Wiener process or called Brownian motion on the underlying probability space to be specified. The stochastic driving terms with the multiplicative noise indicate that the stochastic PDEs (5.1)–(5.3) are in the Stratonovich sense interpreted by the Stratonovich stochastic integrals and the corresponding differential calculus.

5.1 Introduction and Formulation

The existence of random attractors for continuous and discrete random dynamical systems has been studied in the recent three decades by many authors, cf. [2, 4, 10, 14, 17, 18, 34, 60, 61, 65, 71–73, 79–81, 84]. The following theorem is shown in [18, 60].

Theorem 5.1.1. *Given a Banach space X and a universe $\mathscr{D}_X$ of random sets in X, let φ be a continuous random dynamical system on X over the metric dynamical system $(\mathfrak{Q}, \mathcal{F}, P, \{\theta_t\}_{t\in\mathbb{R}})$. If the following two conditions are satisfied:*

(i) *there exists a closed pullback absorbing set $K = \{K(\omega)\}_{\omega\in\mathfrak{Q}} \in \mathscr{D}_X$ for φ,*

(ii) *the cocycle φ is pullback asymptotically compact with respect to $\mathscr{D}_X$,*

then there exists a unique random attractor $\mathcal{A} = \{\mathcal{A}(\omega)\}_{\omega\in\mathfrak{Q}} \in \mathscr{D}_X$ for the cocycle φ and the random attractor is given by

$$\mathcal{A}(\omega) = \bigcap_{\tau\geq 0} \overline{\bigcup_{t\geq\tau} \varphi(t, \theta_{-t}\omega, K(\theta_{-t}\omega))}, \quad \omega \in \mathfrak{Q}.$$

We now formulate the initial-boundary value problem (5.1)–(5.6) of the stochastic Hindmarsh-Rose equations with the multiplicative white noise in the framework of the product Hilbert spaces

$$H = L^2(\Omega, \mathbb{R}^3) \quad \text{and} \quad E = H^1(\Omega, \mathbb{R}^3). \tag{5.7}$$

The norm and inner-product of H or $L^2(\Omega)$ will be denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively. The norm of space E will be denoted by $\|\cdot\|_E$. The norm of $L^p(\Omega)$ or $L^p(\Omega, \mathbb{R}^3)$ will be denoted by

$\|\cdot\|_{L^p}$ for $p \neq 2$. We use $|\cdot|$ to denote a vector norm in Euclidean spaces.

The nonpositive self-adjoint linear differential operator

$$A = \begin{pmatrix} d_1\Delta & 0 & 0 \\ 0 & d_2\Delta & 0 \\ 0 & 0 & d_3\Delta \end{pmatrix} : D(A) \to H, \tag{5.8}$$

where

$$D(A) = \left\{ (\varphi, \phi, \zeta) \in H^2(\Omega, \mathbb{R}^3) : \frac{\partial \varphi}{\partial \nu} = \frac{\partial \phi}{\partial \nu} = \frac{\partial \zeta}{\partial \nu} = 0 \text{ on } \partial\Omega \right\}$$

is the generator of an analytic C_0-semigroup $\{e^{At}\}_{t \geq 0}$ of contraction on the Hilbert space H. By the Sobolev embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$ for space dimension $n \leq 3$, the nonlinear mapping

$$f(u, v, z) = \begin{pmatrix} \varphi(u) + v - z + J \\ \psi(u) - v, \\ q(u - c) - rz \end{pmatrix} : E \longrightarrow H \tag{5.9}$$

is locally Lipschitz continuous. Thus the initial-boundary value problem (5.1)–(5.6) is formulated into an initial value problem of the following stochastic Hindmarsh-Rose evolutionary equation driven by a multiplicative white noise,

$$\begin{aligned} &\frac{dg}{dt} = Ag + f(g) + \varepsilon g \circ \frac{dW}{dt}, \quad t > \tau \in \mathbb{R},\ \omega \in \mathfrak{Q}, \\ &g(\tau) = g_0 = (u_0, v_0, z_0) \in H. \end{aligned} \tag{5.10}$$

Here $g(t, \omega, g_0) = \operatorname{col}(u(t, \cdot, \omega, g_0), v(t, \cdot, \omega, g_0), z(t, \cdot, \omega, g_0))$, where dot stands for the hidden spatial variable x.

Assume that $\{W(t)\}_{t \in \mathbb{R}}$ is a one-dimensional, two-sided standard Wiener process in the probability space $(\mathfrak{Q}, \mathcal{F}, P)$, where the sample space

$$\mathfrak{Q} = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\} \tag{5.11}$$

where $C(\mathbb{R}, \mathbb{R})$ stands for the metric space of continuous functions on the real line, the σ-algebra

$\mathcal{F}$ is generated by the compact-open topology endowed in $\mathfrak{Q}$, and P is the corresponding Wiener measure [2, 14, 54] on $\mathcal{F}$. Define the P-preserving time-shift transformations $\{\theta_t\}_{t\in\mathbb{R}}$ by

$$(\theta_t\omega)(\cdot) = \omega(\cdot + t) - \omega(t), \quad \text{for } \ t \in \mathbb{R}, \ \omega \in \mathfrak{Q}. \tag{5.12}$$

Then $(\mathfrak{Q}, \mathcal{F}, P, \{\theta_t\}_{t\in\mathbb{R}})$ is a metric dynamical system and the stochastic process $\{W(t,\omega) = \omega(t) : t \in \mathbb{R}, \, \omega \in \mathfrak{Q}\}$ is the canonical Wiener process. Accordingly dW/dt in (5.10) denotes the white noise. The results we shall prove in this paper can be extended to a vector white noise with three different but independent scalar noises in the three component equations.

5.2 Random Hindmarsh-Rose Equations and Pullback Dissipativity

The mathematical treatment of the stochastic PDE such as in the form of (5.1)–(5.3) driven by the multiplicative noise will be facilitated by its conversion to a random PDE with coefficients and initial data being random variables instead. For this purpose, one can exploit the following properties of the Wiener process.

Proposition 5.2.1. *Let the MDS $(\mathfrak{Q}, \mathcal{F}, P, \{\theta_t\}_{t\in\mathbb{R}})$ and the Wiener process $W(t)$ be defined as above. Then the following statements hold.*

(1) *The Wiener process $W(t)$ has the asymptotically sublinear growth property,*

$$\lim_{t\to\pm\infty} \frac{|W(t)|}{|t|} = 0, \quad \textit{a.s.} \tag{5.13}$$

(2) *For any given positive constant λ, the stochastic process $X(t) = e^{-\lambda W(t)}$ is a solution of the following stochastic differential equation in the Stratonovich sense,*

$$dX_t = -\lambda X_t \circ dW_t. \tag{5.14}$$

(3) *$W(t)$ is locally Hölder continuous with exponents $\gamma \in \left(0, \frac{1}{2}\right)$. It means that for any integer n,*

$$\sup_{n\le s<t\le n+1} \frac{|W(t) - W(s)|}{|t-s|^\gamma} < \infty, \quad \textit{a.s.} \tag{5.15}$$

Proof. By the law of iterated logarithm [54],

$$\lim_{t\to\pm\infty} \sup \frac{|W(t)|}{\sqrt{2|t| \log \log |t|}} = 1, \quad \text{a.s.}$$

Then (5.13) is valid. Next, from Itô's formula [54] we have

$$dX_t = -\lambda e^{-\lambda W_t} dW_t + \frac{1}{2}\lambda^2 e^{-\lambda W_t} dt.$$

On the other hand, the transformation formula [54] of the stochastic Itô integral and the Stratonovich integral reads

$$h(W_t) \circ dW_t \text{ (Stratonovich sense)} = h(W_t) dW_t \text{ (Itô sense)} + \frac{1}{2} h'(W_t) dt,$$

as long as $h(W_t)$ and $h'(W_t)$ are locally L^2-integrable. Set $h(\omega) = \lambda e^{-\lambda\omega}$ in the above equality. Then

$$-\lambda X_t \circ dW_t = -\lambda e^{-\lambda W_t} dW_t + \frac{1}{2}\lambda^2 e^{-\lambda W_t} dt.$$

Hence (5.14) holds. Finally, (5.15) follows from the Kolmogorov Moment Criterion. □

We now convert the stochastic PDE (5.1)–(5.3) to a system of random PDE by the exponential multiplication of $Q(t,\omega) = e^{-\varepsilon\omega(t)}$:

$$U(t) = Q(t,\omega)u(t), \quad V(t) = Q(t,\omega)v(t), \quad Z(t) = Q(t,\omega)z(t). \tag{5.16}$$

According to the second statement in Proposition 5.2.1, the initial-boundary value problem (5.1)–(5.6) is equivalently converted to the following system of random PDEs:

$$\frac{\partial U}{\partial t} = d_1 \Delta U + \frac{a}{Q(t,\omega)} U^2 - \frac{b}{Q(t,\omega)^2} U^3 + V - Z + JQ(t,\omega), \tag{5.17}$$

$$\frac{\partial V}{\partial t} = d_2 \Delta V + \alpha Q(t,\omega) - \frac{\beta}{Q(t,\omega)} U^2 - V, \tag{5.18}$$

$$\frac{\partial Z}{\partial t} = d_3 \Delta Z + q(U - c\, Q(t,\omega)) - rZ, \tag{5.19}$$

for $\omega \in \mathfrak{Q}$, $t > 0$, $x \in \Omega \subset \mathbb{R}^n$ ($n \leq 3$), with the boundary condition

$$\frac{\partial U}{\partial \nu}(t, x, \omega) = 0, \ \frac{\partial V}{\partial \nu}(t, x, \omega) = 0, \ \frac{\partial Z}{\partial \nu}(t, x, \omega) = 0, \quad t \geq \tau \in \mathbb{R}, \ x \in \partial\Omega, \tag{5.20}$$

and an initial condition for $\omega \in \mathfrak{Q}$,

$$(U(\tau, x, \omega), V(\tau, x, \omega), Z(\tau, x, \omega)) = Q(\tau, \omega)(u_0(x), v_0(x), z_0(x)), \ \ x \in \Omega. \tag{5.21}$$

The equations (5.17)–(5.19) are pathwise nonautonomous random PDEs and (5.17)–(5.21) can be written as the initial value problem of the random evolutionary equation:

$$\begin{aligned} &\frac{\partial G}{\partial t} = AG + F(G, \theta_t \omega), \quad t \geq \tau \in \mathbb{R}, \ \omega \in \mathfrak{Q}, \\ &G(\tau, \omega) = G_\tau(\omega) = Q(\tau, \omega)(u_0, v_0, z_0), \ \omega \in \mathfrak{Q}, \end{aligned} \tag{5.22}$$

for any $g_0 = (u_0, v_0, z_0) \in H$. Here we define the weak solution of the initial value problem (5.22) with the initial state $G_\tau = Q(\tau, \omega) g_0$,

$$G(t, \omega; \tau, G_\tau) = Q(t, \omega) \begin{pmatrix} u \\ v \\ z \end{pmatrix} (t, \cdot, \omega; \tau, G_\tau) = \begin{pmatrix} U \\ V \\ Z \end{pmatrix} (t, \cdot, \omega; \tau, G_\tau),$$

to be the pathwise weak solution [13, page 283] of the nonautonomous initial-boundary problem (5.17)–(5.21), specified as in [78, Definition 2.1].

By conducting *a priori* estimates on the Galerkin approximate solutions of the equations (5.17)–(5.19) and the compactness argument outlined in [13, Chapter II and XV] with some adaptations, we can prove the local existence and uniqueness of the weal solution $G(t, \omega)$ in the space H on a local time interval $t \in [\tau, T(\omega, G_\tau)]$, and the solution is continuously depending on the initial data. Further by the parabolic regularity [62, Theorem 48.5], every weak solution becomes a strong solution in the space E when $t > \tau$ in the time interval of existence. Every weak solution $G(t, \omega)$ of the problem (5.22) on the maximal existence interval has the property

$$G \in C([\tau, T_{max}), H) \cap C^1((\tau, T_{max}), H) \cap L^2_{loc}([\tau, T_{max}), E). \tag{5.23}$$

5.2.1 Global Existence of Pullback Pathwise Solutions

In this section, we first prove the global existence of all the pullback weak solutions of the problem (5.17)–(5.21) and to explore the dissipativity of the generated random dynamical system.

Lemma 5.2.2. *There exists a random variable $r_0(\omega) > 0$ depending only on the parameters such that, for any given random variable $\rho(\omega) > 0$, there is a time $-\infty < \tau(\rho, \omega) \le -1$ and the following statement holds. For any $t_0 \le \tau(\rho, \omega)$ and for any initial data $g_0 = (u_0, v_0, z_0) \in H$ with $\|g_0\| \le \rho(\omega)$, the weak solution $G(t, \omega)$ of the problem* (5.22) *with $G(t_0, \omega) = Q(t_0, \omega) g_0$ uniquely exists on $[t_0, -1]$ and satisfies*

$$\|G(-1,\, \omega;\, t_0,\, Q(t_0, \omega) g_0)\| \le r_0(\omega), \qquad \omega \in \mathfrak{Q}. \tag{5.24}$$

Proof. Take the $L^2(\Omega)$ inner-product $\langle (5.17), c_1 U) \rangle$, $\langle (5.18), V) \rangle$ and $\langle (5.19), Z) \rangle$ with constant $c_1 > 0$ to be determined later, we obtain the following:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(c_1\|U\|^2 + \|V\|^2 + \|Z\|^2\right) + \left(c_1 d_1 \|\nabla U\|^2 + d_2 \|\nabla V\|^2 + d_3 \|\nabla Z\|^2\right) \\
&= \int_\Omega c_1 \left(\frac{a}{Q(t,\omega)} U^3 - \frac{b}{Q(t,\omega)^2} U^4 + UV - UZ + JUQ(t,\omega)\right) dx \\
&\quad + \int_\Omega \left(\alpha V Q(t,\omega) - \frac{\beta}{Q(t,\omega)} U^2 V - V^2 + q(U - cQ(t,\omega))Z - rZ^2\right) dx \\
&\le \int_\Omega c_1 \left(\frac{a}{Q(t,\omega)} U^3 - \frac{b}{Q(t,\omega)^2} U^4 + UV - UZ + JUQ(t,\omega)\right) dx \\
&\quad + \int_\Omega \left\{ \left(2\alpha^2 Q(t,\omega)^2 + \frac{\beta^2}{2Q(t,\omega)^2} U^4 - \frac{3}{8} V^2\right) \right. \\
&\quad \left. + \left[\frac{q^2}{r}(U^2 + c^2 Q(t,\omega)^2) - \frac{1}{2} r Z^2\right]\right\} dx.
\end{aligned} \tag{5.25}$$

Choose the positive constant in (5.25) to be $c_1 = \frac{1}{b}(\beta^2 + 3)$ so that

$$-c_1 \int_\Omega \frac{b}{Q(t,\omega)^2} U^4\, dx + \int_\Omega \frac{\beta^2}{Q(t,\omega)^2} U^4\, dx \le -3 \int_\Omega \frac{U^4}{Q(t,\omega)^2}\, dx.$$

By Young's inequality, we have

$$\begin{aligned}\int_\Omega \frac{c_1 a}{Q(t,\omega)} U^3\, dx &\le \frac{3}{4}\int_\Omega \frac{U^4}{Q(t,\omega)^2}\, dx + \frac{1}{4}(c_1 a Q(t,\omega))^4 |\Omega| \\ &\le \int_\Omega \frac{U^4}{Q(t,\omega)^2}\, dx + (c_1 a\, Q(t,\omega))^4\, |\Omega|,\end{aligned}$$

as well as

$$\begin{aligned}&\int_\Omega c_1(UV - UZ + JUQ(t,\omega))\, dx \\ &\le \int_\Omega \left[2(c_1 U)^2 + \frac{1}{8}V^2 + \frac{(c_1 U)^2}{r} + \frac{1}{4} r Z^2 + \frac{1}{2}(c_1 U)^2 + \frac{1}{2}J^2 Q(t,\omega)^2\right] dx.\end{aligned} \tag{5.26}$$

Collecting those integral terms of U^2 on the right-hand side in (5.25) and in (5.26), we obtain

$$\begin{aligned}&\int_\Omega \left[2(c_1 U)^2 + \frac{(c_1 U)^2}{r} + \frac{1}{2}(c_1 U)^2 + \frac{q^2}{r}U^2\right] dx \\ &\le \int_\Omega \frac{U^4}{Q(t,\omega)^2}\, dx + \left[c_1^2\left(\frac{5}{2} + \frac{1}{r}\right) + \frac{q^2}{r}\right]^2 Q(t,\omega)^2 |\Omega|.\end{aligned}$$

Substitute the above inequalities with respect to the integral terms of U^4, U^3 and U^2 into (5.25).

Then we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(c_1\|U\|^2 + \|V\|^2 + \|Z\|^2\right) + \left(c_1 d_1 \|\nabla U\|^2 + d_2\|\nabla V\|^2 + d_3\|\nabla Z\|^2\right) \\ &\le \int_\Omega \left[\frac{2-3}{Q(t,\omega)^2}U^4 + \left(\frac{1}{8} - \frac{3}{8}\right)V^2 + \left(\frac{1}{4} - \frac{1}{2}\right) r Z^2\right] dx + (c_1 a)^4 Q(t,\omega)^4 |\Omega| \\ &\quad + \left[\frac{1}{2}J^2 + \left(c_1^2\left(\frac{5}{2} + \frac{1}{r}\right) + \frac{q^2}{r}\right)^2 + 2\alpha^2 + \frac{q^2 c^2}{r}\right] Q(t,\omega)^2 |\Omega| \\ &\le -\int_\Omega \left(\frac{1}{Q(t,\omega)^2}U^4(t,x) + \frac{1}{4}V^2(t,x) + \frac{1}{4} r Z^2(t,x)\right) dx \\ &\quad + (c_1 a)^4 Q(t,\omega)^4 |\Omega| + c_2\, Q(t,\omega)^2 |\Omega|,\end{aligned} \tag{5.27}$$

where

$$c_2 = \frac{1}{2}J^2 + \left[c_1^2\left(\frac{5}{2} + \frac{1}{r}\right) + \frac{q^2}{r}\right]^2 + 2\alpha^2 + \frac{q^2 c^2}{r}.$$

Let $d = \min\{d_1, d_2, d_3\}$. Then the inequality (5.27) implies

$$\begin{aligned}
&\frac{d}{dt}(c_1\|U(t)\|^2 + \|V(t)\|^2 + \|Z(t)\|^2) + 2d(c_1\|\nabla u\|^2 + \|\nabla v\|^2 + \|\nabla w\|^2) \\
&+ \int_\Omega \left(\frac{2}{Q(t,\omega)^2} U^4(t,x) + \frac{1}{2} V^2(t,x) + \frac{1}{2} r Z^2(t,x) \right) dx \\
\le\ & 2c_2\, Q(t,\omega)^2|\Omega| + 2(c_1 a)^4 Q(t,\omega)^4 \Omega|.
\end{aligned}$$

Moreover, we have

$$\frac{2}{Q(t,\omega)^2} U^4 \ge \frac{1}{2} \left(c_1 U^2 - \frac{c_1^2 Q(t,\omega)^2}{16} \right).$$

Therefore,

$$\begin{aligned}
&\frac{d}{dt}(c_1\|U(t)\|^2 + \|V(t)\|^2 + \|Z(t)\|^2) + 2d(c_1\|\nabla U\|^2 + \|\nabla V\|^2 + \|\nabla Z\|^2) \\
&+ \frac{1}{2}(c_1\|U(t)\|^2 + \|V(t)\|^2 + r\|Z(t)\|^2) \\
\le\ & \left(2c_2 + \frac{1}{32} c_1^2 \right) Q(t,\omega)^2|\Omega| + 2(c_1 a)^4 Q(t,\omega)^4|\Omega|,
\end{aligned} \tag{5.28}$$

for $t \in [\tau, T_{max})$. Set $\sigma = \frac{1}{2}\min\{1, r\}$. Then the Gronwall inequality is applied to the reduced inequality (5.28) ,

$$\begin{aligned}
&\frac{d}{dt}(c_1\|U(t)\|^2 + \|V(t)\|^2 + \|Z(t)\|^2) + \sigma(c_1\|U(t)\|^2 + \|V(t)\|^2 + \|Z(t)\|^2) \\
\le\ & \left(2c_2 + \frac{1}{32} c_1^2 \right) Q(t,\omega)^2|\Omega| + 2(c_1 a)^4 Q(t,\omega)^4|\Omega|,
\end{aligned}$$

and shows that

$$\begin{aligned}
&c_1\|U(t)\|^2 + \|V(t)\|^2 + \|Z(t)\|^2 \le e^{-\sigma(t-t_0)}(c_1\|U_0\|^2 + \|V_0\|^2 + \|Z_0\|^2) \\
&+ \int_{-\infty}^t e^{-\sigma(t-s)} \left[\left(2c_2 + \frac{1}{32} c_1^2 \right) Q(s,\omega)^2|\Omega| + 2(c_1 a)^4 Q(s,\omega)^4|\Omega| \right] ds,
\end{aligned} \tag{5.29}$$

for $t \in [\tau, T_{max})$. We obtain

$$\begin{aligned}
&\|U(t)\|^2 + \|V(t)\|^2 + \|Z(t)\|^2 \le \frac{\max\{c_1, 1\}}{\min\{c_1, 1\}}\, e^{-\sigma(t-t_0)}(\|U_0\|^2 + \|V_0\|^2 + \|Z_0\|^2) \\
&+ \frac{|\Omega|}{\min\{c_1, 1\}} \int_{-\infty}^t e^{-\sigma(t-s)} \left[\left(2c_2 + \frac{1}{32} c_1^2 \right) Q(s,\omega)^2 + 2(c_1 a)^4 Q(s,\omega)^4 \right] ds.
\end{aligned} \tag{5.30}$$

Hence, the solutions of the initial value problem of the equation (5.22) satisfies the bounded estimate

$$
\begin{aligned}
&\|G(t,\omega;t_0,Q(t_0,\omega)g_0)\|^2 \le \frac{\|Q(t_0,\omega)\|^2 \max\{c_1,1\}}{\min\{c_1,1\}}\, e^{-\sigma(t-t_0)}\, \|g_0\|^2 \\
&+ \frac{|\Omega|}{\min\{c_1,1\}} \int_{-\infty}^{t} e^{-\sigma(t-s)} \left[\left(2c_2 + \frac{1}{32}c_1^2\right) Q(s,\omega)^2 + 2(c_1 a)^4 Q(s,\omega)^4 \right] ds, \;\; t \ge t_0.
\end{aligned}
\tag{5.31}
$$

Take $t=-1$ and substitute $Q(t,\omega)=e^{-\varepsilon\omega(t)}$ into (5.31). We then get

$$
\begin{aligned}
&\|G(-1,\omega;t_0,Q(t_0,\omega)g_0)\|^2 \le \frac{\max\{c_1,1\}}{\min\{c_1,1\}} e^{\sigma-\sigma|t_0|-2\varepsilon\,\omega(t_0)} \|g_0\|^2 \\
&+ \frac{|\Omega|}{\min\{c_1,1\}} \int_{-\infty}^{-1} e^{\sigma+\sigma s} \left[\left(2c_2 + \frac{1}{32}c_1^2\right) e^{-2\varepsilon\omega(s)} + 2(c_1 a)^4 e^{-4\varepsilon\omega(s)} \right] ds.
\end{aligned}
\tag{5.32}
$$

Note that

$$
e^{-\sigma|t_0|-2\varepsilon\omega(t_0)} = \exp\left(-\sigma|t_0|\left[1+\frac{2\varepsilon\,\omega(t_0)}{\sigma|t_0|}\right]\right) = \exp\left(-\sigma|t_0|\left[1-\frac{2\varepsilon\,\omega(t_0)}{\sigma t_0}\right]\right).
$$

By the asymptotically sublinear property (5.13), for any given random variable $\rho(\omega)>0$ and for a.e. $\omega\in\mathfrak{Q}$, there exist a time $\tau(\rho,\omega)\le -1$ such that for any $t_0\le\tau(\rho,\omega)$, we have

$$
1-\frac{2\varepsilon\omega(t_0)}{\sigma t_0} \ge \frac{1}{2} \quad \text{and} \quad e^{\sigma(1-\frac{1}{2}|t_0|)} \frac{\max\{c_1,1\}}{\min\{c_1,1\}} \rho^2(\omega) \le 1. \tag{5.33}
$$

Therefore, from (5.32), we obtain

$$
\|G(-1,\omega;t_0,Q(t_0,\omega)g_0)\| \le r_0(\omega), \quad \text{a.s.} \tag{5.34}
$$

where

$$
r_0(\omega) = \sqrt{1 + \frac{|\Omega|}{\min\{c_1,1\}} \int_{-\infty}^{-1} e^{\sigma+\sigma s} \left[\left(2c_2+\frac{1}{32}c_1^2\right) e^{-2\varepsilon\omega(s)} + 2(c_1 a)^4 e^{-4\varepsilon\omega(s)} \right] ds} \tag{5.35}
$$

in which both integrals

$$
\int_{-\infty}^{-1} e^{\sigma+\sigma s} \left(2c_2+\frac{1}{32}c_1^2\right) e^{-2\varepsilon\omega(s)}\, ds \quad \text{and} \quad \int_{-\infty}^{-1} 2e^{\sigma+\sigma s}(c_1 a)^4 e^{-4\varepsilon\omega(s)}\, ds
$$

are convergent due to the asymptotically sublinear growth property (5.13).

Therefore, the weak solution $G(t, \omega; t_0, Q(t_0, \omega)g_0)$ of the problem (5.22) uniquely exists on $[t_0, -1]$. The proof is completed. $\square$

Lemma 5.2.3. *There exists a random variable $R_0(\omega) > 0$ depending only on the parameters such that, for any given random variable $\rho(\omega) > 0$, the following statement holds. For any $t_0 \le \tau(\rho, \omega)$ specified in Lemma 5.2.2 and any initial data $g_0 = (u_0, v_0, z_0) \in H$ with $\|g_0\| \le \rho(\omega)$, the weak solution $G(t, \omega; t_0, Q(t_0, \omega)g_0)$ of the initial value problem (5.22) with $G(t_0, \omega) = Q(t_0, \omega)g_0$ uniquely exists on $[t_0, \infty)$ and satisfies*

$$\|G(0, \omega; t_0, Q(t_0, \omega)g_0)\|^2 + \int_{-1}^{0} \|\nabla G(s, \omega; t_0, Q(t_0, \omega)g_0)\|^2 \, ds \le R_0^2(\omega), \quad \omega \in \mathfrak{Q} \tag{5.36}$$

Proof. Based on Lemma 5.2.2 and the local extension of the solutions of the problem (5.22) from the time $t_1 = -1$ forward, we can integrate the inequality (5.28) over $[-1, t]$ to get

$$\begin{aligned}
&c_1\|U(t)\|^2 + \|V(t)\|^2 + \|Z(t)\|^2 - (c_1\|U(-1)\|^2 + \|V(-1)\|^2 + \|Z(-1)\|^2) \\
&\quad + 2d \int_{-1}^{t} \left(c_1\|\nabla U(s)\|^2 + \|\nabla V(s)\|^2 + \|\nabla Z(s)\|^2\right) ds \\
&\quad + \sigma \int_{-1}^{t} \left(c_1\|U(s)\|^2 + \|V(s)\|^2 + \|Z(s)\|^2\right) ds \\
&\le |\Omega| \int_{-1}^{t} \left[\left(2c_2 + \frac{1}{32}c_1^2\right) Q(s, \omega)^2 + 2(c_1 a)^4 Q(s, \omega)^4\right] ds, \quad t > -1.
\end{aligned} \tag{5.37}$$

Then

$$\begin{aligned}
&\|G(t, \omega; t_0, Q(t_0, \omega)g_0)\|^2 + 2d \int_{-1}^{t} \|\nabla G(s, \omega; t_0, Q(t_0, \omega)g_0)\|^2 \, ds \\
&\le \frac{\max\{c_1, 1\}}{\min\{c_1, 1\}} \|G(-1, \omega; t_0, Q(t_0, \omega)g_0)\|^2 \\
&\quad + \frac{|\Omega|}{\min\{c_1, 1\}} \int_{-1}^{t} \left[\left(2c_2 + \frac{1}{32}c_1^2\right) Q(s, \omega)^2 + 2(c_1 a)^4 Q(s, \omega)^4\right] ds.
\end{aligned} \tag{5.38}$$

The inequality (5.38) together with Lemma 5.2.2 shows that for $\omega \in \mathfrak{Q}$ and any $T > -1$, the weak solution $G(t, \omega; t_0, Q(t_0, \omega)g_0) \in C[t_0, T; H) \cap L^2(t_0, T; E)$ uniquely exists for $t \in [-1, T]$ and will not blow up. In particular, let $t = 0$ in (5.38) and we obtain

$$\|G(0, \omega; t_0, Q(t_0, \omega) g_0)\|^2 + \int_{-1}^{0} \|\nabla G(s, \omega; t_0, Q(t_0, \omega) g_0)\|^2 \, ds \le R_0^2(\omega), \tag{5.39}$$

where

$$\begin{aligned} R_0^2(\omega) = & \frac{1}{\min\{1, 2d\} \min\{c_1, 1\}} \\ & \times \left\{ \max\{c_1, 1\} |r_0(\omega)|^2 + |\Omega| \int_{-1}^{0} \left[\left(2c_2 + \frac{1}{32} c_1^2 \right) Q(s, \omega)^2 + 2(c_1 a)^4 Q(s, \omega)^4 \right] ds \right\} \end{aligned} \tag{5.40}$$

where $r_0(\omega)$ is defined in (5.35). Note that $r_0(\omega)$ and $R_0(\omega)$ are both independent of random variables $\rho(\omega)$. □

Remark 5.2.4. We can certainly merge the above two lemmas into one which gives rise to the bounded estimate (5.39). Here we split the time interval $[t_0, 0]$ to $[t_0, -1] \cup [-1, 0]$ in order to facilitate the argument in the proof of the pullback asymptotic compactness of the associated random dynamical system later in Section 5.3.

5.2.2 Hindmarsh-Rose Cocycle and Absorbing Property

Now define a concept of stochastic semiflow, which is related to the concept of cocycle in the theory of random dynamical systems.

Definition 5.2.5. Let $(\mathfrak{Q}, \mathcal{F}, P, \{\theta_t\}_{t \in \mathbb{R}})$ be a metric dynamical system. A family of mappings $S(t, \tau, \omega) : X \to X$ for $t \ge \tau \in \mathbb{R}$ and $\omega \in \mathfrak{Q}$ is called a *stochastic semiflow* on a Banach space X, if it satisfies the properties:

(i) $S(t, s, \omega) S(s, \tau, \omega) = S(t, \tau, \omega)$, for all $\tau \le s \le t$ and $\omega \in \mathfrak{Q}$.

(ii) $S(t, \tau, \omega) = S(t - \tau, 0, \theta_\tau \omega)$, for all $\tau \le t$ and $\omega \in \mathfrak{Q}$.

(iii) The mapping $S(t, \tau, \omega) x$ is measurable in (t, τ, ω) and continuous in $x \in X$.

Here in the setting of the stochastic evolutionary equation (5.22) formulated from the stochastic Hindmarsh-Rose equations (5.1)–(5.6), we define $S(t, \tau, \omega) : H \to H$ for $t \ge \tau \in \mathbb{R}$ and $\omega \in \mathfrak{Q}$

by

$$S(t,\tau,\omega)\,g_0 = \frac{1}{Q(t,\omega)}\,G(t,\,\omega;\,\tau,\,G_0) = \begin{pmatrix} u \\ v \\ z \end{pmatrix}(t,\,\omega;\,\tau,g_0) \tag{5.41}$$

and then define a mapping $\Phi : \mathbb{R}^+ \times \mathfrak{Q} \times H \to H$, where $\mathbb{R}^+ = [0,\infty)$, to be

$$\Phi(t-\tau,\,\theta_\tau\omega,\,g_0) = S(t,\tau,\omega)\,g_0 \tag{5.42}$$

which is equivalent to

$$\Phi(t,\,\omega,\,g_0) = S(t,0,\omega)g_0 = \frac{1}{Q(t,\omega)}\,G(t,\,\omega;\,0,\,G_0). \tag{5.43}$$

The following lemma shows that this mapping Φ is a cocycle on the Hilbert space H over the canonical metric dynamical system $(\mathfrak{Q},\mathcal{F},P,\{\theta_t\}_{t\in\mathbb{R}})$ specified in (5.11) and (5.12). Therefore, the following *pullback identity* is validated:

$$\Phi(t,\theta_{-t}\omega,g_0) = S(0,-t,\omega)g_0 = \frac{1}{Q(0,\omega)}G(0,\,\omega;\,-t,\,G_0) = g(0,\,\omega;\,-t,\,g_0) \tag{5.44}$$

for any $t \geq 0$ and $\omega \in \mathfrak{Q}$. We shall call this mapping Φ defined by (5.42) the *Hindmarsh-Rose cocycle*, which is a random dynamical system on H. We call $\{\Phi(t,\theta_{-t}\omega,g_0) : t \geq 0\}$ a *pullback quasi-trajectory* with the initial state g_0 for the Hindmarsh-Rose cocycle.

Remark 5.2.6. Here the pullback quasi-trajectory $\{\Phi(t,\theta_{-t}\omega,g_0), t \geq 0\}$ is not a single trajectory but the set of all the points at time $t = 0$ of the bunch of trajectories started from the same initial state g_0 but at different pullback initial time $-t$.

Lemma 5.2.7. *The mapping* $\Phi : \mathbb{R}^+ \times \mathfrak{Q} \times H \to H$ *defined by* (5.41) *and* (5.42) *is a cocycle on the space* H *over the canonical metric dynamical system* $(\mathfrak{Q},\mathcal{F},P,\{\theta_t\}_{t\in\mathbb{R}})$. *Moreover,*

$$(\Pi_t g)(\omega) = \Phi(t,\theta_{-t}\omega,g(\theta_{-t}\omega)), \quad t \geq 0, \tag{5.45}$$

where $\{g(\omega) : \omega \in \mathfrak{Q}\}$ *can be any* H*-valued random set on the probability space* $(\mathfrak{Q},\mathcal{F},P)$*, turns out to be a semigroup of operators on the* H*-valued random sets.*

Proof. First we check the cocycle property of the mapping Φ,

$$\Phi(t+s,\omega,g_0)=\Phi(t,\theta_s\,\omega,\Phi(s,\omega,g_0)),\quad t\geq 0,\ s\geq 0,\ \omega\in\mathfrak{Q}, \tag{5.46}$$

is satisfied by this mapping Φ. Since we have (5.43),

$$\Phi(t+s,\omega,g_0)=\frac{1}{Q(t+s,\omega)}G(t+s,\omega;\,0,\,G_0)$$

and, on the other hand,

$$\begin{aligned}
\Phi(t,\theta_s\,\omega,\Phi(s,\omega,g_0)) &= \frac{1}{Q(t,\omega)}G(t,\theta_s\omega;\,0,\,G(s,\omega;0,G_0)) \quad \text{(by (5.43))}\\
&= g(t,\theta_s\omega;\,0,\,g(s,\omega;0,g_0)) = S(t,0;\theta_s\omega)\,g(s,\omega;0,g_0) \quad \text{(by (5.41))}\\
&= S(t,0;\theta_s\omega)\,S(s,0;\omega)g_0 = S(t+s-s,0;\theta_s\omega)\,S(s,0;\omega)g_0\\
&= S(t+s,s;\omega)\,S(s,0;\omega)g_0 \qquad \text{(by the 2nd condition of Definition 5.2.5)}\\
&= S(t+s,0;\omega)\,g_0 = \frac{1}{Q(t+s,\omega)}G(t+s,\omega;0,G_0).
\end{aligned}$$

Therefore, the cocycle property (5.46) of the mapping Φ is valid by comparison of the above two equalities.

The second claim that $\{\Pi_t\}_{t\geq 0}$ is a semigroup can be shown as follows,

$$\begin{aligned}
(\Pi_t\,[\Pi_\sigma\,g])(\omega) &= \Phi(t,\,\theta_{-t}\omega,\,[\Pi_\sigma\,g](\theta_{-t}\omega))\\
&= \Phi(t,\,\theta_{-t}\omega,\,\Phi(\sigma,\theta_{-\sigma}(\theta_{-t}\omega),g(\theta_{-\sigma}(\theta_{-t}\omega))))\\
&= \Phi(t,\,\theta_{-t}\omega,\,\Phi(\sigma,\theta_{-(t+\sigma)}\omega,g(\theta_{-(t+\sigma)}\omega)))\\
&= \Phi(t,\,\theta_{-(t+\sigma)}\theta_\sigma\omega,\,\Phi(\sigma,\theta_{-(t+\sigma)}\omega,g(\theta_{-(t+\sigma)}\omega)))\\
&= \Phi(t,\,\theta_\sigma\theta_{-(t+\sigma)}\omega,\,\Phi(\sigma,\theta_{-(t+\sigma)}\omega,g(\theta_{-(t+\sigma)}\omega)))\\
&= \Phi(t+\sigma,\,\theta_{-(t+\sigma)}\omega,\,g(\theta_{-(t+\sigma)}\omega)) = (\Pi_{t+\sigma}\,g)(\omega),\quad t,\sigma\geq 0.
\end{aligned} \tag{5.47}$$

where the final equality follows from the cocycle property of Φ already proved. $\square$

Remark 5.2.8. Apparently when the stochastic PDEs (5.1)–(5.3) are converted to the random

PDEs (5.17)–(5.19) by the exponential multiplication (5.16), we see the coefficients are time-depending random variables instead of constants, which means the system (5.22) is nonautonomous in time. The justification for the corresponding stochastic semiflow (5.41) to be well-defined and satisfy the stationary property in Definition 5.2.5 is due to the stationary property possessed by the underlying Wiender process $\{W(t)\}_{t\in\mathbb{R}}$, which is characterized by the stationary increment $W(t) - W(s)$ of the Gaussian distribution with mean zero and variance $t - s$, in the problem setting.

Theorem 5.2.9. *There exists a pullback absorbing set in the space H with respect to the tempered universe $\mathscr{D}_H$ for the Hindmarsh-Rose cocycle Φ, which is the bounded random ball*

$$B_0(\omega) = B_H(0, R_0(\omega)) = \{\xi \in H : \|\xi\| \le R_0(\omega)\} \tag{5.48}$$

where $R_0(\omega)$ is given in (5.40).

Proof. For any bounded random ball $B(\omega) = B_H(0, \rho(\omega)) \in \mathscr{D}_H$ and any $g_0 \in B(\theta_{-t}\omega)$, by Definition 1.3.14 we have

$$\lim_{t\to-\infty} e^{-\beta t}\rho(\theta_{-t}\omega) = 0, \quad \text{for any } \beta > 0. \tag{5.49}$$

From (5.31), for $-t \le -1$ we have

$$\begin{aligned}
\sup_{g_0\in B(\theta_{-t}\omega)} \|G(-1,\omega;-t,Q(-t,\omega)g_0)\|^2 &\le \frac{\|Q(-t,\omega)\|^2 \max\{c_1,1\}}{\min\{c_1,1\}} e^{\sigma(1-t)}\|g_0\|^2 \\
&\quad + \frac{|\Omega|}{\min\{c_1,1\}} \int_{-\infty}^{-1} e^{\sigma(1+s)} \left[\left(2c_2 + \frac{1}{32}c_1^2\right) Q(t,s)^2 + 2(c_1 a)^4 Q(t,s)^4\right] ds \\
&\le \frac{e^{\sigma}\max\{c_1,1\}}{\min\{c_1,1\}} e^{-2\varepsilon\omega(-t)-\sigma t}\rho^2(\theta_{-t}\omega) \qquad (\text{since } g_0 \in B(\theta_{-t}\omega)) \\
&\quad + \frac{|\Omega|}{\min\{c_1,1\}} \int_{-\infty}^{-1} e^{\sigma(1+s)} \left[\left(2c_2 + \frac{1}{32}c_1^2\right) e^{-2\varepsilon\omega(s)} + 2(c_1 a)^4 e^{-4\varepsilon\omega(s)}\right] ds \\
&\le \frac{e^{\sigma}\max\{c_1,1\}}{\min\{c_1,1\}} \exp\left[-\frac{\sigma t}{2}\left(1 - \frac{4\varepsilon}{\sigma}\left(\frac{\omega(-t)}{-t}\right)\right)\right] e^{-\frac{\sigma t}{2}}\rho^2(\theta_{-t}\omega) \\
&\quad + \frac{|\Omega|}{\min\{c_1,1\}} \int_{-\infty}^{-1} e^{\sigma(1+s)} \left[\left(2c_2 + \frac{1}{32}c_1^2\right) e^{-2\varepsilon\omega(s)} + 2(c_1 a)^4 e^{-4\varepsilon\omega(s)}\right] ds.
\end{aligned}$$

From (5.13), we have

$$\lim_{t\to\infty} \exp\left[-\frac{\sigma t}{2}\left(1-\frac{4\varepsilon}{\sigma}\left(\frac{\omega(-t)}{-t}\right)\right)\right] = 0, \quad \omega \in \mathfrak{Q}.$$

Since $B(\omega) = B_H(0, \rho(\omega)) \in \mathscr{D}_H$, the radius $\rho(\theta_{-t}\omega)$ is a tempered random variable, so that

$$\lim_{t\to\infty} e^{-\frac{\sigma t}{2}}\rho^2(\theta_{-t}\omega) = \lim_{t\to\infty} |e^{-\frac{\sigma t}{4}}\rho(\theta_{-t}\omega)|^2 = 0.$$

Therefore, there exists a finite random variable $T_B(\omega) > 1$ such that for all $t \geq T_B(\omega)$ we have

$$\frac{e^{\sigma}\max\{c_1, 1\}}{\min\{c_1, 1\}}\exp\left[-\frac{\sigma t}{2}\left(1-\frac{4\varepsilon}{\sigma}\left(\frac{\omega(-t)}{-t}\right)\right)\right] \leq 1 \text{ and } e^{-\frac{\sigma t}{2}}\rho^2(\theta_{-t}\omega) \leq 1, \ \omega \in \mathfrak{Q}.$$

Then

$$\sup_{g_0 \in B(\theta_{-t}\omega)} \|G(-1, \theta_{-t}\omega; -t, Q(-t, \omega)g_0)\| \leq r_0(\omega), \tag{5.50}$$

for $t \geq T_B(\omega)$, $\omega \in \mathfrak{Q}$ and where $r_0(\omega)$ is given in (5.35).

Finally, put together (5.39), (5.40) and (5.50). We end up with

$$\begin{aligned} &\sup_{g_0 \in B(\theta_{-t}\omega)} \|\Phi(t, \theta_{-t}\omega, g_0)\| \\ = &\sup_{g_0 \in B(\theta_{-t}\omega)} \|G(0, \theta_{-t}\omega; -t, Q(-t, \omega)g_0)\| \leq R_0(\omega), \end{aligned} \tag{5.51}$$

for $t \geq T_B(\omega)$ a.s. Hence, the random set in (5.48) is a pullback absorbing set for the Hindmarsh-Rose cocycle Φ. The proof is completed. □

5.3 The Existence of Random Attractor with the Multiplicative Noise

In this section, we shall prove that this Hindmarsh-Rose cocycle is pullback asymptotically compact on H through the following two lemmas. Then the main result on the existence of a random attractor for the Hindmarsh-Rose cocycle is established.

Lemma 5.3.1. *Assume that for any random variable $R(\omega) > 0$ and any given $\tau < -2$, there exists a random variable $M(R, \omega) > 0$ such that the following statement is valid*: *If there is a time*

$t^* \in [-2,-1]$ such that $G(t^*,\omega;\,\tau,Q(\tau,\omega)g_0) \in E$ for any $g_0 \in H$ which satisfies

$$\|G(t^*,\,\omega;\,\tau,\,Q(\tau,\omega)g_0)\|_E \le R(\omega),$$

then it holds that

$$\|G(0,\,\omega;\,\tau,\,Q(\tau,\omega)g_0)\|_E \le M(R,\omega). \tag{5.52}$$

Proof. Denote the solution of (5.22) by $G(t,\omega;\tau,Q(\tau,\omega)g_0) = (U(t),V(t),Z(t))$. Take the L^2 inner-product $\langle(5.17), -\Delta U(t)\rangle$ to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla U\|^2 + d_1\|\Delta U\|^2 \\
&= \int_\Omega \left(-\frac{a}{Q(t,\omega)}U^2\Delta U - \frac{b}{Q(t,\omega)^2}U^2|\nabla U|^2 - V\Delta U + Z\Delta U - JQ(t,\omega)\Delta U\right)dx \\
&\le \int_\Omega \left(\frac{2a^2}{d_1 Q(t,\omega)^2}U^4 + \frac{d_1}{8}|\Delta U|^2 + \frac{2}{d_1}V^2 + \frac{d_1}{8}|\Delta U|^2\right)ds \\
&\quad + \int_\Omega \left(\frac{2}{d_1}Z^2 + \frac{d_1}{8}|\Delta U|^2 + \frac{2J^2Q(t,\omega)^2}{d_1} + \frac{d_1}{8}|\Delta U|^2\right)dx - \int_\Omega \frac{b}{Q(t,\omega)^2}U^2|\nabla U|^2\,dx.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\frac{d}{dt}\|\nabla U\|^2 + d_1\|\Delta U\|^2 + \frac{2b}{Q(t,\omega)^2}\|U\nabla U\|^2 \\
&\le \frac{4a^2}{d_1 Q(t,\omega)^2}\|U\|^4_{L^4} + \frac{4}{d_1}\|V\|^2 + \frac{4}{d_1}\|Z\|^2 + \frac{4J^2Q(t,\omega)^2}{d_1}|\Omega|, \quad t > \tau.
\end{aligned} \tag{5.53}$$

Take the L^2 inner-product $\langle(5.18), -\Delta V(t)\rangle$, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla V\|^2 + d_2\|\Delta V\|^2 + \|\nabla V\|^2 \\
&= \int_\Omega \left(-\alpha Q(t,\omega)\Delta V + \frac{\beta}{Q(t,\omega)}U^2\Delta V + V\Delta V\right)dx \\
&\le \int_\Omega \left(\frac{\alpha^2 Q(t,\omega)^2}{d_2} + \frac{d_2}{4}|\Delta V|^2 + \frac{\beta^2}{d_2 Q(t,\omega)^2}U^4 + \frac{d_2}{4}|\Delta V|^2 - |\nabla V|^2\right)dx.
\end{aligned}$$

Then

$$\frac{d}{dt}\|\nabla V\|^2 + d_2\|\Delta V\|^2 + 2\|\nabla V\|^2 \le \frac{2\alpha^2 Q(t,\omega)^2}{d_2}|\Omega| + \frac{2\beta^2}{d_2 Q(t,\omega)^2}\|U\|^4_{L^4}, \quad t > \tau. \tag{5.54}$$

Take the L^2 inner-product $\langle(5.19), -\Delta Z(t)\rangle$, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla Z\|^2 + d_3\|\Delta Z\|^2\\&= \int_\Omega [qcQ(t,\omega)\Delta Z - qU\Delta Z + rZ\Delta Z]\,dx\\&\le \int_\Omega \left(\frac{q^2c^2Q(t,\omega)^2}{d_3} + \frac{d_3}{4}|\Delta Z|^2 + \frac{q^2}{d_3}U^2 + \frac{d_3}{4}|\Delta Z|^2 - r|\nabla Z|^2\right)dx.\end{aligned}$$

It implies

$$\frac{d}{dt}\|\nabla Z\|^2 + d_3\|\Delta Z\|^2 + 2r\|\nabla Z\|^2 \le \frac{2q^2c^2Q(t,\omega)^2}{d_3}|\Omega| + \frac{2q^2}{d_3}\|U\|^2, \quad t > \tau. \tag{5.55}$$

Sum up the above estimates (5.53), (5.54) and (5.55). Then we obtain

$$\begin{aligned}&\frac{d}{dt}(\|\nabla U\|^2 + \|\nabla V\|^2 + \|\nabla Z\|^2) + d_1\|\Delta U\|^2 + d_2\|\Delta V\|^2 + d_3\|\Delta Z\|^2\\&\quad + \frac{2b}{Q(t,\omega)^2}\|U\nabla U\|^2 + 2\|\nabla V\|^2 + r\|\nabla Z\|^2\\&\le \frac{2q^2}{d_3}\|U\|^2 + \frac{4}{d_1}\|V\|^2 + \frac{4}{d_1}\|Z\|^2 + \frac{1}{Q(t,\omega)^2}\left(\frac{4a^2}{d_1} + \frac{2\beta^2}{d_2}\right)\|U\|^4_{L^4}\\&\quad + Q(t,\omega)^2\left(\frac{4J^2}{d_1} + \frac{2\alpha^2}{d_2} + \frac{2q^2c^2}{d_3}\right)|\Omega|.\end{aligned} \tag{5.56}$$

Since $H^1(\Omega) \hookrightarrow L^4(\Omega)$, there is a positive constant $\eta > 0$ associated with the Sobolev imbedding inequality such that

$$\|U\|^4_{L^4} \le \eta(\|U\|^2 + \|\nabla U\|^2)^2 \le 2\eta(\|U\|^4 + \|\nabla U\|^4).$$

For any $t \in [t^*, 0] \subset [\tau, 0]$, the inequality (5.31) implies that

$$\begin{aligned}&\|G(t,\omega;\tau,Q(\tau,\omega)g_0)\|^2 \le \frac{\max\{c_1,1\}}{\min\{c_1,1\}}\|G(t^*,\omega;\tau,Q(\tau,\omega)g_0)\|^2\\&\quad + \frac{|\Omega|}{\min\{c_1,1\}}\int_{-\infty}^0 e^{\sigma s}\left[\left(2c_2 + \frac{1}{32}c_1^2\right)Q(s,\omega)^2 + 2(c_1a)^4Q(s,\omega)^4\right]ds,\end{aligned} \tag{5.57}$$

where the improper integral in (5.57) is convergent due to (5.13) and $\sigma = \frac{1}{2}\min\{1, r\} > 0$, as given after (5.28). Denote by

$$
\begin{aligned}
P_0(R,\omega) = \; & \frac{\max\{c_1,1\}}{\min\{c_1,1\}} R^2(\omega) \\
& + \frac{|\Omega|}{\min\{c_1,1\}} \int_{-\infty}^{0} e^{\sigma s} \left[\left(2c_2 + \frac{1}{32} c_1^2 \right) Q(s,\omega)^2 + 2(c_1 a)^4 Q(s,\omega)^4 \right] ds.
\end{aligned}
$$

Then from (5.56) we obtain

$$
\begin{aligned}
& \frac{d}{dt} \|\nabla G\|^2 + d_1 \|\Delta U\|^2 + d_2 \|\Delta V\|^2 + d_3 \|\Delta Z\|^2 \\
& + \frac{2b}{Q(t,\omega)^2} \|U \nabla U\|^2 + 2\|\nabla V\|^2 + r\|\nabla Z\|^2 \\
\le \; & \max\left\{ \frac{2q^2}{d_3}, \frac{4}{d_1} \right\} P_0(R,\omega) + \frac{2\eta}{Q(t,\omega)^2} \left(\frac{4a^2}{d_1} + \frac{2\beta^2}{d_2} \right) P_0^2(R,\omega) \\
& + \frac{2\eta}{Q(t,\omega)^2} \left(\frac{4a^2}{d_1} + \frac{2\beta^2}{d_2} \right) \|\nabla U\|^4 + Q(t,\omega)^2 \left(\frac{4J^2}{d_1} + \frac{2\alpha^2}{d_2} + \frac{2q^2 c^2}{d_3} \right) |\Omega|.
\end{aligned}
\tag{5.58}
$$

Here we can apply the uniform Gronwall inequality to the following inequality

$$
\begin{aligned}
\frac{d}{dt} \|\nabla G(t)\|^2 \le \; & \frac{2\eta}{Q(t,\omega)^2} \left(\frac{4a^2}{d_1} + \frac{2\beta^2}{d_2} \right) \|\nabla G(t)\|^2 \|\nabla G(t)\|^2 \\
& + \max\left\{ \frac{2q^2}{d_3}, \frac{4}{d_1} \right\} P_0(R,\omega) + \frac{2\eta}{Q(t,\omega)^2} \left(\frac{4a^2}{d_1} + \frac{2\beta^2}{d_2} \right) P_0^2(R,\omega) \\
& + Q(t,\omega)^2 \left(\frac{4J^2}{d_1} + \frac{2\alpha^2}{d_2} + \frac{2q^2 c^2}{d_3} \right) |\Omega|,
\end{aligned}
\tag{5.59}
$$

for $t \ge t^*$, which is written in the form

$$
\frac{d\xi}{dt} \le p\,\xi + h, \tag{5.60}
$$

where

$$
\begin{aligned}
\xi(t) &= \|\nabla G(t)\|^2, \\
p(t) &= \frac{2\eta}{Q(t,\omega)^2} \left(\frac{4a^2}{d_1} + \frac{2\beta^2}{d_2} \right) \|\nabla G(t)\|^2, \\
h(t) &= \max\left\{ \frac{2q^2}{d_3}, \frac{4}{d_1} \right\} P_0(R,\omega) + \frac{2\eta}{Q(t,\omega)^2} \left(\frac{4a^2}{d_1} + \frac{2\beta^2}{d_2} \right) P_0^2(R,\omega) \\
& \quad + Q(t,\omega)^2 \left(\frac{4J^2}{d_1} + \frac{2\alpha^2}{d_2} + \frac{2q^2 c^2}{d_3} \right) |\Omega|.
\end{aligned}
$$

By integration of the inequality (5.28) over $[t, t+1]$ for $t \in [t^*, -1]$, we can deduce that

$$\int_t^{t+1} 2d(c_1\|\nabla U(s)\|^2 + \|\nabla V(s)\|^2 + \|\nabla Z(s)\|^2)\, ds \le c_1\|U(t)\|^2 + \|V(t)\|^2 + \|Z(t)\|^2$$
$$+ \int_t^{t+1} \left[\left(2c_2 + \frac{1}{32}c_1^2\right) Q(s,\omega)^2|\Omega| + 2(c_1 a)^4 Q(s,\omega)^4|\Omega|\right] ds.$$

Since $Q(t,\omega) = e^{-\varepsilon\omega(t)}$, the above inequality implies that, for $t \in [t^*, -1]$,

$$\begin{aligned}\int_t^{t+1} \xi(s)\, ds &\le \frac{\max\{c_1, 1\}}{2d\ \min\{c_1, 1\}} P_0(R,\omega) \\ &+ \frac{1}{2d\ \min\{c_1, 1\}} \int_t^{t+1} \left[\left(2c_2 + \frac{1}{32}c_1^2\right) e^{-2\varepsilon\omega(s)}|\Omega| + (c_1 a)^4 e^{-4\varepsilon\omega(s)}|\Omega|\right] ds.\end{aligned} \tag{5.61}$$

Here $e^{-\varepsilon\omega(t)}$ is continuous function on $[-2, 0]$, so that there is a bound

$$|Q(t,\omega)| = e^{-\varepsilon\omega(t)} \le e^{\varepsilon|\omega(t)|} \le C(\omega) = \exp\left(\varepsilon \sup_{t\in[-2,0]} |\omega(t)|\right), \quad t \in [-2, 0].$$

Then (5.61) implies that for any $\tau < -2$ and $t \in [t^*, -1]$,

$$\int_t^{t+1} \xi(s)\, ds \ \le N_1(R,\omega), \tag{5.62}$$

where

$$\begin{aligned}N_1(R,\omega) = &\frac{1}{2d\ \min\{c_1, 1\}} \\ &\times \left\{\max\{c_1, 1\} P_0(R,\omega) + \left(2c_2 + \frac{1}{32}c_1^2\right) C^2(\omega)|\Omega| + (c_1 a)^4 C^4(\omega)|\Omega|\right\}.\end{aligned}$$

Next we have

$$\begin{aligned}\int_t^{t+1} p(s)\, ds\ &\le \int_t^{t+1} 2\eta \left(\frac{4a^2}{d_1} + \frac{2\beta^2}{d_2}\right) \frac{1}{Q(s,\omega)^2} \|\nabla G(s)\|^2\, ds \\ &\le 2\eta\, C^2(\omega) \left(\frac{4a^2}{d_1} + \frac{2\beta^2}{d_2}\right) \int_t^{t+1} \|\nabla G(s)\|^2\, ds \le N_2(R,\omega),\end{aligned} \tag{5.63}$$

where

$$N_2(R,\omega) = 2\eta\, C^2(\omega) \left(\frac{4a^2}{d_1} + \frac{2\beta^2}{d_2} \right) N_1(R,\omega).$$

Moreover, for any $\tau < -2$ and $t \in [t^*, -1]$, we obtain

$$\begin{aligned}
\int_t^{t+1} h(s)\, ds \;\le& \int_t^{t+1} \Bigg[\max\left\{ \frac{2q^2}{d_3}, \frac{4}{d_1} \right\} P_0(R,\omega) + \frac{2\eta}{Q(s,\omega)^2} \left(\frac{4a^2}{d_1} + \frac{2\beta^2}{d_2} \right) P_0^2(R,\omega) \\
&+ Q(s,\omega)^2 \left(\frac{4J^2}{d_1} + \frac{2\alpha^2}{d_2} + \frac{2q^2c^2}{d_3} \right) |\Omega| \Bigg]\, ds \\
&\le \max\left\{ \frac{2q^2}{d_3}, \frac{4}{d_1} \right\} P_0(R,\omega) + 2\eta C^2(\omega) \left(\frac{4a^2}{d_1} + \frac{2\beta^2}{d_2} \right) P_0^2(R,\omega) \\
&+ C^2(\omega) \left(\frac{4J^2}{d_1} + \frac{2\alpha^2}{d_2} + \frac{2q^2c^2}{d_3} \right) |\Omega| = N_3(R,\omega).
\end{aligned} \tag{5.64}$$

Now we have shown that, for any $\tau < -2$ and $t \in [t^*, -1]$,

$$\int_t^{t+1} \xi(s)\, ds \le N_1, \quad \int_t^{t+1} p(s)\, ds \le N_2, \quad \int_t^{t+1} h(s)\, ds \le N_3. \tag{5.65}$$

Thus the uniform Gronwall inequality [62, Lemma D.3] applied to (5.60) shows that

$$\xi(t) = \|\nabla G(t)\|^2 \le (N_1 + N_3) e^{N_2}, \quad \text{for all } \; t \in [t^* + 1, 0]. \tag{5.66}$$

Finally, the claim (5.52) is proved:

$$\begin{aligned}
&\|G(0,\omega;\tau,Q(\tau,\omega)g_0)\|_E^2 = \|G(0,\omega;\tau,Q(\tau,\omega)g_0)\|^2 + \|\nabla G(0,\omega;\tau,Q(\tau,\omega)g_0)\|^2 \\
&\le M(R,\omega) = P_0(R,\omega) + (N_1(R,\omega) + N_3(R,\omega)) e^{N_2(R,\omega)}.
\end{aligned}$$

The proof is completed. □

Lemma 5.3.2. *For the Hindmarsh-Rose cocycle Φ, there exists a random variable $M^*(\omega) > 0$ with the property that for any given random variable $\rho(\omega) > 0$ there is a finite time $T(\rho,\omega) > 0$ such that if $g_0 = (u_0, v_0, z_0) \in H$ with $\|g_0\| \le \rho(\omega)$, then $\Phi(t, \theta_{-t}\omega, g_0) \in E$ and*

$$\|\Phi(t, \theta_{-t}\omega, g_0)\|_E \le M^*(\omega), \quad \textit{for } \; t > T(\rho,\omega). \tag{5.67}$$

Proof. We have proved in Theorem 5.2.9 the existence of a pullback absorbing set $B_0(\omega) = B_H(0, R_0(\omega))$ for the Hindmarsh-Rose cocycle Φ in H. Thus it suffices to show that the above statement (5.67) holds for $\rho(\omega) = R_0(\omega)$ given in (5.40), namely, for $g_0 \in B_0(\omega)$.

From (5.31), for any $g_0 \in B_0(\omega)$, we obtain

$$\begin{aligned} \|G(t,\omega;\tau,Q(\tau,\omega)g_0)\|^2 &\leq \frac{\max\{c_1,1\}}{\min\{c_1,1\}} e^{-\sigma(t-\tau)} |Q(\tau,\omega)|^2 R_0^2(\omega) \\ &+ \frac{|\Omega|}{\min\{c_1,1\}} \int_{-\infty}^t e^{-\sigma(t-s)} \left[\left(2c_2 + \frac{1}{32}c_1^2 \right) Q(s,\omega)^2 + 2(c_1 a)^4 Q(s,\omega)^4 \right] ds. \end{aligned} \tag{5.68}$$

Now we prove that there exists a time $T^*(R_0(\omega)) < -2$ such that for any $\tau \leq T^*(R_0)$ one has

$$\sup_{t\in[-2,0]} \sup_{g_0 \in B_0(\omega)} \|G(t,\omega;\tau,Q(\tau,\omega)g_0)\| \leq R_1(\omega), \tag{5.69}$$

where $R_1(\omega) > 0$ is a positive random variable given in (5.75) later in this proof.

Take $t = -2$ and recall that $Q(\tau,\omega) = e^{-\varepsilon\omega(\tau)}$. The inequality (5.68) implies

$$\begin{aligned} \|G(-2,\omega;\tau,Q(\tau,\omega)g_0)\|^2 &\leq \frac{\max\{c_1,1\}}{\min\{c_1,1\}} e^{2\sigma - \sigma|\tau| - 2\varepsilon\omega(\tau)} R_0^2(\omega) \\ &+ \frac{|\Omega|}{\min\{c_1,1\}} \int_{-\infty}^{-2} e^{2\sigma+\sigma s} \left[\left(2c_2 + \frac{1}{32}c_1^2 \right) e^{-2\varepsilon\omega(s)} + 2(c_1 a)^4 e^{-4\varepsilon\omega(s)} \right] ds. \end{aligned} \tag{5.70}$$

Note that $\tau \leq T^*(R_0) < -2$ implies

$$e^{-\sigma|\tau| - 2\varepsilon\omega(\tau)} = \exp\left(-\sigma|\tau| \left[1 + \frac{2\varepsilon\omega(\tau)}{\sigma|\tau|} \right] \right) = \exp\left(-\sigma|\tau| \left[1 - \frac{2\varepsilon\omega(\tau)}{\sigma\tau} \right] \right).$$

By the asymptotically sublinear growth property (5.13), for $\omega \in \mathfrak{Q}$, there exist a time $T^*(R_0) \leq -2$ such that for any $\tau \leq T^*(R_0)$, which means τ is very negative, we have

$$1 - \frac{2\varepsilon\omega(\tau)}{\sigma\tau} \geq \frac{1}{2} \quad \text{and} \quad e^{\sigma(2 - \frac{1}{2}|\tau|)} \frac{\max\{c_1,1\}}{\min\{c_1,1\}} R_0^2(\omega) \leq 1. \tag{5.71}$$

Then we get

$$
\begin{aligned}
&\|G(-2,\omega;\tau,Q(\tau,\omega)g_0)\|^2\\
&\le 1+\frac{|\Omega|}{\min\{c_1,1\}}\int_{-\infty}^{-2}e^{2\sigma+\sigma s}\left[\left(2c_2+\frac{1}{32}c_1^2\right)e^{-2\varepsilon\omega(s)}+2(c_1a)^4e^{-4\varepsilon\omega(s)}\right]ds\\
&\le 1+\frac{|\Omega|}{\min\{c_1,1\}}\int_{-\infty}^{-1}e^{2\sigma+\sigma s}\left[\left(2c_2+\frac{1}{32}c_1^2\right)e^{-2\varepsilon\omega(s)}+2(c_1a)^4e^{-4\varepsilon\omega(s)}\right]ds\\
&=r_0^2(\omega),
\end{aligned}
\tag{5.72}
$$

where $r_0(\omega)$ is given in (5.35).

For $t\in[-2,0]$, integrate the inquality (5.28) over $[-2,t]$ to obtain

$$
\begin{aligned}
&\|G(t,\omega;\tau,Q(\tau,\omega)g_0)\|^2+2d\int_{-2}^{t}\|\nabla G(s,\omega;\tau,Q(\tau,\omega)g_0)\|^2\,ds\\
&\le\frac{\max\{c_1,1\}}{\min\{c_1,1\}}\|G(-2,\omega;\tau,Q(\tau,\omega)g_0)\|^2\\
&\quad+\frac{|\Omega|}{\min\{c_1,1\}}\int_{-2}^{t}\left[\left(2c_2+\frac{1}{32}c_1^2\right)Q(s,\omega)^2+2(c_1a)^4Q(s,\omega)^4\right]ds.
\end{aligned}
\tag{5.73}
$$

The inequalities (5.72) and (5.73) imply that (5.69) is valid:

$$
\|G(t,\omega;\tau,Q(\tau,\omega)g_0)\|^2\le R_1(\omega),
\tag{5.74}
$$

for all $t\in[-2,0],\ g_0\in B_0(\omega)$ and where

$$
\begin{aligned}
R_1(\omega)\ &=\frac{\max\{c_1,1\}}{\min\{c_1,1\}}r_0^2(\omega)\\
&\quad+\frac{|\Omega|}{\min\{c_1,1\}}\int_{-2}^{0}\left[\left(2c_2+\frac{1}{32}c_1^2\right)Q(s,\omega)^2+2(c_1a)^4Q(s,\omega)^4\right]ds.
\end{aligned}
\tag{5.75}
$$

Next for $t\ge -2$ and $\tau<T^*(R_0)$, we integrate (5.28) and by (5.69) to get

$$
\begin{aligned}
&\int_t^{t+1}\|\nabla G(s,\omega;\tau,Q(\tau,\omega)g_0)\|^2\,ds\le\frac{\max\{c_1,1\}}{2d\,\min\{c_1,1\}}\|G(t,\omega;\tau,Q(\tau,\omega)g_0)\|^2\\
&+\frac{|\Omega|}{2d\,\min\{c_1,1\}}\int_t^{t+1}\left[\left(2c_2+\frac{1}{32}c_1^2\right)Q(s,\omega)^2+2(c_1a)^4Q(s,\omega)^4\right]ds\le K(\omega),
\end{aligned}
\tag{5.76}
$$

where

$$
\begin{aligned}
K(\omega) = &\frac{1}{2d \min\{c_1, 1\}} \max\left\{c_1,\, 1,\, \left(2c_2 + c_1^2\right)|\Omega|,\, 2(c_1 a)^4 |\Omega|\right\} \\
&\times \left\{ R_1(\omega) + \int_{-2}^{0} \left[Q(s,\omega)^2 + Q(s,\omega)^4\right] ds \right\}.
\end{aligned}
$$

Take $t = -2$ and $\tau < T^*(R_0)$ in (5.76). It implies that there is a time $t^* \in [-2, -1]$ such that

$$
\|\nabla G(t^*, \omega; \tau, Q(\tau,\omega) g_0)\|^2 \leq K(\omega),
$$

so that

$$
\|G(t^*, \omega;\, \tau,\, Q(\tau,\omega) g_0)\|_E^2 \leq R_1(\omega) + K(\omega). \tag{5.77}
$$

Finally, we combine Lemma 5.3.1 and the bound estimate (5.77) to conclude that for all $t > |T^*(R_0(\omega))|$ it holds that

$$
\|\Phi(t, \theta_{-t}\omega, g_0)\|_E = \|G(0, \omega; -t,\, Q(-t,\omega) g_0\|_E \leq M((R_1 + K)^{1/2}, \omega) \tag{5.78}
$$

where $M(R, \omega)$ is specified in (5.52). Thus the claim (5.67) of this lemma is proved for $\rho(\omega) = R_0(\omega)$ with

$$
M^*(\omega) = M((R_1 + K)^{1/2}, \omega) \quad \text{and} \quad T(\rho, \omega) = |T^*(R_0(\omega))|.
$$

Consequently, (5.67) is also proved for any random variable $\rho(\omega)$ as well, by the remark at the beginning of this proof. It completes the proof. □

We complete this chapter to present the main result on the existence of a random attractor for the Hindmarsh-Rose random dynamical system Φ in the space H.

Theorem 5.3.3. *For any positive parameters $d_1, d_2, d_3, a, b, \alpha, \beta, q, r, J, \varepsilon$ and $c \in \mathbb{R}$, there exists a random attractor $\mathcal{A}(\omega)$ in the space $H = L^2(\Omega, \mathbb{R}^3)$ with respect to the universe $\mathscr{D}_H$ for the Hindmarsh-Rose cocycle Φ over the metric dynamical system $(\mathfrak{Q}, \mathcal{F}, P, \{\theta_t\}_{t\in\mathbb{R}})$. Moreover, the random attractor $\mathcal{A}(\omega)$ is a bounded random set in the space E.*

Proof. In Lemma 5.2.9, we proved that there exists a pullback absorbing set $B_0(\omega)$ in H for the Hindmarsh-Rose cocycle Φ. According to Definition 1.3.17, Lemma 5.3.2 and the compact imbedding $E \hookrightarrow H$ show that the Hindmarsh-Rose cocycle Φ is pullback asymptotically compact on H with respect to $\mathscr{D}_H$. Hence, by Theorem 5.1.1, there exists a random attractor in H for this random dynamical system Φ, which is given by

$$\mathcal{A}(\omega) = \bigcap_{\tau \geq 0} \overline{\bigcup_{t \geq \tau} \Phi(t, \theta_{-t}\omega, B_0(\theta_{-t}\omega))}. \tag{5.79}$$

Since $\mathcal{A}(\omega)$ is an invariant set, Lemma 5.3.2 implies that the random attractor $\mathcal{A}(\omega)$ is also a bounded random set in E. $\square$

Chapter 6

Random Attractor for Stochastic Hindmarsh-Rose Equations with Additive Noise

Note to Reader

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In this chapter, we shall study the longtime random dynamics in terms of the existence of a random attractor for the diffusive Hindmarsh-Rose equations driven by the additive noise,

$$du = d_1 \Delta u \, dt + (\varphi(u) + v - z + J) \, dt + h_1(x) \, dW_1, \tag{6.1}$$

$$dv = d_2 \Delta v \, dt + (\psi(u) - v) \, dt + h_2(x) \, dW_2, \tag{6.2}$$

$$dz = d_3 \Delta z \, dt + (q(u - c) - rz) \, dt + h_3(x) \, dW_3, \tag{6.3}$$

for $t > \tau,\ x \in \Omega \subset \mathbb{R}^n$ $(n \leq 2)$.
We impose the homogeneous Neumann boundary condition

$$\frac{\partial u}{\partial \nu}(t, x) = 0, \ \ \frac{\partial v}{\partial \nu}(t, x) = 0, \ \ \frac{\partial z}{\partial \nu}(t, x) = 0, \quad t > \tau \in \mathbb{R},\ x \in \partial\Omega, \tag{6.4}$$

and an initial condition

$$u(\tau, x) = u_0(x),\ v(\tau, x) = v_0(x),\ z(\tau, x) = z_0(x), \quad \tau \in \mathbb{R},\ x \in \Omega. \tag{6.5}$$

The parameters $d_1, d_2, d_3, a, b, \alpha, \beta, q, r$ and J are arbitrary positive constants, and $c \in \mathbb{R}$ is the reference value for the membrane potential of a neuron cell. Moreover, $\{h_i(x) : i = 1, 2, 3\} \subset W^{2,4}(\Omega)$ are given functions and $W(t) = \{W_1(t), W_2(t), W_3(t)\}$, where $W_i(t), i = 1, 2, 3,$ are independent, two-sided, real-valued standard Wiener processes on an underlying probability space $(\mathfrak{Q}, \mathcal{F}, P)$ to be specified later.

6.1 Formulation and Random Environment

The nonpositive self-adjoint linear differential operator

$$A = \begin{pmatrix} d_1\Delta & 0 & 0 \\ 0 & d_2\Delta & 0 \\ 0 & 0 & d_3\Delta \end{pmatrix} : D(A) \to H, \tag{6.6}$$

where

$$D(A) = \left\{ (u, v, z) \in H^2(\Omega, \mathbb{R}^3) : \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial z}{\partial \nu} = 0 \text{ on } \partial\Omega \right\}$$

is the generator of an analytic contraction C_0-semigroup $\{e^{At}\}_{t \geq 0}$ on the Hilbert space H. By the fact that $H^1(\Omega) \hookrightarrow L^6(\Omega)$ is a continuous Sobolev imbedding for space dimension $n \leq 3$, the nonlinear mapping

$$f(u, v, z) = \begin{pmatrix} \varphi(u) + v - z + J \\ \psi(u) - v, \\ q(u - c) - rz \end{pmatrix} : E \longrightarrow H \tag{6.7}$$

is locally Lipschitz continuous. Let $W(t) = \text{col}\,(W_1(t), W_2(t), W_3(t))$ and

$$\Lambda(h) \;=\; \begin{pmatrix} h_1(x) & 0 & 0 \\ 0 & h_2(x) & 0 \\ 0 & 0 & h_3(x) \end{pmatrix}.$$

Then the initial-boundary value problem (6.1)–(6.5) is formulated into an initial value problem of the following stochastic Hindmarsh-Rose evolutionary equation driven by the additive noise:

$$\begin{aligned} &dg = A\, g\, dt + f(g)\, dt + \Lambda(h)\, dW, \quad t > \tau \in \mathbb{R}, \\ &g(\tau, \omega, g_0) = g_0 = (u_0, v_0, z_0) \in H. \end{aligned} \tag{6.8}$$

The solutions of (6.8) is denoted by

$$g(t, \omega, g_0) = \text{col}\,(u(t, \cdot, \omega, g_0), v(t, \cdot, \omega, g_0), z(t, \cdot, \omega, g_0))$$

where (u, v, z) is the vector of solutions to the problem (6.1)–(6.5), and dot stands for the hidden spatial variable x, and $\omega \in \mathfrak{Q}$.

Specifically assume that $\{W_i(t) : i = 1, 2, 3\}_{t \in \mathbb{R}}$ are independent two-sided standard Wiener processes (Brownian motion) in the canonical probability space $(\mathfrak{Q}, \mathcal{F}, P)$, where the sample space

$$\mathfrak{Q} = \{\omega(t) = (\omega_1(t), \omega_2(t), \omega_3(t)) \in C(\mathbb{R}, \mathbb{R}^3) : \omega(0) = 0\}, \tag{6.9}$$

the σ-algebra $\mathcal{F}$ is generated by the compact-open topology endowed in $\mathfrak{Q}$, and P is the corresponding Wiener measure [2,14,18,54] on $\mathcal{F}$. Define a family of P-preserving time-shift transformations $\{\theta_t\}_{t \in \mathbb{R}}$ by

$$(\theta_t \, \omega)(\cdot) = \omega(\cdot + t) - \omega(t), \quad \text{for } \ t \in \mathbb{R}, \ \omega \in \mathfrak{Q}. \tag{6.10}$$

Then $(\mathfrak{Q}, \mathcal{F}, P, \{\theta_t\}_{t \in \mathbb{R}})$ is a metric dynamical system and the stochastic process $\{W(t, \omega) = \omega(t) : t \in \mathbb{R}, \omega \in \mathfrak{Q}\}$ is a three-dimensional canonical Wiener process.

For a given $\kappa > 0$ to be specified, introduce the Ornstein-Uhlenbeck process $\Gamma(\theta_t \omega) = \operatorname{col}(\Gamma_1(\theta_t \omega_1), \Gamma_2(\theta_t \omega_2), \Gamma_3(\theta_t \omega_3))$, which is defined by

$$\begin{aligned}\Gamma_i(t, \omega_i) &= -\kappa \int_{-\infty}^{t} e^{-\kappa(t-s)} dW_i(s, \omega_i) = -\kappa \int_{-\infty}^{0} e^{\kappa \xi} dW_i(t + \xi, \omega_i) \\ &= -\kappa \int_{-\infty}^{0} e^{\kappa s} dW_i(s, \theta_t \omega_i) = -\kappa \int_{-\infty}^{0} e^{\kappa s} (\theta_t \omega_i)(s) ds = \Gamma_i(0, \theta_t \omega_i) := \Gamma_i(\theta_t \omega_i).\end{aligned} \tag{6.11}$$

The Ornstein-Uhlenbeck processes $\Gamma_i(t, \omega_i) = \Gamma_i(\theta_t \omega_i), i = 1, 2, 3,$ satisfy the scalar stochastic differential equation

$$d\Gamma_i = -\kappa \Gamma_i \, dt + dW_i, \quad \Gamma(-\infty) = 0. \tag{6.12}$$

Define $\Gamma^h(\theta_t \omega) = \operatorname{col}(\Gamma_1^h(\theta_t \omega_1), \Gamma_2^h(\theta_t \omega_2), \Gamma_3^h(\theta_t \omega_3))$ to be the corresponding abstract Ornstein-Uhlenbeck process

$$\Gamma_i^h(\theta_t \omega_i) = h_i(x) \Gamma_i(\theta_t \omega_i), \quad 1 \le i \le 3. \tag{6.13}$$

For any $p \ge 2$ and any $\kappa > 0$, the Ornstein-Uhlenbeck process $\Gamma(\theta_t \omega)$ is tempered in $L^p(\mathbb{R}, \mathbb{R}^3)$. It means that for any $\varepsilon > 0$,

$$\lim_{|t| \to \infty} e^{-\varepsilon |t|} \, |\Gamma(\theta_t \omega)|^p = 0. \tag{6.14}$$

Thus the abstract Ornstein-Uhlenbeck process $\Gamma^h(\theta_t\omega)$ satisfies the similar property: if $h_i \in L^p(\Omega), 1 \leq i \leq 3$, then for any $\varepsilon > 0$,

$$\lim_{|t|\to\infty} e^{-\varepsilon|t|} \|\Gamma^h(\theta_t\omega)\|^p_{L^p(\Omega,\mathbb{R}^3)} = 0. \tag{6.15}$$

6.2 Hindmarsh-Rose Cocycle and Pullback Absorbing Property

The first step to treat the stochastic PDE problem (6.1)–(6.5) is to convert the system to random PDE, which has random coefficients and random initial data, by the additive transformation:

$$\begin{aligned} U(t,\omega;\tau,g_0) &= u(t,\cdot,\omega,\tau,g_0) - \Gamma_1^h(\theta_t\omega_1), \\ V(t,\omega;\tau,g_0) &= v(t,\cdot,\omega,\tau,g_0) - \Gamma_2^h(\theta_t\omega_2), \\ Z(t,\omega;\tau,g_0) &= z(t,\cdot,\omega,\tau,g_0) - \Gamma_3^h(\theta_t\omega_3), \end{aligned} \tag{6.16}$$

where $\omega = (\omega_1, \omega_2, \omega_3)$, and dot stands for the hidden spatial variable x.
Then the initial-boundary value problem (6.1)–(6.5) is converted to the following system of random partial differential equations:

$$\begin{aligned} \frac{\partial U}{\partial t} &= d_1\Delta U + d_1\Delta h_1\Gamma_1(\theta_t\omega_1) + a(U + \Gamma_1^h(\theta_t\omega_1))^2 - b(U + \Gamma_1^h(\theta_t\omega_1))^3 \\ &\quad + (V + \Gamma_2^h(\theta_t\omega_2)) - (Z + \Gamma_3^h(\theta_t\omega_3)) + J + \kappa\Gamma_1^h(\theta_t\omega_1), && (6.17) \\ \frac{\partial V}{\partial t} &= d_2\Delta V + d_2\Delta h_2\Gamma_2(\theta_t\omega_2) + \alpha - \beta(U + \Gamma_1^h(\theta_t\omega_1))^2 \\ &\quad - (V + \Gamma_2^h(\theta_t\omega_2)) + \kappa\Gamma_2^h(\theta_t\omega_2), && (6.18) \\ \frac{\partial Z}{\partial t} &= d_3\Delta Z + d_3\Delta h_3\Gamma_3(\theta_t\omega_3) + q(U + \Gamma_1^h(\theta_t\omega_1) - c) \\ &\quad - r(Z + \Gamma_3^h(\theta_t\omega_3)) + \kappa\Gamma_3^h(\theta_t\omega_3), && (6.19) \end{aligned}$$

for $\omega \in \mathfrak{Q}$, $t > \tau$, $x \in \Omega \subset \mathbb{R}^n$ ($n \leq 2$), with the Neumann boundary condition

$$\frac{\partial U}{\partial \nu}(t,x,\omega) = 0,\ \frac{\partial V}{\partial \nu}(t,x,\omega) = 0,\ \frac{\partial Z}{\partial \nu}(t,x,\omega) = 0, \quad t \geq \tau \in \mathbb{R},\ x \in \partial\Omega, \tag{6.20}$$

and an initial condition

$$(U, V, Z)(\tau, \omega) = g_0 - \Gamma^h(\theta_\tau \omega) = (u_0 - \Gamma_1^h(\theta_\tau \omega_1),\ v_0 - \Gamma_2^h(\theta_\tau \omega_2),\ z_0 - \Gamma_3^h(\theta_\tau \omega_3)). \tag{6.21}$$

The initial-boundary value problem (6.17)–(6.21) can be written as an initial value problem of the pathwise non-autonomous random evolutionary equation

$$\begin{gathered} \frac{\partial G}{\partial t} = AG + \mathscr{F}(G, \theta_t\, \omega), \quad t \geq \tau \in \mathbb{R},\ \omega \in \mathfrak{Q}, \\ G(0, \omega; \tau, g_0) = g_0 = (u_0, v_0, z_0) \in H. \end{gathered} \tag{6.22}$$

We define the weak solution of the initial value problem (6.22),

$$G(t, \omega; \tau, g_0) = (U(t, \omega; \tau, g_0), V(t, \omega; \tau, g_0), Z(t, \omega; \tau, g_0)), \tag{6.23}$$

to be the weak solution of the nonautonomous initial-boundary problem (6.17)–(6.21), specified in [78, Definition 2.1].

By conducting estimates on the Galerkin approximate solutions and through the compactness argument outlined in [13, Chapter II and XV] with some adaptations, we can prove the local existence and uniqueness of the weak solution $G(t, \omega) = G(t, \omega; \tau, g_0)$ in the space H on a time interval $[\tau, T_{\max}(\tau, \omega, g_0))$ for some $\tau < T_{\max}(\tau, \omega, g_0) \leq \infty$, and the solution continuously depends on the initial data. Further by the parabolic regularity [62, Theorem 48.5], every weak solution becomes a strong solution in the space E for $t > \tau$ in the existence interval and has the regularity property

$$G \in C([\tau, T_{max}), H) \cap C^1((\tau, T_{max}), H) \cap L^2_{loc}([\tau, T_{max}), E). \tag{6.24}$$

6.2.1 Global Existence of Pullback Pathwise Solutions

The converted system of random partial differential equations (6.17)–(6.19) is non-autonomous by nature and we shall deal with the pullback weak solutions to investigate the random dynamics.

Lemma 6.2.1. *For any $\tau \in \mathbb{R}$, $\omega \in \mathfrak{Q}$, and any given initial data $g_0 = (u_0, v_0, z_0) \in H$, the weak solution $G(t, \omega; \tau, g_0)$ defined in* (6.23) *of the initial boundary-problem of the random PDE* (6.17)–(6.21) *uniquely exists on $[\tau, \infty)$. Consequently, the weak solution $(u, v, z)(t, \theta_\tau \omega; \tau, g_0) =$*

$G(t, \theta_\tau\omega; \tau, g_0) + \Gamma^h(\theta_t\omega)$ *of the original problem* (6.1)–(6.5) *uniquely exists on* $[\tau, \infty)$ *and continuously depends on the initial data.*

Proof. Take the H inner-products $\langle(6.17), c_1U(t)\rangle$, $\langle(6.18), V(t)\rangle$ and $\langle(6.19), Z(t)\rangle$ with a constant $c_1 > 0$ to be specified later and then sum up the resulting equalities. Recall that $U = u - \Gamma_1^h$, $V = v - \Gamma_2^h$ and $Z = z - \Gamma_3^h$. We obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(c_1\|U\|^2 + \|V\|^2 + \|Z\|^2\right) + \left(c_1 d_1\|\nabla U\|^2 + d_2\|\nabla V\|^2 + d_3\|\nabla Z\|^2\right) \\
&= \int_\Omega c_1 U\left[d_1\Delta h_1\Gamma_1(\theta_t\omega_1) + k\Gamma_1^h(\theta_t\omega_1) + \Gamma_2^h(\theta_t\omega_2) - \Gamma_3^h(\theta_t\omega_3)\right]\, dx \\
&\quad + \int_\Omega V\left[d_2\Delta h_2\Gamma_2(\theta_t\omega_2) - \Gamma_2^h(\theta_t\omega_2) + k\Gamma_2^h(\theta_t\omega_2)\right]\, dx \\
&\quad + \int_\Omega Z\left[d_3\Delta h_3\Gamma_3(\theta_t\omega_3) + q\Gamma_1^h(\theta_t\omega_1) - r\Gamma_3^h(\theta_t\omega_3) + k\Gamma_3^h(\theta_t\omega_3)\right]\, dx \\
&\quad + \int_\Omega \left[c_1UV - c_1ZU - V^2 + q(U-c)Z - rZ^2 + c_1JU + \alpha V\right]\, dx \\
&\quad + \int_\Omega \left\{(c_1au^3 - c_1bu^4 - \beta vu^2) + [c_1\Gamma_1^h(\theta_t\omega_1)(bu^3 - au^2) + \beta\Gamma_2^h(\theta_t\omega_2)u^2]\right\} dx.
\end{aligned} \tag{6.25}$$

For the first three integral terms on the right-hand side of equality (6.25), we have

$$\begin{aligned}
&\int_\Omega c_1 U\left[d_1\Delta h_1\Gamma_1(\theta_t\omega_1) + k\Gamma_1^h(\theta_t\omega_1) + \Gamma_2^h(\theta_t\omega_2) - \Gamma_3^h(\theta_t\omega_3)\right]\, dx \\
&+ \int_\Omega V\left[d_2\Delta h_2\Gamma_2(\theta_t\omega_2) - \Gamma_2^h(\theta_t\omega_2) + k\Gamma_2^h(\theta_t\omega_2)\right]\, dx \\
&+ \int_\Omega Z\left[d_3\Delta h_3\Gamma_3(\theta_t\omega_3) + q\Gamma_1^h(\theta_t\omega_1) - r\Gamma_3^h(\theta_t\omega_3) + k\Gamma_3^h(\theta_t\omega_3)\right]\, dx \\
\le{}& c(h)|\Gamma(\theta_t\omega)|^2 + \frac{c_1^2}{2}\int_\Omega U^2 dx + \frac{1}{12}\int_\Omega V^2 dx + \frac{r}{6}\int_\Omega Z^2 dx,
\end{aligned} \tag{6.26}$$

where $c(h) > 0$ is a constant depending on the functions $h(x) = (h_1(x), h_2(x), h_3(x))$. Note that

$$u^4 = \left[\left(U + \Gamma_1^h(\theta_t\omega_1)\right)^2\right]^2 \le \left[2\left(U^2 + \left(\Gamma_1^h(\theta_t\omega_1)\right)^2\right)\right]^2 \le 8\left[U^4 + \left(\Gamma_1^h(\theta_t\omega_1)\right)^4\right].$$

The 5^{th} integral term on the right-hand side of (6.25) is

$$\int_\Omega (c_1au^3 - c_1bu^4 - \beta vu^2)\, dx + c_1\int_\Omega \Gamma_1^h(\theta_t\omega_1)(bu^3 - au^2) dx + \beta\int_\Omega \Gamma_2^h(\theta_t\omega_2)u^2 dx$$

$$
\begin{aligned}
&= \int_\Omega \left[(c_1 a + c_1 b\Gamma_1^h(\theta_t\omega_1))u^3 - c_1 b u^4 - \beta v u^2 + \left(\beta\Gamma_2^h(\theta_t\omega_2) - c_1 a\Gamma_1^h(\theta_t\omega_1)\right) u^2\right] \, dx \\
&\le \int_\Omega \left[\frac{3}{4}u^4 + \frac{1}{4}\left(c_1 a + c_1 b\Gamma_1^h(\theta_t\omega_1)\right)^4 - c_1 b u^4\right] dx \qquad (6.27) \\
&\quad + \int_\Omega \left[2\beta^2 u^4 + \frac{v^2}{8} + \left(\beta\Gamma_2^h(\theta_t\omega_2) - c_1 a\Gamma_1^h(\theta_t\omega_1)\right)^2 + \frac{u^4}{4}\right] \, dx.
\end{aligned}
$$

Choose the positive constant in (6.25) and (6.27) to be

$$
c_1 = \frac{1}{b}\left(2\beta^2 + \frac{11}{8}\right)
$$

so that

$$
\int_\Omega (-c_1 b u^4 + 2\beta^2 u^4) \, dx \le -\frac{11}{8}\int_\Omega u^4 \, dx.
$$

Then (6.27) becomes

$$
\begin{aligned}
&\int_\Omega (c_1 a u^3 - c_1 b u^4 - \beta v u^2) \, dx + c_1 \int_\Omega \Gamma_1^h(\theta_t\omega_1)(b u^3 - a u^2) \, dx + \beta \int_\Omega \Gamma_2^h(\theta_t\omega_2) u^2 \, dx \\
&\le -\frac{3}{8}\int_\Omega u^4 \, dx + \frac{1}{4}\int_\Omega 8\left[(c_1 a)^4 + (c_1 b)^4(\Gamma_1^h(\theta_t\omega_1))^4\right] \, dx + \int_\Omega \frac{v^2}{8} \, dx \\
&\quad + \int_\Omega 2\left[\beta^2(\Gamma_2^h(\theta_t\omega_2))^2 + (c_1 a_1)^2(\Gamma_1^h(\theta_t\omega_1))^2\right] \, dx \\
&\le -\frac{3}{8}\int_\Omega \left[U + \Gamma_1^h(\theta_t\omega_1)\right]^4 \, dx + \frac{1}{8}\int_\Omega \left[V + \Gamma_2^h(\theta_t\omega_2)\right]^2 \, dx + 2(c_1 a)^4|\Omega| \\
&\quad + 2(c_1 b)^4 \int_\Omega \left(\Gamma_1^h(\theta_t\omega_1)\right)^4 \, dx + [2\beta^2 + (c_1 a)^2] \, \|\Gamma^h(\theta_t\omega)\|^2 \\
&\le -3\int_\Omega \left[U^4 + (\Gamma_1^h(\theta_t\omega_1))^4\right] \, dx + \frac{1}{4}\int_\Omega \left[V^2 + (\Gamma_2^h(\theta_t\omega_2))^2\right] \, dx + 2(c_1 a)^4|\Omega| \qquad (6.28) \\
&\quad + 2(c_1 b)^4\|\Gamma^h(\theta_t\omega)\|_{L^4}^4 + [2\beta^2 + (c_1 a)^2] \, \|\Gamma^h(\theta_t\omega)\|^2 \\
&\le -3\int_\Omega U^4 \, dx - 3\|\Gamma^h(\theta_t\omega)\|_{L^4}^4 + \frac{1}{4}\int_\Omega V^2 \, dx \\
&\quad + \frac{1}{4}\|\Gamma^h(\theta_t\omega)\|^2 + 2(c_1 b)^4\|\Gamma^h(\theta_t\omega)\|_{L^4}^4 + [2\beta^2 + (c_1 a)^2] \, \|\Gamma^h(\theta_t\omega)\|^2 + 2(c_1 a)^4|\Omega| \\
&\le -3\int_\Omega U^4 \, dx + \frac{1}{4}\int_\Omega V^2 \, dx + 2(c_1 b)^4\|\Gamma^h(\theta_t\omega)\|_{L^4}^4 \\
&\quad + \left[2\beta^2 + (c_1 a)^2 + \frac{1}{4}\right] \|\Gamma^h(\theta_t\omega)\|^2 + 2(c_1 a)^4|\Omega|.
\end{aligned}
$$

Next, the 4^{th} integral term in (6.25) is estimated,

$$\begin{aligned}
&\int_\Omega [c_1UV - c_1ZU - V^2 + c_1JU + \alpha V + q(U-c)Z - rZ^2]\,dx \\
&\le \int_\Omega \Bigg[3c_1^2U^2 + \frac{V^2}{12} + \frac{3c_1^2}{2r}U^2 + \frac{r}{6}Z^2 - V^2 + \frac{c_1^2}{2}U^2 + \frac{J^2}{2} + 3\alpha^2 + \frac{V^2}{12} \\
&\qquad + \left(\frac{3q^2}{r}(U^2+c^2) + \frac{r}{6}Z^2\right) - rZ^2\Bigg]\,dx.
\end{aligned}$$

Collect all the integral terms with U^2 involved from the above inequality to obtain

$$\begin{aligned}
&\int_\Omega \left(\frac{c_1^2}{2} + 3c_1^2 + \frac{3c_1^2}{2r} + \frac{c_1^2}{2} + \frac{3q^2}{r}\right)U^2\,dx = \int_\Omega \left(4c_1^2 + \frac{3c_1^2}{2r} + \frac{3q^2}{r}\right)U^2\,dx \\
&\le \int_\Omega U^4\,dx + \left(4c_1^2 + \frac{3c_1^2}{2r} + \frac{3q^2}{r}\right)^2 |\Omega|.
\end{aligned} \tag{6.29}$$

Assemble all the estimates (6.26)–(6.29) into (6.25). Then we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(c_1\|U\|^2 + \|V\|^2 + \|Z\|^2\right) + \left(c_1d_1\|\nabla U\|^2 + d_2\|\nabla V\|^2 + d_3\|\nabla Z\|^2\right) \\
&\le \int_\Omega (1-3)U^4dx + \int_\Omega \left(\frac{1}{12} - 1 + \frac{1}{12} + \frac{1}{12} + \frac{1}{4}\right)V^2dx + \int_\Omega \left(\frac{r}{6} + \frac{r}{6} + \frac{r}{6} - r\right)Z^2dx \\
&\quad + c(h)|\Gamma(\theta_t\omega)|^2 + 2(c_1b)^4\|\Gamma^h(\theta_t\omega)\|_{L^4}^4 + \left[2\beta^2 + (c_1a)^2 + \frac{1}{4}\right]\|\Gamma^h(\theta_t\omega)\|^2 \\
&\quad + \left[2(c_1a)^4 + \frac{J^2}{2} + 3\alpha^2 + \frac{3q^2c^2}{r} + \left(4c_1^2 + \frac{3c_1^2}{2r} + \frac{3q^2}{r}\right)^2\right]|\Omega| \\
&\le \int_\Omega -2U^4\,dx - \int_\Omega \frac{V^2}{2}\,dx - \int_\Omega \frac{r}{2}Z^2\,dx + c(h)|\Gamma(\theta_t\omega)|^2 + 2(c_1b)^4\|\Gamma^h(\theta_t\omega)\|_{L^4}^4 \\
&\quad + \left[2\beta^2 + (c_1a)^2 + \frac{1}{4}\right]\|\Gamma^h(\theta_t\omega)\|^2 + N|\Omega|,
\end{aligned}$$

where

$$N = \left[2(c_1a)^4 + \frac{J^2}{2} + 3\alpha^2 + \frac{3q^2c^2}{r} + \left(4c_1^2 + \frac{3c_1^2}{2r} + \frac{3q^2}{r}\right)^2\right].$$

Let $d = \min\{d_1, d_2, d_3\}$. It follows that

$$
\begin{aligned}
&\frac{d}{dt}\left(c_1\|U\|^2 + \|V\|^2 + \|Z\|^2\right) \\
&+ 2d(c_1\|\nabla U\|^2 + \|\nabla V\|^2 + \|\nabla Z\|^2) + \int_\Omega (4U^4 + V^2 + rZ^2)\,dx \\
\leq\ & 2c(h)|\Gamma(\theta_t\omega)|^2 + 4(c_1 b)^4\|\Gamma^h(\theta_t\omega)\|_{L^4}^4 + \left[4\beta^2 + (c_1 a)^2 + \frac{1}{2}\right]\|\Gamma^h(\theta_t\omega)\|^2 + 2N|\Omega|.
\end{aligned}
\tag{6.30}
$$

Since $4U^4 \geq c_1 U^2 - \frac{c_1^2}{16}$, the inequality (6.30) implies that

$$
\begin{aligned}
&\frac{d}{dt}\left(c_1\|U\|^2 + \|V\|^2 + \|Z\|^2\right) \\
&+ 2d(c_1\|\nabla U\|^2 + \|\nabla V\|^2 + \|\nabla Z\|^2) + c_1\|U\|^2 + \|V\|^2 + r\|Z\|^2 \\
\leq\ & 2c(h)|\Gamma(\theta_t\omega)|^2 + 4(c_1 b)^4\|\Gamma^h(\theta_t\omega)\|_{L^4}^4 \\
&+ \left[4\beta^2 + (c_1 a)^2 + \frac{1}{2}\right]\|\Gamma^h(\theta_t\omega)\|^2 + 2N|\Omega| + \frac{c_1^2}{16}|\Omega| \\
\leq\ & \mathscr{C}(h)\left(|\Gamma(\theta_t\omega)|^2 + |\Gamma(\theta_t\omega)|^4\right) + F|\Omega|,
\end{aligned}
\tag{6.31}
$$

for $t \geq \tau$, $\omega \in \mathfrak{Q}$, where the constant $F = 2N + \frac{c_1^2}{16}$ and $\mathscr{C}(h) > 0$ is a constant depending on h. Let $\sigma = \min\{1, r\}$. Gronwall inequality applied to the inequality from (6.31),

$$
\begin{aligned}
&\frac{d}{dt}\left(c_1\|\|U(t)\|^2 + \|V(t)\|^2 + \|Z(t)\|^2\right) + \sigma(c_1\|U(t)\|^2 + \|V(t)\|^2 + \|Z(t)\|^2) \\
&\leq \mathscr{C}(h)\left(|\Gamma(\theta_t\omega)|^2 + |\Gamma(\theta_t\omega)|^4\right) + F|\Omega|,
\end{aligned}
\tag{6.32}
$$

shows that

$$
\begin{aligned}
&c_1\|U(t)\|^2 + \|V(t)\|^2 + \|Z(t)\|^2 \\
\leq\ & e^{-\sigma(t-\tau)}(c_1\|U_0\|^2 + \|V_0\|^2 + \|Z_0\|^2) \\
&+ \int_\tau^t e^{-\sigma(t-s)}\left(\mathscr{C}(h)\left(|\Gamma(\theta_s\omega)|^2 + |\Gamma(\theta_s\omega)|^4\right) + F|\Omega|\right) ds.
\end{aligned}
\tag{6.33}
$$

It means that the weak solutions of the problem (6.22) satisfy

$$
\begin{aligned}
&\|G(t,\theta_\tau\omega;\tau,g_0)\|^2 = \|U(t)\|^2 + \|V(t)\|^2 + \|Z(t)\|^2 \\
&\le \frac{\max\{1,c_1\}}{\min\{1,c_1\}} e^{-\sigma(t-\tau)} \|g_0 - \Gamma^h(\theta_\tau\omega)\|^2 \\
&\quad + \frac{1}{\min\{1,c_1\}} \int_\tau^t e^{-\sigma(t-s)} \left(\mathscr{C}(h)\left(|\Gamma(\theta_s\omega)|^2 + |\Gamma(\theta_s\omega)|^4\right) + F|\Omega|\right) ds \\
&\le \frac{\max\{1,c_1\}}{\min\{1,c_1\}} e^{-\sigma(t-\tau)} \|g_0 - \Gamma^h(\theta_\tau\omega)\|^2 \\
&\quad + \frac{1}{\min\{1,c_1\}} \int_{-\infty}^t e^{-\sigma(t-s)} \left(\mathscr{C}(h)\left(|\Gamma(\theta_s\omega)|^2 + |\Gamma(\theta_s\omega)|^4\right) + F|\Omega|\right) ds
\end{aligned}
\tag{6.34}
$$

for $t \ge \tau \in \mathbb{R}$, $\omega \in \mathfrak{Q}$ and $g_0 \in H$.

Since the Ornstein-Uhlenbeck process $\Gamma(\theta_t\omega)$ is tempered, the last integral in (6.34) is convergent. Therefore, (6.34) shows that the weak solution of the initial value problem (6.17)–(6.21) will never blow up at any finite time $t \ge \tau$. The time interval of maximal existence of any weak solution is always $[\tau, \infty)$. $\square$

Lemma 6.2.2. *There exists a random variable $R_0(\omega) > 0$ depending only on the parameters such that for any tempered random variable $\rho(\omega) > 0$ there exists a random variable $T(\rho,\omega) > 0$ and the following statement holds*: *For any $\tau \le -T(\rho,\omega)$, $\omega \in \mathfrak{Q}$, and any initial data $g_0 = (u_0, v_0, z_0) \in H$ with $\|g_0\| \le \rho(\theta_\tau\omega)$, the weak solution $G(t,\theta_\tau\omega;\tau,g_0)$ of the problem* (6.17)–(6.21) *uniquely exists on $[\tau,\infty)$ and satisfies*

$$
\|G(0,\theta_\tau\omega;\tau,g_0)\|^2 + \int_{-1}^0 \|\nabla G(s,\theta_\tau\omega;\tau,g_0)\|^2\, ds \le R_0(\omega). \tag{6.35}
$$

Proof. Let $t = -1$. From the already shown inequality (6.34), we get

$$
\begin{aligned}
\|G(-1,\theta_\tau\omega;\tau,g_0)\|^2 &\le \frac{\max\{1,c_1\}}{\min\{1,c_1\}} e^{\sigma(1+\tau)} \|g_0 - \Gamma^h(\theta_\tau\omega)\|^2 \\
+ \frac{1}{\min\{1,c_1\}} \int_{\infty}^{-1} & e^{\sigma(1+s)} \left(\mathscr{C}(h)\left(|\Gamma(\theta_s\omega)|^2 + |\Gamma(\theta_s\omega)|^4\right) + F|\mathfrak{Q}|\right) ds.
\end{aligned}
\tag{6.36}
$$

Thus for any given random variable $\rho(\omega) > 0$ and for all $\omega \in \mathfrak{Q}$, there exists a time $T(\rho,\omega) > 1$

such that for any $\tau \le -T(\rho, \omega)$ we have

$$\frac{\max\{1, c_1\}}{\min\{1, c_1\}} e^{\sigma(1+\tau)} \|g_0 - \Gamma^h(\theta_\tau \omega)\|^2 \le 2 \frac{\max\{1, c_1\}}{\min\{1, c_1\}} e^{\sigma(1+\tau)} \left(\rho^2(\omega) + \|\Gamma^h(\theta_\tau \omega)\|^2\right) \le 1, \tag{6.37}$$

since $\Gamma^h(\theta_t \omega)$ is tempered. Substituting the above inequality into (6.36), we obtain

$$\|G(-1, \theta_\tau \omega; \tau, g_0)\|^2 \le r_0(\omega),$$

where

$$r_0(\omega) = 1 + \frac{1}{\min\{1, c_1\}} \int_{\infty}^{-1} e^{\sigma(1+s)} \left(\mathscr{C}(h)(|\Gamma(\theta_s \omega)|^2 + |\Gamma(\theta_s \omega)|^4) + F|\Omega|\right) ds. \tag{6.38}$$

For $t \in [-1, \infty)$, integrate the inequality (6.31) over $[-1, t]$ to get

$$\begin{aligned} & c_1\|U(t)\|^2 + \|V(t)\|^2 + \|Z(t)\|^2 - (c_1\|U(-1)\|^2 + \|V(-1)\|^2 + \|Z(-1)\|^2) \\ & + 2d \int_{-1}^{t} (c_1\|\nabla U(s)\|^2 + \|\nabla V(s)\|^2 + \|\nabla Z(s)\|^2)\, ds \\ & + \sigma \int_{-1}^{t} (c_1\|U(s)\|^2 + \|V(s)\|^2 + \|Z(s)\|^2)\, ds \\ & \le \int_{-1}^{t} \left[\mathscr{C}(h) \left(|\Gamma(\theta_s \omega)|^2 + |\Gamma(\theta_s \omega)|^4\right) + F|\Omega|\right] ds. \end{aligned} \tag{6.39}$$

Thus for $t \in [-1, 0]$ we have

$$\begin{aligned} & \min\{c_1, 1\}\|G(t, \theta_\tau \omega; \tau, g_0)\|^2 + 2d \int_{-1}^{t} \|\nabla G(s, \theta_\tau \omega; \tau, g_0)\|^2\, ds \\ \le\ & \max\{c_1, 1\}\|G(-1, \theta_\tau \omega; \tau, g_0)\|^2 + \int_{-1}^{t} \left[\mathscr{C}(h)\ \left(|\Gamma(\theta_s \omega)|^2 + |\Gamma(\theta_s \omega)|^4\right) + F|\Omega|\right] ds. \end{aligned} \tag{6.40}$$

Let $t = 0$ in (6.40) and we see that the claim (6.35) is proved:

$$\|G(0, \theta_\tau \omega; \tau, g_0)\|^2 + \int_{-1}^{0} \|\nabla G(s, \theta_\tau \omega; \tau, g_0)\|^2\, ds \le R_0(\omega), \tag{6.41}$$

where

$$R_0(\omega) = \frac{1}{\min\{c_1, 1, 2d\}} \times \left\{\max\{c_1, 1\} r_0(\omega) + \int_{-1}^{0} \left[\mathscr{C}(h)\ \left(|\Gamma(\theta_s\omega)|^2 + |\Gamma(\theta_s\omega)|^4\right) + F|\Omega|\right]\ ds\right\}. \tag{6.42}$$

Note that both $r_0(\omega)$ and $R_0(\omega)$ are random variables independent of any initial data. The proof is compldeted. $\square$

The two lemmas that we have shown expose the longtime dissipativity for pullback solution trajectories of the stochastic Hindmarsh-Rose cocycle to be defined in the next subsection.

6.2.2 Hindmarsh-Rose Stochastic Semiflow and Absorbing Property

Now we can define the concept of stochastic semiflow associated with the random PDE (6.17)–(6.19) and then define the cocycle $\Phi : \mathbb{R}^+ \times \mathfrak{Q} \times H \to H$ over MDS $(\mathfrak{Q}, \mathcal{F}, P, \{\theta_t\}_{t\in\mathbb{R}})$ for the stochastic Hindmarsh-Rose equations.

Here in the context of Chapter 5 and this Chapter, we can define $S(t, \tau, \omega) : H \to H$ for all $t \geq \tau \in \mathbb{R}$ and $\omega \in \mathfrak{Q}$ by

$$S(t, \tau, \omega) g_0 = (u, v, z)(t, \omega; \tau, g_0) = G(t, \omega; \tau, g_0) + \Gamma^h(\theta_t\omega). \tag{6.43}$$

Then define the mapping $\Phi : \mathbb{R}^+ \times \mathfrak{Q} \times H \to H$ to be

$$\Phi(t,\ \theta_\tau\omega,\ g_0) = S(t + \tau,\ \tau,\ \theta_\tau\omega)\, g_0 \tag{6.44}$$

which implies that

$$\Phi(t, \omega, g_0) = S(t, 0, \omega)\, g_0 = G(t, \omega;\ 0, g_0) + \Gamma^h(\theta_t\omega). \tag{6.45}$$

Lemma 6.2.3. *The mapping $\Phi : \mathbb{R}^+ \times \mathfrak{Q} \times H \to H$ defined by* (6.44) *is a cocycle on the Hilbert space H over the canonical metric dynamical system* $(\mathfrak{Q}, \mathcal{F}, P, \{\theta_t\}_{t\in\mathbb{R}})$.

It holds that

$$\Phi(t, \theta_{-t}\omega, g_0) = G(0, \theta_{-t}\omega; -t, g_0) + \Gamma^h(\omega) \tag{6.46}$$

for any $g_0 \in H, t \geq 0$ *and* $\omega \in \mathfrak{Q}$. *This random dynamical system* Φ *is called the stochastic Hindmarsh-Rose cocycle.*

Proof. We need to check the cocyle property of the mapping Φ:

$$\Phi(t+s, \omega, g_0) = \Phi(t, \theta_s\,\omega, \Phi(s, \omega, g_0)), \quad t \geq 0,\; s \geq 0,\; \omega \in \mathfrak{Q}. \tag{6.47}$$

Note that, (6.11) and (6.13) imply that for any $\omega \in \mathfrak{Q}$,

$$\Gamma^h(\theta_s\omega)(t) = \Gamma^h(\omega)(t+s), \quad t, s \in \mathbb{R} \quad \text{and} \quad \Gamma^h(\theta_s\omega) = \Gamma^h(\theta_s\omega)(0) = \Gamma^h(\omega)(s).$$

According to (6.45),

$$\Phi(t+s, \omega, g_0) = G(t+s, \omega; 0, g_0) + \Gamma^h(\theta_{t+s}\omega).$$

On the other hand,

$$\begin{aligned}
&\Phi(t, \theta_s\omega, \Phi(s, \omega, g_0)) = G(t, \theta_s\omega; 0, \Phi(s, \omega, g_0)) + \Gamma^h(\theta_t\,\theta_s\,\omega) \\
&= S(t, 0, \theta_s\omega)\,(G(s, \omega; 0, g_0) + \Gamma^h(\theta_s\omega)) \quad \text{(by (6.46))} \\
&= S(t, 0, \theta_s\omega)\,S(s, 0, \omega)g_0 = S(t+s-s, 0, \theta_s\omega)\,S(s, 0, \omega)g_0 \\
&= S(t+s, s, \omega)\,S(s, 0, \omega)g_0 \qquad \text{(by the second condition of Definition 5.2.5)} \\
&= S(t+s, 0, \omega)\,g_0 = G(t+s, \omega; 0, g_0) + \Gamma^h(\theta_{t+s}\omega).
\end{aligned}$$

Therefore, the cocycle property (6.47) of the mapping Φ is proved by comparison of the above two equalities. Moreover, by definition we have

$$\Phi(t, \theta_{-t}\omega, g_0) = S(0, -t, \theta_{-t}\omega)g_0 = G(0, \theta_{-t}\omega; -t, g_0) + \Gamma^h(\theta_t(\theta_{-t}\omega)).$$

Thus the equality (6.46) is valid. $\square$

Theorem 6.2.4. *There exists a pullback absorbing set in the space H with respect to the universe $\mathscr{D}_H$ for the stochastic Hindmarsh-Rose cocycle Φ, which is the bounded ball*

$$K(\omega) = B_H(0, R_H(\omega)) = \{\xi \in H : \|\xi\| \le R_H(\omega)\} \tag{6.48}$$

where $R_H(\omega) = \sqrt{R_0(\omega) + \|\Gamma^h(\omega)\|^2}$ and $R_0(\omega)$ is given in Lemma 6.2.2 *by* (6.42).

Proof. For any bounded random ball $D(\omega) = B_H(0, \rho(\omega)) \in \mathscr{D}_H$, which is centered at the origin with the radius $\rho(\omega)$ in H, and for any initial state $g_0 \in D(\theta_{-t}\omega)$, by Definition 1.3.14 and the definition of the universe $\mathscr{D}_H$, we have

$$\lim_{t\to\infty} e^{-\varepsilon t}\rho(\theta_{-t}\omega) = 0, \quad \text{for any constant } \varepsilon > 0. \tag{6.49}$$

From (6.36), for any $t > 1$ we have

$$\begin{aligned}
&||| \, G(-1, \theta_{-t}\omega; -t, D(\theta_{-t}\omega))||| = \sup_{g_0 \in D(\theta_{-t}\omega)} \|G(-1, \theta_{-t}\omega; -t, g_0)\| \\
&\le 2\,\frac{\max\{1, c_1\}}{\min\{1, c_1\}} e^{\sigma(1-t)} \left(\rho^2(\theta_{-t}\omega) + \|\Gamma^h(\theta_{-t}\omega)\|^2\right) \\
&\quad + \frac{1}{\min\{1, c_1\}} \int_{\infty}^{-1} e^{\sigma(1+s)} \left(\mathscr{C}(h)\left(|\Gamma(\theta_s\omega)|^2 + |\Gamma(\theta_s\omega)|^4\right) + F|\Omega|\right) ds.
\end{aligned}$$

Since $\Gamma^h(\theta_t\omega)$ and $\Gamma(\theta_t\omega)$ are tempered random variables, there exists a time $T_D(\omega) > 1$ such that for any $t \ge T_D(\omega)$ and $\omega \in \mathbf{\Omega}$ we have

$$2\,\frac{\max\{1, c_1\}}{\min\{1, c_1\}} e^{\sigma(1-t)} \left(\rho^2(\theta_{-t}\omega) + \|\Gamma^h(\theta_{-t}\omega)\|^2\right) \le 1. \tag{6.50}$$

Thus

$$\sup_{g_0 \in D(\theta_{-t}\omega)} \|G(-1, \theta_{-t}\omega; -t, g_0)\| \le r_0(\omega), \quad \text{for all } t \ge T_D(\omega),$$

where $r_0(\omega)$ is given in (6.38). By (6.46) and the inequalities (6.40)–(6.41) in Lemma 6.2.2, the above inequality implies that

$$||| \, \Phi(t, \theta_{-t}\omega, D(\theta_{-t}\omega))||| = |||G(0, \theta_{-t}\omega; -t, D(\theta_{-t}\omega)) + \Gamma^h(\omega)|||$$

$$= \sup_{g_0 \in D(\theta_{-t}\omega)} \|G(0, \theta_{-t}\omega; -t, g_0) + \Gamma^h(\omega)\| \le \sqrt{R_0(\omega) + \|\Gamma^h(\omega)\|^2} =: R_H(\omega),$$

for $t \ge T_D(\omega)$, $\omega \in \mathfrak{Q}$, where $R_0(\omega)$ is given in (6.42). It shows that the bounded ball $K(\omega) = B_H(0, R_H(\omega))$ in (6.48) is a pullback absorbing set for Hindmarsh-Rose random dynamical system Φ. □

6.3 The Existence of Random Attractor with the Additive Noise

In this section, we shall prove that the stochastic Hindmarsh-Rose cocycle Φ is pullback asymptotically compact on H through the following theorem. Then the main result on the existence of a random attractor for this random dynamical system is established.

Theorem 6.3.1. *For the Hindmarsh-Rose random dynamical system Φ with the assumption that space dimension $n = \dim(\Omega) \le 2$, there exists a random variable $R_E(\omega) > 0$ independent of any initial time and initial state with the property that for any bounded random set $D \in \mathscr{D}_H$ there is a finite time $T(D, \omega) > 0$ such that*

$$\|\Phi(t, \theta_{-t}\,\omega, D(\theta_{-t}\omega))\|_E = \sup_{g_0 \in D(\theta_{-t}\omega)} \|\Phi(t, \theta_{-t}\,\omega, g_0\|_E \le R_E(\omega). \tag{6.51}$$

for all $t \ge T(D, \omega)$.

Proof. We can just consider any bounded ball $D = B_H(0, \rho(\omega)) \in \mathscr{D}_H$ in this proof.

Step 1. Respectively take the L^2 inner-products $\langle(6.17), -\Delta U(t)\rangle, \langle(6.18), -\Delta V(t)\rangle$ and $\langle(6.19), -\Delta Z(t)\rangle$. Sum up the resulting equalities. For any $t > \tau \in \mathbb{R}$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\nabla U\|^2 + \|\nabla V\|^2 + \|\nabla Z\|^2) + d_1\|\Delta U\|^2 + d_2\|\Delta V\|^2 + d_3\|\Delta Z\|^2 \\
= &-\int_\Omega J\Delta U dx - \int_\Omega \alpha \Delta V dx + \int_\Omega qc\Delta Z dx \\
&- \int_\Omega \Delta U \left[d_1 \Delta h_1 \Gamma_1(\theta_t\omega_1) + \kappa\Gamma_1^h(\theta_t\omega_1)\right] dx \\
&- \int_\Omega \Delta V \left[d_2 \Delta h_2 \Gamma_2(\theta_t\omega_2) + \kappa\Gamma_2^h(\theta_t\omega_2)\right] dx \\
&- \int_\Omega \Delta Z \left[d_3 \Delta h_3 \Gamma_3(\theta_t\omega_3) + \kappa\Gamma_3^h(\theta_t\omega_3)\right] dx \\
&+ \int_\Omega [\Delta U(bu^3 - au^2 - v + z) + \Delta V(\beta u^2 + v) + \Delta Z(rz - qu)]\, dx.
\end{aligned} \tag{6.52}$$

The key last integral on the right-hand side of (6.52) can be written as

$$
\begin{aligned}
&\int_\Omega \big[\Delta U(bu^3 - au^2 - v + z) + \Delta V(\beta u^2 + v) + \Delta Z(rz - qu)\big]\, dx \\
= &\int_\Omega \big[(bu^3 - au^2 - v + z)\Delta u + (\beta u^2 + v)\Delta v + (rz - qu)\Delta z\big]\, dx \\
+ &\int_\Omega [-(bu^3 - au^2 - v + z)\Delta\Gamma_1^h(\theta_t\omega_1) - (\beta u^2 + v)\Delta\Gamma_2^h(\theta_t\omega_2) \\
&- (rz - qu)\Delta\Gamma_3^h(\theta_t\omega_3)]\, dx.
\end{aligned}
\tag{6.53}
$$

The first integral on the right-hand side of (6.53) is estimated as follows.

$$
\begin{aligned}
&\int_\Omega \big[(bu^3 - au^2 - v + z)\Delta u + (\beta u^2 + v)\Delta v + (rz - qu)\Delta z\big]\, dx \\
= &- 3b\int_\Omega u^2|\nabla u|^2\, dx + 2a\int_\Omega u|\nabla u|^2\, dx + \int_\Omega \nabla u\cdot\nabla v\, dx - \int_\Omega \nabla z\cdot\nabla u\, dx \\
&- 2\beta\int_\Omega u\nabla u\cdot\nabla v\, dx - \int_\Omega |\nabla v|^2\, dx - r\int_\Omega |\nabla z|^2\, dx + q\int_\Omega \nabla u\cdot\nabla z\, dx \\
\le &\frac{-3b}{4}\|\nabla(u^2)\|^2 + 2a\|u\|_{L^\infty}\|\nabla u\|^2 + \|\nabla u\|^2 + \|\nabla v\|^2 + \|\nabla z\|^2 + \|\nabla u\|^2 \\
&+ \beta\|u\|_{L^\infty}(\|\nabla u\|^2 + \|\nabla v\|^2) - \|\nabla v\|^2 - r\|\nabla z\|^2 + q(\|\nabla u\|^2 + \|\nabla z\|^2) \\
\le &2C\max\{2a,\beta\}\|u\|_{H^1}(\|\nabla u\|^2 + \|\nabla v\|^2) + \max\{2,q\}(\|\nabla u\|^2 + \|\nabla v\|^2 + \|\nabla z\|^2) \\
\le &\tilde{C}\max\{2a,\beta\}(\|\nabla u\|^3 + \|\nabla u\|\|\nabla v\|^2) + \max\{2,q\}(\|\nabla u\|^2 + \|\nabla v\|^2 + \|\nabla z\|^2) \\
\le &\frac{C_1}{4}\left(\|\nabla G\|^3 + \|\nabla\Gamma^h(\theta_t\omega)\|^3 + \|\nabla G\|^2 + \|\nabla\Gamma^h(\theta_t\omega)\|^2\right) \\
\le &\frac{C_1}{4}\left(\frac{3}{4}\|\nabla G\|^4 + \frac{1}{4} + \frac{3}{4}\|\nabla\Gamma^h(\theta_t\omega)\|^4 + \frac{1}{4} + \frac{1}{2}\|\nabla G\|^4 + \frac{1}{2}\|\nabla\Gamma^h(\theta_t\omega)\|^4\right) \\
= &\frac{C_1}{4}\left(\frac{5}{4}\|\nabla G\|^4 + \frac{5}{4}\|\nabla\Gamma^h(\theta_t\omega)\|^4 + \frac{1}{2}\right) \le \frac{C_1}{2}\left(\|\nabla G\|^4 + \|\nabla\Gamma^h(\theta_t\omega)\|^4 + 1\right),
\end{aligned}
\tag{6.54}
$$

where $C_1 > 0$ is constant and we have used the Young's inequality and (6.43). For the second step of the chain inequalities in (6.54), the Sobolev embedding $H^1(\Omega) \hookrightarrow L^\infty(\Omega)$ under the assumption

$\dim(\Omega) \leq 2$ so that $\|u\|_{L^\infty} \leq C\|u\|_{H^1}$ is used to deal with the integral term

$$-2\beta \int_\Omega u \nabla u \cdot \nabla v \, dx.$$

Next we estimate the second integral in (6.53):

$$\begin{aligned}
&\int_\Omega [(au^2 - bu^3 - v + z)\Delta\Gamma_1^h(\theta_t\omega_1) - (\beta u^2 + v)\Delta\Gamma_2^h(\theta_t\omega_2) - (rz - qu)\Delta\Gamma_3^h(\theta_t\omega_3)] \, dx \\
&= \int_\Omega [u^2 \left(a\Delta\Gamma_1^h(\theta_t\omega_1) - \beta\Delta\Gamma_2^h(\theta_t\omega_2)\right) + bu^3\Delta\Gamma_1^h(\theta_t\omega_1) \\
&\quad - v(\Delta\Gamma_1^h(\theta_t\omega_1) + \Delta\Gamma_2^h(\theta_t\omega_2)) \\
&\quad + z(\Delta\Gamma_1^h(\theta_t\omega_1) - r\Delta\Gamma_3^h(\theta_t\omega_3)) + qu\Delta\Gamma_3^h(\theta_t\omega_3)] \, dx \qquad (6.55) \\
&\leq \int_\Omega \left[\frac{u^4}{4} + \left(a\Delta\Gamma_1^h(\theta_t\omega_1) - \beta\Delta\Gamma_2^h(\theta_t\omega_2)\right)^2 + \frac{3}{4}u^4 + \frac{b^4}{4}(\Delta\Gamma_1^h(\theta_t\omega_1))^4 + \frac{v^2}{2}\right. \\
&\quad + \frac{1}{2}(\Delta\Gamma_1^h(\theta_t\omega_1) + \Delta\Gamma_2^h(\theta_t\omega_2))^2 + \frac{z^2}{2} + \frac{1}{2}(\Delta\Gamma_1^h(\theta_t\omega_1) - r\Delta\Gamma_3^h(\theta_t\omega_3))^2 \\
&\quad \left. + \frac{u^2}{2} + \frac{q^2}{2}(\Gamma_3^h(\theta_t\omega_3))^2\right] dx.
\end{aligned}$$

Step 2. We further treat the integral in the last step of (6.55), which is decomposed into the following two parts. The first part is

$$\begin{aligned}
&\int_\Omega \left(u^4 + \frac{u^2}{2} + \frac{v^2}{2} + \frac{z^2}{2}\right) dx = \int_\Omega \left[\left(U + \Gamma_1^h(\theta_t\omega_1)\right)^4 + \frac{1}{2}\left(U + \Gamma_1^h(\theta_t\omega_1)\right)^2\right. \\
&\quad \left. + \frac{1}{2}\left(V + \Gamma_2^h(\theta_t\omega_2)\right)^2 + \frac{1}{2}\left(Z + \Gamma_3^h(\theta_t\omega_3)\right)^2\right] dx \\
&\leq \int_\Omega \left[8\left(U^4 + (\Gamma_1^h(\theta_t\omega_1))^4\right) + U^2 + (\Gamma_1^h(\theta_t\omega_1))^2\right. \qquad (6.56) \\
&\quad \left. + V^2 + (\Gamma_2^h(\theta_t\omega_2))^2 + Z^2 + (\Gamma_3^h(\theta_t\omega_3))^2\right] dx \\
&\leq \int_\Omega (8U^4(t) + U^2(t) + V^2(t) + Z^2(t)) \, dx + 8\|\Gamma^h(\theta_t\omega)\|_{L^4}^4 + \|\Gamma^h(\theta_t\omega)\|^2.
\end{aligned}$$

According to (6.34) and $D = B_H(0, \rho(\omega))$, for $t \geq \tau \in \mathbb{R}$,

$$
\begin{aligned}
\|U(t)\,\|^2 + \|V(t)\|^2 + \|Z(t)\|^2 \leq \frac{\max\{1, c_1\}}{\min\{1, c_1\}}\, 2e^{-\sigma(t-\tau)}(\rho^2(\theta_\tau\omega) + \|\Gamma^h(\theta_\tau\omega)\|^2) \\
+ \frac{1}{\min\{1, c_1\}} \int_{-\infty}^{t} e^{-\sigma(t-s)} \left(\mathscr{C}(h)\,(|\Gamma(\theta_s\omega)|^2 + |\Gamma(\theta_s\omega)|^4) + F|\Omega|\right) ds.
\end{aligned}
$$

The tempered property of $\rho^2(\theta_\tau\omega) + \|\Gamma^h(\theta_\tau\omega)\|^2$ implies that there is a sufficiently large random variable $T(D,\omega) > 6$ such that if $\tau \leq -T(D,\omega)$, then it holds that $\|U(t)\|^2 + \|V(t)\|^2 + \|Z(t)\|^2 \leq Q_1(\omega)$ for any $t \in [\tau/2, 0]$, where

$$
Q_1(\omega) = 1 + \frac{1}{\min\{1, c_1\}} \int_{-\infty}^{0} e^{\sigma s} \left(\mathscr{C}(h)\,(|\Gamma(\theta_s\omega)|^2 + |\Gamma(\theta_s\omega)|^4) + F|\Omega|\right) ds. \tag{6.57}
$$

By the embedding $H^1(\Omega) \hookrightarrow L^4(\Omega)$, there is a positive constant $\eta > 0$ such that $\|U\|_{L^4}^4 \leq \eta(\|U\|^2 + \|\nabla U\|^2)^2 \leq 2\eta(\|U\|^4 + \|\nabla U\|^4)$. It follows from (6.56) that

$$
\begin{aligned}
&\int_\Omega \left(u^4(t) + \frac{u^2(t)}{2} + \frac{v^2(t)}{2} + \frac{z^2(t)}{2}\right) dx \\
&\leq 16\,\eta\,\|U(t)\|^4 + 16\,\eta\,\|\nabla U(t)\|^4 + Q_1(\omega) + 8\|\Gamma^h(\theta_t\omega)\|_{L^4}^4 + \|\Gamma^h(\theta_t\omega)\|^2 \\
&\leq 16\,\eta\, Q_1^2(\omega) + 16\,\eta\,\|\nabla G(t)\|^4 + Q_1(\omega) + 8\|\Gamma^h(\theta_t\omega)\|_{L^4}^4 + \|\Gamma^h(\theta_t\omega)\|^2
\end{aligned}
\tag{6.58}
$$

provided that $t \in [\tau/2, 0]$ and $\tau \leq -T(D,\omega)$.

For the second part (the rest part) in the last integral of (6.55), we have

$$
\begin{aligned}
&\int_\Omega \Big[\left(a\Delta\Gamma_1^h(\theta_t\omega_1) - \beta\Delta\Gamma_2^h(\theta_t\omega_2)\right)^2 + \frac{b^4}{4}(\Delta\Gamma_1^h(\theta_t\omega_1))^4 \\
&\quad + \frac{1}{2}\left(\Delta\Gamma_1^h(\theta_t\omega_1) + \Delta\Gamma_2^h(\theta_t\omega_2)\right)^2 \\
&\quad + \frac{1}{2}\left(\Delta\Gamma_1^h(\theta_t\omega_1) - r\Delta\Gamma_3^h(\theta_t\omega_3)\right)^2 + \frac{q^2}{2}(\Gamma_3^h(\theta_t\omega_3))^2\Big]\, dx \\
&\leq \int_\Omega \big[2a^2(\Delta h_1(x))^2(\Gamma_1(\theta_t\omega_1))^2 + 2\beta^2(\Delta h_2(x))^2(\Gamma_2(\theta_t\omega_2))^2 \\
&\quad + \frac{1}{2}b^4(\Delta h_1(x))^4(\Gamma_1(\theta_t\omega_1))^4 + 2(\Delta h_1(x))^2(\Gamma_1(\theta_t\omega_1))^2 \\
&\quad + (\Delta h_2(x))^2(\Gamma_2(\theta_t\omega_2))^2 + (r^2 + q^2)(\Delta h_3(x))^2(\Gamma_3(\theta_t\omega_3))^2\big]\, dx
\end{aligned}
\tag{6.59}
$$

$$
\begin{aligned}
&= \int_\Omega \Big[2(a^2+1)(\Delta h_1)^2(\Gamma_1(\theta_t\omega_1))^2 + (2\beta^2+1)(\Delta h_2)^2(\Gamma_2(\theta_t\omega_2))^2 \\
&\qquad + (r^2+q^2)(\Delta h_3)^2(\Gamma_3(\theta_t\omega_3))^2 + \frac{1}{2}b^4(\Delta h_1)^4(\Gamma_1(\theta_t\omega_1))^4\Big]\,dx \\
&\le |\Gamma(\theta_t\omega)|^2\left[2(a^2+1)\|\Delta h_1\|^2 + (2\beta^2+1)\|\Delta h_2\|^2\right] \\
&\quad + |\Gamma(\theta_t\omega)|^2\left[(r^2+q^2)\|\Delta h_3\|^2\right] + \frac{1}{2}\,b^4|\Gamma(\theta_t\omega)|^4\|\Delta h_1\|^4_{L^4}.
\end{aligned}
$$

In (6.59), the assumption that $\{h_i(x) : i = 1, 2, 3\} \subset W^{2,4}(\Omega)$ specified in Section 6.1 is used.

Step 3. Assemble the estimates (6.58) and (6.59) of the two parts in (6.55). Then we have proved that

$$
\begin{aligned}
&\int_\Omega [(au^2 - bu^3 - v + z)\Delta\Gamma_1^h(\theta_t\omega) - (\beta u^2+v)\Delta\Gamma_2^h(\theta_t\omega) - (rz-qu)\Delta\Gamma_3^h(\theta_t\omega)]\,dx \\
&\le 16\,\eta\,\|\nabla G\|^4 + Q_2(t,\omega),
\end{aligned}
\tag{6.60}
$$

where

$$
\begin{aligned}
Q_2(t,\omega) &= 16\,\eta\,Q_1^2(\omega) + Q_1(\omega) + 8\|\Gamma^h(\theta_t\omega)\|^4_{L^4} + \|\Gamma^h(\theta_t\omega)\|^2 \\
&\quad + |\Gamma(\theta_t\omega)|^2\left[2(a^2+1)\|\Delta h_1\|^2 + (2\beta^2+1)\|\Delta h_2\|^2\right] \\
&\quad + |\Gamma(\theta_t\omega)|^2\left[(r^2+q^2)\|\Delta h_3\|^2\right] + \frac{1}{2}\,b^4|\Gamma(\theta_t\omega)|^4\|\Delta h_1\|^4_{L^4}.
\end{aligned}
\tag{6.61}
$$

In turn, substitute the inequalities (6.54) and (6.60) into (6.53), we get

$$
\begin{aligned}
&\int_\Omega \left[(bu^3 - au^2 - v + z)\Delta U + (\beta u^2+v)\Delta V + (rz-qu)\Delta Z\right]dx \\
\le &\left(\frac{C_1}{2} + 16\,\eta\right)\|\nabla G\|^4 + \frac{C_1}{2}\left(\|\nabla\Gamma^h(\theta_t\omega)\|^4 + 1\right) + Q_2(t,\omega).
\end{aligned}
\tag{6.62}
$$

Besides, by the Gauss Divergence theorem and the homogeneous Neumann boundary condition, in (6.52) we have

$$
\int_\Omega J\,\Delta U dx = \int_\Omega \alpha\,\Delta V dx = \int_\Omega q\,c\,\Delta Z dx = 0.
$$

Moreover, the three middle terms in (6.52) satisfy the estimates

$$
\begin{aligned}
& - \int_\Omega \Delta U \left[d_1 \Delta h_1 \Gamma_1(\theta_t \omega_1) + \kappa \Gamma_1^h(\theta_t \omega_1) \right] dx \\
& - \int_\Omega \Delta V \left[d_2 \Delta h_2 \Gamma_2(\theta_t \omega_2) + \kappa \Gamma_2^h(\theta_t \omega_2) \right] dx \\
& - \int_\Omega \Delta Z \left[d_3 \Delta h_3 \Gamma_3(\theta_t \omega_3) + \kappa \Gamma_3^h(\theta_t \omega_3)) \right] dx \\
\le \; & \frac{1}{2} \left(d_1 \|\Delta U\|^2 + d_2 \|\Delta V\|^2 + d_3 \|\Delta Z\|^2 \right) + \frac{1}{2} C_2(h) \, |\Gamma(\theta_t \omega)|^2,
\end{aligned}
\tag{6.63}
$$

where $C_2(h) > 0$ is a constant only depending on the functions $\{h_1, h_2, h_3\}$.

Finally, we substitute (6.62) and (6.63) into the inequality (6.52). It follows that

$$
\begin{aligned}
& \frac{d}{dt} \|\nabla G(t)\|^2 + d_1 \|\Delta U(t)\|^2 + d_2 \|\Delta V(t)\|^2 + d_3 \|\Delta Z(t)\|^2 \\
\le \; & (C_1 + 32\,\eta) \, \|\nabla G(t)\|^4 + C_1 \left(\|\nabla \Gamma^h(\theta_t \omega)\|^4 + 1 \right) + 2 Q_2(t, \omega) + C_2(h) |\Gamma(\theta_t \omega)|^2.
\end{aligned}
\tag{6.64}
$$

Step 4. In the final step of this proof, we apply the uniform Gronwall inequality [62] to the following differential inequality reduced from (6.64),

$$
\begin{aligned}
\frac{d}{dt} \|\nabla G(t)\|^2 \le \; & (C_1 + 32\,\eta) \, \|\nabla G\|^4 + C_1 \left(\|\nabla \Gamma^h(\theta_t \omega)\|^4 + 1 \right) \\
& + 2 Q_2(t, \omega) + C_2(h) |\Gamma(\theta_t \omega)|^2,
\end{aligned}
\tag{6.65}
$$

which can be written in the form

$$
\frac{d\zeta}{dt} \le \lambda \, \zeta + \xi, \quad \text{for } t \in [\tau/2, 0], \;\; \tau \le -T(D, \omega),
\tag{6.66}
$$

where $T(D, \omega) > 6$ is specified before (6.57), and

$$
\begin{aligned}
& \zeta(t) = \|\nabla G(t)\|^2, \\
& \lambda(t) = (C_1 + 32\,\eta) \|\nabla G(t)\|^2, \\
& \xi(t) = C_1 \left(\|\nabla \Gamma^h(\theta_t \omega)\|^4 + 1 \right) + 2 Q_2(t, \omega) + C_2(h) |\Gamma(\theta_t \omega)|^2.
\end{aligned}
$$

To estimate the functions $\zeta(t)$ and $\lambda(t)$, we integrate of the inequality (6.31) over the time interval

$[t-1,t] \subset [\tau/2, 0]$ to get

$$\begin{aligned} &2d \int_{t-1}^{t} \|\nabla G(s, \theta_\tau\omega;\, \tau, g_0 - \Gamma^h(\theta_\tau\omega))\|^2 ds \\ \leq& \frac{\max\{1, c_1\}}{\min\{1, c_1\}} \|G(t-1, \theta_\tau\omega; \tau, g_0 - \Gamma^h(\theta_\tau\omega))\|^2 \\ +& \frac{1}{\min\{1, c_1\}} \int_{t-1}^{t} \left[\mathscr{C}(h) \left(|\Gamma(\theta_s\omega)|^2 + |\Gamma(\theta_s\omega)|^4\right) + F|\Omega|\right] ds. \end{aligned} \tag{6.67}$$

It has been shown in Step 2 that

$$\|G(t-1, \theta_\tau\omega;\, \tau, g_0 - \Gamma^h(\theta_\tau\omega))\|^2 \leq Q_1(\omega) \tag{6.68}$$

and $Q_1(\omega)$ is given in (6.57). It follows from (6.67) and (6.68) that

$$\int_{t-1}^{t} \zeta(s) ds = \int_{t-1}^{t} \|\nabla G(s, \theta_\tau\omega;\, \tau, g_0 - \Gamma^h(\theta_\tau\omega))\|^2 ds \leq R_1(\omega), \tag{6.69}$$

for any $g_0 \in D(\theta_\tau\omega),\ t \in [-2, 0] \subset [\tau/2 + 1, 0],\ \tau \leq -T(D, \omega)$, where

$$\begin{aligned} R_1(\omega) \ =& \frac{1}{2d} \left\{ \frac{\max\{c_1, 1\}}{\min\{c_1, 1\}} Q_1(\omega) \right. \\ &\left. + \frac{1}{\min\{1, c_1\}} \int_{-3}^{0} \left[\mathscr{C}(h) \left(|\Gamma(\theta_s\omega)|^2 + |\Gamma(\theta_s\omega)|^4\right) + F|\Omega|\right] ds \right\}. \end{aligned} \tag{6.70}$$

Then in the same way, for any $g_0 \in D(\theta_\tau\omega),\ t \in [-2, 0],\ \tau \leq -T(D, \omega)$, we have

$$\int_{t-1}^{t} \lambda(s)\, ds \leq (C_2 + 32\,\eta) R_1(\omega). \tag{6.71}$$

Moreover, for $t \in [-2, 0],\ \tau < -T(D, \omega)$, we have

$$\begin{aligned} \int_{t-1}^{t} \xi(s)\, ds \leq& \int_{t-1}^{t} \left[C_1 \left(\|\nabla \Gamma^h(\theta_s\omega)\|^4 + 1\right) + 2Q_2(s, \omega) + C_2(h)|\Gamma(\theta_s\omega)|^2\right] ds \\ \leq& \int_{-3}^{0} \left[C_1 \left(\|\nabla \Gamma^h(\theta_s\omega)\|^4 + 1\right) + 2Q_2(s, \omega) + C_2(h)|\Gamma(\theta_s\omega)|^2\right] ds. \end{aligned} \tag{6.72}$$

Therefore, for any $t \in [-2, 0], \tau \leq -T(D, \omega)$ and $g_0 \in D(\theta_t\omega)$, applying the uniform Gronwall inequality to (6.66) and by (6.69), (6.71) and (6.72), we obtain

$$
\begin{aligned}
&\|\nabla G(t,\omega;\tau,g_0)\|^2 \le e^{R_1(\omega)}\left\{(C_2+32\,\eta)R_1(\omega)\right. \\
&\left. + \int_{-3}^{0}\left[C_1\left(\|\nabla\Gamma^h(\theta_s\omega)\|^4+1\right)+2Q_2(s,\omega)+C_2(h)|\Gamma(\theta_s\omega)|^2\right]\,ds\right\}.
\end{aligned}
\tag{6.73}
$$

Finally, by (6.46) and (6.73), we reach the conclusion that any for $t \ge T(D,\omega)$,

$$
\begin{aligned}
&|||\Phi(t,\theta_{-t}\,\omega, D(\theta_{-t}\,\omega)|||_E^2 \\
&= \sup_{g_0\in D(\theta_{-t}\,\omega)} \|\Phi(t,\theta_{-t}\,\omega,g_0)\|_E^2 = \sup_{g_0\in D(\theta_{-t}\,\omega)} \|G(0,\theta_{-t}\omega;\,-t,\,g_0)+\Gamma^h(\omega)\|_E^2 \\
&\le \sup_{g_0\in D(\theta_{-t}\,\omega)} 2\left(\|G(0,\theta_{-t}\omega;\,-t,\,g_0)\|_E^2+\|\Gamma^h(\omega)\|_E^2\right) \\
&= \sup_{g_0\in D(\theta_{-t}\,\omega)} 2\left(\|G(0,\theta_{-t}\omega;\,-t,\,g_0)\|_H^2+\|\nabla G(0,\theta_{-t}\omega\,;-t,\,g_0)\|_H^2+\|\Gamma^h(\omega)\|_E^2\right) \\
&\le R_E^2(\omega),
\end{aligned}
\tag{6.74}
$$

where

$$
\begin{aligned}
&R_E^2(\omega) = 2Q_1(\omega)+2\|\Gamma^h(\omega)\|_E^2+2e^{R_1(\omega)}\left\{(C_2+32\,\eta)R_1(\omega)\right. \\
&\left. + \int_{-3}^{0}\left[C_1\left(\|\nabla\Gamma^h(\theta_s\omega)\|^4+1\right)+2Q_2(s,\omega)+C_2(h)|\Gamma(\theta_s\omega)|^2\right]\,ds\right\},
\end{aligned}
\tag{6.75}
$$

and $Q_1(\omega)$ is given in (6.57). Note that $R_E(\omega)$ is a random variable independent of any initial time and initial state. Thus the result (6.51) of this theorem is proved. □

We complete this section by proving the main result on the existence of a random attractor for the Hindmarsh-Rose random dynamical system Φ in the space H.

Theorem 6.3.2. *For the spacial domain of dimension $n = \dim(\Omega) \le 2$ and for any positive parameters $d_1, d_2, d_3, a, b, \alpha, \beta, q,\ r, J$ and any $c \in \mathbb{R}$, there exists a unique random attractor $\mathcal{A}(\omega)$ in the space $H = L^2(\Omega,\mathbb{R}^3)$ with respect to $\mathscr{D}_H$ for the Hindmarsh-Rose random dynamical system Φ over the metric dynamical system $(\mathfrak{Q}, \mathcal{F}, P, \{\theta_t\}_{t\in\mathbb{R}})$.*

Proof. In Theorem 6.2.4, we proved that there exists a pullback absorbing set $K(\omega) \subset H$ for the stochastic Hindmarsh-Rose cocycle Φ. According to Definition 1.3.17, Theorem 6.3.1 and the compact imbedding $E \hookrightarrow H$ show that this cocycle Φ is pullback asymptotically compact on H with respect to $\mathscr{D}_H$.

Hence, by Theorem 5.1.1, there exists a unique random attractor in the space H for this Hindmarsh-Rose random dynamical system Φ, which is given by

$$\mathcal{A}(\omega) = \bigcap_{\tau \geq 0} \overline{\bigcup_{t \geq \tau} \varphi(t, \theta_{-t}\omega, K(\theta_{-t}\omega))}, \quad \omega \in \mathfrak{Q}, \tag{6.76}$$

where $K(\omega) = B_H(0, R_H(\omega))$ is defined in (6.48). The proof is completed. $\square$

We make a remark that there is an essential difficulty in proving the pullback asymptotic compactness of the stochastic Hindmarsh-Rose cocycle for the space dimension $n = 3$ of a bounded domain Ω. This is the reason that we reduce the space dimension $n = \dim(\Omega) \leq 2$ in Theorem 6.3.1 and Theorem 6.3.2 for this Hindmarsh-Rose random dynamical system. All the results shown in the Section 2 remain valid for space dimesion $n = \dim(\Omega) \leq 3$. We conjecture that there should exist a random attractor for the random dynamical system generated by the stochastic Hindmarsh-Rose equations with the additive noise also on the 3-dimensional domain space.

Chapter 7

A New Model of Coupled Neurons and Synchronization

Note to Reader

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In this chapter, we present a new model of coupled two neurons in terms of the following system of the coupled partly diffusive Hindmarsh-Rose equations:

$$\begin{aligned}
\frac{\partial u_1}{\partial t} &= d\Delta u_1 + au_1^2 - bu_1^3 + v_1 - w_1 + J + p(u_2 - u_1), \\
\frac{\partial v_1}{\partial t} &= \alpha - v_1 - \beta u_1^2, \\
\frac{\partial w_1}{\partial t} &= q(u_1 - c) - rw_1, \\
\frac{\partial u_2}{\partial t} &= d\Delta u_2 + au_2^2 - bu_2^3 + v_2 - w_2 + J + p(u_1 - u_2), \\
\frac{\partial v_2}{\partial t} &= \alpha - v_2 - \beta u_2^2, \\
\frac{\partial w_2}{\partial t} &= q(u_2 - c) - rw_2,
\end{aligned} \tag{7.1}$$

for $t > 0,\ x \in \Omega \subset \mathbb{R}^n$ ($n \leq 3$), where Ω is a bounded domain with locally Lipschitz continuous boundary. Here $(u_i, v_i, w_i),\ i = 1, 2$, are the state variables for two Hindmarsh-Rose (HR) neurons. The input electrical current $J > 0$ and the coefficient of neuron coupling strength $p > 0$ are treated as constants. For cell biological reason, the coupling terms are only with the two equations of the membrane potential of neuronal cells.

In this system (7.1), the variable $u_i(t, x)$ refers to the membrane electrical potential of a neuronal cell, the variable $v_i(t, x)$ called the spiking variable represents the transport rate of the ions of sodium and potassium through the fast ion channels, and the variable $w_i(t, x)$ called the bursting variable represents the transport rate across the neuronal cell membrane through slow channels of calcium and other ions.

All the involved parameters are positive constants except $c\,(= u_R) \in \mathbb{R}$, which is a reference value of the membrane potential of a neuron cell.

We impose the homogeneous Neumann boundary conditions for the u_i-components,

$$\frac{\partial u_1}{\partial \nu}(t,x) = 0, \quad \frac{\partial u_2}{\partial \nu}(t,x) = 0, \quad \text{for } \ t > 0, \ x \in \partial\Omega, \tag{7.2}$$

and the initial conditions to be specified are denoted by ($i = 1, 2$)

$$u_i(0,x) = u_i^0(x), \quad v_i(0,x) = v_i^0(x), \quad w_i(0,x) = w_i^0(x), \quad x \in \Omega. \tag{7.3}$$

The new model (7.1) in this chapter is composed of the coupled partly diffusive Hindmarsh-Rose equations and it reflects the structural feature of neuronal cells: the central cell body containing the nucleus and intracellular organelles, the dendrites of short branches near the nucleus receiving incoming signals of voltage pulses, the long-branch axon, and the nerve terminals to communicate with other cells. The long axon of neurons propagating outreaching signals and the fact that neurons are immersed in aqueous biochemical solutions with charged ions suggest that the partly diffusive reaction-diffusion equations (7.1) will be more appropriate and realistic to describe the neuronal dynamics of the signal transmission network for ensemble of neurons. It is expected that this new model and the advancing result on the exponential synchronization achieved in this chapter will be exposed to a wide range of researches and applications in neurodynamics. Here we shall present the analysis of absorbing dynamics of this new model and then prove the main result on the synchronization of the coupled Hindmarsh-Rose neurons at a uniform exponential rate with the estimate of a threshold of the coupling strength for realizing the synchronization.

7.1 Partly Diffusive and Coupled Hindmarsh-Rose Equations for Neurons

Define the Hilbert spaces $H = L^2(\Omega, \mathbb{R}^6)$ and $E = [H^1(\Omega) \times L^2(\Omega, \mathbb{R}^2)]^2$. The norm and inner-product of H or $L^2(\Omega)$ will be denoted by $\| \cdot \|$ and $\langle \cdot, \cdot \rangle$, respectively. The norm of E or $H^1(\Omega) \times L^2(\Omega, \mathbb{R}^2)$ will be denoted by $\| \cdot \|_E$. We use $| \cdot |$ to denote a vector norm in $\mathbb{R}^n$.

The initial-boundary value problem (7.1)–(7.3) can be formulated into the initial value problem of the evolutionary equation:

$$\begin{aligned} \frac{\partial g}{\partial t} &= Ag + f(g) + P(g), \quad t > 0, \\ g(0) &= g_0 \in H. \end{aligned} \tag{7.4}$$

Here the column vector $g(t) = \text{col}\,(u_1(t,\cdot), v_1(t,\cdot), w_1(t,\cdot), u_2(t,\cdot), v_2(t,\cdot), w_2(t,\cdot))$ is the unknown function and the initial data function is $g_0 = \text{col}\,(u_1^0, v_1^0, w_1^0, u_2^0, v_2^0, w_2^0)$. The nonpositive self-adjoint operator associated with this problem is

$$A = \begin{pmatrix} d\Delta & 0 & 0 \\ 0 & -I & 0 \\ 0 & 0 & -rI \\ d\Delta & 0 & 0 \\ 0 & -I & 0 \\ 0 & 0 & -rI \end{pmatrix} : D(A) \to H, \tag{7.5}$$

where $D(A) = \{g \in [H^2(\Omega) \times L^2(\Omega, \mathbb{R}^2)]^2 : \partial u_1/\partial \nu = \partial u_1/\partial \nu = 0\}$, is the generator of a C_0-semigroup $\{e^{At}\}_{t \geq 0}$ on the Hilbert space H. Since $H^1(\Omega) \hookrightarrow L^6(\Omega)$ is a continuous imbedding for space dimension $n \leq 3$ and by the Hölder inequality, the nonlinear mapping

$$f(g) = \begin{pmatrix} au_1^2 - bu_1^3 + v_1 - w_1 + J \\ \alpha - \beta u_1^2 \\ q(u_1 - c) \\ au_2^2 - bu_2^3 + v_2 - w_2 + J \\ \alpha - \beta u_2^2 \\ q(u_2 - c) \end{pmatrix} : E \longrightarrow H \tag{7.6}$$

is a locally Lipschitz continuous mapping. The coupling mapping is the vector function

$$P(g) = \begin{pmatrix} p(u_2 - u_1) \\ 0 \\ 0 \\ p(u_1 - u_2) \\ 0 \\ 0 \end{pmatrix} : H \longrightarrow H \tag{7.7}$$

Consider the weak solution of this initial value problem (7.4), cf. [13, Section XV.3], defined below and similar to what is presented in Chapters 2 and 3.

Definition 7.1.1. A six-dimensional vector function $g(t, x), (t, x) \in [0, \tau] \times \Omega$, is called a *weak solution* to the initial value problem of the evolutionary equation (7.4) formulated from (7.1), if the following conditions are satisfied:

(i) $\frac{d}{dt}(g, \zeta) = (Ag, \zeta) + (f(g) + P(g), \zeta)$ is satisfied for a.e. $t \in [0, \tau]$ and any $\zeta \in E$;

(ii) $g(t, \cdot) \in C([0, \tau]; H) \cap L^2([0, \tau]; E)$ and $g(0) = g_0$.

Here $(\cdot, \cdot)$ is the dual product of the dual space E^* versus E.

The following proposition can be proved by the Galerkin approximation method.

Proposition 7.1.2. *For any given initial state $g_0 \in H$, there exists a unique local weak solution $g(t, g_0)$, $t \in [0, \tau]$, for some $\tau > 0$ may depending on g_0, of the initial value problem* (7.4) *associated with the coupled partly diffusive Hindmarsh-Rose equations* (7.1). *The weak solution $g(t, g_0)$ continuously depends on the initial data g_0 and satisfies*

$$g \in C([0, \tau]; H) \cap C^1((0, \tau); H) \cap L^2([0, \tau]; E). \tag{7.8}$$

If the initial data $g_0 \in E$, then the weak solution becomes a strong solution on the existence time interval $[0, \tau]$, which has the regularity

$$g \in C([0, \tau]; E) \cap C^1((0, \tau); E) \cap L^2([0, \tau]; D(A)). \tag{7.9}$$

In the next section, we shall prove the global existence of weak solutions in time for the initial value problem problem (7.4) and present the analysis of the absorbing dynamics of the solution semiflow generated by the weak solutions.

The basics of infinite dimensional dynamical systems, which can be called as semiflow when generated by the autonomous parabolic partial differential equations, can be referred to [13,62,68].

Definition 7.1.3. Let $\{S(t)\}_{t\geq 0}$ be a semiflow on a Banach space $\mathscr{X}$. A bounded set B^* of $\mathscr{X}$ is called an absorbing set for this semiflow, if for any given bounded set $B \subset \mathscr{X}$ there exists a finite time $T_B \geq 0$ depending on B, such that $S(t)B \subset B^*$ for all $t \geq T_B$. The semiflow is called dissipative on $\mathscr{X}$ if there exists an absorbing set in $\mathscr{X}$.

In Section 7.3, we shall prove the main result on asymptotic synchronization of the coupled Hindmarsh-Rose neurons realized by this new model, provided that the coupling strength exceeds a threshold quantified in terms of the involved parameters. Moreover, the synchronization has a uniform exponential rate independent of any initial conditions.

7.2 Absorbing Analysis and Dynamics

First we prove the global existence of weak solutions in time for the initial value problem (7.4) of the coupled partly diffusive Hindmarsh-Rose equations.

Theorem 7.2.1. *For any given initial state $g_0 \in H$, there exists a unique global weak solution in time, $g(t) = \operatorname{col}(u_1(t), v_1(t), w_1(t), u_2(t), v_2(t), w_2(t))$, $t \in [0, \infty)$, of the initial value problem* (7.4).

Proof. Summing up the L^2 inner-product of the u_1-equation with $C_1 u_1(t)$ and the L^2 inner-product of the u_2-equation with $C_1 u_2(t)$, where the adjustable constant $C_1 > 0$ is to be determined later, and by Young's inequality we get

$$
\begin{aligned}
&\frac{C_1}{2}\frac{d}{dt}(\|u_1\|^2 + \|u_2\|^2) + C_1 d(\|\nabla u_1\|^2 + \|\nabla u_2\|^2) \\
= &\int_\Omega C_1(au_1^3 - bu_1^4 + u_1 v_1 - u_1 w_1 + J u_1)\, dx \qquad (7.10)\\
&+ \int_\Omega (C_1(au_2^3 - bu_2^4 + u_2 v_2 - u_2 w_2 + J u_2) - p(u_1 - u_2)^2)\, dx.
\end{aligned}
$$

Summing up the L^2 inner-products of the v_i-equation with $v_i(t)$ and the L^2 inner-products of the w_i-equation with $w_i(t)$ for $i = 1, 2$, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|v_1\|^2 + \|v_2\|^2) = \int_\Omega (\alpha v_1 - \beta u_1^2 v_1 - v_1^2 + \alpha v_2 - \beta u_2^2 v_2 - v_2^2)\, dx \\
&\le \int_\Omega \left(\alpha v_1 + \frac{1}{2}(\beta^2 u_1^4 + v_1^2) - v_1^2 + \alpha v_2 + \frac{1}{2}(\beta^2 u_2^4 + v_2^2) - v_2^2 \right) dx \\
&\le \int_\Omega \left(2\alpha^2 + \frac{1}{8}v_1^2 + \frac{1}{2}\beta^2 u_1^4 - \frac{1}{2}v_1^2 + 2\alpha^2 + \frac{1}{8}v_2^2 + \frac{1}{2}\beta^2 u_2^4 - \frac{1}{2}v_2^2 \right) dx \\
&= \int_\Omega \left(4\alpha^2 + \frac{1}{2}\beta^2(u_1^4 + u_2^4) - \frac{3}{8}(v_1^2 + v_2^2) \right) dx,
\end{aligned}
\tag{7.11}
$$

and

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|w_1\|^2 + \|w_2\|^2) = \int_\Omega (q(u_1 - c)w_1 - r w_1^2 + q(u_2 - c)w_2 - r w_2^2)\, dx \\
&\le \int_\Omega \left(\frac{q^2}{2r}(u_1 - c)^2 + \frac{1}{2}r w_1^2 - r w_1^2 + \frac{q^2}{2r}(u_2 - c)^2 + \frac{1}{2}r w_2^2 - r w_2^2 \right) dx \\
&\le \int_\Omega \left(\frac{q^2}{r}(u_1^2 + u_2^2 + 2c^2) - \frac{1}{2}r(w_1^2 + w_2^2) \right) dx.
\end{aligned}
\tag{7.12}
$$

Now we choose the positive constant in (7.10) to be $C_1 = \frac{1}{b}(\beta^2 + 4)$, so that

$$
\int_\Omega (-C_1 b u_i^4)\, dx + \int_\Omega (\beta^2 u_i^4)\, dx \le \int_\Omega (-4u_i^4)\, dx, \quad i = 1, 2.
$$

Then we estimate all the mixed product terms on the right-hand side of (7.10) by using the Young's inequality in an appropriate way as follows. For $i = 1, 2$,

$$
\int_\Omega C_1 a u_i^3\, dx \le \frac{3}{4}\int_\Omega u_i^4\, dx + \frac{1}{4}\int_\Omega (C_1 a)^4\, dx \le \int_\Omega u_i^4\, dx + (C_1 a)^4 |\Omega|,
$$

and

$$
\begin{aligned}
&\int_\Omega C_1(u_i v_i - u_i w_i + J u_i)\, dx \\
&\le \int_\Omega \left(2(C_1 u_i)^2 + \frac{1}{8}v_i^2 + \frac{(C_1 u_i)^2}{r} + \frac{1}{4} r w_i^2 + C_1 u_i^2 + C_1 J^2 \right) dx,
\end{aligned}
$$

where on the right-hand side of the second inequality we can further treat the three terms involving u_i^2 as follows,

$$\int_\Omega \left(2(C_1 u_i)^2 + \frac{(C_1 u_i)^2}{r} + C_1 u_i^2\right) dx \le \int_\Omega u_i^4\, dx + \left[C_1^2 \left(2 + \frac{1}{r}\right) + C_1\right]^2 |\Omega|.$$

Besides, in (7.12) we have

$$\begin{aligned}\int_\Omega \frac{1}{r} q^2 u_i^2\, dx &\le \int_\Omega \left(\frac{u_i^4}{2} + \frac{q^4}{2r^2}\right) dx \\ &\le \int_\Omega u_i^4\, dx + \frac{q^4}{r^2}|\Omega|.\end{aligned}$$

Substitute the above term estimates into (7.10) and (7.12). Then sum up the resulting inequalities (7.10)–(7.12) to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(C_1(\|u_1\|^2 + \|u_2\|^2) + (\|v_1\|^2 + \|v_2\|^2) + (\|w_1\|^2 + \|w_2\|^2)\right) \\
&\quad + C_1 d\left(\|\nabla u_1\|^2 + \|\nabla u_2\|^2\right) \\
&\le \int_\Omega C_1(a u_1^3 - b u_1^4 + u_1 v_1 - u_1 w_1 + J u_1)\, dx \\
&\quad + \int_\Omega (C_1(a u_2^3 - b u_2^4 + u_2 v_2 - u_2 w_2 + J u_2) - p(u_1 - u_2)^2)\, dx \\
&\quad + \int_\Omega \left(4\alpha^2 + \frac{1}{2}\beta^2(u_1^4 + u_2^4) - \frac{3}{8}(v_1^2 + v_2^2)\right) dx \\
&\quad + \int_\Omega \left(\frac{q^2}{r}(u_1^2 + u_2^2 + 2c^2) - \frac{1}{2} r(w_1^2 + w_2^2)\right) dx \\
&\le \int_\Omega (3-4)(u_1^4 + u_2^4)\, dx + \int_\Omega \left(\frac{1}{8} - \frac{3}{8}\right)(v_1^2 + v_2^2)\, dx + \int_\Omega \left(\frac{1}{4} - \frac{1}{2}\right) r(w_1^2 + w_2^2)\, dx \\
&\quad + |\Omega| \left(2(C_1 a)^4 + 2C_1 J^2 + 2\left[C_1^2\left(2 + \frac{1}{r}\right) + C_1\right]^2 + 4\alpha^2 + \frac{2q^2c^2}{r} + \frac{2q^4}{r^2}\right) \\
&= -\int_\Omega \left((u_1^4 + u_2^4)(t,x) + \frac{1}{4}(v_1^2 + v_2^2)(t,x) + \frac{1}{4} r(w_1^2 + w_2^2)(t,x)\right) dx + C_2|\Omega|,
\end{aligned} \tag{7.13}$$

where $C_2 > 0$ is the constant given by

$$C_2 = 2(C_1 a)^4 + 2C_1 J^2 + 2\left[C_1^2\left(2 + \frac{1}{r}\right) + C_1\right]^2 + 4\alpha^2 + \frac{2q^2c^2}{r} + \frac{2q^4}{r^2}.$$

We see that (7.13) yields the following group estimate,

$$
\begin{aligned}
&\frac{d}{dt}\left(C_1(\|u_1\|^2+\|u_2\|^2)+(\|v_1\|^2+\|v_2\|^2)+(\|w_1\|^2+\|w_2\|^2)\right)\\
&+C_1 d\,(\|\nabla u_1\|^2+\|\nabla u_2\|^2)\\
&+2\int_\Omega\left((u_1^4+u_2^4)(t,x)+\frac{1}{4}(v_1^2+v_2^2)(t,x)+\frac{1}{4}r(w_1^2+w_2^2)(t,x)\right)dx\\
\le\ & 2\,C_2|\Omega|,
\end{aligned}
\tag{7.14}
$$

for $t\in I_{max}=[0,T_{max})$, which is the maximal time interval of solution existence. Note that

$$
2u_i^4\ge\frac{1}{2}\left(C_1u_i^2-\frac{C_1^2}{16}\right),\quad i=1,2.
$$

It follows from (7.14) that

$$
\begin{aligned}
&\frac{d}{dt}\left(C_1(\|u_1\|^2+\|u_2\|^2)+(\|v_1\|^2+\|v_2\|^2)+(\|w_1\|^2+\|w_2\|^2)\right)\\
&+C_1 d\,(\|\nabla u_1\|^2+\|\nabla u_2\|^2)\\
&+\frac{1}{2}\int_\Omega\left(C_1(u_1^2+u_2^2)(t,x)+(v_1^2+v_2^2)(t,x)+r(w_1^2+w_2^2)(t,x)\right)dx\\
\le\ & \left(2C_2+\frac{C_1^2}{16}\right)|\Omega|.
\end{aligned}
$$

Set $r_1=\frac{1}{2}\min\{1,r\}$. Then we have

$$
\begin{aligned}
&\frac{d}{dt}\left(C_1(\|u_1\|^2+\|u_2\|^2)+(\|v_1\|^2+\|v_2\|^2)+(\|w_1\|^2+\|w_2\|^2)\right)\\
&+C_1 d\,(\|\nabla u_1\|^2+\|\nabla u_2\|^2)\\
&+r_1(C_1(\|u_1\|^2+\|u_2\|^2)+(\|v_1\|^2+\|v_2\|^2)+(\|w_1\|^2+\|w_2\|^2))\\
\le\ & \left(2C_2+\frac{C_1^2}{16}\right)|\Omega|.
\end{aligned}
\tag{7.15}
$$

Apply the Gronwall inequality to (7.15) with the term $C_1 d\,(\|\nabla u_1\|^2+\|\nabla u_2\|^2)$ being removed, we obtain

$$
\begin{aligned}
\|g(t)\|^2 &= \|u_1(t)\|^2 + \|u_2(t)\|^2 + \|v_1(t)\|^2 + \|v_2(t)\|^2 + \|w_1(t)\|^2 + \|w_2(t)|^2 \\
&\le \frac{\max\{C_1, 1\}}{\min\{C_1, 1\}} e^{-r_1 t} \|g_0\|^2 + \frac{M}{\min\{C_1, 1\}} |\Omega|
\end{aligned}
\tag{7.16}
$$

for $t \in I_{max} = [0, T_{max})$, where

$$
M = \frac{1}{r_1} \left(2C_2 + \frac{C_1^2}{16} \right).
$$

The estimate (7.16) shows that the weak solution $g(t, x)$ will never blow up at any finite time because it is uniformly bounded. Indeed we have

$$
\|g(t)\|^2 \le \frac{\max\{C_1, 1\}}{\min\{C_1, 1\}} \|g_0\|^2 + \frac{M}{\min\{C_1, 1\}} |\Omega|, \quad \text{for } t \in [0, \infty). \tag{7.17}
$$

Therefore the weak solution of the initial value problem (7.4) for the partly diffusive Hindmarsh-Rose equations (7.1) exists globally in time for any initial state. The time interval of maximal existence is always $[0, \infty)$ for any initial state g_0. $\square$

The global existence and uniqueness of the weak solutions and their continuous dependence on the initial data enable us to define the solution semiflow of the partly diffusive Hindmarsh-Rose equations (7.1) on the space H as follows:

$$
S(t) : g_0 \longmapsto g(t, g_0), \quad g_0 \in H, \ \ t \ge 0,
$$

where $g(t, g_0)$ is the weak solution with the initial status $g(0) = g_0$. We shall call this semiflow $\{S(t)\}_{t \ge 0}$ the *coupling Hindmarsh-Rose semiflow* generated by the evolutionary equation (7.4).

Corollary 7.2.2. *There exists an absorbing set for the coupling Hindmarsh-Rose semiflow $\{S(t)\}_{t\ge 0}$ in the space H, which is the bounded ball*

$$
B_H^* = \{h \in H : \|h\|^2 \le K\} \tag{7.18}
$$

where $K = \frac{M|\Omega|}{\min\{C_1, 1\}} + 1$.

Proof. From the uniform estimate (7.16) in Theorem 7.2.1 we see that

$$\limsup_{t \to \infty} \|g(t, g_0)\|^2 < K = \frac{M|\Omega|}{\min\{C_1, 1\}} + 1 \tag{7.19}$$

for all weak solutions of (7.4) with any initial data $g_0 \in H$. Moreover, for any given bounded set $B = \{h \in H : \|h\|^2 \le R\}$ in H, there exists a finite time

$$T_0(B) = \frac{1}{r_1} \log^+ \left(R \, \frac{\max\{C_1, 1\}}{\min\{C_1, 1\}} \right) \tag{7.20}$$

such that $\|g(t)\|^2 < K$ for all $t > T_0(B)$ and for all $g_0 \in B$. Thus, by Definition 7.1.3, the bounded ball B_H^* shown in (7.18) is an absorbing set and the coupling Hindmarsh-Rose semiflow is dissipative in the phase space H. □

Corollary 7.2.3. *For any initial data $g_0 \in H$, the weak solution $g(t, g_0)$ of the initial value problem* (7.4) *of the coupled partly diffusive Hindmarsh-Rose equations* (7.1) *satisfies the estimate*

$$\int_0^1 \|g(t, g_0)\|_E^2 \, dt \le M_1 \|g_0\|^2 + M_2 |\Omega|, \tag{7.21}$$

where M_1 and M_2 are two positive constants independent of initial data.

Proof. Integrate the differential inequality (7.15) over the time interval $[0, 1]$ to get

$$C_1 d \int_0^1 (\|\nabla u_1(t)\|^2 + \|\nabla u_2(t)\|^2) \, dt \le \max\{C_1, 1\} \|g_0\|^2 + \left(2C_2 + \frac{C_1^2}{16} \right) |\Omega|.$$

And (7.17) means that

$$\int_0^1 \|g(t, g_0)\|^2 \, dt \le \frac{\max\{C_1, 1\}}{\min\{C_1, 1\}} \|g_0\|^2 + \frac{M}{\min\{C_1, 1\}} |\Omega|.$$

Summing up the above two inequalities, we reach the result (7.21). □

In the next result, we show that the coupling Hindmarsh-Rose semiflow $\{S(t)\}_{t \ge 0}$ has also the absorbing property in the space E with the H^1-regularity for the u-components.

Theorem 7.2.4. *For the coupling Hindmarsh-Rose semiflow $\{S(t)\}_{t \geq 0}$, there exists an absorbing set in the space E, which is a bounded ball*

$$B_E^* = \{h \in E : \|h\|_E^2 \leq Q\} \tag{7.22}$$

where $Q > 0$ is a constant. For any given bounded set $B \subset H$, there exists a finite time $T_B > 0$ such that for any initial state $g_0 \in B$, the weak solution $g(t, g_0) = S(t)g_0$ of the initial value problem (7.4) *of the coupled partly diffusive Hindmarsh-Rose equations* (7.1) *enters the ball B_E^* permanently for $t \geq T_B$.*

Proof. We make estimates by taking the L^2 inner-products of the u_i-equation with $-\Delta u_i, i = 1, 2$, and then summing up the inequalities to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\nabla u_1\|^2 + \|\nabla u_2\|^2) + d(\|\Delta u_1\|^2 + \|\Delta u_2\|^2) \\
&\leq \int_\Omega (-au_1^2 \Delta u_1 - 3bu_1^2 |\nabla u_1|^2 - v_1 \Delta u_1 + w_1 \Delta u_1 - J\Delta u_1)\, dx \\
&\quad + \int_\Omega (-au_2^2 \Delta u_2 - 3bu_2^2 |\nabla u_2|^2 - v_2 \Delta u_2 + w_2 \Delta u_2 - J\Delta u_2)\, dx - p\|\nabla(u_1 - u_2)\|^2 \\
&\leq \int_\Omega \left(2au_1 |\nabla u_1|^2 - 3bu_1^2 |\nabla u_1|^2 + \frac{2v_1^2}{d} + \frac{2w_1^2}{d} + \frac{d}{4}|\Delta u_1|^2\right) dx \\
&\quad + \int_\Omega \left((2au_2 |\nabla u_2|^2 - 3bu_2^2 |\nabla u_2|^2 + \frac{2v_2^2}{d} + \frac{2w_2^2}{d} + \frac{d}{4}|\Delta u_2|^2\right) dx - p\|\nabla(u_1 - u_2)\|^2 \\
&\leq \int_\Omega \frac{2}{d}\left(v_1^2 + v_2^2 + w_1^2 + w_2^2\right) dx + \frac{d}{2}(\|\Delta u_1\|^2 + \|\Delta u_2\|^2) - p\|\nabla(u_1 - u_2)\|^2 \\
&\quad + C_3(\|\nabla u_1\|^2 + \|\nabla u_2\|^2),
\end{aligned} \tag{7.23}$$

where $C_3 = a^2/(3b)$ is a constant, because

$$2au_i - 3bu_i^2 = C_3 - (\sqrt{3b}u_i - \sqrt{C_3})^2 \leq C_3, \quad i = 1, 2.$$

Then from (7.23) it follows that

$$\begin{aligned}
&\frac{d}{dt}(\|\nabla u_1\|^2 + \|\nabla u_2\|^2) + d(\|\Delta u_1\|^2 + \|\Delta u_2\|^2) \\
&\leq C_3(\|\nabla u_1\|^2 + \|\nabla u_2\|^2) + \int_\Omega \frac{2}{d}\left(v_1^2 + v_2^2 + w_1^2 + w_2^2\right) dx, \quad t > 0.
\end{aligned} \tag{7.24}$$

By Corollary 7.2.2, for any given bounded set $B = \{h \in H : \|h\|^2 \le R\} \subset H$, there is a finite time $T_0(B) > 0$ such that for all $t > T_0(B)$ and any initial state $g_0 \in B$,

$$\int_\Omega \frac{2}{d}\left(v_1^2(t,x) + v_2^2(t,x) + w_1^2(t,x) + w_2^2(t,x)\right) dx \le \frac{2}{d}\,\|g(t,g_0)\|^2 \le \frac{2K}{d}. \tag{7.25}$$

On the other hand, for a bounded domain Ω in $\mathbb{R}^3$ combined with the homogeneous Neumann boundary condition, the Sobolev imbedding $H^2(\Omega) \hookrightarrow H^1(\Omega) \hookrightarrow L^2(\Omega)$ is continuous and compact. By the interpolation of these Sobolev spaces, for any given $\varepsilon > 0$, there is a constant $C_\varepsilon > 0$ such that

$$\|\nabla u_i(t)\|^2 \le \varepsilon \|\Delta u_i\|^2 + C_\varepsilon \|u_i\|^2, \quad \text{for } i = 1, 2.$$

Therefore, there exists a constant $C_4 > 0$ only depending on the parameters a, b and d such that (with the above $\varepsilon = d$)

$$(C_3 + 1)(\|\nabla u_1\|^2 + \|\nabla u_2\|^2) \le d(\|\Delta u_1\|^2 + \|\Delta u_2\|^2) + C_4(\|u_1\|^2 + \|u_2\|^2) \tag{7.26}$$

for all $t > \tau > 0$.

Substitute (7.25) and (7.26) into (7.24). Then we obtain the inequality

$$\begin{aligned} &\frac{d}{dt}(\|\nabla u_1\|^2 + \|\nabla u_2\|^2) + (\|\nabla u_1\|^2 + \|\nabla u_2\|^2) \\ &\le C_4(\|u_1\|^2 + \|u_2\|^2) + \frac{2K}{d} \le C_4 \|g(t,g_0)\|^2 + \frac{2K}{d} \le C_4 K + \frac{2K}{d} \end{aligned} \tag{7.27}$$

for all $t > \max\{1, T_0(B)\}$.

By Corollary 7.2.3 and (7.21), for any given bounded ball $B = \{h \in H : \|h\|^2 \le R\}$ aforementioned and $g_0 \in B$, the mean value theorem shows that the weak solution $g(t, g_0) \in L^2([0,1], E)$ and there exists a time $0 < \tau \le 1$, such that

$$\|g(\tau, g_0)\|_E^2 = \int_0^1 \|g(t,g_0)\|_E^2\, dt \le M_1 \|g_0\|^2 + M_2|\Omega| \le M_1 R + M_2 |\Omega|. \tag{7.28}$$

Now we can use the Gronwall inequality to (7.27), namely,

$$\frac{d}{dt}(\|\nabla u_1\|^2 + \|\nabla u_2\|^2) + (\|\nabla u_1\|^2 + \|\nabla u_2\|^2) \le C_4 K + \frac{2K}{d}, \quad t \in [\tau, \infty),$$

to reach the uniform estimate

$$
\begin{aligned}
&\|\nabla u_1(t)\|^2 + \|\nabla u_2(t)\|^2 \le e^{-(t-\tau)}(\|\nabla u_1(\tau)\|^2 + \|\nabla u_2(\tau)\|^2) + C_4 K + \frac{2K}{d} \\
\le\, & e^{-(t-1)}\|g(\tau, g_0)\|_E^2 + C_4 K + \frac{2K}{d} \le e^{-(t-1)}(M_1 R + M_2|\Omega|) + C_4 K + \frac{2K}{d} \\
\le\, & e^{-(t-1)} M_1 R + M_2|\Omega| + C_4 K + \frac{2K}{d},
\end{aligned} \tag{7.29}
$$

for $t > \max\{1, T_0(B)\}$ and where $T_0(B)$ is given in (7.20).

Finally, it follows that for any $g_0 \in B$, there exists a finite time

$$T_B = \max\{T_0(B), T_1(B)\},$$

where $T_1(B) = 1 + \log^+(R)$, such that $e^{-(t-1)}R < 1$. Hence,

$$\|g(t, g_0)\|_E^2 = \|\nabla u_1(t)\|^2 + \|\nabla u_2(t)\|^2 + \|g(t, g_0)\|_H^2 \le Q, \quad \text{for } t > T_B, \tag{7.30}$$

where

$$Q = M_1 + M_2|\Omega| + K(1 + C_4 + 2/d). \tag{7.31}$$

Thus the bounded ball B_E^* in (7.22) with Q given in (7.31) is an absorbing set for the coupling Hindmarsh-Rose semiflow $\{S(t)\}_{t\ge 0}$ in the space E. $\square$

7.3 Synchronization of the Coupled Hindmarsh-Rose Neurons

Synchronization of neurons is one of the central topics in neuroscience. Here we shall prove that the new model of the coupled Hindmarsh-Rose neurons proposed in this chapter will yield the asymptotic synchronization of two coupled neurons at a uniform exponential rate, which can be potentially extended to synchronization study for complex neuronal network.

Definition 7.3.1. For the model equations (7.1) of two coupled neurons, we define the *asynchronous degree* of the coupled Hindmarsh-Rose semiflow to be

$$deg_s(\text{HR}) = \sup_{g_1^0, g_2^0 \in L^2(\Omega, \mathbb{R}^3)} \left\{ \limsup_{t\to\infty} \|g_1(t) - g_2(t)\|_{L^2(\Omega,\mathbb{R}^3)} \right\}$$

where $g_1(t) = \text{col}\,(u_1(t), v_1(t), w_1(t))$ and $g_2(t) = \text{col}\,(u_2(t), v_2(t), w_2(t))$ are the two component solutions of (7.1) with any initial state $g_0 = \text{col}\,(g_1^0, g_2^0)$. The semiflow is said to be asymptotically synchronized if $deg_s(\text{HR}) = 0$.

The following synchronization theorem is the main result of this work.

Theorem 7.3.2. *For the coupled Hindmarsh-Rose semiflow generated by the weak solutions of the initial value problem* (7.4) *of the coupled partly diffusive Hindmarsh-Rose equations* (7.1)*,*

$$deg_s(\text{HR}) = 0 \tag{7.32}$$

provided that the coefficient of coupling strength $p > 0$ satisfies

$$p > \frac{4\beta^2}{b} + \frac{a^2}{b} + \frac{b}{32\beta^2 r}\left[q - \frac{8\beta^2}{b}\right]^2. \tag{7.33}$$

Under the condition (7.33)*, the coupled Hindmarsh-Rose neurons are asymptotically synchronized in the space $L^2(\Omega, \mathbb{R}^3)$ at a uniform exponential rate independent of any initial states.*

Proof. Let $g_1(t) = \text{col}\,(u_1(t), v_1(t), w_1(t))$ and $g_2(t) = \text{col}\,(u_2(t), v_2(t), w_2(t))$ be the first three components and the last three components of any solution of (7.1) in H with the initial states $g_1^0 = (u_1^0, v_1^0, w_1^0)$ and $g_2^0 = (u_2^0, v_2^0, w_2^0)$, respectively. Denote by $U(t) = u_1(t) - u_2(t), V(t) = v_1(t) - v_2(t), W(t) = w_1(t) - w_2(t)$. Then

$$g_1(t) - g_2(t) = \text{col}\,(U(t), V(u), W(t)), \quad t \geq 0.$$

By subtraction of the last three equations from the first three equations in (7.1), we obtain the differenced Hindmarsh-Rose equations:

$$\begin{aligned}
\frac{\partial U}{\partial t} &= d\Delta U + a(u_1 + u_2)U - b(u_1^2 + u_1 u_2 + u_2^2)U + V - W - 2pU, \\
\frac{\partial V}{\partial t} &= -V - \beta(u_1 + u_2)U, \\
\frac{\partial W}{\partial t} &= qU - rW.
\end{aligned} \tag{7.34}$$

Conduct estimates by taking the L^2 inner-products of the first equation of (7.34) with $\lambda U(t)$ (the constant $\lambda > 0$ is to be chosen later), the second equation of (7.34) with $V(t)$, and the third equation of (7.34) with $W(t)$ respectively and then sum them up to get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\lambda\|U(t)\|^2 + \|V(t)\|^2 + \|W(t)\|^2) \\
&\quad + d\lambda\,\|\nabla U(t)\|^2 + 2p\lambda\,\|U(t)\|^2 + \|V(t)\|^2 + r\,\|W(t)\|^2 \\
= &\int_\Omega \lambda\left(a(u_1+u_2)U^2 - b(u_1^2 + u_1u_2 + u_2^2)U^2\right)dx \\
&\quad + \int_\Omega (\lambda UV - \beta(u_1+u_2)UV + (q-\lambda)UW)\,dx \\
\le &\int_\Omega \left(\lambda a\,(u_1+u_2)U^2 - \beta(u_1+u_2)UV - \lambda b\,(u_1^2+u_1u_2+u_2^2)U^2\right)dx \\
&\quad + \left(\lambda^2 + \frac{1}{2r}(q-\lambda)^2\right)\|U(t)\|^2 + \frac{1}{4}\|V(t)\|^2 + \frac{r}{2}\|W(t)\|^2, \quad t>0.
\end{aligned}
\tag{7.35}
$$

In the last step of (7.35), we used the following Young's inequalities:

$$
\lambda U(t)V(t) \le \lambda^2U^2(t) + \frac{1}{4}V^2(t),
$$
$$
(q-\lambda)U(t)W(t) \le \frac{1}{2r}(q-\lambda)^2U^2(t) + \frac{r}{2}W^2(t).
$$

The integral terms in the last inequality of (7.35) are treated as follows:

$$
\begin{aligned}
&\int_\Omega \left(\lambda a\,(u_1+u_2)U^2 - \beta(u_1+u_2)UV - \lambda b\,(u_1^2+u_1u_2+u_2^2)U^2\right)dx \\
\le &\int_\Omega \left(\lambda a\,(u_1+u_2)U^2 - \beta(u_1+u_2)UV - \frac{\lambda b}{2}(u_1^2+u_2^2)U^2\right)dx \\
\le &\int_\Omega \left(\lambda a\,(u_1+u_2)U^2 + 2\beta^2(u_1^2+u_2^2)U^2 + \frac{1}{4}V^2 - \frac{\lambda b}{2}(u_1^2+u_2^2)U^2\right)dx.
\end{aligned}
\tag{7.36}
$$

Now we choose the constant multiplier to be

$$
\lambda = \frac{8\beta^2}{b} > 0, \tag{7.37}
$$

so that (7.36) is reduced to

$$
\begin{aligned}
&\int_\Omega \left(\lambda a\,(u_1+u_2)U^2 - \beta(u_1+u_2)UV - \lambda b\,(u_1^2+u_1u_2+u_2^2)U^2\right) dx \\
\le &\int_\Omega \left(\lambda a\,(u_1+u_2)U^2 + \frac{1}{4}V^2 - \frac{\lambda b}{4}(u_1^2+u_2^2)U^2\right) dx \\
= &\frac{1}{4}\|V(t)\|^2 + \int_\Omega \left(\lambda a\,(u_1+u_2)U^2 - \frac{\lambda b}{4}(u_1^2+u_2^2)U^2\right) dx \\
= &\frac{1}{4}\|V(t)\|^2 + \int_\Omega \left(a(u_1+u_2) - \frac{b}{4}(u_1^2+u_2^2)\right) \lambda\, U^2\, dx \\
= &\frac{1}{4}\|V(t)\|^2 + \int_\Omega \left[\frac{2a^2}{b} - \left(\frac{a}{b^{1/2}} - \frac{b^{1/2}}{2}\,u_1\right)^2 - \left(\frac{a}{b^{1/2}} - \frac{b^{1/2}}{2}\,u_2\right)^2\right] \lambda\, U^2\, dx \\
\le &\frac{1}{4}\|V(t)\|^2 + \frac{2\lambda a^2}{b}\|U(t)\|^2.
\end{aligned}
\tag{7.38}
$$

Substitute (7.38) into (7.35). Then we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\lambda\|U(t)\|^2 + \|V(t)\|^2 + \|W(t)\|^2) \\
&+ d\lambda\,\|\nabla U(t)\|^2 + 2p\lambda\,\|U(t)\|^2 + \|V(t)\|^2 + r\,\|W(t)\|^2 \\
\le &\left(\lambda^2 + \frac{2\lambda a^2}{b} + \frac{1}{2r}(q-\lambda)^2\right)\|U(t)\|^2 + \frac{1}{2}\|V(t)\|^2 + \frac{r}{2}\|W(t)\|^2, \quad t>0.
\end{aligned}
\tag{7.39}
$$

From the above inequality we get

$$
\begin{aligned}
&\frac{d}{dt}(\lambda\|U(t)\|^2 + \|V(t)\|^2 + \|W(t)\|^2) \\
&+ 4p\lambda\,\|U(t)\|^2 + \|V(t)\|^2 + r\|W(t)\|^2 \\
\le &\left(2\lambda^2 + \frac{4\lambda a^2}{b} + \frac{1}{r}(q-\lambda)^2\right)\|U(t)\|^2, \quad t>0.
\end{aligned}
\tag{7.40}
$$

Under the condition of this theorem that the coupling coefficient $p>0$ satisfies (7.33), we have

$$
\delta = 4p\lambda - \left(2\lambda^2 + \frac{4\lambda a^2}{b} + \frac{1}{r}(q-\lambda)^2\right) > 0, \tag{7.41}
$$

where λ is given by (7.37). Then (7.40) and (7.41) yield the differential inequality

$$\begin{aligned}
&\frac{d}{dt}(\lambda\|U(t)\|^2 + \|V(t)\|^2 + \|W(t)\|^2) + \min\left\{\frac{\delta}{\lambda}, r\right\}(\lambda\|U(t)\|^2 + \|V(t)\|^2 + \|W(t)\|^2) \\
\le\ &\frac{d}{dt}(\lambda\|U(t)\|^2 + \|V(t)\|^2 + \|W(t)\|^2) + \delta\|U(t)\|^2 + \|V(t)\|^2 + r\|W(t)\|^2 \le 0
\end{aligned}$$

for $t > 0$. This inequality is written as

$$\frac{d}{dt}(\lambda\|U(t)\|^2 + \|V(t)\|^2 + \|W(t)\|^2) + \mu(\lambda\|U(t)\|^2 + \|V(t)\|^2 + \|W(t)\|^2) \le 0, \ \ t > 0, \tag{7.42}$$

where $\mu = \min\{\delta/\lambda, r\}$, for any initial state $g_0 = \operatorname{col}(g_1^0, g_2^0) \in H$. We can solve (7.42) by Gronwall inequality to reach the conclusion that

$$\begin{aligned}
\min\{1, \lambda\}\|g_1(t) - g_2(t)\|^2_{L^2(\Omega,\mathbb{R}^3)} &\le \lambda\|U(t)\|^2 + \|V(t)\|^2 + \|W(t)\|^2 \\
&\le e^{-\mu t}\max\{1, \lambda\}\|g_1^0 - g_2^0\|^2_{L^2(\Omega,\mathbb{R}^3)} \to 0, \ \ \text{as } \ t \to \infty.
\end{aligned} \tag{7.43}$$

Hence it is proved that

$$deg_s(\text{HR}) = \sup_{g_1^0, g_2^0 \in L^2(\Omega,\mathbb{R}^3)} \left\{\limsup_{t\to\infty} \|g_1(t) - g_2(t)\|_{L^2(\Omega,\mathbb{R}^3)}\right\} = 0.$$

It shows that the coupled Hindmarsh-Rose neurons are asymptotically synchronized in the space $L^2(\Omega, \mathbb{R}^3)$ at a uniform exponential rate. The proof is completed. $\square$

As a remark, one can further study the synchronization problem of the coupled neurons in the regular space E. Another interesting question is to find the lower bound of a threshold of the coupling strength $p > 0$ for the self-synchronization in this model.

Chapter 8

Synchronization of Boundary Coupled Hindmarsh-Rose Neuron Network

Note to Reader

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Mathematical testimony for synchronization of coupled neurons by a hybrid model of partly diffusive partial-ordinary differential equations is an open problem. The new model of boundary coupled neuron network presented in this chapter reflects the structural feature of neuron cells, especially the short-branch dendrites receiving incoming signals and the long-branch axon propagating outreaching signals as well as that neurons are immersed in aqueous biochemical solutions with charged ions.

8.1 A New Model of Boundary Coupled Neuron Network

In this chapter, we present a new model of boundary coupled neuron network in terms of the following system of the partly diffusive Hindmarsh-Rose equations,

$$
\begin{aligned}
&\frac{\partial u}{\partial t} = d\Delta u + au^2 - bu^3 + v - w + J,\\
&\frac{\partial v}{\partial t} = \alpha - v - \beta u^2,\\
&\frac{\partial w}{\partial t} = q(u-c) - rw,\\
&\frac{\partial u_i}{\partial t} = d\Delta u_i + au_i^2 - bu_i^3 + v_i - w_i + J, \quad 1 \le i \le m,\\
&\frac{\partial v_i}{\partial t} = \alpha - v_i - \beta u_i^2, \quad 1 \le i \le m,\\
&\frac{\partial w_i}{\partial t} = q(u_i - c) - rw_i, \quad 1 \le i \le m,
\end{aligned}
\tag{8.1}
$$

for $t > 0,\ x \in \Omega \subset \mathbb{R}^n$ $(n \leq 3)$, where Ω is a bounded domain and its boundary

$$\partial\Omega = \Gamma = \bigcup_{i=0}^{m} \Gamma_i$$

is locally Lipschitz continuous, where the boundary pieces $\Gamma_i, i = 0, 1, \cdots, m$, are measurable and mutually non-overlapping. Here $(u_i, v_i, w_i),\ i = 1, \cdots, m$, are the state variables for the *neighbor neurons* denoted by $\mathcal{N}_i, i = 1, \cdots, m$, coupled with the *central neuron* denoted by $\mathcal{N}_c$ whose state variables are (u, v, w).

In this system (8.1), the variables $u(t, x)$ and $u_i(t, x)$ refer to the membrane electrical potential of a neuron cell, the variables $v(t, x)$ and $v_i(t, x)$ called the spiking variables represent the transport rate of the ions of sodium and potassium through the fast ion channels, and the variables $w(t, x)$ and $w_i(t, x)$ called the bursting variables represent the transport rate across the neuron cell membrane through slow channels of calcium and other ions.

The coupling boundary conditions affiliated with the system (8.1) are given by

$$\begin{aligned} &\frac{\partial u}{\partial \nu}(t, x) = 0, \ \text{ for } x \in \Gamma_0, \ \ \frac{\partial u}{\partial \nu}(t, x) + pu = pu_i, \ \text{ for } x \in \Gamma_i, \ 1 \leq i \leq m; \\ &\frac{\partial u_i}{\partial \nu}(t, x) = 0, \ \text{ for } x \in \Gamma \backslash \Gamma_i, \ \ \frac{\partial u_i}{\partial \nu}(t, x) + pu_i = pu, \ \text{ for } x \in \Gamma_i, \ 1 \leq i \leq m. \end{aligned} \tag{8.2}$$

where $\partial/\partial\nu$ stands for the normal outward derivative, $p > 0$ is the coupling strength constant. The initial conditions to be specified are denoted by

$$\begin{aligned} &u(0, x) = u^0(x), \quad v(0, x) = v^0(x), \quad w(0, x) = w^0(x), \quad x \in \Omega, \\ &u_i(0, x) = u_i^0(x), \quad v_i(0, x) = v_i^0(x), \quad w_i(0, x) = w_i^0(x), \quad x \in \Omega, \end{aligned} \tag{8.3}$$

for $1 \leq i \leq m$.

All the parameters in this system (8.1) including the input electrical current J are positive constants except a reference value of the membrane potential of neuron cells $c = u_R \in \mathbb{R}$.

In this study of the neuron network (8.1)–(8.3), we shall work with the following Hilbert spaces for the subsystem of three equations for each involved single neuron:

$$H = L^2(\Omega, \mathbb{R}^3), \quad \text{and} \quad E = H^1(\Omega) \times L^2(\Omega, \mathbb{R}^2).$$

Also define the product spaces

$$\mathbb{H} = [L^2(\Omega, \mathbb{R}^3)]^{1+m} \quad \text{and} \quad \mathbb{E} = [H^1(\Omega) \times L^2(\Omega, \mathbb{R}^2)]^{1+m}$$

for the entire system (8.1)–(8.3). The norm and inner-product of the Hilbert space $\mathbb{H}$, H or $L^2(\Omega)$ will be denoted by $\| \cdot \|$ and $\langle \cdot, \cdot \rangle$, respectively. The norm of $\mathbb{E}$ or E will be denoted by $\| \cdot \|_E$. We use $| \cdot |$ to denote the vector norm or the measure of set in $\mathbb{R}^n$.

The initial-boundary value problem (8.1)–(8.3) can be formulated as an initial value problem of the evolutionary equation:

$$\begin{gathered} \frac{\partial}{\partial t} \begin{pmatrix} g \\ g_i \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & A_i \end{pmatrix} \begin{pmatrix} g \\ g_i \end{pmatrix} + \begin{pmatrix} f(g) \\ f(g_i) \end{pmatrix}, \quad 1 \le i \le m, \quad t > 0, \\ g(0) = g^0 \in H, \quad g_i(0) = g_i^0 \in H. \end{gathered} \tag{8.4}$$

Here $g(t) = \text{col}\,(u(t,\cdot), v(t,\cdot), w(t,\cdot))$ and $g_i(t) = (u_i(t,\cdot), v_i(t,\cdot), w_i(t,\cdot))$. The initial data functions are $g^0 = \text{col}\,(u^0, v^0, w^0)$ and $g_i^0 = \text{col}\,(u_i^0, v_i^0, w_2^0)$, for $1 \le i \le m$. The nonpositive, self-adjoint and diagonal operator $\mathcal{A} = \text{diag}\,(A, A_1, \cdots, A_m)$ is defined by the block operators

$$A = A_i = \begin{bmatrix} d\Delta & 0 & 0 \\ 0 & -I & 0 \\ 0 & 0 & -rI \end{bmatrix}, \quad 1 \le i \le m, \tag{8.5}$$

with the domain

$$D(\mathcal{A}) = \{\text{col}\,(h, h_1, \cdots, h_m) \in [H^2(\Omega) \times L^2(\Omega, \mathbb{R}^2)]^{1+m} : \text{(8.2) satisfied}\}.$$

Due to the continuous Sobolev imbedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$ for space dimension $n \le 3$ and by the Hölder inequality, the nonlinear mapping

$$\begin{pmatrix} f(g) \\ f(g_i) \end{pmatrix} = \begin{pmatrix} au_1^2 - bu_1^3 + v_1 - w_1 + J \\ \alpha - \beta u_1^2 \\ q(u_1 - c) \\ au_2^2 - bu_2^3 + v_2 - w_2 + J \\ \alpha - \beta u_2^2 \\ q(u_2 - c) \end{pmatrix} : E \times E \longrightarrow H \times H \tag{8.6}$$

is a locally Lipschitz continuous mapping for $1 \leq i \leq m$.

We shall consider the weak solution of this initial value problem (8.4), cf. [13, Section XV.3] and the corresponding definition presented in Chapters 2 and 3. The following proposition can be proved by the Galerkin approximation method.

Proposition 8.1.1. *For any given initial state* $(g^0, g_1^0, \cdots, g_m^0) \in \mathbb{H}$, *there exists a unique local weak solution* $(g(t, g^0), g_1(t, g_1^0), \cdots, g_m(t, g_m^0))$, $t \in [0, \tau]$, *for some* $\tau > 0$, *of the initial value problem* (8.4) *formulated from the problem* (8.1)–(8.3). *The weak solution continuously depends on the initial data and satisfies*

$$(g, g_1, \cdots, g_m) \in C([0, \tau]; \mathbb{H}) \cap C^1((0, \tau); \mathbb{H}) \cap L^2([0, \tau]; \mathbb{E}). \tag{8.7}$$

If the initial state is in $\mathbb{E}$, *then the solution is a strong solution with the regularity*

$$(g, g_1, \cdots, g_m) \in C([0, \tau]; \mathbb{E}) \cap C^1((0, \tau); \mathbb{E}) \cap L^2([0, \tau]; D(A) \times D(A_i)^m). \tag{8.8}$$

The basics of infinite dimensional dynamical systems or called semiflow generated by parabolic partial differential equations are referred to [13, 62, 68].

Definition 8.1.2. Let $\{S(t)\}_{t \geq 0}$ be a semiflow on a Banach space $\mathscr{X}$. A bounded set B^* of $\mathscr{X}$ is called an absorbing set of this semiflow, if for any given bounded set $B \subset \mathscr{X}$ there exists a finite time $T_B \geq 0$ depending on B, such that $S(t)B \subset B^*$ permanently for all $t \geq T_B$.

8.2 Global Existence of Solutions and Absorbing Semiflow

First we prove the global existence of weak solutions in time for the initial value problem (8.4) of the boundary coupled partly diffusive Hindmarsh-Rose equations.

Theorem 8.2.1. *For any given initial state* $(g^0, g_1^0, \cdots, g_m^0) \in \mathbb{H}$, *there exists a unique global weak solution in time,* $(g(t), g_1(t), \cdots, g_m(t))$, $t \in [0, \infty)$, *of the initial value problem* (8.4) *formulated from the original initial-boundary value problem* (8.1)–(8.3).

Proof. Sum up the L^2 inner-products of the u-equation with $C_1 u(t)$ and the u_i-equation with $C_1 u_i(t)$ for $1 \leq i \leq m$, the constant $C_1 > 0$ to be chosen, we get

$$
\begin{aligned}
&\frac{C_1}{2}\frac{d}{dt}\left(\|u\|^2 + \sum_{i=1}^m \|u_i\|^2\right) + C_1 d\left(\|\nabla u\|^2 + \sum_{i=1}^m \|\nabla u_i\|^2\right) \\
=& \, C_1 \int_\Omega \left[(au^3 - bu^4 + uv - uw + Ju) + \sum_{i=1}^m (au_i^3 - bu_i^4 + u_i v_i - u_i w_i + J u_i)\right] dx \\
&+ dC_1 \sum_{i=1}^m \int_{\Gamma_i} (p(u_i - u)u + p(u - u_i)u_i)\; dx \\
=& \int_\Omega C_1(au^3 - bu^4 + uv - uw + Ju)\, dx \\
&+ \sum_{i=1}^m \int_\Omega (C_1(au_i^3 - bu_i^4 + u_i v_i - u_i w_i + J u_i)\, dx - dC_1 p \sum_{i=1}^m \int_{\Gamma_i} (u - u_i)^2\, dx \\
\leq& \, C_1 \int_\Omega \left[(au^3 - bu^4 + uv - uw + Ju) + \sum_{i=1}^m (au_i^3 - bu_i^4 + u_i v_i - u_i w_i + J u_i)\right] dx,
\end{aligned}
$$

by the coupling boundary condition (8.2). Then sum up the L^2 inner-products of the v-equation with $v(t)$ and the v_i-equation with $v_i(t)$ for $1 \leq i \leq m$, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|v\|^2 + \sum_{i=1}^m \|v_i\|^2\right) = \int_\Omega \left[(\alpha v - \beta u^2 v - v^2 + \sum_{i=1}^m (\alpha v_i - \beta u_i^2 v_i - v_i^2)\right] dx \\
\leq& \int_\Omega \left[\alpha v + \frac{1}{2}(\beta^2 u^4 + v^2) - v^2 + \sum_{i=1}^m (\alpha v_i + \frac{1}{2}(\beta^2 u_i^4 + v_i^2) - v_i^2)\right] dx \\
\leq& \int_\Omega \left[(1+m)\alpha^2 + \frac{1}{2}\beta^2(u^4 + \sum_{i=1}^m u_i^4) - \frac{3}{8}(v^2 + \sum_{i=1}^m v_i^2)\right] dx,
\end{aligned}
$$

and similarly for the w-equation and w_i-equation, $1 \le i \le m$, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|w\|^2 + \sum_{i=1}^m \|w_i\|^2\right) \\
&= \int_\Omega \left[(q(u-c)w - rw^2) + \sum_{i=1}^m (q(u_i - c)w_i - rw_i^2)\right] dx \\
&\le \int_\Omega \left[\frac{q^2}{2r}(u-c)^2 + \frac{1}{2}rw^2 - rw^2 + \sum_{i=1}^m \left(\frac{q^2}{2r}(u_i - c)^2 + \frac{1}{2}rw_i^2 - rw_i^2\right)\right] dx \\
&\le \int_\Omega \left[\frac{q^2}{r}\left(u^2 + \sum_{i=1}^m u_i^2 + (1+m)c^2\right) - \frac{r}{2}\left(w^2 + \sum_{i=1}^m w_i^2\right)\right] dx.
\end{aligned}
$$

To treat the nonlinear integral terms on the right-hand side of the first inequality above, we choose the positive constant to be $C_1 = \frac{1}{b}(\beta^2 + 4)$. Then

$$
\begin{aligned}
&\int_\Omega (-C_1 b u^4)\, dx + \int_\Omega (\beta^2 u^4)\, dx = \int_\Omega (-4u^4)\, dx, \\
&\int_\Omega (-C_1 b u_i^4)\, dx + \int_\Omega (\beta^2 u_i^4)\, dx = \int_\Omega (-4u_i^4)\, dx, \quad i = 1, \cdots, m.
\end{aligned}
\tag{8.9}
$$

Using the Young's inequality in an appropriate way, we deduce that

$$
\begin{aligned}
&\int_\Omega C_1 a u_i^3\, dx \le \frac{3}{4}\int_\Omega u^4\, dx + \frac{1}{4}\int_\Omega (C_1 a)^4\, dx \le \int_\Omega u^4\, dx + (C_1 a)^4|\Omega|, \\
&\int_\Omega C_1 a u_i^3\, dx \le \frac{3}{4}\int_\Omega u_i^4\, dx + \frac{1}{4}\int_\Omega (C_1 a)^4\, dx \le \int_\Omega u_i^4\, dx + (C_1 a)^4|\Omega|,
\end{aligned}
\tag{8.10}
$$

for $i = 1, \cdots, m$. Moreover, we have

$$
\begin{aligned}
&C_1 \int_\Omega \left((uv - uw + ju) + \sum_{i=1}^m (u_i v_i - u_i w_i + J u_i)\right) dx \\
&\le \int_\Omega \left(2(C_1 u)^2 + \frac{1}{8}v^2 + \frac{(C_1 u)^2}{r} + \frac{1}{4}rw^2 + C_1 u^2 + C_1 J^2\right) dx \\
&+ \int_\Omega \sum_{i=1}^m \left(2(C_1 u_i)^2 + \frac{1}{8}v_i^2 + \frac{(C_1 u_i)^2}{r} + \frac{1}{4}rw_i^2 + C_1 u_i^2 + C_1 J^2\right) dx
\end{aligned}
\tag{8.11}
$$

where on the right-hand side of the inequality (8.11) we can further treat the terms involving u^2

and u_i^2 as follows,

$$\begin{aligned}
&\int_\Omega \left(2(C_1 u)^2 + \frac{(C_1 u)^2}{r} + C_1 u^2 + \sum_{i=1}^m \left[2(C_1 u_i)^2 + \frac{(C_1 u_i)^2}{r} + C_1 u_i^2 \right] \right) dx \\
\le &\int_\Omega \left(u^4 + \sum_{i=1}^m u_i^4 \right) dx + (1+m) \left[C_1^2 \left(2 + \frac{1}{r} \right) + C_1 \right]^2 |\Omega|.
\end{aligned} \tag{8.12}$$

Besides we have

$$\int_\Omega \frac{1}{r} q^2 \left(u^2 + \sum_{i=1}^m u_i^2 \right) dx \le \int_\Omega \left(u^4 + \sum_{i=1}^m u_i^4 \right) dx + \frac{q^4}{r^2}(1+m)|\Omega|. \tag{8.13}$$

Substitute the estimates (8.9)–(8.13) into the first three differential inequalities in this proof and then sum them up to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left[C_1 \left(\|u\|^2 + \sum_{i=1}^m \|u_i\|^2 \right) + \left(\|v\|^2 + \sum_{i=1}^m \|v_i\|^2 \right) + \left(\|w\|^2 + \sum_{i=1}^m \|w_i\|^2 \right) \right] \\
&\quad + C_1 d \left(\|\nabla u\|^2 + \sum_{i=1}^m \|\nabla u_i\|^2 \right) \\
\le\, & C_1 \int_\Omega \left[(au^3 - bu^4 + uv - uw + Ju) + \sum_{i=1}^m (au_i^3 - bu_i^4 + u_i v_i - u_i w_i + Ju_i) \right] dx \\
&\quad + \int_\Omega \left[(1+m)\alpha^2 + \frac{1}{2}\beta^2 (u^4 + \sum_{i=1}^m u_i^4) - \frac{3}{8}(v^2 + \sum_{i=1}^m v_i^2) \right] dx \\
&\quad + \int_\Omega \left[\frac{q^2}{r} \left(u^2 + \sum_{i=1}^m u_i^2 + (1+m)c^2 \right) - \frac{r}{2} \left(w^2 + \sum_{i=1}^m w_i^2 \right) \right] dx \\
\le\, & \int_\Omega (3-4) \left(u^4 + \sum_{i=1}^m u_i^4 \right) dx + \int_\Omega \left(\frac{1}{8} - \frac{3}{8} \right) \left(v^2 + \sum_{i=1}^m v_i^2 \right) dx \\
&\quad + \int_\Omega \left(\frac{1}{4} - \frac{1}{2} \right) r \left(w^2 + \sum_{i=1}^m w_i^2 \right) dx \\
&\quad + (1+m)|\Omega| \left((C_1 a)^4 + C_1 J^2 + \left[C_1^2 \left(2 + \frac{1}{r} \right) + C_1 \right]^2 + 2\alpha^2 + \frac{q^2 c^2}{r} + \frac{q^4}{r^2} \right) \\
=\, & -\int_\Omega \left(\left[u^4 + \sum_{i=1}^m u_i^4 \right] + \frac{1}{4} \left[v^2 + \sum_{i=1}^m v_i^2 \right] + \frac{r}{4} \left[w^2 + \sum_{i=1}^m w_i^2 \right] \right) dx + \frac{C_2}{2}(1+m)|\Omega|,
\end{aligned} \tag{8.14}$$

where

$$C_2 = 2(C_1 a)^4 + 2C_1 J^2 + 2\left[C_1^2\left(2+\frac{1}{r}\right)+C_1\right]^2 + 4\alpha^2 + \frac{2q^2c^2}{r} + \frac{2q^4}{r^2}$$

is a constant.

From (8.14) it follows that

$$\begin{aligned}
&\frac{d}{dt}\left[C_1\left(\|u\|^2+\sum_{i=1}^m\|u_i\|^2\right)+\left(\|v\|^2+\sum_{i=1}^m\|v_i\|^2\right)+\left(\|w\|^2+\sum_{i=1}^m\|w_i\|^2\right)\right]\\
&+2\int_\Omega\left(\left[u^4+\sum_{i=1}^m u_i^4\right]+\frac{1}{4}\left[v^2+\sum_{i=1}^m v_i^2\right]+\frac{r}{4}\left[w^2+\sum_{i=1}^m w_i^2\right]\right)dx \le C_2(1+m)|\Omega|,
\end{aligned} \tag{8.15}$$

for $t \in I_{max} = [0, T_{max})$, the maximal time interval of solution existence. Note that in the first part of the integral term of (8.15) we have

$$\frac{1}{4}\left(C_1 u^2 - \frac{C_1^2}{16}\right) \le u^4 \quad \text{and} \quad \frac{1}{4}\left(C_1 u_i^2 - \frac{C_1^2}{16}\right) \le u_i^4, \quad 1 \le i \le m.$$

Then (8.15) yields the following differential inequality

$$\begin{aligned}
&\frac{d}{dt}\left[C_1\left(\|u\|^2+\sum_{i=1}^m\|u_i\|^2\right)+\left(\|v\|^2+\sum_{i=1}^m\|v_i\|^2\right)+\left(\|w\|^2+\sum_{i=1}^m\|w_i\|^2\right)\right]\\
&+\ r^*\left[C_1\left(\|u\|^2+\sum_{i=1}^m\|u_i\|^2\right)+\left(\|v\|^2+\sum_{i=1}^m\|v_i\|^2\right)+\left(\|w\|^2+\sum_{i=1}^m\|w_i\|^2\right)\right]\\
&\le \frac{d}{dt}\left[C_1\left(\|u\|^2+\sum_{i=1}^m\|u_i\|^2\right)+\left(\|v\|^2+\sum_{i=1}^m\|v_i\|^2\right)+\left(\|w\|^2+\sum_{i=1}^m\|w_i\|^2\right)\right]\\
&+\ \frac{1}{2}\int_\Omega\left(\left[u^2+\sum_{i=1}^m u_i^2\right]+\left[v^2+\sum_{i=1}^m v_i^2\right]+r\left[w^2+\sum_{i=1}^m w_i^2\right]\right)dx\\
&\le \left(C_2+\frac{C_1^2}{32}\right)(1+m)|\Omega|,
\end{aligned} \tag{8.16}$$

where $r^* = \frac{1}{2}\min\{1, r\}$.

Apply the Gronwall inequality to (8.16). Then we obtain the following bounding estimate of the weak solutions:

$$
\begin{aligned}
&\|g(t, g^0)\|^2 + \sum_{i=1}^m \|g_i(t, g_i^0)\|^2 \\
&\le \frac{\max\{C_1, 1\}}{\min\{C_1, 1\}} e^{-r^* t} \left(\|g^0\|^2 + \sum_{i=1}^m \|g_i^0\|^2 \right) + \frac{M}{\min\{C_1, 1\}} |\Omega| \qquad (8.17) \\
&\le \frac{\max\{C_1, 1\}}{\min\{C_1, 1\}} \left(\|g^0\|^2 + \sum_{i=1}^m \|g_i^0\|^2 \right) + \frac{M}{\min\{C_1, 1\}} |\Omega|
\end{aligned}
$$

for $t \in I_{max} = [0, T_{max}) = [0, \infty)$, where

$$
M = \frac{1+m}{r^*} \left(C_2 + \frac{C_1^2}{32} \right). \tag{8.18}
$$

The estimate (8.17) shows that the weak solution $g(t, x)$ will never blow up at any finite time because it is bounded uniformly on the existence time interval. Therefore, for any initial data in $\mathbb{H}$, the unique weak solution of the initial value problem (8.4) of the boundary coupled neuron network (8.1)–(8.3) exists in $\mathbb{H}$ globally in time. □

The global existence and uniqueness of the weak solutions and their continuous dependence on the initial data enable us to define the solution semiflow $\{S(t) : \mathbb{H} \to \mathbb{H}\}_{t \ge 0}$ of the boundary coupled Hindmarsh-Rose neuron network system (8.1)–(8.3) on the space $\mathbb{H}$ as follows,

$$
S(t) : (g^0, g_1^0, \cdots, g_m^0) \longmapsto (g(t, g^0), g_1(t, g_1^0), \cdots, g_m(t, g_m^0)), \quad t \ge 0. \tag{8.19}
$$

We call this semiflow $\{S(t)\}_{t \ge 0}$ the *boundary coupling Hindmarsh-Rose semiflow*.

Theorem 8.2.2. *There exists an absorbing set for the boundary coupling Hindmarsh-Rose semiflow $\{S(t)\}_{t \ge 0}$ in the space $\mathbb{H}$, which is the bounded ball*

$$
B^* = \{h \in \mathbb{H} : \|h\|^2 \le Q\} \tag{8.20}
$$

where $Q = \frac{M|\Omega|}{\min\{C_1, 1\}} + 1$.

Proof. This is the consequence of the uniform estimate (8.17) in Theorem 8.2.1 because

$$
\limsup_{t \to \infty} \left(\|g(t)\|^2 + \sum_{i=1}^m \|g_i(t)\|^2 \right) < Q = \frac{M|\Omega|}{\min\{C_1, 1\}} + 1 \tag{8.21}
$$

for all weak solutions of (8.4) with any initial data in $\mathbb{H}$. Moreover, for any given bounded set $B = \{h \in \mathbb{H} : \|h\|^2 \leq \rho\}$ in $\mathbb{H}$, there exists a finite time

$$T_0(B) = \frac{1}{r^*} \log^+ \left(\rho \, \frac{\max\{C_1, 1\}}{\min\{C_1, 1\}} \right) \tag{8.22}$$

such that all the solution trajectories started from the set B will permanently enter the bounded ball B^* shown in (8.20) for $t \geq T_0(B)$. $\square$

8.3 Synchronization of the Boundary Coupled Neuron Neiwork

Synchronization for ensemble of neurons and for complex neuron network or some artificial neural network is one of the central and significant topics in neuroscience and in the theory of artificial intelligence.

We introduced a new concept of synchronization dynamics for a neuron network.

Definition 8.3.1. For the dynamical system generated by a model differential equation such as (8.4) of multiple neurons with whatever type of coupling, define the *asynchronous degree* in a state space $\mathscr{X}$ to be

$$deg_s(\mathscr{X}) = \sum_j \sum_k \sup_{g_j^0, \, g_k^0 \in \mathscr{X}} \left\{ \limsup_{t \to \infty} \, \|g_j(t) - g_k(t)\|_{\mathscr{X}} \right\},$$

where $g_j(t)$ and $g_k(t)$ are any two solutions of the model differential equation with the initial states g_j^0 and g_k^0, respectively. Then the coupled neuron network is said to be asymptotically synchronized in the space $\mathscr{X}$, if $deg_s(\mathscr{X}) = 0$.

In this section, we shall prove the main result of this work on the asymptotic synchronization of the boundary coupled Hindmarsh-Rose neuron network described by (8.1)–(8.3) in the space H. This result provides a quantitative threshold for the coupling strength and the stimulation signals to reach the asymptotic synchronization.

To address mathematically this synchronization problem of the neuron network specified in Section 8.1, denote by $U_i(t) = u(t) - u_i(t), V_i(t) = v(t) - v_i(t), W_i(t) = w(t) - w_i(t)$, for $i = 1, \cdots, m$. Then for any given initial states g^0 and $g_i^0, \cdots, g_m^0$ in the space H, the difference between the solutions associated with the neuron $\mathcal{N}_c$ and the neuron $\mathcal{N}_i$ is

$$g(t, g^0) - g_i(t, g_i^0) = \text{col}\,(U_i(t), V_i(t), W_i(t)), \quad t \geq 0.$$

By subtraction of the corresponding three pairs of equations of the i-th neuron from the central neuron in (8.1), we obtain the differencing Hindmarsh-Rose equations as follows. For $i = 1, \cdots, m$,

$$\begin{aligned}
\frac{\partial U_i}{\partial t} &= d\Delta U_i + a(u + u_i)U_i - b(u^2 + uu_i + u_i^2)U_i + V_i - W_i, \\
\frac{\partial V_i}{\partial t} &= -V_i - \beta(u + u_i)U_i, \\
\frac{\partial W_i}{\partial t} &= qU_i - rW_i.
\end{aligned} \tag{8.23}$$

Here is the main result on the synchronization of the boundary coupled Hindmarsh-Rose neuron network.

Theorem 8.3.2. *If the threshold condition for stimulation signal strength of the boundary coupled Hindmarsh-Rose neuron network is satisfied that for any given initial conditions $g^0, g_i^0 \in H$,*

$$p \liminf_{t \to \infty} \int_{\Gamma_i} U_i^2(t, x)\, dx > R\,|\Omega|, \quad i = 1, \cdots, m, \tag{8.24}$$

where

$$R = \frac{1 + m}{r^* \min\{C_1, 1\}} \left(\frac{C_1^2}{32} + C_2 \right) \left[\eta_2\, d\,|\Omega| + \left[\frac{8\beta^2}{b} + \frac{a^2}{b} + \frac{b}{16\beta^2 r} \left(q - \frac{8\beta^2}{b} \right)^2 \right] \right] \tag{8.25}$$

with $C_1 = \frac{1}{b}(\beta^2 + 4)$, $\eta_2 > 0$ being the constant in Poincaré inequality (8.33)*, and*

$$C_2 = 2(C_1 a)^4 + 2C_1 J^2 + 2\left[C_1^2 \left(2 + \frac{1}{r} \right) + C_1 \right]^2 + 4\alpha^2 + \frac{2q^2 c^2}{r} + \frac{2q^4}{r^2}, \tag{8.26}$$

then the boundary coupled Hindmarsh-Rose neuron network generated by (8.4) *is asymptotically synchronized in the space H at a uniform exponential rate.*

Proof. Step 1. Take the L^2 inner-products of the first equation in (8.23) with $KU_i(t)$, the second equation in (8.23) with $V_i(t)$, and the third equation in (8.23) with $W_i(t)$, where $K > 0$ to be chosen. Then sum them up and use Young's inequalities to get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(K\|U_i(t)\|^2 + \|V_i(t)\|^2 + \|W_i(t)\|^2) + dK\|\nabla U_i(t)\|^2 + \|V_i(t)\|^2 + r\,\|W_i(t)\|^2 \\
=& \int_\Gamma K\frac{\partial U_i}{\partial \nu}\, U_i\, dx + \int_\Omega K(a(u+u_i)U_i^2 - b(u^2+uu_i+u_i^2)U_i^2)\, dx \\
&+ \int_\Omega (KU_iV_i - \beta(u+u_i)U_iV_i + (q-K)U_iW_i)\, dx \qquad (8.27)\\
\le& \int_\Gamma K\frac{\partial U_i}{\partial \nu}\, U_i\, dx + \int_\Omega \left(Ka(u+u_i)U_i^2 - \beta(u+u_i)U_iV_i - Kb\,(u^2+uu_i+u_i^2)U_i^2\right) dx \\
&+ \left(K^2 + \frac{1}{2r}(q-K)^2\right)\|U_i(t)\|^2 + \frac{1}{4}\|V_i(t)\|^2 + \frac{r}{2}\|W_i(t)\|^2, \quad t > 0.
\end{aligned}
$$

By the the boundary coupling condition (8.2), the boundary integral in (8.27) yields

$$
\begin{aligned}
&\int_\Gamma K\frac{\partial U_i}{\partial \nu}\, U_i\, dx \\
=& K\int_\Gamma \sum_{i=1}^{m} p[(u_i-u)-(u-u_i)]\, U_i\, dx \qquad (8.28)\\
=& -2Kp\int_{\Gamma_i} U_i^2(t,x)\, dx - 2Kp\int_{\Gamma\backslash(\Gamma_0\cup\Gamma_i)} u^2(t,x)\, dx
\end{aligned}
$$

for $1 \le i \le m$. We estimate another integral term on the right-hand side of (8.27),

$$
\begin{aligned}
&\int_\Omega \left(Ka(u+u_i)U_i^2 - \beta(u+u_i)U_iV_i - Kb\,(u^2+uu_i+u_i^2)U_i^2\right) dx \\
\le& \int_\Omega \left(Ka(u+u_i)U_i^2 - \beta(u+u_i)U_iV_i - \frac{Kb}{2}(u^2+u_i^2)U_i^2\right) dx \qquad (8.29)\\
\le& \int_\Omega \left(Ka(u+u_i)U_i^2 + 2\beta^2(u^2+u_i^2)U_i^2 + \frac{1}{4}V_i^2 - \frac{Kb}{2}(u^2+u_i^2)U_i^2\right) dx.
\end{aligned}
$$

Now we choose the constant multiplier K to be

$$K = \frac{8\beta^2}{b} > 0. \qquad (8.30)$$

Then (8.29) is reduced to

$$\begin{aligned}
&\int_\Omega \left(Ka\,(u+u_i)U_i^2 - \beta(u+u_i)U_iV_i - Kb\,(u^2+uu_i+u_i^2)U_i^2\right) dx \\
\le &\int_\Omega \left(Ka(u+u_i)U_i^2 + \frac{1}{4}V_i^2 - \frac{Kb}{4}(u^2+u_i^2)U_i^2\right) dx \\
= &\frac{1}{4}\|V_i(t)\|^2 + \int_\Omega \left(a(u+u_i) - \frac{b}{4}(u^2+u_i^2)\right) KU_i^2\, dx \\
= &\frac{1}{4}\|V_i(t)\|^2 + \int_\Omega \left[\frac{2a^2}{b} - \left(\frac{a}{b^{1/2}} - \frac{b^{1/2}}{2}\,u\right)^2 - \left(\frac{a}{b^{1/2}} - \frac{b^{1/2}}{2}\,u_i\right)^2\right] KU_i^2\, dx \\
\le &\frac{1}{4}\|V_i(t)\|^2 + \frac{2Ka^2}{b}\|U_i(t)\|^2.
\end{aligned} \tag{8.31}$$

Substitute (8.28) and (8.31) into (8.27). Then for $1 \le i \le m$ it holds that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(K\|U_i(t)\|^2 + \|V_i(t)\|^2 + \|W_i(t)\|^2) + 2Kp\int_{\Gamma_i} U_i^2(t,x)\, dx \\
&+ 2Kp\int_{\Gamma\backslash(\Gamma_0\cup\Gamma_i)} u^2(t,x)\, dx + dK\,\|\nabla U_i(t)\|^2 + \|V_i(t)\|^2 + r\,\|W_i(t)\|^2 \\
\le &\left(K^2 + \frac{Ka^2}{b} + \frac{1}{2r}(q-K)^2\right)\|U_i(t)\|^2, \quad t > 0.
\end{aligned} \tag{8.32}$$

Step 2. By Poincaré inequality, there exist positive constants η_1 and η_2 depending only on the spatial domain Ω and its dimension such that

$$\eta_1\|U_i(t)\|^2 \le \|\nabla U_i(t)\|^2 + \eta_2\left(\int_\Omega U_i(t,x)\, dx\right)^2, \quad 1 \le i \le m. \tag{8.33}$$

On the other hand, Theorem 8.2.2 with (8.18) and (8.21) confirm that

$$\limsup_{t\to\infty}\left[\|g(t)\|^2 + \sum_{i=1}^m \|g_i(t)\|^2\right] \le \frac{1+m}{r^*\min\{C_1,1\}}\left(C_2 + \frac{C_1^2}{32}\right)|\Omega|. \tag{8.34}$$

Note that

$$\|U_i(t)\|^2 \le 2(\|u(t)\|^2 + \|u_i(t)\|^2) \le 2\left(\|g(t)\|^2 + \sum_{i=1}^m \|g_i(t)\|^2\right).$$

Then it follows from (8.33) and (8.34) that, for any given bounded set $B \subset H$ and any initial data

$g^0, g_i^0 \in B$, we have

$$
\begin{aligned}
&\frac{d}{dt}(K\|U_i(t)\|^2 + \|V_i(t)\|^2 + \|W_i(t)\|^2) + 4Kp\int_{\Gamma_i} U_i^2(t,x)\,dx \\
&+ 2\,\eta_1 dK\,\|U_i(t)\|^2 + \|V_i(t)\|^2 + r\|W_i(t)\|^2 \\
\le\, & 2\eta_2\, dK\left(\int_\Omega U_i(t,x)\,dx\right)^2 + \left(K^2 + \frac{Ka^2}{b} + \frac{1}{2r}(q-K)^2\right)\|U_i(t)\|^2 \\
\le\, & 2\eta_2\, dK|\Omega|\|U_i(t)\|^2 + 2\left(K^2 + \frac{Ka^2}{b} + \frac{1}{2r}(q-K)^2\right)\|U_i(t)\|^2. \\
\le\, & \frac{4(1+m)}{r^*\min\{C_1,1\}}\left(C_2 + \frac{C_1^2}{32}\right)|\Omega|\left[\eta_2\, dK|\Omega| + \left(K^2 + \frac{Ka^2}{b} + \frac{1}{2r}(q-K)^2\right)\right]
\end{aligned}
\tag{8.35}
$$

for $t > T_B$, where $T_B > 0$ is a constant depending only on the set B. The differential inequality (8.35) is written as

$$
\begin{aligned}
&\frac{d}{dt}(K\|U_i(t)\|^2 + \|V_i(t)\|^2 + \|W_i(t)\|^2) + 4Kp\int_{\Gamma_i} U_i^2(t,x)\,dx \\
&+ 2\,\eta_1 dK\,\|U_i(t)\|^2 + \|V_i(t)\|^2 + r\|W_i(t)\|^2 < 4KR\,|\Omega|, \quad t > T_B.
\end{aligned}
\tag{8.36}
$$

The constants $K = 8\beta^2/b$ in (8.30) and $R > 0$ in (8.25) are independent of initial data.
Since the absorbing property (8.21) of the solution semiflow implies that there exists a sufficiently large

$$\tau = \tau(g^0, g_1^0, \cdots, g_m^0) > 0$$

depending on the initial data such that

$$\|g(\tau, g^0)\|^2 \le \frac{M|\Omega|}{\min\{C_1,1\}}, \quad \|g_i(\tau, g_i^0)\|^2 \le \frac{M|\Omega|}{\min\{C_1,1\}}, \quad 1 \le i \le m, \tag{8.37}$$

we see that the condition (8.24) with (8.25) on the stimulation signal strength of the boundary coupling $p\int_{\Gamma_i} U_i^2(t,x)\,dx, 1 \le i \le m$, is the threshold crossing inequality

$$4Kp\int_{\Gamma_i} U_i^2(t,x)\,dx > 4KR\,|\Omega|, \quad t > \tau. \tag{8.38}$$

It signifies that the stimulation signal strength of boundary coupling $p\int_{\Gamma_i} U_i^2(t,x)\,dx$ exceeds the synchronization threshold $R|\Omega|$. Therefore, from (8.36) we obtain the following differential inequalities: For $i = 1, \cdots, m$,

$$\begin{aligned}
&\frac{d}{dt}(K\|U_i(t)\|^2 + \|V_i(t)\|^2 + \|W_i(t)\|^2) \\
&+ \min\{2\eta_1 d, 1, r\}(K\,\|U_i(t)\|^2 + \|V_i(t)\|^2 + \|W_i(t)\|^2) \\
\le\; &\frac{d}{dt}(K\|U_i(t)\|^2 + \|V_i(t)\|^2 + \|W_i(t)\|^2) \\
&+ 2\,\eta_1 dK\,\|U_i(\tau)\|^2 + \|V_i(\tau)\|^2 + r\|W_i(\tau)\|^2 < 0, \quad t > \tau.
\end{aligned} \tag{8.39}$$

Finally we apply the Gronwall inequality to (8.39) and reach the conclusion that for all $i = 1, \cdots, m$,

$$\begin{aligned}
K\|U_i(t)\|^2 + \|V_i(t)\|^2 + \|W_i(t)\|^2 \le &e^{-\mu(t-\tau)}(K\|U_i(\tau)\|^2 + \|V_i(\tau)\|^2 + \|W_i(\tau)\|^2) \\
\le &\,2e^{-\mu(t-\tau)}\max\{K, 1\}Q \to 0, \quad \text{as } t \to \infty,
\end{aligned} \tag{8.40}$$

where $\mu = \min\{2\eta_1 d, 1, r\}$ is the uniform exponential convergence rate. Thus for any $j, k = 1, \cdots, m$, we have

$$\begin{aligned}
&\sup_{g_j^0, g_k^0 \in H} \left\{ \limsup_{t\to\infty} \|g_j(t, g_j^0) - g_k(t, g_k^0)\|_H \right\} \\
\le\; &\sup_{g_j^0, g^0 \in H} \left\{ \limsup_{t\to\infty} \|g_j(t, g_j^0) - g(t, g^0)\|_H \right\} \\
&+ \sup_{g_k^0, g^0 \in H} \left\{ \limsup_{t\to\infty} \|g(t, g^0) - g_k(t, g_k^0)\|_H \right\} \to 0, \quad \text{as } t \to \infty.
\end{aligned} \tag{8.41}$$

Therefore, it is proved that

$$deg_s(\mathrm{H}) = \sum_{j=0}^{m}\sum_{k=0}^{m} \sup_{g_j^0, g_k^0 \in L^2(\Omega, \mathbb{R}^3)} \left\{ \limsup_{t\to\infty} \|g_j(t) - g_k(t)\|_{L^2(\Omega,\mathbb{R}^3)} \right\} = 0.$$

Here we denote $g_0(t, g_0^0) = g(t, g^0)$ for $i = 0$. It shows that the boundary coupled Hindmarsh-Rose neuron network generated by (8.4) is asymptotically synchronized in the space $H = L^2(\Omega, \mathbb{R}^3)$ at a uniform exponential rate. The proof is completed. $\square$

Remark 8.3.3. The main result of this paper shows the asymptotic synchronization of a boundary coupled Hindmarsh-Rose neuron network *in the local sense* of multiple neurons around a central neuron. The biological interpretation of the assumption (8.24) is that the product of the *boundary coupling strength* represented by the coefficient p and the *accumulated stimulating signals on the coupling piece* Γ_i *in a long run* represented by

$$\liminf_{t\to\infty} \int_{\Gamma_i} U_i^2(t,x)\, dx \quad \text{for } i = 1, \cdots, m$$

exceeds the threshold constant $R|\Omega|$.

Remark 8.3.4. Mathematical models for neuron dynamics such as the Hodgkin-Huxley equations (HHE) and the FitzHugh-Nagumo equations (FHN) for number of neurons may include intercellular voltage-currrent coupling by the Kirchhoff law inside the interior domain in the distributional sense for depolarized and hyperpolarized neuron cells. But biologically a partly diffusive model with the boundary coupling given by the Robin boundary condition (8.2) would be more realistic for the neuron network dynamics for two reasons. First, this boundary coupling represents exactly the Kirchhoff law across the cell membrane of two neurons through biological synapses. Second, the bio-electric potential signals are mainly related to the first component u-equations. Certainly this work can be extended to HHE and FHN neuron networks.

Remark 8.3.5. For space dimension $n = 1$, the synchronization result in this work can be biologically interpreted as the chain-like or ring-like neuron networks, in which each neuron is coupled with the two neighbor neurons at the end of the axon. For space dimension $n = 2$ or $n = 3$, reasonably the mathematical domain in a neuron network model needs not to exactly reflect the mostly unknown ensemble configurations of real biological neuron cells. Here the essential information we acquired is the quantitative synchronization threshold expressed by the biological parameters, which may give us better understanding or insight regarding the roles of key parameters and how to improve or control the synchronization or desynchronization for different purposes.

Remark 8.3.6. The main theorem in this paper provides a sufficient condition for realization of the asymptotic synchronization of this kind boundary coupled neuron network. The threshold for triggering the synchronization may possibly be reduced through further investigations. Besides, one can explore the cases for the neurons in a network to have different parameter values by the same approach and expect to reach the same type of result.

Remark 8.3.7. As a corollary of Theorem 8.3.2, the proof of (8.35) to (8.40) shows that the neuron network present in this work can be partly synchronized if the condition (8.24) is satisfied only for a subset of the neighbor neurons indexed by $i \in I_{sub} \subset \{1, \cdots, m\}$.

Chapter 9

Future Research Directions

There is an essential difficulty in proving the pullback asymptotic compactness of the Hindmarsh-Rose random dynamical system with the additive noise for the space dimension $n = 3$. The proof in Theorem 6.3.1 and Theorem 6.3.2 are constrained to $n \leq 2$. I will continue to work on the case of three-dimensional domain toward the conjecture that there should exist a random attractor by alternative approaches.

My future research will include some of the following problems and topics:

- To investigate the deterministic neural networks described by the FitzHugh-Nagumo equations widely exposed to potential and eminent applications.
- To explore the internal structure of the global attractor and its connection to phenotype neuron bursting patterns for the partly diffusive Hindmarsh-Rose equations starting from the one-dimensional domain by the approaches of pattern recognition, multi-scale approximation, topological degree, semi-discretization, and other tools.
- Synchronization of the random neural networks with models of SDE or SPDE.
- To generalize the asymptotic synchronization results of the neuron networks to the complex and artificial neural networks with computational estimation and control of the convergence rate.
- To study the synchronization dynamics of the considered Hindmarsh-Rose neuron network but the coupling terms involve time delay, which is a difficult and meaningful problem.

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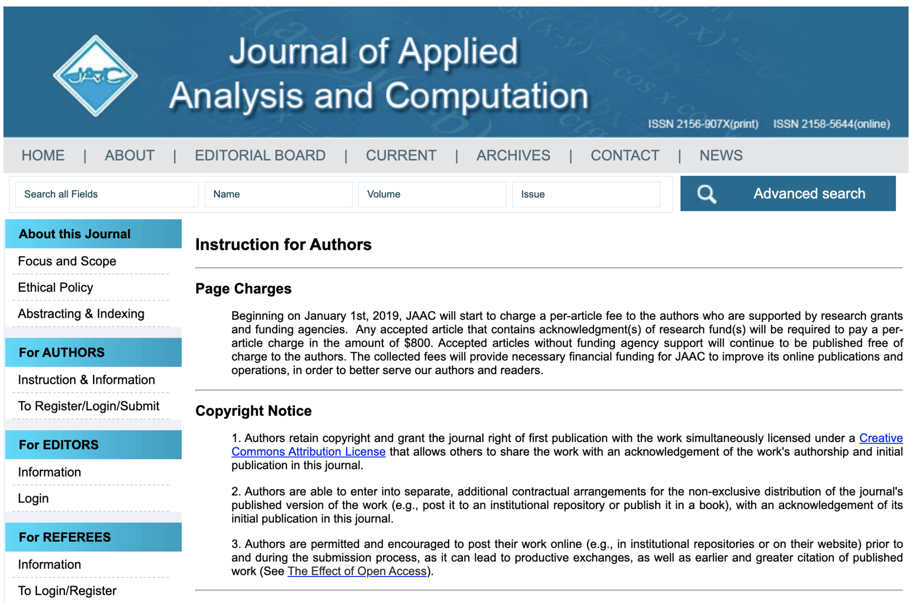

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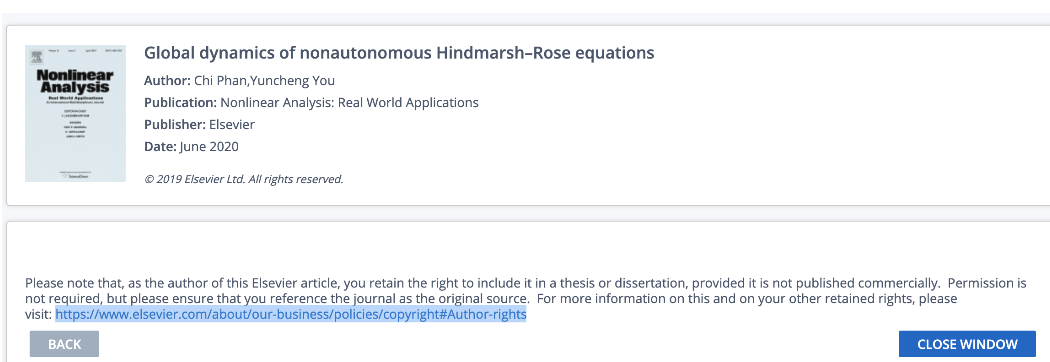

Global dynamics of nonautonomous Hindmarsh–Rose equations

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Appendix B
Publications

[1] C. Phan, Random Attractor for Stochastic Hindmarsh-Rose Equations with Multiplicative Noise, *Discrete and Continuous Dynamical Systems, Series B*, Vol. 22 (2020), doi: 10.3934/dcdsb.2020060.

[2] C. Phan and Y. You, Random Attractor for Stochastic Hindmarsh-Rose Equations with Additive Noise, *Journal of Dynamics and Differential Equations*, online published (2019). http://doi.org/10.1007/s10884-019-09816-4.

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Appendix C
Conference Presentations

[1] *The* 39^{th} *Southeastern-Atlantic Regional Conference on Differential Equations*, Random Attractor for Stochastic Hindmarsh-Rose Equations with Multiplicative Noise, Daytona Beach, FL, October 26, 2019.

[2] *AMS Sectional Meeting* $\#1152$, Special Session on Applications of Differential Equations in Mathematical Biology, Global Attractors for Hindmarsh-Rose Equations in Neurodynamics, Gainesville, FL, November 3, 2019.

[3] *Joint Mathematics Meeting 2020*, AMS Special Session SS74A on Stochastic Differential Equations and Application of Mathematical Biology, Random Dynamics for Stochastic Hindmarsh-Rose Equations with Multiplicative Noise, Denver, CO, January 17, 2020.