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15. Practice and Mentoring

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Practice and Mentoring

© Cymra Haskell,
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1 Introduction

Explaining is a basic human activity, in which we all engage to a greater or lesser extent. How many of us have given directions to a newcomer in town? How many of us have helped someone with their homework or explained to a friend how we cook one of our favorite recipes? In this chapter we will explore the specific art and practice of explaining quantitative and mathematical material and discuss how to encourage tutors in a QMaSC to develop and refine their craft.

Explaining seems like a natural activity, so some might assume that competent practitioners of an art will be competent tutors of that art. However, think of a time when you were given directions but were still unable to find your way. The problem may not have been that the person you asked didn't know how to get to your desired destination, but they were not able to communicate that information to you. Perhaps they gave you too little information, so you missed a turn or got lost. Alternatively, they may have given you too much information, so you forgot some of it or simply couldn't see the big picture. If something as simple as giving someone directions can require knowledge and skills beyond that of knowing how to get to the desired destination, it is not hard to understand the potentially large disconnect between being a competent practitioner and a competent tutor of complex quantitative or mathematical material. Section 2 will examine what good tutoring of quantitative and mathematical material looks like and the skills and knowledge a person needs to be an effective and masterful tutor.

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The art and practice of tutoring involves both strong interpersonal skills as well as quantitative or mathematical habits of mind and deep content knowledge. Although a QMaSC is not a school of education, there are things it can do to enhance the effectiveness of its tutors. Section 3 will explore some of these and, in particular, will discuss mechanisms for mentoring tutors and how mentoring can accelerate the progress of tutors toward becoming masterful tutors. By providing its tutors with the framework, time, and space to refine their craft, a QMaSC is actually providing them with a very rich opportunity to develop both personally and professionally. In particular, tutors will deepen their content knowledge by asking questions they might never otherwise have thought to ask and by exploring connections between different areas in their subject they might never otherwise have explored.

2 The Art and Practice of Tutoring

Tutoring quantitative and mathematical material is an incredibly complex activity that draws upon both interpersonal skills as well knowledge, skills, and attitudes of mind specific to the subject. In this section we attempt to identify more precisely what these entail. The act of articulating them by necessity considers them distinct, however, each skill reinforces and draws upon the others, so in reality they do not fall into neat packages.

It is unrealistic to expect every tutor to possess all the knowledge and skills listed here. Instead, we attempt to articulate the knowledge and skills of masterful tutors. In the next section we will discuss mechanisms for mentoring tutors to increase their knowledge and skills and accelerate their progress toward becoming a masterful tutor.

Interpersonal Skills

To learn new material students need to venture outside of their comfort zone in order to broaden it. They can only do this if they feel secure and confident. It is the responsibility of the tutor to create an environment conducive to this. This involves a delicate juggling act between providing encouragement and validation while simultaneously addressing and correcting errors and misinterpretations. It also requires sensitivity and the ability to read body language in order to assess how the student is feeling. One mechanism tutors may use to create this environment is interacting with students at other levels; for example, they may discuss their hobbies or hopes and aspirations.

masterful tutors solicit input from students and listen carefully in return in order to assess students' understanding. This kind of listening is a little different from the kind of listening we do when chatting with our friends. Accurately articulating thoughts and understanding is very difficult at the best of times; it is all the more difficult for a student who is learning a new subject. As a consequence, students often express themselves imprecisely. masterful tutors listen empathically,

expecting to find correct understanding but also being very careful not to let incorrect understanding go unnoticed. This is another delicate balancing act involving a host of skills, including familiarity with cultural norms and expressions as well as familiarity with common misunderstandings and misinterpretations of the material. One mechanism tutors may use to ensure they are assessing a student's understanding correctly is to rephrase what they have understood and verify with the student this was their intention.

masterful tutors are able to juggle the needs of students with the demands of the discipline. Students, learning new material for the first time, will often say things imprecisely or use incorrect mathematical notation. Constantly correcting their language and notation can undermine their confidence. A tutor will balance the needs of the discipline as regards attention to detail with the needs of maintaining student confidence. One mechanism for doing this is to let imprecisions ride while a general discussion is happening but then insist on correct notation as the student gains understanding and confidence.

masterful tutors do not let themselves be drawn into competition with students. Although this may sound almost ridiculous to have to articulate, it can occur. It is likely to happen when a tutor is explaining something to two or more students, one of which already has a relatively firm grasp of the content. This student may interrupt and try to guide the conversation to use an approach that makes sense to them but is different from the tutor's approach. Some tutors may feel the need to demonstrate their understanding is as great as this student's understanding and will react to this by skipping over the explanation, only dwelling on the technical points, or diminishing this student's approach or contribution. The effect is the other student has been abandoned and the confidence of the student that had a firm grasp of the material may have been undermined. Instead, masterful tutors do not feel any need to compare their understanding of the material with students' understanding and have the self-confidence to recognize and relish occasions when a student's understanding surpasses their own. These tutors remember that their obligation is to ensure all students understand the material and all students are validated in their own thinking. They will carefully consider the approach suggested by the student. If they think it is a better approach than their own, they will adopt it. If they think it is problematic, inferior to their own, or is an approach that does not permit them to address an issue that needs addressing, they will still validate the student's contribution, but continue with their explanation and possibly return to the student's approach when they are done. Of course, being able to do this requires the tutor to have a solid enough command of the subject to effectively evaluate alternative approaches and their implications.

Masterful tutors do not talk too much. It is a common complaint of students that they are able to follow the work of a tutor or teacher, but they are not able to do problems by themselves. To address this issue, masterful tutors make sure students drive discussions as much as possible. Some mechanisms tutors use to ensure they are doing this is to have the student be the one holding the

pencil or to simply stop themselves, even in midstream, and consciously ask the student for input. To be effective in this, tutors have to be comfortable enough themselves with the material so, when a student takes an unexpected, but not incorrect, approach, they are able to assess the situation and allow the student to continue. They also need to reflect on how and why they do problems the way they do and be able to assess when other approaches are mathematically sound and will lead to correct solutions.

Masterful tutors are able to motivate students and engage them in their learning. They are able to generate excitement in the student regarding the material and a desire to perform well. One mechanism they may use to do this is to show excitement themselves or to talk about how the subject has been relevant to them, be it in 'real' life or in their other classes.

Knowledge, Skills and Attitudes of Mind Specific to the Subject

Masterful tutors have a sound knowledge of the subject they are tutoring. They are able to do the exercises required of the students and have performed or could perform well on a test on the content. However, competence in a subject as measured by the ability to excel in a test on the material does not constitute sufficient knowledge for masterful tutoring. In addition to competency, masterful tutors have both an attitude of mind similar to researchers and a profound understanding of the specific content they are tutoring. In the next few paragraphs we attempt to articulate more precisely what this attitude of mind and understanding encompasses.

Quantitative subjects tend to be very precise subjects in which every idea builds on previous ideas. Students do not usually learn in such a linear way; they have jumps and holes in their understanding of the big picture, and they go back later and fill in the holes. When explaining material, masterful tutors accommodate human learning styles by painting the big picture but also accommodate the needs of the discipline by carefully and accurately filling in the details. Most tutors who perform well on tests can do the latter but may have trouble with the former. Masterful tutors have done more than develop excellent problem-solving skills in the subject. They have reflected on the nature of the processes and procedures used, why they are valid, how they relate to other procedures, and why they are helpful in the situation at hand. They can articulate the larger goal and combine all the little steps into groups of steps, each of which has an intermediary goal. In this way they are able to help the student see the bigger picture.

Masterful tutors can solve a single problem in multiple ways and are cognizant of the principles on which these different ways are based. They understand the advantages and disadvantages of the different methods and when those methods are applicable. They have open minds, realize there are doubtless other methods of solution they have not yet entertained, and they are willing and able to assess the advantages and disadvantages of these other methods.

Masterful tutors are familiar with common misconceptions held by students who are learning

the subject for the first time and with common errors made by students. Moreover, they understand the sources of these misconceptions and errors. In particular, when the source of misunderstanding is an unconscious comparison of the new practice with another, previously learned practice, the tutor is able to identify the source, draw it to the attention of the student, and enter into a dialogue with the student comparing the two practices.

Masterful tutors are able to analyze the conceptual steps needed to master a particular quantitative practice and can identify the concept with which a student is having difficulty. Moreover, they can formulate examples and exercises on the spur of the moment that illustrate the concept and provide students with a mechanism for internalizing it and putting it into practice.

Masterful tutors are familiar with different models and metaphors used for understanding a mathematical practice. They are able to identify the metaphor to which the student is relating the practice. They are able to come up with examples that employ that metaphor and, if appropriate, are able to enter into a discussion with the student about other metaphors that also capture the practice.

Masterful tutors understand how the particular class relates to other areas of the discipline or to other disciplines. Not only does this give them the tools to generate excitement for the subject and motivate the questions being asked; but, by comparing a particular mathematical practice with a practice in another field of mathematics or another discipline, they can solidify students' understandings of that practice.

To aid in understanding the differences between competency in a mathematical subject and the more profound understanding needed to be a masterful tutor of that subject, we draw on an analogy inspired by Richard Skemp [1]. Consider the city of Los Angeles. This is a large city, situated in an even larger megalopolis covering hundreds of square miles. Some of the cities in the megalopolis (like Beverly Hills and Santa Monica) have completely merged with each other and a driver cruising in Los Angeles could pass from one to the other without even being aware of it. Other cities (like Temecula or Oceanside) are clearly separate from Los Angeles though there are many roads that connect these cities to Los Angeles and to each other. The megalopolis is analogous to the whole field of mathematics and Los Angeles is analogous to the subject area of mathematics being tutored. Students learning the subject for the first time are like tourists visiting Los Angeles; this may or may not be their first time visiting the city, so the city may be totally new to them or they may be somewhat familiar with some parts of it. Tutors who have mastered the subject as measured by their ability to excel on a typical test on the material but do not yet have the profound knowledge of a masterful tutor are like typical residents of Los Angeles. They are familiar with their neighborhood and have learned the quickest routes to get from home to work and from home to their favorite haunts in the city. However, when they venture out of these familiar stomping grounds they can get lost and they may not even know the most direct route from one of their favorite haunts to another without going via home. They do not have the big

picture of the city and they are not aware of how the city is laid out and how the different parts of the city are interconnected. masterful tutors are like taxi drivers in the city. They are very familiar with the whole city and, given any two points in the city they can find many routes from one to the other; to a tourist wanting to sample the delights of the city they will give one set of directions and to the business man trying to get there in the most efficient way possible they will give another. They can probably also find their way from Los Angeles to one of the neighboring cities like Temecula, though they won't have the same familiarity with Temecula as they do with Los Angeles.

In Appendix B, three hypothetical tutoring sessions are presented and discussed. The three sessions differ primarily in the tutors' knowledge, skills and attitudes of mind specific to the subject and illustrate how important these are for effective tutoring.

3 Mentoring Tutors in Order to Improve their Art

In whatever way your QMaSC decides to train and mentor tutors, it is important to remember and practice all the principles of teaching you are seeking to elicit from the tutors. In particular, you should endeavor to build their self-confidence, not break it, you should make them feel you are their ally and not their judge, you should encourage them to critically evaluate their own effectiveness as tutors, and you should empower them to own their learning and try out new techniques.

We have identified three different (but interrelated) activities that will accelerate the progress of tutors toward becoming masterful tutors. We recommend that QMaSCs find ways to ensure their tutors are engaging in these three activities.

Identifying the Knowledge, Skills, Attitudes of Mind, and Behavioral Traits of a Masterful Tutor

Unlike mathematics in which every step is carefully constructed and can be pondered for several minutes, hours, days, or even years, tutoring happens in real time. When we tutor we make thousands of largely unconscious decisions every minute. Will we speak or remain silent? Will we smile or frown? What timbre of voice will we use? Will we take hold of the pencil ourselves or insist that the student write? Will we interrupt to address an issue or let the student continue? Which misunderstanding will we address first? Will we insist on proper notation or let it ride? Will we address an issue at a purely procedural level or will we embark on a lengthier conceptual discussion of it? In order to adopt the largely unconscious behavioral traits of a masterful tutor it can help enormously to articulate exactly what it is that constitutes these traits. Since these traits form a complex set of contextual behaviors, articulating them is a far from easy task and is something that should be an ongoing process. We can help tutors in this process by directing

them to literature on the subject as well as providing the framework, time, and space for in-depth discussions. Some examples of literature on the subject include ‘Knowing and Teaching Elementary Mathematics,’ by Liping Ma [2], ‘Relational Understanding and Instrumental Understanding,’ by Richard Skemp [1], ‘Measuring Teacher Quality in Practice,’ by Deborah Ball and Heather Hill [3], a report put out by the Conference Board of the Mathematical Sciences, ‘The Mathematical Education of Teachers II’ [4], and Appendix B of this chapter. Periodic assignments of such readings followed by group meetings of the tutors for discussion can be valuable for advancing tutors toward becoming masterful tutors.

Watching, Analyzing, and Reflecting on Tutoring Sessions

Articulating the traits of a masterful tutor is one thing, internalizing and adopting those traits is another. Watching, analyzing, and reflecting on tutoring sessions can accelerate the adoption of these traits. The challenge here is to provide tutors with these sessions. Three ways to do this are given below.

All tutors have experienced their own tutoring sessions, so they should be encouraged to reflect critically on these sessions. To this end a QMaSC might consider requiring its tutors to keep a journal cataloging their tutoring sessions including reflections on the sessions. To stimulate deeper reflections, the tutors could be provided with prompts and there could be regular group meetings in which the tutors discuss each other’s experiences. We give two examples of such prompts below.

Reflective Prompt 1: Choose one of your tutoring sessions in the QMaSC and answer the following questions.

- a) Describe the student’s problem.
- b) What are the potential misunderstandings that a similar student might have when trying to solve such a problem?
- c) What questions did you ask the student in order to assess their level of understanding and any misunderstandings they may have? What were their responses? Did you anticipate most of their responses or were you surprised by any of them? What were the main things that prevented the student from being able to solve the problem independently?
- d) How did you address these issues in your session with the student?
- e) What other problems did you develop in order to cement the student’s understanding of the concepts that you worked on with him or her?
- f) In hindsight, can you think of any ways you may have improved this session?

Reflective Prompt 2: Choose one of your tutoring sessions in the QMaSC in which your initial approach to solving the problem was not the approach you finally settled on with the student. Describe the student’s problem. Explain your initial idea of how to solve the problem. Explain how

you ended up solving the problem with the student. Describe why and how you ended up doing the problem in a different way with the student. Did you also explain your way to the student? Why or why not?

A second way to provide tutors with tutoring sessions to watch, analyze and discuss is to have tutors observe each others' sessions. This can be very valuable though its success depends significantly on the way it is implemented. The problem is three-fold. If the observer is not a masterful tutor, she may not notice missed opportunities or imagine alternative approaches the tutor could have taken. Secondly, the observer is often not critical enough for fear of hurting the other tutor's feelings. Thirdly, since tutoring involves a huge number of split-second decisions that can be based on the tutor's highly nuanced assessments of the student's level of understanding, the observer may forget the particular criticisms she had and, even if she does remember, the observer and/or the tutor may have forgotten the exact contexts in which they arose. This method works best when the observer is herself a masterful tutor, the tutor trusts the observer and can constructively handle criticism from her, and no time elapses between the session and the discussion of the session.

A third way to provide tutors with tutoring sessions is to use videos of sessions. There are many advantages to watching, analyzing, and discussing a tutoring session on video as compared to one that has just been observed. One big advantage is that the viewers do not need to rely on their memories when discussing particular events in a video; instead, they can zoom in on those few minutes. A second advantage is that you can ensure you are watching a session that provides a rich source of material for discussion. If the video does not involve tutors in your QMaSC, a third advantage is that tutors watching the video won't feel compelled to mitigate their criticism in order not to hurt the tutor's feelings. However, using videos requires the QMaSC to have a rich source of video material. A QMaSC can explore the possibilities of making videos of sessions in their own QMaSC; however, because of issues relating to student confidentiality, this may not be possible at all or may require the students involved to sign permission forms. Commercial or online sources of videos are mostly of classrooms rather than tutoring sessions but these can still be highly valuable for discussing many of the issues that arise in tutoring. One commercial source is the book and DVD by Jo Boaler and Cathy Humphreys 'Connecting Mathematical Ideas' [5]. The DVD contains videos of Cathy Humphreys teaching in a middle school classroom and the book has written commentary by both her and the mathematics educator, Jo Boaler. On the Max Ray Blog [6] there is a collection of links to videos of people doing and teaching mathematics.

Engaging in Profound, Professionally Oriented Explorations of the Subject

As we have seen, the profound, professionally oriented understanding of the master tutor goes beyond that of a person who can excel in a test on the subject. Furthermore, it is unrealistic to

expect tutors who are newly hired in the QMaSC to have this kind of understanding. If tutors do not obtain this understanding in their classes, when, where, and how will they obtain it?

In her comparison of elementary school teachers in China and the United States [2], Liping Ma found that the Chinese teachers in her study who had profound understanding had obtained it primarily on the job through the study of teaching materials, mathematical and pedagogical discussions with their colleagues, and deep reflection on the material they were teaching and the students' responses to their teaching. If a QMaSC wants its tutors to gain this understanding, it is important that it provide them with the time, space, and framework for engaging in such study, discussions with their colleagues, and reflections. To that end we recommend it organize professional development workshops for its tutors. A good format for such a workshop is to divide the tutors into small groups (say, three or four tutors each) and give each group a mathematical question to ponder and discuss. The question could be the same for each group or different for the different groups. After a period of time the groups reconvene into one large group, each group presents their thoughts and findings to everyone else, and a discussion follows. (Chart paper can be useful for this.)

Appendix C provides examples illustrating what such workshops might look like. The modules in this appendix were developed by the author of this chapter for a class in the Department of Mathematics at the University of Southern California called 'The Foundations of Mathematics and the Acquisition of Mathematical Knowledge.' In this class K-12 mathematics is used as a conduit for a deeper exploration of mathematics itself and for an exploration of how we acquire and learn mathematics. Most of the activities in this class are suitable for professional development workshops for tutors in mathematics. Some of the modules you'll find in the appendix are based on elementary mathematics and might not, at first, seem relevant to math tutors in a QMaSC situated in a college. However, they have been included for a variety of reasons. First of all, these workshops should be accessible to all tutors in the QMaSC, so they could be done in a group meeting of all the tutors together. Secondly, they certainly do not preclude very rich mathematical discussions and they provide a good starting point for all tutors; some of the issues that arise in more advanced mathematics stem from these issues in elementary mathematics and starting with examples in more advanced mathematics may be too intimidating for some tutors. Thirdly, for QMaSCs that service courses outside of mathematics departments, the issues raised in these workshops might indeed be relevant to the material they are tutoring. Finally, since elementary mathematics will be very familiar to all the tutors in your QMaSC, providing professional development in this area will serve the purpose of highlighting the kind of profound and professionally-oriented knowledge for which tutors should be aiming. This will encourage them to look for that kind of knowledge in the more advanced material they are tutoring. There are also other modules in the appendix that involve college level mathematics.

For many years, mathematics educators at the University of Michigan have been studying the

mathematical knowledge needed for teaching mathematics at the elementary and middle school levels. Their work is described in [7]. More information can be found at the website of the Study of Instructional Improvement [8] or the website of the Learning Mathematics for Teaching project [9]. They have released a series of questions that are posted on these websites that could be used as the framework for one or more professional development workshops. Although the mathematical material in many of these questions is probably less advanced than the subject matter being dealt with in a QMaSC that is housed in a college, as we noted in the paragraph above, having professional development workshops on such material can be very valuable and we recommend their use. In particular, many of the questions concerned with the appropriateness of examples or whether examples provide illustrations of particular metaphors, can provide good food for thought and stimulate tutors to start thinking about the more advanced mathematics they are tutoring in these terms.

4 Summary

Tutoring in a QMaSC is a highly sophisticated activity that involves well-developed interpersonal skills as well as a profound understanding of the subject matter. As such it provides rich growth and learning opportunities, and tutors will undoubtedly develop both personally and professionally through their employment in a QMaSC. In this chapter we have attempted to articulate some of the skills and knowledge needed to be a masterful tutor and have suggested some steps QMaSCs can take in order to accelerate the progress of their tutors toward becoming masterful tutors.

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6 Appendix A: The QMaSC as a Community

In this chapter we discussed the art of tutoring and saw that by providing tutors with the framework, time, and space to refine their craft, a QMaSC not only improves the tutoring services it provides but also provides the tutors with a valuable opportunity to develop personally and professionally. This, of course, benefits the institution, its students, and, most especially, the tutors themselves.

Some QMaSCs may hire students in administrative roles as well as, or instead of, as tutors. Although employment in this capacity doesn’t provide students with the same learning opportunities as tutoring, it still has advantages both for the students and for the institution or department in which the QMaSC is housed. In particular, it provides students with the opportunity to do a job that is meaningful to them (rather than, for example, wash dishes at the cafeteria), it provides them with an opportunity to demonstrate responsibility, and it is a valuable work experience they can put on their resume. Students working in a QMaSC will feel they are contributing to the mission of the institution (which, indeed, they are) and, as a result, will be more invested in the institution and their own degree. This will probably result in greater retention and recruitment of students. Most importantly, through their employment, the students may get to know the faculty and other students (including graduate students, should the institution have a graduate school). These connections can be invaluable to them. Not only will they draw upon them when making important life decisions, including what courses they should take, whether or not they should apply to graduate school (for undergraduates), and to what kinds of jobs they should apply, but through them they will learn what it means to be a mathematician or other academic professional. This can help them imagine themselves in one of these roles and can provide motivation in their classes. Of course, it also benefits the institution when students are happy, motivated, and successful.

Notice that essentially all of these benefits arise out of a strong sense of community within the QMaSC and the department in which it is housed, and the best way for the QMaSC to enhance these benefits is to enhance the sense of community. The easiest way to do this is to have social gatherings with food! The power of these gatherings cannot be overemphasized. They could be

gatherings for just the employees of the QMaSC, or they could include faculty and/or graduate students in the departments served by the QMaSC. Employees should also be encouraged to suggest improvements in how the QMaSC is run, and regular group meetings of the employees to discuss the running of the QMaSC can be very valuable both for ensuring the employees have a common vision and for soliciting feedback from the people involved in the day-to-day affairs of the QMaSC. It is good to blur the lines between the responsibilities of different types of employees; however, this has to be done with care. It can be problematic if an employee who is hired essentially as an administrator is not qualified to act as a tutor takes on the role of tutor. Also be careful not to oblige employees to take on responsibilities that are not in their job description and might make them feel overwhelmed. Soliciting employees' help and allowing them to have a say in their responsibilities is a good idea so all employees have a sense of being valued for any contribution they can make to the QMaSC. If you do not package your employees in a box you are more likely to access the personality traits, unique to them, that will benefit the QMaSC, while also giving them the sense of being valued.

7 Appendix B: Examples Showing Levels of Tutor Understanding

To illustrate some of the knowledge, skills and attitudes of mind of a masterful tutor, we'll consider a scenario in which a student walks into a QMaSC with a question and we compare the hypothetical responses of different tutors. Since many readers of this chapter may not be mathematicians we have deliberately chosen an example involving elementary mathematics to which essentially all readers will be able to relate. This serves the added purpose of highlighting the sophisticated mathematical thinking in which masterful tutors engage even when tutoring elementary material. We will look at examples involving more advanced mathematics in Section 3.

Example: To set the stage, imagine the QMaSC situated at a high school. A new student at the school and recent immigrant to the country has come to the QMaSC with a question about how to perform subtraction. He is attempting to evaluate $43 - 25$. His work is shown in the picture below.

$$\begin{array}{r} 543 \\ - 25 \\ \hline 38 \end{array}$$

The student knows he has the wrong answer because when he adds 38 and 25 he gets 63 instead of 43, but he doesn't know what he did incorrectly.

Here are the hypothetical responses of three different tutors in the QMaSC.

Tutor 1

Tutor: Can you draw a picture to illustrate what you did?

Student: What do you mean?

Tutor: Well, can you make up a story of when you might want to do this subtraction problem?

Student: How about, I have 43 candies and I give 25 of them to my friend. How many do I have left?

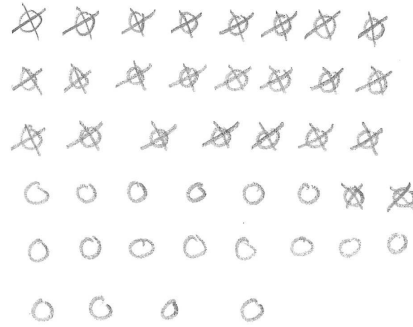
Tutor: Excellent! Can you draw me a picture now?

Student: You mean of the candies?

Tutor: Yes.

Student: Okay.

Student draws the picture below. (It takes some time.) He counts off 25 of the candies crossing them out as he goes indicating that he's giving them to his friend. He loses count a couple of times in the process but eventually goes back and corrects himself.



Tutor: So how many do you have left?

Student: 18!

Tutor: Excellent!

The session ends.

Tutor 2

Tutor: Oh, you're just making a small mistake; you need to take 1 away from the 4 instead of adding it.

Tutor points at the student's work.

Student: You mean like this?

Student works the problem as shown below.

$$\begin{array}{r} 3/4 \ 13 \\ 25 \\ \hline 18 \end{array}$$

Student: That would give me 18. Yes, that's right because $18 + 25 = 43$! Hmm, but I always thought we added.

Tutor: No, you should take 1 away. Let's do another problem to reinforce the idea. How about $64 - 37$?

The student works the problem correctly and the session ends.

Tutor 3

Tutor: Let's see. It is great that you noticed it was incorrect! Can you explain to me what you did?

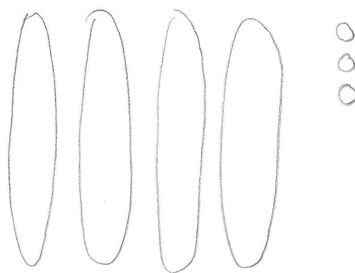
Student: Well, I want to take 5 from 3 but 5 is bigger than 3 so I can't. So I go to the ten's place. Since I'm giving a ten to the 3, I need to give it to the 4 as well. That gives me 5 here and 13 over here. Now 5 from 13 is 8 and 2 from 5 is 3.

Tutor: Good. You seem to have a really good understanding of what you're doing but you're making a mistake. Let's see if I can't help you find your mistake. As you noticed, the number forty-three is 4 tens and 3 ones. So, can you draw me a picture of 4 tens and 3 ones?

Student: You mean like rods?

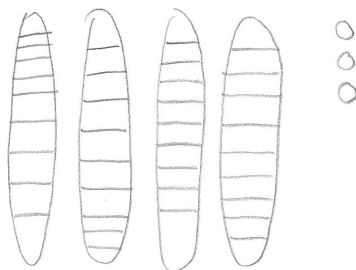
Tutor: Yes, exactly.

Student draws the picture below.



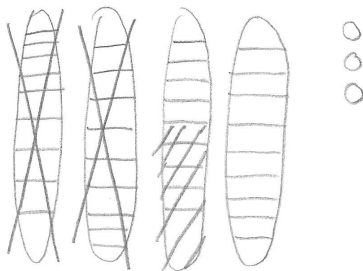
Tutor: Let's put some lines in the rods so that we can see that they are each composed of ten ones. Try to do it carefully so that all the ones look the same size.

Student modifies the picture as shown below.



Tutor: Okay, that'll work. Just remember that all the little pieces are the same size as each other. Now, we want to take away twenty-five which is 2 tens and 5 ones. How can we do that?

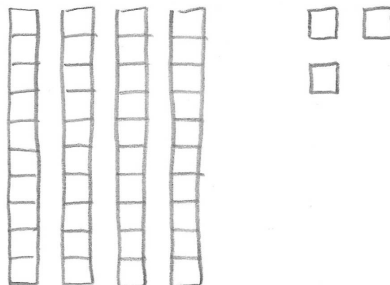
Student crosses out 2 tens and 5 ones as shown below.



Student: I have 18 left!

Tutor: All right. Good. We've got the correct answer. But we want to understand what's going on when we do the problem on paper so that we can find your mistake. So, let's try to copy the work you did on paper in our picture. Let's start again with our picture of 4 tens and 3 ones. Can you draw it again? Since we're doing it again anyway, try to draw it a little more carefully. Here, maybe we can use some graph paper.

Student draws the picture below.



Tutor: Now, what's the first thing you did when you tried to do this problem on paper?

Tutor points to the student's calculation on paper.

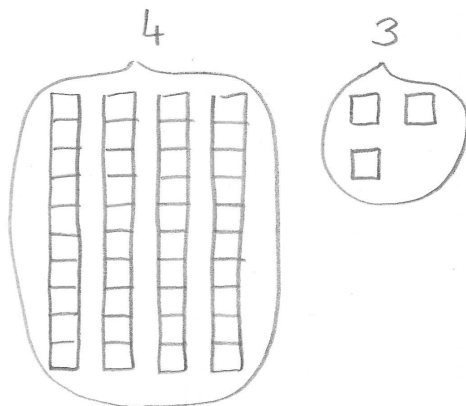
Student: I tried to take 5 from 3.

Tutor: That's right. Notice, that means that you're trying to take 5 ones from these 3 ones here.

Tutor points at the 3 single ones in the picture.

Tutor: Actually, let's label our picture so that we can really see how it relates to what we're doing on paper.

Tutor labels the picture as shown below.



Tutor: Now we can see that it is labeled as 4 tens and 3 ones which is how it looks on paper. Okay, since we can't take 5 ones from 3 ones, where are the extra ones going to come from?

Student: From the longer rods; we need to break up one of the longer rods.

Tutor: Exactly! We're going to break one of these long rods into 10 ones. Can you draw it?

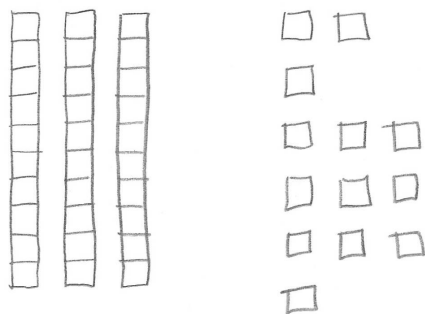
Student: What do you mean?

Tutor: Well, imagine taking this rod here and breaking it up into ten little pieces.
Tutor is pointing at one of the rods in the picture.

Tutor: All those little pieces need to be put over here with the ones. Can you draw what it'll look like now?

Student: Yes, I think so. Like this?

The student draws the picture below.



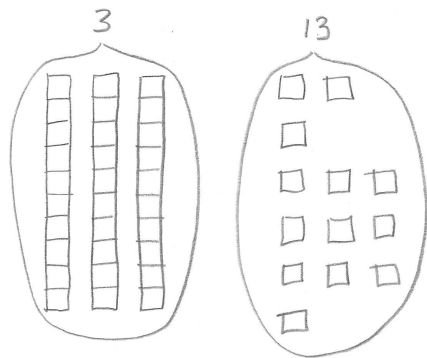
Tutor: Great! So how many tens and how many ones do we have now?

Student: We have 3 tens and 13 ones.

Tutor: Can you label that on the picture the way we did before?

Student: Like this?

Student labels the picture as shown below.



Tutor: Awesome! All right. Remember what we're doing. We're taking 2 tens and 5 ones away from these 3 tens and 13 ones. Let's compare that with what you wrote on your paper.

Tutor points to the student's original work.

Student: What do you mean?

Tutor: Well, we started with 4 tens and 3 ones. That's over here in the picture.

Tutor points to the 4 tens and 3 ones in the calculation on paper and in the picture.

Tutor: We wanted to take away 2 tens and 5 ones. On paper we started by trying to take 5 from 3.

Tutor points to the calculation on paper.

Tutor: That corresponded to trying to take 5 ones from the 3 ones here.

Tutor points to the 3 ones in the picture.

Tutor: We couldn't do that so we went over to the long rods and turned one of those long rods into ten ones.

Tutor is pointing at the picture.

Tutor: So over here on the paper, we crossed out the 4 and put a 1 in front of the 3 to indicate that we now had thirteen ones.

Tutor points to the calculation.

Tutor: But look at your picture. When you broke up that long rod into 10 pieces how many long rods did you have left?

Tutor is pointing at the picture.

Student: There are 3 left.

Tutor: Now look at what you wrote. How many tens did you say that you had left?

Tutor is pointing at the calculation.

Student: Oh, I see. Over here I have 3 long rods but over here I wrote 5.

Tutor: Yes. Can you correct your work now?

Student: So I should put 3 here instead of 5?

Tutor: Exactly! Can you finish it?

Student: Yes. Over here I take 5 ones from the 13 ones leaving 8 ones, and 2 tens from 3 tens leaving 1 ten. That's 18!

Tutor: Shall we try another example?

Student: Okay.

They do another example drawing both the picture and writing the algorithm down on paper. The student realizes that they should subtract 1 when they 'borrow' from the ten's place instead of adding one.

Student: But I'm confused. I've always added 1 to the ten's place.

Tutor: Really?

Student: Yes, I'm sure I always added 1 and I never seemed to have a problem before.

Tutor: Hmm. Were you using a different method? You're new here, aren't you? Did you learn how to subtract somewhere else?

Student: Yeah, I just moved here. I learned to subtract at home.

Tutor: Can you remember what you used to do?

Student: Yes, I think so. I did it like this.

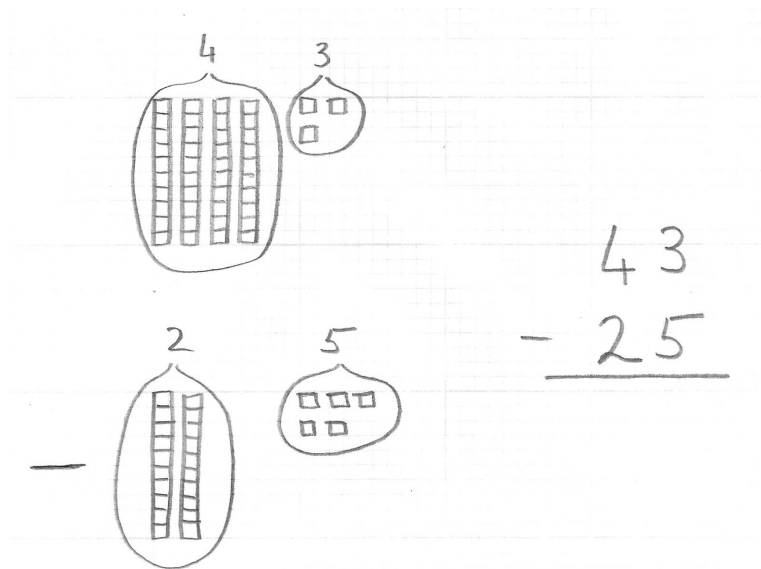
The student shows the example below.

$$\begin{array}{r} 43 \\ 325 \\ \hline 18 \end{array}$$

Student: We always added 1. I started adding to the top number when I came here, because that's what I saw people doing here. But it gives a different answer.

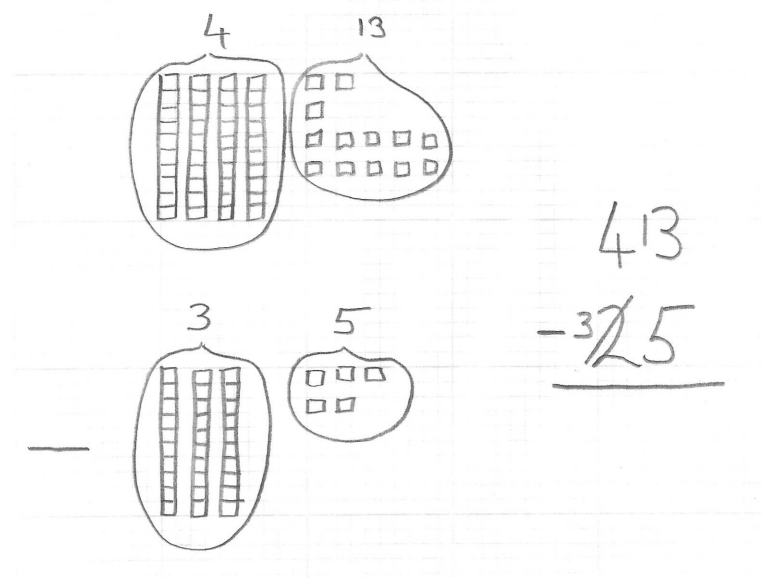
Tutor: Oh, that's interesting. Let's see if we can't work out what you're doing by drawing some pictures. We have the 4 tens and 3 ones that we had before. This time, since we're changing what we're taking away, we'd better draw a picture of what we're taking away too.

Tutor draws the picture below.



Tutor: Now let's see. You cross out the 2 in what you're taking away and add 1. That's in the tens place so you'll now have 3 tens. To compensate you add ten ones to the other number to get thirteen ones.

Tutor modifies the picture to obtain the picture below.



Tutor: Oh, that's cool! I see what we're doing. Do you understand how this works? Why can I add 1 ten over here and 10 ones over here?

Tutor is pointing at the picture.

Student: Oh yes! If we add ten to both terms then the difference is still the same!
That's cool.

Tutor: Right. We've changed the problem to a new problem that has the same answer! The new problem is to subtract 3 tens and 5 ones from 4 tens and 13 ones.

Student: Cool.

Tutor: Let's try to do another problem this way. Can you think of another problem for us to do?

Student: How about $83 - 57$?

Student and tutor work this problem using this other method, drawing pictures as they go along. The session ends.

Let's take a closer look at the three approaches taken by the different tutors to this situation. All three tutors had pretty good people skills and in no way made the student feel uncomfortable or stupid. All three tutors involved the student in the solution; they didn't just show the student how to do the problem but made sure the student was working the problem himself. All three tutors knew how to do the problem. (Although this is not obvious from the dialogue in the case of Tutor 1, we will assume it was the case in this hypothetical situation.) In summary, the tutors were similar in terms of their interpersonal skills and their ability to perform well on a test on the subject matter. Where the tutors differed was in the mathematical approaches they took to the problem and the degrees to which they engaged the student in a mathematical discussion.

Tutor 1 had excellent intentions in getting the student to think about the problem concretely. With mathematically less sophisticated students than the student in this example, this may have been an appropriate starting point and it probably wouldn't have been harmful as a starting point even with this student. What is problematic is that the tutor stopped there. The picture that was drawn and the method of calculation bore almost no relationship to how the calculation is represented and performed on paper. Indeed, the mathematical technique employed was that of counting. Although it is important for students to realize that counting is a valid approach to solving subtraction problems and is the theoretical underpinning of the subtraction algorithm, it is impractical to perform subtraction problems by counting when the numbers are large and that is the reason we use the subtraction algorithm. Indeed, this was highlighted by the fact that the student made multiple errors when trying to perform the calculation this way the first time. By stopping where he did, this tutor may have sent a dangerous message by suggesting that counting is the appropriate approach to take towards subtraction problems of this type. Moreover, the tutor missed an important cue; the fact that the student knew how to check his answer and actually did so is a strong indication that the student was more mathematically adept than one might first suppose in a student having trouble with the subtraction algorithm. This meant that the tutor's approach

was probably inappropriate for such a student. In the final analysis, although the student left with the correct answer to that particular problem, he gained essentially nothing from the interaction; he learned no tools for tackling subtraction problems in general and gained no understanding of the mathematics underlying the algorithm. In fact, the interaction may have been unintentionally harmful; by not attempting to access the student's mathematical thinking, this tutor may have sent a message that such thinking will only lead to confusion and it is better to do problems in a simplistic way.

Tutor 2 had a completely procedural approach to the problem. He identified the student's mistake and was effective at correcting the mistake without making the student feel stupid. He was adept at teaching the algorithm procedurally, as indicated by his ability to create an appropriate problem to solidify the student's mastery of the procedure, and the student probably left the QMaSC knowing how to perform the algorithm. However, the student learned little or no mathematics. Moreover, what drew the student to the QMaSC in the first place was confusion arising from having seen two different approaches to the algorithm, and this was never addressed. Like Tutor 1, this tutor missed the cue indicating the mathematical sophistication of the student. Moreover, the student gave him another cue that he also missed; when the student said, "I always thought we added," the tutor jumped to the conclusion that the student was remembering the standard algorithm used in this country incorrectly. In fact, the student had learned a different algorithm in which you do add instead of subtract; his problem arose when he came to this country, saw the algorithm used here, and mixed it up with what he had learned before. By not catching this cue, Tutor 2 missed the opportunity to validate the student's mathematical thinking and engage in a mathematical discussion with the student on the nature of the two algorithms. He also missed the opportunity to deepen his own mathematical thinking and understanding of the subtraction algorithm.

We can only speculate about why the first two tutors missed these important cues (especially since the example is purely hypothetical!). It was probably a combination of reasons. One reason may be that these tutors felt the need to be authority figures in the room and, as a result, were not mentally prepared to learn from the student. Another reason may be they did not think mathematical flexibility is important; once they had learned one way to solve a problem they didn't think it interesting or instructive to explore other ways. Yet another reason may be they were not very confident in their own mathematical abilities and were not accustomed to engaging in mathematical exploration of a problem; although they were good at learning procedural techniques they had not developed mathematically to the level where they were able to analyze and explore those techniques.

It is interesting to observe that although Tutor 1's approach in its use of manipulatives (in the form of pictures) was more modern than Tutor 2's, in this case he was probably less effective than Tutor 2, since his choice of manipulatives didn't relate to the way the problem is performed on

paper. At least with Tutor 2 the student left the QMaSC able to perform the subtraction algorithm correctly. That was not the case with Tutor 1.

With Tutor 3 we see masterful tutoring taking place and it is immediately recognizable as such. He started by congratulating the student on noticing his error thereby reinforcing the student's sense of his own mathematical competence. Before proceeding with any kind of explanation, he asked the student to explain his method of approach and listened carefully to the student's answer. This was instrumental in assessing the mathematical maturity of the student and his level of understanding of the subtraction algorithm. The student's explanation revealed he had a relatively sophisticated understanding of place value and he knew place value played a role in the calculation. The tutor hooked onto this immediately by asking the student to draw a picture of the number forty-three that illustrated how the number appears on paper. They then worked through the algorithm reinforcing each step with the picture, so that the pictures illustrated and motivated the algorithm. The tutor knew when to enforce precision and when not to do so; the first time the student drew 4 tens and 3 ones, the tutor did not insist that all the ones be the same size. The second time though, when the student had gained confidence, the tutor suggested they pay more attention and make it more precise. After completing the problem the tutor realized that, although the student had understood the error he had made in that particular problem, that didn't necessarily mean the student could recognize the algorithm he had in his head was incorrect. However, the tutor didn't feel compelled to articulate this himself for the student. Instead they worked through another problem together with the pictures and on paper, so the student saw again the mistake he was making when doing the problem on paper. Doing this permitted the student to identify the cause of his mistake himself. It also made space for the student to articulate the source of his confusion; namely that he had modified his method to account for what he saw people doing in this country. By this time the tutor knew the student was quite sophisticated mathematically, so rather than assume the student was incorrectly remembering the subtraction algorithm, he drew on his knowledge that this student was a recent immigrant and questioned him to find out what he might have been doing in the past. This led to the incredibly rich discussion and discovery on part of both the tutor and the student of this alternative way to do subtraction. The student left this encounter not only competent at performing the subtraction algorithm correctly, but with an increased sense of his own mathematical competence, a richer sense of the power of analytical and mathematical thinking, and a much deeper understanding of place value and the role it plays in subtraction.

There was a potentially rich discussion in which Tutor 3 did not engage with the student, be it consciously or unconsciously. The student said, "since I'm giving a ten to the 3, I need to give it to the 4 as well." What was the source of this misconception? Since we now know the student was accustomed to a different algorithm in which you give a ten to both the minuend and the subtrahend, this misconception probably originated in the explanation his teacher in his home

country gave for the algorithm. However, this student is a high school student probably studying algebra, so this misconception was no doubt reinforced by the idea that whatever you do to one side of an equation you should do to the other. There could have been a whole discussion of why you need to do this when you are working with an equation but you do not need to do this in this setting. Following the student's explanation of the algorithm, the tutor needed to make a choice; embark on this discussion or investigate the algorithm directly. The tutor chose the latter and that was probably the best choice. Without a good understanding of the algorithm, the student would probably have had difficulty understanding why you perform different actions to the number in the ten's place and the number in the one's place, so a discussion of this misconception at that point in time would probably have been confusing and unproductive. Unfortunately, after the student understood the algorithm, it is unlikely he would have had the patience to confront his former misconception and the source of that misconception. So, that was a discussion they didn't get to have. Tutors have to make these kinds of decisions all the time.

8 Appendix C: Examples of Professional Development Workshops for Tutors

This appendix contains examples of professional development workshops for tutors. They are modules developed by the author of this chapter for a class taught in the Department of Mathematics at the University of Southern California called ‘The Foundations of Mathematics and the Acquisition of Mathematical Knowledge.’ Engagement in such workshops will increase tutors’ profound, professionally oriented understanding of the subjects in which they tutor and accelerate their progress toward becoming masterful tutors. QMaSCs should adapt the workshops to fit the mathematical sophistication of their tutors and their time constraints.

Models for Multiplication and Division of Positive Numbers

Materials Needed: Ample supplies of chart paper, markers, and whatever is needed to attach the chart paper to the wall of the room for students to display their work. Moderator should be equipped with plenty of examples of story problems both using fractions and not using fractions that illustrate each interpretation of multiplication and division for distribution to the groups for item (11). These stories should contain multiple examples of each interpretation of multiplication and of each interpretation of division. They should also contain many examples involving multiplication and division by fractions. They could be written on individual cards or could be put together on a sheet of paper to be distributed to the groups.

Mathematics covered: We use models and metaphors to understand the arithmetic operations we do with numbers and, in applications, to know when we should use these operations. The primary model we use to understand addition of positive numbers is *combine* and the corresponding model for subtraction is *take away*. Notice these models break down when we expand our number system to include negative numbers. This module is about the models we use for multiplication and division. There are two primary models for multiplication, in one of which the two products play very different roles ($a \times b$ means a copies of b things) and in the other of which they play the same role (the area of a rectangle whose one side has length a and other has length b). The consequence of the second is that multiplication is commutative; when combined with the first this means there are actually *three* ways to interpret a specific expression $a \times b$. The three ways are distinguished by the *units* of the numbers involved. This in turn means there are three models for division.

Multiplication: The product $c = a \times b$ has three different interpretations.

1. a) It is the total size of a objects each of which has size b . In this case a is unitless, b can have units, and c has the same units as b or, more generally, a has units, the units of b are some (other) units divided by the units of a , and the units of c are those other units.
- b) It is the total size of b objects each of which has size a . In this case a has units, b is unitless, and c has the same units as a or, more generally, b has some units, the units of a are some (other) units divided by the units of b , and the units of c are those other units.
2. It is the area of a region that is a units in one direction and b units in the other direction. In this case a and b have identical units and the units of c are those units squared.

Notice how a and b play very different roles in (1)(a) and (1)(b).

Division: The quotient $a = c \div b$ has three different interpretations.

1. It is the the number of ‘pieces’ when something of size c is divided into pieces each of which has size b . In this case c has whatever units it has, the units of b are some (other) units divided by the units of c , and the units of a are those other units. This is called the *measurement model of division*. It is the ‘inverse’ of (1)(a) of multiplication.
2. It is the size of each piece when an object of size c is divided into b pieces. In this case, c and b have whatever units they have and the units of a are the units of c divided by the units of b . This is called the *partitive model of division*. It is the ‘inverse’ of model (1)(b) of multiplication.
3. It is the length of one side of a rectangle when the area is c and the width is b . In this case a and b have the same units and the units of c are the square of these units. This is called the *area model of division*.

Adults who have successfully mastered multiplication and division often do not consciously distinguish between the different meanings of $a \times b$ and $c \div b$ given above. Also, they may not distinguish between multiplying by, say, $1/2$ and dividing by 2. However, to children learning this material for the first time, the different interpretations of these expressions can be very different and they may have mastered one interpretation but not another.

Notice that the interpretations (1)(a) and (1)(b) for multiplication are the same; the only difference is which number plays which role; one number measures the *size* of each object and the other number measures *how much or how many* of the objects we have. In interpretation (2), the two numbers are playing equal roles. Here are examples analyzing the interpretations used in different multiplicative expressions.

Story Problem: Chengxi has 6 bags of cookies each of which contains 4 cookies. How many cookies does he have total?

Discussion: The answer is 6×4 . This uses interpretation (1); the 6 is a measure of how many bags we have, the 4 is the size of each bag and the product is the size of the total. In terms of units the 6 is measured in bags, the 4 is measured cookies per bag, and the product is measured in cookies.

Story Problem: Megan runs at a speed of 12 miles per hour. How far can she run in half-an-hour?

Discussion: The answer is $(1/2) \times 12$. This uses interpretation (1); the half measures how much time passes, the 12 measures the ‘size’ of one unit of time measured in miles, and the product is the size of the resulting combination in miles. In terms of units, the half is measured in hours, the 12 is measured in miles per hour, and the product is measured in miles.

Story Problem: A farmer has a field that is 40 meters long by 30 meters wide. What is the area of her field?

Discussion: The answer is 40×30 . This uses interpretation (2); each number has units of meters and the product has units of square meters.

Here are examples analyzing the interpretations used in different expressions involving a division.

Story Problem: Sylvie has 24 cookies total that she packs in bags with 4 cookies per bag. How many bags of cookies does she have?

Discussion: The answer is $24 \div 4$. This is the measurement model of division (1); the answer is the number of collections obtained when 24 cookies are divided into sets each of which has size 4 cookies. In terms of units, the 24 is measured in cookies, the 4 is measured in cookies per bag, and the quotient is measured in bags.

Story Problem: If Lisbeth strolls at a rate of half-a-mile per hour, how long does it take her to go one third of a mile?

Discussion: The answer is $(1/3) \div (1/2)$. This is the measurement model of division (1); it tells us how many pieces of size $1/2$ mile lie in a piece of size $1/3$ mile. In terms of units, the $1/3$ is measured in miles, the $1/2$ is measured in miles per hour, and the quotient is measured in hours.

Story Problem: Aditya has 24 cookies total that he evenly distributes into 6 bags. How many cookies are in each bag?

Discussion: The answer is $24 \div 6$. This is the partitive model of division (2); the answer is the size of each bag when 24 cookies are divided into 6 bags. In terms of units, the 24 is measured in cookies, the 6 is measured in bags and the quotient is measured in cookies per bag.

Story Problem: A rectangular cake has one side that is 9 inches long and has an area of 99 square inches. How long is the other side?

Discussion: The answer is $99 \div 9$. This is model (3); the answer is the length of one side of a rectangle whose area is 99 and whose width is 9.

When students have completed this module they should be cognizant of the two models that are used for multiplication and the three models that are used for division and be able to identify when each model is being used. Moreover, given a calculation and a model for the calculation, they should also be able to come up with a story in which that calculation would be performed based on that model. Furthermore, they should be able to identify the difference between a problem involving division by an integer and a problem involving multiplication by its reciprocal.

Accessibility: This module should be accessible to any student that has completed high school mathematics.

1. The moderator asks the students to write down a short story problem whose solution is the product 6×4 . Each student should write their story on chart paper.
2. The students share their stories. If one of the categories is not represented, the moderator should supply a story problem in that role. Also, if not already included in the students' stories, the moderator should present stories where both the numbers 6 and 4 clearly have units (as in the example of Megan above), since these examples are a little harder to interpret. The moderator categorizes the stories into three categories: those that have 6 groups of size 4, those that have 4 groups of size 6, and those in which 4 and 6 play similar roles. The moderator should not explain the rationale behind the categories; to divine it will be the next task assigned the students.
3. The moderator divides the students into groups. The groups are asked to identify what is similar about the stories in each of the categories and what is different about the stories in different categories. For each story they should also work out the *units* of each of the numbers 6, 4, and the product 24. Have them display the units in a chart as shown below. They should say what's similar about the units in stories that are in the same category and what's different about the units in stories that are in different categories.

Story	units of 6	units of 4	units of 24
Halloween	bags	candies per bag	candies
⋮	⋮	⋮	⋮

4. The class reconvenes and the groups report their discussions back to the whole class. In the discussion that follows it should be made explicit that there are three models for the multiplication $a \times b$ as identified above. These models should be understood intuitively as the size of a copies of b things, the size of b copies of a things and the area model. Students should also understand that these three ways are characterized by the *units* assigned to the

numbers 6, 4, and 24; in the first model the 6 has simple units, the units of 4 are some (other) units *per* units of 6, and the units of 24 are these other units etc.

5. Students go back to their groups charged with creating 3 stories for the multiplication 15×7 (one of each type) and 3 stories for the multiplication $20 \times (1/4)$ (one of each type). When the moderator visits the different groups she should check that their stories for $20 \times (1/4)$ are really for that and not for $20 \div 4$ instead (the result is the same, of course, but the concept is different).
6. Class reconvenes and students share their stories. In the discussion that follows, the difference between $20 \times (1/4)$ and $20 \div 4$ should be discussed. The stories of the different groups should be pooled and displayed in categories according to how the different numbers are used.
7. Students are asked to write down a story in which you'd want to do the calculation $24 \div 6$. Students write their story on chart paper.
8. Students share their stories. The stories should be displayed on the wall where everyone can see them. If any of the three models of division are not represented, the moderator should provide a story with that model. Also, if not already included in the students' stories, the moderator should include examples in which both numbers 24 and 6 clearly have units, as in the example of Lisbeth above.
9. In groups, students are asked to chart the units of each number 24, 6, and 4, in each story (just as they did for the multiplication stories) and to categorize all the stories according to the model they illustrate of division (similar to the way the moderator had categorized the multiplication stories). They shouldn't be told *how* to categorize the stories; that is for them to work out.
10. Class reconvenes and groups report back. Hopefully they will have found the three categories based on an analysis of the units involved. In the discussion that follows the moderator should try to make them articulate that one category consists of all the stories in which the quotient $24 \div 6$ is the number of groups when 24 things are divided into groups each of size 6 and another category consists of all the stories in which the quotient is the size of each group when 24 things are divided into 6 groups. There should be a discussion of why the number of things has plain units but the size of each group has units of one thing *per* units of another.
11. Students divide into groups. Each group is given a collection of, say, 10 story problems whose solution is given by multiplication or division of numbers. In the case of multiplication, students are asked to identify if the model being used is (1) or (2) and if it is (1) which number has units and which is indicating how *many* or how *much* of that thing is being combined. In the case of division, they should identify which of the three models is being used and the units of all the numbers involved. Here are some examples of stories that could

be used. (The stories listed here are in order; they should be mixed up when presented to the students.)

- *Story:* If one scoop of ice cream has 260 calories, how many calories do 5 scoops of ice cream have? (*Answer:* 5×260 . Category (1); 5 has units of scoops and 260 has units of calories per scoop.)
- *Story:* If a recipe calls for 3 cups of flour and you want to make half that recipe, how much flour should you use? (*Answer:* $(1/2) \times 3$. Category (1); $(1/2)$ has units of recipes and 3 has units of cups of flour per recipe.)
- *Story:* If lumber costs \$1.50 per board foot, how much does it cost to buy 30 board feet? (*Answer:* 30×1.5 . Category (1); 30 has units of board feet and 1.5 has units of dollars per board foot.)
- *Story:* If Sally runs at 12 miles per hour, how far can she run in half-an-hour? (*Answer:* $12 \times (1/2)$. Category (1); $(1/2)$ has units of hours and 12 has units of miles per hour.)
- *Story:* If a rectangular piece of paper has a width of 5 inches and a length of 8 inches, how much area does it cover? (*Answer:* 8×5 . Category (2); both 8 and 5 are measured in inches.)
- *Story:* If you have 14 lb of turkey and each serving is $1/4$ of a pound, how many servings of turkey do you have? (*Answer:* $14 \div (1/4)$. Category (1).)
- *Story:* Benny has 2 cups of sugar and his recipe for cookies calls for one-and-a-half cups of sugar. How many recipes of cookies can he make with the sugar he has? (*Answer:* $2 \div (3/2)$. Category (1).)
- *Story:* If you divide 15 candies between 5 friends, how many candies will each friend get? (*Answer:* $15 \div 5$. Category (2).)
- *Story:* If Sally walks one third of a mile in half-an-hour, how fast is she walking? (*Answer:* $(1/3) \div (1/2)$. Category (2).)
- *Story:* If a farmer has a rectangular field that is 1200 square feet and one side has a length of 30 feet, how long is the other side? (*Answer:* $1200 \div 30$. Category (3).)

Stories should include many examples with multiplication and division by fractions, since it is easy for students to get confused and think they are, for example, dividing by 2 when actually they are multiplying by $1/2$ (the result, of course, is the same, but the concept is different). They may also confuse division by $1/2$ with multiplication by $1/2$.

12. Class reconvenes and groups report back. The discussion should include the difference between story problems whose solution is 12×2 and story problems whose solution is $12 \div (1/2)$. Many students won't distinguish between these two though they are quite different for children

learning the subject for the first time and it is an important and non-trivial result that $12 \div (1/2) = 12 \times 2$.

13. Students go back to their groups. Each group is asked to create three story problems whose solution is the division $36 \div 4$ (one for each model) and three story problems whose solution is the division $12 \div (1/2)$ (also one for each model). Tell the students to be careful in the second case; it is easy to come up with stories that illustrate, say, $12 \div 2$ or 12×2 instead of $12 \div (1/2)$.
14. Class reconvenes and groups share their stories. Discussion follows.

Area

Materials needed:

- Ample supplies of chart paper, markers, and whatever is needed to attach the chart paper to the wall of the room for students to display their work.
- A blackboard in the room could be useful but is not absolutely essential.
- Enough rulers for every shape to have at least one.
- Pictures of the two triangles in item (14) to distribute to the groups (one per group).
- Copies of the triangle in item (16) to distribute to the different groups (one per group).
- Sheets of graph paper with shapes drawn on them for item (18); on each sheet the shapes should include a regular pentagon, a regular hexagon, an irregular convex polygon, a non-convex polygon, and a shape with curved edges. Each group gets one sheet of shapes.
- Extra graph paper with smaller squares than the squares used for the sheets of shapes would be desirable but not absolutely essential. (This is so the students can use the smaller squares to get a better estimate of the area of the shape with curved edges.)

Mathematics covered: This module is about area. Area is a quantity we associate with regions in the plane and surface area is a quantity we associate with regions in three-dimensional space. The area of a region is somehow a measure of how much two-dimensional ‘stuff’ is needed to cover that region. When we *measure* area we assign a *number* to this rather intangible quality of a shape. There are many subtle issues involved with how we make this assignment and a full investigation of this topic belongs to the fields of differential geometry and measure theory which are usually graduate level classes in mathematics.

In this module and at this level, the goal is to get students to gain a better understanding of the familiar area formulae that are used in K-12 mathematics; namely the areas of a rectangle, triangle, and circle, and the surface areas of a cylinder, cone, and sphere. In particular, with only three conditions that describe how we want addition of numbers to correspond to the intangible quality that we call area, and a choice of unit, we obtain these familiar formulae. The three conditions are:

- The area of a region is non-negative.
- The area of a region is a property of its shape only and not its location in space. In other words, if you have a shape that is located in two or three dimensional space and you move it (i.e. you perform a Euclidean transformation) then its area will remain unchanged.
- The area of the union of two disjoint regions should be equal to the sum of their individual areas. The common metaphor of addition of numbers is that it should be what you get when

you *combine* things, so we get this condition by insisting that metaphor continue to hold in the case of areas.

The usual choice of unit is that a square whose side has length 1 and whose width has length 1 has an area of 1.

After completing this module students should have a better understanding of what area is and how we use numbers to measure areas. They should understand why the standard formula for the area of a rectangle is length times width, why the area of a triangle is equal to one half its base times its height, and should understand how to calculate the area of any polygonal region by dividing it up into triangles. They should have the concept of finding the areas of regions with curved edges by approximating them by polygons and should have an intuitive understanding of why the area of a circle is equal to πr^2 based on the fact that the ratio of the diameter to the circumference is π . They should be able to derive the formulae for the surface area of a cone and of a cylinder. They will see that the concept of the surface area of curved surfaces in three dimensional space is different from that of area in the plane, will have explored its meaning and will have seen a derivation of the formula for the surface area of a sphere that does not use calculus. They should also have gained some consciousness of what we mean by a feature of an object and how numbers are abstract objects that we use to measure the feature of an object.

Accessibility: This module should be accessible to all students that have successfully completed high school mathematics and will be relevant to all undergraduate students, math majors or not. However, this module is very challenging because it requires students to question the meaning of area and many students may have no idea how to start thinking about this. For this reason, this module shouldn't be done too early; students will need to have developed a thirst and delight for mathematical inquiry otherwise they are likely to shut down.

Note: This is a long module that might best be done over (at least) two days. A recommended breaking place is given.

1. The moderator divides the students into groups and charges them with discussing, 'What do we mean by the area of an object?' The moderator should go around to the different groups to ensure they are thinking constructively about the question. In particular, many groups will probably define area in terms of itself, saying things such as, 'the area of an object is the number of square feet that it takes up.' The moderator should bring this to their attention. For example, the moderator might ask, 'What is a square foot?' If they say that it is a square whose length and width are both equal to 1 then the next question should be, "What do we mean by the area of some non-rectangular object?" (The point here is that you can't fill the object up with squares of length 1 and side 1, instead you have to chop things up into smaller pieces.)

2. The class reconvenes and the groups report their discussions back to the whole class. In the class discussion that follows the moderator should bring their attention to any *properties* they said area should have. For example, they might somehow articulate that if a shape is moved from one location to another its area shouldn't change, or they might say the area of a stick-like object should be equal to zero, or areas should be positive, or if two shapes butt up next to each other, then the area of the combined shape should be the sum of the two areas.
3. After the discussion, have students go back to groups. They should be charged with discussing *why* the area of a rectangle is equal to its length times its width. For example, why is not the area of a rectangle equal to any of the following? Or, maybe we *could* use one or more of these as our notion of the area of a rectangle? Why or why not?
 - a) the length of the diagonal
 - b) the length of the perimeter
 - c) the length of the diagonal multiplied by the width of the rectangle
 - d) twice the base times width
 - e) the base times twice the width
 - f) the base times the width squared

The moderator should circulate among the groups. If a group is ruling the first two out based on units, then prod the group to think why the units of area should be length squared. Hopefully the groups will come up with things similar to the following.

- a) A long, skinny rectangle doesn't have much area compared to a square but its diagonal is much longer, so that's why the diagonal doesn't work.
 - b) If we rotate a rectangle its area doesn't change but its base times the width squared will change because its base and width will be different.
4. Class reconvenes and the groups report back to the whole class. In the discussion that follows, if a group says, for example, that one rectangle has a larger area than another but its diagonal is shorter than the other's, then the moderator should ask them how they know that the area of the one is bigger than the area of the other. If they are using properties of area in their explanations, then the moderator should highlight the fact they are doing this and the property that they are using.
 5. The moderator should now talk a little about how we use numbers to measure *features* of objects. For example, we use numbers to count how many objects there are in a collection (the objects are the different collections and the feature is how many things the collection contains), we use numbers to measure lengths of line segments (the objects are line segments and the feature is how far the segment extends), and we use numbers to measure the weight

of an object (the objects here are physical objects and the feature is how heavy the object is). Students can come up with other examples, no doubt. Area is a feature of a region in the plane (the objects are regions in the plane and how much ‘stuff’ the region occupies is the feature). The discussion the class has been engaged in is, how should we assign a number to the amount of stuff there is? This is a little more subtle than how we do it for lengths, so, first the class will explore how we do this for lengths.

6. The students go back to their groups charged with discussing how we decide whether a line segment has length 10 or length 200 or length 127.3?
7. The class reconvenes and groups report back. Hopefully they will say things such as, “We choose what 1 unit is and then, when we want to measure something else, we take that unit length and move it to see how many of those fit into our length.” Moderator can ask questions such as, “What if more than 4 fit in but less than 5?” Hopefully they will respond with something to the effect, “We can chop our unit up into little pieces that are all the same length.” Moderator should ask how they know they all have the same length. Hopefully the response will be if you put them all next to each other you can see they have the same length. The details of the discussion are not too important, this is just an illustration of the kinds of discussions that should be going on at this point.
8. The moderator summarizes. To associate a number with the length of a line segment we need to do the following.
 - We choose a line segment whose length will be 1. This is our choice of *unit*. When we measure in centimeters this is different from when we measure in feet.
 - The lengths of all line segments should be *non-negative*. (Length 0 is okay since dots have length 0.)
 - If a line segment is moved from one place to another then its length doesn’t change.
 - If we butt two line segments up next to each other then the length of the resulting line segment should be the sum of their lengths.

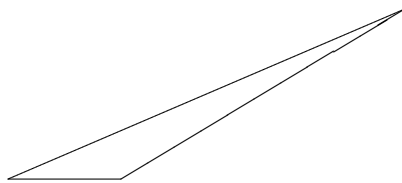
These properties are enough to determine the length of *any* line segment. There could be a discussion here of how, in general, when we associate a number to a feature we want to do that in such a way that *combining objects* should correspond to *addition of numbers*.

9. Now it is time for the class to return to the method of assigning areas to regions in the plane. How do we do this? The class should understand that area is a *numerical* assignment to a *shape* in the plane that has the following properties
 - We choose a region whose area will be 1.
 - The area of any region should be non-negative.

- When a region is moved from one place to another its area doesn't change.
 - The area of the union of two disjoint regions should be equal to the sum of their two areas.
10. The students go back to their groups to discuss, based on this, what possible expressions we could come up with that would capture the area of a rectangle in terms of the dimensions of the rectangle. In particular, recall the expressions suggested above. Would any of these work as the area of a rectangle?
 11. The class reconvenes and the groups report back. From the discussion that follows it should come out, given these properties of area, the area of a rectangle *must be* equal to a constant times its base times its width. The constant corresponds to the choice of unit. If a square of length 1 and width 1 is said to have 1 unit of area then the constant is equal to 1.
 12. The students go back to their groups charged with *proving* that the area of a triangle is equal to one half its length times its height.
 13. The groups report back. Hopefully most groups will be able to come up with a relatively convincing argument probably based on taking two congruent triangles, chopping them up, and reconstructing the pieces to form a rectangle. However, it is likely their arguments may not work for some triangles. The discussion that follows should focus on what we mean by the base of the triangle and what we mean by its height. Could *any* side be the base or did the proofs assume that one particular side was the base (namely the side that connected the two angles that were both less than 90°). If we're assuming the base is the side that connects these two angles (which they almost certainly all were doing) does every triangle always have such a side? (The answer is yes, because the sum of angles of a triangles is 180° : if there is at most one angle whose measure is less than 90° then there must be two angles with measure greater than 90° which would contradict the fact that the sum is 180° .)

This may be an appropriate place to break for the day.

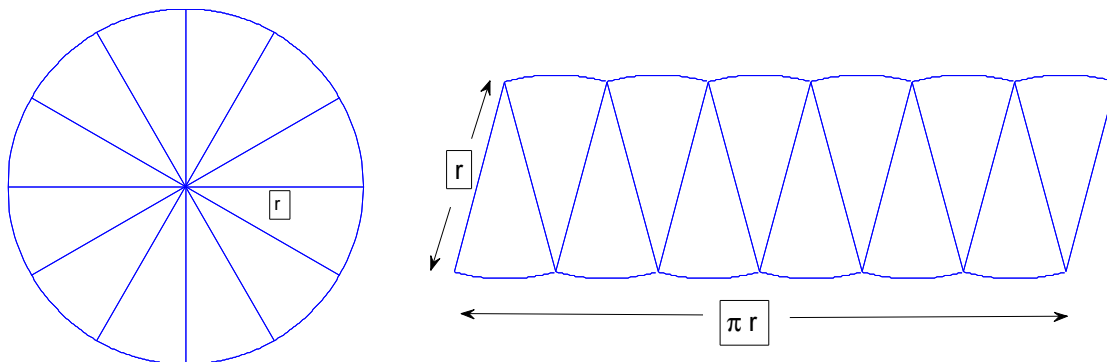
14. The students return to their groups charged with discussing why the area of a triangle is one half the base times the height, even when the base is taken to be a side that is adjacent to an angle that is greater than 90° . The moderator should give them a triangle to work with as shown below. Notice that the triangle is chosen so that when you double it, the parallelogram that is formed has to be chopped up into more than two pieces in order to form a rectangle whose width is the base and whose height is the height of the triangle.



15. The class reconvenes and the groups report back.
16. The groups are given a sheet of shapes drawn on graph paper. The shapes should include a regular pentagon, a regular hexagon, an irregular convex polygon, a non-convex polygon, and a shape that has curved edges. The groups are asked to find the area of each shape. They are also charged with determining if they can come up with *formulae* for the areas of a regular pentagon and a regular hexagon. Can they come up with a formula for the area of a general quadrilateral?
17. The groups report back. In the discussion that follows the following points should be drawn out.
 - a) We find the areas of other polygonal shapes by chopping them up into regions whose shape we know (triangles and rectangles).
 - b) We *can* find formulae for the areas of regular pentagons and hexagons. The formula for a hexagon is quite nice; the formula for the pentagon includes the value of a trigonometric function. Since the shape of one of these objects is determined by knowing what kind of object it is and the length of one of its sides, it makes sense there should be formulae for their areas based solely on the length of one side.
 - c) For a general quadrilateral it is harder to come up with a formula for its area. The reasoning behind this is that the shape and area of the quadrilateral are not determined solely on the lengths of the four sides, so any formula for its area would have to include some other lengths (not dissimilar to the way that the formula for the area of a triangle contains its height). However, it may be hard to describe what those other lengths might be. There *are* ways to do this but a) the formulae you get are not particularly memorable and b) interesting issues arise as to whether you can define what you mean by, say, the *height* of the quadrilateral. This discussion could touch on many interesting mathematical points including what it means to have a formula for something and properties of quadrilaterals (for example, every quadrilateral must have at least two acute angles; why?)
 - d) To find the area of a shape with curved edges is much harder. Discussion should center around sandwiching the error between two numbers (an upper estimate and a lower estimate) and how we could get a better sandwich if we took finer graph paper, so that in theory we could estimate the area to any desired degree of accuracy. There could be

a rich discussion here about what the difference is between this and actually knowing the area exactly. In particular, suppose the area is equal to π . What do we know about π ? (The point is that we know the value of π only to a degree of accuracy.)

18. The moderator should show them the picture below that illustrates why the area of a circle is equal to πr^2 . (Notice this derivation rests on the fact that the ratio of the circumference of a circle to its diameter is equal to π .)

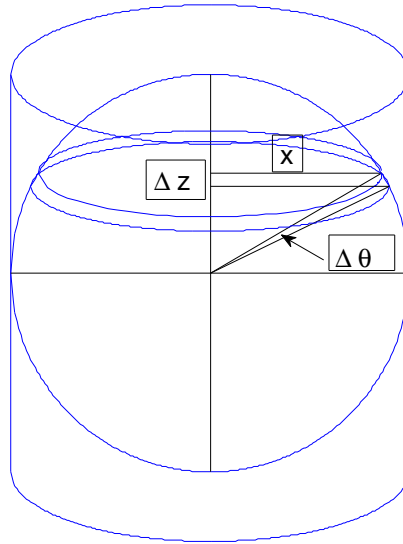


19. The students return to their groups to calculate the area of the region that is bounded by $y = x^2$, $x = 0$, $x = 1$ and the x -axis. Groups should use the ideas of area we've found so far and not use calculus. For students who have taken calculus this shouldn't be too hard; it will be a familiar Riemann sum calculation. For other students this is a very hard problem. They should be prodded to break the region up into n pieces and estimate the area in terms of n . Then look at what happens as n gets larger and larger.
20. The class reconvenes and the groups report back.
21. Now it is time to discuss what we mean by the area of a region that lies in three dimensional space and not in the plane. The groups are first charged with discussing what the surface area of the curved surface of a cylinder is and what the surface area of the curved surface of a cone is. Note: these two surfaces can be formed from planar regions, so this is simply a matter of bending a flat region in three dimensional space. Can the groups prove the familiar formulae for these two surface areas?
22. The groups reconvene and report back.
23. The groups are sent back to discuss what we mean by the surface area of a sphere and how we can find it. Can they find the surface area of a sphere *without* using calculus?
24. The groups report back. The discussion that follows should bring out the following points.
- a) Since a sphere cannot be formed by deforming a flat piece of paper, this is really a *new* sense of area. To relate it to areas in a plane and to the surface area of a cone, we

approximate the sphere with many flat pieces (like a geodesic dome) and say that the sum of the areas of the flat pieces should be close to the area of the sphere. The smaller the pieces are (and therefore the more of them we use) the closer we should get to the actual area of the sphere.

b) How can we *find* the surface area of a sphere? The meaning of its area tells us how to find it in theory. However, is there some way we can see the area is equal to $4\pi r^2$? Here are some possible approaches.

- This method is similar to the picture of the circle. Take a hemisphere and cut it up in thin slivers that stretch from the top of the dome to its edge. Lay the pieces top to bottom. One might at first think that it would look a bit like a rectangle whose height is $\pi r/2$ and whose width is πr . From this you'd get the surface area of the sphere to be equal to $2(\pi r/2)(\pi r) = \pi\pi r^2$ instead of $4\pi r^2$. The problem with the argument is that the pieces are not *flat*, so when you lay them top to bottom, they do not fit snugly together, and even in the limit as they get thinner and thinner, the shape won't look like a rectangle.
- Archimedes calculated the surface area of a sphere by slicing it into thin strips each of which looks like the frustrum of a cone. His argument is close to the argument used in calculates today but doesn't involve evaluating an integral. This argument is easily found online.
- We can calculate the surface area of a sphere by considering the projection of the sphere onto the curved surface of a cylinder that has the same radius as the sphere and circumscribes it (see the picture below). The important observation is that this projection map is area preserving, so the surface area of the sphere is the same as the area of the curved surface of the cylinder. Below is a way to see this.



The surface area of the projection of the strip at height z onto the cylinder is exactly equal to $2\pi r\Delta z$. The surface area of the same strip of the sphere is approximately equal to $2\pi xr\Delta\theta$. But

$$\begin{aligned}
 \Delta z &= r \sin((n+1)\Delta\theta) - r \sin(n\Delta\theta) \\
 &= r \sin((n+1)\Delta\theta) - z \\
 &= r [\sin(n\Delta\theta) \cos(\Delta\theta) + \cos(n\Delta\theta) \sin(\Delta\theta)] - z \\
 &= z \cos(\Delta\theta) + x \sin(\Delta\theta) - z \\
 &= z (\cos(\Delta\theta) - 1) + x \sin(\Delta\theta).
 \end{aligned}$$

It follows that

$$\frac{\Delta\theta}{\Delta z} = \frac{1}{z \left(\frac{\cos(\Delta\theta)-1}{\Delta\theta} \right) + x \left(\frac{\sin(\Delta\theta)}{\Delta\theta} \right)},$$

so the surface area of the strip of the sphere is approximately equal to

$$2\pi xr \frac{1}{z \left(\frac{\cos(\Delta\theta)-1}{\Delta\theta} \right) + x \left(\frac{\sin(\Delta\theta)}{\Delta\theta} \right)} \Delta z.$$

As $n \rightarrow \infty$ (and the strips get narrower and narrower) the approximation gets better and better and $(\cos(\Delta\theta) - 1)/\Delta\theta \rightarrow 0$ and $\sin(\Delta\theta)/\Delta\theta \rightarrow 1$. Thus, the surface area of the strip of the sphere is closer and closer to $2\pi r\Delta z$ which is the same same as the surface area of the projection of the strip onto the cylinder. Thus, the projection of the sphere onto the cylinder is an area preserving map and the surface area of the sphere must be equal to the curved surface area of the cylinder. Since the cylinder has height $2r$ and radius r , its curved surface

has area $(2\pi r)(2r) = 4\pi r^2$. Thus we get the formula for the surface area of the sphere without calculus (although the derivation did depend on the knowledge that $(\cos(\Delta\theta) - 1)/\Delta\theta \rightarrow 0$ and $\sin(\Delta\theta)/\Delta\theta \rightarrow 1$ as $\Delta\theta \rightarrow 0$).

Of course, finding the surface area of a sphere without using calculus is a very hard problem and we can't expect any of the students to come up with these arguments or even anything close to them. However, it is interesting for the students to *see* them and there are plenty of things to discuss. For example, why, exactly, is the area of the strip of the sphere at height z approximately equal to $2\pi r \Delta\theta$? Why does this approximation improve as $n \rightarrow \infty$?

Variables

Mathematics covered: We use letters in mathematics in many different ways; here are *some* examples.

- i) A letter can be used to represent a number or mathematical object that can take on any value in a given set of numbers or set of objects. This is how letters are used when we are describing properties that various mathematical constructs have. For example, we write

$$a(b + c) = ab + ac,$$

to state the distributive law for real numbers. In this statement, a , b , and c can be any real numbers. Similarly, when stating the distributive law in set theory, we write

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C),$$

where A , B , and C can be any sets that are subsets of the same universal set. In a more complicated example we write,

$$(kf + g)'(x) = kf'(x) + g'(x)$$

where f and g are any differentiable functions, x is any number that lies in both the domain of f and the domain of g , and k is any real number.

- ii) A letter can be used to represent a specific mathematical object that is a known but complex object, in order to be able to refer to it by name. This is the case, for example, in statements such as ‘consider the function f that is given by

$$f(x) = \frac{x^2}{3x - 1}.'$$

Here, f refers to this very specific function. The letter x in this example, is being used in the sense of i); it represents any number that lies in the domain of the function f .

- iii) A letter can be used to represent a quantity whose value is (presently) unknown. This is how letters are used in many elementary algebra problems. For example, suppose Abigail scored 72, 67, 82 and 79 on her first four math tests and her average score on these tests including the final was 77. To find her score on the final we may let x denote this score and write down the equation

$$\frac{72 + 67 + 82 + 79 + x}{5} = 77$$

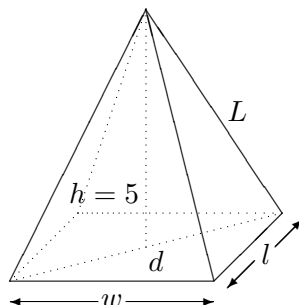
which we can solve for x .

- iv) A letter can be used to represent the measurement of a feature associated with a set of objects. As such it is a function defined on a set of objects whose target space is a set of numbers (be they integers, real numbers, complex numbers etc.) This is how letters are used in many applied problems in calculus and in modeling problems.

In cases iii) and iv), the letters are often called *variables*, though this is something of a misnomer in the case of iii). (Sometimes letters that are used as in i) may also be called variables but in these modules we'll restrict the term 'variable' to refer to letters that are used as in iii) and iv).) In elementary algebra problems, variables generally refer to letters used in the sense of iii). The switch to using them as in iv) usually happens in algebra II or pre-calculus classes and is often not made explicit, so students can find themselves manipulating algebraic expressions in which variables in the sense of iv) are being used, but their concept of a variable is that of iii). This causes problems. To resolve the cognitive disconnect, they may start to think of a variable as something that varies. This is, indeed, true (usually), but it lacks specificity. In the following discussion we show how the variables we use in many problems in calculus should be interpreted as functions on an underlying set of objects as in iv). We shall also see how making this interpretation explicit can aid in solving problems and in understanding the method of solution.

In pre-calculus classes, students explore the relations between variables and, in particular, they explore *functional* relations. They learn y is a function of x if the graph of the relation between x and y passes the vertical line test. To test their understanding of this concept they are typically given various curves in the plane and asked to identify in each case if y is a function of x . However, this is completely abstract. If the variables x and y are 'unknowns,' what is meant by the *graph of the relation between x and y* ? Indeed, in the most common examples of the relation between two variables that are not functionally related, the graphs are *regions* in the plane and not curves in the plane, so the examples that are given are not ones that encourage a concrete interpretation of the relation between variables. On the other hand, if you think of x and y as each measuring a feature associated with an underlying set of objects, then the graph of the relation between x and y has a very concrete meaning; (a, b) lies on the graph if and only if there is an object in the set for which $x = a$ and $y = b$. Functional relations are intimately connected with *causality* and *control*. In particular, suppose the underlying set is our universe and y is a function of x . If the value of x is known then this means that we are dealing with that subset of the universe consisting of all those elements for which x has the given value. Since y is a function of x the fact that we are dealing with this subset *causes* y to take on a particular value. Similarly, if we can *control* the value of x then we can control the value of y by choosing the value of x appropriately.

To clarify this, let's take a look at an example. Consider the set of all polyhedra that have a rectangular base and whose sides are triangles that meet at a point 5 units vertically above the center of the base (see the picture below).



Some variables associated with this set include the width of the base w , the length of the base l , the length of the diagonal of the base d , the height h , and the slant height L . Notice each variable associates a number with each polyhedron; in other words, each variable is a function on this set of polyhedra. Since the height h is identically equal to 5 it may seem strange to call it a ‘variable’ but, as a function defined on the set of objects, it is a variable; just as we can have constant functions, so we can have variables that are constants. The graph of the relation between w and l includes everything in the first quadrant. The graph of the relation between l and h is a horizontal half-line at height 5 and restricted to $l > 0$. This passes the vertical line test, so h is a function of l (though it is rather trivially a function of l). The graph of the relation between d and L is that quarter of the hyperbola defined by $L^2 = d^2/4 + 25$ that lies in the first quadrant. This passes the vertical line test so the value of d determines the value of L and L is a function of d . The graph of the relation between w and L includes everything in the first quadrant that satisfies $L > \sqrt{(w^2/4) + 25}$. This graph does not satisfy the vertical line test, so L is not a function of w .

These kinds of variables in algebra are the same as variables in statistics and random variables in probability. In statistics there is a population of interest and variables are measurements of features associated with the set of interest; in other words, they are functions on the population. Random variables in probability are functions on the sample space of the experiment. Notice there is often freedom in exactly what you choose your population and/or sample space to be. Similarly we shall see that there is freedom in what you consider to be the underlying set of variables in algebra. Despite that, statisticians and probabilists still recognize the value in identifying *some* set as the population or sample set of the experiment and we argue that, similarly, when considering algebraic variables, we should have an underlying set in mind. The main difference between variables in statistics, random variables in probability and variables in algebra, is that in the former two cases we endow the population and sample space with the added structure of a probability measure. This means that the kinds of questions we ask in probability and statistics are a little different from the questions we ask in algebra; in the latter we are interested if one variable is a *function* of one or more of the others, in the former we are interested *to what extent* one variable is determined by one or more of the others. Another difference is many variables in statistics are not *numerical*; the target space of the function is not a set of numbers but instead a set of objects. (Examples include

eye color or race.)

To explore the concept of variables and see how we can think of them as functions on sets of objects and why this is useful, let's look at a number of examples. Let's first consider the following problem that is a typical optimization problem covered in calculus classes. In this example, apart from seeing how various variables can be thought of as functions on an underlying set of objects, we shall also see that what we normally think of as *functions* are, in fact, variables. Indeed, any quantity that is a function of a variable, is itself a variable. Indeed, some of the confusion about variables no doubt arises from the common tendency to confuse a function with its output variable. Often this is benign, but there are many situations in which it can be problematic.

Problem A: A closed box with a square base is to have a volume of 12 m^3 . The material for the top and sides costs \$2 per square meter and the material for the bottom costs \$4 per square meter. Find the dimensions of the least expensive box that can be constructed.

In this problem we are interested in the set of all boxes that might conceivably be constructed. These boxes all have square bases and volumes that are equal to 12 m^3 . For example, one of the boxes in the set is 2 m by 2 m by 3 m and another box is 4 m by 4 m by 0.75 m . We are charged with finding the box in the set that is cheapest to construct.

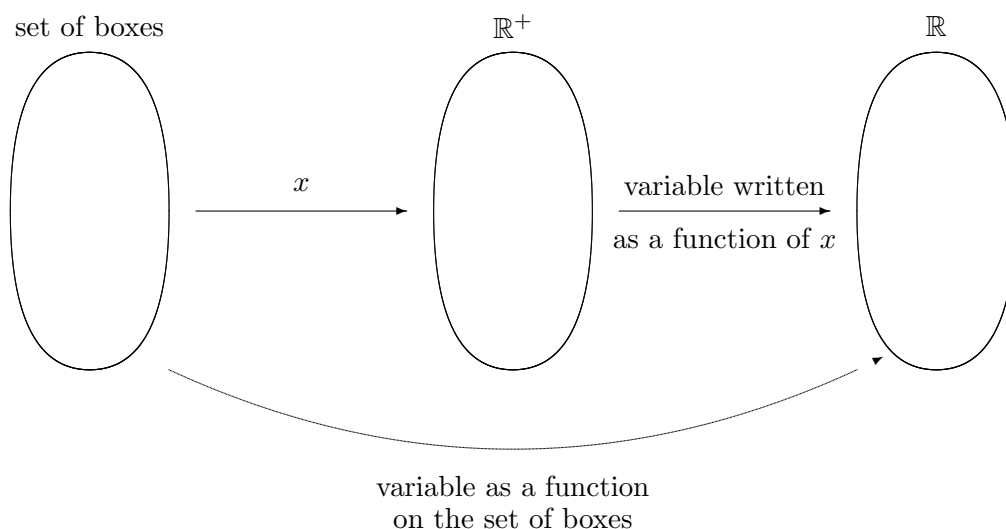
There are many variables associated with this set, most of which are uninteresting in the solution of the problem. Here are a few of them.

- a) The height h in meters.
- b) The height l in inches.
- c) The length of the base x in meters.
- d) The cost of construction C in dollars.
- e) The length of the diagonal d in meters. (In other words, the length from one corner of the box to the diagonally opposite corner.)
- f) The girth g in meters. (In other words, twice the sum of the two shortest dimensions of the box.)
- g) The rate at which the cost C changes as x is increased, measured in dollars per meter. This is the variable dC/dx .
- h) The rate at which the length of the base changes as the height is increased, measured in inches per meter. This is the variable $r = 39.3701(dx/dh)$.
- i) The rate at which dx/dh changes as x is increased, measured in meters per square meter. This is the variable $d^2x/dxdh$ (which is not, by the way, the same as $\frac{\partial^2 x}{\partial x \partial h}$).
- j) The average cost of all the boxes as the height ranges from 1 meter to the height of the box, measured in dollars. This is the variable $\left(\frac{1}{h-1}\right) \int_1^h C dh$.

Every one of these variables measures a physical characteristic of the boxes in the set and is a function on the set of boxes; in other words, to each box it associates a number.

Notice that x , h and l are rather special; if we know the value of any of them then we can identify the particular box in the set to which that value corresponds. In one of the modules below we call these variables *characteristic* variables of the set. What it means is that every variable defined on this set of boxes can be thought of as a function of x or as a function of l or as a function of h . Indeed, this is how most people would think of most of these variables.

Let's take the variable x as an example. When we write another variable as a function of x , we are introducing the variable x as an intermediary, so that the variable is now a composition of its representations as a function of x and x as a function on the set of boxes. This is illustrated in the picture below.



This process of introducing an intermediary variable is actually not an uncommon practice in mathematics. For example, when we evaluate an integral by substitution, we introduce a variable u that is a function of x and consider the quotient of the integrand divided by du/dx . When we write this quotient as a function of u in order to evaluate the integral, we are introducing the variable u as an intermediary variable, so the quotient becomes a function of u which in turn is a function of x .

To see how it can be useful to think of variables in this problem as functions on the set of boxes, imagine a student who has recognized that she wants to optimize the construction cost but is having difficulty writing the construction cost as a function of some other variable in the problem. If you can get the student to identify the underlying set that is of interest (namely, the set of boxes) then you can take an example of a box in the set, and find its construction cost. For example, consider the box that is $2 \times 2 \times 3$. Now you have a concrete box, the student will probably be able to find

the construction cost for this box. Now take another example; say a box that is $3 \times 3 \times ?$. What is the third dimension? Well, you can calculate it because you know $3^2h = 12$ so $h = 3/4$. Now you have a concrete box again, so the student will probably be able to calculate the construction cost. Having done these two examples, you now take a box that is $x \times x \times ?$. The example shows you how to find the third dimension and, once you have that, the example again shows you how to find the cost. Notice this whole approach rests on the fact that the student has identified the underlying set of objects of interest and can consider examples of objects in that set.

There are other benefits to thinking about variables in optimization problems as functions on an underlying set of objects. In particular, it is easier to make sense of the process by which we perform the optimization (see the third module below) and it can help clear the fog created by the plethora of notations for derivatives and integrals.

The next example is a typical related rates problem that appears in calculus classes.

Problem B: An airplane, flying at an altitude of 6 miles, passes directly over a radar antenna. When the airplane is 10 miles from the antenna, the radar detects that its distance from the antenna is decreasing at the rate of 240 mph. What is the speed of the airplane?

The set of objects that should be considered in this problem is the collection of points in time before and after that point in time when the airplane is 10 miles from the antenna. There are many variables associated with this set. Here are some of them:

- a) The (straight line) distance in miles of the airplane from the antenna, D .
- b) The horizontal distance in miles of the airplane from the point that is 6 miles vertically above the antenna, x .
- c) The time in hours from when the airplane is directly above the antenna, t .
- d) The rate at which x is changing in miles per hour, dx/dt .
- e) The rate at which D is changing in miles per hour, dD/dt .
- f) The rate at which x is changing as D is increased in miles per mile, dx/dD .
- g) The temperature at the antenna in degrees Celsius, T .
- h) The vertical height of the airplane above the ground in miles, h .

Each of these variables is a measurement of a feature associated with the objects in the set. (Notice h is a constant variable.) The variable t is special, since the value of t is enough to determine the object in the set under consideration. In this problem, most people would think of all the variables above as functions of t rather than as functions of the underlying set of objects. Since t is so closely associated with the objects in the set, the two points of view are really equivalent.

To solve this problem, students need to do the following.

- a) Identify the variable x whose derivative with respect to time is equal to the rate that is given.

- b) Identify the variable D whose derivative with respect to time is the rate that we are trying to find.
- c) Identify the relationship between x and D , namely $D^2 = x^2 + 36$. In particular, when determining this relationship a common mistake is to substitute $D = 10$ and write $100 = x^2 + 36$. This equation is not a relation; instead it is an equation that is satisfied by two objects in the set (those objects for which $D = 10$).

Students taking calculus are generally relatively comfortable with introducing a variable that is an unknown in the problem (as in standard algebra problems). However, they are less comfortable with introducing variables that are measurements of features of objects on an underlying set. This is the root of the difficulty doing a) and b). In particular, at this point in the course, students often think of speed as the derivative of distance without understanding there may be many different distances of interest. The root of the difficulty doing c) is that their notion of a variable is that of an ‘unknown,’ so they do not realize equalities involving variables need to be interpreted contextually; some equalities refer to particular values of the variables whereas others refer to an identity of variables. For example, the statement, $D^2 = x^2 + 36$ is an equality between two variables; the variable D^2 and the variable $x^2 + 36$. For every element in the set these two variables have the same value. On the other hand, consider the statement ‘When $D = 10$, $100 = x^2 + 36$.’ This is shorthand for the more complete statement, ‘For all those objects in the set for which the value of D is 10, the value of the variable $x^2 + 36$ is 100.’ The equalities here are not equalities of variables, instead they are equalities of particular outputs of the variables (thought of as functions) for particular items in the underlying set. Notice when we talk about the value of a variable, we are using the same language that we use with functions (as in the value of a function). This is no surprise, since variables *are* functions.

The last example is a typical algebra problem. Unlike the two problems above, it is not clear thinking about variables arising in this problem as functions on an underlying set of objects is helpful for solving the problem. However, we are including the example in order to fully explore the concept of a variable and to see it is possible to think of variables that are unknowns as functions on an underlying set of objects. This is analogous to what happens to models we have for various arithmetic operations when we extend our number system. For example, when working with whole numbers, the primary model for the subtraction $a - b$ is that of taking b things away from a things. When we extend to integers this model breaks down and instead we think of $a - b$ as the distance and direction you need to go to get from the tip of vector b to the tip of vector a . This second model encompasses the first in that it still works with whole numbers though we do not *need* this model when we are just dealing with whole numbers. Similarly, thinking of a variable as a function on an underlying set of objects is a model that is important when discussing the relations between variables; it is still valid for problems that do not concern the relations between variables though

it is not *necessary* in these problems.

Problem C: A painter has a paint mixture that contains 1% blue dye and another mixture that contains 6% blue dye. How many liters are needed of each in order to make 50 liters of a solution that contains 4% blue dye?

The standard way to solve this problem would be to let x denote the amount (in liters) that should be used of the 1% mixture and y denote the amount (in liters) that should be used of the 6% mixture. These ‘variables’ are not really variables at all. Instead, they are letters used in the sense of iii) above, each one represents a particular number whose value is (presently) unknown. We can think of x and y as functions on an underlying set, but the underlying set just consists of the single element that is the 50 liters of solution that contains 4% blue dye. Notice, since x and y take on single values only (indeed, $x = 20$ and $y = 30$), the graph of the relation between x and y consists of a single point (namely, $(20, 30)$).

To find the values of x and y we write down the simultaneous equations

$$x + y = 50 \tag{1}$$

$$\frac{0.01x + 0.06y}{x + y} = 0.04, \tag{2}$$

which we can solve by any number of methods. To help students gain an understanding of what they are doing when they solve these equations, we may have them graph each line and notice that the solution to the simultaneous system of equations is the point of intersection of the two lines. What is interesting here is the graphs of these lines do not correspond to anything physical in the problem, so by using this method of solution we are performing operations in our world of mathematics that are not physically meaningful.

Another approach to this problem is to consider the set of all possible mixtures (of any amount) that could be made from the 6% and 1% mixtures. Some variables associated with this set include:

- a) The amount (in liters) used of the 1% mixture, x .
- b) The amount (in liters) used of the 6% mixture, y .
- c) The ratio of the amount of 1% mixture to the amount of 6% mixture, $r = x/y$.
- d) The amount (in liters) of blue dye in the mixture, $D = 0.01x + 0.06y$.
- e) The fraction of blue dye in the mixture, $f = (0.01x + 0.06y)/(x + y)$.
- f) The total amount of paint in the mixture (in liters), $P = x + y$.

These are all variables in the sense of iv). The variables x and y do not depend on each other in this set; knowing the value of x gives you no information about the value of y and the graph of the relation between x and y includes everything in the first quadrant. Together, the values of x and y constitute enough information to determine the object in the set. That is why all the other

variables can be written in terms of x and y . We are interested in that object in the set for which $P = 50$ and $f = 0.04$. Each of these equations defines a subset of the set of objects in question; we are interested in that object which lies in the intersection of these two subsets. In each of these subsets, the values of x and y are related; equation (1) is the relation in the set $\{P = 50\}$ and equation (2) is the relation in the set $\{f = 0.04\}$. Now when we sketch the graphs of these lines, they are physically meaningful (provided we limit the graphs to those parts of the lines that lie in the first quadrant); each point (a, b) on the graph of, say, $x + y = 50$ corresponds to an object in the set $\{P = 50\}$ for which $x = a$ and $y = b$. The point of intersection corresponds to that object which lies in the intersection of the two sets and therefore its x and y coordinates constitute the solution to the problem.

In these modules students will explore the concept of a variable. They will see the term is used differently in different situations and the concept of a variable as an ‘unknown’ is insufficient to cover all of the ways in which variables are used. They will look at optimization problems and related rates problems in calculus where variables are used in the sense of iv) and explicitly determine the underlying sets of objects and see how interpreting the variables involved as functions on these sets can be helpful both in terms of solving the problems and in terms of understanding the method of solution. They will also see that variables as used in iv) are, in fact, the same as variables in statistics and random variables in probability.

Variables 1: Variables and Relations between Variables

Materials Needed: Ample supplies of chart paper, markers, and whatever is needed to attach the chart paper to the wall of the room for students to display their work. At least 30 cards with mathematical vignettes for item (2). Some story lines with pairs of variables prepared and ready to hand out if the students' own problems do not work well for item (7). One card for each group with a mathematical vignette with open-ended questions about variables for item (10).

Mathematics covered: In this module students will explore their own concept of a variable and of the relation between two variables. They will recall what we mean by a relation and explore different equalities involving variables and think about which of these equalities are relations between variables and which of them are not. From the exploration it will become apparent, and also made explicit, that algebraic variables measure features of objects and as such are functions on sets of objects.

Accessibility: This module is written assuming students have had at least one semester of calculus. However, the material is important for any student that has had high school mathematics, and the module could be adapted for students who haven't yet had calculus.

1. The moderator should tell the class the focus of the day will be *variables*: what we mean by them in mathematics and what we mean by the *relation between two variables*. There could be a short preliminary discussion about what students understand by the word variable.
2. The students are divided into groups. Each group is given, say, thirty cards. On each card is written a mathematical vignette that contains letters with one of those letters or an expression involving those letters written below the vignette. The students are asked whether or not they consider the expression to be a *variable*. Using these as examples they are asked to come up with a definition of what is meant by a variable in mathematics. Examples of vignettes with different expressions that could be used are given below.

Vignette: Consider the function f that is given by

$$f(x) = \frac{x}{x^2 - 4}.$$

By the quotient rule we see that

$$f'(x) = -\frac{x^2 + 4}{(x^2 - 4)^2}.$$

Examples of expressions include: f , f' , x , $f(x)$, $f'(x)$, and $\frac{x}{x^2 - 4}$.

Vignette: Suppose $y = 3x^2 + 7x + 4$. Then

$$\frac{dy}{dx} = 6x + 7.$$

Examples of expressions include: x , y , $\frac{dy}{dx}$, $3x^2 + 7x + 4$ and $7x + 4$.

Vignette: The associative law of addition states that

$$(a + b) + c = a + (b + c)$$

for all real numbers a , b and c .

Examples of expressions include: a , b , c , and $a + b$.

Vignette: For all sets A , B , and C ,

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

Examples of expressions include: A , B , C , $A \cap B$ and $B \cup C$.

Vignette: For all functions f and g , $(f \circ g)(x) = f(g(x))$.

Examples of expressions include: f , $f \circ g$, x , $g(x)$ and $f(g(x))$.

Vignette: Abigail scored 72, 67, 82 and 79 on her first four math tests and her average score on these tests including the final was 77. Let x denote her score on the final. Then

$$\frac{72 + 67 + 82 + 79 + x}{5} = 77.$$

Examples of expressions include: x , $79 + x$, and $\frac{72 + 67 + 82 + 79 + x}{5}$.

Vignette: A painter has a paint mixture containing 1% blue dye and another mixture containing 6% blue dye. In order to determine how many liters she needs of each in order to make 50 liters of a solution containing 4% blue dye, let x be the number of liters she needs of the 1% mixture and let y be the number of liters she needs of the 6% mixture. Then

$$\begin{aligned} x + y &= 50 \\ \text{and} \quad \frac{0.01x + 0.06y}{x + y} &= 0.04. \end{aligned}$$

Examples of expressions include: x , y , $x + y$ and $0.01x + 0.06y$.

Vignette: A closed box with a square base is to have a volume of $12 m^3$. The material for the top and sides costs \$2 per square meter and the material for the bottom costs \$4 per square meter. In order to find the dimensions of the least expensive box that can be constructed, let h denote the height in meters, let x denote the length of the base in meters, let C denote the cost in dollars, and let V denote the volume in cubic meters. Notice that

$$C = 2(4xh + x^2) + 4x^2 = 8xh + 6x^2.$$

Moreover, since $V = 12$, it follows that $x^2h = 12$, so

$$h = \frac{12}{x^2}.$$

Thus

$$C = \frac{96}{x} + 6x^2.$$

Examples of expressions include: x , h , C , V , x^2h , $8xh + 6x^2$, and $\frac{96}{x} + 6x^2$.

Vignette: An airplane, flying at an altitude of 6 miles, passes directly over a radar antenna. When the airplane is 10 miles from the antenna, the radar detects that its distance from the antenna is decreasing at the rate of 240 mph. In order to determine the speed of the airplane, let x denote the distance of the airplane from the point 6 miles vertically above the antenna and let D denote the straight line distance of the airplane from the antenna. Notice that

$$D^2 = x^2 + 36.$$

It follows that

$$D \frac{dD}{dt} = x \frac{dx}{dt}$$

and when $D = 10$, $x = 8$. Thus, when $D = 10$,

$$\frac{dx}{dt} = \frac{(10)(240)}{8} = 300,$$

and we see that the airplane is traveling at 300 mph.

Examples of expressions include: x , D , t , D^2 , $\frac{dx}{dt}$, and $x^2 + 36$.

There should be at least 30 cards total including at least 5 different examples for each of the 5 kinds of expressions listed below. This will allow for a rich discussion about the ways in which we use letters in mathematics and a rich enough set of examples to be able to make comparisons and classify the different meanings associated with the word ‘variable.’ The same vignette can be used on more than one card with different expressions to be considered on the different cards.

- i) An expression representing the name of a known and explicitly described mathematical object.
- ii) An expression representing any number or object in a set of numbers or set of mathematical objects (as in the statement of a theorem).
- iii) An expression representing a particular quantity that is an ‘unknown.’
- iv) An expression representing a measurable quantity that can assume different values.
- v) An expression representing a measurable quantity that is constant on the set in question.

If the moderator notices some groups are having a hard time coming up with a definition of a variable, she could prompt the group by saying a) the definition could be multiple sentences long and b) the definition could encompass different things as long as all of those things are explicitly included.

3. The class reconvenes and the groups report back. There is material here for a very rich discussion and there is not a ‘right’ or ‘wrong’ answer. Discussion should center around whether or not each group has categorized the expressions correctly according to their definition. In particular, some students might insist a variable be a single letter, so that, for example, in the vignette about boxes, they may try to claim C , x , and h are variables but $8xh + 6x^2$ is not. The moderator should ask them if it makes sense for a variable to be equal to something that is not a variable? Different groups will probably come up with different definitions and this is okay. It would be good to bring this to the students’ attention, however, since it means different people mean different things when talking about a ‘variable.’ After all the groups have reported back it would be good to try to come up with 3 groups of cards; those in which everyone agrees the expression is a variable, those in which everyone agrees the expression is not a variable, and those in which there is disagreement about whether or not that constitutes a variable. From this, perhaps the class as a whole can come up with a definition encompassing everything falling into the first group. The definition will hopefully encompass variables as in the sense of a fixed but unknown numerical measure and a quantity that can take on different values. The definition will probably also encompass some of the things that fall in the other two groups. These should be gone through one by one and re-evaluated as to whether or not they are variables. If students are resistant to calling, for example, $8xh + 6x^2$ a variable, the moderator should insist that it is a quantity that can take on different values and therefore should be considered a variable as well. If they are still resistant, ask them to give you the benefit of the doubt for a while and see how they feel at the end of the module. If the definition includes expressions whose value is non-numerical (e.g. the value may be a function or it may be a set) then the moderator should explain that in what follows we’ll only be interested in variables whose values are numerical. Having explored everyone’s notion of

a variable, the next focus will be on determining what we mean by the relation between two variables.

4. Students are divided into groups and asked to come up with examples of ‘word’ problems in which they would use at least *two* variables to solve the problem. Tell the students to be creative; the problems could come from algebra or calculus or differential equations etc. For each example, the students should write down the example, define the variables that they would use in its solution, and *very briefly* say how they would use those variables to solve the problem. (It is not important that they solve the problem; indeed, most of the problems probably won’t have ‘nice’ numbers). The moderator should walk around to the different groups and make sure there is a rich set of examples. It would be helpful if there was at least one regular problem in algebra, at least one regular calculus I optimization problem and at least one calculus I related rates problem. It would be great if there were other types of problems as well; for example, there could be modeling problems, problems from differential equations, problems from geometry etc. Depending on how many groups there are, the moderator may want the groups to come up with more than one example each. It would be good to have at least 10 examples total.
5. The class reconvenes and the groups share their problems. Hopefully some of the examples will include variables having fixed numerical values and variables that are more clearly functions on an underlying set and can assume many different values. If this is not the case, the moderator should include some problems of his or her own. When the groups share their problems, the moderator should press them, if necessary, on the exact definition of the variables they use. (The variable x is the length *of what*? Starting where and ending where? Notice the question *of what* is priming students to realize that x is measuring a feature *of an object*. This set of objects is the underlying set.) Discussion should center around whether or not everyone agrees with the presenters that the ‘variables’ that they used are, indeed, variables as the class had decided.
6. The moderator should ask students if they recall what we mean by the *relation* between two (numerical) variables and the graph of the relation. The goal of the discussion that follows is to have students understand that the relation between two variables x and y is the set of all pairs of values of x and y . In particular, the relation between x and y is a subset of the plane; (a, b) lies in the relation if and only if we can simultaneously have $x = a$ and $y = b$. If students have no recollection of what is meant by the relation between two variables, the moderator should prod them to remember back to their pre-calculus class when this was discussed. If that doesn’t work, ask them what it means to say that y is a *function* of x . They will probably remember this but there are different things that they could say. They may say that the graph of x and y should pass the vertical line test. If they say this, the next

question should be what is meant by the graph of x and y ; hopefully they will say something to the effect that it consists of all the possible pairs of values of x and y . The moderator can then explain that this is what is meant by the relation between x and y . Another thing they might say is that the value of x determines the value of y . In this case the moderator should affirm this and ask them to come up with examples of variables x and y where y is *not* a function of x . (If they are stymied here the moderator may have to send them back to their groups to discuss this between themselves.) Having the example(s), the next question is why is y not a function of x ? Hopefully they will say there is more than one possible value of x for some values of y . How do they know this? They may give an example of two objects in the underlying set for which the value of x is the same but the value of y is different. From this it will be easy for them to understand that the relation is all pairs of values of x and y . Or possibly, they will write down an equation that describes the relation between x and y . In this case the moderator can explain that this equation is the relation between x and y and the graph of the equation is the graph of the relation. In particular, the relation is the set of all pairs of values of x and y .

However the discussion goes, the outcome should be that students understand the relation between x and y is a subset of the plane and (a, b) lies in the relation if and only if it is possible to have both $x = a$ and $y = b$. The goal in what follows will be for students to be able to say explicitly what it means to be ‘possible to have both $x = a$ and $y = b$.’ In particular this means x and y are measurements of features of objects in an underlying set and there is an object in the set for which both $x = a$ and $y = b$.

7. The students divide into groups. Each group is given a story with a collection of pairs of variables and is asked to sketch the relation between the two variables in each case and to determine in which cases the relation is a functional relation and in which it is not. The moderator could use the students’ own examples or give them other examples. Below is an example of what could be given to one of the groups.

Boxes with square bases cost \$2 per square meter for the material for the sides and top and \$4 per square meter for the material for the bottom.

- a) The length of the base and the height of the boxes.
- b) The length of the base and the surface area of the boxes.
- c) The surface area and the construction cost of the boxes.
- d) The length of the base and the area of the bottom of the boxes.

Notice the pairs of variables include two cases where the relation is a region in space; one in which knowing the value of one variable gives *no* information about the value of the other

(as in a) above) and one in which it gives *some* information but not enough to determine its value (as in b) above). It also includes a case where there is a functional relation between the two variables (as in d) above). Ideally the fourth pair would be a case where the graph of the relation is a curve but the relation is not a functional relation. However, it is hard to come up with such examples. Failing that, the fourth example could be a pair of variables where the graph is more of a challenge to determine (as in c) above).

8. Class reconvenes and groups report back. Discussion that follows should center around how they worked out whether or not a point (a, b) was or was not in the relation. The moderator should press them to say what they mean by it is possible for x to be equal to a and y to be equal to b . In each case they should end up saying there is an object in the underlying set for which $x = a$ and $y = b$. Moderator summarizes the discussion by noticing in each case there is an underlying set of objects under consideration and the two variables under consideration are both measurements of some feature of those objects; as such they are *functions* defined on the set of objects. A point (a, b) is in the graph of the relation between x and y if there is an object in the set for which $x = a$ and $y = b$.
9. The next item is to discuss different kinds of equalities involving variables. The moderator presents the following equalities to the class.

- $(x - 1)(x + 1) = x^2 - 1$
- $x^2 + 1 = 2x$
- $\sqrt{x^2} = x$
- Let $y = x^2 - 1$.

Discussion centers around what the equalities mean. However the discussion goes, it should come out that the first equality is an identity of variables; for every value of x the left-hand side is the same as the right-hand side, the second equality is only true for a particular value of x , the third equality is true on a large subset of the domain of each expression, and the fourth equality is being used to *define* the variable y . The moderator should ask students for which of the equalities does it make sense to add 3 to both sides, to multiply both sides by x , to compose each side with another variable t (supposing x is a function of t), to differentiate both sides, and to integrate both sides? Since the equalities are not appearing in context, the discussion is necessarily open-ended; there are not right and wrong answers but just an exploration of what each of these might mean in some context. The idea is simply to prompt students to think about the nature of equalities; for complex objects, such as variables that are functions on an underlying set of objects, there are different ways that an equality can be interpreted, since there are different aspects of the objects that can be equated, and operations we perform may depend on the nature of the equalities involved.

10. The class is divided into groups and each group is given a mathematical vignette that contains equalities involving variables and open-ended questions about the nature of those equalities. Different groups are given different cards. Each group is asked to prepare a presentation for the rest of the class discussing the issues raised on their group's card. Here are some examples for the cards.

Vignette: Consider the curve in the plane that is described by the equation

$$x^3 - 3y^3 - x + e^y = 1.$$

Notice the point $(1, 0)$ lies on this curve. To find the slope of the curve at the point $(1, 0)$, we differentiate both sides of the equality with respect to x to obtain

$$3x^2 - 9y^2 \frac{dy}{dx} - 1 + e^y \frac{dy}{dx} = 0.$$

Substituting $x = 1$ and $y = 0$ we then get

$$3 - 1 + \frac{dy}{dx} = 0.$$

It follows that the slope is -2 .

Questions: In what sense is $x^3 - 3y^3 - x + e^y$ equal to 1? Is this only true when $x = 1$ and $y = 0$ or is it true more generally? Is this the relation between the variables x and y ? What does it mean to differentiate both sides with respect to x ; if $x^3 - 3y^3 - x + e^y = 1$ is the relation between x and y , why do we say that y is a function of x ? Does this graph pass the vertical line test? What is the meaning of the variable dy/dx ? In what sense is $3x^2 - 9y^2 \frac{dy}{dx} - 1 + e^y \frac{dy}{dx}$ equal to 0? Is this true only when $x = 1$ and $y = 0$ or is it true more generally? How generally? Think about the two equations

$$\begin{aligned} x^3 - 3y^3 - x + e^y &= 1 \\ \text{and} \quad 3x^2 - 9y^2 \frac{dy}{dx} - 1 + e^y \frac{dy}{dx} &= 0 \end{aligned}$$

in the three variables x , y , and dy/dx . Do these two equations describe the relation between the three variables? What do we mean by the relation between three variables anyway? Consider the equation

$$3 - 1 + \frac{dy}{dx} = 0.$$

What is the nature of this equality? If we wanted to find the concavity of the curve at $(1, 0)$ could we differentiate both sides to get

$$\frac{d^2y}{dx^2} = 0?$$

Vignette: A painter has a paint mixture that contains 1% blue dye and another mixture that contains 6% blue dye that she mixes to make other shades of blue. Let x be the number of liters that she uses of the 1% mixture and let y be the number of liters that she uses of the 6% mixture. If she wants to make 50 liters of a solution that contains 4% blue dye, then she'll need

$$x + y = 50$$

$$\text{and } 0.01x + 0.06y = 0.04(x + y).$$

Questions: In what sense is $x + y = 50$? In what sense is $0.01x + 0.06y = 0.04(x + y)$? Do these equations show the relation between x and y ? What is the relation between x and y ? What does the graph of the line $x + y = 50$ represent physically? What about the graph of $0.01x + 0.06y = 0.04(x + y)$? Would it make sense to differentiate both sides of $x + y = 50$ to obtain

$$1 + \frac{dy}{dx} = 0?$$

Why or why not?

Vignette: An airplane, flying at an altitude of 6 miles, passes directly over a radar antenna. When the airplane is 10 miles from the antenna, the radar detects that its distance from the antenna is decreasing at the rate of 240 mph. In order to determine the speed of the airplane, let x denote the distance of the airplane from the point 6 miles vertically above the antenna. Since the airplane is 10 miles from the antenna,

$$100 = x^2 + 36.$$

Differentiating, we get

$$0 = 2x \frac{dx}{dt} + 0$$

so the airplane is not stationary.

Questions: What mistake did this student make? How could you help this student understand her error? Under what circumstances are we allowed to differentiate both sides of an equation and under what circumstances is it nonsense to do this? Can you come up with examples to explain this to the student?

Vignette: A closed box with a square base is to have a volume of 12 m^3 . The material for the top and sides costs \$2 per square meter and the material for the bottom costs \$4 per square meter. In order to find the dimensions of the least expensive box that can be constructed, let h denote the height in meters, let x denote the length of the base in meters, let C denote the cost in dollars, and let V denote the volume in cubic meters. Notice that

$$C = 2(4xh + x^2) + 4x^2 = 8xh + 6x^2.$$

Moreover, since $V = 12$, it follows that $x^2h = 12$, so

$$h = \frac{12}{x^2}.$$

Thus

$$C = \frac{96}{x} + 6x^2.$$

Questions: What is the nature of the equality $C = 8xh + 6x^2$? Is this a relation between variables? Compare this equality with the equalities $V = 12$ and $x^2h = 12$. Can we differentiate these equalities to obtain

$$\begin{aligned}\frac{dC}{dx} &= 8 \left(h + x \frac{dh}{dx} \right) + 12x \\ \frac{dV}{dx} &= 0? \\ 2xh + x^2 \frac{dh}{dx} &= 0?\end{aligned}$$

What about the following. Are they meaningful?

$$\begin{aligned}\frac{dC}{dh} &= 8 \left(\frac{dx}{dh} h + x \right) + 12x \frac{dx}{dh} \\ \text{or } 1 &= 8 \left(\frac{dx}{dC} h + x \frac{dh}{dC} \right) + 12x \frac{dx}{dC}\end{aligned}$$

Does it make sense to talk about $\frac{dC}{dx}$? What about $\frac{dC}{dh}$? What about $\frac{d^2C}{dx dh}$?

11. The class reconvenes and groups report back. Discussion follows.

Variables 2: Comparing Variables in Algebra with Variables in Statistics and Random Variables in Probability

Materials Needed: Ample supplies of chart paper, markers, and whatever is needed to attach the chart paper to the wall of the room for students to display their work. Collections of word problems for item (6). Each group needs a collection of problems.

Mathematics Covered: Variables in statistics and random variables in probability are functions defined on the population and sample space of the experiment respectively. Variables in algebra can also be thought of in this way. In this module, through a comparison of variables in statistics and random variables in probability, students will gain an understanding of how variables in algebra can be thought of as functions defined on a set of objects. They will also contrast variables in algebra with random variables in probability and variables in statistics. They will see that it is the

additional structure of a probability measure on the underlying set that leads us to ask different kinds of questions in probability and statistics.

Accessibility: This module is designed for students that have taken a class in probability and therefore know what a random variable is. Since random variables in probability are defined as functions on the sample space of the experiment, students will be primed for thinking about a variable in algebra as a function on a set of objects. This will allow a rich discussion and comparison of these two kinds of variables.

1. The moderator should tell students in this module the class will be comparing variables in probability, variables in statistics and variables in algebra. In particular, the goal is to understand what variables *are* in these different contexts and to determine in what ways they are the same and in what ways they are different in these different contexts.
2. In groups, the class is charged with discussing what *random* variables in probability are. They should be asked to remember the definition (if they can), come up with a rich set of examples illustrating the different *types* of variables that there are, and to compare random variables in probability with non-random variables in statistics.
3. The class reconvenes and groups report back. Hopefully at least some of them will have remembered the definition of a random variable. If not, the moderator should make sure that the definition emerges from the discussion. Most of them will probably be able to come up with examples, though the majority of their examples will probably be of discrete numerical variables, so the moderator should ensure examples of categorical variables and numerical continuous variables are also presented to the class. They may have difficulty articulating what a non-random variable in statistics is, because this is often not discussed explicitly. (A variable in statistics is a function defined on the population of interest. Different kinds of variables are characterized by the target space of the function; if it is a set of objects then the variable is said to be categorical and if it is a set of numbers with which it makes sense to do arithmetic then the variable is said to be numerical.) If nobody is able to come up with a definition and examples, the moderator should present some examples. To contrast these examples with random variables in probability, the moderator should have students consider the experiment of taking a sample from the population and consider some *random* variables associated with that experiment. It is important the following things emerge from the discussion and the students are relatively comfortable with them.
 - a) The fact that a random variable X is a *function* whose domain is the sample space of an experiment and that $\{X = 5\}$, for example, is a subset of the sample space.

- b) Examples of variables including both numerical continuous, numerical discrete, and categorical variables and an explicit discussion of how variables are categorized according to the kinds of values they assume.
 - c) The fact that non-random variables in statistics are functions on the population of interest.
 - d) A comparison of random variables and non-random variables where the experiment is ‘take a sample’ from the population of interest. For example, if we consider the set of all voters, a non-random variable is the party affiliation of each voter. If we take a random sample from the set of all voters, then a random variable is the number of republicans in the sample. (Notice, by the way, the party of affiliation of voters in the sample is not a random variable since this is a function on the specific sample obtained and not on the sample space of the experiment.)
4. Students return to their groups charged with comparing variables in algebra with random variables in probability and variables in statistics. They should come up with examples of problems in which (algebraic) variables are used in the solution. What do the variables mean? Is the meaning of the variable the same as the meaning in probability and statistics? As the moderator visits the groups she should listen to the kinds of examples they are coming up with. If all the examples are of variables that represent an ‘unknown,’ the moderator should encourage students to think about optimization problems and related rates problems in calculus in which the variables used are more obviously similar to variables in probability and statistics.
5. The class reconvenes and groups report back. Hopefully they will have come up with a rich set of examples for discussion. They may or may not have come to the conclusion that variables in algebra are the same as variables in probability and statistics. In the discussion that follows the moderator should use their examples to encourage them to think of variables as a function on a set of objects. Take the easier examples first, and ask them if there is some way that the variable that is used *could be thought of* as a function on a set of objects. Go through each example in this way, so that in every case they can see that, indeed, a variable is a function on a set of objects.
6. Students return to their groups. Each group is given a collection of ‘word’ problems in algebra and calculus and they are asked, in each case, to identify the underlying set of objects of interest and to define 5 variables associated with this set of objects. Ask them to be as creative as possible when defining the variables. Can they come up with variables that are undoubtedly variables but are irrelevant to solving the problem at hand? Can they come up with variables that are a little different from the usual things that they think of as variables? Here are a collection of problems that could be used along with some examples of variables.

- a) A closed box with a square base is to have a volume of $12 m^3$. The material for the top and sides costs \$2 per square meter and the material for the bottom costs \$4 per square meter. Find the dimensions of the least expensive box that can be constructed.
 - b) An airplane, flying at an altitude of 6 miles, passes directly over a radar antenna. When the airplane is 10 miles from the antenna, the radar detects that its distance from the antenna is decreasing at the rate of 240 mph. What is the speed of the airplane?
 - c) A painter has a paint mixture that contains 1% blue dye and another mixture that contains 6% blue dye. How many liters does she need of each in order to make 50 liters of a solution that contains 4% blue dye?
7. The class reconvenes and the groups report their discussions.
 8. The next step is to contrast variables in algebra with variables in statistics and random variables in probability. The students return to their groups and are asked to formulate six questions total that we might be interested in; two involving algebraic variables, two involving random variables, and two involving variables in statistics. Examples of questions in the former might be simple algebra I or algebra II problems, optimization problems and related rates problems. Examples of questions in probability might have to do with finding the probability a random variable takes on a particular value or finding the distribution of a random variable. Examples of questions in statistics might be the degree to which two variables are correlated or the degree to which one variable can be used to predict the value of another.
 9. The class reconvenes and groups share their stories. Discussion should center around the kinds of questions we are interested in about algebraic variables, the kinds of questions we are interested in about random variables, and the kinds of questions we are interested in about variables in statistics. As the class brainstorms, someone should be assigned the task of writing down on chart paper or the blackboard (where everyone can see) the kinds of questions that are asked in each case. Here are examples of things that might be said.
 - *Algebraic variables:* What is the relation between two variables? Is one variable a function of another and, if so, can we find a formula for that function? What is the largest possible value of a variable? How fast does a variable change as time moves on?
 - *Random variables:* What is the probability that the variable assumes a particular set of values? What is the distribution of the variable? What is the joint distribution of two variables? What is the correlation between two variables? What is the expected value and variance of a variable?
 - *Statistical variables:* What is the distribution of the variable? How much information does one variable give us about another? What is the expected value and variance of a

variable?

10. The next question is whether it makes sense to take a question that we ask about algebraic variables and ask it of a random variable or a variable in statistics? For example, does it make sense to ask what the largest value of a random variable or a variable in statistics is? Does it make sense to ask what the relation between two random variables in probability or two variables in statistics is? Does it make sense to ask how fast a random variable in probability or a variable in statistics is changing? These questions can be asked concretely by referring to the student's stories. The moderator should summarize that most of these questions certainly make sense but they may not be very interesting questions.

Similarly, does it make sense to take a question that we ask about random variables or variables in statistics and ask it of an algebraic variable? If we have a variable in algebra, does it make sense to ask what its distribution is? Does it make sense to ask the probability of it assuming a particular value? If you have two variables in algebra, does it make sense to ask to what extent one of the variables is determined by the other? Again, these questions can be asked more concretely by referring to the students' stories. The moderator should summarize that most of these questions do not really make sense, because they require the additional structure of a *probability measure* on the underlying set on which the variables are defined. It is the assumption of this additional structure that drives the kinds of questions that are asked.

Variables 3: The Power of Thinking of Variables as Functions on Sets of Objects in Optimization Problems

Materials Needed: Ample supplies of chart paper, markers, and whatever is needed to attach the chart paper to the wall of the room for students to display their work. Optimization problems written out with pictures and a wire frame (depending on the problems chosen) for item (1). Optimization problems for each group for item (7).

Mathematics covered: This module is about optimization. Optimization problems encountered in first semester calculus problems are typically very challenging for students. The main challenge is setting up an expression for the variable to be optimized in terms of another variable in the problem. Some of the reasons for this difficulty are the following.

- Students have a poor understanding of what variables are and that a variable is the measurement of a *feature* associated with a *set of objects*. In particular, this means a variable is a *function* whose domain is a set of objects.

- The choice of independent variable is left to the student. Since students have a poor understanding of what variables are, they have difficulty pulling variables out of thin air (so to speak).

In this module students will reflect on optimization as a process. They will look closely at the variables involved and how the nature of these variables and their relationship to the underlying set determine the method of optimization to be used.

When optimization problems are covered in first semester calculus classes there is a natural underlying question that is rarely addressed directly; why is it that writing the variable to be optimized y as a function of another variable x , helps us find the maximum and minimum values of y ? When you first think about this it may, indeed, seem a little mysterious. However, when you consider a variable is actually a function on a set of objects, it becomes less mysterious; by writing y as a function of x (where the values of x range over some interval), rather than thinking of it as a function on a set of objects that has no structure, we are endowing the domain space with the structure of the real numbers. This makes it easier to perform the optimization, because it allows us to systematically run through all the values that y can attain by letting x increase from its lowest value to its highest value.

In these optimization problems we are interested in the optimum value of y and/or the object in the set for which that optimum is attained. The independent variable x is chosen by the person solving the problem as a *tool* for solving the problem and is generally not of interest in its own right. As a tool, it has some important properties. First of all, knowing the value of x is enough to identify the object in the set being considered. In particular, this means there is a one-to-one relationship between the values of x and the objects in the set. Moreover, in calculus I problems, the values of x range over an interval. In this module we will say variables having these properties are *characteristic* of the set. Not all sets have characteristic variables; for example, if the cardinality of the set is greater than the cardinality of the real numbers no such variable can exist. Calculus I optimization problems are problems where the set of objects under consideration has at least one characteristic variable. Optimization problems encountered in multivariable calculus classes are problems where the underlying set is best characterized by more than one variable and other kinds of optimization problems in which one might use, say, simulated annealing, or problems in the calculus of variations are concerned with more complex sets that are not easily characterized by even a handful of variables.

In contrast to the independent variable x , in optimization problems studied in first semester calculus classes, there generally is not a one-to-one relationship between the values of the variable to be optimized y and the objects in the set. (In those rare cases where there is a one-to-one relationship, it is far from obvious.) This variable is generally not characteristic of the set, so the range of values it takes on are more difficult to determine.

In this module students will think deeply about what we mean by a variable. They will practice

identifying the underlying set of objects being considered in various kinds of problems involving variables, will practice identifying variables associated with these sets, and will learn to classify variables as characteristic or uncharacteristic. They will explore the optimization process used in Calculus I optimization problems and gain some appreciation of what it is about those optimization problems that means they can be solved using the technique that is generally employed.

Accessibility: This module is written for students who have taken a calculus class. Moreover, it is assumed in this module that students already have an understanding that variables can be thought of as functions on an underlying set of objects.

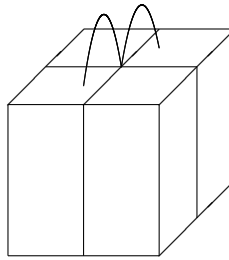
1. The students are divided into groups. Each group is given one or more optimization problems to do. (The moderator shouldn't tell them they are optimization problems; they should simply be asked to do the problems.) Some of the optimization problems should be typical Calculus I optimization problems but others should involve optimizing a characteristic variable and others should involve optimization over a set that has less structure. Here are some sample problems that could be used. The moderator should ensure there are at least 2 problems from each type of problem. It might be appropriate to give each group three problems; one of each type. Notice it is quite unlikely they will be able to solve the Type C problems, but that's okay; this exercise is not about finding the solutions, it is about the processes we use to find the solutions.

Type A: Optimizing a characteristic variable.

- a) If a taco costs \$1.25 and you have \$62, what is the maximum number of tacos you can buy?
- b) A company wants to package its product in a cylindrical can whose volume is 16π cubic inches and whose radius is between 1 and 2 inches. What is the smallest possible height of the can? What is the smallest possible radius?
- c) A church is organizing free thanksgiving meals for the homeless. One meal consists of 4 ounces of turkey, $\frac{1}{4}$ cup of cranberry sauce, 1 cup of mashed potatoes, $\frac{1}{2}$ cup of yams, 5 brussel sprouts, a cup of green beans and a cornbread muffin. If the church has 280 lb of turkey, 600 pints of cranberry sauce, 1250 pints of mashed potatoes, 500 pints of yams, 6000 brussel sprouts, 400 pints of green beans and 1200 cornbread muffins, what is the maximum number of people that the church can feed?

Type B: Typical Calculus I optimization problems.

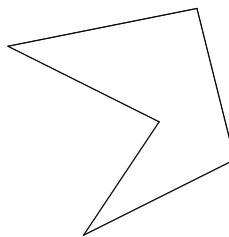
- a) What are the dimensions of the box of largest volume that has a square base and can be wrapped with 106 cm of ribbon if 10 cm are left for the bow? (See the picture below.)



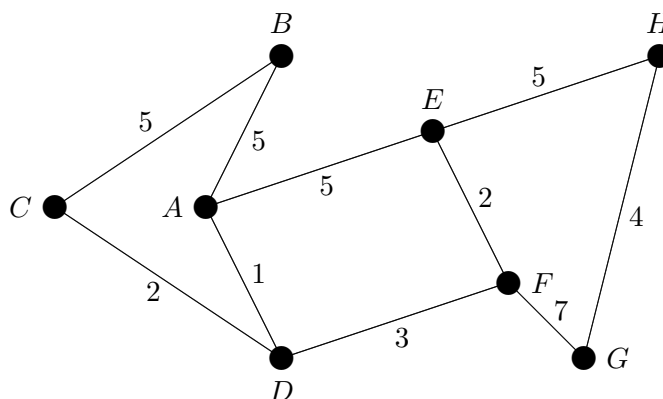
- b) A cable is to be run from a power plant on one side of a river 900 meters wide to a factory on the other side 3000 meters downstream. If it costs \$5 per meter to run the cable under water and \$4 per meter to run the cable over land, what is the most economical route by which to run the cable?
- c) You operate a tour service that can be booked by private parties. For each excursion you require a minimum of 50 people and you can accommodate up to 80 people total. For 50 people you charge \$200 per person and for each additional person beyond 50 the rate is reduced (for everybody) by \$2 per person. It costs you \$6000 plus \$32 per person to conduct the tour. What are the maximum and minimum profits you can make on each tour? How many people do you need to maximize your profit?

Type C: Optimizing over a set with less structure.

- a) Consider the set of all quadrilaterals that are completely contained in the figure below and all of whose vertices lie on the edges of the figure. What is the largest perimeter that such a quadrilateral could have? What is the smallest perimeter that the quadrilateral could have?



- b) The group should be given a wire frame living in three dimensional space. They should consider all those surfaces that include all edges of the frame. Which one of these surfaces has the smallest possible area? What does that surface look like?
- c) Shown below is a graph with ‘costs’ associated with different edges. Find the least expensive route through the graph that starts and ends at A and visits every node at least once.



2. The class regroupes and students share their solutions with each other. The moderator asks the class to think about what's similar about all the problems. Hopefully it will emerge in all of the problems we are trying to make some quantity as large or as small as we possibly can; in other words, these are optimization problems.
3. The students go back to their groups and are asked to discuss what's different about the different problems. In particular, what is it about the problems that requires the *method* by which the optimization is done to be different in the different cases? How do the methods differ? Can the students categorize all the problems according to the method used to solve them? Students should try to articulate in what ways the methods of solution for the problems in each category are similar and what it is about the problems in each category that means such an approach would work.
4. The class reconvenes and the groups report back. In the discussion that follows the moderator should try to steer the class to noticing some problems were solved by looking at simple inequalities, some problems were solved by writing the variable to be optimized as a function of another variable (this is the standard Calculus I technique), and other problems were not solved at all or were solved by ad-hoc methods. The class should discuss what it is about the problems and variables in the first group that means they can be solved by looking at simple inequalities, what it is about the problems and variables in the second group that means they can't be solved by looking at simple inequalities, and what it is about the problems and variables in the third group that means they can't be solved using Calculus I techniques. The moderator should summarize the discussion as follows. All the problems are concerned with making some *quantity* either as small or as large as we can. Since this quantity has different possible values, it is a *variable*. Depending on the nature of the variable to be optimized and its relation to the underlying set, different approaches may be needed to solve the different problems.

5. Students return to their groups. For each problem they should first identify the underlying set of objects under consideration, the variable that is to be optimized, and any other variables that are used in solving the problem. Then they should explore how all these variables are related to the underlying set on which they are defined; in which problems and for which variables is there a one-to-one relationship between the values of the variables and the objects in the set? If you know the value of the variable, can you identify the object in the set to which it corresponds?
6. The class reconvenes and the groups report their discussions. The moderator should summarize that in the Type A problems (the ones that were easy to solve), the variable to be optimized had a one-to-one relationship between its values and the objects in the set. In the Type B problems (the calculus I problems), the variable x that was used as a tool for the optimization had a one-to-one relationship between its values and the objects in the set but the variable to be optimized did not. In the Type C problems there were no easily identifiable variables having a one-to-one relationship between its values and the objects in the set. Let's label those variables having a one-to-one relationship between their values and the objects in the set as *characteristic*.

The next question on the table for discussion is *why* in the Type B Calculus I optimization problems, it was helpful to write the variable to be optimized as a function of another variable x . In other words, what is it about doing this that makes it possible to determine the maximum and minimum values of y ? Why can't we determine the maximum and minimum values of y the same way that we determined it in the other problems? In the discussion that follows, the moderator should ensure that it emerges that writing y as a function of x allows us to systematically explore all the values of y by running through the values of x from its minimum value to its maximum value. In the harder optimization problems there are no natural characteristic variables so this is not a viable method of solution.

7. Students return to their groups. Each group is given a bunch of optimization problems and asked, not to solve the problems, but in each case, to identify the underlying set being considered, identify at least 5 different variables, and for each variable determine if it is a characteristic variable or not. Ask them to be creative when identifying variables.
8. The class reconvenes and groups report back.
9. Now the focus will be on using this new way of looking at optimization problems as a way to help others solve optimization problems. To this end, the groups are asked to *solve* the optimization problems they just considered by doing the following.
 - a) First they should choose a characteristic variable for each problem and find the range of values of that variable.

- b) Then they should choose a value of the variable in the range, consider the corresponding object in the set, and determine the value of the variable to be optimized, y , for that object.
- c) Repeat with a different value of the characteristic variable.
- d) Use their work in the two concrete examples to now write the variable y as a function of x .
- e) Complete the optimization.