Stable Adaptive Control Systems in the Presence of Unmodeled and Actuator Dynamics

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Stable Adaptive Control Systems in the Presence of Unmodeled and Actuator Dynamics

by

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A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mechanical Engineering
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Dedication

This dissertation is dedicated to my brother and his wife, Mustafa and Ozge Dogan, and my wonderful aunt, Pakize Nuray Koseli.
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Abstract

In adaptive control of physical systems, it is well-known that the presence of actuator and/or unmodeled dynamics in feedback loops can yield to unstable closed-loop system trajectories. Motivated by this standpoint, this dissertation presents novel model reference and distributed adaptive control architectures with stability and performance guarantees for uncertain sole and multiagent dynamical systems with unmodeled and/or actuator dynamics.

Specifically, model reference and distributed adaptive control architectures are powerful theoretical tools for both sole and multiagent systems, where they have the capability to suppress the effect of exogenous disturbances and system uncertainties for achieving a desired level of closed-loop system response. However, the closed-loop system stability with these methods can be challenged for a wide array of applications that involve unmodeled dynamics (e.g., rigid body systems coupled with flexible appendages, airplanes with high aspect ratio wings, high speed vehicles with rigid body and flexible dynamics coupling, and flexible dynamics as in lightweight agents and/or flexible appendages as in freight carrying operations), actuator dynamics (e.g., cooperation of low and high speed autonomous vehicles and nonidentical agent actuation capabilities), and system uncertainties (e.g., unknown parameters in dynamics due to modelling errors and/or structural damage due to adverse conditions).

The challenges associated with the system uncertainties and unmodeled dynamics are first addressed using six novel approaches that determine and relax the stability limits (e.g., conditions and trade-offs) as well as the improve transient performance. In particular, it is known that a closed-loop dynamical system subject to an adaptive controller remains stable either if there does not exist significant unmodeled dynamics or the effect of system uncertainties is negligible. This implies that these controllers cannot tolerate large system uncertainties even when the unmodeled dynamics satisfy a set of limits. The first approach is predicated on a novel model reference adaptive control architecture that is augmented with an adaptive robustifying term. Unlike standard adaptive controllers, the proposed architecture allows the closed-loop dynamical system to remain stable in the presence of large system uncertainties when the unmodeled
system dynamics satisfy a set of conditions. The second, third, fourth, fifth, and sixth approaches of this dissertation are the generalizations of the first one. These approaches respectively consider an experimental verification, a theoretical extension to a class of nonlinear unmodeled dynamics, an architecture to achieve a guaranteed performance, a theoretical extension for dynamical systems with unstructured uncertainties, and an asymptotic decoupling approach for the problem of presence of unmodeled dynamics in the dynamical system. In particular, the second approach presents an experimental result for the purpose of demonstrating the efficacy of the first approach, where a benchmark mechanical system setup is used involving an inverted pendulum on a cart coupled with another cart through a spring in the presence of unknown frictions. The third approach presents an extension of the first approach to a wider class of unmodeled dynamics involving nonlinear functions. Moreover, the fourth approach presents a direct uncertainty minimization framework with the added term in the control signal and the update law, which is developed through a gradient descent procedure with a new cost function involving a cost function gain, in order to minimize the effect of both system uncertainties and unmodeled dynamics on the closed-loop system response. The fifth approach presents stability conditions of model reference adaptive control architectures in the presence of unstructured system uncertainties subject to (residual) approximating errors satisfying the linear growth inequality and unmodeled dynamics. Note that the fourth and fifth approaches are also experimentally validated on the benchmark mechanical system setup. Finally, the last approach presents a new framework guaranteeing asymptotic convergence between the trajectories of an uncertain dynamical system and a given reference model without relying on any measurements from the coupled dynamics.

The challenges associated with the uncertain dynamical systems in the presence of both unmodeled and actuator dynamics are second addressed using four novel approaches that determine and relax the stability limits. Specifically, model reference adaptive control architectures with standard, hedging-based (that alters the ideal reference model dynamics of each agent in order to ensure correct adaptation in the presence of actuator dynamics), and expanded reference models (uses a reference model predicated on the weight estimation and a copy of the actuator dynamics) are analyzed for this class of uncertain dynamical systems and sufficient stability conditions are developed. A robustifying term is then synthesized for the latter architecture and to analytically show that this term can allow for a relaxed sufficient stability condition.

The challenges associated with the uncertain multiagent systems in the presence of unmeasurable unmodeled dynamics are third addressed using a novel distributed adaptive architecture. Specifically, standard distributed adaptive control method with system uncertainties and coupled dynamics in a leader-
follower setting is analyzed, where local stability conditions are developed. An additional feedback term within the control signal of each agent is also proposed for relaxing the local stability conditions.

The challenges associated with the uncertain multiagent systems in the presence of heterogeneous actuator dynamics are finally addressed using novel distributed adaptive architectures. First, a distributed adaptive control architecture in a leader-follower setting for the class of both scalar and high-order multiagent systems is proposed. The proposed architectures use a hedging method for ensuring correct adaptation in the presence of heterogeneous actuator dynamics of these agents. Second, sufficient stability conditions are showed, where evaluation of these conditions with respect to a given graph topology allows stability verification of the controlled multiagent system.

To summarize, verifiable model reference adaptive control architectures for both sole and multiagent systems are introduced in this dissertation, where the stability limits of these architectures in the presence of unmodeled and actuator dynamics as well as system uncertainties are shown. The proposed theoretical treatments involve Lyapunov stability theory, linear matrix inequalities, and matrix mathematics. For bridging the gap between theory and practice, several simulation and experimental results are also presented.
Chapter 1: Introduction

1.1 Adaptive Control

In the literature, adaptive control architectures are classified as either direct or indirect [2–8]. Specifically, indirect adaptive control architectures estimate the unknown system parameters (e.g., modeling errors, structural damages, idealized assumptions, linearization, degraded modes of operation, unknown control effectiveness, and perturbed information exchange) and use these estimations within the control structure. On the other hand, direct adaptive control architectures adapt feedback gains in response to system variations to suppress the effect of system uncertainties without requiring a parameter estimation algorithm. The control architecture of this dissertation builds on direct adaptive controllers, namely, model reference [8–10] and distributed adaptive controllers [11–16].

In direct adaptive control of physical systems, it is well-known that the presence of unmodeled and/or actuator dynamics in feedback loops can yield unstable closed-loop system trajectories [17, 18]. Motivated by this standpoint, this dissertation presents novel model reference and distributed adaptive control architectures with stability and performance guarantees for uncertain dynamical systems (sole and multiagent) with unmodeled and/or actuator dynamics.

1.2 Adaptive Control of Uncertain Systems with Unmodeled and/or Actuator Dynamics

Obtaining system stability in the presence of not only system uncertainties but also unmodeled dynamics is a challenging problem. Motivated by [17], several studies considered unmodeled dynamics in their adaptive control formulation to achieve stability [19–23]. Specifically, in [2, 24, 25] (also see references therein), direct adaptive control architectures are proposed, where these approaches guarantee system stability in the presence of unmodeled dynamics with respect to a set of initial conditions or under the assumption of persistency of excitation. The authors of [18, 26–29] guarantee the stability without making assumptions on initial conditions and persistency of excitation with a projection operator-based approach. Moreover, their resulting stability conditions are determined for model reference adaptive control systems.
Practically, these conditions imply that the closed-loop dynamical system subject to a model reference adaptive control architecture remains stable either if there does not exist significant unmodeled dynamics or the effect of system uncertainties is negligible. This in turn means that standard model reference adaptive control architectures cannot deal with large system uncertainties even when unmodeled dynamics satisfy a set of conditions. However, it is known that, one of the main advantages of using model reference adaptive controllers versus fixed-gain robust controllers is the capability of dealing with large system uncertainties in the absence of unmodeled dynamics. Thus, it is an important problem to relax this stability condition to allow model reference adaptive controllers for dealing large system uncertainties even in the presence of unmodeled dynamics.

On the other hand, in adaptive control of uncertain systems, the presence of actuator and/or unmodeled dynamics in feedback loops can yield to instability. In particular, most research results are focused on either actuator or unmodeled dynamics. In [30, 31], architectures that involve analysis of controlled uncertain dynamical systems with actuator dynamics are studied. In [32–34], a hedging-based architectures for model reference adaptive control architectures is studied. Here, this architecture changes the trajectories of the ideal reference model to allow for correct adaptation that is not affected by the presence of actuator dynamics. Then, from the results in [32–34], sufficient stability conditions are determined to guarantee the stability of the hedged reference model trajectories and the stability of the closed-loop system trajectories [35, 36]. Finally, the studies in [35, 36] are extended with a new approach based on an expanded reference model, which can guarantee a better closed-loop system performance in the presence of actuator dynamics than the hedging method [37].

While the authors of [2, 18, 24, 25, 28] study unmodeled dynamics problem and the authors of [30–37] study actuator dynamics problem for model reference adaptive control architectures, the presence of unmodeled and actuator dynamics together is not considered in the literature. In practical applications, it should be noted that the effects of both dynamics are strictly present in feedback loops (e.g., flight control problem of flexible aerospace vehicles [38–40]). Thus, it is an important problem to address the presence of actuator and unmodeled dynamics together with all sufficient stability conditions.

1.3 Adaptive Control of Uncertain Multiagent Systems with Unmodeled or Actuator Dynamics

Multiagent systems such as aerial, ground, water, and underwater vehicles can exchange information within each other to cooperatively accomplish given tasks. A major research area for the civilian
and military applications of multiagent systems is the designing distributed control architectures such that
multiagent system perform given tasks through local interactions [41–43]. On the other hand, agents can
be subject to system uncertainties (e.g., unknown parameters in agent dynamics due to modeling errors
and/or structural damages due to adverse conditions of the environments multiagent systems operate in)
and coupled dynamics (flexible dynamics as in lightweight agents and/or flexible appendages as in freight
carrying operations) in applications. In the multiagent systems literature, there exist studies for controlling
uncertain multiagent systems (see, for example [12, 13, 44–50] and references therein). Yet, these studies
do not consider the agent dynamics coupled with other dynamics resulting from, for example, unmodeled
dynamics. Thus, it is important problem to address the stability conditions for adaptive control of multiagent
systems in the presence of not only system uncertainties but also coupled dynamics.

On the other hand, obtaining system stability and performance with nonidentical agent actuation
capabilities (e.g., unmanned vehicles working together with low and high bandwidth actuators) and unknown
parameters in the agent dynamics is an important problem. Specifically, when a multiagent system has agents
with nonidentical actuation capabilities, they cannot execute given distributed control approaches identically.
In [51, 52], it is shown that this can lead to the instability of the controlled multiagent system. Thus, it is an
important problem to address the stability conditions for distributed adaptive control of multiagent systems
in the presence of not only system uncertainties but also heterogeneous actuator dynamics.

1.4 Contributions

Motivated by the above problems, the main contributions of this dissertation are the analysis and
synthesis of multiple adaptive architectures for control of uncertain sole and multiagent systems in the
presence of unmodeled and/or actuator dynamics. Specifically:

i) A model reference adaptive control architecture is proposed for sole systems in the presence of
unmodeled dynamics. In contrast to standard model reference adaptive controllers, the key feature
of the proposed architecture, given in Chapter 2, allows the closed-loop dynamical system to remain
stable in the presence of large system uncertainties when the unmodeled dynamics, which depend on
the control signal and the system state vector, satisfy a relaxed stability limit.

ii) With several system-theoretical approaches, the first contribution is then generalized in Chapter 3.
Specifically, one approach involving an experimental validation presents the efficacy of the first
approach, where a benchmark mechanical system setup is used involving an inverted pendulum on a cart coupled with another cart through a spring in the presence of unknown frictions (Section 3.1). The other approach presents a generalization of the first approach to a wider class of unmodeled dynamics involving nonlinear functions (Section 3.2). Moreover, another approach presents a direct uncertainty minimization framework with the added term in the control signal and the update law, which is developed through a gradient descent procedure with a new cost function involving a cost function gain, in order to minimize the effect of both system uncertainties and unmodeled dynamics on the closed-loop system response (Section 3.3). The other approach presents stability conditions of model reference adaptive control architectures in the presence of unstructured system uncertainties subject to (residual) approximating errors satisfying the linear growth inequality and unmodeled dynamics (Section 3.4). Finally, the last approach presents a novel framework guaranteeing asymptotic convergence between the trajectories of an uncertain dynamical system and a given reference model without relying on any measurements from the coupled dynamics (Section 3.5).

iii) Several model reference adaptive control architectures for sole systems in the presence of unmodeled and actuator dynamics are proposed in Chapter 4. These approaches involve standard model reference adaptive control that uses an ideal reference model along with the projection operator-based weight update law, hedging-based model reference adaptive control that modifies the trajectories of an ideal reference model to allow for correct projection operator-based adaptation that is not affected by the presence of actuator dynamics, expanded model reference adaptive control that uses a reference model predicated on the weight estimation and a copy of the actuator dynamics in order to offer a better closed-loop system performance as compared with the hedging method for the actuator dynamics problem alone, and relaxed expanded model reference adaptive control architecture that uses a robustifying term for the previous architecture. Here, sufficient stability conditions are determined that guarantees the stability of this architecture in terms of boundedness. Specifically, it is found that added robustifying term for the latter architecture allows for a relaxed sufficient stability condition.

iv) A standard distributed adaptive control method with system uncertainties and coupled dynamics in a leader-follower setting is analyzed in Chapter 5. Here, local stability conditions, which guarantee boundedness of the closed-loop trajectories of the overall multiagent system, are developed. Then,
an additional feedback term is presented within the control signal of each agent in order to relax the aforementioned local stability conditions.

v) For a class of uncertain networked multiagent systems with single integrator dynamics in the context of a leader-follower problem, a novel distributed adaptive control design procedure for guaranteeing overall stability in the presence of agents having different actuator bandwidths is considered in Chapter 6. Specifically, distributed adaptive control architectures are implemented for agent uncertainties and a hedging method, which modifies ideal reference models of each agent, is utilized to allow for correct adaptation that does not get affected due to the presence of actuator bandwidths. Then, the stability of the networked multiagent systems are analyzed and the actuator bandwidth limits of each agent is computed.

The theoretical treatments in all the above system-theoretic approaches involve Lyapunov stability theory, linear matrix inequalities, and matrix mathematics. For bridging the gap between theory and practice, several simulation and experimental results are also presented in detail throughout this dissertation.
Chapter 2: Relaxing the Stability Limit of Adaptive Control Systems in the Presence of Unmodeled Dynamics

It is known that a closed-loop dynamical system subject to an adaptive controller remains stable either if there does not exist significant unmodeled dynamics or the effect of system uncertainties is negligible. This implies that these controllers cannot tolerate large system uncertainties even when the unmodeled dynamics satisfy a set of conditions. In this paper, we present an adaptive control architecture such that the proposed adaptive controller is augmented with an adaptive robustifying term. Unlike standard adaptive controllers, the proposed architecture allows the closed-loop dynamical system to remain stable in the presence of large system uncertainties when the unmodeled system dynamics satisfy a set of conditions. A numerical example is provided to demonstrate the efficacy of the proposed approach.

2.1 Introduction

Adaptive control systems, which can be broadly classified as either direct or indirect, have the capability to guarantee system stabilization and a prescribed level of command following performance in the presence of system uncertainties that can result from idealized assumptions, linearization, model order reduction, exogenous disturbances, and degraded modes of operation [2–7]. In particular, direct adaptive controllers adapt feedback gains in response to system variations to suppress the effect of system uncertainties without requiring a parameter estimation algorithm. This property distinguishes them from indirect adaptive controllers that employ an estimation algorithm to approximate unknown system parameters and adapt controller gains. The control framework of this paper builds on a well-known class of direct adaptive controllers, namely, model reference adaptive controllers [9, 10].

A challenge in the design of model reference adaptive control architectures is to achieve system stability in the presence of not only system uncertainties but also unmodeled dynamics. In particular, the authors of the seminal paper [17] investigate the stability of the closed-loop dynamical system subject to an adaptive controller and reveal that the presence of unmodeled dynamics can result in system instability.

This chapter is previously published in [1]. Permission is included in Appendix C.
Motivated from this phenomenon, several approaches are proposed in the literature. From an intelligent (i.e., neural networks-based or fuzzy logic-based) adaptive control perspective, the authors of, for example, [19–23] consider unmodeled dynamics in their formulation to achieve robust stability. From a direct adaptive control perspective, the authors of, for example, [2, 24, 25] (also see references therein) propose approaches toward robust model reference adaptive control designs. Although these approaches can achieve system stability in the presence of unmodeled dynamics with respect to a set of initial conditions or under the assumption of persistency of excitation, the authors of [18, 26–29] utilize a projection operator-based approach [53] to show the stability without making assumptions on initial conditions and persistency of excitation — when a stability limit holds for model reference adaptive control systems [18].

From a practical standpoint, this stability limit implies that the closed-loop dynamical system subject to a model reference adaptive controller remains stable either if there does not exist significant unmodeled dynamics or the effect of system uncertainties is negligible. This means that these controllers cannot tolerate large system uncertainties even when unmodeled dynamics satisfy a set of conditions. Considering one of the main advantages of using adaptive controllers versus fixed-gain robust controllers, which is the capability of tolerating large system uncertainty levels (in the absence of unmodeled dynamics), it is of theoretical and practical interest to relax this stability limit to allow model reference adaptive controllers to tolerate large system uncertainty levels even in the presence of unmodeled dynamics.

2.1.1 Contribution

In this paper, a model reference adaptive control architecture is proposed for system stabilization and command following. Unlike standard adaptive controllers (see Theorem 2.2.1 and Corollary 2.2.1), the key feature of the proposed architecture allows the closed-loop dynamical system to remain stable in the presence of large system uncertainties when the unmodeled dynamics, which depend on the control signal and the system state vector, satisfy a relaxed stability limit. This is accomplished by adding an adaptive robustifying term to the control signal that augments a projection operator-based adaptive control approach. Specifically, without making assumptions on initial conditions and persistency of excitation, we first show a stability condition (see Theorem 2.3.1) using Lyapunov stability for uniform boundedness of the closed-loop dynamical system trajectories. We then present a theoretical interpretation of this condition (see Corollary 2.3.1), which gives the aforementioned relaxed stability limit. The efficacy of the proposed approach is further demonstrated through an illustrative numerical example on a coupled dynamical system.
2.1.2 Organization and Notation

The organization of this paper is as follows. Specifically, Section 2.2 presents the standard model
reference adaptive control problem formulation in the presence of unmodeled dynamics and shows a version
of the stability limit revealed in [18], where this limit is relaxed in Section 2.3 with the proposed approach
to the model reference adaptive control problem. The illustrative numerical example is provided in Section
2.4 and concluding remarks are summarized in Section 2.5.

Throughout this paper, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^n$ denotes the set of $n \times 1$ real column
vectors, $\mathbb{R}^{n \times m}$ denotes the set of $n \times m$ real matrices, $\mathbb{R}_+$ (resp., $\mathbb{R}_+$) denotes the set of positive (resp.,
nonnegative) real numbers, $\mathbb{R}_+^{n \times n}$ (resp., $\mathbb{R}_+^{n \times n}$) denotes the set of $n \times n$ positive-definite (resp., nonnegative-
definite) real matrices, $\mathbb{S}_+^{n \times n}$ denotes the set of $n \times n$ symmetric real matrices, $\mathbb{D}_+^{n \times n}$ denotes the set of $n \times n$
real matrices with diagonal scalar entries, and “$\triangleq$” denotes the equality by definition. In addition, we use
$\cdot^T$ for the transpose operator, $(\cdot)^{-1}$ for the inverse operator, $\text{tr}(\cdot)$ for the trace operator, $\lambda (A)$ (resp., $\lambda (A)$)
for the maximum (resp., minimum) eigenvalue of the matrix $A \in \mathbb{R}^{n \times n}$, $\|\cdot\|_2$ for the Euclidean norm, and
$\|A\|_2 \triangleq (\lambda (A^T A))^{\frac{1}{2}}$ for the induced 2-norm of the matrix $A \in \mathbb{R}^{n \times m}$.

2.2 Problem Formulation and the Stability Limit of Standard Model Reference Adaptive Control

In this section, we present the standard model reference adaptive control problem formulation in the
presence of unmodeled dynamics and show a version of the stability limit revealed in [18]. Specifically, we
first introduce a definition of the projection operator [7] used in the main results of this paper.

**Definition 2.2.1** Consider a convex hypercube in $\mathbb{R}^n$ given by

$$
\Omega = \{ \theta \in \mathbb{R}^n : (\theta_i^{\min} \leq \theta_i \leq \theta_i^{\max})_{i=1,2,\ldots,n} \},
$$

(2.1)

where $\theta_i^{\min}$ and $\theta_i^{\max}$ respectively represent the minimum and maximum bounds for the $i$th component of the
$n$-dimensional parameter vector $\theta$. For a sufficiently small positive constant $\varepsilon$, another hypercube is given
by

$$
\Omega_\varepsilon = \{ \theta \in \mathbb{R}^n : (\theta_i^{\min} + \varepsilon \leq \theta_i \leq \theta_i^{\max} - \varepsilon)_{i=1,2,\ldots,n} \},
$$

(2.2)
where $\Omega_\varepsilon \subset \Omega$. Then, the projection operator $\text{Proj}: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is defined by

$$\text{Proj}(\theta, y) \triangleq \begin{cases} 
\left( \frac{\theta^\text{max}_i - \theta_i}{\varepsilon} \right) y_i, & \text{if } \theta_i > \theta^\text{max}_i - \varepsilon \text{ and } y_i > 0 \\
\left( \frac{\theta_i - \theta^\text{min}_i}{\varepsilon} \right) y_i, & \text{if } \theta_i < \theta^\text{min}_i + \varepsilon \text{ and } y_i < 0 \\
y_i, & \text{otherwise}
\end{cases} \quad (2.3)$$

componentwise, where $y \in \mathbb{R}^n$.

It follows from the Definition 2.2.1 that

$$(\theta - \theta^*)^T (\text{Proj}(\theta, y) - y) \leq 0, \quad \theta^* \in \mathbb{R}^n, \quad (2.4)$$

holds [7, 53]. The definition of projection operator can be generalized to matrices as $\text{Proj}_m(\Theta, Y) = (\text{Proj}(\text{col}_1(\Theta), \text{col}_1(Y)), \ldots, \text{Proj}(\text{col}_m(\Theta), \text{col}_m(Y)))$, where $\Theta \in \mathbb{R}^{n \times m}$, $Y \in \mathbb{R}^{n \times m}$ and $\text{col}_i(\cdot)$ denotes $i$th column operator. In this case, for a given $\Theta^* \in \mathbb{R}^{n \times m}$, it follows from (2.4) that

$$\text{tr} \left[ (\Theta - \Theta^*)^T (\text{Proj}_m(\Theta, Y) - Y) \right] = \sum_{i=1}^{m} \text{col}_i(\Theta - \Theta^*)^T (\text{Proj}(\text{col}_i(\Theta), \text{col}_i(Y)) - \text{col}_i(Y)) \leq 0. \quad (2.5)$$

Next, we introduce the problem formulation considered throughout this paper. Specifically, we consider the uncertain dynamical system subject to a class of unmodeled system dynamics that depends on the control signal and the system state vector given by

$$\dot{x}(t) = Ax(t) + B\Lambda u(t) + B\delta(x(t)) + Bp(t), \quad x(0) = x_0, \quad (2.6)$$

$$\dot{q}(t) = Fq(t) + G_1\Lambda u(t) + G_2x(t), \quad q(0) = q_0, \quad (2.7)$$

$$p(t) = Hq(t), \quad (2.8)$$

where $x(t) \in \mathbb{R}^n$ is the measurable state vector, $u(t) \in \mathbb{R}^m$ is the control input, $q(t) \in \mathbb{R}^p$ and $p(t) \in \mathbb{R}^n$ respectively are the unmodeled dynamics state and output vectors, $A \in \mathbb{R}^{n \times n}$ is a known system matrix, $B \in \mathbb{R}^{n \times m}$ is a known input matrix such that the pair $(A, B)$ is controllable, $\Lambda \in \mathbb{R}^{m \times m} \cap \mathbb{D}^{m \times m}$ is unknown control effectiveness matrix, $\delta: \mathbb{R}^n \to \mathbb{R}^m$ is a system uncertainty, and $F \in \mathbb{R}^{p \times p}$, $G_1 \in \mathbb{R}^{p \times m}$, $G_2 \in \mathbb{R}^{p \times n}$, and $H \in \mathbb{R}^{m \times p}$ are matrices associated with unmodeled dynamics such that $F$ is Hurwitz. Note that since
Λ ∈ \( \mathbb{R}^m \times m \cap \mathbb{D}^m \times m \), it follows that there exists \( \lambda_L \in \mathbb{R}_+ \) and \( \lambda_U \in \mathbb{R}_+ \) such that \( \lambda_L \leq \| \Lambda \|_2 \leq \lambda_U \) holds. In addition, note that since \( F \) is Hurwitz, there exists \( S \in \mathbb{R}^p \times p \cap \mathbb{S}^p \times p \) such that \( 0 = F^T S + SF + I \).

**Remark 2.2.1** Let \( z(t) = \beta q(t), \) \( z(t) \in \mathbb{R}^p \), where \( \beta \in \mathbb{R}_+ \) is a free variable to be used later in analysis of this paper. Then, the unmodeled dynamics given by (2.7) and (2.8) can equivalently be represented as

\[
\dot{z}(t) = Fz(t) + \beta G_1 \Lambda u(t) + \beta G_2 x(t), \quad z(0) = \beta q_0. \tag{2.9}
\]

\[
p(t) = \beta^{-1} H z(t). \tag{2.10}
\]

Furthermore, using this state transformation, the uncertain dynamical system subject to the considered class of unmodeled system dynamics (2.6), (2.7), and (2.8) can be equivalently written as

\[
\dot{x}(t) = Ax(t) + B \Lambda u(t) + B \delta(x(t)) + \beta^{-1} B H z(t), \quad x(0) = x_0. \tag{2.11}
\]

\[
\dot{z}(t) = Fz(t) + \beta G_1 \Lambda u(t) + \beta G_2 x(t), \quad z(0) = \beta q_0. \tag{2.12}
\]

We are now ready to overview a standard adaptive control formulation and show a version of the stability limit revealed in [18]. We begin with the following standard assumption on system uncertainty parameterization [7].

**Assumption 2.2.1** The system uncertainty \( \delta : \mathbb{R}^n \to \mathbb{R}^m \) can be parameterized as

\[
\delta(x) = W_0^T \sigma_0(x), \quad x \in \mathbb{R}^n, \tag{2.13}
\]

where \( W_0 \in \mathbb{R}^{s \times m} \) is an unknown weight matrix and \( \sigma_0 \in \mathbb{R}^n \to \mathbb{R}^s \) is a known basis function of the form

\[
\sigma_0(x) = [\sigma_{0_1}(x), \sigma_{0_2}(x), ..., \sigma_{0_s}(x)]^T. \tag{2.14}
\]

In addition, the basis function satisfies the inequality

\[
\| \sigma_0(x(t)) \|_2 \leq l_0 \| x(t) \|_2 + l_c, \quad x(t) \in \mathbb{R}^n,
\]

where \( l_0 \in \mathbb{R}_+ \) and \( l_c \in \mathbb{R}_+ \).
Now, consider the reference system, which captures a desired closed-loop dynamical system performance, given by
\[
\dot{x}_r(t) = A_r x_r(t) + B_r c(t), \quad x_r(0) = x_{r0},
\]
(2.15)
where \(x_r(t) \in \mathbb{R}^n\) is the reference state vector, \(c(t) \in \mathbb{R}^m\) is a given uniformly continuous bounded command, \(A_r \in \mathbb{R}^{n \times n}\) is the Hurwitz reference system matrix, and \(B_r \in \mathbb{R}^{n \times m}\) is the command input matrix. Since \(A_r\) is Hurwitz, there exists \(P \in \mathbb{R}^{n \times n}_+ \cap \mathbb{S}^{n \times n}\) such that \(0 = A_r^T P + PA_r + I\).

The next assumption is standard and known as matching condition in the adaptive control literature; e.g., see [2, 3, 7, 54–56].

**Assumption 2.2.2** There exist \(K_1 \in \mathbb{R}^{m \times n}\) and \(K_2 \in \mathbb{R}^{m \times m}\) such that \(A_r \triangleq A - BK_1\) and \(B_r \triangleq BK_2\).

It now follows from Assumptions 2.2.1 and 2.2.2 that (2.11) can be rewritten as
\[
\dot{x}(t) = A_r x(t) + B_r c(t) + B \Lambda [u(t) + W^T \sigma (\cdot)] + \beta^{-1} B H z(t),
\]
(2.16)
where \(W \triangleq [\Lambda^{-1} W_0^T, \Lambda^{-1} K_1, -\Lambda^{-1} K_2]^T \in \mathbb{R}^{q \times m}\) and \(\sigma (\cdot) \triangleq [\sigma_0^T (x(t)), x^T (t), c^T (t)]^T \in \mathbb{R}^q\) with \(q \triangleq s + n + m\). In addition, letting \(e(t) \triangleq x(t) - x_r(t)\) be the system error, the system error dynamics can be written using (2.15) and (2.16) as
\[
\dot{e}(t) = A_r e(t) + B \Lambda [u(t) + W^T \sigma (\cdot)] + \beta^{-1} B H z(t), \quad e(0) = e_0.
\]
(2.17)

Considering (2.17), let the adaptive control law be given by
\[
u(t) = -\hat{W}^T (t) \sigma (\cdot)
\]
(2.18)
where the \(\hat{W}(t) \in \mathbb{R}^{q \times m}\) is an estimate of the unknown weight \(W\) satisfying the projection operator based weight update law
\[
\dot{\hat{W}}(t) = \gamma \text{Proj}_m [\hat{W}(t), \sigma (\cdot) e^T (t) PB], \quad \hat{W}(0) = \hat{W}_0.
\]
(2.19)
with $\gamma \in \mathbb{R}_+$ being the learning rate. Note that since we utilize a projection bound in (2.19), it follows that $\|\hat{W}(t)\|_2 \leq w^*, w^* \in \mathbb{R}_+$. Now, using (2.18) respectively in (2.17) and (2.12), one can write

$$
\dot{e}(t) = A_r e(t) - B\hat{W}^T(t)\sigma(\cdot) + \beta^{-1} BHz(t), \quad e(0) = e_0,
$$

$$
\dot{z}(t) = Fz(t) - \beta G_1\hat{W}^T(t)\sigma(\cdot) + \beta G_2x(t), \quad z(0) = \beta q_0,
$$

where $\hat{W} \triangleq \tilde{W}(t) - W \in \mathbb{R}^{q \times m}$. The next theorem is on the stability of the closed-loop dynamical system subject to a standard, projection operator-based adaptive controller.

**Theorem 2.2.1** Consider the uncertain dynamical system subject to control signal and system state dependent unmodeled dynamics given by (2.11) and (2.12), the reference system given by (2.15), the adaptive control law given by (2.18) and (2.19), and assume that Assumptions 2.2.1 and 2.2.2, and the following condition

$$
\mathcal{R} \triangleq \begin{bmatrix} 1 & \eta \\ \eta & \alpha \end{bmatrix} > 0
$$

hold, where $\alpha \in \mathbb{R}_+$, $\eta \triangleq -\beta^{-1} \|PB\|_2 \|H\|_2 - \alpha \beta \lambda_U w^* l \|SG_1\|_2 - \alpha \beta \|SG_2\|_2$, and $l \triangleq (1 + l_0)$. Then, the solution $(e(t), z(t), \hat{W}(t))$ of the closed-loop dynamical system is uniformly bounded.

**Proof.** To show uniform boundedness of the solution $(e(t), z(t), \hat{W}(t))$, consider the Lyapunov-like function given by

$$
\mathcal{V}(e, \hat{W}, z) = e^T Pe + \gamma^{-1} \text{tr}(\hat{W}\Lambda^{1/2})^T(\hat{W}\Lambda^{1/2}) + \alpha z^T Sz.
$$

Note that $\mathcal{V}(0, 0, 0) = 0$ and $\mathcal{V}(e, z, \hat{W}) > 0$ for all $(e, z, \hat{W}) \neq (0, 0, 0)$. Differentiating (2.23) along the closed-loop dynamical system trajectories yields

$$
\dot{\mathcal{V}}(e(t), \hat{W}(t), z(t)) \leq -\|e(t)\|_2^2 - \alpha \|z(t)\|_2^2 + 2\beta^{-1} \|e(t)\|_2 \|PB\|_2 \|H\|_2 \|z(t)\|_2
$$

$$
+ 2\alpha \beta \|z(t)\|_2 \|SG_1\|_2 \|\Lambda\|_2 \|\hat{W}(t)\|_2 \|\sigma(\cdot)\|_2
$$

$$
+ 2\alpha \beta \|z(t)\|_2 \|SG_2\|_2 \|x(t)\|_2.
$$

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Next, an upper bound for $\|\sigma(\cdot)\|_2$ can be given using Assumption 2.2.1 as

$$\|\sigma(\cdot)\|_2 \leq le(t) + d^*, \quad (2.25)$$

where $d^* \in \mathbb{R}_+$ is an upper bound for $l \|x_1(t)\|_2 + l_\epsilon$; that is, $l \|x_1(t)\|_2 + l_\epsilon \leq d^*$, since $x_1(t)$ and $c(t)$ are both bounded by their definitions. Then, an upper bound for $\|x(t)\|_2$ can be given $\|x(t)\|_2 \leq \|e(t)\|_2 + \|c(t)\|_2$, Now, it follows from (2.24) and (2.25) that

$$\dot{V}(e(t), \tilde{W}(t), z(t)) \leq -\xi^T(t)R\xi(t) + 2\alpha\beta \|z(t)\|_2 \|SG_1\|_2 \lambda_U w^* d^*$$

$$+ 2\alpha\beta \|z(t)\|_2 \|SG_2\|_2 \|x_1(t)\|_2, \quad (2.26)$$

where $\xi(t) \triangleq [\|e(t)\|_2, \|z(t)\|_2]^T$. Finally, (2.26) can be rewritten as

$$\dot{V}(e(t), \tilde{W}(t), z(t)) \leq -\Lambda(R) \|\xi(t)\|_2 \left[ \|\xi(t)\|_2 - \frac{r_0}{\Lambda(R)} \right], \quad (2.27)$$

where $r_0 \triangleq 2\alpha\beta \|SG_1\|_2 \lambda_U w^* d^* + 2\alpha\beta \|SG_2\|_2 \|x_1(t)\|_2 \leq x_2^*$, and hence, there exists a compact set such that $\dot{V}(e(t), z(t), \tilde{W}(t)) < 0$ outside of this set, which proves uniform boundedness of solution $(e(t), z(t), \tilde{W}(t))$ [7, 57].

Note that the condition given by (2.22) imposes a stability limit for standard adaptive control systems and a version of this stability limit is highlighted in [18]. An interpretation of this stability limit is given in the next corollary.

**Corollary 2.2.1** There exists $\alpha \in \mathbb{R}_+$ such that (2.22) holds if

$$\|SG_1\|_2 \|H\|_2 \lambda_U w^* l + \|SG_2\|_2 \|H\|_2 < \frac{1}{4 \|PB\|_2}. \quad (2.28)$$

**Proof.** Let $m_1$ and $m_2$ be the leading principle minors [58] of the matrix $R$ given by

$$m_1 = 1, \quad (2.29)$$

$$m_2 = \alpha - (\beta^{-1} \|PB\|_2 \|H\|_2 + \alpha \beta \|SG_1\|_2 \lambda_U w^* l + \alpha \beta \|SG_2\|_2)^2. \quad (2.30)$$

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Note that $m_1 \in \mathbb{R}_+$ holds automatically. In addition, there always exists a positive $\alpha$ such that $m_2 \in \mathbb{R}_+$ holds when (2.28) is true. Hence, since in this case all the leading principle minors of $\mathcal{R}$ are positive, it follows that (2.22) holds.

\[\blacksquare\]

**Remark 2.2.2** It follows from the results in Theorem 2.2.1 and Corollary 2.2.1 that the solution $(e(t), z(t), \hat{W}(t))$ of the closed-loop dynamical system is uniformly bounded when the stability limit given by (2.28) holds. In particular, it can be readily seen from (2.28) that the closed-loop dynamical system remains bounded either if there does not exist significant unmodeled dynamics or the effect of system uncertainties is negligible. Hence, the standard model reference adaptive control formulation considered in this section cannot tolerate large system uncertainties even when unmodeled dynamics satisfy a set of conditions, which violates the main purpose of utilizing an adaptive controller in the feedback loop, as discussed.

### 2.3 Relaxing the Stability Limit of Model Reference Adaptive Control Systems

The main contribution of this paper is to propose an adaptive control architecture to relax the stability limit revealed in Section 2.2 and discussed in Remark 2.2.2. For this purpose, we introduce an adaptive robustifying term to the standard adaptive control formulation presented in Section 2.2. Specifically, let the adaptive control law be given by

\[u(t) = -\hat{W}^T(t)\sigma(\cdot) - \hat{\mu}(t)B^TPe(t),\quad (2.31)\]

where $\hat{W}(t)$ satisfies the weight update law given by (2.19) and $\hat{\mu}(t)$ is a projection operator-based adaptive robustifying term given by

\[\dot{\hat{\mu}}(t) = \mu_0\text{Proj}\left(\hat{\mu}(t), \|B^TPe(t)\|^2 - \sigma_\mu \hat{\mu}(t)\right), \quad \hat{\mu}(0) = \hat{\mu}_0, \quad \hat{\mu}(0) \in \mathbb{R}_+, \quad (2.32)\]

with $\mu_0 \in \mathbb{R}_+$ and $\sigma_\mu \in \mathbb{R}_+$ being design parameters. Note that since $\hat{\mu}(0) \in \mathbb{R}_+, \hat{\mu}(t) \in \mathbb{R}_+$ holds automatically. In addition, we select the projection bound for (2.32) as

\[\hat{\mu}(t) \leq \mu \psi, \quad \mu = \lambda U^2l^2w^+ + \lambda_L, \quad \psi > 1, \quad (2.33)\]
where $\psi > 1$ allows one to choose a sufficient large projection bound as necessary. It should be noted that the results of this section also hold without the leakage term (i.e., $-\sigma_\mu \hat{\mu}(t)$) in (2.32); however, we utilize this term to drive $\hat{\mu}(t)$ closer to zero for instants when the effect of $\|B^T Pe(t)\|_2^2$ in (2.32) becomes small.

Next, using (2.31) respectively in (2.17) and (2.12), one can write

\[
\begin{align*}
\dot{e}(t) &= A_re(t) - B\hat{\Lambda}\hat{W}(t)\sigma(\cdot) - \hat{\mu}(t)B\hat{\Lambda}^T Pe(t) + \beta^{-1}BHz(t), \quad e(0) = e_0, \quad (2.34) \\
\dot{z}(t) &= Fz(t) - \beta G_1\hat{\Lambda}\hat{W}(t)\sigma(\cdot) - \beta \hat{\mu}(t)G_1\hat{\Lambda}^T Pe(t) + \beta G_2x(t), \quad z(0) = z_0. \quad (2.35)
\end{align*}
\]

In the reminder of this section, we show boundedness of the closed-loop dynamical system with the proposed adaptive robustifying term and show that the proposed approach relaxes the stability limit given by (2.28).

**Theorem 2.3.1** Consider the uncertain dynamical system subject to control signal and system state dependent unmodeled dynamics given by (2.11) and (2.12), the reference system given by (2.15), the adaptive control law given by (2.31), (2.19), and (2.32), and assume that Assumptions 2.2.1 and 2.2.2, and the following condition

\[
R \triangleq \begin{bmatrix}
1 & 0 & \eta_1 \\
0 & 2\mu\lambda_L & \eta_2 \\
\eta_1 & \eta_2 & \alpha \end{bmatrix} > 0,
\]

(2.36)

hold, where $\alpha \in \mathbb{R}^+$, $\eta_1 \triangleq -\alpha\beta\lambda_L\|w^*l\|_2 - \alpha\beta\|SG_1\|_2$, and $\eta_2 \triangleq -\beta^{-1}\|H\|_2 - \alpha\beta\mu\psi\lambda_L\|SG_1\|_2$. Then, the solution $(e(t), \hat{W}(t), z(t), \hat{\mu}(t))$, $\hat{\mu}(t) \triangleq \hat{\mu}(t) - \mu$, of the closed-loop dynamical system is uniformly bounded.

**Proof.** To show uniform boundedness of the solution $(e(t), \hat{W}(t), z(t), \hat{\mu}(t))$, consider the Lyapunov-like function given by

\[
\mathcal{V}(e, \hat{W}, z, \hat{\mu}) = e^T Pe + r^{-1}\tr((\hat{W}\Lambda^{1/2})^T(\hat{W}\Lambda^{1/2}) + \alpha z^T Sz + \mu_0^{-1}\hat{\mu}^2\lambda_L.
\]

(2.37)

Note that $\mathcal{V}(0, 0, 0, 0) = 0$ and $\mathcal{V}(e, \hat{W}, z, \hat{\mu}) > 0$ for all $(e, \hat{W}, z, \hat{\mu}) \neq (0, 0, 0, 0)$. Differentiating (2.37) along the closed-loop dynamical system trajectories yields
\[ \mathcal{V}(e(t), \hat{W}(t), z(t), \hat{\mu}(t)) = -e^T(t)e(t) - 2e^T(t)PBA\hat{W}^T(t)\sigma(\cdot) - 2\hat{\mu}(t)e^T(t)PBA^T Pe(t) \]
\[ + 2\beta^{-1} e^T(t)PBHz(t) - \alpha \sigma^T(t)z(t) - 2\alpha \beta z^T(t)SG_1 \hat{W}^T(t)\sigma(\cdot) \]
\[ - 2\alpha \beta \hat{\mu}(t) z^T(t)SG_1 \Lambda B^T Pe(t) + 2\alpha \beta z^T(t)SG_2 x(t) \]
\[ + 2\gamma^{-1} \text{tr}\hat{W}^T(t)\hat{W}(t)\Lambda + 2\mu_0^{-1} \hat{\mu}(t) \hat{\mu}(t) \lambda_L. \quad (2.38) \]

Now, we can calculate an upper bound for (2.38) as

\[ \mathcal{V}(e(t), \hat{W}(t), z(t), \hat{\mu}(t)) \]
\[ \leq -\|e(t)\|_2^2 - \alpha \|z(t)\|_2^2 - 2\mu \lambda_L \|B^T Pe(t)\|_2^2 \]
\[ - 2e^T(t)PBA\hat{W}^T(t)\sigma(\cdot) + 2\text{tr}\hat{W}^T(t)\text{Proj}_m[\hat{W}(t), \sigma(\cdot)] e^T(t)PB \Lambda \]
\[ + 2\beta^{-1} \|B^T Pe(t)\|_2 \|H\|_2 \|z(t)\|_2 + 2\alpha \beta \|\Lambda\|_2 \|\hat{W}(t)\|_2 \|SG_1\|_2 \|z(t)\|_2 \|\sigma(\cdot)\|_2 \]
\[ + 2\alpha \beta \hat{\mu}(t) \|SG_1\|_2 \|\Lambda\|_2 \|z(t)\|_2 \|B^T Pe(t)\|_2 + 2\alpha \beta \|z(t)\|_2 \|SG_2\|_2 \|x(t)\|_2 \]
\[ - 2\hat{\mu}(t) \lambda_L \|B^T Pe(t)\|_2^2 + 2\hat{\mu}(t) \text{Proj}(\hat{\mu}(t), \|B^T Pe(t)\|_2^2 - \sigma_\mu \hat{\mu}(t)) \lambda_L \]
\[ + 2\sigma_\mu \lambda_L \hat{\mu}(t) \hat{\mu}(t) - 2\sigma_\mu \lambda_L \hat{\mu}(t) \hat{\mu}(t) \]
\[ \leq -\|e(t)\|_2^2 - \alpha \|z(t)\|_2^2 - 2\mu \lambda_L \|B^T Pe(t)\|_2^2 \]
\[ + 2\beta^{-1} \|B^T Pe(t)\|_2 \|H\|_2 \|z(t)\|_2 + 2\alpha \beta \|\Lambda\|_2 \|\hat{W}(t)\|_2 \|SG_1\|_2 \|z(t)\|_2 \|\sigma(\cdot)\|_2 \]
\[ + 2\alpha \beta \hat{\mu}(t) \|SG_1\|_2 \|\Lambda\|_2 \|z(t)\|_2 \|B^T Pe(t)\|_2 + 2\alpha \beta \|z(t)\|_2 \|SG_2\|_2 \|x(t)\|_2 \]
\[ + 2\sigma_\mu \lambda_L \hat{\mu}(t) \mu - 2\sigma_\mu \lambda_L \mu^2(t). \quad (2.39) \]

Similar to the proof of Theorem 2.2.1, (2.39) can be rewritten as

\[ \mathcal{V}(e(t), \hat{W}(t), z(t), \hat{\mu}(t)) \]
\[ \leq -\|e(t)\|_2^2 - \alpha \|z(t)\|_2^2 - 2\mu \lambda_L \|B^T Pe(t)\|_2^2 \]
\[ + 2\beta^{-1} \|B^T Pe(t)\|_2 \|H\|_2 \|z(t)\|_2 + 2\alpha \beta \lambda_L w^* \|SG_1\|_2 \|z(t)\|_2 \|e(t)\|_2 + d^* \]
\[ + 2\alpha \beta \mu \psi \|SG_1\|_2 \lambda_L \|z(t)\|_2 \|B^T Pe(t)\|_2 + 2\alpha \beta \|z(t)\|_2 \|SG_2\|_2 \|e(t)\|_2 \|x(t)\|_2 \]
\[ - 2\sigma_\mu \lambda_L \hat{\mu}^2(t) + 2\sigma_\mu \lambda_L \mu^2 \psi \]
\[ \leq -\xi^T(t)\mathcal{R}\xi(t) + r_0 \|z(t)\|_2 + c^*, \quad (2.40) \]
where $\xi(t) \triangleq \begin{bmatrix} \|e(t)\|_2, \|B^T Pe(t)\|_2, \|z(t)\|_2 \end{bmatrix}^T$, $r_0 \triangleq 2\alpha\beta \|SG_1\|_2 \lambda_U w^* d^* + 2\alpha\beta \|SG_2\|_2 x_t^*$, and $c^* \triangleq 2\lambda_L \sigma_\mu \mu^2 \psi$. Finally, (2.40) can be further rewritten as

$$\dot{V}(e(t), \tilde{W}(t), z(t), \tilde{\mu}(t)) \leq -\left[ \sqrt{\lambda(R)} \|\xi(t)\|_2 - \frac{r_0}{2\sqrt{\lambda(R)}} \right]^2 + \left( \frac{r_0^2}{4\lambda(R)} + c^* \right), \quad (2.41)$$

and hence, there exists a compact set such that $\dot{V}(e(t), \tilde{W}(t), z(t), \tilde{\mu}(t)) < 0$ outside of this set, which proves uniform boundedness of solution $(e(t), \tilde{W}(t), z(t), \tilde{\mu}(t))$ [7, 57].

In the proof of Theorem 2.3.1, (2.41) presents an upper bound of the time derivative of the Lyapunov function given by (2.37), which is used to conclude uniform boundedness of the solution $(e(t), \tilde{W}(t), z(t), \tilde{\mu}(t))$. In order to reach this conclusion, a sufficient condition is given by (2.36) — the positive-definiteness of $R$. Similar to Corollary 2.2.1 and under the assumption of $\lambda_L > \frac{1}{2}$, we are now ready to state a theoretical interpretation of (2.36) in the following corollary. In particular, this corollary shows conditions for the positive-definiteness of $R$, where this reveals the aforementioned relaxed stability limit, i.e., when unmodeled system dynamics satisfy these conditions, then closed-loop dynamical system remains stable even in the presence of large system uncertainties unlike standard adaptive controllers (see Section 2.2).

**Corollary 2.3.1** If the conditions

$$\|SG_1\|_2 \|H\|_2 < \frac{\sqrt{2\lambda_L} - 1}{\psi \lambda_U}, \quad (2.42)$$

$$\|SG_2\|_2 \|H\|_2 + \|SG_1\|_2 \|H\|_2 \lambda_U w^* l$$

$$< \frac{1}{\sqrt{2}} \sqrt{\frac{(\lambda_U^2 w^* l^2 + \lambda_L) \left( 2\lambda_L - 1 - \psi^2 \lambda_U^2 \|SG_1\|_2^2 \|H\|_2^2 - 2\psi \lambda_U \|SG_1\|_2 \|H\|_2 \right)}{\lambda_L}}, \quad (2.43)$$

hold, then (2.36) is satisfied.

**Proof.** Let $m_1$, $m_2$, and $m_3$ be the leading principle minors [58] of the matrix $R$ given by

$$m_1 = 1, \quad (2.44)$$

$$m_2 = 2\mu \lambda_L, \quad (2.45)$$

$$m_3 = \frac{1}{\lambda_L} \left( \lambda^2 - \frac{\lambda_U^2 w^* l^2}{\lambda_L} \right). \quad (2.46)$$
\[
m_3 = 2\alpha\mu\lambda_L - \beta^2 \|H\|_2^2 - 2\alpha\lambda_U \mu \psi \|H\|_2 \|SG_1\|_2 \\
- \alpha^2 \beta^2 \lambda_U^2 \mu^2 \psi^2 \|SG_1\|_2^2 - 2\alpha^2 \beta^2 \lambda_L \lambda_U^2 \mu \psi^2 \|SG_1\|_2^2 \\
- 2\mu \lambda_L \alpha^2 \beta^2 \|SG_2\|_2^2 - 4\mu \lambda_L \alpha^2 \beta^2 \lambda_U \psi \|SG_1\|_2 \|SG_2\|_2. \tag{2.46}
\]

Note that \(m_1 \in \mathbb{R}_+\) and \(m_2 \in \mathbb{R}_+\) hold automatically. To show \(m_3 \in \mathbb{R}_+\), we let \(\beta \triangleq \frac{\|H\|_2}{\sqrt{\alpha \mu}}\) such that

\[
m_3 = \alpha \mu \left[2\lambda_L - 1 - \lambda_U^2 \psi^2 \|H\|_2 \|SG_1\|_2 - 2\lambda_U \psi \|H\|_2 \|SG_1\|_2 \\
- 2\lambda_L \left(\frac{\lambda_U^2}{\mu} \|H\|_2 \|SG_1\|_2^2 + \|H\|_2 \|SG_2\|_2^2 + \frac{2\lambda_U \psi \|H\|_2 \|SG_1\|_2 \|SG_2\|_2}{\lambda_L}\right)\right]. \tag{2.47}
\]

Now, using the definition of \(\mu\) in (2.33), (2.47) can be equivalently written as

\[
m_3 = \alpha \mu \left[2\lambda_L - 1 - 2\lambda_U \psi \|SG_1\|_2 \|H\|_2 - \lambda_U^2 \psi^2 \|SG_1\|_2 \|H\|_2^2 \\
- \left(2\lambda_L \left(\frac{\lambda_U^2}{\lambda_L} \|H\|_2 \|SG_1\|_2 \|SG_2\|_2 + \frac{2\lambda_U \psi \|H\|_2 \|SG_1\|_2 \|SG_2\|_2}{\lambda_L}\right)\right)\right], \tag{2.48}
\]

and hence, \(m_3 \in \mathbb{R}_+\) holds when (2.42) and (2.43) are true. Finally, since in this case all the leading principle minors of \(\mathcal{R}\) are positive, it follows that (2.36) holds. 

\[\Box\]

**Remark 2.3.1** Since \(\lambda_L > \frac{1}{2}\) and (2.42) are assumed for the results in Corollary 2.3.1, one can rewrite the upper bound for (2.43) as

\[
\|SG_1\|_2 \|H\|_2 \lambda_U \psi \|SG_2\|_2 \|H\|_2 < \frac{1}{\sqrt{2}} \sqrt{\left(\frac{\lambda_U^2 \psi \|SG_1\|_2 \|SG_2\|_2}{\lambda_L}\right)} \varphi + \varphi, \tag{2.49}
\]

or equivalently

\[
\frac{\|SG_1\|_2 \|H\|_2 \lambda_U \psi \|SG_2\|_2 \|H\|_2}{\sqrt{\left(\frac{\lambda_U^2 \psi \|SG_1\|_2 \|SG_2\|_2}{\lambda_L}\right)}} \varphi + \varphi < \frac{1}{\sqrt{2}}, \tag{2.50}
\]

where \(\varphi \triangleq (2\lambda_L - 1 - \rho^2(2\lambda_L + 1 - 2\sqrt{2\lambda_L}) - 2\rho(\sqrt{2\lambda_L} - 1))\) and \(\rho \in (0, 1)\). Unlike the stability limit for the standard model reference adaptive control problem given by (2.28) whose left hand side grows
unbounded with respect to \( w^* \); that is,

\[
\lim_{w^* \to \infty} \left[ \|S\|_2 \|H\|_2 \lambda U w^* I + \|S\|_2 \|H\|_2 \right] = \infty,
\]

(2.51)

the left hand side of (2.50) is bounded with respect to \( w^* \); that is,

\[
\lim_{w^* \to \infty} \left[ \frac{\|S\|_2 \|H\|_2 \lambda U w^* I + \|S\|_2 \|H\|_2}{\sqrt{\left( \frac{\lambda L}{\lambda L} \right) \varphi \lambda U}} \right] = \frac{\|S\|_2 \|H\|_2}{\sqrt{\varphi / \lambda L}}.
\]

(2.52)

Thus, the proposed adaptive control framework of this section with the adaptive robustifying term allows the closed-loop dynamical system to remain bounded in the presence of large system uncertainties when the unmodeled dynamics satisfy the condition given by (2.49) or

\[
\frac{\|S\|_2 \|H\|_2}{\sqrt{\varphi / \lambda L}} < \frac{1}{\sqrt{2}}.
\]

(2.53)

as a worst-case scenario for sufficiently large system uncertainties. Finally, if both \( \lambda L \) and \( \lambda U \) are sufficiently close to 1 and \( \rho \) is sufficiently close to 0, then (2.53) further implies that

\[
\|S\|_2 \|H\|_2 < \frac{1}{\sqrt{2}}.
\]

(2.54)

2.4 Illustrative Numerical Example

Consider the coupled dynamical system in Figure 2.1 with the dynamics given by

\[
\dot{x}(t) = \begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix} x(t) + \begin{bmatrix}
\rho_1/m_1
\end{bmatrix} \Lambda u(t) + \begin{bmatrix}
0 \\
\rho_1/m_1
\end{bmatrix} z(t) + \begin{bmatrix}
(k_2/\rho_1) \\
(-k_1 - k_2)/\rho_1
\end{bmatrix}^T \delta(x(t)) x(t), \quad x(0) = 0,
\]

(2.55)
\[
\dot{z}(t) = \begin{bmatrix}
0 & 1 \\
(-k_2 - k_3)/m_2 & -c/m_2
\end{bmatrix} z(t) + \beta \begin{bmatrix}
0 \\
 k_2/m_2
\end{bmatrix} x(t) + \beta \begin{bmatrix}
0 \\
 k_2/m_2
\end{bmatrix} \Lambda u(t), \quad z(0) = 0,
\]  
(2.56)
for (2.28)). That is, it is clear from Figure 2.2 that the standard model reference adaptive controller cannot tolerate the given unmodeled dynamics and the system uncertainty at the same time.

For the proposed model reference adaptive control approach in Theorem 2.3.1 with $\gamma = 0.4$, $\mu_0 = 1.1$, $\sigma_\mu = 0.05$, and the projection bounds $\tilde{\mu}_{\text{max}} = 1$ and $\tilde{W}_{\text{max}} = 0.35$, Figure 2.3 shows that the system state vector closely tracks the reference state vector in time, where the closed-loop dynamical system is clearly stable. This result is expected from Corollary 2.3.1. Because, the stated relaxed stability limit of this corollary holds in this case (to compute this limit, we use $\lambda_L = 0.95$, $\lambda_U = 1.05$ and $\psi = 1.1$, which results in

$$\|S_G\|_2 \|H\|_2 $\lambda_U w^* l + \|S_G\|_2 \|H\|_2 = 0.4633,$$

(2.60)

and

$$\frac{1}{\sqrt{2}} \sqrt{\left( \frac{\lambda_U^2 w^* l^2 + \lambda_L}{2\lambda_L - 1 - \psi^2 \lambda_U^2} \right) \left( \|S_G\|_2 \|H\|_2 - 2\psi \lambda_U \|S_G\|_2 \|H\|_2 \right)} = 0.7831,$$

(2.61)

for (2.43)). That is, this figure shows that the proposed model reference adaptive controller tolerates the given unmodeled dynamics and the system uncertainty at the same time.

2.5 Conclusions

For standard model reference adaptive controllers, the existence of unmodeled dynamics imposes a strict stability limit. This limit implies that these controllers cannot tolerate large system uncertainties even when the unmodeled dynamics satisfy a set of conditions. From a practical standpoint, this contradicts the main advantage of utilizing adaptive control approaches, which is the capability of tolerating large system uncertainty levels. To address this challenge, we introduced and analyzed an adaptive control framework, which allows the closed-loop dynamical system to remain stable in the presence of large system uncertainties when the unmodeled system dynamics satisfy a relaxed stability limit. This was accomplished by adding an adaptive robustifying term to the control signal that augments a projection operator-based adaptive control approach. An illustrative numerical example complemented the proposed theoretical contribution. For future research, we will focus on generalizations of the proposed framework to a broader class of unmodeled dynamics involving nonlinearities. Moreover, we will also consider extensions when the state vector of
the uncertain dynamical system is not measurable for applications including, for example, active noise suppression, active control of flexible structures, fluid flow control, and combustion control processes. To
address this future research direction, observer-based [7, 59, 60] and observer-free [61, 62] approaches can be pursued. Finally, we will apply the proposed framework to high-fidelity dynamical system models and real-world vehicles.
Chapter 3: Generalizations and Applications of Adaptive Control Laws to Uncertain Systems with Unmodeled Dynamics

This chapter provides several generalizations of the framework presented in Chapter 2. Specifically, these generalizations include i) an experimental verification (Section 3.1), ii) a theoretical extension to a class of nonlinear unmodeled dynamics (Section 3.2), iii) an architecture to achieve guaranteed closed-loop system performance (Section 3.3), iv) a theoretical extension for dynamical systems with unstructured uncertainties (Section 3.4), and v) an asymptotic decoupling approach (Section 3.5) for the problem of presence of unmodeled dynamics in the dynamical system.

3.1 Experimental Results of a Model Reference Adaptive Control Law on an Uncertain System with Unmodeled Dynamics

Model reference adaptive control is a powerful tool that has a capability to suppress the effect of system uncertainties for achieving a desired level of closed-loop system performance. Yet, for a wide array of applications including unmodeled dynamics such as coupled rigid body systems with flexible interconnection links, airplanes with high aspect ratio wings, and high speed vehicles with strong rigid body and flexible dynamics coupling, the closed-loop system stability with model reference adaptive control laws can be challenged. To this end, the authors have recently studied the stability interplay between a class of unmodeled dynamics and system uncertainties for model reference adaptive control laws, and proposed a robustifying term to relax the resulting interplay. The contribution of this paper is to present experimental results for the purpose of demonstrating the efficacy of this proposed term, where we use a benchmark mechanical system setup involving an inverted pendulum on a cart coupled with another cart through a spring in the presence of unknown frictions.

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2This section is previously published in [63]. Permission is included in Appendix C.
3.1.1 Introduction

Model reference adaptive control is a powerful theoretical method that has a natural capability to suppress the effect of system uncertainties for achieving a desired level of closed-loop system performance. For a wide array of applications including unmodeled dynamics however, the closed-loop system stability with model reference adaptive control laws can be challenged. These applications include coupled rigid body systems with flexible interconnection links, airplanes with high aspect ratio wings, and high speed vehicles with strong rigid body and flexible dynamics coupling, to name but a few examples. To this end, we have recently studied in [1] the stability interplay between a class of unmodeled dynamics and system uncertainties for model reference adaptive control laws, and proposed a robustifying term to relax the resulting interplay. The contribution of this paper is to present experimental results in order to complement these recent theoretical studies and demonstrate the efficacy of this proposed term. For this purpose, a benchmark mechanical system setup is considered that involves an inverted pendulum on a cart coupled with another cart through a spring in the presence of unknown frictions.

Control of an inverted pendulum itself on a cart, where force is applied to the cart in horizontal direction to keep the pendulum balanced upright position, is a well-respected benchmark problem. While this paper is not intended to be a survey article on this benchmark problem, one can refer to the studies in, for instance, [64–69]. For example, the authors of [64] utilize Lyapunov stability analysis and the authors of [65] utilize a small gain theorem to guarantee stability of this benchmark problem. Furthermore, in [66] by the transfer of energy from the actuated (cart) to the underactuated degrees of freedom (pendulum), partial feedback linearization is utilized to linearize the cart dynamics. These approaches can achieve system stability but their performance can be deteriorated in the presence of significant system uncertainties. Specifically, on this benchmark problem, friction (and cables attached to the cart) can create considerable amount of system uncertainties for feedback control. To deal with the effect of friction on the design of feedback control laws to stabilize the cart with an inverted pendulum, the authors of [67] create friction models and they reduce the small amplitude oscillations due to friction. To address system uncertainties, the authors of [68] consider an integral sliding-mode control and the authors of [69] utilize a combination of extended high-gain observers and dynamic inversion with a multiple time-scale structure. We also note here that the above studies do not consider the setup we consider in this paper, which includes unmodeled dynamics (i.e., the second cart connected to the inverted pendulum on the first cart through a spring).
As discussed above, we resort to model reference adaptive control theory in this paper to address feedback control of the cart coupled with another cart through a spring in the presence of unknown frictions. Specifically, Section 3.1.2 first presents a concise overview of the theoretical results documented in [1] for completeness on the stability interplay between a class of unmodeled dynamics and system uncertainties for model reference adaptive control laws as well as the proposed robustifying term to relax the resulting interplay. Note that our overview given in Section 3.1.2 also includes discussions on how to make necessary theoretical alterations to make these results fit to the considered benchmark setup. Section 3.1.3 then presents our experimental study. Finally, Section 3.1.4 summaries our conclusions.

Consistent with [1], \( \mathbb{R} \) denotes the set of real numbers, \( \mathbb{R}^n \) denotes the set of \( n \times 1 \) real column vectors, \( \mathbb{R}^{n \times m} \) denotes the set of \( n \times m \) real matrices, \( \mathbb{R}_+ \) (resp., \( \mathbb{R}_+^\times \)) denotes the set of positive (resp., nonnegative) real numbers, \( \mathbb{R}^{n \times n}_+ \) (resp., \( \mathbb{R}_+^{n \times n} \)) denotes the set of \( n \times n \) positive-definite (resp., nonnegative-definite) real matrices, \( \mathbb{D}^{n \times n} \) denotes the set of \( n \times n \) real matrices with diagonal scalar entries, \( I_n \) denote the \( n \times n \) identity matrix \( 0_n \) denote the \( n \times 1 \) vector of all zeros, \( 0_{n \times n} \) denote the \( n \times n \) zero matrix, and “\( \triangleq \)” denotes the equality by definition throughout this paper. In addition, we use \((\cdot)^T\) for the transpose operator, \((\cdot)^{-1}\) for the inverse operator, \(\text{tr}(\cdot)\) for the trace operator, \(\lambda_\max(A)\) (resp., \(\lambda_\min(A)\)) for the maximum (resp., minimum) eigenvalue of the matrix \( A \in \mathbb{R}^{n \times n} \), \(\|\cdot\|_2\) for the Euclidean norm, and \(\|A\|_2 \triangleq (\lambda_\max(A^TA))^{\frac{1}{2}}\) for the induced 2-norm of the matrix \( A \in \mathbb{R}^{n \times m} \). We also refer to Appendices A and B, respectively for a necessary definition and a solution procedure based on dead-zone operator.

3.1.2 Model Reference Adaptive Control of Uncertain Systems with Unmodeled Dynamics: Overview of the Results in [1]

We start with introducing the problem formulation. Specifically, we consider the uncertain dynamical system subject to a class of unmodeled dynamics given by

\[
\dot{x}_p(t) = A_p x_p(t) + B_p \Lambda u(t) + B_p W_0^T x_p(t) + B_p p(t), \quad x_p(0) = x_{p0}, \tag{3.1}
\]

\[
\dot{q}(t) = F q(t) + G_1 \Lambda u(t) + G_2 x_p(t), \quad q(0) = q_0, \tag{3.2}
\]

\[
p(t) = H q(t). \tag{3.3}
\]
Here, \( x_p(t) \in \mathbb{R}^{n_p} \) is a measurable (i.e., accessible) state vector of the modeled dynamics, \( u(t) \in \mathbb{R}^m \) is the control input, \( q(t) \in \mathbb{R}^p \) and \( p(t) \in \mathbb{R}^m \) are the unmodeled dynamics unmeasurable state and output vectors, respectively, \( A_p \in \mathbb{R}^{n_p \times n_p} \) is a known system matrix, \( B_p \in \mathbb{R}^{n_p \times m} \) is a known input matrix such that the pair \((A_p, B_p)\) is controllable, \( \Lambda \in \mathbb{R}^{m \times m} \cap \mathbb{D}^{m \times m} \) is unknown control effectiveness matrix\(^3\), \( W_0 \in \mathbb{R}^{n_p \times m} \) is an unknown weight matrix, and \( F \in \mathbb{R}^{p \times p}, G_1 \in \mathbb{R}^{p \times m}, G_2 \in \mathbb{R}^{p \times n_p}, \) and \( H \in \mathbb{R}^{m \times p} \) are matrices associated with unmodeled dynamics such that \( F \) is Hurwitz\(^4\). A graphical representation of the considered class of physical systems is shown in Figure 3.1.

For introducing an additional flexibility in the stability analysis of the considered setup (3.1), (3.2), and (3.3) subject to model reference adaptive control laws to be summarized below, a state transformation \( z(t) = \beta q(t), z(t) \in \mathbb{R}^p \), is used in [1] with \( \beta \in \mathbb{R}_+ \) being a free variable. In particular, with this state transformation, one can equivalently write (3.1), (3.2), and (3.3) as

\[
\begin{align*}
\dot{x}_p(t) & = A_p x_p(t) + B_p \Lambda u(t) + B_p \Lambda W^T x_p(t) + \beta^{-1} B_p H z(t), \quad x_p(0) = x_{p0}, \quad (3.4) \\
\dot{z}(t) & = F z(t) + \beta G_1 \Lambda u(t) + \beta G_2 x_p(t), \quad z(0) = \beta q_0, \quad (3.5) \\
p(t) & = \beta^{-1} H z(t). \quad (3.6)
\end{align*}
\]

\(^3\)There exists \( \lambda_L \in \mathbb{R}_+ \) and \( \lambda_U \in \mathbb{R}_+ \) satisfying \( \lambda_L \leq \|\Lambda\|_2 \leq \lambda_U \).

\(^4\)There exists \( S \in \mathbb{R}_+^{p \times p} \) such that \( 0 = F^T S + SF + I_p \).
Here, $W = [\Lambda^{-1}W_0^T]^T \in \mathbb{R}^{n_p \times m}$.

In what follows, we choose a standard integrator-based structure for our nominal control law. For this purpose, let $c(t) \in \mathbb{R}^{n_c}$ be a given piecewise continuous reference command and $x_i(t) \in \mathbb{R}^{n_i}$ be the integrator state satisfying

$$
\dot{x}_i(t) = E_p x_p(t) - c(t), \quad x_i(0) = x_{i0}.
$$

Here, $E_p \in \mathbb{R}^{n_c \times n_p}$ allows one to select a subset of $x_p(t)$ to allow following the command $c(t)$. Considering (3.4) and (3.7), the augmented dynamics can now be written as

$$
\dot{x}(t) = \begin{bmatrix} A_p & 0_{n_p \times n_c} \\ E_p & 0_{n_c \times n_p} \end{bmatrix} x(t) + \begin{bmatrix} B_p \\ 0_{n_c \times m} \end{bmatrix} \Lambda [u(t) + W^T x_p(t)]
$$

$$
+ \begin{bmatrix} B_p \\ 0_{n_c \times m} \end{bmatrix} \beta^{-1} H z(t) + \begin{bmatrix} 0_{n_p \times n_c} \\ -I_{n_c} \end{bmatrix} c(t), \quad x(0) = x_0,
$$

(3.8)

with $x(t) = [x_p^T(t), x_i^T(t)]^T \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, and $B_r \in \mathbb{R}^{n \times n_c}$, $n = n_c + n_p$.

Next, consider the reference system (model), which captures a desired closed-loop dynamical system performance, given by

$$
\dot{x}_r(t) = A_r x_r(t) + B_r c(t), \quad x_r(0) = x_{r0}.
$$

(3.9)

Here, $x_r(t) \in \mathbb{R}^n$ is the reference state vector and $A_r \in \mathbb{R}^{n \times n}$ is the Hurwitz5 reference system matrix6. Now, the uncertain dynamical system (3.8) with the unmodeled dynamics (3.5) can be rewritten as

$$
\dot{x}(t) = A_r x(t) + B_r c(t) + BA \left[ u(t) + K x(t) + W^T N x(t) \right] + \beta^{-1} B H z(t),
$$

(3.10)

$$
\dot{z}(t) = F z(t) + \beta G_1 A u(t) + \beta G_2 N x(t),
$$

(3.11)

with $N = [I_{n_p \times n_p}, 0_{n_p \times n_c}] \in \mathbb{R}^{n \times n}$.

5There exists $P \in \mathbb{R}^{n \times n}$ such that $0 = A_r^T P + PA_r + I_n$.

6There exist $K \in \mathbb{R}^{m \times n}$ such that $A_r \equiv A - BK \in \mathbb{R}^{n \times n}$. 

28
The rest of this section is divided into two parts. In the first subsection, we summarize the closed-loop system stability results of standard model reference adaptive control laws for uncertain dynamical systems subject to unmodeled dynamics (Figure 3.1). In the second subsection, we then provide a summary on how a robustifying term can be added to the standard model reference adaptive control laws for the purpose of relaxing the closed-loop stability interplay between system uncertainties and unmodeled dynamics.

3.1.2.1 Stability Limit of Standard Model Reference Adaptive Control Laws

We begin with the overview of the stability limit of standard model reference adaptive control laws in the presence of not only system uncertainties but also unmodeled dynamics. Mathematically speaking, consider the adaptive control law given by

\[ u(t) = -Kx(t) - \hat{W}^T(t)Nx(t). \] (3.12)

Here, \( \hat{W}(t) \in \mathbb{R}^{n_p \times m} \) is an estimate of the unknown weight \( W \) satisfying the projection operator based (see Appendix A) weight update law

\[ \dot{\hat{W}}(t) = \gamma \text{Proj}_m[\hat{W}(t), Nx(t)e^T(t)PB], \quad \hat{W}(0) = \hat{W}_0, \] (3.13)

where \( \gamma \in \mathbb{R}_+ \) being the learning rate and \( e(t) = x(t) - x_r(t) \in \mathbb{R}^n \) being the system error state vector.

Considering the above standard model reference adaptive control law, one can show that if

\[ \mathcal{R} \triangleq \begin{bmatrix} 1 & \eta \\ \eta & \alpha \end{bmatrix} > 0 \] (3.14)

holds, then boundedness of the closed-loop dynamical system trajectories \((e(t), z(t), \hat{W}(t))\), where \( \hat{W}(t) = \hat{W}(t) - W \in \mathbb{R}^{n \times m} \), is guaranteed [1] (we also refer to [18]). In (3.14), \( \alpha \in \mathbb{R}_+ \) and \( \eta \triangleq -\beta^{-1} \|PB\|_2 \|H\|_2 - \alpha \beta \lambda \|S\|_2 - \beta \|SG\|_2 \) with \( \bar{w} \triangleq w^* + \|K\|_2 \). Moreover, it is shown in [1] that there exists \( \alpha \in \mathbb{R}_+ \)

---

7While we have the triple \((\sigma(\cdot), W = [\Lambda^{-1}W_0^T, -\Lambda^{-1}K_1, -\Lambda^{-1}K_2]^T, u(t) = -\hat{W}(t)\sigma(\cdot))\) in [1], we now have the triple \((x(t), W = [\Lambda^{-1}W_0^T]^T, u(t) = -Kx(t) - \hat{W}^T(t)Nx(t))\) in this paper. This is owing to the fact that a different nominal control law is considered here as compared with the one presented in [1].

8Since we utilize a projection bound in (3.13), it follows that \( \|\hat{W}(t)\|_2 \leq w^*, w^* \in \mathbb{R}_+ \).
such that (3.14) holds when

\[ \|SG_1\|_2 \|H\|_2 \lambda \bar{w} + \|SG_2\|_2 \|H\|_2 < \frac{1}{4\|PB\|_2}. \]  

(3.15)

Clearly, (3.15) illustrates that the closed-loop dynamical system remains bounded either if there does not exist significant unmodeled dynamics or the effect of system uncertainties is negligible.

3.1.2.2 Stability Limit of Proposed Model Reference Adaptive Control Laws with a Robustifying Term

We next present a concise overview of the stability limit with the proposed model reference adaptive control law in [1]. In particular, let the adaptive control law be given by

\[ u(t) = -Kx(t) - \hat{W}^T(t)Nx(t) - \hat{\mu}(t)B^TPe(t). \]  

(3.16)

Here, \( \hat{W}(t) \) satisfies the weight update law given by

\[ \dot{\hat{W}}(t) = \gamma \text{Proj}_m \left[ \hat{W}(t), Nx(t)e^T(t)PB \right], \quad \hat{W}(0) = \hat{W}_0, \]  

(3.17)

which is equivalent to (3.13). In addition, \( \hat{\mu}(t) \) is a projection operator-based adaptive robustifying term with the update law given by

\[ \dot{\hat{\mu}}(t) = \mu_0 \text{Proj} \left( \hat{\mu}(t), \|B^TPe(t)\|_2^2 - \sigma \hat{\mu}(t) \right), \quad \hat{\mu}(0) = \mu_0, \quad \hat{\mu}(0) \in \mathbb{R}_+, \]  

(3.18)

with \( \mu_0 \in \mathbb{R}_+ \) and \( \sigma \mu \in \mathbb{R}_+ \) being design parameters\(^9\). Here, we select the projection bound for (3.18) as

\[ \hat{\mu}(t) \leq \mu \psi, \quad \mu \triangleq \lambda \bar{w}^2 + \lambda_L, \]  

(3.19)

with \( \psi > 1 \). In other words, since \( \psi \) is strictly greater than one and arbitrary, one can select a large projection bound as necessary without resorting to precise bounds on system uncertainties.

\(^9\)Since \( \hat{\mu}(0) \in \mathbb{R}_+, \hat{\mu}(t) \in \mathbb{R}_+ \) holds automatically.
Considering the above model reference adaptive control law of [1], one can now show that if
\[
R \triangleq \begin{bmatrix}
1 & 0 & \eta_1 \\
0 & 2\mu\lambda_L & \eta_2 \\
\eta_1 & \eta_2 & \alpha
\end{bmatrix} > 0
\] (3.20)
holds, then boundedness of the closed-loop dynamical system trajectories \((e(t), z(t), \tilde{W}(t), \tilde{\mu}(t))\), \(\tilde{\mu}(t) = \hat{\mu}(t) - \mu \in \mathbb{R}\), is guaranteed [1]. In (3.20), \(\alpha \in \mathbb{R}_+\), \(\eta_1 \triangleq -\alpha\beta\lambda_U\bar{w}\|SG_1\|_2 - \alpha\beta\|SG_2\|_2\), and \(\eta_2 \triangleq -\beta^{-1}\|H\|_2 - \alpha\beta\mu\psi\lambda_U\|SG_1\|_2\). Moreover, it is shown in [1] that there exists \(\alpha \in \mathbb{R}_+\) such that (3.20) holds when
\[
\|SG_1\|_2\|H\|_2 < \frac{\sqrt{2\lambda_L} - 1}{\psi\lambda_U}, \quad \lambda_L > 0.5,
\] (3.21)
\[
\|SG_2\|_2\|H\|_2 + \|SG_1\|_2\|H\|_2 \lambda_U\bar{w} \\
< \frac{1}{\sqrt{2}} \sqrt{\left(\lambda_U^2\bar{w}^2 + \lambda_L\right) \left(2\lambda_L - 1 - \psi^2\lambda_U^2\|SG_1\|_2^2\|H\|_2^2 - 2\psi\lambda_U\|SG_1\|_2\|H\|_2\right)} \lambda_U.
\] (3.22)

Unlike the standard model reference adaptive control law summarized in the above subsection, it can be seen from (3.22) that the closed-loop dynamical system remains bounded even when we have significant system uncertainties (see [1] for details).

We now give the following remarks on the closed-loop system stability interplay between system uncertainties and unmodeled dynamics, which are not directly included in [1]. These remarks are also important on how to make necessary theoretical alterations for applying the results of [1] summarized above to the considered benchmark mechanical system setup discussed in the next section. Note that a version of these discussions can be also revealed by following the results in Section II of [70] for another approach.

**Remark 3.1.1** With the selection of \(\mu \triangleq \lambda_U^2\bar{w}^2 + \lambda_L\) given by (3.19), we obtain the stability limit for proposed model reference adaptive control law in (3.22). Moreover, (3.22) can be further written as
\[
\|SG_2\|_2\|H\|_2 + \|SG_1\|_2\|H\|_2 \lambda_U\bar{w} \\
< \frac{1}{\sqrt{2}} \sqrt{\frac{\mu \left(2\lambda_L - 1 - \psi^2\lambda_U^2\|SG_1\|_2^2\|H\|_2^2 - 2\psi\lambda_U\|SG_1\|_2\|H\|_2\right)}{\lambda_U}}.
\] (3.23)
What one can infer from (3.23) is that a large $\mu$ eases its satisfaction in the presence of unmodeled dynamics.

**Remark 3.1.2** Considering the benchmark mechanical system setup discussed in the next section, we have $G_1 = 0$. In other words, the unmodeled dynamics in this setup does not depend on the control signals. In the light of this fact, one can respectively obtain standard and proposed model reference adaptive control laws stability limits from (3.15) and (3.22) as

\[
\|SG_2\|_2 \|H\|_2 < \frac{1}{4 \|PB\|_2},
\]

(3.24)

\[
\|SG_2\|_2 \|H\|_2 < \sqrt{\frac{\mu(2\lambda_L - 1)}{2\lambda_L}},
\]

(3.25)

for the considered benchmark setup.

3.1.3 Experimental Studies on a Benchmark Mechanical System Setup

3.1.3.1 System Setup

Consider the benchmark mechanical system setup shown in Figure 3.2, which consists of parts produced by Quanser and assembled in our research laboratory. This setup involves an inverted pendulum on a cart coupled with another cart through a spring in the presence of unknown frictions.

In particular, this setup consists of two carts\(^{10}\). A single rod mounted on the first cart whose axis of rotation is perpendicular to the direction of the motion of this cart. The pendulum is attached to this rod. Note that second cart does not involve a pendulum as it is clear from Figure 3.2. First cart is driven by a rack and pinion mechanism using a 6 Volt DC motor, which ensures consistent and continuous traction. Both carts slide along a steel shaft using linear bearings. While the first cart’s position is measured using a sensor coupled to the rack via an additional pinion, the second cart’s position is assumed to be unmeasured for the control design purposes. Moreover, the second cart is attached to the first cart through a spring with spring constant 160 N/m. The pendulum is instrumented using a quadrature incremental encoder and it is mounted in front of the cart for enabling 360 degree rotation. The proposed control method discussed in Section 3.1.2.2 is implemented on the computer and run on MATLAB/Simulink by using

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\(^{10}\) Among several Quanser carts we have in our research laboratory, it is observed that possibly due to the DC motors having different frictions there were differences in the responses under the same nominal control law. To be precise, the cart with serial number 38297 is utilized as the first cart and the cart with serial number 37452 is utilized as the second cart for the experimental results presented in this paper.
Quanser Quarc Real Time Windows Target (Win64). The data transmission between the computer and the drivers is carried out with digital to analog converter Quanser Q8-USB data acquisition board. The linear voltage controlled power amplifier VoltPAQ-X1 is utilized to drive our experiments. The experimental studies run on MATLAB/Simulink with a sampling rate of 1000[Hz]. To summarize, the first cart with the pendulum represents the modeled dynamics and the second cart without any pendulum represents the unmodeled dynamics in our experimental study.

### 3.1.3.2 Results of the Experiments

Following the discussion given in the previous subsection, the linearized dynamics of the considered benchmark mechanical system setup satisfies

\[
\dot{x}_\text{p}(t) = A_p x_\text{p}(t) + B_p \begin{pmatrix}
\frac{m_p^2 l_p^2}{J_T} \\
\frac{\tau_\alpha K_p^2 \tau_\alpha K_m m_p l_p^2}{J_T} - \frac{B_\alpha (J_p + m_p l_p^2)}{J_T} \\
-\frac{(J_\alpha + m_p) m_p l_p}{J_T} \\
\end{pmatrix}
\]
+ $\begin{bmatrix}
0 & 0 \\
J_p + \frac{m_pl^2}{J_p} & \frac{m_pl^2}{J_p}
\end{bmatrix} u(t) + \begin{bmatrix}
0 \\
J_p + \frac{m_pl^2}{J_p}
\end{bmatrix} k^* \Delta V(t) + \begin{bmatrix}
0 \\
J_p + \frac{m_pl^2}{J_p}
\end{bmatrix} k^* W_0^T \sigma_p(\cdot) + \begin{bmatrix}
0 \\
J_p + \frac{m_pl^2}{J_p}
\end{bmatrix} k^* Hq(t),
(3.26)

\dot{q}(t) = \begin{bmatrix}
0 & 1 \\
-\frac{k}{M_z} & -\frac{B_{eq}}{M_z}
\end{bmatrix} q(t) + \begin{bmatrix}
0 & 0 & 0 & 0 & 0
\end{bmatrix} x_p(t),
(3.27)

where $J_T = (J_p(J_p + J_p m_p + J_eq m_p l_p^2)), J_eq = M_1 + \frac{\tau_{eq} K_c^2}{r_{mp}^2}$, $k^* = \frac{\tau_{eq} K_c \dot{\sigma}_u}{r_{mp} K_u}$, $x_p(t) = [x_c^T(t), \dot{x}_c^T(t), \dot{x}_c^T(t)]^T$, and $q(t) = [q_q^T(t), q_q^T(t)]^T$. In connection to (3.1), (3.2), and (3.3), we do not have a $G_1$ term in the above dynamics. Here, $W_0^T = k^*^{-1} \dot{W}_0^T$ and “$\dot{W}_0^T \sigma_p(\cdot)$” is included to capture the uncertainty due to matched unknown friction effects (see, for example, [67]), where it also includes the term $H = k^*^{-1} \begin{bmatrix} k & 0 \end{bmatrix}$ and “$-k x_c(t)$” (spring coefficient “$k$” is assumed to be unknown in this study). Table 3.1 includes the parameters and their values in the above dynamics [71]. The inverted pendulum is initialized at its upward position that is consistent with the linearized dynamics given above. Here, we consider the benchmark mechanical system setup involving an inverted pendulum on a cart coupled with another cart through a spring in the presence of unknown frictions. While the authors in [67] considered modeling these friction effects to improve their closed-loop system stability and performance, we let the proposed model reference adaptive control law to suppress these effects in this paper owing to their matched nature.

We first show the nominal control law response. For the experiments, we select a command that changes between $\pm 0.25$ m calculate $k^* = 1.7235$. The nominal control law gain matrix $K$ is obtained using a linear quadratic regulator based design with the weighting matrices $Q = \text{diag}[220, 400, 1, 1, 100]$ to penalize the states and $R = 0.5$ to penalize the control input. Figures 3.3, 3.4, 3.5, and 3.6 show the nominal control performance and control input in the presence of unmodeled dynamics.

We now show the proposed model reference adaptive control law response discussed in Section 3.1.2.2. In particular, Figures 3.7, 3.8, 3.9, 3.10, 3.11, and 3.12, respectively show the proposed model reference adaptive control performance, control input, and estimated weights and estimated $\mu$ term in the
Table 3.1: Values of the physical parameters in (3.26) and (3.27).

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Name</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau_m$</td>
<td>Motor Efficiency</td>
<td>1</td>
</tr>
<tr>
<td>$\tau_g$</td>
<td>Planetary Gearbox Efficiency</td>
<td>1</td>
</tr>
<tr>
<td>$K_g$</td>
<td>Planetary Gearbox Gear Ratio</td>
<td>3.71</td>
</tr>
<tr>
<td>$K_t$</td>
<td>Motor Current Torque Gear Constant</td>
<td>$7.68 \times 10^{-3}\ [\text{Nm/A}]$</td>
</tr>
<tr>
<td>$K_m$</td>
<td>Motor Back-emf Constant</td>
<td>$7.68 \times 10^{-3}\ [\text{V/(rad/s)}]$</td>
</tr>
<tr>
<td>$R_m$</td>
<td>Motor Armature Resistance</td>
<td>2.6 $[\Omega]$</td>
</tr>
<tr>
<td>$g$</td>
<td>Gravitational Constant on USF</td>
<td>$9.79\ [\text{m/s}^2]$</td>
</tr>
<tr>
<td>$J_p$</td>
<td>Pendulum Inertia</td>
<td>$1.20 \times 10^{-3}\ [\text{kg m}^2]$</td>
</tr>
<tr>
<td>$m_p$</td>
<td>Mass of Pendulum with T fitting</td>
<td>0.23 $[\text{kg}]$</td>
</tr>
<tr>
<td>$M_1, M_2$</td>
<td>Mass of Carts</td>
<td>0.507 and 1.0675 $[\text{kg}]$</td>
</tr>
<tr>
<td>$B_{eq}$</td>
<td>Equivalent Viscous Damping Coefficient at the cart 1</td>
<td>4.3 $[\text{Nms/rad}]$</td>
</tr>
<tr>
<td>$B_{eqz}$</td>
<td>Equivalent Viscous Damping Coefficient at the cart 2</td>
<td>1.1 $[\text{Nms/rad}]$</td>
</tr>
<tr>
<td>$B_p$</td>
<td>Equivalent Viscous Damping Coefficient at the pendulum</td>
<td>0.0024 $[\text{Nms/rad}]$</td>
</tr>
<tr>
<td>$l_p$</td>
<td>Pendulum Length</td>
<td>0.6413 $[\text{m}]$</td>
</tr>
<tr>
<td>$r_{mp}$</td>
<td>Motor Pinion Radius</td>
<td>$6.35 \times 10^{-3}\ [\text{m}]$</td>
</tr>
<tr>
<td>$J_m$</td>
<td>Rotor Moment of Inertia</td>
<td>$3.9 \times 10^{-7}\ [\text{kg m}^2]$</td>
</tr>
<tr>
<td>$k$</td>
<td>Spring Constant</td>
<td>160 $[\text{N/m}]$</td>
</tr>
</tbody>
</table>

Figure 3.3: Nominal control law cart tracking result with unmodeled dynamics.
Figure 3.4: Nominal control law pendulum tracking result with unmodeled dynamics.

Figure 3.5: Nominal control input with unmodeled dynamics.

presence of unmodeled dynamics. Moreover, Figure 3.13 shows the comparison on tracking responses of nominal and proposed model reference adaptive control laws and it can be seen from this figure that
the proposed method can successfully suppresses the uncertainties resulting from unknown frictions in the presence of unmodeled dynamics. Note that in Section 3.1.3, we have the triple $(x_p(t), u(t) = -Kx(t) - \hat{W}^T(t)Nx(t), \hat{w} \triangleq w^* + \|K\|^2_2)$, we now have the triple $(\sigma_p(\cdot) = [-x^T(t)K^T x^T_p(t)]^T, u(t) = -Kx(t) - \hat{W}^T(t)\sigma_p(\cdot), \bar{w} \triangleq \|K\|^2_2 (w^*l^* + 1)$ with $l^* > 1$ because of the selection of the basis function here, which does not change stability limits given by (3.24) and (3.25). In addition, it should be also noted that the projection operator with a dead-zone function in the update law and sigma modification for the adaptive control law (3.17) are utilized in the experimental results of this paper. We refer to Appendix B for theoretical details.

For the control law given by (3.13), we select the learning rate $\gamma = 12$, $\sigma_w = 0.0001$, and the element wise projection bound as $W_{\text{max},1} = 1.1$ and $W_{\text{max},i} = 165$, where $i = 2, 3, 4, 5$. Error bounds for dead-zone function is selected as $e_0 = 0.01(\text{diag}[2, 1, 1, 2, 1])$. Moreover, for the update law given by (3.18), we select learning rate $\mu_0 = 1000$, $\sigma_\mu = 0.00001$, and the projection bound is selected as $\hat{\mu}_{\text{max}} = 1.750005 \times 10^6$ and we calculate $\mu = 6.53 \times 10^5$.

Note that the fundamental stability limit given by (3.24) does not hold for the standard model reference adaptive control law (to compute this limit, we use $\lambda_L = 0.9$), which results in

$$\|SG_2\|_2 \|H\|_2 = 330.4310,$$

(3.28)
Thus, there does not exist guarantees on the boundedness of the closed-loop signals for the standard model reference adaptive control law (we do not include its figures here for this reason). In the proposed case, the fundamental stability limit given by (3.25) holds, which results in

\[ \frac{1}{4\|PB\|_2} = 0.4163. \]  

(3.29)

That is, the proposed approach guarantees the boundedness of the closed-loop signals in the presence of considered unmodeled dynamics.
Figure 3.8: Proposed model reference control law pendulum tracking result with unmodeled dynamics.

Figure 3.9: Proposed model reference control input with unmodeled dynamics.
Figure 3.10: Zoomed proposed model reference control input with unmodeled dynamics.

Figure 3.11: Proposed model reference control of estimated weights.
Figure 3.12: Proposed model reference control of estimated $\hat{\mu}(t)$.

Figure 3.13: Nominal (blue) vs proposed model reference control (green) law cart tracking result with unmodeled dynamics.
3.1.4 Conclusion

To complement our recent theoretical studies on the stability of model reference adaptive control laws with unmodeled dynamics, we presented experimental results on a coupled rigid body system with a flexible interconnection link. Specifically, this benchmark setup involved an inverted pendulum on a cart coupled with another cart through a spring in the presence of unknown frictions. We first overviewed our theoretical results documented in [1] on the stability interplay between a class of unmodeled dynamics and system uncertainties for model reference adaptive control laws as well as the proposed robustifying term to relax the resulting interplay. We then discussed how to make necessary theoretical alterations to make these results fit to the considered benchmark setup and presented our experimental study. Consistent with the findings in [1], the conducted experiments showed that the proposed robustifying term guarantees overall stability of closed-loop system trajectories.

3.2 A Generalization of Fundamental Stability Limits of Model Reference Adaptive Control

the Presence of a Class of Nonlinear Unmodeled Dynamics\textsuperscript{11}

A fundamental trade-off in the design of model reference adaptive controllers is to obtain a stable closed-loop dynamical system in the presence of not only system uncertainties but also unmodeled dynamics, which can be either linear or nonlinear. Specifically, there exist stability limits for these controllers, where the closed-loop dynamical system subject to an adaptive controller preserves stability either if there does not exist significant unmodeled dynamics or the effect of system uncertainties is negligible. To address this problem we recently propose an adaptive control architecture to relax these stability limits for linear unmodeled system dynamics. Generalizations of the proposed adaptive control architecture are made in this paper to a broader class of unmodeled dynamics including nonlinear functions. Specifically, we utilize tools and methods from Lyapunov stability to relax stability limits for model reference adaptive controllers in the presence of nonlinear unmodeled dynamics. An illustrative numerical example is provided to demonstrate the efficacy of the proposed approach.

\textsuperscript{11}This section is previously published in [72]. Permission is included in Appendix C.
3.2.1 Introduction

A fundamental trade-off in the design of model reference adaptive controllers is to obtain a stable closed-loop dynamical system in the presence of not only system uncertainties but also unmodeled dynamics (see the seminal paper [17]), which can be either linear or nonlinear. Specifically, there exist stability limits for these controllers, where the closed-loop dynamical system subject to an adaptive controller preserves stability either if there does not exist significant unmodeled dynamics or the effect of systems uncertainties is negligible [1, 18, 73]. We refer to, for example, [1] for a literature review on model reference adaptive control of uncertain dynamical systems in the presence of unmodeled dynamics.

In particular, in order to address this problem, we recently proposed model reference adaptive control approaches which obtain relaxed fundamental stability limits in the presence of linear unmodeled dynamics [1, 73]. To this end, the contribution of this paper is to generalize of the proposed model reference adaptive control architecture in [1, 73] to a broader class of unmodeled dynamics containing nonlinear functions depending on the control signal and the system state vector. Specifically, this generalization is accomplished under minimal structural assumptions on the given nonlinear unmodeled system dynamics and by utilizing tools and methods from Lyapunov stability. Similar to the contributions in [1, 73], the key feature of our generalized framework allows the closed-loop dynamical system to preserve stability in the presence of large system uncertainties when the class of nonlinear unmodeled dynamics satisfy a set of conditions.

The organization of this paper is as follows. Section 3.2.2 presents the fundamental stability limits for standard model reference adaptive controllers in the presence of a class of nonlinear unmodeled dynamics, where these limits are relaxed in Section 3.2.3 with the proposed approach. Once again, the content of Sections 3.2.2 and 3.2.3 can be viewed as a generalization of our previous results in [1, 73]. An illustrative numerical example is provided in Section 3.2.4 to demonstrate the efficacy of the proposed approach and concluding remarks are summarized in Section 3.2.5.

The notation used in this paper is standard and similar to, for example, [1]. For self-containedness, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^n$ denotes the set of $n \times 1$ real column vectors, $\mathbb{R}^{n \times m}$ denotes the set of $n \times m$ real matrices, $\mathbb{R}_+^n$ (resp. $\mathbb{R}_{++}^n$) denotes the set of positive (resp., nonnegative) real numbers, $\mathbb{R}_{++}^{n \times n}$ (resp., $\mathbb{R}_+^{n \times n}$) denotes the set of $n \times n$ positive-definite (resp. nonnegative-definite) real matrices, $\mathbb{S}_{++}^{n \times n}$ denotes the set of $n \times n$ symmetric real matrices, $\mathbb{D}^{n \times n}$ denotes the set of $n \times n$ real matrices with diagonal scalar entries,
\((\cdot)^T\) denotes the transpose operator, \((\cdot)^{-1}\) denotes the inverse operator, \(\text{tr}(\cdot)\) denotes the trace operator, 
\(\|\cdot\|_2\) denotes the Euclidian norm, and \(\lambda_{\min}(A)\) (resp., \(\lambda_{\max}(A)\)) denotes the minimum (resp., maximum) eigenvalue of the real matrix \(A \in \mathbb{R}^{n \times n}\). For \(x \in \mathbb{R}^n\), \(\text{sat}(x) = [\text{sat}(x_1), \text{sat}(x_2), \ldots, \text{sat}(x_n)]^T\), \(\sin(x) = [\sin(x_1), \sin(x_2), \ldots, \sin(x_n)]^T\), and \(\cos(x) = [\cos(x_1), \cos(x_2), \ldots, \cos(x_n)]^T\) represent the saturation, sine, and cosine functions, respectively.

### 3.2.2 Fundamental Stability Limit of Standard Model Reference Adaptive Controllers

In this section, we present the effect of a class of unmodeled dynamics on standard model reference adaptive controllers and the resulting fundamental stability limit. Specifically, consider the uncertain dynamical system subject to a class of nonlinear unmodeled dynamics given by

\[
\dot{x}(t) = Ax(t) + Bu(t) + B\delta(x(t)) + Bp(t), \quad x(0) = x_0, \tag{3.32}
\]

\[
\dot{q}(t) = f(q(t), u(t)), \quad q(0) = q_0, \tag{3.33}
\]

\[
p(t) = Hq(t), \tag{3.34}
\]

where \(x(t) \in \mathbb{R}^n\) is the measurable state vector, \(u(t) \in \mathbb{R}^m\) is the control input, \(q(t) \in \mathbb{R}^p\) and \(p(t) \in \mathbb{R}^m\) are respectively the unmodeled dynamics state and output vectors, \(A \in \mathbb{R}^{n \times n}\) is a known system matrix, \(B \in \mathbb{R}^{n \times m}\) is a known input matrix such that the pair \((A, B)\) is controllable, \(\Lambda \in \mathbb{R}^{m \times m} \cap \mathbb{D}^{m \times m}\) is an unknown control effectiveness matrix, \(\delta: \mathbb{R}^n \rightarrow \mathbb{R}^m\) is a system uncertainty, \(f: \mathbb{R}^p \times \mathbb{R}^m \rightarrow \mathbb{R}^p\) is the unmodeled dynamics function that satisfies \(f(0, 0) = 0\), and \(H \in \mathbb{R}^{m \times p}\) is output matrix associated with the unmodeled dynamics. Note that since \(\Lambda \in \mathbb{R}^{m \times m} \cap \mathbb{D}^{m \times m}\), it follows that there exists \(\lambda_L \in \mathbb{R}_+\) and \(\lambda_U \in \mathbb{R}_+\) such that \(\lambda_L \leq \|\Lambda\|_2 \leq \lambda_U\) holds.

**Remark 3.2.1** Let \(z(t) = \beta q(t), z(t) \in \mathbb{R}^p\), where \(\beta \in \mathbb{R}_+\) is a free variable to be used later in this paper. Then, the nonlinear unmodeled dynamics given by (3.33) and (3.34) can be equivalently represented as

\[
\dot{z}(t) = \beta f(\beta^{-1}z(t), \Lambda u(t)), \quad z(0) = \beta q_0, \tag{3.35}
\]

\[
p(t) = \beta^{-1}H z(t). \tag{3.36}
\]
Using the state transformation given in Remark 3.2.1, the uncertain dynamical system nonlinear unmodeled dynamics can be equivalently written as

\[
\dot{x}(t) = Ax(t) + B\Lambda u(t) + B\delta(x(t)) + \beta^{-1}BH z(t), \quad x(0) = x_0, \tag{3.37}
\]

\[
\dot{z}(t) = \beta f(\beta^{-1}z(t), \Lambda u(t)), \quad z(0) = \beta q_0. \tag{3.38}
\]

**Assumption 3.2.1** The system uncertainty \(\delta : \mathbb{R}^n \to \mathbb{R}^m\) can be parameterized as

\[
\delta(x) = W_0^T \sigma_0(x), \quad x \in \mathbb{R}^n, \tag{3.39}
\]

where \(W_0 \in \mathbb{R}^{s \times m}\) is an unknown weight matrix and \(\sigma_0 \in \mathbb{R}^n \to \mathbb{R}^s\) is a known basis function on the form \(\sigma_0(x) = [\sigma_0_1(x), \sigma_0_2(x), ..., \sigma_0_s(x)]^T\). In addition, the basis function satisfies the inequality given by

\[
\|\sigma_0(x(t))\|_2 \leq l_0 \|x(t)\|_2 + l_c, \quad x(t) \in \mathbb{R}^n, \tag{3.40}
\]

where \(l_0 \in \mathbb{R}_+\) and \(l_c \in \mathbb{R}_+\).

We next consider the reference system given by

\[
\dot{x}_r(t) = A_r x_r(t) + B_r c(t), \quad x_r(0) = x_{r0}, \tag{3.41}
\]

where \(x_r(t) \in \mathbb{R}^n\) is the reference state vector, \(c(t) \in \mathbb{R}^m\) is a given uniformly continuous bounded command, \(A_r \in \mathbb{R}^{n \times n}\) is the Hurwitz reference system matrix, and \(B_r \in \mathbb{R}^{n \times m}\) is the command input matrix. Note that since \(A_r\) is Hurwitz, there exists \(P \in \mathbb{R}^{n \times n}_+ \cap \mathbb{S}^{n \times n}\) such that

\[
0 = A_r^T P + PA_r + I \tag{3.42}
\]

holds. We now introduce the following standard assumption.

**Assumption 3.2.2** There exist \(K_1 \in \mathbb{R}^{m \times n}\) and \(K_2 \in \mathbb{R}^{m \times m}\) such that \(A_r \triangleq A - BK_1\) and \(B_r \triangleq BK_2\) hold.

It now follows from Assumption 3.2.1 and Assumption 3.2.2 that (3.37) can be rewritten as

\[
\dot{x}(t) = A_r x(t) + B_r c(t) + B \Lambda [u(t) + W^T \sigma(\cdot)] + \beta^{-1}BH z(t), \tag{3.43}
\]
where \( W \triangleq [\Lambda^{-1}W_0^T, -\Lambda^{-1}K_1, -\Lambda^{-1}K_2]^T \in \mathbb{R}^{q \times m}, \sigma(\cdot) \triangleq [\sigma_0^T(x(t)), x^T(t), c^T(t)]^T \in \mathbb{R}^q \), and \( q \triangleq s + n + m \).

In addition, letting \( e(t) \triangleq x(t) - x_r(t) \) be the system error, the system error dynamics can be given using (3.41) and (3.43) as

\[
\dot{e}(t) = A_r e(t) + B \Lambda [u(t) + W^T \sigma(\cdot)] + \beta^{-1} B H z(t), \quad e(0) = e_0.
\] (3.44)

Considering (3.44), let the adaptive control law be given by

\[
u(t) = -\hat{W}^T(t) \sigma(\cdot)
\] (3.45)

where an upper bound for \( \|\sigma(\cdot)\|_2 \) can be written using Assumption 3.2.1 as \( \|\sigma(\cdot)\|_2 \leq l \|e(t)\|_2 + d^* \) with \( d^* \in \mathbb{R}_+ \) being an upper bound for \( l \|x_r(t)\|_2 + \|e(t)\|_2 + l_c \), and \( \hat{W}(t) \in \mathbb{R}^{q \times m} \) is an estimate of the unknown weight \( W \) satisfying the projection operator-based weight update law

\[
\dot{\hat{W}}(t) = \gamma \text{Proj} [\hat{W}(t), \sigma(\cdot) e^T(t) P B] , \quad \hat{W}(0) = \hat{W}_0 \geq 0,
\] (3.46)

where \( \gamma \in \mathbb{R}_+ \) is the learning rate and \( \text{Proj} \) is the projection operator defined next [7].

**Definition 3.2.1** Let \( \Omega = \{ \theta \in \mathbb{R}^n : (\theta_{i}^{\min} \leq \theta_i \leq \theta_{i}^{\max}) \}, \ i = 1, 2, \cdots ,n, \) be a convex hypercube in \( \mathbb{R}^n \), where \( (\theta_{i}^{\min}, \theta_{i}^{\max}) \) represent the minimum and maximum bounds for the \( i \)th component of the \( n \)-dimensional parameter vector \( \theta \). In addition, let \( \Omega_\varepsilon = \{ \theta \in \mathbb{R}^n : (\theta_{i}^{\min} + \varepsilon \leq \theta_i \leq \theta_{i}^{\max} - \varepsilon) \}, \ i = 1, 2, \cdots ,n, \) be a second hypercube for a sufficiently small positive constant \( \varepsilon \), where \( \Omega_\varepsilon \subset \Omega \). Then, the projection operator \( \text{Proj} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) is defined component-wise by

\[
\text{Proj}(\theta, y) \triangleq \begin{cases} 
\left( \frac{\theta^{\max} - \theta}{\varepsilon} \right) y_i, & \text{if } \theta_i > \theta_{i}^{\max} - \varepsilon \text{ and } y_i > 0, \\
\left( \frac{\theta - \theta_{i}^{\min}}{\varepsilon} \right) y_i, & \text{if } \theta_i < \theta_{i}^{\min} + \varepsilon \text{ and } y_i < 0, \\
y_i, & \text{otherwise},
\end{cases}
\] (3.47)

where \( y \in \mathbb{R}^n \).

Based on Definition 3.2.1,

\[
(\theta - \theta^*)^T (\text{Proj}(\theta, y) - y) \leq 0,
\] (3.48)
holds \[53, 74\]. Here, we use the generalization of this definition to matrices as

\[
\text{Proj}_m(\Theta, Y) = (\text{Proj}(\text{col}_1(\Theta), \text{col}_1(Y)), \ldots, \text{Proj}(\text{col}_m(\Theta), \text{col}_m(Y))),
\]

(3.49)

where \(\Theta \in \mathbb{R}^{n \times m}, Y \in \mathbb{R}^{n \times m}\), and \(\text{col}_i(\cdot)\) denotes the \(i\)-th column operator. For a given \(\Theta^* \in \mathbb{R}^{n \times m}\), it now follows from (3.48) that \(\text{tr}\left[\left(\Theta - \Theta^*\right)^{T}(\text{Proj}_m(\Theta, Y) - Y)\right] = \sum_{i=1}^{m}\left[\text{col}_i(\Theta - \Theta^*)^{T}(\text{Proj}(\text{col}_i(\Theta), \text{col}_i(Y)) - \text{col}_i(Y))\right] \leq 0\). Note that as a result of using a projection operator-based update law in (3.46), it follows that

\[
\|\dot{W}(t)\|_2 \leq w^*, \quad w^* \in \mathbb{R}_+.
\]

Using (3.45) respectively in (3.44) and (3.38), one can write

\[
\dot{e}(t) = A_r e(t) - B\Lambda \tilde{W}^T(t)\sigma(\cdot) + \beta^{-1}BH z(t), \quad e(0) = e_0, \quad (3.50)
\]

\[
\dot{z}(t) = \beta f(\beta^{-1}z(t), -\Lambda \tilde{W}^T(t)\sigma(\cdot)), \quad z(0) = \beta q_0, \quad (3.51)
\]

where \(\tilde{W}(t) \triangleq \hat{W}(t) - W \in \mathbb{R}^{q \times m}\).

**Assumption 3.2.3** There exists a continuously differentiable function \(\Phi : \mathbb{R}^p \to \mathbb{R}\) such that \(\Phi(\cdot)\) is positive definite, radially unbounded, \(\Phi(0) = 0\), and

\[
\alpha \Phi'(z(t))\beta f(\beta^{-1}z(t), -\Lambda \tilde{W}^T(t)\sigma(\cdot)) \leq -\alpha \|z(t)\|_2^2 + 2\alpha \beta \phi_0 \|z(t)\|_2 \|\Lambda\|_2 \|\tilde{W}(t)\|_2 \|\sigma(\cdot)\|_2
\]

\[+ \alpha z_0 \|z(t)\|_2, \quad (3.52)\]

holds for a given \(\alpha \in \mathbb{R}_+\) and for all \(z(t)\). In (3.52), \(\Phi' \triangleq \partial \Phi / \partial z\), \(\phi_0\) and \(z_0\) are positive definite bounded constants.

In the reminder of this section, we show boundedness of the closed-loop dynamical system, when a fundamental stability limit holds for the highlighted standard model reference adaptive control problem formulation under the above structural condition assumption.

**Theorem 3.2.1** Consider the uncertain dynamical system given by (3.32), the nonlinear unmodeled dynamics given by (3.33) and (3.34), the reference system given by (3.41), and the adaptive control law given by (3.45) and (3.46). Assume that Assumption 3.2.1, Assumption 3.2.2, and Assumption 3.2.3 hold. Assume, in
addition, there exists an \( \alpha \in \mathbb{R}_+ \) such that

\[
\mathcal{R} \triangleq \begin{bmatrix} 1 & \eta \\ \eta & \alpha \end{bmatrix} > 0, \quad (3.53)
\]

holds, where \( \eta \triangleq -\beta^{-1} \|PB\|_2 \|H\|_2 - \alpha \beta \phi_0 \lambda_U w^* l \) and \( l \triangleq (1 + l_0) \). Then, the solution \((e(t), z(t), \tilde{W}(t))\) of the closed-loop dynamical system is uniformly ultimately bounded.

**Proof.** To show uniform ultimate boundedness of the solution \((e(t), z(t), \tilde{W}(t))\), consider the Lyapunov-like function given by

\[
\mathcal{V}(e, z, \tilde{W}) = e^T P e + \alpha \Phi(z) + \gamma^{-1} \text{tr}(\tilde{W} \Lambda^* z^T \tilde{W} \Lambda^*) \quad (3.54)
\]

Note that \( \mathcal{V}(0, 0, 0) = 0 \) and \( \mathcal{V}(e, z, \tilde{W}) > 0 \) for all \((e, z, \tilde{W}) \neq (0, 0, 0)\). Differentiating (3.54) along the closed-loop dynamical system trajectories and using Assumption 3.2.3 yields

\[
\dot{\mathcal{V}}(e(t), z(t), \tilde{W}(t)) \leq -\|e(t)\|^2_2 + 2\beta^{-1} e(t)^T P B H z(t) \\
+ \alpha \Phi'(z(t)) \beta f(\beta^{-1} z(t), -\Lambda \tilde{W}^T(t) \sigma(\cdot)) \\
\leq -\|e(t)\|^2_2 - \alpha \|z(t)\|^2_2 + 2\beta^{-1} \|e(t)\|_2 \|P B\|_2 \|H\|_2 \|z(t)\|_2 \\
+ 2\alpha \beta \phi_0 \|z(t)\|_2 \|\Lambda\|_2 \|\tilde{W}(t)\|_2 \|\sigma(\cdot)\|_2 + \alpha \zeta_0 \|z(t)\|_2. \quad (3.55)
\]

Using similar steps as the proof of Theorem 1 in [73], it follows from (3.55) that

\[
\dot{\mathcal{V}}(e(t), z(t), \tilde{W}(t)) \leq -\xi^T(t) \check{\mathcal{R}} \xi(t) + 2\alpha \beta \phi_0 \|z(t)\|_2 \lambda_U w^* d^* + \alpha \zeta_0 \|z(t)\|_2, \quad (3.56)
\]

where \( \xi(t) \triangleq [\|e(t)\|_2, \|z(t)\|_2]^T \). Finally, (3.56) can be rewritten as

\[
\mathcal{V}(e(t), z(t), \tilde{W}(t)) \leq -\lambda(\mathcal{R}) \|\xi(t)\|_2 \left[\|\xi(t)\|_2 - \frac{r_0}{\lambda(\mathcal{R})}\right], \quad (3.57)
\]

where \( r_0 \triangleq 2\alpha \beta \phi_0 \lambda_U w^* d^* + \alpha \zeta_0 \). It follows from (3.57) that there exists a compact set such that \( \dot{\mathcal{V}}(e(t), z(t), \tilde{W}(t)) < 0 \) outside of this set, and hence, the solution \((e(t), z(t), \tilde{W}(t))\) is uniformly ultimately bounded [7, 57]. \[\square\]
We note that as shown in [73] (also see [18]), there exists $\alpha \in \mathbb{R}_+$ such that (3.53) is satisfied if
\[
\phi_0 \|H\|_2 < \frac{1}{4w^*l\lambda_U \|PB\|_2}, \tag{3.58}
\]
holds. This condition which depends on the allowable system uncertainty and the amount of unmodeled dynamics present, corresponds to all the leading principle minors of $\mathcal{R}$ being positive.

**Remark 3.2.2** It follows from the results in Theorem 3.2.1 and the existence of $\alpha$, then the solution $(e(t), z(t), \hat{W}(t))$ of the closed-loop dynamical system is uniformly ultimately bounded when the fundamental stability limit given by (3.58) holds. Specifically, the standard model reference adaptive control formulation considered in this section cannot tolerate large system uncertainties even when nonlinear unmodeled dynamics satisfy a set of conditions. A version of this fundamental stability limit is also revealed in [18],[1, 73].

The following result presents an extension of the above theorem.

**Corollary 3.2.1** Consider the uncertain dynamical system given by (3.32), the nonlinear unmodeled dynamics given by $\dot{q}(t) = f(q(t), x(t), \Lambda u(t))$ and (3.34), the reference system given by (3.41), and the adaptive control law given by (3.45) and (3.46). Assume that Assumption 3.2.1 and Assumption 3.2.2 hold, and there exists a continuously differentiable function $\Phi : \mathbb{R}^p \to \mathbb{R}$ such that $\Phi(\cdot)$ is positive definite, radially unbounded, $\Phi(0) = 0$, and
\[
\alpha \Phi'(z(t))\beta f(\beta^{-1}z(t), x(t), -\Lambda \hat{W}(t)\sigma(\cdot)) \leq -\alpha \|z(t)\|_2^2 + 2\alpha \beta \phi_1 \|z(t)\|_2 \|\Lambda\|_2 \|\hat{W}(t)\|_2 \|\sigma(\cdot)\|_2 + 2\alpha \beta \phi_2 \|z(t)\|_2 \|x(t)\|_2 + \alpha z_0 \|z(t)\|_2, \tag{3.59}
\]
holds, where $\phi_1$ and $\phi_2$ are positive bounded constants. Assume, in addition, there exists an $\alpha \in \mathbb{R}_+$ such that
\[
\phi_1 \|H\|_2 \lambda U w^* l + \phi_2 \|H\|_2 < \frac{1}{4 \|PB\|_2}, \tag{3.60}
\]
holds. Then, the solution $(e(t), z(t), \hat{W}(t))$ of the closed-loop dynamical system is uniformly ultimately bounded (note that $\dot{q}(t) = f(q(t), x(t), \Lambda u(t))$ can be written as $\dot{z}(t) = \beta f(\beta^{-1}z(t), x(t), \Lambda u(t))$ with using the state transformation given in Remark 3.2.1).

**Proof.** The result follows using similar steps in our recent work Section 2 of [1] as well as the proof of Theorem 3.2.1, and hence, is omitted. \qed
As it can be readily seen from (3.58) and (3.60) that the closed-loop dynamical system remains bounded either if there does not exist significant nonlinear unmodeled dynamics or the effect of system uncertainties is negligible. In the next section, we resort to our recent methods in [1, 73] to relax this fundamental limit.

3.2.3 Relaxing Fundamental Stability Limit of Model Reference Adaptive Controllers

In this section, we resort to an adaptive robustifying term proposed recently in our papers [1, 73], which is augmented with (3.45) to relax the fundamental stability limit given by (3.58). In particular, let the adaptive control law be given by

$$u(t) = -\hat{W}^T(t)\sigma(\cdot) - \hat{\mu}(t)B^TPe(t),$$  \hspace{1cm} (3.61)

where $\hat{W}(t)$ satisfies the weight update law given by (3.46) and $\hat{\mu}(t)$ is a projection operator-based adaptive robustifying term that satisfies [1, 73]

$$\dot{\hat{\mu}}(t) = \mu_0 \text{Proj}\left(\hat{\mu}(t), \left\|B^TPe(t)\right\|^2 - \sigma_\mu \hat{\mu}(t)\right), \quad \hat{\mu}(0) = \mu_0, \quad \hat{\mu}(t) \in \mathbb{R}_+,$$  \hspace{1cm} (3.62)

with $\mu_0 \in \mathbb{R}_+$ and $\sigma_\mu \in \mathbb{R}_+$ being design parameters. Note that since $\hat{\mu}(0) \in \mathbb{R}_+$, then $\hat{\mu}(t) \in \mathbb{R}_+$ holds. In addition, we select the projection bound for (3.62) as

$$\hat{\mu}(t) \leq \mu \psi, \quad \mu = \frac{2\lambda_\psi \lambda_\psi L^2 w^2}{\epsilon - \lambda_\psi^2 \psi^2}, \quad \psi > 1,$$  \hspace{1cm} (3.63)

where $\epsilon \in \mathbb{R}_+$ is chosen such that $\epsilon > \lambda_\psi^2 \psi^2$ holds.

Next, using (3.61) respectively in (3.44) and (3.38), one can write

$$\dot{e}(t) = A_e e(t) - B\Lambda \hat{W}^T(t)\sigma(\cdot) - \hat{\mu}(t)BAB^T Pe(t) + \beta^{-1}BH_\zeta(t), \quad e(0) = e_0,$$  \hspace{1cm} (3.64)

$$\dot{z}(t) = \beta \dot{f}(\beta^{-1}\zeta(t), -\Lambda(\hat{W}^T(t)\sigma(\cdot) + \hat{\mu}(t)B^T Pe(t))), \quad \zeta(0) = z_0.$$  \hspace{1cm} (3.65)

**Assumption 3.2.4** There exists a continuously differentiable function $\Phi : \mathbb{R}^p \rightarrow \mathbb{R}$ such that $\Phi(\cdot)$ positive definite, radially unbounded, $\Phi(0) = 0$, and
\[
\alpha \Phi'(z(t)) \beta f(\beta^{-1}z(t), -\Lambda(\hat{W}^T(t) \sigma(\cdot) + \hat{\mu}(t) B^T Pe(t)))
\leq -\alpha \|z(t)\|_2^2 + 2 \alpha \beta \phi_0 \|z(t)\|_2 \|\Lambda\|_2 \|\hat{W}(t)\|_2 \|\sigma(\cdot)\|_2
+ 2 \alpha \beta \phi_0 \|\hat{\mu}(t)\| \|\Lambda\|_2 \|z(t)\|_2 \|B^T Pe(t)\|_2 + \alpha z_0 \|z(t)\|_2,
\] (3.66)

holds for a given \(\alpha \in \mathbb{R}_+\) and for all \(z(t)\).

In the reminder of this section, we show uniform ultimate boundedness of the closed-loop dynamical system with the above adaptive robustifying term and show that the proposed approach relaxes the fundamental stability limit given by (3.58) in the presence the considered nonlinear unmodeled dynamics.

**Theorem 3.2.2** Consider the uncertain dynamical system given by (3.32), the nonlinear unmodeled dynamics given by (3.33) and (3.34), the reference system given by (3.41), and the adaptive control law given by (3.61), (3.46), and (3.62). Assume that Assumption 3.2.1, Assumption 3.2.2, and Assumption 3.2.4 hold.

Assume, in addition, there exists an \(\alpha \in \mathbb{R}_+\) such that

\[
\mathcal{R} \triangleq \begin{bmatrix} 1 & 0 & \eta_1 \\ 0 & 2 \mu \lambda \eta_2 & \eta_2 \\ \eta_1 & \eta_2 & \alpha \end{bmatrix} > 0,
\] (3.67)

holds, where \(\eta_1 \triangleq -\alpha \beta \phi_0 \lambda \psi \|H\|_2 \) and \(\eta_2 \triangleq \beta^{-1} \|H\|_2 - \alpha \beta \phi_0 \mu \psi \lambda \). Then, the solution \((e(t), z(t), \hat{W}(t), \hat{\mu}(t))\), where \(\hat{\mu}(t) \triangleq \hat{\mu}(t) - \mu\), of the closed-loop dynamical system is uniformly ultimately bounded.

**Proof.** To show uniform ultimate boundedness of the solution \((e(t), z(t), \hat{W}(t), \hat{\mu}(t))\), consider the Lyapunov-like function given by

\[
\mathcal{V}(e, z, \hat{W}, \hat{\mu}) = e^T Pe + \alpha \Phi(z) + \gamma^{-1} \text{tr}(\hat{W} \Lambda^\frac{1}{2})^T(\hat{W} \Lambda^\frac{1}{2}) + \mu_0^{-1} \hat{\mu}^2 \lambda \eta.
\] (3.68)

Note that \(\mathcal{V}(0, 0, 0, 0) = 0\) and \(\mathcal{V}(e, z, \hat{W}, \hat{\mu}) > 0\) for all \((e, z, \hat{W}, \hat{\mu}) \neq (0, 0, 0, 0)\). Differentiating (3.68) along the closed-loop dynamical system trajectories and utilizing Assumption 3.2.4 yields
\[ \mathcal{V}(e(t), z(t), \tilde{W}(t), \tilde{\mu}(t)) = -e^T(t)e(t) - 2e^T(t)PBA\tilde{W}^T(t)\sigma(\cdot) - 2\tilde{\mu}(t)e^T(t)PB\lambda B^T Pe(t) \]
\[ + \alpha \Phi' (z(t)) \beta f(\beta^{-1} z(t), -\Lambda(\tilde{W}^T(t)\sigma(\cdot) + \tilde{\mu}(t)B^T Pe(t))) \]
\[ + 2\beta^{-1} e^T(t)PBHz(t) + 2\gamma^{-1} \tau \tilde{W}^T(t)\hat{W}(t)\Lambda + 2\mu_0^{-1} \tilde{\mu}(t)\tilde{\mu}(t)\lambda_L \]
\[ \leq -\|e(t)\|^2_2 - \alpha \|z(t)\|^2_2 - 2\mu_0 \|B^T Pe(t)\|^2_2 \]
\[ - 2\sigma_\mu \lambda_L \tilde{\mu}^2(t) + 2\sigma_\mu \lambda_L \tilde{\mu}(t) + 2\beta^{-1} \|B^T Pe(t)\|_2^2 \|H\|_2^2 \|z(t)\|_2 \]
\[ + 2\alpha \beta \phi_0 \|\Lambda\|_2 \|\tilde{W}(t)\|_2 \|z(t)\|_2 \|\sigma(\cdot)\|_2 + \alpha \xi_0 \|z(t)\|_2 \]
\[ + 2\alpha \beta \phi_0 \|\tilde{\mu}(t)\| \|\Lambda\|_2 \|z(t)\|_2 \|B^T Pe(t)\|_2. \quad (3.69) \]

We can obtain an upper bound for (3.69) using similar steps as the proof of Theorem 2 in our recent work [73] which follows as

\[ \mathcal{V}(e(t), z(t), \tilde{W}(t), \tilde{\mu}(t)) \leq -\xi^T(t)\mathcal{R}\xi(t) + r_0 \|z\|_2 + c^*, \quad (3.70) \]

where \( \xi(t) \triangleq \left[\|e(t)\|_2, \|B^T Pe(t)\|_2, \|z(t)\|_2\right]^T \), \( r_0 \triangleq 2\alpha \beta \phi_0 \lambda_U w^* d^* + \alpha \xi_0 \), and \( c^* \triangleq 2\lambda_L \sigma_\mu \tilde{\mu} \). Finally, (3.70) can be rewritten as

\[ \mathcal{V}(e(t), z(t), \tilde{W}(t), \tilde{\mu}(t)) \leq -\left[\sqrt{\mathcal{R}}\|\xi(t)\|_2 - \frac{r_0}{2\sqrt{\mathcal{R}}} \right]^2 + \left(\frac{r_0^2}{4\mathcal{R}} + c^* \right), \quad (3.71) \]

and hence, there exists a compact set such that \( \mathcal{V}(e(t), z(t), \tilde{W}(t), \tilde{\mu}(t)) < 0 \) outside of this set, which proves uniform ultimate boundedness of the solution \( (e(t), z(t), \tilde{W}(t), \tilde{\mu}(t)) \) [57, 74].

We note that as shown in [73], there exists \( \alpha \in \mathbb{R}_+ \) such that (3.67) is satisfied if

\[ \phi_0 \|H\|_2 < \frac{-\lambda_U \psi + \sqrt{\lambda_U^2 \psi^2 + \varepsilon (2\lambda_L - 1) \choose \varepsilon}}, \quad (3.72) \]

holds, where \( \lambda_L > \frac{1}{2} \). This condition which does not depend on the system uncertainty, corresponds to all the leading principle minors of \( \mathcal{R} \) being positive.

**Remark 3.2.3** Unlike the fundamental stability limit given by (3.58) for the standard model reference adaptive control problem, it is of practical importance to note that the new fundamental stability limit given by (3.72) does not depend on upper bounds of the system uncertainty, and hence, it is relaxed in
this sense. That is, the above framework allows the closed-loop dynamical system to remain bounded in the presence of large system uncertainties when the nonlinear unmodeled dynamics satisfy the condition given by (3.72). This result is consistent with our recent results of [1, 73], but holds for a broader class of unmodeled dynamics.

In the next result, we now show the boundedness of the closed-loop dynamical system with the aforementioned adaptive control law, which relaxes the fundamental stability limit given by (3.60) when the considered class of nonlinear unmodeled dynamics contains both control and state dependent terms. For this purpose, we select the projection bound for (3.62) as

\[ \hat{\mu}(t) \leq \mu_{\psi}, \]  

where \( \mu \equiv \lambda_{U} l^{2} w^{*} l^{2} + \lambda_{L} \) and \( \psi > 1 \).

**Corollary 3.2.2** Consider the uncertain dynamical system given by (3.32), the nonlinear unmodeled dynamics given by \( \dot{q}(t) = f(q(t), x(t), \Lambda u(t)) \) and (3.34), the reference system given by (3.41), and the adaptive control law given by (3.61), (3.46), and (3.62). Assume that Assumption 3.2.1 and Assumption 3.2.2 hold, and there exists a continuously differentiable function \( \Phi : \mathbb{R}^{p} \rightarrow \mathbb{R} \) such that \( \Phi(\cdot) \) is positive definite, radially unbounded, \( \Phi(0) = 0 \), and

\[
\alpha \Phi'(z(t)) \beta f(\beta^{-1} z(t), x(t), -\Lambda \hat{w}^{T}(t) \sigma(\cdot)) \leq -\alpha \left\| z(t) \right\|^{2} + 2\alpha \beta \phi_{1} \| \Lambda \|_{2} \left\| \hat{w}(t) \right\|_{2} \left\| z(t) \right\|_{2} \left\| \sigma(\cdot) \right\|_{2} + 2\alpha \beta \phi_{1} \| \Lambda \|_{2} \left\| z(t) \right\|_{2} \left\| \hat{w}(t) \right\|_{2} \left\| B^{T} P e(t) \right\|_{2} + 2\alpha \beta \phi_{2} \| z(t) \|_{2} \left\| x(t) \right\|_{2} + \alpha \phi_{0} \| z(t) \|_{2},
\]  

(3.73)

holds, where \( \phi_{1} \) and \( \phi_{2} \) are positive bounded constants. Assume, in addition, there exists an \( \alpha \in \mathbb{R}_{+} \) such that

\[
\phi_{1} \left\| H \right\|_{2} < \frac{\sqrt{2} \lambda_{U} - 1}{\psi \lambda_{U}},
\]  

(3.74)

\[
\phi_{1} \left\| H \right\|_{2} \lambda_{U} w^{*} l + \phi_{2} \left\| H \right\|_{2} < \frac{1}{\sqrt{2}} \left( \frac{\lambda_{L}^{2} w^{*} l^{2} + \lambda_{L}}{2 \lambda_{L} - 1 - \psi^{2} \phi_{1} \left\| H \right\|_{2}^{2} - 2 \psi \lambda_{U} \phi_{1} \left\| H \right\|_{2}} \right),
\]  

(3.75)
hold. Then, the solution \((e(t), z(t), \tilde{W}(t), \tilde{\mu}(t))\) of the closed-loop dynamical system is uniformly ultimately bounded (note that \(\dot{q}(t) = f(q(t), x(t), \Lambda u(t))\) can be written as \(\dot{z}(t) = \beta f(\beta^{-1}z(t), x(t), \Lambda u(t))\) with using the state transformation given in Remark 3.2.1).

**Proof.** The result follows using similar steps in our recent work Section 3 of [1] as well as the proof of Theorem 3.2.2, and hence, is omitted. ■

Unlike the stability limit for the standard model reference adaptive control problem given by (3.60) whose left hand side grows unbounded with respect to \(w^\ast\), the proposed adaptive control framework of this section allows the closed-loop dynamical system to remain bounded in the presence of large system uncertainties when the control and state dependent nonlinear unmodeled dynamics satisfy the conditions given by (3.74) and (3.75). Once again, this result is consistent with our recent results of [1, 73], but holds for a broader class of unmodeled dynamics.

**Remark 3.2.4** To highlight the allowable unmodeled dynamics such that Assumption 3.2.3 and Assumption 3.2.4 hold, we discuss the following cases.

i) Let the term on the right hand side in \(\dot{z}(t) = \beta f(\beta^{-1}z(t), \Lambda u(t))\) be linear such that one can consider

\[
\dot{z}(t) = Fz(t) + \beta G \Lambda u(t),
\]

where \(G \in \mathbb{R}^{p \times m}\) and \(F \in \mathbb{R}^{p \times p}\) are the unmodeled dynamics matrices where \(F\) is Hurwitz. In this case, if we select \(\Phi(z) = z^T S z\) where \(S \in \mathbb{R}^{p \times p}_+ \cap \mathbb{S}^{p \times p}_+\) is the solution of the Lyapunov equation given by \(0 = F^T S + S F + I\), the same stability limits given in (3.58) and (3.72) are obtained with \(\phi_0 \triangleq \|SG\|_2\) for the standard model reference adaptive controller and the proposed adaptive controller, respectively. This is the case considered in our previous work [73].

ii) Similarly, let the term on the right hand side in \(\dot{z}(t) = \beta f(\beta^{-1}z(t), x(t), \Lambda u(t))\) be linear such that one can consider

\[
\dot{z}(t) = Fz(t) + \beta G_1 \Lambda u(t) + \beta G_2 x(t),
\]

where \(G_1 \in \mathbb{R}^{p \times m}, G_2 \in \mathbb{R}^{p \times n},\) and \(F \in \mathbb{R}^{p \times p}\) are the unmodeled dynamics matrices where \(F\) is Hurwitz. In this case, \(\Phi(z) = z^T S z\) can again be selected with \(S \in \mathbb{R}^{p \times p}_+ \cap \mathbb{S}^{p \times p}_+\) being the solution of
Lyapunov equation in $0 = F^T S + S^T F + I$, such that the same stability limits given in (3.60) and (3.75) are obtained with $\phi_1 \triangleq \|SG_1\|_2$ and $\phi_2 \triangleq \|SG_2\|_2$ for the standard model reference adaptive control and proposed adaptive controller, respectively. This is the case considered in our previous work [1].

iii) For the nonlinear case, we can consider locally Lipschitz, bounded nonlinear terms (i.e., $\text{sat}(q(t))$, $\cos(q(t))$, $\sin(q(t))$) in the unmodeled dynamics, such that Remark 3.2.1, Assumption 3.2.3, (3.59), Assumption 3.2.4, and (3.73) can be satisfied. Moreover, the effect of these locally Lipschitz, bounded nonlinear terms end up inside the terms $r_0$ as $\alpha_{z_0}$ in our analysis, such that they do not affect the structure (i.e., positive definiteness) of $R$. This further implies that they do not necessarily affect the fundamental stability limits for both the standard model reference adaptive controller and the proposed adaptive controller.

The above results show that the proposed adaptive controller relaxes the fundamental stability limit present in standard model reference adaptive controllers subject to nonlinear unmodeled dynamics with not only control dependent terms, but also both control and state dependent terms. In the next section, we consider nonlinear unmodeled dynamics with both control and state dependent terms where the nonlinear terms fit the example as in iii) of Remark 3.2.4.

### 3.2.4 Illustrative Numerical Example

Consider the coupled dynamical system as shown in Figure 3.14 with the dynamics given by

$$
\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \rho_1/m_1 \end{bmatrix} \Lambda u(t) + \beta^{-1} \begin{bmatrix} 0 \\ \rho_1/m_1 \end{bmatrix} z(t) + \begin{bmatrix} 0 \\ \rho_1/m_1 \end{bmatrix} \begin{bmatrix} (-k_1 - k_2)/\rho_1 \\ 0 \end{bmatrix}^T \delta(x(t)) x(t), \quad x(0) = 0,
$$

(3.78)
\[
\dot{z}(t) = \begin{bmatrix} 0 & 1 \\ -k_2/m_2 & -c_2/m_2 \end{bmatrix} z(t) + \beta \begin{bmatrix} 0 & 0 \\ k_2/m_2 & 0 \end{bmatrix} x(t) + \beta \begin{bmatrix} 0 \\ \rho_2/m_2 \end{bmatrix} \Lambda u(t) + \beta \begin{bmatrix} 0 \\ -k_3/m_2 & -c_1/m_2 \end{bmatrix} \text{sat}(z(t))
\]

where \(c_1, c_2, k_1, k_2, k_3, m_1, m_2, \rho_1, \rho_2, \beta\) and \(\Lambda\) are unknown parameters. Note that (3.78) can be equivalently represented in the form given by (3.37), where the second mass affects the first mass as unmodeled dynamics.

For our study, we consider \(c_1 = 7, c_2 = 7, k_1 = 2, k_2 = 1.5, k_3 = 2, m_1 = 1, m_2 = 0.2, \rho_1 = 2, \rho_2 = 0.5, \beta = 0.05, \Lambda = 1\), and a saturation bound of 0.3. We select a reference model subject to zero initial conditions with a natural frequency of \(w_n = 0.25\) rad/s and a damping ratio \(r_n = 0.9\) such that

\[
A_r = \begin{bmatrix} 0 & 1 \\ -w_n^2 & -2w_n r_n \end{bmatrix}, \quad B_r = \begin{bmatrix} 0 \\ w_n^2 \end{bmatrix}.
\]

For the standard model reference adaptive control approach with \(\gamma = 0.5\) and the projection bound \(\hat{W}_{\text{max}} = 0.275\), Figure 3.15 shows that the system state vector diverges from the reference state vector in time, and hence, the closed-loop dynamical system becomes unstable. This result can be expected from Corollary 3.2.1. In particular, the stated stability limit of this corollary does not hold in this case (to compute this limit,
we use $\lambda_L = 0.95$, $\lambda_U = 1.05$, $l = 1$, and $w^* = 0.4763$, which results in

$$
\phi_1 \|H\|_2 \lambda_U w^* l + \phi_2 \|H\|_2 = 0.4502,
$$

(3.81)

and

$$
1/(4 \|PB\|_2) = 0.0061,
$$

(3.82)

for (3.60)). The inability to satisfy the stability limit can be seen from Figure 3.15 in that the standard model reference adaptive controller cannot tolerate the given unmodeled dynamics and the system uncertainty at the same time.

For the proposed model reference adaptive control approach with $\gamma = 0.5$, $\mu_0 = 1.1$, $\sigma_\mu = 0.9$, $\psi = 1.1$, and the projection bounds $\hat{\mu}_{max} = 1$ and $\hat{W}_{max} = 0.275$ (note that $\gamma$ and $\hat{W}_{max}$ are the same as for the standard model reference adaptive control approach), Figure 3.16 shows that the system state vector closely tracks the reference state vector in time, where the closed-loop dynamical system is clearly stable. This
result is expected from Corollary 3.2.2, since the stated relaxed stability limit holds in this case (to compute this limit, we use $\lambda_L = 0.95$, $\lambda_U = 1.05$ and $\psi = 1.1$, which results in

$$\phi_1 \|H\|_2 \lambda_U w^* l + \phi_2 \|H\|_2 = 0.4502,$$

(3.83)

and

$$\frac{1}{\sqrt{2}} \left[ \frac{\lambda_U w^* l^2 + \lambda_L}{\lambda_L} \left( 2\lambda_L - 1 - \psi^2 \lambda_L \phi_1^2 \|H\|_2^2 - 2\psi \lambda_U \phi_1 \|H\|_2 \right) \right] = 0.6394,$$

(3.84)

for (3.75)). That is, this shows that the proposed model reference adaptive controller tolerates the given class of nonlinear unmodeled dynamics and the system uncertainty at the same time.

3.2.5 Conclusion

In this paper, we generalized our recent work [1, 73] to model reference adaptive control of uncertain dynamical systems subject to a class of nonlinear unmodeled dynamics. Specifically, under certain structural conditions (3.52), (3.59), (3.66), and (3.73), it was shown that the results presented in [1, 73] for linear
unmodeled dynamics can be applied to the case of nonlinear unmodeled dynamics considered in this paper. As we compare the results in Sections 3.2.2 and 3.2.3 of this paper, it was theoretically shown that the results of the latter section for the proposed framework allows the closed-loop dynamical system to preserve stability in the presence of large system uncertainties when the nonlinear unmodeled system dynamics satisfy a relaxed stability limit. Finally, an illustrative numerical example demonstrated the results of this paper.

3.3 Performance Guarantees in Adaptive Control of Uncertain Systems with Unmodeled Dynamics

The presence of unmodeled dynamics degrades the stability and performance of adaptive control architectures. While there are studies that focus on the stability aspects of adaptive control architectures for uncertain systems with unmodeled dynamics, methods that offer performance guarantees in the sense of predictably minimizing the difference between uncertain system trajectories and given reference model trajectories are not available in the literature. In this paper, we address this gap through a recently developed direct uncertainty minimization framework. Specifically, a model reference adaptive control architecture is proposed and mathematically analyzed for uncertain systems with unmodeled dynamics. The key feature of our architecture is the added term in the control signal and the update law, which is developed through a gradient descent procedure with a new cost function involving a cost function gain in order to minimize the effect of both system uncertainties and unmodeled dynamics on the closed-loop system response. Therefore, the proposed architecture can be effective in achieving performance guarantees, where an illustrative numerical example shows the offered predictable performance as a function of the cost function gain. Finally, we also provide an experimental study on a physical system involving two carts connected with each other through a spring in order to demonstrate the efficacy of the proposed architecture.

3.3.1 Introduction

In the absence of unmodeled dynamics, adaptive control architectures guarantee Lyapunov stability of the closed-loop system and asymptotic convergence of uncertain system trajectories to given reference model trajectories (see, for example, [2, 3, 6–8]). However, the presence of unmodeled dynamics, which result from the coupling between rigid body and flexible appendages in applications such as airplanes with high aspect ratio wings and slung load systems, degrades the stability and performance of adaptive control.

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12This section is previously published in [75]. Permission is included in Appendix C.
architectures to uniform boundedness. While there are studies establishing sufficient stability conditions for uniform boundedness of the closed-loop system in adaptive control of uncertain systems with unmodeled dynamics (see, for example, [1, 18, 28, 29, 73, 76] and references therein), the following question remains unanswered:

*How can adaptive control architectures offer performance guarantees through predictably minimizing the difference between uncertain system trajectories subject to unmodeled dynamics and given reference model trajectories?*

In adaptive control of uncertain systems without unmodeled dynamics, there are several well-documented methods that offer performance guarantees (see, for example, [5, 31, 54–56, 77–84] and references therein). In particular, an optimal control modification approach is developed in [77], input low-pass filter approaches are developed in [31, 78], a low-frequency learning approach is developed in [54], state predictor approaches are developed in [55, 79–83], a command governor approach is developed in [5], an artificial basis function approach is developed in [56], and a direct uncertainty minimization approach is developed in [84]. While these approaches have the capability in minimizing the difference between uncertain system trajectories and given reference model trajectories for addressing predictable performance, their common denominator is that they do not consider the presence of unmodeled dynamics in their problem formulations.

In this paper, we address the above practical question through the recently developed direct uncertainty minimization framework [84]. Specifically, a model reference adaptive control architecture is proposed and mathematically analyzed for uncertain systems with unmodeled dynamics. The key feature of our architecture is the added term in the control signal and the update law, where it is developed through a gradient descent procedure with a new cost function involving a cost function gain in order to minimize the effect of both system uncertainties and unmodeled dynamics on the closed-loop system response. Therefore, the proposed architecture can be effective in achieving performance guarantees, where an illustrative numerical example shows the offered performance guarantees as a function of the cost function gain. In particular, this example demonstrates that the difference between uncertain system trajectories subject to unmodeled dynamics and given reference model trajectories becomes predictably smaller when the cost function gain gets larger. As an important byproduct, we also show that the fundamental sufficient stability condition of standard model reference adaptive control architectures for uncertain systems with unmodeled
dynamics is recovered when this cost function gain is sufficiently large. This means that the proposed approach delivers the offered performance without degrading the standard stability properties of model reference adaptive control architectures. Finally, we also provide an experimental study on a physical system involving two carts connected with each other through a spring in order to demonstrate the efficacy of the proposed architecture.

The remainder of this paper is as follows. Section 3.3.2 states the notation used in this paper and Section 3.3.3 formulates the considered problem. Sections 3.3.4, 3.3.5, and 3.3.6 then present respectively the proposed model reference adaptive control architecture for uncertain systems with unmodeled dynamics, an illustrative numerical example, and an experimental study. Finally, concluding remarks are summarized in Section 3.3.7.

### 3.3.2 Notation

In this paper, we use a fairly standard notation. In particular, \( \mathbb{R} \) denotes the set of real numbers, \( \mathbb{R}^n \) denotes the set of \( n \times 1 \) real column vectors, \( \mathbb{R}^{n \times m} \) denotes the set of \( n \times m \) real matrices, \( \mathbb{R}_+ \) (resp., \( \mathbb{R}_+^n \)) denotes the set of positive (resp., nonnegative) real numbers, \( \mathbb{R}_+^{n \times n} \) (resp., \( \mathbb{R}_+^{n \times n} \)) denotes the set of \( n \times n \) positive-definite (resp., nonnegative-definite) real matrices, \( I_n \) denotes the \( n \times n \) identity matrix, and “\( \triangleq \)” denotes the equality by definition. We also use \( (\cdot)^T \) for the transpose operator, \( (\cdot)^{-1} \) for the inverse operator, \( \text{tr}(\cdot) \) for the trace operator, \( \lambda(A) \) (resp., \( \lambda_-(A) \)) for the maximum (resp., minimum) eigenvalue of the matrix \( A \in \mathbb{R}^{n \times n} \), \( \|\cdot\|_2 \) for the Euclidean norm, and \( \|A\|_2 \triangleq (\lambda(A^TA))^{\frac{1}{2}} \) for the induced 2-norm of the matrix \( A \in \mathbb{R}^{n \times m} \).

We now state the definition of the rectangular projection operator from Exercise 11.3 of [7] and [85] for the results of this paper. Specifically, consider a convex hypercube \( \Omega = \{\theta \in \mathbb{R}^n : (\theta_i^{\text{min}} \leq \theta_i \leq \theta_i^{\text{max}})_{i=1,2,\ldots,n}\} \), \( \Omega \in \mathbb{R}^n \) with \( \theta_i^{\text{min}} \) and \( \theta_i^{\text{max}} \) respectively denoting the minimum and maximum bounds for the \( i \)-th component of the parameter vector \( \theta \in \mathbb{R}^n \). Without loss of generality, we consider each bound to be symmetric as \( \theta_i^{\text{min}} = -\theta_i^{\text{max}} \) in this paper. For a sufficiently small constant \( \varepsilon_0 \in \mathbb{R}_+ \), in addition, consider another convex hypercube \( \Omega_{\varepsilon_0} = \{\theta \in \mathbb{R}^n : (\theta_i^{\text{min}} + \varepsilon_0 \leq \theta_i \leq \theta_i^{\text{max}} - \varepsilon_0)_{i=1,2,\ldots,n}\} \), where \( \Omega_{\varepsilon_0} \subset \Omega \). Then, the component-wise projection operator \( \text{Proj} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) is defined as
\[
\text{Proj}(\theta, y) = (\theta_i^{\text{max}} - \theta_i)y_i/\varepsilon_0 \quad \text{when} \quad \theta_i > \theta_i^{\text{max}} - \varepsilon_0 \quad \text{and} \quad y_i > 0,
\]
\[
\text{Proj}(\theta, y) = (\theta_i^{\text{min}} - \theta_i)y_i/\varepsilon_0 \quad \text{when} \quad \theta_i < \theta_i^{\text{min}} + \varepsilon_0 \quad \text{and} \quad y_i < 0,
\]
and \( \text{Proj}(\theta, y) = y_i \) otherwise, where \( y \in \mathbb{R}^n \). This definition yields \( (\theta - \theta^*)^T(\text{Proj}(\theta, y) - y) \leq 0 \) [53], where \( \theta^* \in \Omega_{\varepsilon_0} \). One can also generalize this definition to matrices using \( \text{Proj}_m(\Theta, Y) = (\text{Proj}(\text{col}_1(\Theta), \text{col}_1(Y)), \ldots, \text{Proj}(\text{col}_m(\Theta)), \ldots) \).
col_m(Y)), which yields \( \text{tr} \left[ (\Theta - \Theta^*)^T (\text{Proj}_m(\Theta, Y) - Y) \right] = \sum_{i=1}^{m} \left[ \text{col}_i(\Theta - \Theta^*)^T (\text{Proj}(\text{col}_i(\Theta), \text{col}_i(Y)) - \text{col}_i(Y)) \right] \leq 0 \) with \( n \times m \) matrices \( Y, \Theta, \) and \( \Theta^* \) (here, \( \text{col}_i(\cdot) \) denotes \( i \)-th column function).

### 3.3.3 Problem Formulation

In this section, we first introduce the considered class of uncertain systems with unmodeled dynamics and then state the adaptive control objective. Specifically, consider the dynamics of a physical system given by

\[
\dot{x}(t) = Ax(t) + Bu(t) + BW^T x(t) + By_q(t), \quad x(0) = x_0, \tag{3.85}
\]

\[
\dot{q}(t) = Fq(t) + Gx(t), \quad q(0) = q_0, \tag{3.86}
\]

\[
y_q(t) = Hq(t), \tag{3.87}
\]

where \( x(t) \in \mathbb{R}^n \) is a measurable state vector of the modeled dynamics, \( u(t) \in \mathbb{R}^m \) is a control input, \( q(t) \in \mathbb{R}^p \) and \( y_q(t) \in \mathbb{R}^m \) are respectively an unmeasurable state vector and an unmeasurable output vector of the unmodeled dynamics, \( A \in \mathbb{R}^{n \times n} \) and \( B \in \mathbb{R}^{n \times m} \) are respectively a known system matrix and a known input matrix with the pair \((A, B)\) being controllable and \( B \) having full column rank, \( W \in \mathbb{R}^{n \times m} \) is an unknown weight matrix, and \( F \in \mathbb{R}^{p \times p}, G \in \mathbb{R}^{p \times n}, \) and \( H \in \mathbb{R}^{m \times p} \) are matrices associated with unmodeled dynamics such that \( F \) is Hurwitz. Note that \( F \) being Hurwitz implies the existence of the matrix \( S \in \mathbb{R}^{p \times p}_+ \) such that the Lyapunov equation \( 0 = F^T S + SF + I_p \) holds.

As it is often adopted in the literature (see, for example, [1, 73, 76]), we now introduce a state transformation \( z(t) = \beta q(t), z(t) \in \mathbb{R}^p, \) in order to have a flexibility in the adaptive control analysis performed later in this paper (see Section 3.3.4). In particular, with this state transformation, one can equivalently write (3.85), (3.86), and (3.87) in the form

\[
\dot{x}(t) = Ax(t) + Bu(t) + BW^T x(t) + \beta^{-1} BH z(t), \quad x(0) = x_0, \tag{3.88}
\]

\[
\dot{z}(t) = Fz(t) + \beta G x(t), \quad z(0) = \beta q_0, \tag{3.89}
\]

\[
y_q(t) = \beta^{-1} Hz(t). \tag{3.90}
\]
For capturing a user-defined, desired closed-loop system response, we next consider the reference model given by

$$\dot{x}_r(t) = A_r x_r(t) + B_r c(t), \quad x_r(0) = x_{r0}, \quad (3.91)$$

where $x_r(t) \in \mathbb{R}^n$ is a reference state vector, $c(t) \in \mathbb{R}^m$ is a bounded command, $A_r \in \mathbb{R}^{n \times n}$ is a Hurwitz reference system matrix, and $B_r \in \mathbb{R}^{n \times m}$ is a command input matrix. Note that $A_r$ being Hurwitz implies the existence of the matrix $P \in \mathbb{R}^{n \times n}$ such that the Lyapunov equation $0 = A_r^T P + PA_r + I_n$ holds. In (3.88), since the pair $(A, B)$ is known and controllable, one can select matrices $K_1 \in \mathbb{R}^{m \times n}$ and $K_2 \in \mathbb{R}^{m \times m}$ to construct the pair $(A_r, B_r)$ in (3.91) according to $A_r \triangleq A - BK_1$ and $B_r \triangleq BK_2$ (see, for example, Section III.A of [8] for details). Finally, one can equivalently rewrite (3.88) and (3.89) as

$$\dot{x}(t) = A_r x(t) + B_r c(t) + B [u(t) + K_1 x(t) - K_2 c(t) + W^T x(t)] + \beta^{-1} B H z(t), \quad (3.92)$$
$$\dot{z}(t) = F z(t) + \beta G x(t). \quad (3.93)$$

We are now ready to state the model reference adaptive control objective of this paper. Specifically, consider the uncertain system given by (3.92) subject to unmodeled dynamics given by (3.93) and the reference model given by (3.91). Design a feedback control law $u(t)$ that not only guarantees stability of the closed-loop system in terms of uniform boundedness as in, for example, [1, 18, 28, 29, 73, 76], but also offers performance guarantees through predictably minimizing the difference between the state vector of the uncertain system $x(t)$ and the state vector of the reference model $x_r(t)$. This objective is addressed in the next section.

### 3.3.4 Proposed Adaptive Control Architecture

To address the model reference adaptive control objective stated in Section 3.3.3, consider the feedback control law given by

$$u(t) = u_n(t) + u_a(t) + u_p(t), \quad (3.94)$$

with the nominal control law $u_n(t)$ satisfying

$$u_n(t) = -K_1 x(t) + K_2 c(t), \quad (3.95)$$
the adaptive control law $u_a(t)$ satisfying

$$u_a(t) = -\hat{W}^T(t)x(t),$$

(3.96)

and the direct uncertainty minimization mechanism $u_p(t)$ satisfying

$$u_p(t) = -\phi(t).$$

(3.97)

In (3.96), $\hat{W}(t) \in \mathbb{R}^{n \times m}$ is an estimate of the unknown weight $W$ that satisfies weight update law predicated on the projection operator given by

$$\dot{\hat{W}}(t) = \gamma \text{Proj}_m\left[\hat{W}(t), x(t)\left(e^T(t)PB + \frac{k}{\mu}\phi^T(t)\right)\right], \quad \hat{W}(0) = \hat{W}_0,$$

(3.98)

with $\gamma \in \mathbb{R}_+$ being the learning rate, $k \in \mathbb{R}_+$ and $\mu \in \mathbb{R}_+$ being the design gains, and $e(t) \triangleq x(t) - x_r(t) \in \mathbb{R}^n$ being the system error. The added term $\phi(t) \in \mathbb{R}^m$ in the control signal (3.97) and in the weight update law (3.98) is the key feature of the proposed architecture here for achieving performance guarantees that has the form

$$\phi(t) = \phi(0) + k\left(B((e(t) - e(0)) - \int_0^t A_r e(\tau) d\tau)\right),$$

(3.99)

where $B \triangleq (B^T B)^{-1}B^T$ with $(B^T B)^{-1}$ is well-defined since $B$ has full column rank.

The added term $\phi(t)$ in (3.97) and (3.98) is introduced in [84]. However, this paper considers uncertain systems with unmodeled dynamics, where this is not considered in [84]. Therefore, not only the stability analysis of this paper (see Theorem 3.3.1) differs from the stability analysis in Theorem 2 of [84] but also the gradient descent procedure of this paper, which gives the form of this added term for performance guarantees, utilizes a new cost function (see Theorem 3.3.2) that differs from the cost function in Theorem 1 of [84].

Next, using (3.94), one can equivalently rewrite (3.92) as

$$\dot{x}(t) = A_r x(t) + B_r c(t) - B\hat{W}^T(t)x(t) + \beta^{-1}BHz(t) - B\phi(t),$$

(3.100)
where \( \tilde{W}(t) \triangleq \hat{W}(t) - W \in \mathbb{R}^{n \times m} \) is the weight estimation error. The system error dynamics and the weight estimation error dynamics can also be given by

\[
\dot{e}(t) = A_r e(t) - B \tilde{W}^T(t)x(t) + \beta^{-1} BH z(t) - B\phi(t), \quad e(0) = e_0, \tag{3.101}
\]
\[
\dot{\tilde{W}}(t) = \gamma \text{Proj}_m \left[ \hat{W}(t), x(t) \left( e^T(t)PB + \frac{k}{\mu} \phi^T(t) \right) \right], \quad \tilde{W}(0) = \tilde{W}_0. \tag{3.102}
\]

For following theorem, note that (3.99) can equivalently be written as

\[
\dot{\phi}(t) = k \left( B(\dot{e}(t) - A_r e(t)) \right), \tag{3.103}
\]

where (3.99) results from (3.103) using integration by parts. Note also that (3.101) can be rewritten as

\[
\left( \dot{e}(t) - A_r e(t) \right) = -B(\tilde{W}^T(t)x(t) - \beta^{-1} H z(t) + \phi(t)), \tag{3.104}
\]

which yields

\[
B(\dot{e}(t) - A_r e(t)) = -(\tilde{W}^T(t)x(t) - \beta^{-1} H z(t) + \phi(t)). \tag{3.105}
\]

Hence, (3.103) can be reorganized using (3.105) as

\[
\dot{\phi}(t) = -k(\tilde{W}^T(t)x(t) - \beta^{-1} H z(t) + \phi(t)). \tag{3.106}
\]

**Theorem 3.3.1** Consider the uncertain system with unmodeled dynamics given by (3.88) and (3.89), the reference model given by (3.91), and the feedback control architecture given by (3.94), (3.95), (3.96), (3.97), (3.98), and (3.99). In addition, assume that

\[
\mathcal{R} = \begin{bmatrix}
I_n & \eta_1 & \eta_2 \\
\eta_1^T & dI_p & \eta_3 \\
\eta_2^T & \eta_3^T & 2\mu^{-1}kI_m
\end{bmatrix} > 0, \tag{3.107}
\]

holds, where \( d \in \mathbb{R}^+, k \in \mathbb{R}^+, \mu \in \mathbb{R}^+, \eta_1 \triangleq -\beta^{-1} PBH - dBG^TS, \eta_2 \triangleq PB, \) and \( \eta_3 \triangleq -\mu^{-1} \beta^{-1} kH^T. \) The solution \((e(t), \phi(t), z(t), \tilde{W}(t))\) of the closed-loop dynamical system is then uniformly bounded.
Proof. To show uniform boundedness of the solution \((e(t), \phi(t), z(t), \tilde{W}(t))\), consider the Lyapunov-like function given by

\[
V(e, \phi, z, \tilde{W}) \triangleq e^T P e + \mu^{-1} \phi^T \phi + \gamma^{-1} \text{tr} \tilde{W}^T \tilde{W} + dz^T S z.
\] (3.108)

Note that \(V(0, 0, 0, 0) = 0\) and \(V(e, \phi, z, \tilde{W}) > 0\) for all \((e, \phi, z, \tilde{W}) \neq (0, 0, 0, 0)\). Differentiating (3.108) along the closed-loop dynamical system trajectories and utilizing the property of the projection operator from Section 3.3.2 yields

\[
\dot{V}(e(t), \phi(t), z(t), \tilde{W}(t)) \leq -e^T(t)e(t) + 2\beta^{-1}e^T(t)P\tilde{B}H z(t) - 2e^T(t)PB\phi(t) + 2\mu^{-1}\beta^{-1}k\phi^T(t)Hz(t) - 2\mu^{-1}k\phi^T(t)\phi(t) - dz^T(t)z(t) + 2d\beta z^T(t)SG x(t),
\]

\[
= -\xi^T(t)R\xi(t) + 2d\beta z^T(t)SG x(t),
\] (3.109)

where \(\xi(t) = [e^T(t), z^T(t), \phi^T(t)]^T\). Now, we rewrite (3.109) as

\[
\dot{V}(e(t), \phi(t), z(t), \tilde{W}(t)) \leq -\left[ \sqrt{\lambda(R)} \left\| \xi(t) \right\|_2 - \frac{r_0}{2\sqrt{\lambda(R)}} \right]^2 + \frac{r_0^2}{4\lambda(R)},
\] (3.110)

where \(r_0 \triangleq 2d\beta \left\| SG \right\|_2 x^*_t, \| x_t(t) \|_2 \leq x^*_t\). Hence, there exists a compact set such that

\[
\dot{V}(e(t), \phi(t), z(t), \tilde{W}(t)) < 0
\]

outside of this set, which proves uniform boundedness of solution \((e(t), \phi(t), z(t), \tilde{W}(t))\) [7, 57]. The proof is now complete.

While Theorem 3.3.1 addresses the stability of the closed-loop system with the proposed model reference adaptive control architecture, the terms \(\tilde{W}^T(t)x(t) - \beta^{-1}Hz(t) + \phi(t)\)” in the system error dynamics given by (3.101) can yield poor system performance when their magnitude is large unless \(\phi(t)\) suppresses the effect of \(\tilde{W}^T(t)x(t) - \beta^{-1}Hz(t)\). In the next theorem, we show that \(\phi(t)\) is specifically designed to suppress the effect of \(\tilde{W}^T(t)x(t) - \beta^{-1}Hz(t)\)” through a gradient descent procedure.
Theorem 3.3.2  The added term $\phi(t)$ given by (3.99) satisfies the negative gradient of the cost function given by

$$J(\phi) \triangleq \frac{k}{2} \| \phi(t) + \tilde{W}^T x(t) - \beta^{-1} H z(t) \|^2_2.$$  \hspace{1cm} (3.111)

Proof. The negative gradient of the cost function given by (3.111) with respect to $\phi(t)$ satisfies

$$\dot{\phi}(t) = -\frac{\partial J(\phi)}{\partial \phi(t)}$$
$$= -k(\phi(t) + \tilde{W}^T x(t) - \beta^{-1} H z(t))$$
$$= k(B(\dot{e}(t) - A_r e(t))), \quad \phi(0) = \phi_0,$$  \hspace{1cm} (3.112)

which gives (3.99) by applying integration by parts, as discussed. The proof is now complete. \hfill \blacksquare

As it is well-known in algorithm designs based on the gradient descent method, the positive cost function gain $k$ in (3.111), which also appears in (3.98) and (3.99) of the proposed model reference adaptive control architecture as a tuning parameter, helps to minimize the total effect of the term “$\tilde{W}^T x(t) - \beta^{-1} H z(t) + \phi(t)$” in (3.101) better when it gets larger. Therefore, it can be increased to predictably minimize (that is, as a function of $k$) the difference between uncertain system trajectories and given reference model trajectories. The question stated in Section 3.3.1 is now addressed through Theorems 3.3.1 and 3.3.2.

Finally, we are interested to evaluate the assumption (3.107) of Theorem 3.3.1 for large values of $k$ since the proposed model reference adaptive control architecture offers better performance guarantees for such values. This is addressed in the next corollary.

Corollary 3.3.1  Let the cost function gain $k$ in (3.111) be sufficiently large such that $A_0 - (2\mu^{-1}k)^{-1}B_0B_0^T > 0$ holds, where

$$A_0 = \begin{bmatrix} I_n & \eta_1 \\ \eta_1^T & dI_p \end{bmatrix}, \quad B_0 = [B^T P, H]^T,$$  \hspace{1cm} (3.113)
and let $\beta = \mu^{-1} k$. Then, if
\[
\|SG\|_2 \|H\|_2 < \frac{1}{4\|PB\|_2},
\] (3.114)
holds, then (3.107) holds.

Proof. From a special case in Corollary 1 of [1], there is $d \in \mathbb{R}_+$ such that $A_0 > 0$ when (3.114) holds. Now, (3.107) holds with $\beta = \mu^{-1} k$ when $A_0 > 0$ and $A_0 - (2\mu^{-1} k)^{-1}B_0B_0^T > 0$, which follows from the Schur complement of (3.107) [86]. Since $A_0 > 0$ and $k$ is selected such that $A_0 - (2\mu^{-1} k)^{-1}B_0B_0^T > 0$, the result follows. ■

Note that (3.114) is a version of the fundamental sufficient stability condition of standard model reference adaptive control architectures for uncertain systems with unmodeled dynamics [1, 18]. This means that the proposed approach delivers the offered performance without degrading the standard stability properties of model reference adaptive control architectures.

3.3.5 Illustrative Numerical Example

Consider a second-order dynamical system subject to unmodeled dynamics given by
\[
\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} -0.0082 \\ -0.0626 \end{bmatrix} q(t), \quad x(0) = 0, \quad (3.115)
\]
\[
\dot{q}(t) = \begin{bmatrix} -0.25 & 3.1524 \\ -3.1524 & -0.25 \end{bmatrix} q(t) + \begin{bmatrix} 0.1246 & 0 \\ -0.0164 & 0 \end{bmatrix} x(t), \quad q(0) = 0. \quad (3.116)
\]

Note that (3.115) and (3.116) can be equivalently represented in the form given by (3.88) and (3.89). For the results in this paper, we select a reference system subject to zero initial conditions with natural frequency of $w_n = 0.3 \text{rad/s}$ and a damping ratio $r_n = 0.9$ such that
\[
A_r = \begin{bmatrix} 0 & 1 \\ -w_n^2 & -2w_nr_n \end{bmatrix}, \quad B_r = \begin{bmatrix} 0 \\ w_n^2 \end{bmatrix}. \quad (3.117)
\]

Figure 3.17 shows the closed-loop system response with the standard model reference adaptive control architecture with $\gamma = 0.4$ and the projection bound $\hat{W}_{\text{max}} = 2$, and Figure 3.18 shows the closed-
Figure 3.17: System state response, reference state response, and control signal for the standard model reference adaptive control architecture.

loop system response with the proposed model reference adaptive control architecture in Theorem 3.3.1 with $\gamma = 0.4$ and the same projection bound. Note that the stability condition given by (3.107) holds with $\mu = 1$ and $d = 6 \times 10^{-9}$ when $k \geq 108$. Comparing the closed-loop system responses in these two figures, it is clear that the proposed architecture offers a desirable performance through minimizing the difference between uncertain system trajectories and given reference model trajectories. To elucidate the fact that this difference becomes predictably smaller when the cost function gain gets larger we also include Figure 3.19, where as we increase $k$ from 0.1 to 110 the closed-loop system response becomes more desirable.

3.3.6 Experimental Study

In this section, we provide an experimental study on a physical system involving two carts connected with each other through a spring in order to demonstrate the efficacy of the proposed architecture.
3.3.6.1 Experimental Setup

Consider the physical system shown in Figure 3.20 in the presence of unknown frictions, where the left cart represents the modeled dynamics and the right cart represents the unmodeled dynamics. Note that both carts slide along a steel shaft using linear bearings. The left cart is driven by a rack and pinion mechanism using a 6 Volt DC motor. The position of the left cart is measured using a sensor coupled to the rack via an additional pinion, while the position of the right cart is considered to be unmeasured since it represents the unmodeled dynamics. In addition, the data transmission between the computer and the drivers is carried out with digital to analog converter Quanser Q8-USB data acquisition board. The linear voltage controlled power amplifier VoltPAQ-X1 is utilized to drive our experiments. Finally, the control laws are implemented on the computer with MATLAB/Simulink by using Quanser Quarc Real Time Windows Target (Win64) with a sampling rate of 1000 [Hz].
Figure 3.19: System state response, reference state response, and control signal for the proposed model reference adaptive control architecture with $k = 0.1, 0.5, 1, 5, 20, 110$ (the line colour variations from light to dark indicate the direction that $k$ is increased from 0.1 to 110).

### 3.3.6.2 Experimental Results

In this experimental study, we utilize an integrator-based nominal control architecture. Mathematically speaking, let $x_i(t) \in \mathbb{R}^m$ be the integrator state satisfying

$$
\dot{x}_i(t) = Ex(t) - c(t), \quad x_i(0) = x_{i0}.
$$

(3.118)

Here, $E \in \mathbb{R}^{m \times n}$ allows one to select a subset of $x(t)$ to allow following the command $c(t)$. Considering (3.85) and (3.118), the linearized uncertain augmented dynamics of the considered physical system can be written as
Figure 3.20: A benchmark physical system involving two carts connected with each other through a spring.

\[
\dot{x}_a(t) = \begin{bmatrix}
0 & 1 & 0 \\
0 & -\frac{\tau_c K_m^2}{R_c e^{\alpha t}} & -\frac{B_{eq}}{J_{eq}} & 0 \\
1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} x_a(t) + \begin{bmatrix}
0 \\
\frac{1}{J_{eq}} \\
0 \\
0
\end{bmatrix} u(t)
\] (3.119)

\[
\dot{q}(t) = \begin{bmatrix}
0 & 1 \\
-\frac{k_s}{M_2} & -\frac{B_{eq}}{M_2} & \frac{L_2}{M_2} & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} q(t) + \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} x(t)
\] (3.120)
Table 3.2: Values of the physical parameters in (3.115) and (3.120).

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Name</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau_m$</td>
<td>Motor Efficiency</td>
<td>1</td>
</tr>
<tr>
<td>$\tau_g$</td>
<td>Planetary Gearbox Efficiency</td>
<td>1</td>
</tr>
<tr>
<td>$K_g$</td>
<td>Planetary Gearbox Gear Ratio</td>
<td>3.71</td>
</tr>
<tr>
<td>$K_t$</td>
<td>Motor Current Torque Constant</td>
<td>$7.68 \times 10^{-3}$ [Nm/A]</td>
</tr>
<tr>
<td>$K_m$</td>
<td>Motor Back-emf Constant</td>
<td>$7.68 \times 10^{-3}$ [V/(rad/s)]</td>
</tr>
<tr>
<td>$R_m$</td>
<td>Motor Armature Resistance</td>
<td>2.6 [Ω]</td>
</tr>
<tr>
<td>$g$</td>
<td>Gravitational Constant on USF</td>
<td>9.79 [m/s²]</td>
</tr>
<tr>
<td>$M_1, M_2$</td>
<td>Mass of Carts</td>
<td>0.57 and 0.6975 [kg]</td>
</tr>
<tr>
<td>$B_{eq}$</td>
<td>Equivalent Viscous Damping Coefficient at the cart 1</td>
<td>4.3 [Nms/rad]</td>
</tr>
<tr>
<td>$B_{eq2}$</td>
<td>Equivalent Viscous Damping Coefficient at the cart 2</td>
<td>1.1 [Nms/rad]</td>
</tr>
<tr>
<td>$r_{mp}$</td>
<td>Motor Pinion Radius</td>
<td>$6.35 \times 10^{-3}$ [m]</td>
</tr>
<tr>
<td>$J_m$</td>
<td>Rotor Moment of Inertia</td>
<td>$3.9 \times 10^{-7}$ [kg m²]</td>
</tr>
<tr>
<td>$k$</td>
<td>Spring Constant</td>
<td>160 [N/m]</td>
</tr>
</tbody>
</table>

where $J_{eq} = M_1 + \frac{\tau_{Ja}K^2}{r_{mp}}$, $x_a(t) = [x^T_c(t), x^T_i(t), x^T(t)]^T$, and $q(t) = [q^T_c(t), q^T_i(t), q^T(t)]^T$. Moreover, $V(t) = \tau^{-1}u(t)$ being the applied voltage with $\tau = \frac{\tau_{Ja}K \tau_m}{r_{mp}K_m}$. Here, $W$ captures the uncertainty due to matched unknown friction effects (see, for example, [67]) and the term $H = [k_s, 0]$ (spring coefficient $k_s$ is considered to be unknown in this study). Table 3.2 includes the parameters and their values in the above dynamics.

We select a command that changes between $\pm 0.1$ [m] and calculate $\tau = 1.7235$. The nominal control architecture gain matrix $K$ is obtained using a linear quadratic regulator based design with the weighting matrices $Q = \text{diag}[400, 50, 5]$ to penalize the states and $R = 2$ to penalize the control input. Now, for the reference model given by (3.91), $A_r = A_a - B_a K$ and $B_r = [0, 0, -1]^T$ with the matrices in (3.119). For the control architecture update law given by (3.98), we select the learning rate as $\gamma = 1$ and the elementwise projection bound as $\hat{W}_{\text{max}} = 1.55$. We also choose $\mu = 1$ and $k = 10$.

Figure 3.21 shows the comparison on command following responses of the standard and the proposed model reference adaptive control architectures, where it can be clearly seen that the proposed architecture offers a desirable performance through minimizing the difference between uncertain system trajectories (i.e., uncertainties resulting from unknown frictions) and the given reference model trajectories. Finally, Figure 3.22 shows the history of $\phi(t)$ term utilized in the proposed model reference adaptive control

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13While the nominal value of $k_s$ is given in Table 3.2, it changes with respect to how the spring is connected to the cart.
Figure 3.21: Position of the left cart and its control signal with the standard model reference adaptive control architecture (blue) and with the proposed model reference adaptive control architecture (green) in the presence of unmodeled dynamics due to the presence of the right card and unknown frictions.

architecture. Note that Figures 3.21 and 3.22 present experimental results with the considered physical system.

3.3.7 Conclusion

While adaptive control architectures guarantee Lyapunov stability of the closed-loop system and asymptotic convergence of uncertain system trajectories to given reference model trajectories in the absence of unmodeled dynamics, their presence degrades the stability and performance of adaptive control architectures to uniform boundedness. To this end, there are studies that address stability aspects of adaptive control architectures for uncertain systems with unmodeled dynamics in terms of developing sufficient stability conditions for uniform boundedness. However, methods that offer performance guarantees in the sense of predictably minimizing the difference between uncertain system trajectories in the presence of unmodeled dynamics and given reference model trajectories are not available in the literature. We addressed this gap in this paper through proposing and analyzing a model reference adaptive control architecture predicated on the direct uncertainty minimization framework. In particular, the proposed architecture involves an added term both in the control signal and the update law that is developed through a gradient descent procedure with a
Figure 3.22: The history of the performance improvement term $\phi(t)$ utilized in the proposed model reference adaptive control architecture in Figure 3.21.

new cost function involving a cost function gain in order to minimize the effect of both system uncertainties and unmodeled dynamics on the closed-loop system response. An illustrative numerical example was also presented to show the offered predictable performance of the proposed architecture as a function of the cost function gain. Finally, we also provided an experimental study on a physical system involving two carts connected with each other through a spring in order to demonstrate the efficacy of the proposed architecture.

3.4 Adaptive Control Systems with Unstructured Uncertainty and Unmodeled Dynamics:

A Relaxed Stability Tradeoff$^{14}$

In this paper, we first analyze stability tradeoffs of model reference adaptive control architectures in the presence of unstructured system uncertainties subject to (residual) approximating errors satisfying the linear growth inequality and unmodeled dynamics. We then synthesize adaptive robustifying terms to relax the aforementioned stability tradeoff, which presents our first contribution. Specifically, these terms in the feedback loop guarantee overall system stability even in the presence of significant system uncertainties when unmodeled dynamics satisfy a condition. As our second contribution, we demonstrate

$^{14}$This section has been submitted to the IEEE Transactions on Control Systems Technology for possible publication.
our theoretical findings in an experiment involving an inverted pendulum on a cart (modeled dynamics) coupled with another cart through a spring (unmodeled dynamics).

3.4.1 Introduction

System uncertainties (e.g., disturbances and unknown coefficients in system models) and unmodeled dynamics (e.g., flexible appendages in rigid body structures) can deteriorate stability properties of model reference adaptive control systems. Thus, developing analytical conditions to reveal stability tradeoffs between system uncertainties and unmodeled dynamics has received a considerable attention in the adaptive control literature; see, for example, [1, 17, 18, 28, 29, 63, 73, 76, 87] and references therein. In particular, these studies pave a way to implement adaptive control architectures reliably in real-world applications involving unmodeled dynamics. To this end, it is now known that adaptive control architectures, which are predicated on a projection operator-based approach, can directly guarantee closed-loop system stability in terms of boundedness without relying on, for example, initial condition or persistency of excitation assumptions. Yet, a common conclusion is that closed-loop system stability may not be still guaranteed against significant system uncertainties in the presence of unmodeled dynamics; see Theorem 2 in [18] and Corollary 1 in [1].

This paper focuses on model reference adaptive control systems in the presence of unstructured system uncertainties subject to (residual) approximating errors satisfying the linear growth inequality and unmodeled dynamics (unlike [18] and [1] that focus on structured system uncertainties and unmodeled dynamics). Although physical systems that can be modeled by the first principles can have structured system uncertainties, a wide array of system uncertainties can be unstructured [8]; for example, when an aerial robot performs an aggressive maneuver at high angles of attack. Regardless of how rigid a physical system is, in addition, a considerable set of real-world applications involve unmodeled dynamics.

To this end, we start with analyzing stability tradeoffs of model reference adaptive control architectures when both unstructured system uncertainties and unmodeled dynamics are present. We then synthesize adaptive robustifying terms to relax the aforementioned stability tradeoff, which presents our first contribution. Specifically, these terms in the feedback loop guarantee overall system stability even in the presence of significant system uncertainties when unmodeled dynamics satisfy a condition. Note that this conclusion is consistent with the results focusing on a simpler adaptive control problem involving only structured system uncertainties and unmodeled dynamics; see Corollary 2 in [1]. As our second contribution,
we demonstrate our theoretical findings in an experiment (in addition to an illustrative numerical example) involving an inverted pendulum on a cart (modeled dynamics) coupled with another cart through a spring (unmodeled dynamics) in the presence of (unstructured) friction uncertainty.

Note that there are several approaches in the literature that addresses unstructured system uncertainties in adaptive control; see, for example, [7, 88–90]. However, they do not consider the presence of unmodeled dynamics in their formulation. The studies that consider both the presence of unstructured system uncertainties and unmodeled dynamics in adaptive control include, for example, [91–93]. However, the authors of these papers do not consider the practical fact that the unmodeled dynamics can be affected by the control signal applied to the modeled portion of the uncertain dynamical system. In this paper, we take into account the presence of the control signal in the unmodeled dynamics, which is a much harder theoretical problem in contrast to its absence. A fairly standard notation is used in this paper, where we refer to, for example, [1] due to page limitations. Finally, [94] can be viewed as a preliminary conference version of this paper. The present paper considerably expands on [94] by providing proofs of our results in detail and an experimental study along with additional key discussions.

3.4.2 Problem Formulation

This section presents the problem formulation and the mathematical preliminaries needed for our results in Sections 3.4.3 and 3.4.4. Consider the class of physical systems with unstructured system uncertainties and unmodeled dynamics

$$\dot{x}_p(t) = A_p x_p(t) + B_p \Lambda u(t) + B_p \delta_p(x_p(t)) + B_p p(t), \quad x_p(0) = x_{p0}, \quad (3.121)$$
$$\dot{q}(t) = F q(t) + G_{1} \Lambda u(t) + G_{2} x_p(t), \quad q(0) = q_{0}, \quad (3.122)$$
$$p(t) = H q(t), \quad (3.123)$$

where $x_p(t) \in \mathbb{R}^{n_p}$ is a measurable state vector, $u(t) \in \mathbb{R}^{m}$ is a control signal, $q(t) \in \mathbb{R}^{p}$ and $p(t) \in \mathbb{R}^{m}$ respectively are unmodeled dynamics unmeasurable state and output vectors, $A_p \in \mathbb{R}^{n_p \times n_p}$ and $B_p \in \mathbb{R}^{n_p \times m}$ respectively are known system and input matrices$^{15}$, $\Lambda \in \mathbb{R}^{m \times m} \cap \mathbb{R}^{m \times m}$ is an unknown control effectiveness matrix$^{16}$, and $\delta_p : \mathbb{R}^{n_p} \rightarrow \mathbb{R}^{m}$ is an unstructured system uncertainty satisfying the parameterization (e.g., see

---

$^{15}$Throughout this paper, it is assumed that the pair $(A_p, B_p)$ is controllable.

$^{16}$Although $\Lambda$ is unknown, existence of $\lambda_L \in \mathbb{R}_+$ and $\lambda_U \in \mathbb{R}_+$ are guaranteed by definition such that $\lambda_L \leq ||\Lambda||_2 \leq \lambda_U$ holds.
\[ \delta_p(x_p(t)) = W_p^T \sigma_p(x_p(t)) + \epsilon(x_p(t)), \quad x_p \in \mathbb{R}^{n_p}, \] (3.124)

with \(W_p \in \mathbb{R}^{s \times m}\) being an unknown weight matrix, \(\sigma_p(x_p(t)) \in \mathbb{R}^{n_p} \to \mathbb{R}^s\) being a basis function of the form \(\sigma_p(x_p(t)) = [\sigma_{p_1}(x_p(t)), \sigma_{p_2}(x_p(t)), \ldots, \sigma_{p_k}(x_p(t))]^T\) that satisfies the linear growth inequality \(\|\sigma_p(x_p(t))\|_2 \leq l_0 \|x_p(t)\|_2 + l_c, \quad l_c \in \mathbb{R}_+, \quad l_0 \in \mathbb{R}_+\), and \(\epsilon(x_p(t)) \in \mathbb{R}^{n_p} \to \mathbb{R}^m\) being an unknown (residual) approximation error that also satisfies the linear growth inequality \(\|\epsilon(x_p(t))\|_2 \leq k_0 \|x_p(t)\|_2 + k_c, \quad k_0, k_c \in \mathbb{R}_+\). In addition, \(F \in \mathbb{R}^{p \times p}, G_1 \in \mathbb{R}^{p \times m}, G_2 \in \mathbb{R}^{p \times n_p}, \) and \(H \in \mathbb{R}^{m \times p}\) are the matrices associated with unmodeled dynamics such that \(F\) is Hurwitz\(^{18}\).

Next, let \(\beta \in \mathbb{R}_+\) be a free variable\(^{19}\) and consider the state transformation \(z(t) \triangleq \beta q(t), \quad z(t) \in \mathbb{R}^p\). One can then equivalently rewrite (3.121), (3.122), and (3.123) as

\[
\begin{align*}
\dot{x}_p(t) &= A_p x_p(t) + B_p \Lambda [u(t) + \delta_p(x_p(t))] + \beta^{-1} B_p H z(t), \\
x_p(0) &= x_{p0}, \quad \text{(3.125)} \\
\dot{z}(t) &= F z(t) + \beta G_1 \Lambda u(t) + \beta G_2 x_p(t), \\
z(0) &= \beta q_0. \quad \text{(3.126)}
\end{align*}
\]

As often done in the literature (e.g., see [8] and [7]), the feedback control architectures studied in Sections 3.4.3 and 3.4.4 of this paper includes nominal and adaptive components. For the nominal control components, in particular, let \(c(t) \in \mathbb{R}^{n_c}\) be a given bounded command\(^{20}\) and \(x_i(t) \in \mathbb{R}^{n_i}\) be the integrator state satisfying

\[
\dot{x}_i(t) = E_p x_p(t) - c(t), \quad x_i(0) = x_{i0}, \quad \text{(3.127)}
\]

where \(E_p \in \mathbb{R}^{n_c \times n_p}\) allows one to select a subset of \(x_p(t)\) to be followed by the command \(c(t)\). Considering (3.125) and (3.127), the augmented system dynamics can now be written as

\(^{17}\)The first linear growth inequality on the basis function is common in the adaptive control literature (e.g., see [1]). The second linear growth inequality on the unknown approximation error resulting from the unstructured part of the system uncertainty is often utilized in the literature with \(k_0 = 0\) on a compact subset of the Euclidean space (e.g., see Chapter 12 in [7]). Here, the reason we consider a case with \(k_0 \in \mathbb{R}_+\) is owing to the fact that we do not constrain (3.124) to hold on such a compact subset.

\(^{18}\)Since \(F\) is Hurwitz, which is a common assumption in the adaptive control systems literature addressing unmodeled dynamics (e.g., see [18] and [1]), there exists \(S \in \mathbb{R}^{p \times p}\) such that \(0 = F^T S + S F + I\) holds.

\(^{19}\)As in the results of [1], we introduce this free variable to facilitate the stability analysis given later in this paper.

\(^{20}\)Note that the existence of \(x^*_0 \in \mathbb{R}_+\) is guaranteed here by definition such that \(\|c(t)\|_2 \leq x^*_0\) holds.
3.4.3 and 3.4.4 under the standard assumption on the rank
feedback gain matrix

where $x(t) = [x_p^T(t), x_l^T(t)]^T \in \mathbb{R}^n, A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, B_1 \in \mathbb{R}^{n \times n_1}, n = n_p + n_c$.

For the model reference adaptive control components of Sections 3.4.3 and 3.4.4, we next consider the reference model

$$
\dot{x}_r(t) = A_rx_r(t) + B_rc(t), \quad x_r(0) = x_{r0},
$$

where $x_r(t) \in \mathbb{R}^n$ is the reference model state vector and $A_r \triangleq A - BK \in \mathbb{R}^{n \times n}$ is a Hurwitz reference model system matrix\(^{21}\). While $A$ may not be Hurwitz, $A_r$ can be made Hurwitz through a proper selection of the feedback gain matrix $K \in \mathbb{R}^{m \times n}$ used in the nominal components of the control signals given in Sections 3.4.3 and 3.4.4 under the standard assumption on the rank $\begin{bmatrix} A_p & B_p \\ E_p & 0 \end{bmatrix} = n_p + m$ Chapter 13.4 of [7]. By letting the system error be $e(t) \triangleq x(t) - x_r(t)$, we can now write the system error dynamics and (3.126) respectively as

$$
\dot{e}(t) = A_e e(t) + B\Lambda [u(t) + Kx(t) + W^T \sigma(\cdot) + \epsilon(\cdot)] + \beta^{-1} BH_z(t), \quad e(0) = e_0, \quad (3.130)
$$

$$
\dot{z}(t) = Fz(t) + \beta G_1Au(t) + \beta G_2Nx(t), \quad z(0) = \beta q_0, \quad (3.131)
$$

where $W \triangleq [W_p^T, (\Lambda^{-1} - I)K^T]^T \in \mathbb{R}^{(s+n) \times m}, \sigma(\cdot) \triangleq [\sigma_p^T(x_p(t)), x^T(t)]^T \in \mathbb{R}^{(s+n)},$ and $N \triangleq [I_{n_p \times n_p}, 0_{n_p \times n_c}] \in \mathbb{R}^{n_p \times n}$. Consistent with the literature, the objective of this paper is to establish (sufficient) stability tradeoff conditions in terms of boundedness of the system error dynamics given by (3.130) coupled with the unmodeled dynamics given by (3.131) in the presence of model reference adaptive control laws in the feedback loop, in order to pave a way to implement these laws reliably in real-world applications. Motivated by this standpoint, we first analyze stability tradeoffs of standard adaptive control laws in Section 3.4.3, then synthesize adaptive robustifying terms for a relaxed stability tradeoff in Section 3.4.4 (the first contribution

\(^{21}\)Since $A_r$ is Hurwitz, there exists $P \in \mathbb{R}^{n \times n}_+$ such that $0 = A_r^T P + PA_r + I$ holds.
of this paper), and finally demonstrate our theoretical findings through an illustrative numerical example in Section 3.4.5 and in an experiment in Section 3.4.6 (the second contribution of this paper).

3.4.3 Stability Tradeoff of Standard Adaptive Control Laws

Consider the feedback control architecture given by

\[ u(t) = -Kx(t) - \hat{W}^T(t)\sigma(\cdot). \] (3.132)

Here, "\(Kx(t)\)" denotes the nominal control component and "\(\hat{W}^T(t)\sigma(\cdot)\)" denotes the adaptive control component, where \(\hat{W}(t) \in \mathbb{R}^{(s+n) \times m}\) is an estimate of the unknown weight \(W\) satisfying a standard (e.g., see [18] and [1]) projection operator-based parameter adjustment mechanism\(^{22,23}\) given by

\[
\dot{\hat{W}}(t) = \gamma \text{Proj}_m[\hat{W}(t), \sigma(\cdot)e^T(t)PB], \quad \hat{W}(0) = \hat{W}_0, \tag{3.133}
\]

with \(\gamma \in \mathbb{R}_+\) being the learning rate (adaptation gain). Using (3.132) both in (3.130) and (3.131), one can write

\[
\dot{e}(t) = A_1e(t) - B\hat{W}^T(t)\sigma(\cdot) + B\hat{\Lambda}e(\cdot) + \beta^{-1}BH\tilde{z}(t), \quad e(0) = e_0, \tag{3.134}
\]

\[
\dot{\tilde{z}}(t) = F\tilde{z}(t) - \beta G_1\Lambda Kx(t) - \beta G_1\Lambda\hat{W}^T(t)\sigma(\cdot) + \beta G_2Nx(t), \quad \tilde{z}(0) = \beta q_0, \tag{3.135}
\]

where \(\tilde{W} \triangleq \hat{W}(t) - W \in \mathbb{R}^{(s+n) \times m}\).

The following theorem now establishes a stability tradeoff condition for guaranteeing boundedness of the standard model reference adaptive control architecture (3.132) utilized in (3.134) and (3.135), where it can be considered as a generalization\(^{24}\) of the results in, for example, Theorem 2 in [18] and Corollary 1 in [1] to the case of unstructured system uncertainties.

**Theorem 3.4.1** Consider the dynamical system subject to a class of unstructured system uncertainties and unmodeled dynamics given by (3.128) and (3.131), the reference model given by (3.129), and the adaptive

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\(^{22}\)We refer to [7] and [53] with regard to the details on the projection operator. Concisely, \(\text{tr}[(\hat{\Theta} - \Theta)^T(\text{Proj}_m(\hat{\Theta}, Y) - Y)] = \sum_{i=1}^{m} |\text{col}_i(\hat{\Theta} - \Theta)^T(\text{Proj}(\text{col}_i(\hat{\Theta}), \text{col}_i(Y)) - \text{col}_i(Y))| \leq 0\) holds owing to the projection operator, where \(\hat{\Theta}\) denoting an estimate of parameter \(\Theta\), \(Y\) being function, and \(\text{Proj}_m(\hat{\Theta}, Y) = (\text{Proj}(\text{col}_1(\hat{\Theta}), \text{col}_1(Y)), \ldots, \text{Proj}(\text{col}_m(\hat{\Theta}), \text{col}_m(Y)))\).

\(^{23}\)Owing to the projection operator existence of \(w^* \in \mathbb{R}_+\) is guaranteed such that \(||\hat{W}(t)||_2 \leq w^*\) holds in (3.133).

\(^{24}\)Here, we refer to the point iii) in Remark 3.4.1.
control law given by (3.132) and (3.133). If the stability tradeoff condition given by

\[ \|SG_1\|_2 \|H\|_2 \lambda_U \bar{w} + \|SG_2\|_2 \|H\|_2 < \frac{\rho^*}{4 \|PB\|_2}, \]  

(3.136)

holds\(^{25}\), then the solution \((e(t), z(t), \hat{W}(t))\) is uniformly bounded, where \(\rho^* \triangleq 1 - 2\lambda_d k_0 \|PB\|_2\), \(\bar{w} \triangleq w^* l + \|K\|_2\), and \(l \triangleq (1 + l_0)\).

**Proof.** Consider the Lyapunov-like function given by

\[ V(e, \hat{W}, z) = e^T Pe + \gamma^{-1} \text{tr}(\hat{W} \Lambda^\frac{1}{2})^T (\hat{W} \Lambda^\frac{1}{2}) + \alpha z^T S z, \]  

(3.137)

where \(\alpha \in \mathbb{R}_+\). The time-derivative of (3.137) satisfies

\[ \dot{V}(e(t), \hat{W}(t), z(t)) \leq -\|e(t)\|_2^2 - \alpha \|z(t)\|_2^2 + 2\beta^{-1} \|e(t)\|_2 \|PB\|_2 \|H\|_2 \|z(t)\|_2 \]

\[ + 2 \|e(t)\|_2 \|PB\|_2 \|\Lambda\|_2 \|\epsilon(\cdot)\|_2 + 2\alpha \beta \|z(t)\|_2 \|SG_1\|_2 \|\Lambda\|_2 \|\hat{W}(t)\|_2 \|\sigma(\cdot)\|_2 \]

\[ + 2\alpha \beta \|z(t)\|_2 \|SG_1\|_2 \|\Lambda\|_2 \|K\|_2 \|x(t)\|_2 + 2\alpha \beta \|z(t)\|_2 \|SG_2\|_2 \|x(t)\|_2. \]  

(3.138)

For \(\|\epsilon(\cdot)\|_2\) and \(\|\sigma(\cdot)\|_2\) in (3.138), note that

\[ \|\epsilon(\cdot)\|_2 \leq k_0 \|e(t)\|_2 + d^*_1, \]  

(3.139)

\[ \|\sigma(\cdot)\|_2 \leq l \|e(t)\|_2 + d^*_2, \]  

(3.140)

hold with \(d^*_1 \in \mathbb{R}_+\) is an upper bound for \(k_0 \|x_r(t)\|_2 + k_c\) (e.g., \(k_0 \|x_r(t)\|_2 + k_c \leq d^*_1\)) and \(d^*_2 \in \mathbb{R}_+\) is an upper bound for \(l \|x_r(t)\|_2 + \|c(t)\|_2 + l_c\) (e.g., \(l \|x_r(t)\|_2 + \|c(t)\|_2 + l_c \leq d^*_2\)^{26}. With (3.139) and (3.140), (3.138) can be upper bounded by

\[ \dot{V}(e(t), \hat{W}(t), z(t)) \leq -\bar{\Lambda}(\mathcal{R}) \|\xi(t)\|_2 \left[ \|\xi(t)\|_2 - \frac{r_0}{\bar{\Lambda}(\mathcal{R})} \right], \]  

(3.141)

where \(\xi(t) \triangleq [\|e(t)\|_2, \|z(t)\|_2]^T\),

---

\(^{25}\) Notice that if (3.136) holds, then \(\rho^* > 0\) must hold.

\(^{26}\) Existence of \(d^*_1 \in \mathbb{R}_+\) and \(d^*_2 \in \mathbb{R}_+\) are guaranteed owing to the boundedness of \(x_r(t)\) and \(c(t)\).
\[ R \triangleq \begin{bmatrix} \rho^* & \eta \\ \eta & \alpha \end{bmatrix}, \quad (3.142) \]

\[ \eta \triangleq -\beta^{-1} \|PB\|_2 \|H\|_2 - \alpha \beta \lambda_U \hat{w} \|SG_1\|_2 - \alpha \beta \|SG_2\|_2, \quad r_0 \triangleq 2 \|PB\|_2 \lambda_U d_1^* + 2 \alpha \beta \|SG_1\|_2 \lambda_U (w^* d_2^* + \|K\|_2 x^*_T) + 2 \alpha \beta \|SG_2\|_2 x^*_T, \text{ and } \|x_t(t)\|_2 \leq x^*_T. \]

Here, uniform boundedness of the solution \((e(t), z(t), \hat{W}(t))\) follows from (3.141) owing to the projection operator (e.g., see Section 11.5 of [7]) when \( R \) given by (3.142) is positive-definite; that is, \( \lambda(R) \in \mathbb{R}_+ \).

Next, we investigate the positive-definiteness of \( R \) in (3.142) to obtain (3.136). To this end, the leading principle minors [58] of \( R \) are given by \( m_1 = 1 - 2 \lambda_w k_0 \|PB\|_2 \|H\|_2 + \alpha \beta \|SG_1\|_2 \lambda_U \hat{w} + \alpha \beta \|SG_2\|_2 \) and \( m_2 = \rho^* \alpha - \left( \beta^{-1} \|PB\|_2 \|H\|_2 + \alpha \beta \|SG_1\|_2 \lambda_U \hat{w} + \alpha \beta \|SG_2\|_2 \right)^2 \). Note that positive \( \rho^* \) yields \( m_1 \in \mathbb{R}_+ \). There also exists a positive \( \alpha \) that yields \( m_2 \in \mathbb{R}_+ \) when (3.136) holds. Hence, the result is now immediate since all the leading principle minors of \( R \) are positive; that is, \( R \) is positive-definite based on (3.136).

\[ \textbf{Remark 3.4.1} \] From (3.136), one can make the following observations: i) If the linear growth inequality constant \( k_0 \) on the unknown approximation error resulting from the unstructured part of the system uncertainty is sufficiently large, then \( \rho^* > 0 \) does not hold. This alone further implies that (3.136) does not hold. ii) For cases when \( \rho^* > 0 \) holds, (3.136) does not hold against significant system uncertainties (i.e., sufficiently large \( w^* \)) in the presence of unmodeled dynamics. In the next section presenting the first contribution of this paper, we relax the stability tradeoff condition given by (3.136) (i.e., \( \rho^* > 0 \)) to address the issues i) and ii) by adding robustifying terms to the adaptive control law given by (3.132). iii) Finally, if the unknown approximation error \( \varepsilon(x_p(t)) \) is always bounded for all \( x_p \in \mathbb{R}^{n_p} \) (i.e., \( k_0 = 0 \)), which can be restrictive in real-world applications, one recovers the stability tradeoff condition given in Corollary 1 of [1], as expected.

3.4.4 Synthesis of Adaptive Robustifying Terms for Relaxed Stability Tradeoff

We propose the feedback control architecture given by

\[ u(t) = -Kx(t) - \hat{W}^T(t) \sigma(\cdot) - \hat{\mu}(t) B^T Pe(t) - \hat{\zeta}(t) \tanh(B^T Pe(t)) \|e(t)\|_2. \quad (3.143) \]

Here, while “\( Kx(t) \)” and “\( \hat{W}^T(t) \sigma(\cdot) \)” with \( \hat{W}(t) \) satisfying (3.133) denote the terms used in (3.132) of the previous section, the proposed feedback control architecture in (3.143) now also includes adaptive
robustifying terms “$\hat{\mu}(t)B^TPe(t)$” and “$\hat{\zeta}(t)\tanh(B^TPe(t))\|e(t)\|_2$” for addressing the limitations discussed in $i$) and $ii$) of Remark 3.4.1\textsuperscript{27}. Specifically, let $\hat{\mu}(t)$ and $\hat{\zeta}(t)$ respectively satisfy the projection operator-based parameter adjustment mechanisms given by

$$\dot{\hat{\mu}}(t) = \mu_0 \text{Proj} \left( \hat{\mu}(t), \|B^TPe(t)\|_2^2 \right), \quad \hat{\mu}(0) = \hat{\mu}_0, \quad (3.144)$$

$$\dot{\hat{\zeta}}(t) = \zeta_0 \text{Proj} \left( \hat{\zeta}(t), e^T(t)PB \tanh(B^TPe(t)) \|e(t)\|_2 \right), \quad \hat{\zeta}(0) = \hat{\zeta}_0, \quad (3.145)$$

with $\hat{\mu}(0) \in \mathbb{R}_+$ and $\hat{\zeta}(0) \in \mathbb{R}_+$, where $\mu_0 \in \mathbb{R}_+$ and $\zeta_0 \in \mathbb{R}_+$ are design parameters\textsuperscript{28}. Let also the projection bounds for (3.144) and (3.145) be selected as\textsuperscript{29}

$$\hat{\mu}(t) \leq \mu \psi_\mu, \quad \mu \triangleq \lambda_U \frac{k^2 + \lambda_L}{2}, \quad \psi_\mu > 1, \quad (3.146)$$

$$\hat{\zeta}(t) \leq \zeta \psi_\zeta, \quad \zeta \triangleq \frac{k_0 \lambda_U}{\lambda_L}, \quad \psi_\zeta > 1, \quad (3.147)$$

where $k^* \triangleq \tilde{w} + \frac{k_0 \lambda_U \psi_\tan \psi_\tan}{\lambda_L}$ and $\psi_\tan \geq \tanh(\cdot)$. Using (3.143) both in (3.130) and (3.131), one can write

$$\dot{e}(t) = A_t e(t) - B\Lambda \tilde{w}^T(t)\sigma(\cdot) - B\Lambda e(\cdot) - \hat{\mu}(t)BAB^TPe(t) - \hat{\zeta}(t)B\Lambda \tanh(B^TPe(t))\|e(t)\|_2 + \beta^{-1}BHz(t), \quad e(0) = e_0, \quad (3.148)$$

$$\dot{z}(t) = Fz(t) - \beta G_1\Lambda Kx(t) - \beta G_1\Lambda \tilde{w}^T(t)\sigma(\cdot) - \beta \hat{\mu}(t)G_1\Lambda B^TPe(t) - \beta \hat{\zeta}(t)G_1\Lambda \tanh(B^TPe(t))\|e(t)\|_2 + \beta G_2Nx(t), \quad z(0) = \beta q_0. \quad (3.149)$$

The following theorem presents the first contribution of this paper. In particular, it establishes a new stability tradeoff condition, which removes the condition of $\rho^* > 0$ and relaxes (3.136) of the previous section, for guaranteeing the boundedness of the proposed model reference adaptive control architecture (3.143) utilized in (3.148) and (3.149).

\textsuperscript{27}In (3.143), $\tanh(\cdot)$ stands for the vector tangent hyperbolic function of the form $\tanh(x) = [\tanh(x_1), \tanh(x_2), \ldots, \tanh(x_n)]^T$ with $x \in \mathbb{R}^n$.

\textsuperscript{28}Since $\hat{\mu}(0) \in \mathbb{R}_+$ and $\hat{\zeta}(0) \in \mathbb{R}_+$, then $\mu(\cdot) \in \mathbb{R}_+$ and $\zeta(\cdot) \in \mathbb{R}_+$ hold for positive design parameters $\mu_0$ and $\zeta_0$. From a practical standpoint, one should also select zero as the projection operator lower bound and a proper positive finite number as the projection operator upper bound in both (3.144) and (3.145).

\textsuperscript{29}Since $\psi_\mu > 1$ and $\psi_\zeta > 1$, one can choose projection bounds as large as desired theoretically.
**Theorem 3.4.2** Consider the dynamical system subject to a class of unstructured system uncertainties and unmodeled dynamics given by (3.128) and (3.131), the reference model given by (3.129), the proposed adaptive control law given by (3.143), (3.133), (3.144), and (3.145). If the stability tradeoff condition given by

\[
\|SG_2\|_2 \|H\|_2 + \|SG_1\|_2 \|H\|_2 \lambda_U k^* < \frac{1}{\sqrt{2}} \sqrt{\frac{(\lambda_U^2 k^2 + \lambda_L) \phi_0}{\lambda_L}},
\]  

(3.150)

holds\(^{30}\), then the solution \((e(t), \dot{W}(t), z(t), \tilde{\mu}(t), \tilde{\xi}(t))\) is uniformly bounded, where \(\phi_0 \triangleq (2\lambda_L - 1 - \psi_\mu^2 \lambda_U^2 \phi_1^2 - 2\psi_\mu \lambda_U \phi_1), \phi_1 \triangleq \|SG_1\|_2 \|H\|_2, \tilde{\mu}(t) \triangleq \tilde{\mu}(t) - \mu, and \tilde{\xi}(t) \triangleq \tilde{\xi}(t) - \xi\).

**Proof.** Consider the Lyapunov-like function given by

\[
V(e, \dot{W}, z, \tilde{\mu}, \tilde{\xi}) = e^T Pe + \gamma^{-1} \text{tr}(\dot{W} A \dot{z})^T (\dot{W} A \dot{z}) + \alpha z^T S z + \mu_0^{-1} \mu^2 \lambda_L + \zeta_0^{-1} \xi^2 \lambda_L.
\]  

(3.151)

The time-derivative of (3.151) satisfies

\[
\dot{V}(e(t), \dot{W}(t), z(t), \tilde{\mu}(t), \tilde{\xi}(t)) = -e^T(t) e(t) - 2e^T(t) P B A \dot{W}^T(t) \sigma(\cdot) - 2\tilde{\mu}(t) e^T(t) P B A \dot{W}^T(t) e(t)
\]

\[
- 2e^T(t) P B A \tilde{\xi}(t) \tanh(B^T Pe(t)) \|e(t)\|_2 + 2e^T(t) P B A \dot{e}(\cdot) + 2\beta^{-1} e^T(t) P B H z(t)
\]

\[
- \alpha z^T(t) z(t) - 2\alpha \beta z^T(t) S G_1 A \dot{W}^T(t) \sigma(\cdot) - 2\alpha \beta \dot{z}^T(t) S G_1 A K x(t)
\]

\[
- 2\alpha \beta \tilde{\mu}(t) \dot{z}^T(t) S G_1 A B \dot{W}(t) + 2\alpha \beta z^T(t) S G_2 N x(t) - 2\alpha \beta \dot{z}^T(t) S G_1 A \dot{\xi}(t) \tanh(B^T Pe(t)) \|e(t)\|_2
\]

\[
+ 2\gamma^{-1} \text{tr} \dot{W}^T(t) \dot{W}(t) A + 2\mu_0^{-1} \tilde{\mu}(t) \tilde{\mu}(t) \lambda_L + 2\zeta_0^{-1} \tilde{\xi}(t) \tilde{\xi}(t) \lambda_L.
\]  

(3.152)

With (3.139) and (3.140), (3.152) can be upper bounded by

\[
\dot{V}(e(t), \dot{W}(t), z(t), \tilde{\mu}(t), \tilde{\xi}(t)) \leq -\|e(t)\|_2^2 - \alpha \|z(t)\|_2^2 - 2\mu \lambda_L \|B^T Pe(t)\|_2^2 - 2\tilde{\xi}(t) e^T(t) P B A \tanh(B^T Pe(t)) \|e(t)\|_2
\]

\[
+ 2 \|B^T Pe(t)\|_2 \|A\|_2 \|e(\cdot)\|_2 - 2e^T(t) P B A \dot{W}^T(t) \sigma(\cdot) + 2 \text{tr} \dot{W}^T(t) \text{Proj}_n [\dot{W}(t), \sigma(\cdot) e^T(t) P B] \|A\|_2
\]

\[
+ 2\beta^{-1} \|B^T Pe(t)\|_2 \|H\|_2 \|z(t)\|_2 + 2\alpha \beta \|A\|_2 \|\dot{W}(t)\|_2 \|SG_1\|_2 \|z(t)\|_2 \|\sigma(\cdot)\|_2
\]

\[30\) Notice that if (3.150) holds, then \(\|SG_1\|_2 \|H\|_2 < \frac{2\sqrt{2}}{\psi_\mu \lambda_U}\) with \(\lambda_L > 0.5\) must hold.

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\[ +2\alpha\beta \|A\|_2 \|K\|_2 \|SG_1\|_2 \|z(t)\|_2 \|x(t)\|_2 + 2\alpha\beta \hat{\mu}(t) \|SG_1\|_2 \|\Lambda\|_2 \|z(t)\|_2 \|B^T Pe(t)\|_2 \]
\[ +2\alpha\beta \hat{\zeta}(t) \|z(t)\|_2 \|SG_1\|_2 \|\Lambda\|_2 \psi_{\tan} \|e(t)\|_2 + 2\alpha\beta \|z(t)\|_2 \|SG_2\|_2 \|x(t)\|_2 \]
\[ -2\hat{\mu}(t)\hat{\lambda}_L \|B^T Pe(t)\|_2^2 + 2\hat{\mu}(t) \text{Proj} \left( \hat{\mu}(t), \|B^T Pe(t)\|_2^2 \right) \hat{\lambda}_L \]
\[ +2\hat{\xi}(t) \text{Proj} \left( \hat{\xi}(t), e^T(t) PB \tanh \left( B^T Pe(t) \right) \|e(t)\|_2 \right) \hat{\lambda}_L, \]

Based on \( \zeta \) given by (3.147), we can now rearrange (3.153) as

\[
\hat{V}(e(t), \hat{W}(t), z(t), \hat{\mu}(t), \hat{\xi}(t)) \leq -\|e(t)\|_2^2 - \alpha \|z(t)\|_2^2 - 2\mu \hat{\lambda}_L \|B^T Pe(t)\|_2^2 + 2\beta^{-1} \|B^T Pe(t)\|_2 \|H\|_2 \|z(t)\|_2 \\
+2\alpha\beta \hat{\lambda}_U \hat{w} \|SG_1\|_2 \|z(t)\|_2 \|e(t)\|_2 + 2\alpha\beta \hat{\lambda}_U \hat{w}^* \|SG_1\|_2 \|z(t)\|_2 \|e(t)\|_2 \\
+2\alpha\beta \|K\|_2 \|SG_1\|_2 \|z(t)\|_2 x^*_t + 2\alpha\beta \mu \|SG_1\|_2 \hat{\lambda}_U \|z(t)\|_2 \|B^T Pe(t)\|_2 \\
+2\alpha\beta \|z(t)\|_2 \|SG_2\|_2 \|e(t)\|_2 + 2\|B^T Pe(t)\|_2 \hat{\lambda}_U d^*_t \\
+2 \|e(t)\|_2 \hat{k}_0 \hat{\lambda}_U \left[ \|B^T Pe(t)\|_2 - e^T(t) PB \tanh \left( B^T Pe(t) \right) \right].
\]

In (3.154), note that \( \|\phi B^T Pe(t)\|_2 - \phi e^T(t) PB \tanh(\phi B^T Pe(t)) \leq \mathcal{L}_t = 0.2785 \) holds for all \( \phi \in \mathbb{R}_+ \). Based on this fact and following similar steps to the ones given in the proof of Theorem 3.4.1, (3.154) can be rewritten as

\[
\hat{V}(e(t), \hat{W}(t), z(t), \hat{\mu}(t), \hat{\xi}(t)) \leq -\hat{\lambda} \left( \mathcal{R} \right) \|\hat{\xi}(t)\|_2 \left[ \|\hat{\xi}(t)\|_2 - \frac{r_0}{\hat{\lambda} \left( \mathcal{R} \right)} \right],
\]

where \( \hat{\xi}(t) \triangleq \left[ \|e(t)\|_2, \|B^T Pe(t)\|_2, \|z(t)\|_2 \right]^T \),

\[
\mathcal{R} \triangleq \begin{bmatrix}
1 & 0 & \eta_1 \\
0 & 2\mu \hat{\lambda}_L & \eta_2 \\
\eta_1 & \eta_2 & \alpha
\end{bmatrix}, \tag{3.156}
\]

where \( \alpha \in \mathbb{R}_+, \eta_1 \triangleq -\alpha\beta \hat{\lambda}_U k^2 \|SG_1\|_2 - \alpha\beta \|SG_2\|_2, \eta_2 \triangleq -\beta^{-1} \|H\|_2 - \alpha\beta \mu \hat{\lambda}_U \|SG_1\|_2, \) and \( r_0 \triangleq 2\alpha\beta \cdot \|SG_1\|_2 \hat{\lambda}_U (\hat{w}^* d^*_t + \|K\|_2 x^*_t) + 2\alpha\beta \|SG_2\|_2 x^*_t + 2\hat{\lambda}_U d^*_t + 2\hat{k}_0 \hat{\lambda}_U L_i \).
Next, we investigate the positive-definiteness of $\mathcal{R}$ in (3.155) to obtain (3.150). To this end, the leading principle minors [58] of $\mathcal{R}$ are given by $m_1 = 1$, $m_2 = 2\mu\lambda_L$, and $m_3 = 2\alpha\mu\lambda_L - \beta^{-2}\|H\|_2^2 - \alpha^2\beta^2\lambda_U^2\mu^2\psi_{\mu}^2\|S_G1\|_2^2 - 2\alpha\lambda_U\mu\psi_{\mu}\|H\|_2\|S_G1\|_2 - 2\alpha^2\beta^2\lambda_L\lambda_U^2\mu^2k^2\|S_G1\|_2 - 2\mu\lambda_L\alpha^2\beta^2\|S_G2\|_2^2 - 4\mu\lambda_L - \alpha^2\beta^2\lambda_Uk^2\|S_G1\|_2\|S_G2\|_2$. Note that $m_1 \in \mathbb{R}^+$ and $m_2 \in \mathbb{R}^+$. In addition, by letting $\beta \triangleq \frac{|H|}{\sqrt{\mu}}$, one obtains

$$m_3 = \alpha\mu\left[2\lambda_L - 1 - \lambda_U^2\psi_{\mu}^2\phi_1^2 - 2\lambda_U\psi_{\mu}\phi_1 - 2\lambda_L\left(\frac{\lambda_U^2k^2\phi_1^2 + \phi_2^2 + 2\lambda_Uk^2\phi_1\phi_2}{\mu}\right)\right], \quad (3.157)$$

where $\phi_2 \triangleq \|S_G2\|_2\|H\|_2$ and $\mu$ is as in (3.146). Note also that $m_3 \in \mathbb{R}^+$ from (3.150). Hence, the result is now immediate since all the leading principle minors of $\mathcal{R}$ are positive; that is, $\mathcal{R}$ is positive-definite from (3.150).

**Remark 3.4.2** The proposed feedback control architecture (3.143) including the adaptive robustifying terms guarantees boundedness of the closed-loop error dynamics (3.148) coupled with the unmodeled dynamics (3.149) subject to a new stability tradeoff condition (3.150). First, we note that this new condition removes the requirement of $\rho^* > 0$. Second, it relaxes (3.136) in that overall system stability in the presence of significant system uncertainties (i.e., sufficiently large $w^*$) can now be ensured. To elucidate this point, recall that $k^* \triangleq \bar{w} + \frac{k_0\lambda_U\psi_{\mu}\psi_{\mu}}{\lambda_L}$, where $k^* \to \infty$ as $w^* \to \infty$. Specifically, at the limit when $k^*$ becomes sufficiently large and $\phi_0 > 0$, (3.150) yields

$$\frac{\|S_G1\|_2\|H\|_2}{\sqrt{\phi_0}/\lambda_L} < \frac{1}{\sqrt{2}}. \quad (3.158)$$

Hence, when unmodeled system dynamics satisfy (3.158), overall system stability holds in this case. Finally, this conclusion is consistent with Corollary 2 in [1] that only considers structured system uncertainties along with unmodeled dynamics; that is, the results of this paper captures a broader class of physical systems with unstructured system uncertainties along with unmodeled dynamics that cannot be captured with Corollary 2 in [1] (e.g., see Sections 3.4.5 and 3.4.6)\(^{31}\).

**Remark 3.4.3** Condition (3.150) can be further rewritten as follows. Specifically, recall $\phi_0 > 0$ with $\|S_G1\|_2\|H\|_2 < \frac{2\lambda_L^{-1}}{\sqrt{\psi_{\mu}\lambda_U}}$ and $\lambda_L > 0.5$. Now, let $\rho_0 \in (0, 1)$, where one can write $\|S_G1\|_2\|H\|_2 \triangleq \rho_0 \frac{2\lambda_L^{-1}}{\sqrt{\psi_{\mu}\lambda_U}}$.

\(^{31}\)Using the proposed model reference adaptive control method utilized in [1] alone, which does not have the last term on the right hand side of (3.143), one cannot obtain a relaxed stability tradeoff condition. Specifically, with the proposed architecture in [1], the condition on $\rho^* \triangleq 1 - 2\lambda_k\|PB\|_2$ being positive remains and it is hard to satisfy this condition in practice as $k_0$ gets larger; see also ii) in Remark 3.4.1.
It now follows that
\[ \phi_0 \triangleq (2\lambda L - 1 - \psi^2 \lambda_U \phi_1^2 - 2\psi \lambda_U \phi_1) = 2\lambda L - (\psi \lambda_U \phi_1 + 1)^2 = 2\lambda L - (\sqrt{2\lambda_U \rho_0})^2 = 2\lambda L(1 - \rho_0^2). \]
Note that the expression for \( \phi_0 \) is positive since \( \rho_0 \in (0, 1) \). Finally, using the expression for \( \phi_0 \) in (3.150), we have
\[ \|SG_2\|_2 \|H\|_2 + \|SG_1\|_2 \|H\|_2 \lambda_U k^* < \sqrt{(\lambda_U^2 k^2 + \lambda_L)(1 - \rho_0^2)}. \]

### 3.4.5 Illustrative Numerical Example

Consider the coupled mechanical system depicted in Figure 3.23 (i.e., \( x(t) = [\theta(t), \dot{\theta}(t), q_1(t), \dot{q}_1(t)]^T \)) is the state for the modeled part and \( z(t) = [q_2(t), \dot{q}_2(t)]^T \) is the state for the unmodeled part) with the dynamics given by

\[
\dot{x}(t) = \begin{bmatrix}
0 & 1 & 0 & 0 \\
\frac{(M_1 + m_0)g}{MI_{fp}} & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
-\frac{m_0 g}{M_1} & 0 & 0 & 0
\end{bmatrix} x(t) + \begin{bmatrix}
0 \\
-\frac{1}{MI_{fp}} \\
0 \\
\frac{1}{M_1}
\end{bmatrix} u(t)
\]
\[
\begin{bmatrix}
0 & -\frac{1}{M_l l_p} \\
\frac{1}{M_l} & 0
\end{bmatrix} \Lambda \varepsilon(\cdot) + \begin{bmatrix}
0 & -\frac{1}{M_l l_p} \\
\frac{1}{M_l} & 0
\end{bmatrix} \Lambda x + 0 \cdot \beta^{-1} + \begin{bmatrix}
0 & k_1 \\
-1 & -c_1
\end{bmatrix} H z(t),
\]
(3.159)

\[
\dot{z}(t) = \begin{bmatrix}
0 & 1 \\
-k_1 + k_2 & -c_1 + c_2
\end{bmatrix} z(t) + \begin{bmatrix}
0 & 1 \\
\frac{1}{M_l} & 0
\end{bmatrix} \Lambda u + \begin{bmatrix}
0 & k_1 \\
0 & c_1
\end{bmatrix} \begin{bmatrix}
0 & 0 \\
G_1 & G_2
\end{bmatrix} x(t),
\]
(3.160)

In (3.160), the unstructured system uncertainties with \( \varepsilon(\cdot) = x^T(t) \cos(x(t)) + 0.1 \). Here, spring constants are \( k_1 = 0.1N/m \) and \( k_2 = 0.75N/m \), damping coefficients are \( c_1 = 0.15 \) and \( c_2 = 0.5 \), inverted pendulum and cart masses are \( m_0 = 0.5kg, M_1 = 0.25kg, \) and \( M_2 = 1kg \), length of the inverted pendulum is \( l_p = 1m \), and acceleration due to gravity is \( g = 9.81m/s^2 \). We select learning rate as \( \gamma = 45 \), unmodeled dynamics free variable as \( \beta = 10 \) (for analysis purposes), constant command as \( c(t) = 1.2m \), unknown control effectiveness matrix as \( \Lambda = 0.98 \), and the projection bound as \( \hat{W}_{\text{max}} = |W| + 0.5 \). We calculate free variable \( \zeta = 2.2105 \).

The controller gain matrix \( K_1 \) is obtained through a linear quadratic regulator design with weighting matrices \( Q = \text{diag}[2, 2, 1, 1] \) to penalize the modeled states and \( R = 5 \) to penalize the control input (i.e., this results in \( K_1 = [-17.7366, -3.8328, 0.4472, 1.0913] \) and the matrix \( K_2 \) is selected as \( K_2 = -0.4472 \) satisfying \( K_2 = -(C(A - BK_1)^{-1}) \) with \( C = [0, 0, 1, 0] \) such that a translational position command for the cart can be followed.

The system performance of the standard model reference adaptive controller in Theorem 3.4.1 is not included to this paper due to the page limitations; however, we note that all the state and control signals grow unboundedly. This is due to the fact that its stability tradeoff condition does not hold in this case.

In contrast, the system performance of the proposed model reference adaptive controller in Theorem 3.4.2 shown in Figure 3.24 is bounded, where we use \( \mu_0 = 90, \xi_0 = 2, \hat{\mu}_{\text{max}} = 51.4704, \hat{\xi}_{\text{max}} = 2.4316, \) and

\[32\] For computing this condition, we use \( \lambda_L = 0.95, \lambda_U = 1.05, l = 2.1, k_0 = 2 \), and we calculate \( k^* = 6.4482 \) that results in \( \rho^* = -36.1316 \) and \( \|\Sigma G_1\| \|H\| \| \Sigma G_2 \| \|H\| \|w^* l\| + \|\Sigma G_2\| \|H\| \|w^* l\| = 1.4069 \) and \( \rho^* / (4 \|PB\|) = -1.0217 \) for (3.136). Here, we have \( w^* l \) instead of \( \hat{\nu} \); this is because of the fact that we have not use an integrator-based nominal controller in this numerical example.
Figure 3.24: Stable simulation results for the proposed model reference adaptive controller given in Theorem 3.4.2.

\[ \hat{W}_{\text{max}} = 0.9663 \] in the design of this architecture. The boundedness of the closed-loop dynamical system trajectories in this case is expected since the proposed approach satisfies its stability tradeoff condition\(^{33}\).

### 3.4.6 Experimental Studies

In this section, we apply the standard model reference adaptive control architecture given by (3.132) (Theorem 3.4.1) and the proposed model reference adaptive control architecture given by (3.143) (Theorem 3.4.2) to a physical system, where this is the second contribution of this paper. This physical system, produced by Quanser, is shown in Figure 3.25. Specifically, it involves an inverted pendulum on a cart (left in Figure 3.25; modeled dynamics subject to a measurable state vector) coupled with another cart through a spring (right in Figure 3.25; unmodeled dynamics subject to an unmeasurable state vector) in the presence of velocity-dependent friction uncertainty, where both carts slide on a shaft using linear bearings. Due to the presence of the inverted pendulum, the friction primarily affects the cart on the left and it is negligible for the cart on the right (e.g., see [96] for further details). For the cart on the left with inverted pendulum, while

\(^{33}\)For computing this condition, we use \( \lambda_L = 0.95 \), \( \lambda_U = 1.05 \), \( \psi_\mu = 1.1 \), and \( \psi_\zeta = 1.1 \), which results in \( \|S_{G_1}\|_2 \|H\|_2 = 0.3199 < 0.3276 \), and \( \|S_{G_1}\|_2 \|H\|_2 \lambda_k^* + \|S_{G_2}\|_2 \|H\|_2 = 2.2237 < 4.6612 \).
the authors of, for example, [67] develop friction models, these models are not only approximate but also do not consider the presence of unmodeled dynamics; hence, here we treat the friction as an unstructured system uncertainty. The inverted pendulum is instrumented using a quadrature incremental encoder and it is attached in front of the cart through a rod for enabling 360 [deg] rotation and the axis of rotation of this rod is perpendicular to the direction of the motion of the cart with the inverted pendulum. In order to obtain consistent and continuous traction, the cart on the left is driven by a rack and pinion mechanism using a 6 [V] DC motor. In addition, VoltPAQ-X1 is utilized to run the experiments that is a linear voltage controlled power amplifier. Using Quanser Quarc Real Time Windows Target (Win64), the results of this paper are implemented on the computer through MATLAB/Simulink with a sampling rate of 1000 [Hz]. Finally, Quanser Q8-USB data acquisition board is utilized to obtain the data transmission between the computer and the drivers with digital to analog converter.

Mathematically, the linearized dynamics of this physical system with the inverted pendulum being its upward position has the form given by
for the proposed model reference adaptive control law becomes

\[
\begin{aligned}
\dot{x}_p(t) &= \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & \kappa_1 & \kappa_2 & \kappa_3 \\
0 & \kappa_4 & \kappa_5 & \kappa_6
\end{bmatrix} \begin{bmatrix} x_p(t) \end{bmatrix} + \begin{bmatrix} 0 \\
0 \\
I_p + mp_t^2 & J_T \\
J_T & m_p & J_T
\end{bmatrix} \begin{bmatrix} u(t) \end{bmatrix} + \begin{bmatrix} 0 \\
0 \\
\delta_p(x_p(t)) \end{bmatrix} + \begin{bmatrix} 0 \\
0 \\
0 \end{bmatrix} \begin{bmatrix} k \\
0 \end{bmatrix} q(t), \\
\dot{q}(t) &= \begin{bmatrix} 0 & 1 \end{bmatrix} q(t) + \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_p(t) \end{bmatrix},
\end{aligned}
\]

(3.161)

\[
\begin{aligned}
\dot{\hat{q}}(t) &= \begin{bmatrix}
0 & 1 \\
-k/m_p & -B_{eq}/m_p
\end{bmatrix} q(t) + \begin{bmatrix} 0 & 0 & 0 & 0 \\
-k/m_p & 0 & 0 & 0
\end{bmatrix} \begin{bmatrix} x_p(t) \end{bmatrix},
\end{aligned}
\]

(3.162)

where \(v(t) = \tau^{-1}u(t)\) with \(v(t)\) being the applied voltage, \(\kappa_1 \triangleq \frac{m_p^2 g_K}{J_T}, \kappa_2 \triangleq \frac{\tau_a K_z K_m J_p + m_p^2 l_p}{J_T} - \frac{B_{eq}(J_p + m_p l_p)}{J_T}, \kappa_3 \triangleq -\frac{m_p B_p}{J_T}, \kappa_4 \triangleq \frac{(J_{eq} + m_p) m_p g l_p}{J_T}, \kappa_5 \triangleq -\frac{m_p B_p}{J_T} - \frac{\tau_a K_z K_m m_p}{J_T} - \frac{m_p}{J_T} - \frac{B_{eq}(J_{eq} + m_p)}{J_T}, \kappa_6 \triangleq -\frac{B_p}{J_T} + J_{eq} (J_{eq} + m_p + J_{eq} m_p^2), J_{eq} = M_1 + \frac{\tau_a J_m K_z}{l_m}, \tau \triangleq \frac{\tau_a K_z K_m}{l_m}, x_p(t) = [x_c(t), \theta(t), \dot{x}_c(t), \dot{\theta}(t)]^T, \text{ and } q(t) = [q_c(t), \dot{q}_c(t)]^T. \text{ We refer to Table 3.3 for the values of all these physical parameters. Note that (3.161) and (3.162) satisfies the form given by (3.121), (3.122), and (3.123) with } G_1 = 0 \text{ and } \Lambda = I.}

**Remark 3.4.4** Since \(G_1 = 0 \text{ and } \Lambda = I\) in the experimental setup of this section, the stability tradeoff conditions given by (3.136) for the standard model reference adaptive control law and given by (3.150) for the proposed model reference adaptive control law becomes

\[
\|SG_2\|_2 \|H\|_2 < \frac{\rho^*}{4 \|PB\|_2},
\]

(3.163)

\[
\|SG_2\|_2 \|H\|_2 < \sqrt{\frac{\mu(2\lambda_L - 1)}{2\lambda_U}},
\]

(3.164)

respectively, where \(\lambda_L = 1\) and \(\lambda_U = 1\).
Table 3.3: Values of the physical parameters in (3.161) and (3.162).

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Name</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau_m$</td>
<td>Motor Efficiency</td>
<td>1</td>
</tr>
<tr>
<td>$\tau_g$</td>
<td>Planetary Gearbox Efficiency</td>
<td>1</td>
</tr>
<tr>
<td>$K_g$</td>
<td>Planetary Gearbox Gear Ratio</td>
<td>3.71</td>
</tr>
<tr>
<td>$K_t$</td>
<td>Motor Current Torque Gear Constant</td>
<td>$7.68 \times 10^{-3}$ [Nm/A]</td>
</tr>
<tr>
<td>$K_m$</td>
<td>Motor Back-emf Constant</td>
<td>$7.68 \times 10^{-3}$ [V/(rad/s)]</td>
</tr>
<tr>
<td>$R_m$</td>
<td>Motor Armature Resistance</td>
<td>2.6 [Ω]</td>
</tr>
<tr>
<td>$g$</td>
<td>Gravitational Constant on USF</td>
<td>9.79 [m/s²]</td>
</tr>
<tr>
<td>$J_p$</td>
<td>Pendulum Inertia</td>
<td>1.20 $\times 10^{-3}$ [kg m²]</td>
</tr>
<tr>
<td>$m_p$</td>
<td>Mass of Pendulum with T fitting</td>
<td>0.127 [kg]</td>
</tr>
<tr>
<td>$M_1, M_2$</td>
<td>Mass of Carts</td>
<td>0.57 and 0.695 [kg]</td>
</tr>
<tr>
<td>$B_{eq}$</td>
<td>Equivalent Viscous Damping Coefficient at the cart 1</td>
<td>4.3 [Nms/rad]</td>
</tr>
<tr>
<td>$B_{eqz}$</td>
<td>Equivalent Viscous Damping Coefficient at the cart 2</td>
<td>1.1 [Nms/rad]</td>
</tr>
<tr>
<td>$B_p$</td>
<td>Equivalent Viscous Damping Coefficient at the pendulum</td>
<td>0.0024 [Nms/rad]</td>
</tr>
<tr>
<td>$l_p$</td>
<td>Pendulum Length</td>
<td>0.1778 [m]</td>
</tr>
<tr>
<td>$r_{mp}$</td>
<td>Motor Pinion Radius</td>
<td>$6.35 \times 10^{-3}$ [m]</td>
</tr>
<tr>
<td>$J_m$</td>
<td>Rotor Moment of Inertia</td>
<td>$3.9 \times 10^{-7}$ [kg m²]</td>
</tr>
<tr>
<td>$k$</td>
<td>Spring Constant</td>
<td>160 [N/m]</td>
</tr>
</tbody>
</table>

**Remark 3.4.5** Since $\hat{\mu}(t)$ and $\hat{\zeta}(t)$ are monotonically increasing functions (up to the projection operator upper bound), one can practically utilize leakage terms in (3.144) and (3.145) as

\[
\dot{\hat{\mu}}(t) = \mu_0 \text{Proj} \left( \hat{\mu}(t), \|B^T Pe(t)\|_2^2 - \sigma_{\hat{\mu}} \hat{\mu}(t) \right),
\]

\[
\hat{\mu}(0) = \mu_0, \ \hat{\mu}(0) \in R_+,
\]

\[
\dot{\hat{\zeta}}(t) = \zeta_0 \text{Proj} \left( \hat{\zeta}(t), e^T(t)PB \right)
\]

\[
\cdot\tanh(B^T Pe(t)) \|e(t)\|_2 - \sigma_{\hat{\zeta}} \hat{\zeta}(t),
\]

\[
\hat{\zeta}(0) = \zeta_0, \ \hat{\zeta}(0) \in R_+.
\]

Note that the presence of these added leakage terms neither change the system-theoretical conclusion in Theorem 3.4.2 nor its stability tradeoff condition (e.g., see [1]).
For the results in Theorems 3.4.1 and 3.4.2, we utilize radial basis functions (e.g., see [7]) to form the basis function appearing in the unstructured system uncertainty parameterization (3.124) as

$$\sigma_p(x_p(t)) = [\sigma_{p_1}(\dot{x}_c(t)), \ldots, \sigma_{p_5}(\dot{x}_c(t)), x_p(t)],$$  \tag{3.167}

with $\sigma_{p_i}(\dot{x}_c) = \exp(-b_\delta(\dot{x}_c(t) - b_{c_i})^2), i = 1, 2, \ldots, 5, b_\delta = 10^3$. Here, all radial basis functions depend on the cart velocity, since the friction uncertainty is velocity-dependent. Furthermore, “$\dot{x}_c(t) - b_{c_i}$” is the weighted distance for uniform distribution of all radial basis functions over the velocity interval ±0.125 [m/s] with $b_{c_i}$ being weights\(^{34}\). Note that we also include the term “$x_p(t)$” to the basis function in order to practically capture other potential system uncertainties that may linearly depend on the measurable state vector. For the feedback control architectures in both theorems, in addition, we choose free variable $\beta = 1$, command $c(t)$ that changes smoothly from 0 [m] to 1 [m], learning rates as $\gamma = 15, \mu_0 = 5 \times 10^3, \zeta_0 = 15 \times 10^3$, and leakage gains discussed in Remark 3.4.4 as $\sigma_\mu = 5 \times 10^{-6}$ and $\sigma_\zeta = 1 \times 10^{-5}$ (here, $\zeta$ is calculated as $\zeta = 1 \times 10^3$). The nominal control gain matrix $K$ is also obtained through a linear quadratic regulator design with weighting matrices $Q = \text{diag}([120, 20, 3, 1.5, 120])$ to penalize the modeled states and $R = 0.1$ to penalize the control signal.

Figure 3.26 show the experimental results with the standard model reference adaptive control architecture given by (3.132) (Theorem 3.4.1) and Figures 3.27 and 3.28 show the experimental results with the proposed model reference adaptive control architecture given by (3.143) (Theorem 3.4.2)\(^{35}\). Specifically, the stability tradeoff condition given by (3.163) does not hold for the standard model reference adaptive control architecture; hence, this is the reason why its performance in Figure 3.26 is unstable\(^{36}\). On the other hand, the stability tradeoff condition given by (3.164) holds for the proposed standard model reference architecture in this experimental study; hence, the overall closed-loop system response is stable\(^{37}\).

\(^{34}\)In experiments, we observe that the cart velocity can change over the velocity interval ±0.2 [m/s]. Such potential practical deviations between the actual velocity interval and the velocity interval we distribute the radial basis functions is captured within the theoretical framework of this paper. Because, we do not restrict the unknown (residual) approximation error to be bounded on a compact set.

\(^{35}\)All figures are given from 30 [sec] owing to the fact that the first 30 [sec] denotes the initialization period of this physical system (i.e., putting the pendulum to its upward position).

\(^{36}\)We use $\bar{c}_0 = 10^3$ and $\bar{W}_{\text{max}} = 1.5$ to compute the stability tradeoff condition for the standard model reference adaptive control architecture. From (3.163), we obtain $\|S_G\|_2 \|H\|_2 = 1.1687 \times 10^4$ and $\rho^*/(4\|PB\|_2) = -499.7204$; hence, this condition doesn’t hold.

\(^{37}\)We use $\bar{c}_0 = 10^3, \bar{W}_{\text{max}} = 1.5, \psi_\mu = 1.1, \psi_\epsilon = 1100, \psi_\zeta = 1.1, \mu_{\text{max}} = 184 \times 10^{10}$, and $\zeta_{\text{max}} = 121580$ to compute the stability tradeoff condition for the proposed model reference adaptive control architecture. From (3.164), we obtain $\|S_G\|_2 \|H\|_2 = 1.1687 \times 10^4$ and $\sqrt{\mu(2\lambda_d - 1)/2\lambda_d} = 7.8683 \times 10^4$; hence, this condition holds.
3.4.7 Conclusion

We studied model reference adaptive control systems in the presence of a class of unstructured system uncertainties and unmodeled dynamics. In Section 3.4.3, Theorem 3.4.1 revealed a stability tradeoff.
Figure 3.28: Stable experimental results for the proposed model reference adaptive controller given in Theorem 3.4.2 (adaptive parameter estimates).

condition that shows closed-loop system stability cannot be necessarily guaranteed against significant system uncertainties in the presence of unmodeled dynamics (see Remark 3.4.1). Building on the results of this section, Section 3.4.4 proposed a new adaptive control architecture with robustifying terms and Theorem 3.4.2 showed a relaxed stability tradeoff condition that can tolerate significant system uncertainties when unmodeled dynamics satisfy a set of conditions (see Remark 3.4.2). Both numerical and experimental results demonstrated our theoretical findings on an inverted pendulum on a cart (modeled dynamics) coupled with another cart through a spring (unmodeled dynamics) in the presence of (unstructured) friction uncertainty.

3.5 An Asymptotic Decoupling Approach for Adaptive Control with Unmeasurable Coupled Dynamics

While adaptive control methods have the capability to suppress the effect of system uncertainties without excessive reliance on dynamical system models, their stability can be adversely affected in the presence of coupled dynamics. Motivated by this standpoint, the contribution of this paper is a decoupling approach for model reference adaptive control algorithms. The key feature of the proposed framework is that it guarantees asymptotic convergence between the trajectories of an uncertain dynamical system and a given reference model without relying on any measurements from the coupled dynamics under a tight sufficient

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38This section has been submitted to the *International Journal of Adaptive Control and Signal Processing* for possible publication.
stability condition. We also provide a generalization to address the uncertainty in the control effectiveness matrix, where the resulting sufficient stability condition in this case relies on linear matrix inequalities. Finally, numerical examples are provided to illustrate the efficacy of the presented theoretical results.

3.5.1 Introduction

3.5.1.1 Literature Review and Contribution

Adaptive control methods have the capability to suppress the effect of system uncertainties without excessive reliance on dynamical system models. However, their stability can be challenged by coupled dynamics. There exist a broad class of practical adaptive control applications involving coupled dynamics such as feedback control of rigid body dynamical systems with flexible appendages (i.e., airplanes with high aspect ratio wings and slung load systems) and large-scale physically interconnected dynamical systems (i.e., cooperative robot manipulator dynamical systems and actuated-unactuated dynamical systems) [63, 97–100] (also see references therein). Dating back to the seminal paper by the authors of [17], it has been well-documented in the adaptive control literature that the presence of coupled dynamics can result in system instability. Therefore, there is a practical need to synthesize adaptive control methods that provide closed-loop system stability not only in the absence of coupled dynamics as in standard adaptive control approaches but also in their presence.

Notable approaches that provide a remedy to this adaptive control problem include [1, 2, 18, 24, 25, 101] (also see references therein). In particular, the authors of [2, 24, 25] propose adaptive control approaches when coupled dynamics are presented in feedback loops. However, their drawback is that they can guarantee system stability with respect to a set of initial conditions or under a persistency of excitation assumption. In addition, the authors of [18] reveal a stability condition for adaptive control systems with coupled dynamics. While their results are promising, they only guarantee boundedness of uncertain dynamical system trajectories under a (excessively) conservative condition obtained through a Lyapunov stability analysis. The authors of [1] relax the stability condition given in [18] by developing an adaptive robustifying term. Yet, similar to [18], only boundedness of uncertain dynamical system trajectories is established in [1]. Finally, [101] investigates feedback control of coupled actuated-unactuated dynamical systems. While they can guarantee stability and asymptotic convergence between the trajectories of an uncertain dynamical system and a reference model, their approach requires measurements from the
coupled dynamics (unlike the results in [1, 18]) that is not always practically available in feedback control applications (i.e., control of high-speed airlanes).

This paper studies model reference adaptive control in the presence of coupled dynamics (Section 3.5.2). Specifically, we propose a decoupling approach that guarantees stability and asymptotic convergence between the trajectories of an uncertain dynamical system and a given reference model (Section 3.5.3). The key feature of the proposed framework is that it does not rely on any measurements from the coupled dynamics and the resulting sufficient stability condition is tight owing to the fact that it is derived directly from the coupled dynamics. In addition, we also provide a generalization to address the uncertainty in the control effectiveness matrix, where the resulting sufficient stability condition in this case relies on linear matrix inequalities (Section 3.5.4). Finally, numerical examples are provided to illustrate the efficacy of the presented theoretical results (Section 3.5.5).

3.5.1.2 Notation

In this paper, \( \mathbb{R} \) denotes the set of real numbers, \( \mathbb{R}_+ \) (respectively, \( \mathbb{R}_+ \)) denotes the set of positive (respectively, nonnegative) real numbers, \( \mathbb{R}^n \times \mathbb{R}^n \) (respectively, \( \mathbb{R}^n \times \mathbb{R}^n \)) denotes the set of \( n \times n \) positive-definite (respectively, nonnegative-definite) real matrices, \( \mathbb{R}^{n \times n} \) denotes the set of \( n \times n \) real matrices with diagonal scalar entries, \( \text{col}_i(\cdot) \) denotes the \( i \)-th column of a real matrix, and \( \triangleq \) denotes the equality by definition. We also use the rectangular projection operator from [85] and Exercise 11.3 of [7]. Mathematically, consider a convex hypercube \( \Omega = \{ \theta \in \mathbb{R}^n : (\theta_i^{\min} \leq \theta_i \leq \theta_i^{\max})_{i=1,2,\ldots,n} \} \), \( \Omega \subseteq \mathbb{R}^n \), with \( \theta_i^{\min} \) and \( \theta_i^{\max} \) respectively denoting the minimum and maximum bounds\(^{39}\) for the \( i \)-th component of the parameter vector \( \theta \in \mathbb{R}^n \). For a sufficiently small constant \( \varepsilon_0 \in \mathbb{R}_+ \), in addition, consider another convex hypercube \( \Omega_{\varepsilon_0} = \{ \theta \in \mathbb{R}^n : (\theta_i^{\min} + \varepsilon_0 \leq \theta_i \leq \theta_i^{\max} - \varepsilon_0)_{i=1,2,\ldots,n} \} \), where \( \Omega_{\varepsilon_0} \subseteq \Omega \). Then, the component-wise projection operator \( \text{Proj} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) is defined as \( \text{Proj}(\theta, y) = (\theta_i^{\max} - \theta_i)y_i/\varepsilon_0 \) if \( \theta_i > \theta_i^{\max} - \varepsilon_0 \) and \( y_i > 0 \), \( \text{Proj}(\theta, y) = (\theta_i - \theta_i^{\min})y_i/\varepsilon_0 \) if \( \theta_i < \theta_i^{\min} + \varepsilon_0 \) and \( y_i < 0 \), and \( \text{Proj}(\theta, y) = y_i \) otherwise, where \( y \in \mathbb{R}^n \). From [7] and [53], this definition gives \( (\Theta - \Theta^\star)^T(\text{Proj}(\Theta, y) - y) \leq 0 \), where \( \Theta^\star \in \Omega_{\varepsilon_0} \). Finally, one can also generalize this definition to matrices using \( \text{Proj}_m(\Theta, Y) = (\text{Proj}(\text{col}_1(\Theta), \text{col}_1(Y)), \ldots, \text{Proj}(\text{col}_m(\Theta), \text{col}_m(Y))) \), which gives \( \text{tr}[(\Theta - \Theta^\star)^T(\text{Proj}_m(\Theta, Y) - Y)] = \sum_{i=1}^m[\text{col}_i((\Theta - \Theta^\star)^T(\text{Proj}(\text{col}_i(\Theta), \text{col}_i(Y)) - \text{col}_i(Y)))] \leq 0 \) with \( n \times m \) matrices \( Y, \Theta \), and \( \Theta^\star \).

\(^{39}\)Without loss of generality, we consider each bound to be symmetric as \( \theta_i^{\min} = -\theta_i^{\max} \) for the results in this paper.
3.5.2 Problem Formulation

This section presents the problem formulation and the feedback control objective considered in this paper. Specifically, consider the uncertain dynamical system given by

\[
\dot{x}(t) = Ax(t) + B(u(t) + W_0^T \sigma_0(x(t)) + p(t)), \quad x(0) = x_0, \tag{3.168}
\]

where \(x(t) \in \mathbb{R}^n\) is a measurable state vector, \(u(t) \in \mathbb{R}^m\) is a control vector, \(A \in \mathbb{R}^{n \times n}\) and \(B \in \mathbb{R}^{n \times m}\) are matrices such that the pair \((A, B)\) is controllable, \(W_0 \in \mathbb{R}^{s \times m}\) is an unknown weight matrix, and \(\sigma_0 : \mathbb{R}^n \to \mathbb{R}^s\) is an available basis function composed of locally Lipschitz functions. In addition, \(p(t) \in \mathbb{R}^m\) in (3.168) is an unmeasurable output vector of the coupled dynamics given by

\[
\dot{q}(t) = Fq(t) + G_1u(t) + G_2x(t), \quad q(0) = q_0, \tag{3.169}
\]

\[
p(t) = Hq(t), \tag{3.170}
\]

where \(q(t) \in \mathbb{R}^p\) is an unmeasurable state vector of the coupled dynamics and \(F \in \mathbb{R}^{p \times p}, G_1 \in \mathbb{R}^{p \times m}\), \(G_2 \in \mathbb{R}^{p \times n}\), and \(H \in \mathbb{R}^{m \times p}\) are matrices with \(F\) being Hurwitz. Since \(F\) is Hurwitz, note that there exists \(S \in \mathbb{R}_+^{p \times p}\) satisfying the Lyapunov equation given by \(0 = F^TS + SF + I\).

**Remark 3.5.1** Since the state and output vectors of the coupled dynamics given by (3.169) and (3.170) are both unmeasurable, the consideration of \(F\) being Hurwitz is fairly standard in the literature (see, for example, [18] and [1]).

Next, consider the reference model given by

\[
\dot{x}_r(t) = A_rx_r(t) + B_rc(t), \quad x_r(0) = x_{r0}, \tag{3.171}
\]

which captures a desired closed-loop dynamical system behavior. In (3.171), \(x_r(t) \in \mathbb{R}^n\) is a state vector of the reference model, \(c(t) \in \mathbb{R}^m\) is a given uniformly continuous bounded command, and \(A_r \triangleq A - BK_1 \in \mathbb{R}^{n \times n}\).
and \( B_r \triangleq B K_2 \in \mathbb{R}^{n \times m} \) are matrices with \( A_r \) being Hurwitz\(^{41}\). Since \( A_r \) is Hurwitz, note that there exists \( P \in \mathbb{R}^{n \times n}_+ \) satisfying the Lyapunov equation given by \( 0 = A_r^T P + PA_r + I \).

We are now ready to state the adaptive control objective considered in this paper: Design feedback control algorithms to asymptotically drive the state vector of the uncertain dynamical system given by (3.168), which is interconnected with the coupled dynamics given by (3.169) and (3.170), to the state vector of the reference model given by (3.171). In addition, establish sufficient stability conditions to guarantee boundedness of the state vector of the coupled dynamics. To achieve this objective, Section 3.5.3 proposes an asymptotic decoupling approach. We then provide a generalization in Section 3.5.4 to address the uncertainty in the control effectiveness matrix.

**Remark 3.5.2** The authors of Section III of [18] and Section 2 of [1] also consider a similar problem formulation. Based on the problem formulation presented in this section, in particular, they present model reference adaptive control algorithms that only guarantee the boundedness of the closed-loop system trajectories when the conservative sufficient stability condition

\[
\| S G_1 \|_2 \| H \|_2 w^* l + \| S G_2 \|_2 \| H \|_2 < 1/(4 \| P B \|_2)
\]  

(3.172)

holds, where \( w^* \in \mathbb{R}_+ \) is an upper bound of their weight estimate of \( W_0 \) predicated on the projection operator and \( l = 1 + l_0 \) with \( l_0 \) resulting from their assumption \( \| \sigma_0(x(t)) \|_2 \leq l_0 \| x(t) \|_2 + l_c \), \( l_0, l_c \in \mathbb{R}_+ \). From (3.172), one can observe that the algorithms presented in Section III of [18] and Section 2 of [1] cannot tolerate large system uncertainties, which violates one of the main advantages of utilizing adaptive control laws in feedback loops. While Section 3 of [1] provides a remedy to this problem subject to a similar but relaxed conservative sufficient stability condition, it still only guarantees the boundedness of the closed-loop system trajectories. In contrast to the results in [18] and [1], this paper proposes feedback control architectures that guarantee not only boundedness but also asymptotic convergence between the state vector of the uncertain dynamical system given by (3.168) and the state vector of the reference model given by (3.171) under tight and less conservative sufficient stability conditions.

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\(^{41}\)In Sections 3.5.3 and 3.5.4, the matrices \( K_1 \in \mathbb{R}^{m \times n} \) and \( K_2 \in \mathbb{R}^{m \times m} \) are used in the nominal part of the proposed feedback control algorithms. Note that \( K_1 \) can be always selected to make \( A_r \) Hurwitz since the pair \((A, B)\) is considered to be controllable and \( K_2 \) needs to be chosen to allow a subset of the reference model state vector to be followed by the given command. We refer to, for example, Section III of [8] for details.
3.5.3 Asymptotic Decoupling Approach

This section first presents the proposed model reference adaptive control architecture predicated on a decoupling term for the problem formulation stated in Section 3.5.2 and then shows the boundedness of all closed-loop system trajectories as well as the asymptotic convergence between the state vector of the uncertain dynamical system given by (3.168) and the state vector of the reference model given by (3.171).

3.5.3.1 Proposed Architecture

To address the model reference adaptive control objective stated in Section 3.5.2, we propose the feedback control architecture given by

$$u(t) = -K_1 x(t) + K_2 c(t) - \hat{W}_0^T(t) \sigma_0(x(t)) - \hat{p}(t).$$

(3.173)

In (3.173), the term “$-K_1 x(t) + K_2 c(t)$” denotes the nominal control algorithm with the matrices $K_1 \in \mathbb{R}^{m \times n}$ and $K_2 \in \mathbb{R}^{m \times m}$. In addition, the term “$-\hat{W}_0^T(t) \sigma_0(x(t))$” denotes the adaptive control algorithm, where $\hat{W}_0(t) \in \mathbb{R}^{n \times m}$ is an estimate of the unknown weight $W_0$ satisfying the projection operator-based weight update law given by

$$\dot{\hat{W}}_0(t) = \gamma_1 \text{Proj}_m [\hat{W}_0(t), \sigma_0(x(t)) e^T(t) P B], \quad \hat{W}_0(0) = \hat{W}_{00},$$

(3.174)

with $\gamma_1 \in \mathbb{R}_+$ being the learning rate, $e(t) \triangleq x(t) - x_r(t)$ being the system error between the dynamical system given by (3.168) and the reference model given by (3.171), and $P$ being the solution to the Lyapunov equation defined in Section 3.5.2. Furthermore, the term “$-\hat{p}(t)$” denotes the decoupling mechanism, where $\hat{p}(t) \in \mathbb{R}^m$ is the estimate of the unmeasurable output vector of the coupled dynamics satisfying the observer dynamics\(^{42}\)

$$\dot{\hat{q}}(t) = F \hat{q}(t) + G_1 u(t) + G_2 x(t) + \frac{1}{\alpha} S^{-1} H^T B^T P e(t), \quad \hat{q}(0) = \hat{q}_0,$$

(3.175)

$$\hat{p}(t) = H \hat{q}(t),$$

(3.176)

\(^{42}\)While similar observer dynamics are used in [102] and [103], the considered model reference adaptive control objective is entirely different than the ones in these references. In other words, this paper focuses on the coupled dynamics problem, whereas this is not considered in both [102] and [103].
with \( \hat{q}(t) \in \mathbb{R}^p \) being the estimate of the unmeasurable state vector of the coupled dynamics, \( \alpha \in \mathbb{R}_+ \) being a design parameter, and \( S \) being the solution to the Lyapunov equation defined in Section 3.5.2.

Considering the feedback control architecture given by (3.173), one can rewrite the uncertain dynamical system in (3.168) as

\[
\dot{x}(t) = A_r x(t) + B_r c(t) - B \hat{W}_0^T(t) \sigma_0(x(t)) + B \hat{p}(t),
\]

where \( \hat{W}_0(t) \triangleq \hat{W}_0(t) - W_0 \in \mathbb{R}^{n \times m} \) denotes the weight estimation error and \( \hat{p}(t) \triangleq p(t) - \hat{p}(t) = Hq(t) - H \hat{q}(t) \in \mathbb{R}^m \) denotes the unmeasurable output vector estimation error with \( \hat{q}(t) \triangleq q(t) - \hat{q}(t) \in \mathbb{R}^p \) being the unmeasurable state vector estimation error. Now, from (3.168) and (3.171) and from (3.169) and (3.175), respectively, one can write the system error dynamics and the unmeasurable state vector estimation error dynamics as

\[
\dot{e}(t) = A_r e(t) - B \hat{W}_0^T(t) \sigma_0(x(t)) + B H \hat{q}(t), \quad e(0) = e_0,
\]
\[
\dot{\hat{q}}(t) = F \hat{q}(t) - \frac{1}{\alpha} S^{-1} H^T B^T Pe(t), \quad \hat{q}(0) = \hat{q}_0.
\]

The error dynamics given by (3.178) and (3.179) are utilized in the next section for stability and convergence analysis.

### 3.5.3.2 Stability and Convergence Analysis

To begin with, we show the Lyapunov stability of the error dynamics given by (3.178) and (3.179) as well as the convergence of the pair \((e(t), \hat{q}(t))\) to \((0,0)\).

**Lemma 3.5.1** Consider the uncertain dynamical system given by (3.168), the coupled dynamics given by (3.169) and (3.170), and the reference model given by (3.171). The feedback control law given by (3.173), (3.174), (3.175), and (3.176) then guarantees the Lyapunov stability of the solution \((e(t), \hat{q}(t), \hat{W}_0(t))\) along with \( \lim_{t \to \infty} (e(t), \hat{q}(t)) = (0,0) \).

**Proof.** We first show the Lyapunov stability of the solution \((e(t), \hat{q}(t), \hat{W}_0(t))\). Specifically, consider the Lyapunov function candidate \( V(e, \hat{q}, \hat{W}_0) = e^T P e + \alpha \hat{q}^T S \hat{q} + \gamma T \text{tr} \hat{W}_0^T \hat{W}_0 \), where \( P \) and \( S \) satisfy the Lyapunov equations defined in Section 3.5.2. Note that \( V(0,0,0) = 0 \) and \( V(e, \hat{q}, \hat{W}_0) > 0 \) for all \((e, \hat{q}, \hat{W}_0) \neq (0,0,0)\). The time derivative of this Lyapunov function candidate satisfies
\[ \dot{V}(\cdot) = 2e^T(t)P\left[A_r e(t) - B\tilde{W}_0^T(t)\sigma_0(x(t)) + BH\tilde{q}(t)\right] + 2\alpha q^T(t)S\left[F\tilde{q}(t) - \frac{1}{\alpha}S^{-1}H^TB^T Pe(t)\right] + 2\text{tr}\tilde{W}_0^T(t)\left[\text{Proj}_m\left[\tilde{W}_0(t), \sigma_0(x(t))e^T(t)PB\right]\right] \leq -e^T(t)e(t) - \alpha q^T(t)\tilde{q}(t) \leq 0. \] (3.180)

Hence, (3.180) guarantees the Lyapunov stability of the solution \((e(t), \tilde{q}(t), \tilde{W}_0(t))\).

We next apply the Barbalat’s lemma (see, for example, [7]) to show \(\lim_{t \to \infty}(e(t), \tilde{q}(t)) = (0, 0)\). To this end, note that the Lyapunov stability of the solution \((e(t), \tilde{q}(t), \tilde{W}_0(t))\) implies the boundedness of the triple \((e(t), \tilde{q}(t), \tilde{W}_0(t))\) for all \(t \in \mathbb{R}_+\). Note also that the boundedness of the system error \(e(t)\) along with the boundedness of the reference model state vector \(x_r(t)\) implies the boundedness of the uncertain system state vector \(x(t)\) for all \(t \in \mathbb{R}_+\). Now, consider

\[ \dot{V}(\cdot) \leq -2e^T(t)\left[A_r e(t) - B\tilde{W}_0^T(t)\sigma_0(x(t)) + BH\tilde{q}(t)\right] - 2\tilde{q}^T(t)\left[F\tilde{q}(t) - \frac{1}{\alpha}S^{-1}H^TB^T Pe(t)\right], \] (3.181)

where all the terms on the right hand side of (3.181) are bounded for all \(t \in \mathbb{R}_+\). Therefore, it now follows from the Barbalat’s lemma that \(\lim_{t \to \infty}(e(t), \tilde{q}(t)) = (0, 0)\).

**Remark 3.5.3** While Lemma 3.5.1 establishes the Lyapunov stability of the solution \((e(t), \tilde{q}(t), \tilde{W}_0(t))\) along with \(\lim_{t \to \infty}(e(t), \tilde{q}(t)) = (0, 0)\), it does not guarantee the boundedness of neither \(q(t)\) nor \(\tilde{q}(t)\) for all \(t \in \mathbb{R}_+\). This in turn implies that the proposed feedback control architecture given by (3.173) can be unbounded due to the presence of the decoupling term “\(-\hat{p}(t) = -H\tilde{q}(t)\)” inside this architecture.

Based on Lemma 3.5.1, the next theorem addresses the issue highlighted in Remark 3.5.3, which presents our first main result.

**Theorem 3.5.1** Consider the uncertain dynamical system given by (3.168), the coupled dynamics given by (3.169) and (3.170), and the reference model given by (3.171). In addition, consider that \(\mathcal{F}_1 \triangleq F - G_1H\) is Hurwitz. The feedback control law given by (3.173), (3.174), (3.175), and (3.176) then guarantees the Lyapunov stability of the solution \((e(t), \tilde{q}(t), \tilde{W}_0(t))\), \(\lim_{t \to \infty}(e(t), \tilde{q}(t)) = (0, 0)\), and the boundedness of all closed-loop system signals including the proposed feedback control architecture given by (3.173) for all \(t \in \mathbb{R}_+\).

**Proof.** From Lemma 3.5.1, the Lyapunov stability of the solution \((e(t), \tilde{q}(t), \tilde{W}_0(t))\), \(\lim_{t \to \infty}(e(t), \tilde{q}(t)) = (0, 0)\), and the boundedness of the uncertain dynamical system state vector \(x(t)\) for all \(t \in \mathbb{R}_+\) is
immediate. Now, in order to show the boundedness of the remaining closed-loop signals, one can rewrite (3.175) using (3.173) as

$$
\dot{\hat{q}}(t) = F_1 \hat{q}(t) + \Phi(\cdot),
$$

(3.182)

$$
\Phi(\cdot) = G_1 [-K_1(x_r(t) + e(t)) + K_2 c(t)] - G_1 [\hat{W}_0^T(t) \sigma_0(x_r(t) + e(t))]
+ G_2(x_r(t) + e(t)) + \frac{1}{\alpha} S^{-1} H^T B^T P e(t).
$$

(3.183)

Note that all the terms on the right hand side of (3.183) are bounded for all $t \in \mathbb{R}_+$. Hence, since $F_1$ is Hurwitz in (3.182), it follows from the input-to-state stability Chapter 4.9 of [57] that $\hat{q}(t)$ is bounded for all $t \in \mathbb{R}_+$. This in turn implies the boundedness of $u(t)$ for all $t \in \mathbb{R}_+$, which addresses the issue highlighted in Remark 3.5.3. Finally, since $\tilde{q}(t)$ and $\hat{q}(t)$ are both bounded for all $t \in \mathbb{R}_+$, then $q(t)$ is bounded for all $t \in \mathbb{R}_+$.

The proof is now complete.

Remark 3.5.4 If the coupled dynamics given by (3.169) does not depend on the control vector $u(t)$ (i.e., $G_1 = 0$), then $F_1$ stated in Theorem 3.5.1 becomes Hurwitz automatically owing to the fact that $F$ is considered to be Hurwitz (see also Remark 3.5.1).

Remark 3.5.5 Following the discussion in Remark 3.5.2, the closely related papers to our results; that is, [18] and [1], only guarantee the boundedness of the closed-loop system trajectories subject to conservative and relaxed conservative sufficient stability conditions, respectively. In this section, however, we also guarantee the asymptotic convergence between the state vector of the uncertain dynamical system and the state vector of the reference model under the sufficient stability condition stated in Theorem 3.5.1 (i.e., $F_1$ being Hurwitz), which is tight since it is derived directly from the coupled dynamics as shown in (3.182) and (3.183). We also refer to Section 3.5.5 on this point.

Remark 3.5.6 The results presented in this section also hold without utilizing the projection operator in the weight update law given by (3.174). Considering real-world implementations, however, the projection operator is needed in adaptive control applications to ensure boundedness, for example, when the unknown weight matrix in (3.168) varies with respect to time.

Theorem 3.5.1 addresses the model reference adaptive control objective stated in Section 3.5.2. Building on the results of this section, the next section provides a generalization to address the uncertainty in the control effectiveness matrix.
3.5.4 Generalization to Address the Uncertainty in the Control Effectiveness Matrix

We now generalize the results in Section 3.5.3 to address the uncertainty in the control effectiveness matrix; that is, when \( u(t) \) in (3.168) is replaced by \( \Lambda u(t) \) with \( \Lambda \in \mathbb{R}_{+}^{m \times m} \cap \mathbb{D}^{m \times m} \) being an unknown control effectiveness matrix (see, for example, [7] and [8]). Mathematically, consider the uncertain dynamical system given by

\[
\dot{x}(t) = Ax(t) + B(\Lambda u(t) + W_0^T \sigma_0(x(t)) + p(t)), \quad x(0) = x_0,
\]

(3.184)

which is interconnected with the coupled dynamics in (3.169) and (3.170). Here, we consider the parameterization given by \( \Lambda = I + \delta\Lambda \) with \( \delta\Lambda \in \mathbb{R}_{+}^{m \times m} \cap \mathbb{D}^{m \times m} \) being an unknown matrix subject to the assumption \( \delta\Lambda > -I \), where this is fairly well-adopted in the model reference adaptive control literature (see, for example, [35, 104–106]).

3.5.4.1 Proposed Architecture

To address the model reference adaptive control objective stated in Section 3.5.2 in the presence of uncertainty in the control effectiveness matrix, we propose the feedback control architecture given by

\[
u(t) = (I + \hat{\delta}\Lambda(t))^{-1}[-K_1 x(t) + K_2 c(t) - \hat{W}_0^T \sigma_0(x(t)) - \hat{p}(t)].
\]

(3.185)

As compared to (3.173), the term \((I + \hat{\delta}\Lambda(t))^{-1}\) is utilized in (3.185) with \( \hat{\delta}\Lambda(t) \in \mathbb{R}_{+}^{m \times m} \cap \mathbb{D}^{m \times m} \) (i.e., \( \hat{\delta}\Lambda(t) = \text{diag}([\hat{\delta}_{\lambda 1}(t), \hat{\delta}_{\lambda 2}(t), \ldots, \hat{\delta}_{\lambda m}(t)]) \)) by definition being an estimate of the unknown part of the control effectiveness matrix satisfying the projection operator-based weight update law\(^{43}\)

\[
\hat{\delta}_{\lambda i}(t) = \gamma_{\lambda i} \text{Proj} \left[ \hat{\delta}_{\lambda i}(t), u(t)e^T(t)PB_i \right], \quad \hat{\delta}_{\lambda i}(0) = \hat{\delta}_{\lambda i 0}, \quad i = 1, 2, \ldots, m.
\]

(3.186)

In (3.186), \( \gamma_{\lambda i} \in \mathbb{R}_{+}, i = 1, 2, \ldots, m \), is the learning rate and \( B_i, i = 1, 2, \ldots, m \), is a standard basis.

\(^{43}\)For the well-posedness of the inversion of the term \((I + \hat{\delta}\Lambda(t))^{-1}\) we consider that the elemental projection bounds are defined such that \(-1 < \hat{\delta}_{\lambda,\min} \leq \hat{\delta}_{\lambda i}(t) \leq \hat{\delta}_{\lambda,\max}, i = 1, 2, \ldots, m.\)
Next, based on the feedback control architecture given by (3.185), one can write the uncertain dynamical system in (3.184) as

\[
\dot{x}(t) = A_r x(t) + B_r c(t) - B \tilde{W}_0^T(t) \sigma_0(x(t)) - B \tilde{\delta}_\Lambda(t) u(t) + B \bar{p}(t),
\]

(3.187)

where \( \tilde{\delta}_\Lambda(t) \triangleq \delta_\Lambda - \hat{\delta}_\Lambda(t) \) is estimation error of unknown part of control effectiveness matrix. Owing to the diagonal structure of \( \tilde{\delta}_\Lambda(t) \), (3.187) can be further written as

\[
\dot{x}(t) = A_r x(t) + B_r c(t) - B \tilde{W}_0^T(t) \sigma_0(x(t)) - B \sum_{i=1}^m b_i \tilde{\delta}_{\lambda_i}(t) u(t) + B \bar{p}(t).
\]

(3.188)

From (3.188) and (3.171) and from (3.169) and (3.175), respectively, one can now write the system error dynamics and the unmeasurable state vector estimation error dynamics as

\[
\dot{e}(t) = A_r e(t) - B \tilde{W}_0^T(t) \sigma_0(x(t)) - B \sum_{i=1}^m b_i \tilde{\delta}_{\lambda_i}(t) u(t) + B H \bar{q}(t),
\]

(3.189)

\[
\dot{\tilde{q}}(t) = F \tilde{q}(t) - \frac{1}{\alpha} S^{-1} H^T B^T P e(t),
\]

\( \tilde{q}(0) = \tilde{q}_0. \)

(3.190)

The error dynamics given by (3.189) and (3.190) are utilized in the next section for stability and convergence analysis.

### 3.5.4.2 Stability and Convergence Analysis

To begin with, we show the Lyapunov stability of the error dynamics given by (3.189) and (3.190).

**Lemma 3.5.2** Consider the uncertain dynamical system given by (3.184), the coupled dynamics given by (3.169) and (3.170), and the reference model given by (3.171). The feedback control law given by (3.185), (3.174), (3.175), (3.176), and (3.186) then guarantees the Lyapunov stability of the solution \((e(t), \tilde{q}(t), \tilde{W}_0(t), \tilde{\delta}_{\lambda_i}(t)).\)

**Proof.** To show the Lyapunov stability of the solution \((e(t), \tilde{q}(t), \tilde{W}_0(t), \tilde{\delta}_{\lambda_i}(t)).\), consider the Lyapunov function candidate \(V(e, \tilde{q}, \tilde{W}_0, \tilde{\delta}_{\lambda_i}) = e^T P e + \alpha \tilde{q}^T S \tilde{q} + \gamma^{-1} \text{tr} \tilde{W}_0^T \tilde{W}_0 + \sum_{i=1}^m \gamma_{2i}^{-1} \tilde{\delta}_{\lambda_i}^2,\) where \(P\) and \(S\) satisfy the Lyapunov equations defined in Section 3.5.2. Note that \(V(0, 0, 0, 0) = 0\) and \(V(e, \tilde{q}, \tilde{W}_0, \tilde{\delta}_{\lambda_i}) > 0\) for all
(e, \bar{q}, \bar{W}_0, \bar{\delta}_\lambda) \neq (0, 0, 0, 0). The time derivative of this Lyapunov function candidate satisfies

\[ \dot{V}(\cdot) = 2e^T(t)P\left[A_we(t) - B\bar{W}_0^T(t)\sigma_0(x(t)) + BH\bar{q}(t)\right] - 2e^T(t)PB\sum_{i=1}^m b_i\bar{\delta}_\lambda(t)u(t) \\
+ 2\alpha\bar{q}^T(t)S\left[F\bar{q}(t) - \frac{1}{\alpha}S^{-1}H^TB^TPe(t)\right] + 2tr\bar{W}_0^T(t)\left[\text{Proj}_m[\bar{W}_0(t), \sigma_0(x(t))e^T(t)PB]\right] \\
+ 2\sum_{i=1}^m \bar{\delta}_\lambda(t)\left[\text{Proj}\left[\bar{\delta}_\lambda(t), u(t)e^T(t)PBb_i\right]\right] \\
\leq -e^T(t)e(t) - \alpha\bar{q}^T(t)\bar{q}(t) = 0. \quad (3.191) \]

Hence, (3.191) guarantees the Lyapunov stability of the solution \((e(t), \bar{q}(t), \bar{W}_0(t), \bar{\delta}_\lambda(t))\).

**Remark 3.5.7** Unlike Lemma 3.5.1, Lemma 3.5.2 does not imply \(\lim_{t \to \infty}(e(t), \bar{q}(t)) = (0, 0)\) from Barbalat’s lemma. To elucidate this point, consider

\[ \dot{V}(\cdot) \leq -2e^T(t)\left[A_we(t) - B\bar{W}_0^T(t)\sigma_0(x(t)) + BH\bar{q}(t)\right] - 2e^T(t)B\sum_{i=1}^m b_i\bar{\delta}_\lambda(t)u(t) \\
- 2\bar{q}^T(t)\left[F\bar{q}(t) - \frac{1}{\alpha}S^{-1}H^TB^TPe(t)\right]. \quad (3.192) \]

The Lyapunov stability of \((e(t), \bar{q}(t), \bar{W}_0(t), \bar{\delta}_\lambda(t))\) implies the quadruple \((e(t), \bar{q}(t), \bar{W}_0(t), \bar{\delta}_\lambda(t))\) to be bounded for all \(t \in \mathbb{R}_+\). In addition, the boundedness of the system error \(e(t)\) along with the boundedness of the reference model state vector \(x_0(t)\) implies the boundedness of the uncertain dynamical system state vector \(x(t)\) for all \(t \in \mathbb{R}_+\). Thus, all terms on the right hand side of (3.192) are bounded except \(u(t)\) in (3.185) since it depends on the decoupling term \(-\bar{p}(t) = -H\bar{q}(t)\), where the boundedness of \(\bar{q}(t)\) does not follow from Lemma 3.5.2.

To address the problem stated in Remark 3.5.7, we now give the following lemma that guarantees the boundedness of all closed-loop system signals including the proposed feedback control architecture given by (3.185) for all \(t \in \mathbb{R}_+\).

**Lemma 3.5.3** Under the conditions stated in Lemma 3.5.2, all closed-loop system signals including the proposed feedback control architecture given by (3.185) are bounded for all \(t \in \mathbb{R}_+\) when \(F_2(\hat{\delta}_\lambda(t)) = (F - G_1M(\hat{\delta}_\lambda(t))H + \frac{1}{\epsilon}I)\) is quadratically stable with \(\epsilon \in \mathbb{R}_+\) being a design parameter. Here \(M(\hat{\delta}_\lambda(t)) \triangleq (I + \hat{\delta}_\lambda(t))^{-1} \in \mathbb{R}_+^{n \times m} \cap \mathbb{D}^{m \times m} \).
Proof. From Lemma 3.5.2 and Remark 3.5.7, the boundedness of the quadruple \((e(t), \bar{q}(t), \bar{W}_0(t), \bar{\delta}_{\lambda i}(t))\) and the boundedness of \(x(t)\) are guaranteed for all \(t \in \mathbb{R}_+\). Now, in order to show the boundedness of the remaining closed-loop signals, one can rewrite (3.175) using (3.185) as

\[
\dot{q}(t) = \mathcal{F}_2(\hat{\delta}_\lambda(t))\dot{q}(t) + \Phi(\cdot), \\
\Phi(\cdot) = G_1\mathcal{M}(\hat{\delta}_\lambda(t))[-K_1(x_i(t) + e(t)) + K_2c(t)] - G_1\mathcal{M}(\hat{\delta}_\lambda(t))[\bar{W}_0^T(t)\sigma_0(x_i(t) + e(t))] \\
+ G_2(x_i(t) + e(t)) + \frac{1}{\alpha}S^{-1}H^TB^TPe(t).
\]  

(3.193)

(3.194)

Note that \(\mathcal{M}(\hat{\delta}_\lambda(t))\) is well defined and all the terms on the right hand side of (3.194) are bounded for all \(t \in \mathbb{R}_+\). With \(\mathcal{F}_2(\hat{\delta}_\lambda(t))\) being quadratically stable in (3.193), in addition, it follows from the input-to-state stability Chapter 4.9 of [57] that \(\dot{q}(t)\) is bounded for all \(t \in \mathbb{R}_+\). This in turn implies the boundedness of \(u(t)\) given by (3.185) for all \(t \in \mathbb{R}_+\). Finally, since \(\ddot{q}(t)\) and \(\ddot{q}(t)\) are bounded for all \(t \in \mathbb{R}_+\), then \(q(t)\) is bounded for all \(t \in \mathbb{R}_+\). The proof is now complete. \(\blacksquare\)

Remark 3.5.8 By definition, \(\mathcal{F}_2(\hat{\delta}_\lambda(t))\) being quadratically stable means the existence of \(\mathcal{P} \in \mathbb{R}^{p \times p}_+\) satisfying \(\mathcal{F}_2^T(\hat{\delta}_\lambda(t))\mathcal{P} + \mathcal{P}\mathcal{F}_2(\hat{\delta}_\lambda(t)) < 0\) [107]. Since \(\mathcal{F}_2(\hat{\delta}_\lambda(t))\) is affine with respect to \(\mathcal{M}(\hat{\delta}_\lambda(t))\), one can use linear matrix inequalities to check the quadratic stability of \(\mathcal{F}_2(\hat{\delta}_\lambda(t))\) for given lower and upper bounds \(\mathcal{M}_{\text{min},i} \triangleq (I + \hat{\delta}_{\lambda \text{max},i})^{-1}\) and \(\mathcal{M}_{\text{max},i} \triangleq (I + \hat{\delta}_{\lambda \text{min},i})^{-1}\), \(i = 1, \ldots, m\), on the each diagonal element of \(\mathcal{M}(\hat{\delta}_\lambda(t))\). Mathematically, let \(\mathcal{M}_{i_1,\ldots,i_l} = \text{diag}(i_1\mathcal{M}_{\text{max},1} + (1 - i_1)\mathcal{M}_{\text{min},1}, \ldots, i_m\mathcal{M}_{\text{max},m} + (1 - i_m)\mathcal{M}_{\text{min},m})\) be the corners of the hypercube defining the variation of \(\mathcal{M}(t)\), where \(i_l \in \{0,1\}\) and \(l = 1,\ldots,2^m\). Then, if \(\mathcal{F}_2^T_{i_1,\ldots,i_l}\mathcal{P} + \mathcal{P}\mathcal{F}_2_{i_1,\ldots,i_l} < 0\), where \(\mathcal{F}_2_{i_1,\ldots,i_l} = [F - G_1\mathcal{M}_{i_1,\ldots,i_l}H + \frac{\xi}{\alpha}]\), then the quadratic stability of \(\mathcal{F}_2(\hat{\delta}_\lambda(t))\) is immediate.

Based on the quadratic stability condition given in Lemma 3.5.3, one can now address the problem stated in Remark 3.5.7 to conclude \(\lim_{t \to \infty}(e(t), \bar{q}(t)) = (0,0)\). Therefore, we are now ready to state the following theorem that presents the second main result of this paper.

Theorem 3.5.2 Consider the uncertain dynamical system given by (3.184), the coupled dynamics given by (3.169) and (3.170), and the reference model given by (3.171). In addition, consider that \(\mathcal{F}_2(\hat{\delta}_\lambda(t))\) is quadratically stable. The feedback control law given by (3.185), (3.174), (3.175), and (3.176) then guarantees the Lyapunov stability of the solution \((e(t), \bar{q}(t), \bar{W}_0(t), \bar{\delta}_{\lambda i}(t)), \lim_{t \to \infty}(e(t), \bar{q}(t)) = (0,0)\), and
the boundedness of all closed-loop system signals including the proposed feedback control architecture
given by (3.185) for all \( t \in \mathbb{R}_+ \).

Proof. From Lemmas 3.5.2 and 3.5.3, all closed-loop system signals including the proposed feed-
back control architecture given by (3.185) are bounded for all \( t \in \mathbb{R}_+ \). This in turn implies the boundedness
of the terms on the right hand side of (3.192) for all \( t \in \mathbb{R}_+ \). Thus, Barbalat’s lemma now implies that
\( \lim_{t \to \infty} (e(t), \tilde{q}(t)) = (0, 0) \). The proof is now complete. ■

Remark 3.5.9 Following the discussion in Remark 3.5.4, the quadratic stability of \( \mathcal{F}_2(\hat{\delta}_A(t)) \) reduces to \( F \)
being Hurwitz when the coupled dynamics given by (3.169) does not depend on the control vector \( u(t) \) (i.e.,
\( G_1 = 0 \)). Following the discussion in Remark 3.5.5, in addition, we guarantee the asymptotic convergence
between the state vector of the uncertain dynamical system and the state vector of the reference model
under the sufficient stability condition stated in Lemma 3.5.3 (i.e., \( \mathcal{F}_2(\hat{\delta}_A(t)) \) being quadratically stable),
which is derived directly from the coupled dynamics as shown in (3.193) and (3.194). Finally, similar to the
discussion given in Remark 3.5.6, the results of this section also hold without the projection operator-based
weight update law for \( \hat{W}_0(t) \) in (3.174); however, the projection operator is still required for the weight
update law for \( \hat{\delta}_{\lambda}(t) \) in (3.186) to ensure well-posedness of the proposed feedback control architecture
given by (3.185).

In the presence of uncertainty in the control effectiveness matrix, Theorem 3.5.2 addresses the
model reference adaptive control objective stated in Section 3.5.2. The next section now provides numerical
examples to illustrate the efficacy of the results documented in Theorems 3.5.1 and 3.5.2.

3.5.5 Illustrative Numerical Examples

This section presents two numerical examples to respectively illustrate the results in Theorems
3.5.1 and 3.5.2. For the proposed feedback control architecture in Theorem 3.5.1, in particular, consider the uncertain dynamical system given by

\[
\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} \frac{1}{2} \\ -1 \end{bmatrix} W_0^T x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} \gamma \gamma \end{bmatrix} \begin{bmatrix} 0 & -0.0313 & 0.4063 & -1.25 \end{bmatrix} q(t),
\]

(3.195)
which is coupled with the nonminimum phase dynamics

\[
\dot{q}(t) = \begin{bmatrix}
-1 & -3.8125 & -0.9375 & -3.125 \\
4 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 \\
0 & 0 & 2 & 0 \\
\end{bmatrix} F q(t) + \begin{bmatrix}
1 \\
0 \\
0 \\
0 \\
\end{bmatrix} u(t) + \begin{bmatrix}
4 \\
0 \\
0 \\
0 \\
\end{bmatrix} x(t).
\]

(3.196)

Here, \( \gamma_G \) appearing in matrix \( H \) is the coupling strength, and \( x(0) = [0.1, -0.2]^T \) and \( q(0) = [-0.2, 0, -0.1, 0.1]^T \). In addition, we consider the reference model subject to zero initial conditions given by (3.171) with

\[
A_r = \begin{bmatrix}
0 & 1 \\
-w_n^2 & -2w_n r_n \\
\end{bmatrix}, \quad B_r = \begin{bmatrix}
0 \\
w_n^2 \\
\end{bmatrix},
\]

(3.197)

where its natural frequency \( w_n \) and damping ratio \( r_n \) are respectively chosen as 0.3 rad/sec and 0.95. This reference model selection yields \( K_1 = [1.09, 1.57] \) and \( K_2 = 0.09 \). Furthermore, we consider the weight update law subject to zero initial conditions given by (3.174) with \( \gamma_1 = 1 \) and \( \hat{W}_{\text{max}} = 2.1 \) being the bound for the projection operator. Finally, we consider the observer dynamics subject to \( \dot{q}(0) = [0.15, -0.1, 0.2, -0.25]^T \) given by (3.175) with \( \alpha = 0.5 \).

Figures 3.29 and 3.30 show the closed-loop dynamical system performance with the proposed model reference adaptive control framework in Theorem 3.5.1. While both figures demonstrate \( \lim_{t \to \infty}(e(t), \hat{q}(t)) = (0, 0) \) as stated in this theorem, the fundamental stability condition \( F_1 \) is Hurwitz for the coupling strength \( \gamma_{G_c} = 2.0 \) in the former figure and it is not Hurwitz for the coupling strength \( \gamma_{G_c} = 2.2 \) in the latter figure. This is the reason why the proposed feedback control signal as well as the coupled dynamics state remains bounded in the former figure, which validates the results in Theorem 3.5.1.

**Remark 3.5.10** One can obtain the standard model reference adaptive control architecture in Section III of [18] and Section 2 of [1] when the proposed decoupling mechanism “\( -\hat{p}(t) \)” is removed from the feedback control signal given by (3.173). In this case, however, all closed-loop system signals become unbounded for the considered example (figure is not included).

We next demonstrate the proposed feedback control architecture in Theorem 3.5.2 for the uncertainty in the control effectiveness matrix. To this end, we use the same setup and parameter selections given
above with \( u(t) \) in (3.195) being replaced by \( \Lambda u(t) \), where \( \Lambda = 1 + \delta \Lambda \). Here, we consider the uncertain term \( \delta \Lambda \) to be 0.2. We also consider the weight update law subject to zero initial conditions given by (3.186) with \( \gamma_2 = 1 \) and \( \hat{\delta}_{\Lambda_{\text{max}}} = 0.25 \) being the projection operator bound.

Figures 3.31 and 3.32 show the closed-loop dynamical system performance with the proposed model reference adaptive control framework in Theorem 3.5.2. Note that the fundamental stability condition \( \mathcal{F}_2(\hat{\delta}_\Lambda(t)) \) becomes quadratically stable with \( \varepsilon = 0.001 \) for coupling strengths \( \gamma_{G_c} \leq 0.9 \). This is the reason why the former figure with the coupling strength \( \gamma_{G_c} = 0.9 \) validates the results in Theorem 3.5.2 and the latter figure with the coupling strength \( \gamma_{G_c} = 2.5 \) results in unbounded control signal as well as unbounded coupled dynamics state.

**Remark 3.5.11** Similar to the discussion in Remark 3.5.10, all closed-loop system signals become unbounded for this example when the proposed decoupling mechanism \( -\hat{\rho}(t) \) is removed from the feedback control signal given by (3.185) (figure is not included). Note also that the fundamental stability condition \( \mathcal{F}_2(\hat{\delta}_\Lambda(t)) \) becomes quadratically stable with \( \varepsilon = 0.001 \) for coupling strengths \( \gamma_{G_c} \leq 0.9 \) as noted here; however, we numerically observe that coupling strengths \( \gamma_{G_c} \geq 2.5 \) yield to unbounded control signal and coupled dynamics state; see Figure 3.32 for \( \gamma_{G_c} = 2.5 \). To this end, we have two observations. First, Figure 3.32 is obtained for specific learning rates, initial conditions, and command profile, and no theoretical stability guarantees exist for coupling strengths \( \gamma_{G_c} > 0.9 \). Second, one may have conservatism here due to the utilization of linear matrix inequalities in checking quadratic stability of this fundamental stability condition (see, for example, [35, 102]). Yet, this condition is directly derived from the coupled dynamics (not from a Lyapunov analysis that can involve more conservatism) and it is still tight enough from this standpoint.

3.5.6 Conclusion

It is well-known that adaptive control of uncertain dynamical systems in the presence of coupled dynamics can result in system instability. To address this problem, we presented a decoupling approach. Without relying on any measurements from the coupled dynamics and under a tight sufficient stability condition, in particular, the proposed approach allows not only stability but also asymptotic convergence between the trajectories of an uncertain dynamical system and a given reference model (Theorem 3.5.1). We then generalized our framework to address the uncertainty in the control effectiveness matrix, where
the resulting sufficient stability condition in this case relies on linear matrix inequalities (Theorem 3.5.2).

Finally, two numerical examples were given that demonstrate the efficacy of our theoretical findings.
Figure 3.31: Closed-loop dynamical system performance for the feedback control architecture in Theorem
3.5.2 with the coupling strength $\gamma_{G_c} = 0.9$.

Figure 3.32: Closed-loop dynamical system performance for the feedback control architecture in Theorem
3.5.2 with the coupling strength $\gamma_{G_c} = 2.5$. 
Chapter 4: Adaptive Architectures for Control of Uncertain Dynamical Systems with Actuator and Unmodeled Dynamics

In adaptive control of uncertain dynamical systems, it is well-known that the presence of actuator and/or unmodeled dynamics in feedback loops can yield to unstable closed-loop system trajectories. Motivated by this standpoint, this paper focuses on the analysis and synthesis of multiple adaptive architectures for control of uncertain dynamical systems with both actuator and unmodeled dynamics. Specifically, we first analyze model reference adaptive control architectures with standard, hedging-based, and expanded reference models for this class of uncertain dynamical systems and develop sufficient stability conditions. We then synthesize a robustifying term for the latter architecture and analytically show that this term can allow for a relaxed sufficient stability condition. The proposed theoretical treatments involve Lyapunov stability theory, linear matrix inequalities, and matrix mathematics. Finally, we compare the resulting sufficient stability conditions of the considered adaptive control architectures on a benchmark mechanical system subject to actuator and unmodeled dynamics.

4.1 Introduction

4.1.1 Literature Review

In adaptive control of uncertain dynamical systems, it is well-known that the presence of actuator and/or unmodeled dynamics in feedback loops can yield to unstable closed-loop system trajectories. In particular, most studies in the existing adaptive control literature focus on either actuator or unmodeled dynamics. To begin with, the authors of [30, 31] present methods that involve analysis of controlled uncertain dynamical systems with actuator dynamics. The authors of [32–34] propose a hedging method for model reference adaptive control architectures such that this method modifies the trajectories of the ideal reference model to allow for correct adaptation that is not affected by the presence of actuator dynamics. Building on the results in [32–34], the authors of [35, 36] develop sufficient stability conditions that guarantee the...
stability of the hedged, modified reference model trajectories, and therefore, the stability of the closed-loop system trajectories. The authors of [37] further extend the results in [35, 36]. Specifically, they propose an approach based on an extended reference model to the adaptive control literature that can achieve a better closed-loop system performance in the presence of actuator dynamics when it is compared with the hedging method.

For adaptive control of uncertain dynamical systems with unmodeled dynamics, the authors of the seminal paper [17] show the possibility of closed-loop system instability. Motivated by this drawback, the authors of [24] present an approach relying on a persistency of excitation assumption to achieve closed-loop system stability in the presence of unmodeled dynamics. In addition, the authors of [2] use weight update laws involving leakage terms and the authors of [18, 25, 28] utilize the projection operator in the weight update laws to maintain system stability in the face of unmodeled dynamics. From the results documented in [18, 28], one can conclude that the closed-loop stability can be guaranteed either if there does not exist significant unmodeled dynamics or the effect of system uncertainties is negligible. Building on the results of [18, 28], the authors of [1] also utilize the projection operator in order to develop sufficient stability conditions for model reference adaptive control architectures when unmodeled dynamics present. Different from the results documented in [18, 28], however, [1] synthesizes a robustifying term to relax these conditions such that the controlled dynamical system is guaranteed to remain stable in the presence of large system uncertainties when the unmodeled dynamics satisfy a set of conditions.

While the authors of [30–37] study actuator dynamics problem and the authors of [1, 2, 18, 24, 25, 28] study unmodeled dynamics problem for model reference adaptive control architectures, the effects of both dynamics are strictly present in feedback loops for a wide array of applications. To elucidate this point, for example, consider the flight control problem of flexible aerospace vehicles (e.g., see [38–40]). For this problem, it is a challenge to control especially the rigid body longitudinal dynamics of the vehicle in the presence of wing bending and torsion unmodeled flexible dynamics, limited bandwidth elevator surface actuation, and exogenous disturbances and system uncertainties. Therefore, there is a practical need to address the presence of actuator and unmodeled dynamics together with all sufficient stability conditions. The overarching goal of this paper is to provide system-theoretical remedies to this important problem.
4.1.2 Contribution

The main contribution of this paper is the analysis and synthesis of multiple adaptive architectures for control of uncertain dynamical systems with both actuator and unmodeled dynamics. Specifically:

i) For completeness, we first begin with the analysis of the standard model reference adaptive control (MRAC) architecture for this class of uncertain dynamical systems. In particular, this architecture uses an ideal (i.e., not modified) reference model along with the projection operator-based weight update law (Section 4.2.1). Here, we show a sufficient stability condition (Assumption 4.2.1) that guarantees the stability of this architecture in terms of boundedness (Section 4.3.1).

ii) We next study the hedging-based MRAC architecture for this class of uncertain dynamical systems, where this architecture is fairly well-adopted method for control of uncertain dynamical systems with actuator dynamics alone. In particular, this architecture modifies the trajectories of an ideal reference model to allow for correct projection operator-based adaptation that is not affected by the presence of actuator dynamics (Section 4.2.2). Here, we show a sufficient stability condition (Assumption 4.2.2) that guarantees the stability of this architecture in terms of boundedness in the face of both actuator and unmodeled dynamics (Section 4.3.2). Through an illustrative numerical example on a benchmark mechanical system (Example 4.3.1), in addition, we discuss that the sufficient stability condition of this architecture can be satisfied more easily compared with the sufficient stability condition of the standard MRAC architecture.

iii) We then focus on the expanded MRAC architecture. In particular, this method uses a reference model predicated on the weight estimation and a copy of the actuator dynamics (Section 4.2.3) in order to offer a better closed-loop system performance as compared with the hedging method for the actuator dynamics problem alone [37]. Here, once again, we show sufficient stability conditions (Assumptions 4.2.3 and 4.2.4) that yield bounded closed-loop system trajectories in the presence of both actuator and unmodeled dynamics (Section 4.3.3). On the same benchmark mechanical system discussed above, we also provide an illustrative numerical example (Example 4.3.2) that shows the sufficient stability conditions for the expanded MRAC architecture can be satisfied more easily as compared with the sufficient stability condition of the hedging-based MRAC architecture in ii).
Finally, since the MRAC architecture in iii) can offer better performance characteristics and its resulting sufficient stability conditions can be satisfied more easily, we synthesize a robustifying term for this particular architecture (Section 4.2.4). Based on this robustifying term, in particular, we show that this new MRAC architecture results in an added nonnegative matrix in one of the sufficient stability condition, which can allow for a relaxed stability condition (Assumption 4.2.5). That is, by relaxing this condition, our results can be further effective in practical applications when the related sufficient stability condition of the architecture iii) does not hold. Consistent with the above MRAC methods, we utilize tools and methods from Lyapunov stability theory, linear matrix inequalities, and matrix mathematics in the analysis of this new proposed architecture (Section 4.3.4). On the same benchmark example discussed above, the relaxed sufficient stability condition as well as time responses of each MRAC architecture is also provided (Section 4.4).

Note that the authors’ earlier conference papers [109, 110] can be viewed as preliminary works related to the adaptive control architectures discussed in iii) and iv). The present paper not only considerably expands on [109, 110] by providing detailed proofs with additional motivation and discussion on iii) and iv), but also comprehensively contains system-theoretical results for multiple adaptive control architectures for uncertain dynamical systems with both actuator and unmodeled dynamics.

4.1.3 Notation

In this paper, $\mathbb{R}$, $\mathbb{R}^n$, and $\mathbb{R}^{n\times m}$ respectively denote the set of real numbers, $n \times 1$ real column vectors, and $n \times m$ real matrices; $\mathbb{R}_+$ (respectively, $\mathbb{R}_+$) denotes the set of positive (respectively, nonnegative) real numbers; $\mathbb{R}_+^{n\times n}$ (respectively, $\mathbb{R}_+^{n\times n}$) denotes the set of $n \times n$ positive-definite (respectively, nonnegative-definite) real matrices; $\mathbb{D}^{n\times n}$ denotes the set of $n \times n$ real matrices with diagonal scalar entries; and $I_n$, $0_n$, and $0_{n\times n}$ respectively denote the $n \times n$ identity matrix, the $n \times 1$ vector of all zeros, and the $n \times n$ zero matrix. We also write $(\cdot)^T$ for the transpose, $(\cdot)^{-1}$ for the inverse, $\text{tr}(\cdot)$ for the trace, $\text{diag}(a)$ for the diagonal matrix with the vector $a$ on its diagonal, $\text{spec}(A)$ for the eigenvalues of the matrix $A \in \mathbb{R}^{n\times n}$, $\overline{\lambda}(A)$ (respectively, $\underline{\lambda}(A)$) for the maximum (respectively, minimum) eigenvalue of the matrix $A \in \mathbb{R}^{n\times n}$, $\|\cdot\|_F$ for the Frobenius matrix norm, $\|\cdot\|_2$ for the Euclidean norm; and $\|A\|_2 = (\overline{\lambda}(A^TA))^{\frac{1}{2}}$ for the induced two-norm of the matrix $A \in \mathbb{R}^{n\times m}$. 
Finally, we use the following (rectangular) projection operator definition from Exercise 11.3 of [7] and [85]. Consider a convex hypercube in the form \( \Omega = \{ \theta \in \mathbb{R}^n : (\theta_i^{\min} \leq \theta_i \leq \theta_i^{\max}) \} \), where \( i = 1, 2, \cdots, n, \Omega \in \mathbb{R}^n, \) and \( \theta_i^{\min} \) and \( \theta_i^{\max} \) respectively represent the minimum and maximum bounds for the \( i \)th component of the \( n \)-dimensional parameter vector \( \theta \) (we set \( \theta_i^{\min} = -\theta_i^{\max} \) for the results of this paper and without loss of generality). For a sufficiently small positive constant \( \varepsilon \), in addition, consider another hypercube given by \( \Omega_{\varepsilon} = \{ \theta \in \mathbb{R}^n : (\theta_i^{\min} + \varepsilon \leq \theta_i \leq \theta_i^{\max} - \varepsilon) \}_{i=1,2,\cdots,n} \), where \( \Omega_{\varepsilon} \subset \Omega \). The projection operator \( \text{Proj} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \) is then component-wise defined by \( \text{Proj}(\theta, y) = (\theta_i^{\max} - \theta_i) y_i / \varepsilon \) if \( \theta_i > \theta_i^{\max} - \varepsilon \) and \( y_i > 0 \), \( \text{Proj}(\theta, y) = (\theta_i - \theta_i^{\min}) y_i / \varepsilon \) if \( \theta_i < \theta_i^{\min} + \varepsilon \) and \( y_i < 0 \), and \( \text{Proj}(\theta, y) = y_i \) otherwise, where \( y \in \mathbb{R}^n \). From this definition, note that \( (\theta - \Theta^*)^T (\text{Proj}(\theta, y) - y) \leq 0 \) holds [7, 53]. The definition of projection operator can be also generalized to matrices as \( \text{Proj}_m(\Theta, Y) = (\text{Proj}(\text{col}_1(\Theta), \text{col}_1(Y)), \ldots, \text{Proj}(\text{col}_m(\Theta), \text{col}_m(Y))) \), where \( \Theta \in \mathbb{R}^{n \times m}, Y \in \mathbb{R}^{n \times m}, \) and \( \text{col}_i(\cdot) \) denotes \( i \)th column operator. In this case, for \( \Theta^* \in \mathbb{R}^{n \times m} \), \( \text{tr} \left[ (\Theta - \Theta^*)^T (\text{Proj}_m(\Theta, Y) - Y) \right] = \sum_{i=1}^{m} \text{col}_i(\Theta - \Theta^*)^T (\text{Proj}(\text{col}_i(\Theta), \text{col}_i(Y)) - \text{col}_i(Y)) \leq 0 \) follows.

### 4.2 Problem Formulation

As discussed in Section 4.1.2, this paper presents analysis and synthesis methods for multiple adaptive architectures for control of physical systems with uncertainties, actuator dynamics, and unmodeled dynamics. For this purpose, we consider the class of physical systems in the form given by

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + B[v(t) + W^T x(t) + y_q(t)], \quad x(0) = x_0, \\
\dot{q}(t) &= Fq(t) + Gx(t), \quad q(0) = q_0, \\
y_q(t) &= Hq(t).
\end{align*}
\]

Here, \( x(t) \in \mathbb{R}^n \) is the measurable state vector of the modeled dynamics and \( q(t) \in \mathbb{R}^p \) and \( y_q(t) \in \mathbb{R}^m \) are respectively the unmeasurable state and output vectors of the unmodeled dynamics. In addition, \( A \in \mathbb{R}^{n \times n} \) and \( B \in \mathbb{R}^{n \times m} \) are matrices associated with the modeled dynamics with the pair \((A, B)\) being controllable, \( W \in \mathbb{R}^{n \times m} \) is an unknown weight matrix, and \( F \in \mathbb{R}^{p \times p}, G \in \mathbb{R}^{p \times n}, \) and \( H \in \mathbb{R}^{m \times p} \) are matrices associated with the unmodeled dynamics. Since the state and output vectors of the unmodeled dynamics are unmeasurable, we consider (as in, for example, [1, 18, 109]) that \( F \) is Hurwitz for the solvability of the problem, and therefore, there is a unique \( S \in \mathbb{R}^{p \times p}^+ \) satisfying the Lyapunov equation \( 0 = F^T S + SF + I_p \). Furthermore,
Figure 4.1: Graphical representation of the considered uncertain dynamical system with actuator and unmodeled dynamics (4.1), (4.2), (4.3), and (4.4).

\[ v(t) \in \mathbb{R}^m \text{ in (4.1)} \] is the output of the actuator dynamics given by

\[ \dot{v}(t) = -\lambda (v(t) - u(t)), \quad v(0) = v_0, \quad (4.4) \]

with \( u(t) \in \mathbb{R}^m \) being the control input and \( \lambda \in \mathbb{R}_+ \) being the actuator bandwidth of all control channels.

A graphical representation of the considered uncertain dynamical system with actuator and unmodeled dynamics given in Figure 4.1.

**Remark 4.2.1** For the clarity of the exposition, the same bandwidth for all actuator channels is considered in this paper. This is without loss of generality since one can readily replace \( \lambda \) in (4.4) with a positive-definite diagonal matrix (see, for example, [35, 37]) by following the system-theoretical analysis steps given later in this paper. In addition, one can also consider an unknown control effectiveness matrix \( \Lambda \in \mathbb{R}_+^{m \times m} \cap \mathbb{D}^{m \times m} \) in (4.1) by replacing the term “\( v(t) \)” with “\( \Lambda v(t) \)”. This is also without loss of generality by following the system-theoretical analysis steps of this paper along with the results in, for example, [1].

In the remainder of this section, we present the MRAC architectures discussed in (i)-iv) of Section 4.1.2 as well as their sufficient stability conditions for the considered class of dynamical systems given by (4.1), (4.2), (4.3), and (4.4). All the theorems and proofs with regard to each architecture are given in Section 4.3 that utilize these conditions to conclude stability guarantees of these architectures.
In a standard MRAC architecture \cite{7, 8}, the objective is to (approximately or asymptotically) drive the state vector $x(t)$ of the uncertain dynamical system given by (4.1) to the state vector $x_{ri}(t) \in \mathbb{R}^n$ of a reference model capturing an ideal, desired closed-loop dynamical system performance given by

$$\dot{x}_{ri}(t) = A_r x_{ri}(t) + B_r c(t).$$

(4.5)

In (4.5), $c(t) \in \mathbb{R}^m$ is a given uniformly continuous bounded command, $A_r \triangleq A - BK_1 \in \mathbb{R}^{n \times n}$ is the Hurwitz reference system matrix, and $B_r \triangleq BK_2 \in \mathbb{R}^{n \times m}$ is the command input matrix with $K_1 \in \mathbb{R}^{m \times n}$ and $K_2 \in \mathbb{R}^{m \times m}$ are respectively the nominal feedback and feedforward gain matrices. To this end, consider the feedback control algorithm given by

$$u(t) = -K_1 x(t) + K_2 c(t) - \hat{W}_s^T(t) x(t).$$

(4.6)

In (4.6), $\hat{W}_s(t) \in \mathbb{R}^{n \times m}$ is an estimate of $W$ satisfying a projection operator based weight update law

$$\dot{\hat{W}}_s(t) = \gamma \text{Proj}_m [\hat{W}_s(t), x(t) e_s^T(t) P B], \quad \hat{W}_s(0) = \hat{W}_s0,$$

(4.7)

with $\gamma \in \mathbb{R}_+$ being the learning rate, $P \in \mathbb{R}^{n \times n}_+$ is the unique solution to the Lyapunov equation $0 = A_r^T P + PA_r + I_n$ owing to $A_r$ being Hurwitz, and $e_s(t) \triangleq x(t) - x_{ri}(t)$ being the system error state vector for standard MRAC architecture. In addition, $\hat{W}_{s, \max, i+(j-1)n} \in \mathbb{R}_+$ denote symmetric element-wise projection bound defined such that $|[\hat{W}_s(t)]_{ij}| \leq \hat{W}_{s, \max, i+(j-1)n}$, $i = 1, \ldots, n$ and $j = 1, \ldots, m$. The following sufficient condition is needed for the stability of the standard MRAC architecture subject to actuator and unmodeled dynamics (see Section 4.3.1).

**Assumption 4.2.1** Consider

$$\mathcal{R}_s(t) \triangleq \begin{bmatrix} \eta_0 & \tilde{\eta}_s^T(t) \\ \tilde{\eta}_s(t) & \eta(t) \end{bmatrix} \geq \varepsilon I_{n+m+p},$$

(4.8)

\footnote{As it is known \cite{2, 7, 8}, one needs to choose the nominal gain matrices $K_1$ and $K_2$ such that $A_r$ is Hurwitz and $-E A_r^{-1} B_r = I_n$, where $E \in \mathbb{R}^{m \times n}$ is a matrix that allows a user to select a subset of $x(t)$ to be ideally followed by $c(t)$.}
where \( \eta_0 = \begin{bmatrix} 2\alpha_1 \lambda I_m & 0_{m \times p} \\ 0_{m \times p}^T & \alpha_2 I_p \end{bmatrix} \), \( \bar{\eta}_n(t) = \begin{bmatrix} \eta_1(t) & \eta_2(t) \end{bmatrix} \), \( \eta_1(t) = -PB + \alpha_1 \lambda (K_1^T + \hat{W}_s(t)) \), \( \eta_2 = -\beta^{-1}PBH - \alpha_2 \beta G^T S \), \( \eta(t) = I_n - 2PB(K_1 + \hat{W}_s^T(t)) \), \( \beta \in \mathbb{R}_+ \), \( \alpha_1 \in \mathbb{R}_+ \), \( \alpha_2 \in \mathbb{R}_+ \), and \( \varepsilon \in \mathbb{R}_+ \).

**Remark 4.2.2** One can utilize a quadratic stability argument [111] in order to assess the inequality given by (4.8). To elucidate this point, let

\[
\mathcal{R}_s(t) = -I_n - \varepsilon I_{n+m+p} + \bar{\eta}_n(t) - \eta_n(t) \]

be a quadratically stable matrix and note that \( \mathcal{R}_s(t) \) is symmetric. Hence, there must exist a \( P \) satisfying

\[
\mathcal{R}_s(t) P_s + P_s \mathcal{R}_s(t) < 0. \tag{4.10}
\]

Note that (4.10) can be equivalently rewritten as \( (-\mathcal{R}_s(t) + \varepsilon I_{n+m+p}) P_s + P_s (-\mathcal{R}_s(t) + \varepsilon I_{n+m+p}) < 0 \).

Similar to Lemma 3 of [112], by letting \( P_s = I_{n+m+p} \), \( \mathcal{R}_s(t) < 0 \). Hence, (4.10) implies the condition (4.8) given in Assumption 4.2.1. Note also that (4.10) holds when \( (-\mathcal{R}_s(t) + \varepsilon I_{n+m+p}) + (-\mathcal{R}_s(t) + \varepsilon I_{n+m+p}) < 0 \) holds (i.e., \( -\mathcal{R}_s(t) < -\varepsilon I_{n+m+p} \)), where this implies \( \mathcal{R}_{s_1,\ldots,i_l} < 0 \) and therefore, \( -\mathcal{R}_{s_1,\ldots,i_l} < -\varepsilon I_{n+m+p} \).

Here, \( \mathcal{R}_{s_1,\ldots,i_l} \) and \( \mathcal{R}_{s_{1,\ldots,i_l}} \) represent the corners of the hypercube constructed from all the permutations of \( W_{s_{1,\ldots,i_l}}, i_l \in \{1,2\}, l \in \{1,\ldots,2^{mn}\} \) given by

\[
W_{s_{l_1,\ldots,l_l}} = \begin{bmatrix}
(-1)^{i_1} \hat{W}_{s_1,max_1} & (-1)^{i_{1+n}} \hat{W}_{s_{1,\ldots,1+n}} & \cdots & (-1)^{i_{1+(m-1)n}} \hat{W}_{s_{1,\ldots,1+(m-1)n}} \\
(-1)^{i_2} \hat{W}_{s_2,max_2} & (-1)^{i_{2+n}} \hat{W}_{s_{2,\ldots,2+n}} & \cdots & (-1)^{i_{2+(m-1)n}} \hat{W}_{s_{2,\ldots,2+(m-1)n}} \\
\vdots & \vdots & \ddots & \vdots \\
(-1)^{i_m} \hat{W}_{s_m,max_m} & (-1)^{i_{m+n}} \hat{W}_{s_{m,\ldots,m+n}} & \cdots & (-1)^{i_{m+(m-1)n}} \hat{W}_{s_{m,\ldots,m+(m-1)n}}
\end{bmatrix}. \tag{4.11}
\]

That is, it represents the corners of the hypercube based on the maximum variation of \( \hat{W}_s(t) \) using given projection bounds for the elements of \( \hat{W}_s(t) \) [111]. As a consequence, the linear matrix inequality given by \( \mathcal{R}_{s_1,\ldots,i_l} < 0 \) can be utilized to solve (4.10). Finally, when this solution procedure is feasible, then (4.9) is quadratically stable.
4.2.2 Hedging-Based MRAC Architecture

In the hedging-based MRAC architecture [32, 34, 113], the objective is to drive the state vector $x(t)$ of uncertain dynamical system given by (4.1) to the state vector $x_{rh}(t) \in \mathbb{R}^n$ of a reference model, which includes the deficit term “$B[v(t) - u(t)]$”, given by

$$\dot{x}_{rh}(t) = A_rx_{rh}(t) + B_rc(t) + B[v(t) - u(t)].$$

(4.12)

To this end, consider the feedback control algorithm given by

$$u(t) = -K_1x(t) + K_2c(t) - \hat{W}_h^T(t)x(t).$$

(4.13)

In (4.13), $\hat{W}_h(t) \in \mathbb{R}^{n \times m}$ is the estimate of $W$ satisfying a projection operator based weight update law

$$\dot{\hat{W}}_h(t) = \gamma \text{Proj}_m[\hat{W}_h(t), x(t)e_h^T(t)PB], \quad \hat{W}_h(0) = \hat{W}_{h0},$$

(4.14)

with $e_h(t) \triangleq x(t) - x_{rh}(t)$ being the system error state vector for hedging-based MRAC architecture ($\gamma$ and $P$ are as defined in Section 4.2.1). In addition, $\hat{W}_{h \max,i+(j-1)n} \in \mathbb{R}_+$ denote symmetric element-wise projection bound defined such that $|\hat{W}_h(t)|_{ij} \leq \hat{W}_{h \max,i+(j-1)n}$, $i = 1, \ldots, n$ and $j = 1, \ldots, m$. The following sufficient condition is needed for the stability of the hedging-based MRAC architecture subject to actuator and unmodeled dynamics (see Section 4.3.2).

**Assumption 4.2.2** Consider

$$\mathcal{R}_h(t) \triangleq \begin{bmatrix} \eta_0 & \tilde{\eta}_{h1}^T(t) \\ \tilde{\eta}_{h1}(t) & \tilde{\eta}_{h2}(t) \end{bmatrix} \geq \varepsilon I_{2n+m+p},$$

(4.15)

where $\tilde{\eta}_{h1} = \begin{bmatrix} \eta_3(t) & \eta_2 \\ \eta_4(t) & \eta_5 \end{bmatrix}$, $\tilde{\eta}_{h2}(t) = \begin{bmatrix} I_n & \eta_6(t) \\ \eta_6^T(t) & \eta_7(t) \end{bmatrix}$, $\eta_3(t) = \alpha_1 \lambda(K_1^T + \hat{W}_h(t))$, $\eta_4(t) = \alpha_1 \lambda(K_1^T + \hat{W}_h(t)) + \alpha_5 PB$, $\eta_5 = -\alpha_2 \beta G^T S$, $\eta_6(t) = -\alpha_3 PB(K_1 + \hat{W}_h^T(t))$, $\eta_7(t) = \alpha_3 I_n - \eta_6 - \eta_6^T$, and $\alpha_3 \in \mathbb{R}_+$.46

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46In these works, hedging-based MRAC architectures are utilized in the absence of unmodeled dynamics. The analysis given in this work is new with regard to the hedging-based adaptive control of uncertain dynamical systems in the presence of both actuator and unmodeled dynamics.
Remark 4.2.3 Similar to Remark 4.2.2, one can utilize a quadratic stability argument in order to assess the inequality given by (4.15). To elucidate this point, let

\[
\overline{R}(t) \triangleq \begin{bmatrix}
\eta_0 - \varepsilon I_{m+p} & \tilde{\eta}_h(t) \\
\tilde{\eta}_h(t) & \tilde{\eta}_h(t) - \varepsilon I_{2n}
\end{bmatrix} = -\overline{R}(t) + \varepsilon I_{2n+m+p}.
\] (4.16)

be a quadratically stable matrix and note that \(\overline{R}(t)\) is symmetric. Hence, there must exist a \(P_h\) satisfying

\[
\overline{R}(t)P_h + P_h\overline{R}(t) < 0.
\] (4.17)

Now, following the same steps given in Remark 4.2.2, one can readily show that with \(P_h = I_{2n+m+p}\) (4.17) implies (4.15). Once again, \(\overline{R}_{hi_1,...,i_l}\) and \(\overline{R}_{hi_1,...,i_l}\) represent the corners of the hypercube constructed from all the permutations of \(\tilde{W}_{hi_1,...,i_l}\), \(i_t \in \{1, 2\}, l \in \{1, ..., 2^m\}\) given by

\[
\tilde{W}_{hi_1,...,i_l} = \begin{bmatrix}
(-1)^{i_1}\tilde{W}_{hi,\max,1} & (-1)^{i_1+i_2}\tilde{W}_{hi,\max,1+n} & \cdots & (-1)^{i_1+i_2+...+(m-1)}\tilde{W}_{hi,\max,1+(m-1)n} \\
(-1)^{i_1}\tilde{W}_{hi,\max,2} & (-1)^{i_1+i_2}\tilde{W}_{hi,\max,2+n} & \cdots & (-1)^{i_1+i_2+...+(m-1)}\tilde{W}_{hi,\max,2+(m-1)n} \\
\vdots & \vdots & \ddots & \vdots \\
(-1)^{i_1}\tilde{W}_{hi,\max,n} & (-1)^{i_1+i_2}\tilde{W}_{hi,\max,2n} & \cdots & (-1)^{i_1+i_2+...+(m-1)}\tilde{W}_{hi,\max,nn}
\end{bmatrix},
\] (4.18)

and therefore, the linear matrix inequality given by \(\overline{R}_{hi_1,...,i_l} < 0\) can be utilized.

4.2.3 Expanded MRAC Architecture

In the expanded MRAC architecture [37], the objective is to drive the augmented state vector \([\chi^T(t), v^T(t)]^T \in \mathbb{R}^{n+m}\) to the state vector \(\chi(t) = [\chi^T(t), v^T(t)]^T \in \mathbb{R}^{n+m}\) of the expanded reference model\(^\text{47}\) given by

\[
\begin{bmatrix}
\dot{\chi}_t(t) \\
\dot{v}_t(t)
\end{bmatrix} =
\begin{bmatrix}
A + B\tilde{W}^T(t) & B \\
-\lambda(K_1 + \tilde{W}^T(t)) & -\lambda I_m
\end{bmatrix}
\begin{bmatrix}
\chi_t(t) \\
v_t(t)
\end{bmatrix} +
\begin{bmatrix}
0_{n \times m} \\
\lambda K_2 \end{bmatrix} c(t).
\] (4.19)

\(^\text{47}\)Since a deficit term is applied to the ideal reference model in the hedging-based MRAC architecture, this term can significantly alter its trajectories. A better closed-loop system performance can be achieved by utilizing (4.19) as compared with the modified reference model used in the hedging-based architecture given by (4.12). For details on the structure of (4.19) and its comparison with the modified reference model used in the hedging-based architecture, we refer the readers to [37].
In (4.19), $\mathcal{A}(\cdot) \in \mathbb{R}^{(n+m)\times(n+m)}$ is the expanded reference system matrix and $\mathcal{B}(\cdot) \in \mathbb{R}^{(n+m)\times m}$ is the expanded reference system command input matrix. To this end, consider the feedback control algorithm given by

$$u(t) = -K_1x(t) + K_2c(t) - \hat{W}^T(t)x(t).$$  \hspace{1cm} (4.20)

In (4.20), the weight update law for $\hat{W}(t) \in \mathbb{R}^{n\times m}$ is the estimate of $W$ satisfying a projection operator based weight update law

$$\dot{\hat{W}}(t) = \gamma \text{Proj}_m[\hat{W}(t), x(t)e^T(t)PB_0]$$

with $\gamma \in \mathbb{R}_+$ being the learning rate, $e(t) \triangleq [\hat{x}^T(t), \hat{v}^T(t)]^T \in \mathbb{R}^{n+m}$ being the system augmented error vector with $\hat{x}(t) \triangleq x(t) - x_r(t)$, and $\hat{v}(t) \triangleq v(t) - v_r(t)$. In (4.21), $\mathcal{P} \in \mathbb{R}^{(n+m)\times(n+m)}_+$ is the solution of a linear matrix inequality for which further details are given below and $B_0 = [B^T, 0_{m\times m}]^T \in \mathbb{R}^{(n+m)\times m}$. In addition, $\hat{W}_{\text{max}}_{,i+(j-1)n} \in \mathbb{R}_+$ denotes symmetric element-wise projection bound defined such that $|\hat{W}(t)_{ij}| \leq \hat{W}_{\text{max}}_{,i+(j-1)n}$, $i = 1, \ldots, n$ and $j = 1, \ldots, m$. The following sufficient conditions are needed to obtain stability of the expanded MRAC architecture (see Section 4.3.3).

**Assumption 4.2.3** Let

$$A(\hat{W}(t), \lambda, \varepsilon) \triangleq \begin{bmatrix}
A + B\hat{W}^T(t) + \frac{\varepsilon}{2}I_n & B \\
-\lambda(K_1 + \hat{W}^T(t)) & -\lambda I_m + \frac{\varepsilon}{2}I_m
\end{bmatrix},$$  \hspace{1cm} (4.22)

be a quadratically stable matrix.

**Remark 4.2.4** Assumption 4.2.3 implies the existence of $\mathcal{P} \in \mathbb{R}^{(n+m)\times(n+m)}_+$ satisfying

$$\mathcal{A}^T(t)\mathcal{P} + \mathcal{P}\mathcal{A}(t) < 0.$$  \hspace{1cm} (4.23)

Therefore, one can utilize linear matrix inequalities to assess (4.23) (see also Section V of [109]). In particular, let
be the corners of the hypercube constructed from all the permutations of \( W_{i_1, \ldots, i_l} \) given by

\[
W_{i_1, \ldots, i_l} = \begin{bmatrix}
(-1)^{\hat{W}_{i_1,1}} W_{i_1, \max, 1} & (-1)^{\hat{W}_{i_1,1} + n} W_{i_1, \max, 1+n} & \cdots & (-1)^{\hat{W}_{i_1,1} + (m-1)n} W_{i_1, \max, 1+(m-1)n} \\
(-1)^{\hat{W}_{i_2,1}} W_{i_2, \max, 2} & (-1)^{\hat{W}_{i_2,1} + n} W_{i_2, \max, 2+n} & \cdots & (-1)^{\hat{W}_{i_2,1} + (m-1)n} W_{i_2, \max, 2+(m-1)n} \\
\vdots & \vdots & \ddots & \vdots \\
(-1)^{\hat{W}_{i_l,1}} W_{i_l, \max, n} & (-1)^{\hat{W}_{i_l,1} + n} W_{i_l, \max, 2n} & \cdots & (-1)^{\hat{W}_{i_l,1} + (m-1)n} W_{i_l, \max, mn}
\end{bmatrix},
\]

(4.25)

It can then be shown that

\[
A^T_{i_1, \ldots, i_l} P + P A_{i_1, \ldots, i_l} < 0, \quad P > 0,
\]

(4.26)

implies \( A(\hat{W}(t), \lambda, \varepsilon) P + P A(\hat{W}(t), \lambda, \varepsilon) < 0 \) \cite{114, 115}. As a consequence, the linear matrix inequality given by (4.26) can be utilized to solve (4.23) and calculate \( P \).

**Assumption 4.2.4** Consider

\[
\mathcal{R}_1 \triangleq \begin{bmatrix}
\varepsilon P & \tilde{n}_2^T \\
\tilde{n}_2 & \alpha_2 I_p
\end{bmatrix} > 0,
\]

(4.27)

or, equivalently

\[
\mathcal{F}_\varepsilon = \alpha_2 \varepsilon P - \tilde{n}_2^T \tilde{n}_2 > 0,
\]

(4.28)

where \( \tilde{n}_2 = -\beta^{-1} H^T B_0^T P - \beta \alpha_2 \text{SGN} \) with \( N = [I_n, 0_{n \times m}] \in \mathbb{R}^{n \times (n+m)} \).

**Remark 4.2.5** Assumption 4.2.3, which is also stated in (25) of \cite{37}, captures the system-theoretic tradeoff between system uncertainties and actuator dynamics. Assumption 4.2.4, in addition, is the sufficient condition that depends on Assumption 4.2.3 as well as the matrices resulting from the unmodeled dynamics.

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48Note that the parameter \( \varepsilon \) here is the same parameter used in Assumption 4.2.3.

49This follows from the Schur complement \cite{86}. Specifically, \( \mathcal{R}_1 \) is positive-definite if and only if \( \alpha_2 I_p \) is positive-definite, which automatically holds since \( \alpha_2 \) is positive, and \( \mathcal{F}_\varepsilon \) is positive-definite.

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Therefore, different from the previous adaptive control laws given in Sections 4.2.1 and 4.2.2 this architecture provides two relatively separated sufficient conditions, where one needs to satisfy first Assumption 4.2.3 and then assess Assumption 4.2.4.

4.2.4 Relaxed Expanded MRAC Architecture

In this section, we focus on relaxing the second sufficient stability condition of the expanded MRAC architecture given by Assumption 4.2.4. Building on the results of Section 4.2.3, the main contribution here is to utilize a robustifying term for this purpose with this term inspired by the results in Section 3 of [1]. Note that our control procedure here is different than the one in [1] in the sense that this paper considers actuator dynamics whereas this was not considered in [1]. To begin with, consider the feedback control algorithm given by

\[ u(t) = -K_1x(t) + K_2c(t) - \hat{W}^T(t)x(t) - \mu B_0^T B_0 P_m^T e(t). \]  \hspace{1cm} (4.29)

In (4.29), \( \hat{W}(t) \in \mathbb{R}^{n \times m} \) is the estimate of \( W \) satisfying a weight update law given by (4.21). Furthermore, the term \( -\mu B_0^T B_0 P_m^T e(t) \in \mathbb{R}^m \), where \( P_m \) results from partitioning \( P \) as \( P \triangleq [P_n \ P_m] \) with \( P_n \in \mathbb{R}^{(n+m) \times n} \) and \( P_m \in \mathbb{R}^{(n+m) \times m} \), represents the aforementioned robustifying term with \( \mu \in \mathbb{R}_+ \) being a design parameter. Next, Assumption 4.2.3 and the following sufficient condition are needed to obtain stability of the new MRAC architecture of this section (see Section 4.3.4).

Assumption 4.2.5 Consider

\[ \mathcal{R}_2 \triangleq \begin{bmatrix} \epsilon \mathcal{P} + 2\lambda \mu P_n B_0^T B_0 P_m^T & \tilde{\eta}_2^T \\ \tilde{\eta}_2 & \alpha_2 I_p \end{bmatrix} > 0, \]  \hspace{1cm} (4.30)

or, equivalently

\[ \mathcal{F}_{re} = \alpha_2 \epsilon \mathcal{P} + 2\alpha_2 \lambda \mu P_n B_0^T B_0 P_m^T - \tilde{\eta}_2^T \tilde{\eta}_2 > 0. \]  \hspace{1cm} (4.31)

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\textsuperscript{50}Note that the parameter \( \epsilon \) here is the same parameter used in Assumption 4.2.3.

\textsuperscript{51}This follows from the Schur complement [86]. Specifically, \( \mathcal{R}_2 \) is positive-definite if and only if \( \alpha_2 I_p \) is positive-definite, which automatically holds since \( \alpha_2 \) is positive, and \( \mathcal{F}_{re} \) is positive-definite.
Remark 4.2.6 While Assumption 4.2.3 is necessary for both expanded and relaxed expanded MRAC architectures, Assumption 4.2.4 is only necessary for the former architecture whereas Assumption 4.2.5 is only necessary for the latter one. As compared to Assumption 4.2.4, Assumption 4.2.5 has an additional nonnegative matrix \(2\alpha_m\mu B_0^T B_0 P_m^T\) that results owing to the robustifying term \(-\mu B_0^T B_0 P_m^T e(t)\) in the new adaptive feedback control law given by (4.29). As also demonstrated later in this paper (Section 4.4), this additional nonnegative matrix can allow for a relaxed sufficient stability condition.

4.3 Stability Analyses

In this section, we provide the stability analyses of the adaptive control architectures given in Section 4.2. In order to have an additional level of flexibility in our following system-theoretical analyses, we first introduce the state transformation \(z(t) = \beta q(t)\) with \(\beta \in \mathbb{R}^{1, 109}\). In this case, (4.1) and (4.2) can be equivalently written as

\[
\dot{x}(t) = Ax(t) + B\left[v(t) + W^T x(t)\right] + \beta^{-1} BHz(t), \quad x(0) = x_0, \tag{4.32}
\]

\[
\dot{z}(t) = Fz(t) + \beta Gx(t), \quad z(0) = \beta q_0. \tag{4.33}
\]

4.3.1 Stability of the Standard MRAC Architecture

First, we show the stability of the standard MRAC architecture given in Section 4.2.1. Specifically, based on the feedback control law given by (4.6), one can write the system error dynamics between the uncertain dynamical system (4.32) and the ideal reference model (4.5) as

\[
\dot{e}_s(t) = A_r e_s(t) - B\hat{W}_s^T(t) x(t) + \beta^{-1} BHz(t) + B\left[v(t) - u(t)\right], \tag{4.34}
\]

where \(\hat{W}_s(t) = \hat{W}_s(t) - W \in \mathbb{R}^{n \times m}\) is the estimation error. Using (4.6), once again, the system error dynamics given by (4.34) and actuator dynamics given by (4.4) can be further rewritten as

\[
\dot{e}_s(t) = A_r e_s(t) - B\hat{W}_s^T(t) x(t) + \beta^{-1} BHz(t) + B\left[v(t) + K_1 e_s(t) + K_1 x_{ri}(t) - K_2 c(t) + \hat{W}_s^T(t) e_s(t) + \hat{W}_s^T(t) x_{ri}(t)\right]
\]
\[
= A_t e_s(t) - B \hat{W}_s^T(t)x(t) + \beta^{-1}BH \hat{z}(t) + Bv(t)
+ B(K_1 + \hat{W}_s^T(t))e_s(t) - BK_2c(t) + B(K_1 + \hat{W}_s^T(t))x_{\xi}(t).
\]

(4.35)

\[
\hat{v}(t) = -\lambda (v(t) + (K_1 + \hat{W}_s^T(t))e_s(t) + (K_1 + \hat{W}_s^T(t))x_{\xi}(t) - K_2c(t)).
\]

(4.36)

Using the sufficient stability condition for the standard MRAC architecture in Assumption 4.2.1, the next theorem presents the main results of this section, which guarantees boundedness of all closed-loop signals in the presence of system uncertainties, actuator dynamics, and unmodeled dynamics.

**Theorem 4.3.1** Consider the uncertain dynamical system with actuator and unmodeled dynamics given by (4.1), (4.2), (4.3), and (4.4). Consider, in addition, the reference model given by (4.5) and the standard adaptive feedback control law given by (4.6) with (4.7). If Assumption 4.2.1 holds, then the solution \((e_s(t), \hat{W}_s(t), v(t), z(t))\) is uniformly ultimately bounded.

**Proof.** To show the boundedness of the solution \((e_s(t), \hat{W}_s(t), v(t), z(t))\), consider the Lyapunov-like function candidate given by

\[
\mathcal{V}_s(e_s, \hat{W}_s, v, z) = e_s^T Pe_s + \gamma^{-1} \text{tr} \hat{W}_s^T \hat{W}_s + \alpha_1 v^T v + \alpha_2 z^T S z.
\]

(4.37)

Here, \(\mathcal{V}_s(0, 0, 0, 0) = 0\) and \(\mathcal{V}_s(e_s, \hat{W}_s, v, z) > 0\) for all \((e_s, \hat{W}_s, v, z) \neq (0, 0, 0, 0)\). Differentiating \(\mathcal{V}_s(e_s, \hat{W}_s, v, z)\) along the closed-loop dynamical system trajectories and using the property of the projection operator yields

\[
\dot{\mathcal{V}}_s(e_s(t), \hat{W}_s(t), v(t), z(t)) \leq -e_s^T(t) e_s(t) + 2\beta^{-1} e_s^T(t) PBH \hat{z}(t) + 2e_s^T(t) PBv(t)
+ 2e_s^T(t) PB(K_1 + \hat{W}_s^T(t))e_s(t) + 2e_s^T(t) PB(K_1 + \hat{W}_s^T(t))x_{\xi}(t)
- 2e_s^T(t) PBc(t) - 2\alpha_1 \lambda v^T(t)v(t) + 2\alpha_1 \lambda v^T(t)K_2c(t)
- 2\alpha_1 \lambda v^T(t)(K_1 + \hat{W}_s^T(t))e_s(t) - 2\alpha_1 \lambda v^T(t)(K_1 + \hat{W}_s^T(t))x_{\xi}(t)
- \alpha_2 z^T(t)z(t) + 2\alpha_2 \beta z^T(t)SGe_s(t) + 2\alpha_2 \beta z^T(t)SGx_{\xi}(t)
= -e_s^T(t) \left[ I_n - 2PB(K_1 + \hat{W}_s^T(t)) \right] e_s(t)
+ e_s^T(t) \left[ 2\beta^{-1} PB + 2\alpha_2 \beta G^T S \right] z(t)
+ e_s^T(t) \left[ 2PB - 2\alpha_1 \lambda (K_1 + \hat{W}_s(t)) \right] v(t)
\]

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\(-2\alpha_1 \lambda v^T(t)v(t) - \alpha_2 z^T(t)z(t) - 2e_s^t(t)PB_t c(t) + 2\alpha_1 \lambda v^T(t)K_2 c(t) + 2e_s^t(t)PB_t K_1 \dot{W}_t(t) + 2\alpha_2 \beta z^T(t)SGx_{\tilde{t}}(t)\)

\(\begin{aligned}
\xi_s(t) &= -\xi_s^T R_s(t) \xi_s(t) - 2e_s^t(t)PB_t c(t) + 2\alpha_1 \lambda v^T(t)K_2 c(t) + 2e_s^t(t)PB_t K_1 \dot{W}_t(t) + 2\alpha_2 \beta z^T(t)SGx_{\tilde{t}}(t) \\
\end{aligned}\)

where \(\xi_s(t) \triangleq [v^T(t), z^T(t), e_s^T(t)]^\top\). Since \(R_s(t) \succeq \epsilon I_{m+p}\) by Assumption 4.2.1, it follows from (4.38) that

\(\begin{aligned}
\dot{V}_s(e_s(t), \dot{W}_s(t), v(t), z(t)) & \leq -\epsilon \|\xi_s(t)\|_2^2 - 2e_s^t(t)PB_t c(t) + 2\alpha_1 \lambda v^T(t)K_2 c(t) + 2e_s^t(t)PB_t K_1 \dot{W}_t(t) + 2\alpha_2 \beta z^T(t)SGx_{\tilde{t}}(t) \\
& \leq -\epsilon \|\xi_s(t)\|_2^2 - \epsilon \frac{r_s^*}{2} \left( \|\tilde{x}_s(t)\|_2 - \frac{r_s^*}{2} \right),
\end{aligned}\)

where \(r_s^* = 2\|PB_t\|_2 c^* + 2\alpha_1 \lambda \|K_2\|_2 c^* + 2\|PB_t\|_2 + \alpha_1 \lambda (\|K_1\|_2 + \|w_s^*\|_2)\|x_{\tilde{t}}^*\| + 2\alpha_2 \beta \|SG\|_2 \|x_{\tilde{t}}^*\|\) with upper bounds \(c^*, w_s^*\) (this follows from using the projection operator in (4.7)), and \(x_{\tilde{t}}^*\) respectively for \(c(t), \dot{W}_s(t), \) and \(x_{\tilde{t}}(t)\).

Finally, (4.39) can equivalently be written as

\(\begin{aligned}
\dot{V}_s(e_s(t), \dot{W}_s(t), v(t), z(t)) & \leq -\epsilon \|\xi_s(t)\|_2^2 - \epsilon \|\tilde{x}_s(t)\|_2 + r_s^* \|\xi_s(t)\|_2 - k_1 \|\tilde{W}_s(t)\|_F^2 + k_1 \|\tilde{W}_s(t)\|_F^2, \\
\end{aligned}\)

for some \(k_1 \in \mathbb{R}_+\). Applying now the Young’s inequality to the second term at the right side of (4.40) yields

\(\begin{aligned}
\dot{V}_s(e_s(t), \dot{W}_s(t), v(t), z(t)) & \leq -\epsilon \|\xi_s(t)\|_2^2 + r_s^* \|\xi_s(t)\|_2 - k_1 \|\tilde{W}_s(t)\|_F^2 + k_1 \|\tilde{W}_s(t)\|_F^2 + k_3 \\
& = - \left( \epsilon - \frac{k_3}{2} \right) \|\tilde{x}_s(t)\|_2^2 - k_1 \|\tilde{W}_s(t)\|_F^2 + k_4,
\end{aligned}\)

(4.41)
where $k_3 = k_1 w_s^4$ and $k_4 = k_3 + r_s^2/(2k_2)$. Now, by setting $k_2 = \varepsilon$, we have
\[
\dot{V}_s(e_s(t), \hat{W}_s(t), v(t), z(t)) \leq -\frac{\varepsilon}{2} \|\dot{z}_s(t)\|^2 - k_1 \|\dot{W}_s(t)\|^2 + k_4
\]
\[
= -\frac{\varepsilon}{2} \|v(t)\|^2 - \frac{\varepsilon}{2} \|z(t)\|^2 - \frac{\varepsilon}{2} \|e_s(t)\|^2 - k_1 \|\dot{W}_s(t)\|^2 + k_4. \tag{4.42}
\]

Note that (4.42) implies $\dot{V}_s(e_s(t), \hat{W}_s(t), v(t), z(t)) \leq 0$ outside the compact set given by
\[
\mathcal{S}_k = \left\{ (e_s(t), \hat{W}_s(t), v(t), z(t)) : \|e_s(t)\|_2 \leq \sqrt{\frac{2k_4}{\varepsilon}} \right\} \cap \left\{ (e_s(t), \hat{W}_s(t), v(t), z(t)) : \|\hat{W}_s(t)\|_2 \leq \sqrt{\frac{2k_4}{\varepsilon}} \right\} \cap \left\{ (e_s(t), \hat{W}_s(t), v(t), z(t)) : \|v(t)\|_2 \leq \sqrt{\frac{2k_4}{\varepsilon}} \right\} \cap \left\{ (e_s(t), \hat{W}_s(t), v(t), z(t)) : \|z(t)\|_2 \leq \sqrt{\frac{2k_4}{\varepsilon}} \right\}.
\]

As a consequence, the evolution of $\dot{V}_s(e_s(t), \hat{W}_s(t), v(t), z(t))$ is upper bounded by
\[
\dot{V}_s(e_s(t), \hat{W}_s(t), v(t), z(t)) \leq \max_{(e_s(t), \hat{W}_s(t), v(t), z(t)) \in \mathcal{S}_k} V_s(\cdot) = \frac{2k_4}{\varepsilon} + \frac{\gamma^{-1} k_4}{k_1} + \frac{\alpha_1 2k_4}{\varepsilon} + \frac{\alpha_2 \lambda}{\varepsilon} \cdot \frac{2k_4}{\varepsilon}, \tag{4.43}
\]

since $\dot{V}_s(e_s(t), \hat{W}_s(t), v(t), z(t))$ cannot grow outside $\mathcal{S}_k$. Hence, this conclusion then yields to the uniform ultimate boundedness of the solution $(e_s(t), \hat{W}_s(t), v(t), z(t))$ \[7, 57\].

4.3.2 Stability of the Hedging-Based MRAC Architecture

Next, we show the stability of the hedging-based MRAC architecture given in Section 4.2.2. In particular, based on the feedback control law given by (4.13), one can write the system error dynamics between the uncertain dynamical system (4.32) and the hedging-based reference model (4.12) as
\[
\dot{e}_h(t) = A_r e_h(t) - B \hat{W}_h(t) x(t) + B^{-1} B H z(t), \tag{4.44}
\]
where $\hat{W}_h(t) = \hat{W}_h(t) - W \in \mathbb{R}^{n \times m}$ is the estimation error. Moreover, one can write reference system error dynamics between the hedging-based reference model (4.12) and the ideal reference model (4.5) as
\[
\dot{\tilde{x}}_h(t) = A_r \tilde{x}_h(t) + B [v(t) - u(t)], \tag{4.45}
\]
where $\tilde{x}_h(t) \triangleq x_{ih}(t) - x_{ti}(t)$ is the reference model error state. Using the sufficient stability condition for the hedging-based MRAC architecture in Assumption 4.2.2, the next theorem presents the main results of this
section, which guarantees boundedness of all closed-loop signals in the presence of system uncertainties, actuator dynamics, and unmodeled dynamics.

**Theorem 4.3.2** Consider the uncertain dynamical system with actuator and unmodeled dynamics given by (4.1), (4.2), (4.3), and (4.4). Consider, in addition, the ideal reference model given by (4.5), the reference model with deficit term given by (4.12), and the hedging-based adaptive feedback control law given by (4.13) with (4.14). If Assumption 4.2.2 holds, then the solution \( (e_h(t), \tilde{W}_h(t), v(t), z(t), \tilde{x}_h(t)) \) is uniformly ultimately bounded.

**Proof.** To show the boundedness of the solution \( (e_h(t), \tilde{W}_h(t), v(t), z(t), \tilde{x}_h(t)) \), consider the Lyapunov-like function candidate given by

\[
V_h(e_h, \tilde{W}_h, v, z, \tilde{x}_h) = e_h^T P e_h + \gamma^{-1} \text{tr} \tilde{W}_h^T \tilde{W}_h + \alpha_1 v^T v + \alpha_2 z^T S z + \alpha_3 \tilde{x}_h^T P \tilde{x}_h. \tag{4.46}
\]

Here, \( V_h(0, 0, 0, 0, 0) = 0 \) and \( V_h(e_h, \tilde{W}_h, v, z, \tilde{x}_h) > 0 \) for all \( (e_h, \tilde{W}_h, v, z, \tilde{x}_h) \neq (0, 0, 0, 0, 0) \). Differentiating \( V_h(e_h, \tilde{W}_h, v, z, \tilde{x}_h) \) along the closed-loop dynamical system trajectories and using the property of the projection operator yields

\[
\begin{align*}
V_h(e_h(t), \tilde{W}_h(t), v(t), z(t), \tilde{x}_h(t)) & \leq -e_h^T(t)e_h(t) + 2\beta^{-1}e_h^T(t)PBHz(t) - 2\alpha_1 \lambda v^T(t)v(t) \\
& + 2\alpha_1 \lambda v^T(t)K_2 c(t) - 2\alpha_1 \lambda v^T(t)(K_1 + \tilde{W}_h^T(t))e_h(t) \\
& - 2\alpha_1 \lambda v^T(t)(K_1 + \tilde{W}_h^T(t))\tilde{x}_h(t) - 2\alpha_1 \lambda v^T(t)(K_1 + \tilde{W}_h^T(t))x_i(t) \\
& - \alpha_2 \tilde{z}(t)z(t) + 2\alpha_2 \tilde{z}(t)SGe_h(t) + 2\alpha_2 \beta \tilde{z}(t)SG\tilde{x}_h(t) \\
& + 2\alpha_2 \beta \tilde{z}(t)SGx_i(t) - \alpha_3 \tilde{x}_h^T(t)\tilde{x}_h(t) + 2\alpha_3 \tilde{x}_h^T(t)PBv(t) \\
& + 2\alpha_3 \tilde{x}_h^T(t)PB(K_1 + \tilde{W}_h^T(t))e_h(t) \\
& + 2\alpha_3 \tilde{x}_h^T(t)PB(K_1 + \tilde{W}_h^T(t))\tilde{x}_h(t) \\
& + 2\alpha_3 \tilde{x}_h^T(t)PB(K_1 + \tilde{W}_h^T(t))x_i(t) - 2\alpha_3 \tilde{x}_h^T(t)PBK_2 c(t) \\
& = -e_h^T(t)e_h(t) + e_h^T(t) \left[ 2\beta^{-1}PBH + 2\alpha_2 \beta G^T S \right] z(t) \\
& + e_h^T(t) \left[ 2\alpha_3 (K_1^T + \tilde{W}_h(t))B^T P \right] \tilde{x}_h(t) - 2\alpha_1 \lambda v^T(t)v(t) \\
& + v^T(t) \left[ -2\alpha_1 \lambda (K_1 + \tilde{W}_h^T(t)) + 2\alpha_3 B^T P \right] \tilde{x}_h(t)
\end{align*}
\]
\[-2\alpha_1 \lambda v^T(t)(K_1 + \hat{W}_h^T(t))x_{\text{rh}}(t) + 2\alpha_1 \lambda v^T(t)K_2 c(t)\]
\[-\alpha_2 z^T(t)z(t) + 2\alpha_2 \beta z^T(t)SG\hat{x}_h(t) + 2\alpha_2 \beta z^T(t)SGx_{\text{ri}}(t)\]
\[+ 3\xi_h^2(t) \left[ -\alpha_3 + 2\alpha_3 PB(K_1 + \hat{W}_h^T(t)) \right] \hat{x}_h(t)\]
\[+ 2\alpha_3 \hat{x}_h^T(t)PB(K_1 + \hat{W}_h^T(t))x_{\text{ri}}(t) - 2\alpha_3 \hat{x}_h^T(t)PBK_2 c(t)\]
\[= -\xi_h^T(t)R_h \xi_h(t) - 2\alpha_1 \lambda v^T(t)(K_1 + \hat{W}_h^T(t))x_{\text{ri}}(t)\]
\[+ 2\alpha_1 \lambda v^T(t)K_2 c(t) + 2\alpha_2 \beta z^T(t)SGx_{\text{ri}}(t)\]
\[+ 2\alpha_3 \hat{x}_h^T(t)PB(K_1 + \hat{W}_h^T(t))x_{\text{ri}}(t) - 2\alpha_3 \hat{x}_h^T(t)PBK_2 c(t),\]
\[(4.47)\]

where \(\xi_h(t) \triangleq [v^T(t), z^T(t), e^T_h(t), \hat{x}_h^T(t)]^T\). Since \(R_h \geq \epsilon I_{2n+m+p}\) by Assumption 4.2.2, it follows from (4.47) that

\[\dot{\gamma}_h(e_h(t), \hat{W}_h(t), v(t), z(t), \hat{x}_h(t)) \leq -\epsilon \xi_h^T(t) R_h \xi_h(t) - 2\alpha_1 \lambda v^T(t)(K_1 + \hat{W}_h^T(t))x_{\text{ri}}(t)\]
\[+ 2\alpha_1 \lambda v^T(t)K_2 c(t) + 2\alpha_2 \beta z^T(t)SGx_{\text{ri}}(t)\]
\[+ 2\alpha_3 \hat{x}_h^T(t)PB(K_1 + \hat{W}_h^T(t))x_{\text{ri}}(t) - 2\alpha_3 \hat{x}_h^T(t)PBK_2 c(t)\]
\[\leq -\epsilon \|\xi_h(t)\|_2 \left[ \|\xi_h(t)\|_2 - \frac{r_h^*}{\epsilon} \right],\]
\[(4.48)\]

where \(r_h^* = 2\alpha_3 \|PB_i\|_2 c^* + 2\alpha_1 \lambda \|K_2\|_2 c^* + 2(\alpha_3 \|PB\|_2 + \alpha_1 \lambda)(\|K_1\|_2 + w_h^*)x_{\text{ri}}^* + 2\alpha_2 \beta \|SG\|_2 x_{\text{ri}}^*\) with upper bound \(w_h^*\)(this follows from using the projection operator in (4.14)) for \(\hat{W}_h(t)\).

Finally, (4.48) can equivalently be written as

\[\dot{\gamma}_h(e_h(t), \hat{W}_h(t), v(t), z(t), \hat{x}_h(t)) \leq -\epsilon \|\xi_h(t)\|_2^2 + r_h^* \|\xi_h(t)\|_2 - k_1 \|\hat{W}_h(t)\|_F^2 + k_1 \|\hat{W}_h(t)\|_F^2,\]
\[(4.49)\]

Applying now the Young’s inequality to the second term at the right side of (4.49) yields

\[\dot{\gamma}_h(e_h(t), \hat{W}_h(t), v(t), z(t), \hat{x}_h(t)) \leq -\epsilon \|\xi_h(t)\|_2^2 + \frac{r_h^2}{2k_2} + \frac{k_2}{2} \|\xi_h(t)\|_2^2 - k_1 \|\hat{W}_h(t)\|_F^2 + k_3\]
\[= - \left[ \epsilon - \frac{k_2}{2} \right] \|\xi_h(t)\|_2^2 - k_1 \|\hat{W}_h(t)\|_F^2 + k_4,\]
\[(4.50)\]
where $k_3 = k_1 w_0^2$ and $k_4 = k_3 + r_0^2/(2k_2)$. Now, by setting $k_2 = \varepsilon$, we have

$$\dot{V}_h(e_h(t), \tilde{W}_h(t), v(t), z(t), \tilde{x}_h(t)) \leq -\frac{\varepsilon}{2} \|\tilde{x}_h(t)\|^2_2 - k_1 \|\tilde{W}_h(t)\|^2_F + k_4$$

$$= -\frac{\varepsilon}{2} \|v(t)\|^2_2 - \frac{\varepsilon}{2} \|z(t)\|^2_2 - \frac{\varepsilon}{2} \|\tilde{x}_h(t)\|^2_2$$

$$- \frac{\varepsilon}{2} \|e_h(t)\|^2_2 - k_1 \|\tilde{W}_h(t)\|^2_F + k_4. \tag{4.51}$$

Note that (4.51) implies $\dot{V}_h(e_h(t), \tilde{W}_h(t), v(t), z(t), \tilde{x}_h(t)) \leq 0$ outside the compact set given by

$$S_h = \left\{(e_h(t), \tilde{W}_h(t), v(t), z(t), \tilde{x}_h(t)) : \|v(t)\|_2 \leq \sqrt{\frac{2k_4}{\varepsilon}} \right\} \cap \left\{(e_h(t), \tilde{W}_h(t), v(t), z(t), \tilde{x}_h(t)) : \|z(t)\|_2 \leq \sqrt{\frac{2k_4}{\varepsilon}} \right\} \cap \left\{(e_h(t), \tilde{W}_h(t), v(t), z(t), \tilde{x}_h(t)) : \|\tilde{x}_h(t)\|_2 \leq \sqrt{\frac{2k_4}{\varepsilon}} \right\} \cap \left\{(e_h(t), \tilde{W}_h(t), v(t), z(t), \tilde{x}_h(t)) : \|\tilde{x}_h(t)\|_2 \leq \sqrt{\frac{2k_4}{\varepsilon}} \right\} \cap \left\{(e_h(t), \tilde{W}_h(t), v(t), z(t), \tilde{x}_h(t)) : \|\tilde{x}_h(t)\|_2 \leq \sqrt{\frac{2k_4}{\varepsilon}} \right\}.$$

As a consequence, the evolution of $\dot{V}_h(e_h(t), \tilde{W}_h(t), v(t), z(t), \tilde{x}_h(t))$ is upper bounded by

$$\dot{V}_h(e_h(t), \tilde{W}_h(t), v(t), z(t), \tilde{x}_h(t)) \leq \max_{(e_h(t), \tilde{W}_h(t), v(t), z(t), \tilde{x}_h(t)) \in S_h} V_h(\cdot)$$

$$= \overline{\mathcal{X}}(P) \frac{2k_4}{\varepsilon} + \gamma^{-1} \frac{k_4}{k_1} + \alpha_1 \frac{2k_4}{\varepsilon} + \alpha_2 \overline{\mathcal{X}}(S) \frac{2k_4}{\varepsilon} + \alpha_3 \overline{\mathcal{X}}(P) \frac{2k_4}{\varepsilon}, \tag{4.52}$$

since $\dot{V}_h(e_h(t), \tilde{W}_h(t), v(t), z(t), \tilde{x}_h(t))$ cannot grow outside $S_h$. Hence, this conclusion then yields to the uniform ultimate boundedness of the solution $(e_h(t), \tilde{W}_h(t), v(t), z(t), \tilde{x}_h(t))$ [7, 57].

**Remark 4.3.1** Since $x_{\text{ref}}(t)$ and $\tilde{x}_h(t)$ are bounded for all $t \in \mathbb{R}_+$, it follows from the definition of $\tilde{x}_h(t) = x_{\text{hf}}(t) - x_{\text{ref}}(t)$ that $x_{\text{hf}}(t)$ is bounded. Furthermore, since $e_h(t)$ is bounded and $x_{\text{hf}}(t)$ is now bounded for all $t \in \mathbb{R}_+$, $x(t)$ is also bounded from the definition of $e_h(t) = x(t) - x_{\text{hf}}(t)$.

Note that hedging-based approach modifies the ideal reference model given by (4.5) dynamics with a signal “$B[v(t) - u(t)]$” that enables design adaptive controllers without affected by the presence of actuator dynamics [32–35, 116]. We now provide an example that compares the sufficient stability condition for the standard MRAC architecture (Assumption 4.2.1) with the one for the hedging-based MRAC architecture (Assumption 4.2.2).

**Example 4.3.1** Consider an uncertain coupled mechanical system subject to actuator and unmodeled dynamics (Figure 4.2) given by
where $x(t) = [\theta(t), \dot{\theta}(t), q_1(t), \dot{q}_1(t)]^T$ and $q(t) = [q_2(t), \dot{q}_2(t)]^T$. In this example, we consider $k_1 = 0.02N/m$ and $k_2 = 12N/m$ for the spring constants; $c_1 = 0.01Ns/m$ and $c_2 = 0.95Ns/m$ for the damping constants; $m_0 = 0.1kg$, $M_1 = 0.8kg$, and $M_2 = 8kg$ for the masses; $l = 0.2m$ for the inverted pendulum length; and $g = 9.81m/s^2$ for the acceleration due to gravity. In addition, we choose $\lambda = 200$ for the actuator bandwidth, $\gamma = 0.6$ for the learning rate, and $c = 1.5m$ for the command. The projection bound is also selected as
\( \dot{W}_{\text{max}} = |W| + 0.2 \). Moreover, the controller gain matrix \( K_1 \) is obtained using a linear quadratic regulator-based design Section 3.3 on [117] with the weighting matrices \( Q = \text{diag}[1,1,15,10] \) to penalize the state vector and \( R = 0.55 \) to penalize the control input as \( K_1 = [-34.7935,-5.007,-5.2223,-6.7695] \). The gain \( K_2 \) is also selected using a pre-filter design as \( K_2 = -(E(A-BK_1)^{-1}B)^{-1} = -5.2223 \) with \( E = [0,0,1,0] \) such that the translational position command for the cart is aimed to be followed.

First, following the linear matrix inequalities-based computation steps outlined in Remark 4.2.2, the sufficient stability condition for the standard MRAC architecture given by Assumption 4.2.1 could not be satisfied\(^{52}\). Next, following similar computation steps outlined in Remark 4.2.3, the sufficient stability condition for the hedging-based MRAC architecture given by Assumption 4.2.2 is satisfied with parameters \( \beta = 1.1, \varepsilon = 10^{-12}, \alpha_1 = 10^{-10}, \alpha_2 = 10^{-3}, \text{and } \alpha_3 = 10^{-6} \) (i.e., the related linear matrix inequality results in a feasible solution). To summarize, the boundedness of the closed-loop system trajectories is guaranteed for the hedging-based MRAC architecture.

The above illustrative numerical example demonstrates that sufficient stability condition for the hedging-based MRAC can hold for the case when the sufficient stability condition for the standard MRAC architecture does not hold. Therefore, from the closed-loop stability (i.e., boundedness) standpoint, the latter architecture can be more important than the former one. In the next section, we first perform a stability analysis for the expanded MRAC architecture and then provide a similar example for comparison purposes.

### 4.3.3 Stability of the Expanded MRAC Architecture

We now show the stability of the expanded MRAC architecture given in Section 4.2.3. Specifically, based on the feedback control law (4.20) and by adding and subtracting \( "B\dot{W}^T(t)x(t)" \), one can rewrite the uncertain dynamical system (4.32) as

\[
\dot{x}(t) = (A + B\dot{W}^T(t))x(t) + Bv(t) - B\dot{W}^T(t)x(t) + \beta^{-1}BHz(t),
\]

where \( \dot{W}(t) = \dot{W}(t) - W \in \mathbb{R}^{n \times m} \) is the estimation error. Moreover, (4.4) can be further rewritten using (4.20) as

\(^{52}\)We have used various range combinations for \( \beta, \varepsilon, \alpha_1, \alpha_2, \) and \( \alpha_3 \). However, none of these selections satisfied Assumption 4.2.1 (e.g., an interested reader can try \( \beta = 1.1, \varepsilon = 10^{-12}, \alpha_1 = 10^{-10}, \alpha_2 = 10^{-3}, \text{and } \alpha_3 = 10^{-6} \)).
\[ \dot{v}(t) = -\lambda v(t) - \lambda (K_1 + \hat{W}^T(t)) x(t) + \lambda K_2 c(t). \] (4.56)

Now, one can compactly write the error dynamics between the uncertain dynamical system and the actuator dynamics respectively given by (4.55) and (4.56) and expanded reference model given by (4.19) as

\[
\begin{bmatrix}
\dot{\hat{x}}(t) \\
\dot{\hat{v}}(t) \\
\dot{e}(t)
\end{bmatrix}
= \begin{bmatrix}
A + B\hat{W}^T(t) & B & 0 \\
-\lambda (K_1 + \hat{W}^T(t)) & -\lambda I_m & 0 \\
A_r(\hat{W}(t), \lambda) & -\lambda I_m & 0
\end{bmatrix}
\begin{bmatrix}
\hat{x}(t) \\
\hat{v}(t) \\
e(t)
\end{bmatrix}
+ \beta^{-1} \begin{bmatrix}
B \\
0_{m \times m}
\end{bmatrix}
H_z(t) - \begin{bmatrix}
B \\
0_{m \times m}
\end{bmatrix}
\hat{W}^T(t)x(t). \] (4.57)

Using the sufficient stability condition for the expanded MRAC architecture in Assumptions 4.2.3 and 4.2.4, the next theorem presents the main results of this section, which guarantees boundedness of all closed-loop signals in the presence of system uncertainties, actuator dynamics, and unmodeled dynamics.

**Theorem 4.3.3** Consider the uncertain dynamical system with actuator and unmodeled dynamics given by (4.1), (4.2), (4.3), and (4.4). Consider, in addition, the reference model given by (4.19), and the feedback control law given by (4.20) and (4.21). If Assumptions 4.2.3 and 4.2.4 hold, then the solution \((e(t), \hat{W}(t), z(t))\) of the closed-loop dynamical system is uniformly ultimately bounded.

**Proof.** First, to show the boundedness of the proposed reference model (4.19), consider \(A_r(\hat{W}(t), \lambda, \epsilon) = A_r(\hat{W}(t), \lambda) + \frac{\epsilon}{2} I_{(a+m)}\), which is quadratically stable (i.e., \(A_r^T(\hat{W}(t), \lambda) P + PA(\hat{W}(t), \lambda, \epsilon) < 0\)) by Assumption 4.2.3. It follows further by compactness that \(A_r^T(\hat{W}(t), \lambda) P + PA_r(\hat{W}(t), \lambda, \epsilon) \leq -\epsilon P < 0\) holds. This implies \(A_r(\hat{W}(t), \lambda)\) is quadratically stable, therefore, the boundedness of reference model (4.19) is guaranteed since \(c(t)\) is bounded.

Next, to show the boundedness of the solution \((e(t), \hat{W}(t), z(t))\), consider the Lyapunov-like function candidate given by

\[
V_0(e, \hat{W}) = e^T Pe + \gamma^{-1} \text{tr} \hat{W}^T \hat{W}. \] (4.58)

In (4.58), \(V_0(0, 0) = 0\) and \(V_0(e, \hat{W}) > 0\) for all \((e, \hat{W}) \neq (0, 0)\). Differentiating (4.58) along the closed-loop dynamical system trajectories and using the property of the projection operator yields
\[
V_0(e(t), \hat{W}(t)) = 2e^T(t)P \left[ A_r(\hat{W}(t), \lambda) e(t) - B_0\hat{W}^T(t)x(t) \right] + 2\gamma^{-1}r \hat{W}^T(t)\hat{W}(t) \\
+ 2\beta^{-1}e^T(t)PB_0Hz(t)
\]

\[
= e^T(t) \left[ A_r^T(\hat{W}(t), \lambda) P + P A_r(\hat{W}(t), \lambda) \right] e(t) \\
+ 2\gamma^{-1}r \hat{W}^T(t) \left[ \hat{W}(t) - \gamma(t)e^T(t)PB_0 \right] + 2\beta^{-1}e^T(t)PB_0Hz(t)
\]

\[
\leq e^T(t) \left[ A_r^T(\hat{W}(t), \lambda) P + P A_r(\hat{W}(t), \lambda) \right] e(t) + 2\beta^{-1}e^T(t)PB_0Hz(t).
\] (4.59)

By adding and subtracting “\(\varepsilon e^T(t)P e(t)\)” to (4.59), we can now write

\[
\dot{V}_0(e(t), \hat{W}(t)) \leq e^T(t) \left[ \left( A_r(\hat{W}(t), \lambda) + \frac{\varepsilon}{2}I_{(n+m)} \right)^T P + P \left( A_r(\hat{W}(t), \lambda) + \frac{\varepsilon}{2}I_{(n+m)} \right) \right] e(t) \\
- \varepsilon e^T(t)P e(t) + 2\beta^{-1}e^T(t)PB_0Hz(t)
\]

\[
= e^T(t) \left[ A_r^T(\hat{W}(t), \lambda, \varepsilon) P + P A_r(\hat{W}(t), \lambda, \varepsilon) \right] e(t) \\
- \varepsilon e^T(t)P e(t) + 2\beta^{-1}e^T(t)PB_0Hz(t).
\] (4.60)

Since \(A(\hat{W}(t), \lambda, \varepsilon)\) is quadratically stable as noted above, it follows further from (4.60) that

\[
\dot{V}_0(e(t), \hat{W}(t)) \leq -\varepsilon e^T(t)P e(t) + 2\beta^{-1}e^T(t)PB_0Hz(t).
\] (4.61)

Next, we rewrite the unmodeled dynamics given by (4.33) as

\[
\dot{z}(t) = Fz(t) + \beta GNe(t) + \beta Gx_t, \quad z(0) = \beta q_0,
\] (4.62)

and consider the Lyapunov-like function candidate given by

\[
V_1(e, \hat{W}, z) = V_0 + \alpha_2 z^TSz,
\] (4.63)

where note that \(V_1(0, 0, 0) = 0\) and \(V_1(e, \hat{W}, z) > 0\) for all \((e, \hat{W}, z) \neq (0, 0, 0)\). Differentiating (4.63) along the closed-loop system trajectories and using (4.62) yields

\[
\dot{V}_1(e(t), \hat{W}(t), z(t)) \leq -\varepsilon e^T(t)P e(t) + 2\beta^{-1}e^T(t)PB_0Hz(t) + 2\alpha_2 z^TS[Fz(t) + \beta GNe(t) + \beta Gx_t].
\] (4.64)

Now, it follows from (4.64) that
\[ \dot{V}_1(e(t), z(t), \bar{W}(t)) \leq -\xi_1^T(t)R_1\xi_1(t) + 2\alpha_2 \beta z^T S G x_t, \] 

(4.65)

where \( \xi_1(t) = [e^T(t), z^T(t)]^T \). Here, \( R_1 \) is positive-definite by Assumption 4.2.4. Hence,

\[ \dot{V}_1(e(t), \bar{W}(t), z(t)) \leq -\lambda_1(R_1) \| \xi_1(t) \|_2^2 - r_0^* + r_0 \| \xi_1(t) \|_2 - k_1 \| \bar{W}(t) \|_F^2 + k_1 \| \bar{W}(t) \|_F^2, \]

(4.66)

where \( r_0^* = 2\alpha_2 \beta \| S G \|_2 x_t^* \) with \( x_t^* \) being an upper bound for \( x_t(t) \) since \( x_t(t) \) is bounded owing to the boundedness of the proposed reference model given by (4.19).

Finally, (4.66) can equivalently be written as

\[ \dot{V}_1(e(t), \bar{W}(t), z(t)) \leq -\lambda_1(R_1) \| \xi_1(t) \|_2^2 + r_0^* \| \xi_1(t) \|_2 - k_1 \| \bar{W}(t) \|_F^2 + k_1 \| \bar{W}(t) \|_F^2. \]

(4.67)

Applying now the Young’s inequality to the second term at the right side of (4.67) yields

\[ \dot{V}_1(e(t), \bar{W}(t), z(t)) \leq -\lambda_1(R_1) \| \xi_1(t) \|_2^2 + \frac{r_0^*}{2} \| \xi_1(t) \|_2^2 - k_1 \| \bar{W}(t) \|_F^2 + k_3 = -\lambda_1(R_1) - \frac{k_2}{2} \| \xi_1(t) \|_2^2 - k_1 \| \bar{W}(t) \|_F^2 + k_4, \]

(4.68)

where \( k_3 = k_1 w^2 \) and \( k_4 = k_3 + r_0^* / (2k_2) \). In addition, by setting \( k_2 = \lambda_1(R_1) \), we have

\[ \dot{V}_1(e(t), \bar{W}(t), z(t)) \leq -\lambda_1(R_1) \| \xi_1(t) \|_2^2 - k_1 \| \bar{W}(t) \|_F^2 + k_4. \]

(4.69)

Note that (4.69) implies \( \dot{V}_1(e(t), \bar{W}(t), z(t)) \leq 0 \) outside the compact set given by

\[ S_1 = \left\{ (e(t), \bar{W}(t), z(t)) : \| e(t) \|_2 \leq \sqrt{\frac{2k_4}{\lambda_1(R_1)}} \right\} \cap \left\{ (e(t), \bar{W}(t), z(t)) : \| \bar{W}(t) \|_F \leq \sqrt{\frac{2k_4}{\lambda_1(R_1)}} \right\}. \]

As a consequence, the evolution of \( \dot{V}_1(e(t), \bar{W}(t), z(t)) \) is upper bounded by

\[ \dot{V}_1(e, \bar{W}(t), z(t)) \leq \max_{(e(t), \bar{W}(t), z(t)) \in S_1} V_1(\cdot) = \bar{K}(P) \frac{2k_4}{\lambda_1(R_1)} + \gamma^{-1} k_4 k_1 + \alpha_2 \bar{K}(S) \frac{2k_4}{\lambda_1(R_1)}. \]

(4.70)
since $\mathcal{V}_1(e(t), \dot{W}(t), z(t))$ cannot grow outside $S_1$; hence, this conclusion then yields to the uniform ultimate boundedness of the solution $(e(t), z(t), \dot{W}(t))$ \cite{7, 57}.

Similar to the purpose of Example 4.3.1, we now provide a numerical study that compares the sufficient stability conditions for the expanded MRAC architecture (Assumptions 4.2.3 and 4.2.4) with the one for the hedging-based MRAC architectures.

**Example 4.3.2** Once again, consider the uncertain coupled mechanical system subject to actuator and unmodeled dynamics (Figure 4.2) given by (4.53) and (4.54). Here, we consider $k_1 = 0.03N/m$ and $k_2 = 12N/m$ for the spring constants; $c_1 = 0.01Ns/m$ and $c_2 = 0.95Ns/m$ for the damping constants; $m_0 = 0.1kg$, $M_1 = 0.8kg$, and $M_2 = 8kg$ for the masses; $l = 0.2m$ for the inverted pendulum length; and $g = 9.81m/s^2$ for the acceleration due to gravity. In addition, we choose $\lambda = 200$ for the actuator bandwidth, $\gamma = 0.6$ for the learning rate, and $c = 1.5m$ for the command. The projection bound is also selected as $\hat{W}_{\max} = |W| + 0.2$. Moreover, we use the same $K_1$ and $K_2$ as given in Example 4.3.1.

First, following the linear matrix inequalities-based computation steps outlined in Remark 4.2.3, the sufficient stability condition for the hedging-based MRAC architecture given by Assumption 4.2.2 could not be satisfied\(^{53}\). Next, following similar computation steps outlined in Remark 4.2.4, the first sufficient stability condition for the expanded MRAC architecture given by Assumption 4.2.3 is satisfied with the parameters $\beta = 1.1$, $\epsilon = 2.5$, $\alpha_1 = 10^{-10}$, $\alpha_2 = 10^{-3}$, and $\alpha_3 = 10^{-6}$ (i.e., the related linear matrix inequality results in a feasible solution $\mathcal{P}$). In addition, the second stability condition given by (4.28) of Assumption 4.2.4 is also satisfied for this architecture with the same parameters given above\(^{54}\). To summarize, the boundedness of the closed-loop system trajectories is guaranteed for the expanded MRAC architecture.

The above illustrative numerical example demonstrates that the sufficient stability conditions for the expanded MRAC architecture can hold for the case when the sufficient stability condition for the hedging-based MRAC architecture does not hold. Therefore, from the closed-loop system stability (i.e., boundedness) standpoint, the latter architecture can be more important than the former one.

With regard to the sufficient stability conditions for the expanded MRAC architecture for the class of dynamical systems considered throughout this paper, Assumption 4.2.3, which is also stated in \cite{37}, directly

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\(^{53}\)We have used various range combinations for $\beta$, $\epsilon$, $\alpha_1$, $\alpha_2$, and $\alpha_3$. However, none of these selections satisfied Assumption 4.2.2 (e.g., an interested reader can try $\beta = 1.1$, $\epsilon = 2.5$, $\alpha_1 = 10^{-10}$, $\alpha_2 = 10^{-3}$, and $\alpha_3 = 10^{-6}$).

\(^{54}\)Here, all eigenvalues of $\mathcal{F}_e$ are positive; that is, $\text{spec}(\mathcal{F}_e) = \{4.2659 \times 10^{-9}, 3.1918 \times 10^{-5}, 0.0003, 0.0054, 0.0538\}$. 

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results from analyzing given dynamics, and therefore, it may not be relaxed further. However, Assumption 4.2.4 results from a Lyapunov stability analysis. Motivated by this standpoint, the next section provides a stability analysis on how Assumption 4.2.4 can be relaxed as in Assumption 4.2.5 based on the added term \(-\mu B_0^T B_0 P_m e(t)\) in the proposed feedback control law given by (4.28).

### 4.3.4 Stability of the Proposed MRAC Architecture

Finally, we show the stability of the proposed architecture given in Section 4.2.4. In particular, based on the feedback control law given by (4.29), one can write the output of the actuator dynamics given by (4.4) as

\[
\dot{v}(t) = -\lambda v(t) - \lambda (K_1 + \hat{W}^T(t))x(t) + \lambda K_2 e(t) - \lambda \mu B_0^T B_0 P_m e(t).
\]  

(4.71)

In addition, one can compactly write the error dynamics between the uncertain dynamical system and the actuator dynamics respectively given by (4.55) and (4.71) and expanded reference model given by (4.19) as

\[
\begin{bmatrix}
\dot{x}(t) \\
\dot{\tilde{v}}(t) \\
\dot{e}(t)
\end{bmatrix} =
\begin{bmatrix}
A + B\hat{W}^T(t) & B \\
-\lambda (K_1 + \hat{W}^T(t)) & -\lambda I_m
\end{bmatrix}
\begin{bmatrix}
x(t) \\
\tilde{v}(t)
\end{bmatrix} + \beta^{-1}
\begin{bmatrix}
B \\
0_{m \times m}
\end{bmatrix} H z(t)
\]

\[
-\begin{bmatrix}
B \\
0_{m \times m}
\end{bmatrix}
\hat{W}^T(t) x(t) - \lambda \mu
\begin{bmatrix}
0_{n \times (n+m)} \\
B_0^T B_0 P_m
\end{bmatrix}
\begin{bmatrix}
\tilde{x}(t) \\
\tilde{v}(t)
\end{bmatrix} + \beta^{-1}
\begin{bmatrix}
B \\
0_{m \times m}
\end{bmatrix} H z(t)
\]

(4.72)

Using the sufficient stability conditions for the relaxed expanded MRAC architecture in Assumptions 4.2.3 and 4.2.5, the following and last theorem of this paper shows boundedness of all closed-loop signals in the presence of not only system uncertainties but also actuator and unmodeled dynamics.

**Theorem 4.3.4** Consider the uncertain dynamical system with actuator and unmodeled dynamics given by (4.1), (4.2), (4.3), and (4.4). Consider, in addition, the expanded reference model given by (4.19) and the new adaptive feedback control law given by (4.29) with (4.21). If Assumptions 4.2.3 and 4.2.5 hold, then the solution \((e(t), \hat{W}(t), z(t))\) of the closed-loop dynamical system is uniformly ultimately bounded.
Proof. First, consider the Lyapunov-like function candidate in (4.58). Differentiating (4.58) along the closed-loop dynamical system trajectories and using the property of the projection operator yields now

\[ \dot{\mathcal{V}}_0(e(t), \hat{W}(t)) = 2e^T(t)\mathcal{P}[A_r(\hat{W}(t), \lambda)e(t) - B_0\hat{W}^T(t)x(t) - \lambda \mu Me(t)] \]

\[ + 2\gamma^{-1}\text{tr} \hat{W}^T(t)\hat{W}(t) + 2\beta^{-1}e^T(t)\mathcal{P}B_0Hz(t) \]

\[ = e^T(t)\left[A_r^T(\hat{W}(t), \lambda)\mathcal{P} + \mathcal{P}A_r(\hat{W}(t), \lambda)\right]e(t) \]

\[ + 2\gamma^{-1}\text{tr} \hat{W}^T(t)\left[\hat{W}(t) - \gamma x(t)\right]e^T(t)\mathcal{P}B_0 \]

\[ - 2\lambda \mu e^T(t)P_mB_0^TB_0P_m^Te(t) + 2\beta^{-1}e^T(t)\mathcal{P}B_0Hz(t) \]

\[ \leq e^T(t)\left[A_r^T(\hat{W}(t), \lambda)\mathcal{P} + \mathcal{P}A_r(\hat{W}(t), \lambda)\right]e(t) \]

\[ - 2\lambda \mu e^T(t)P_mB_0^TB_0P_m^Te(t) + 2\beta^{-1}e^T(t)\mathcal{P}B_0Hz(t). \]  

Now, adding and subtracting "\(\varepsilon e^T(t)\mathcal{P}e(t)\)" to (4.73), we can now write

\[ \dot{\mathcal{V}}_0(e(t), \hat{W}(t)) \leq e^T(t)\left[A_r(\hat{W}(t), \lambda) + \frac{\varepsilon}{2}I_{n+m}\right]e(t) \]

\[ - \varepsilon e^T(t)\mathcal{P}e(t) - 2\lambda \mu e^T(t)P_mB_0^TB_0P_m^Te(t) + 2\beta^{-1}e^T(t)\mathcal{P}B_0Hz(t) \]

\[ = e^T(t)\left[A_r^T(\hat{W}(t), \lambda, \varepsilon)\mathcal{P} + \mathcal{P}A_r(\hat{W}(t), \lambda, \varepsilon)\right]e(t) \]

\[ - \varepsilon e^T(t)\mathcal{P}e(t) - 2\lambda \mu e^T(t)P_mB_0^TB_0P_m^Te(t) + 2\beta^{-1}e^T(t)\mathcal{P}B_0Hz(t). \]  

Since \(A(\hat{W}(t), \lambda, \varepsilon)\) is quadratically stable by Assumption 4.2.3, it follows from (4.74) that

\[ \dot{\mathcal{V}}_0(e(t), \hat{W}(t)) \leq -\varepsilon e^T(t)\mathcal{P}e(t) - 2\lambda \mu e^T(t)P_mB_0^TB_0P_m^Te(t) + 2\beta^{-1}e^T(t)\mathcal{P}B_0Hz(t). \]  

Second, we utilize (4.62) and consider the same Lyapunov-like function candidate given by (4.63). The time derivative of this Lyapunov-like function candidate has the form given by

\[ \dot{\mathcal{V}}_1(e(t), z(t), \hat{W}(t)) \leq -\varepsilon e^T(t)\mathcal{P}e(t) - 2\lambda \mu e^T(t)P_mB_0^TB_0P_m^Te(t) + 2\beta^{-1}e^T(t)\mathcal{P}B_0Hz(t) \]

\[ + 2\alpha_2z^T(t)S[z(t) + \beta GNe(t) + \beta G\Delta x(t)]. \]
\[
\begin{align*}
\dot{V}_1(e(t), z(t), \tilde{W}(t)) & = -\varepsilon e^T(t)\mathcal{P}e(t) - 2\lambda \mu e^T(t)P_mB_0^T B_0 P_m^T e(t) \\
& \quad + 2\beta^{-1} e^T(t) \mathcal{P} B_0 H z(t) - \alpha_2 z^T(t) z(t) + 2\alpha_2 \beta z^T(t) S G N e(t) \\
& \quad + 2\alpha_2 \beta z^T(t) S G x_\tau(t).
\end{align*}
\]

(4.76)

Now, it follows from (4.76) that

\[
\dot{V}_1(e(t), z(t), \tilde{W}(t)) \leq -\xi_2^T(t) R_2 \xi_2(t) + 2\alpha_2 \beta \|z(t)\|_2 \|S G\|_2 x_\tau^*,
\]

(4.77)

where \(\xi_2(t) = [e^T(t), z^T(t)]^T\). Here, \(R_2\) is positive-definite by Assumption 4.2.5. Hence,

\[
\dot{V}_1(e(t), \tilde{W}(t), z(t)) \leq -\tilde{\lambda}(R_2) \|\xi_2(t)\|_2^2 + r_0 \|\xi_2(t)\|_2 - k_1 \|\tilde{W}(t)\|_F^2 + k_1 \|\tilde{W}(t)\|_F^2,
\]

(4.78)

Finally, (4.78) can equivalently be written as

\[
\dot{V}_1(e(t), \tilde{W}(t), z(t)) \leq -\tilde{\lambda}(R_2) \|\xi_2(t)\|_2^2 + r_0 \|\xi_2(t)\|_2 - k_1 \|\tilde{W}(t)\|_F^2 + k_1 \|\tilde{W}(t)\|_F^2,
\]

(4.79)

with \(k_1 \in \mathbb{R}_+\). Applying now the Young’s inequality to the second term at the right side of (4.79) yields

\[
\begin{align*}
\dot{V}_1(e(t), \tilde{W}(t), z(t)) & \leq -\tilde{\lambda}(R_2) \|\xi_2(t)\|_2^2 + r_0 \|\xi_2(t)\|_2^2 - k_1 \|\tilde{W}(t)\|_F^2 + k_2 \\
& = -\left[\tilde{\lambda}(R_2) - \frac{k_2}{2}\right] \|\xi_2(t)\|_2^2 - k_1 \|\tilde{W}(t)\|_F^2 + k_3
\end{align*}
\]

(4.80)

where \(k_3 = k_1 w^*^2\), \(\|\tilde{W}(t)\|_F \leq w^*\) (this follows from using the projection operator in (4.21)), and \(k_4 = k_3 + r_0^2/(2k_2)\). In addition, by setting \(k_2 = \tilde{\lambda}(R_2)\), we have

\[
\begin{align*}
\dot{V}_1(e(t), \tilde{W}(t), z(t)) & \leq -\frac{\lambda(R_2)}{2} \|\xi_2(t)\|_2^2 - k_1 \|\tilde{W}(t)\|_F^2 + k_4 \\
& = -\frac{\lambda(R_2)}{2} \|e(t)\|_2^2 - \frac{\lambda(R_2)}{2} \|z(t)\|_2^2 - k_1 \|\tilde{W}(t)\|_F^2 + k_4.
\end{align*}
\]

(4.81)

Note that (4.81) implies \(\dot{V}_1(e(t), \tilde{W}(t), z(t)) \leq 0\) outside the compact set given by

\[
S_2 = \left\{ (e(t), \tilde{W}(t), z(t)) : \|e(t)\|_2 \leq \sqrt{\frac{2k_4}{\lambda(R_2)}}, \|\tilde{W}(t)\|_2 \leq \sqrt{\frac{2k_4}{\lambda(R_2)}} \right\} \cap \left\{ (e(t), \tilde{W}(t), z(t)) : \|z(t)\|_2 \leq \sqrt{\frac{2k_4}{\lambda(R_2)}} \right\} \cap \left\{ (e(t), \tilde{W}(t), z(t)) : \|\tilde{W}(t)\|_2 \leq \sqrt{\frac{k_4}{k_1}} \right\}.
\]
As a consequence, the evolution of $\mathcal{V}_1(e(t), \dot{W}(t), z(t))$ is upper bounded by

$$
\mathcal{V}_1(e(t), \dot{W}(t), z(t)) \leq \max_{(e(t), \dot{W}(t), z(t)) \in S} \mathcal{V}_1(\cdot) = \mathcal{X}(\mathcal{P}) \frac{2k_4}{\lambda(R_2^2)} + \alpha_2 \mathcal{X}(S) \frac{2k_4}{\lambda(R_2^2)} + \gamma^{-1} k_4 k_1,
$$

(4.82)

since $\mathcal{V}_1(e(t), \dot{W}(t), z(t))$ cannot grow outside $S_2$; hence, this conclusion then yields to the uniform ultimate boundedness of the solution $(e(t), \dot{W}(t), z(t))$ [7, 57].

**Remark 4.3.2** As also noted in Remark 4.2.6, the main difference between Assumption 4.2.4 and Assumption 4.2.5 is that the latter has an additional nonnegative matrix $2\alpha_2 \lambda \mu P_m B_0^T B_0 P_m$ that results from the utilized robustifying term $-\mu B_0^T B_0 P_m e(t)$ in the proposed adaptive feedback control law given by (4.29). As demonstrated in the next section, this nonnegative term in Assumption 4.2.5 can be useful in cases when Assumption 4.2.4 does not hold.

### 4.4 Illustrative Numerical Examples

In this section, once again, we consider the uncertain coupled mechanical system subject to actuator and unmodeled dynamics (Figure 4.2) given by (4.53) and (4.54). Here, we consider $k_1 = 5.5$N/m and $k_2 = 12$N/m for the spring constants; $c_1 = 0.1$Ns/m and $c_2 = 0.95$Ns/m for the damping constants; $m_0 = 0.1$kg, $M_1 = 0.8$kg, and $M_2 = 8$kg for the masses; $l = 0.2$m for the inverted pendulum length; and $g = 9.81$m/s$^2$ for the acceleration due to gravity. In addition, we choose $\lambda = 60$ for the actuator bandwidth, $\gamma = 0.6$ for the learning rate, and $c = 1.5$m for the command. The projection bound is also selected as $\dot{W}_{\max} = |W| + 0.2$. Moreover, the controller gain matrix $K_1$ is obtained using a linear quadratic regulator-based design Section 3.3 of [117] with the weighting matrices $Q = \text{diag}[1, 1, 100, 10]$ to penalize the state vector and $R = 0.55$ to penalize the control input as $K_1 = [-41.6511, -5.9568, -13.4840, -10.4121]$. The gain $K_2$ is also selected using a pre-filter design as $K_2 = -(E(A - BK_1)^{-1}B)^{-1} = -13.4840$ with $E = [0, 0, 1, 0]$ such that the translational position command for the cart is aimed to be followed.

First, we note that the sufficient stability conditions for the standard MRAC architecture given by Assumption 4.2.1, for the hedging-based MRAC architecture given by Assumption 4.2.2, and for the expanded MRAC architecture given by Assumption 4.2.4 could not be able to satisfied\(^{55}\). Figures 4.3, 4.4,

\(^{55}\)We have used various range combinations for $\beta$, $\varepsilon$, $\alpha_1$, $\alpha_2$, and $\alpha_3$. However, none of these selections satisfied Assumptions 4.2.1, 4.2.2, and 4.2.4 (e.g., an interested reader can try $\beta = 3.16$, $\varepsilon = 3.4$, $\alpha_1 = 1$, $\alpha_2 = 0.1$, and $\alpha_3 = 10^{-5}$). Note also that the
The first sufficient stability condition for the expanded and proposed MRAC architectures given by Assumption 4.2.3 is satisfied with same parameters given above (i.e., the related linear matrix inequality results in a feasible solution $\mathcal{P}$) while the second sufficient stability condition for the expanded MRAC architecture given by Assumption 4.2.4 results one negative eigenvalue of $F_e$; that is, $\text{spec}(F_e) = \{-58.8524, 0.4143, 0.0592, 0.0011, 3.5225 \times 10^{-6}\}$.
Figure 4.5: Unstable time responses for the expanded MRAC architecture given in Section 4.3.3.

Figure 4.6: Stable time responses for the proposed MRAC architecture given in Section 4.3.4.
and 4.5 illustrate the unstable time responses of all these architectures. Next, the second stability condition given by (4.31) of Assumption 4.2.5 is satisfied for the proposed, relaxed expanded MRAC architecture with parameters $\beta = 3.16$, $\varepsilon = 3.4$, $\alpha_1 = 1$, $\alpha_2 = 0.1$, and $\alpha_3 = 10^{-6}$ as well as $\mu = 100$ here\footnote{With the effect of the robustifying term, all eigenvalues of $\mathcal{F}_{re}$ are now positive; that is, $\text{spec}(\mathcal{F}_{re}) = \{ 77.8158, 0.1106, 0.0174, 0.0003, 2.2309 \times 10^{-8} \}$}. Figure 4.6 illustrates the stable time response of this result. To summarize, the boundedness of the closed-loop system trajectories is guaranteed for the proposed MRAC architecture.

Finally, Figure 4.7 presents the performance of the proposed MRAC architecture in Figure 4.6 with different levels of measurement noise $w(t)$ added to the measured state vector $x(t)$ (i.e., $x(t) + w(t)$ is used for feedback). The first, second, and third columns in Figure 4.7 respectively show the cases with low, moderate, and high levels of measurement noise, where the proposed controller remains robust in all cases. As expected, the control signal’s magnitude gets larger proportional to the increase in measurement noise levels. We also observe that the proposed controller achieves a similar cart position following performance in all cases.
4.5 Conclusions

Using tools and methods from Lyapunov stability theory, linear matrix inequalities, and matrix inequalities, we studied the analysis and synthesis of multiple adaptive architectures for control of physical systems with not only uncertainties but also actuator and unmodeled dynamics. First, we revealed sufficient stability conditions for the boundedness of closed-loop system trajectories for the standard MRAC, the hedging-based MRAC, and the expanded MRAC methods. Then, we synthesized a robustifying term for the latter method and analytically proved that this term can yield to a relaxed sufficient stability condition. In addition to our system-theoretical contributions (Theorems 4.3.1, 4.3.2, 4.3.3, and 4.3.4), several illustrative numerical examples demonstrated our results (Examples 4.3.1 and 4.3.2 as well as the study in Section 4.4). A recommended future research direction can focus on applications of our findings to real-world uncertain experimental platforms, where actuator and unmodeled dynamics are both significantly present.
Chapter 5: Distributed Adaptive Control of Uncertain Multiagent Systems with Coupled Dynamics

A major research area in the multiagent systems field is the development of distributed control architectures such that agent teams perform given tasks through local interactions. However, considering a wide array of critical civilian and military applications, each agent can be subject to system uncertainties (e.g., unknown parameters in agent dynamics due to modeling errors and/or structural damages due to adverse conditions of the environments multiagent systems operate in) and coupled dynamics (e.g., flexible dynamics as in lightweight agents and/or flexible appendages as in freight carrying operations). Motivated by this standpoint, this paper studies distributed adaptive architectures for controlling uncertain multiagent systems with unmeasurable coupled dynamics. Specifically, our first contribution is to analyze a standard distributed adaptive control method with system uncertainties and coupled dynamics in a leader-follower setting, where we develop local stability conditions. Our second contribution is to propose an additional feedback term within the control signal of each agent in order to relax the aforementioned local stability conditions. Finally, an illustrative numerical example is given in order to demonstrate our theoretical contributions.

5.1 Introduction

5.1.1 Background and Contributions

Teams of agents (e.g., aerial, ground, water, and underwater vehicles) that exchange information within each other to cooperatively accomplish given tasks is called multiagent systems. Considering a wide array of critical civilian and military applications of multiagent systems, a major research area is the development of distributed control architectures such that agent teams perform given tasks through local interactions (e.g., see [41–43] and references therein). However, each agent can be subject to system uncertainties and coupled dynamics in real-world applications. With regard to system uncertainties, they can result from unknown parameters in agent dynamics due to modeling errors and/or structural damages due

57 A version of this chapter has been submitted to the IEEE American Control Conference for possible publication.
to adverse conditions of the environments multiagent systems operate in. With regard to coupled dynamics, in addition, they can result from flexible dynamics as in lightweight agents and/or flexible appendages as in freight carrying operations.

In the multiagent systems literature, there exists a considerable research effort toward controlling uncertain multiagent systems through system-theoretic adaptive or robust control methods. Since adaptive control methods have the ability to cope with the effects of system uncertainties in an online fashion and require less modeling information as compared to robust control methods, the authors of, for example, [12, 13, 44–50] (also see references therein) utilize these methods. The common denominator of these results is that they do not consider the presence of coupled dynamics in agents. However, as it is well-known in the adaptive control field, the effect of coupled dynamics can yield to unstable system responses (e.g., see [1, 17, 18] and references therein). Therefore, there is a scientific gap on developing stability conditions for adaptive control of multiagent systems in the presence of not only system uncertainties but also coupled dynamics.

The purpose of this paper is to address this fundamental gap, where we study distributed adaptive architectures for controlling uncertain multiagent systems with unmeasurable coupled dynamics (see Section 5.2 for problem formulation). Specifically, our first contribution, which is presented in Section 5.3, is to analyze a standard distributed adaptive control method with system uncertainties and coupled dynamics in a leader-follower setting, where we develop local stability conditions that guarantee boundedness of the closed-loop trajectories of the overall multiagent system. Our second contribution, which is presented in Section 5.4, is to propose an additional feedback term within the control signal of each agent in order to relax the aforementioned local stability conditions. In both Sections 5.3 and 5.4, we use Lyapunov stability theory for our system-theoretic analyzes. Finally, an illustrative numerical example is also given in Section 5.5 for demonstrating the theoretical contributions of this paper. While a leader-follower setting is adopted to present our research results, we note that these results can be readily applied to other classes of multiagent system problems including but not limited to consensus algorithms, formation algorithms, and containment algorithms.

5.1.2 Notation and Mathematical Preliminaries

For the notation used in this paper, we refer to, for example, [50], since it is fairly standard. We now recall necessary basic definitions from graph theory, where we refer to, for example, [43, 118] for
details. In particular, an undirected graph $G$ is defined by a set of nodes $V_G = \{1, \ldots, N\}$ and a set of edges $E_G \subset V_G \times V_G$. When $(i, j) \in E_G$, the nodes $i$ and $j$ are said to be neighbors and neighboring relation is denoted by $i \sim j$. A path $i_0 i_1 \ldots i_L$ is a nodes' finite sequence such that $i_{k-1} \sim i_k$, $k = 1, \ldots, L$ and a graph $G$ is said to be connected when there is a path between any pair of distinct nodes. The degree of a node $d_i$ is given by the number of its neighbors, where the degree matrix of a graph $G$, $D(G) \in \mathbb{R}^{N \times N}$, is defined by $D(G) \triangleq \text{diag}(d)$ with $d = [d_1, \ldots, d_N]^T$. In addition, $A(G) \in \mathbb{R}^{N \times N}$ denotes the adjacency matrix of a graph $G$ and is defined by $[A(G)]_{ij} = 1$ when $(i, j) \in E_G$ and $[A(G)]_{ij} = 0$ otherwise. Finally, $L(G) \triangleq D(G) - A(G)$ denotes the Laplacian matrix of a graph. Throughout this paper, we consider a connected and undirected graph $G$ and utilize the following lemma from, for example, [43] and [119].

**Lemma 5.1.1** Consider a connected and undirected graph $G$. The spectrum of its corresponding Laplacian matrix can be ordered as $0 = \lambda_1(L(G)) < \lambda_2(L(G)) \leq \ldots \leq \lambda_N(L(G))$ with $1_n$ being the eigenvector corresponding to the zero eigenvalue $\lambda_1(L(G))$ and $L(G)1_N = 0_N$. In addition, consider $K = \text{diag}(k)$, $k = [k_1, k_2, \ldots, k_N]^T$, $k_i \in \mathbb{Z}_+$, $i = 1, \ldots, N$, and let at least one element of $k$ is nonzero. Then, $F(G) \triangleq L(G) + K \in \mathbb{R}^{N \times N}$ is a positive-definite matrix (i.e., $-F(G)$ is Hurwitz) and $F(G)1_N = K1_N$.

Next, we utilize the projection operator from, for example, [57] and Exercise 11.3 of [7] for the results of this paper. Specifically, let $\Omega = \{\theta \in \mathbb{R}^n : (\theta^\text{min} \leq \theta_i \leq \theta^\text{max})_{i=1,2,\ldots,n}\}$, $\Omega \subset \mathbb{R}^n$ be a convex hypercube with $\theta^\text{min}$ and $\theta^\text{max}$ respectively denoting the minimum and maximum bounds for the $i$th component of the parameter vector $\theta \in \mathbb{R}^n$ (without loss of generality, we consider $\theta^\text{min} = -\theta^\text{max}$ for our results). In addition, for a sufficiently small constant $\epsilon_0 \in \mathbb{R}_+$, let $\Omega_{\epsilon_0} = \{\theta \in \mathbb{R}^n : (\theta^\text{min} + \epsilon_0 \leq \theta_i \leq \theta^\text{max} - \epsilon_0)_{i=1,2,\ldots,n}\}$ be another convex hypercube (i.e., $\Omega_{\epsilon_0} \subset \Omega$). One can then define the component-wise projection operator $\text{Proj} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ as $\text{Proj}(\theta, y) = (\theta_i^\text{max} - \theta_i)y_i/\epsilon_0$ when $\theta_i > \theta_i^\text{max} - \epsilon_0$ and $y_i > 0$, $\text{Proj}(\theta, y) = (\theta_i - \theta_i^\text{min})y_i/\epsilon_0$ when $\theta_i < \theta_i^\text{min} + \epsilon_0$ and $y_i < 0$, and $\text{Proj}(\theta, y) = y_i$ otherwise, where $y \in \mathbb{R}^n$. As a consequence of this definition, we have $(\theta - \theta^*)^T(\text{Proj}(\theta, y) - y) \leq 0$, where $\theta^* \in \Omega_{\epsilon_0}$ (e.g., see [74]). One can also generalize the projection operator definition to matrices using $\text{Proj}_{m}(\Theta, Y) = (\text{Proj}(\text{col}_1(\Theta), \text{col}_1(Y)), \ldots, \text{Proj}(\text{col}_m(\Theta), \text{col}_m(Y)))$ that gives $\text{tr} \left[((\Theta - \Theta^*)^T(\text{Proj}_{m}(\Theta, Y) - Y)\right] = \sum_{i=1}^m [\text{tr}(\text{col}_i(\Theta - \Theta^*)^T(\text{Proj}(\text{col}_i(\Theta), \text{col}_i(Y))) - \text{col}_i(Y))] \leq 0$ with $n \times m$ dimensional matrices $Y$, $\Theta$, and $\Theta^*$.  

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5.2 Problem Formulation

This paper focuses on control of uncertain multiagent systems with unmeasurable coupled dynamics. To this end, consider a multiagent system consisting of \(N\) agents:

\[
\dot{x}_i(t) = Ax_i(t) + B[u_i(t) + W_i^T \sigma_i(x_i(t)) + H_{ui} \xi_i(t)], \quad x_i(0) = x_{i0},
\]

\[
y_i(t) = Cx_i(t), \tag{5.1}
\]

\[
\dot{\xi}_i(t) = F_{ui} \xi_i(t) + G_{ui}x_i(t), \quad \xi_i(0) = \xi_{i0},
\]

\(i = 1, \ldots, N\), where \(A \in \mathbb{R}^{n \times n}\) is a system matrix, \(B \in \mathbb{R}^{n \times m}\) is an input matrix, \(C \in \mathbb{R}^{p \times n}\) is an output matrix, \(x_i(t) \in \mathbb{R}^n\) is a state vector of agent \(i\) that is only available to this agent (i.e., not to its neighbors), \(y_i(t) \in \mathbb{R}^p\) is an output vector of agent \(i\) that is shared with its neighbors locally through a connected and undirected graph \(G\), \(u_i(t) \in \mathbb{R}^m\) is a control signal of agent \(i\), \(W_i \in \mathbb{R}^{s \times m}\) is an unknown weight matrix of agent \(i\) capturing uncertain parameters in the dynamics of this agent, and \(\sigma_i(x_i(t)) \in \mathbb{R}^n \to \mathbb{R}^s\) is a basis function of agent \(i\) composed of locally Lipschitz functions. Here, (5.3) refers to coupled dynamics of agent \(i\), where \(\xi_i(t) \in \mathbb{R}^q\) is an unmeasurable state vector of coupled dynamics and \(F_{ui} \in \mathbb{R}^{q \times q}\), \(G_{ui} \in \mathbb{R}^{q \times n}\), and \(H_{ui} \in \mathbb{R}^{m \times q}\) are matrices related to coupled dynamics. As standard, we consider that \(F_{ui}\) is Hurwitz such that there exists \(S_i \in \mathbb{R}^{q \times q}_+\) satisfying the Lyapunov equation given by \(0 = F_{ui}^T S_i + S_i F_{ui} + I_q\).

We are now ready to state the two objectives of this paper:

- For (5.1), (5.2), and (5.3), consider first a standard distributed adaptive control method (Section 5.3.1) and analyze this method in order to understand the effect of coupled dynamics on overall system stability (Section 5.3.2). Specifically, we consider a distributed adaptive control algorithm in a leader-follower setting and develop local stability conditions that guarantee boundedness of the closed-loop system trajectories.

- Building on the results of Section 5.3.2, we next propose an additional feedback term within the control signal of each agent (Section 5.4). Specifically, we show relax local stability conditions with the proposed architecture, where they can be satisfied agent-wise by judiciously selecting the resulting design parameters.
5.3 Analysis of a Standard Distributed Adaptive Control Method

5.3.1 Standard Distributed Adaptive Control in the Absence of Coupled Dynamics

In this section, we overview a key research result on standard distributed adaptive control method in the absence of coupled dynamics. In this case, we have

\[
\begin{align*}
\dot{x}_i(t) &= Ax_i(t) + Bu_i(t) + W_i^T \sigma_i(x_i(t)), & x_i(0) = x_{i0}, \\
y_i(t) &= Cx_i(t),
\end{align*}
\]

(5.4)

\(i = 1, \ldots, N\). To drive the outputs of all agents to a given command input only available to leader agents, one can consider the standard distributed adaptive control method given by

\[
\begin{align*}
u_i(t) &= -K_1 x_i(t) + K_2 z_i(t) - \hat{W}_i^T(t) \sigma_i(x_i(t)), \\
\dot{z}_i(t) &= -\sum_{j \sim i} (y_i(t) - y_{ri}(t)) - k_i (y_i(t) - c), & z_i(0) = z_{i0},
\end{align*}
\]

(5.5)

where \(z_i(t) \in \mathbb{R}^p\) is an integral state vector, \(y_{ri}(t) \triangleq Cx_{ri}(t)\) is a reference output vector (details below), \(c \in \mathbb{R}^p\) is a constant command input, and \(k_i \in \{0, 1\}\) denotes whether the agent is a leader or a follower; that is, \(k_i = 0\) for follower agents and \(k_i = 1\) for leader agents. With regard to (5.6) and (5.7), we refer to, for example, [50]. In (5.7), in addition, \(K_1 \in \mathbb{R}^{m \times n}\) and \(K_2 \in \mathbb{R}^{m \times p}\) are feedback control gain matrices and \(\hat{W}_i(t) \in \mathbb{R}^{s \times m}\) is an estimate of unknown parameter in \(W_i\) that satisfies local (i.e., agent-wise) update law

\[
\dot{\hat{W}}_i(t) = \alpha_i \text{ Proj} [\hat{W}_i(t), \sigma_i(x_i(t)) \tilde{q}_i(t)] P_i G
\]

(5.8)

with \(G \triangleq [B^T, 0_{m \times p}^T] \in \mathbb{R}^{(n+p) \times m}\), \(\alpha_i \in \mathbb{R}_+\) being the learning rate gain for each agent \(i\), and

\[
\tilde{q}_i(t) \triangleq \begin{bmatrix} \tilde{x}_i(t) \\ \tilde{z}_i(t) \end{bmatrix} = \begin{bmatrix} x_i(t) \\ z_i(t) \end{bmatrix} - \begin{bmatrix} x_{ri}(t) \\ z_{ri}(t) \end{bmatrix},
\]

(5.9)

being the augmented agent error state and details below for both \(x_{ri}(t)\) and \(z_{ri}(t)\). Considering (5.8), the projection bounds are defined as \(|\hat{W}_i(t)| \leq \hat{W}_{i,\text{max}}\). Moreover, \(P_i \in \mathbb{R}_+^{(n+p) \times (n+p)}\) in (5.8) satisfies the local Lyapunov equation given by

\[
0 = \mathcal{H}_i^T P_i + P_i \mathcal{H}_i + I_{n+p}
\]

with
\( \mathcal{H}_{i1} \triangleq \begin{bmatrix} A - BK_1 & BK_2 \\ -(d_i + k_i)C & 0_{p \times p} \end{bmatrix}, \) (5.10)

when the following assumption holds (e.g., see [50]).

**Assumption 5.3.1** \( \mathcal{H}_{i1} \) is Hurwitz for \( i = 1, \ldots, N \).

We now present the reference model of agents, which is utilized in (5.7) and (5.9), given by

\[
\begin{align*}
\dot{x}_i(t) &= A_{r}x_i(t) + B_{r}z_i(t), \quad x_i(0) = x_{i0}, \\
\dot{z}_i(t) &= -\sum_{i\neq j}(y_i(t) - y_j(t)) - k_i(y_i(t) - c), \quad z_i(0) = z_{i0}, \\
y_i(t) &= Cx_i(t),
\end{align*}
\]

(5.11), (5.12), and (5.13) \( x_i(t) \in \mathbb{R}^n \) is the reference state vector, \( z_i(t) \in \mathbb{R}^p \) is the reference integral state vector, \( y_i(t) \in \mathbb{R}^p \) is the reference output vector, and \( A_r \triangleq A - BK_1 \in \mathbb{R}^{n \times n} \) and \( B_r \triangleq BK_2 \in \mathbb{R}^{n \times p} \). For the next assumption, which is related to the stability of the reference model of agents, we define

\[
\mathcal{H}_{2i} \triangleq \begin{bmatrix} A - BK_1 & BK_2 \\ -\rho_i C & 0_{p \times p} \end{bmatrix}, \quad (5.14)
\]

where \( \rho_i \in \text{spec}(\mathcal{F}(\mathbf{G})) \).

**Assumption 5.3.2** \( \mathcal{H}_{2i} \) is Hurwitz for \( i = 1, 2, \ldots, N \).

Before we proceed further, we present the following lemma on the stability of the reference model of agents given by (5.11), (5.12), and (5.13).

**Lemma 5.3.1** Consider the reference model given by (5.11), (5.12), and (5.13). If Assumptions 5.3.2 holds, then the pair \( (x_i(t), z_i(t)) \) is bounded and \( \lim_{t \to \infty} y_i(t) = c \).

**Proof.** Let \( x_r(t) \triangleq [x_{r1}(t), \ldots, x_{rN}(t)]^T \in \mathbb{R}^{Nn}, z_r(t) \triangleq [z_{r1}(t), \ldots, z_{rN}(t)]^T \in \mathbb{R}^{Np} \), and \( q_r \triangleq [x_r^T(t), z_r^T(t)]^T \in \mathbb{R}^{N(p+n)} \). We can then compactly write

\[
\dot{q}_r(t) = \begin{bmatrix} I_n \otimes (A - BK_1) & I_n \otimes BK_2 \\ -\mathcal{F}(\mathbf{G}) \otimes C & 0_{p \times p} \end{bmatrix} q_r(t) + \begin{bmatrix} 0_{Nn \times Np} \\ K \otimes I_p \end{bmatrix} p_c, \quad (5.15)
\]
where \( p_c \triangleq 1_N \otimes c \in \mathbb{R}^{Np} \). Now let \( \hat{q}_c(t) \triangleq q(t) + \tilde{A}^{-1} \hat{p}_c \) that yields \( \dot{\hat{q}}_c(t) = \tilde{A} \hat{q}_c(t) \). Assumption 5.3.2 implies that \( \tilde{A} \) is Hurwitz; hence, the pair \((x_r(t), z_r(t))\) is bounded and \( \lim_{t \to \infty} \hat{q}_c(t) = 0 \), or equivalently, \( \lim_{t \to \infty} (q(t) + \tilde{A}^{-1} \hat{p}_c) = 0 \). Note that the latter expression also yields \( \lim_{t \to \infty} (\tilde{A} q_r(t) + \tilde{B} \hat{p}_c) = 0 \). Based on the structures of \( \tilde{A} \) and \( \tilde{B} \), one can now write \( \lim_{t \to \infty} (\tilde{A}^{-1}(G) K \otimes I_{Np}) p_c = 0 \). Finally, based on Lemma 5.1.1, it follows that \( \lim_{t \to \infty} y_i(t) = c \). \( \square \)

The above lemma shows that the reference output vector of all agents converges to a given constant command input. Next, we show that the standard distributed adaptive control method given by (5.6), (5.7), and (5.8), which is predicated on the reference model of agents defined above, allows the output vector of all agents to approach to this command input. To this end, one can write the local error dynamics from (5.4), (5.7), (5.11), and (5.12) as

\[
\begin{align*}
\dot{x}_i(t) &= A_{x} x_i(t) + B_{x} \tilde{z}_i(t) - B \tilde{W}_i^T(t) \sigma_i(x_i(t)), \\
\dot{\tilde{z}}_i(t) &= -(d_i + k_i) C \tilde{x}_i(t),
\end{align*}
\]

where \( \tilde{W}_i(t) \triangleq \tilde{W}_i(t) - W_i \in \mathbb{R}^{s \times m} \). Using (5.9), these local error dynamics can now be written in the compact form

\[
\dot{\hat{q}}_i(t) = \mathcal{H}_i \hat{q}_i(t) - G \hat{W}_i^T(t) \sigma_i(x_i(t)).
\]

The next lemma is now immediate.

**Lemma 5.3.2** Consider the uncertain multiagent system given by (5.4) and (5.5). In addition, consider the reference model given by (5.11), (5.12), and (5.13). Then, the standard distributed adaptive control architecture given by (5.6), (5.7), and (5.8) subject to Assumptions 5.3.1 and 5.3.2 guarantees the boundedness of the pair \((\hat{q}(t), \hat{W}(t))\), \( \lim_{t \to \infty} \hat{q}(t) = 0 \), and \( \lim_{t \to \infty} y_i(t) = c \).

**Proof.** To show the boundedness of the pair \((\hat{q}(t), \hat{W}(t))\), consider

\[
\mathcal{V}_i(\hat{q}_i, \hat{W}_i) = \hat{q}_i^T P \hat{q}_i + \alpha_i^{-1} \text{tr} \hat{W}_i^T \hat{W}_i,
\]

where \( P = 1_N \otimes P \in \mathbb{R}^{Np \times Np} \) and \( \alpha_i \geq 0 \). Using (5.9), one can now write the local error dynamics from (5.4), (5.7), (5.11), and (5.12) as

\[
\begin{align*}
\dot{x}_i(t) &= A_{x} x_i(t) + B_{x} \tilde{z}_i(t) - B \tilde{W}_i^T(t) \sigma_i(x_i(t)), \\
\dot{\tilde{z}}_i(t) &= -(d_i + k_i) C \tilde{x}_i(t),
\end{align*}
\]

where \( \tilde{W}_i(t) \triangleq \tilde{W}_i(t) - W_i \in \mathbb{R}^{s \times m} \). Using (5.9), these local error dynamics can now be written in the compact form

\[
\dot{\hat{q}}_i(t) = \mathcal{H}_i \hat{q}_i(t) - G \hat{W}_i^T(t) \sigma_i(x_i(t)).
\]

The next lemma is now immediate.
and note that $V(0,0) = 0$, $\mathcal{V}_i(\tilde{q}_i, \tilde{W}_i) > 0$ for all $(\tilde{q}_i, \tilde{W}_i, \tilde{\xi}_i) \neq (0,0)$, and $\mathcal{V}_i(\tilde{q}_i, \tilde{W}_i)$ is radially unbounded. The time-derivative of (5.19) yields

$$
\dot{\mathcal{V}}_i(\tilde{q}_i(t), \tilde{W}_i(t)) = 2\tilde{q}_i^T(t)P_i[\mathcal{H}_i \tilde{q}_i(t) - G\tilde{W}_i^T(t)\sigma_i(x_i(t)))] + 2\text{tr}\tilde{W}_i^T(t)\text{Proj}[\tilde{W}_i(t), \sigma_i(x_i(t))\tilde{q}_i^T(t)P_iG].
$$

(5.20)

Using the Assumption 5.3.1 and the projection operator property $\text{tr}\tilde{W}_i^T(t)(\text{Proj}(\tilde{W}_i(t), \sigma_i(x_i(t))\tilde{q}_i^T(t)P_iG) - \sigma_i(x_i(t))\tilde{q}_i^T(t)P_iG) \leq 0$ in (5.20) yields

$$
\dot{\mathcal{V}}_i(\tilde{q}_i(t), \tilde{W}_i(t)) \leq -\tilde{q}_i^T(t)\tilde{q}_i(t).
$$

(5.21)

Based on (5.21), one can write the Lyapunov function candidate $\mathcal{V}(\tilde{q}, \tilde{W}) = \sum_{i=1}^{N} \mathcal{V}_i(\tilde{q}_i, \tilde{W}_i)$ with $\dot{\mathcal{V}}(\cdot) \leq -\sum_{i=1}^{N} \tilde{q}_i^T(t)\tilde{q}_i(t)$. Then, the pair $(\tilde{q}_i(t), \tilde{W}_i(t))$ is bounded for all $i = 1, \ldots, N$. Since the trajectories of the reference model is bounded by Lemma 5.3.1, it now follows from Barbalat’s lemma [57] that $\dot{\mathcal{V}}(\cdot) \to 0$ as $t \to \infty$; hence, $\lim_{t \to \infty} \tilde{q}_i(t) = 0$, or equivalently, $x_i(t) - x_r(t) \to 0$ as $t \to \infty$, $z_i(t) - z_r(t) \to 0$ as $t \to \infty$, and $y_i(t) - y_r(t) \to 0$ as $t \to \infty$. Finally, from Lemma 5.3.1, $\lim_{t \to \infty} y_i(t) = c$ is immediate.

**Remark 5.3.1** The purpose of using $y_i(t)$ in (5.7) as also adopted in [49] is to make the adaptive control design process decentralized as illustrated in the above lemma.

To summarize, the standard distributed adaptive control method overviewed above guarantees the boundedness of the closed-loop system trajectories and $\lim_{t \to \infty} y_i(t) = c$ in the absence of coupled dynamics. In the remainder of this section, we analyze this method in order to understand the effect of coupled dynamics on overall system stability.

### 5.3.2 Standard Distributed Adaptive Control Method in the Presence of Coupled Dynamics

Consider the uncertain multiagent system subject to coupled dynamics given by (5.1), (5.2), and (5.3). Consider also the standard distributed adaptive control method given by (5.6), (5.7), and (5.8) with the reference model given by (5.11), (5.12), and (5.13). One can then write the local error dynamics as

$$
\dot{x}_i(t) = A_ix_i(t) + B_i\tilde{z}_i(t) - BW_i^T(t)\sigma_i(x_i(t)) + BH_\mu\tilde{\xi}_i(t),
$$

(5.22)

$$
\dot{\tilde{z}}_i(t) = -(d_i + k_i)C\tilde{x}_i(t).
$$

(5.23)
Using (5.9), these local error dynamics and coupled dynamics can now be written in compact forms

\[
\begin{align*}
\dot{\bar{q}}_i(t) &= H_{i1} \bar{q}_i(t) - GW_i^T(t) \sigma_i(x_i(t)) + GH_{ai} \bar{\xi}_i(t) , \\
\dot{\bar{\xi}}_i(t) &= F_{ai} \bar{\xi}_i(t) + G_{ai} M[q_{ri}(t) + \bar{q}_i(t)],
\end{align*}
\]  

(5.24)

(5.25)

where \( M = [I_n, 0] \) and \( q_{ri}(t) \triangleq [x_{ri}^T(t), z_{ri}^T(t)]^T \).

In the next theorem, which presents the first contribution of this paper, we develop local stability conditions that guarantee boundedness of the closed-loop system trajectories with the standard distributed adaptive control method.

**Theorem 5.3.1** Consider the uncertain multiagent system given by (5.1), (5.2), and (5.3). In addition, consider the reference model given by (5.11), (5.12), and (5.13). Then, the standard distributed adaptive control architecture given by (5.6), (5.7), and (5.8) subject to Assumptions 5.3.1 and 5.3.2 guarantees the boundedness of the triple \((\bar{q}(t), \bar{W}(t), \bar{\xi}(t))\) when the local stability conditions for each agent given by

\[
\mathcal{R}_{si} = \begin{bmatrix} I_{n+p} & v_i \\ v_i^T & \beta_i^{-1} I_q \end{bmatrix} > 0,
\]  

(5.26)

hold, where \( \beta_i \in \mathbb{R}_+ \) and \( v_i = -P_i G_{ai} - \beta_i^{-1} M^T G_{ai}^T S_i \).

**Proof.** To show the boundedness of the triple \((\bar{q}(t), \bar{W}(t), \bar{\xi}(t))\), consider

\[
\mathcal{V}_i(\bar{q}_i, \bar{W}_i, \bar{\xi}_i) = \bar{q}_i^T P_i \bar{q}_i + \alpha_i^{-1} \text{tr} \bar{W}_i^T \bar{W}_i + \beta_i^{-1} \bar{\xi}_i^T \bar{\xi}_i,
\]  

(5.27)

and note that \( \mathcal{V}(0,0,0) = 0 \), \( \mathcal{V}_i(\bar{q}_i, \bar{W}_i, \bar{\xi}_i) > 0 \) for all \( (\bar{q}_i, \bar{W}_i, \bar{\xi}_i) \neq (0,0,0) \), and \( \mathcal{V}_i(\bar{q}_i, \bar{W}_i, \bar{\xi}_i) \) is radially unbounded. The time-derivative of (5.27) yields

\[
\dot{\mathcal{V}}_i(\bar{q}_i(t), \bar{W}_i(t), \bar{\xi}_i(t))
\]

\[
= 2\bar{q}_i^T(t) P_i \left[ H_{i1} \bar{q}_i(t) - GW_i^T(t) \sigma_i(x_i(t)) + GH_{ai} \bar{\xi}_i(t) \right] + 2 \text{tr} \bar{W}_i^T(t) \text{Proj} \left[ \bar{W}_i(t), \sigma_i(x_i(t)) \bar{q}_i(t) \right] P_i G \]

\[
+ 2\bar{q}_i^T(t) P_i G_{ai} \bar{\xi}_i(t) + 2 \beta_i^{-1} \bar{\xi}_i^T(t) [F_{ai} \bar{\xi}_i(t) + G_{ai} M[q_{ri}(t) + \bar{q}_i(t)]] .
\]  

(5.28)
Using Assumption 5.3.1 and the projection operator property $\text{tr} \tilde{W}_i^T(t) (\text{Proj}(\tilde{W}_i(t), \sigma_i(x_i(t)) \tilde{q}_i^T(t)) P G) \leq 0$ in (5.28) yields

$$V_i(\tilde{q}_i(t), \tilde{W}_i(t), \zeta_i(t)) \leq -\tilde{q}_i^T(t) \tilde{q}_i(t) - \beta_i^{-1} \tilde{\xi}_i^T(t) \tilde{\xi}_i(t) + 2\tilde{q}_i^T(t) P_i G H_{ui} \tilde{\xi}_i(t) + 2\beta_i^{-1} \tilde{\xi}_i^T(t) S_i G_{ui} M \tilde{q}_i(t)$$

$$= -\tilde{\xi}_i^T(t) \mathcal{R}_{si} \tilde{\xi}_i(t) + 2\beta_i^{-1} \tilde{\xi}_i^T(t) S_i G_{ui} M \tilde{q}_i(t), \quad (5.29)$$

where $\zeta_i(t) \triangleq [\tilde{q}_i^T(t), \tilde{\xi}_i^T(t)]^T$. Based on (5.29), one can write the Lyapunov function candidate $V(\tilde{q}, \tilde{W}, \tilde{\xi}) = \sum_{i=1}^N V_i(\tilde{q}_i, \tilde{W}_i, \tilde{\xi}_i)$. Since $\mathcal{R}_{si}$ is positive definite by the statement of the this theorem, it now follows that the triple $(\tilde{q}_i(t), \tilde{W}_i(t), \tilde{\xi}_i(t))$ is bounded for all $i = 1, \ldots, N$. ■

Note that the local stability conditions given by (5.26) may not always hold (details in Remark 5.4.1). Motivated by this standpoint, the next section proposes an additional feedback term, a robustifying term, within the control signal of each agent in order to relax these agent-wise conditions.

### 5.4 Proposed Distributed Adaptive Control Method

Building on the results of previous section, consider the proposed distributed adaptive control method given by

$$u_i(t) = -K_1 x_i(t) + K_2 z_i(t) - \tilde{W}_i^T(t) \sigma_i(x_i(t)) - \mu_i G^T P_i \tilde{q}_i(t), \quad \mu_i \triangleq \frac{\beta_i^2}{2} \quad (5.30)$$

$$\dot{z}_i(t) = -\sum_{i \neq j} (y_i(t) - y_j(t)) - k_i (y_i(t) - c), \quad z_i(0) = z_0, \quad (5.31)$$

with (5.8). In (5.30), “$\mu_i G^T P_i \tilde{q}_i(t) \in \mathbb{R}^n$” is a robustifying term\(^{58}\) for each agent with $\beta_i \in \mathbb{R}_+$ being a design parameter. Now, using (5.30) in (5.1) with (5.11), one can write

$$\dot{x}_i(t) = A_i x_i(t) + B_i z_i(t) - B \tilde{W}_i^T(t) \sigma_i(x_i(t)) + BH_{ui} \tilde{\xi}_i(t). \quad (5.32)$$

\(^{58}\)The structure of this term is inspired by the results in, for example, Section 3 of [1]. Note that the robustifying term utilized in this paper is a fixed-gain controller in contrast to Section 3 of [1]. Note also that our control method here is different than the one in [1], since we here focus on multiagent systems whereas this is not considered in [1].
In addition, using (5.23), (5.12) with (5.32), these local error dynamics can now be written in the compact form

\[
\dot{\tilde{q}}_i(t) = \mathcal{H}_i \tilde{q}_i(t) - G \dot{W}_i^T(t) \sigma_i(x_i(t)) + GH_u \xi_i(t) - \mu_i G G^T P \tilde{q}_i(t).
\] (5.33)

Comparing (5.33) with (5.24), notice that “\(\mu_i G G^T P \tilde{q}_i(t)\)” arises based on the introduced robustifying terms in the control signals of agents. In the next theorem, which presents the second and the main contribution of this paper, we show relax local stability conditions (details in Remark 5.4.1).

**Theorem 5.4.1** Consider the uncertain multiagent system given by (5.1), (5.2), and (5.3). In addition, consider the reference model given by (5.11), (5.12), and (5.13). Then, the proposed adaptive control architecture given by (5.30), (5.31), and (5.8) subject to Assumption 5.3.1 and 5.3.2 guarantees the boundedness of the triple \((\tilde{q}(t), \dot{W}(t), \xi(t))\) when the local stability conditions for each agent given by

\[
\mathcal{R}_{pi} = I_i - \beta_i^{-1}(H_u^T H_u + 2S_i G u M M^T G_u^T S_i) > 0,
\] (5.34)

hold. Finally, there always exist \(\beta_i\) for each agent \(i = 1, \ldots, N\) that satisfies (5.34).

**Proof.** To show the boundedness of the triple \((\tilde{q}(t), \dot{W}(t), \xi(t))\), consider (5.27). Then, the time-derivative of (5.27) yields

\[
\mathcal{V}_i(\tilde{q}_i(t), \dot{W}_i(t), \xi_i(t)) = 2\tilde{q}^T_i(t) P_i [\mathcal{H}_i \tilde{q}_i(t) - G \dot{W}_i^T(t) \sigma_i(x_i(t)) + GH_u \xi_i(t)]
\]

\[
-2\mu_i \tilde{q}^T_i(t) P_i G G^T P_\tilde{q}_i(t) + 2\text{tr}\dot{W}_i^T(t) \text{Proj}[\dot{W}_i(t), \sigma_i(x_i(t)) \tilde{q}^T_i(t) P_i G]
\]

\[
+2\tilde{q}^T_i(t) P_i G H_u \xi_i(t) + 2\beta_i^{-1} \xi^T_i(t) [F_u \xi_i(t) + G_u M[q_r(t) + \tilde{q}_i(t)]]
\] (5.35)

Using the Assumption 5.3.1 and the projection operator property \(\text{tr}\dot{W}_i^T(t) (\text{Proj}[\dot{W}_i(t), \sigma_i(x_i(t)) \tilde{q}_i^T(t) P_i G] - \sigma_i(x_i(t)) \tilde{q}_i^T(t) P_i G) \leq 0\) in (5.35) gives

\[
\mathcal{V}_i(\tilde{q}_i(t), \dot{W}_i(t), \xi_i(t))
\]

\[
\leq -\tilde{q}^T_i(t) \tilde{q}_i(t) - \beta_i^{-1} \xi^T_i(t) \xi_i(t) - 2\mu_i \tilde{q}^T_i(t) P_i G G^T P_\tilde{q}_i(t) + 2\tilde{q}^T_i(t) P_i G H_u \xi_i(t)
\]

\[
+2\beta_i^{-1} \xi^T_i(t) S_i G_u M \tilde{q}_i(t) + 2\beta_i^{-1} \xi^T_i(t) S_i G_u M q_r(t).
\] (5.36)
In addition, using the Young’s inequality in the third and fourth lines of (5.36) yields

\[
2\tilde{q}_i(t)P_iGH_{ui}\xi_i(t) \leq \gamma_i\tilde{q}_i(t)P_iGG^TP_i\tilde{q}_i(t) + \gamma_i^{-1}\xi_i(t)H_{ui}^TH_{ui}\xi_i(t),
\]

(5.37)

\[
2\beta_i^{-1}\xi_i(t)S_iG_{ui}M\tilde{q}_i(t) \leq \phi_i\beta_i^{-1}\tilde{q}_i(t)\tilde{q}_i(t) + \phi_i^{-1}\beta_i^{-1}\xi_i(t)S_iG_{ui}MM^TG_{ui}S_i\xi_i(t),
\]

(5.38)

where \(\gamma_i \in \mathbb{R}_+\) and \(\phi_i \in \mathbb{R}_+\). Using (5.37) and (5.38) in (5.36), we now write

\[
\dot{V}_i(\tilde{q}_i(t), \tilde{W}_i(t), \tilde{\xi}_i(t)) \leq -\tilde{q}_i(t)(I_{n+p} - \gamma_iP_iGG^TP_i + 2\mu_iP_iGG^TP_i - \phi_i\beta_i^{-1}I_{n+p})\tilde{q}_i(t)
\]

\[
-\tilde{\xi}_i(t)(\beta_i^{-1}I_n - \gamma_i^{-1}H_{ui}^TH_{ui} - \phi_i^{-1}\beta_i^{-1}S_iG_{ui}MM^TG_{ui}S_i)\tilde{\xi}_i(t)
\]

\[
+2\beta_i^{-1}\xi_i(t)S_iG_{ui}Mq_i(t).
\]

(5.39)

Letting \(\gamma_i = \beta_i^2\) and \(\phi_i = 0.5\beta_i\), and with using the definition of the robustifying term \(\mu_i\) given in (5.30), one can rewrite (5.39) in the form given by

\[
\dot{V}_i(\tilde{q}_i(t), \tilde{W}_i(t), \tilde{\xi}_i(t)) \leq -0.5\tilde{q}_i(t)\tilde{q}_i(t) - \beta_i^{-1}\tilde{\xi}_i(t)\tilde{\xi}_i(t)I_n - \beta_i^{-1}(H_{ui}^TH_{ui} + 2\mu_iG_{ui}MM^TG_{ui}S_i)\tilde{\xi}_i(t)
\]

\[
+2\beta_i^{-1}\xi_i(t)S_iG_{ui}Mq_i(t)
\]

\[
\leq -0.5\tilde{q}_i(t)\tilde{q}_i(t) - \beta_i^{-1}\tilde{\xi}_i(t)\tilde{\xi}_i(t)R_{pi}\tilde{\xi}_i(t) + 2\beta_i^{-1}\xi_i(t)S_iG_{ui}Mq_i(t).
\]

(5.40)

Based on (5.40), once again, one can write the Lyapunov function candidate \(V(\tilde{q}, \tilde{W}, \tilde{\xi}) = \sum_{i=1}^N V_i(\tilde{q}_i, \tilde{W}_i, \tilde{\xi}_i)\). Since \(R_{pi}\) is positive by the statement of this theorem, it now follows that the triple \((\tilde{q}_i(t), \tilde{W}_i(t), \tilde{\xi}_i(t))\) is bounded for all \(i = 1, \ldots, N\). Finally, from the structure of \(R_{pi}\) given by (5.34), it can be readily seen that it is always possible to find a large enough \(\beta_i\) to ensure positive-definiteness of \(R_{pi}\).

\[\blacksquare\]

**Remark 5.4.1** We now compare the agent-wise local stability conditions given by (5.26) and (5.34), where the former ensures the boundedness of the standard distributed adaptive control method and the latter ensures the boundedness of the proposed distributed adaptive control method. For checking the positive-definiteness of (5.26), one can equivalently study, for example, the positive-definiteness of its Schur complement given by

\[
I_{n+p} - \beta_i[P_iGH_{ui} + \beta_i^{-1}M^TG_{ui}S_i][P_iGH_{ui} + \beta_i^{-1}M^TG_{ui}S_i]^T.
\]

(5.41)
While $\beta_i$ is the positive free design parameter in this expression, it can be readily seen that one cannot always find a range for $\beta_i$ to ensure its positive-definiteness. In contrast, as shown in the above theorem, there always exist $\beta_i$ to ensure positive-definiteness of (5.34). From this standpoint, the proposed method of this section predicated on the robustifying terms for agents provides relaxed agent-wise local stability conditions.

5.5 Illustrative Numerical Example

For illustrating the results in Theorems 5.3.1 and 5.4.1, where the former is for the standard distributed adaptive control method and the latter is for the proposed distributed adaptive control method, consider a group of 5 agents on a path graph satisfying (5.1), (5.2), and (5.3) with the matrices given by

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad (5.42)$$

$$F_{u1} = \begin{bmatrix} -0.15 & -1 \\ 2 & 0 \end{bmatrix}, \quad G_{u1} = \begin{bmatrix} 0.4 & 0 \\ 0 & 0 \end{bmatrix}, \quad H_{u1} = \begin{bmatrix} 0 & 5.5 \end{bmatrix}, \quad (5.43)$$

$$F_{u2} = \begin{bmatrix} -0.1 & -1 \\ 1 & 0 \end{bmatrix}, \quad G_{u2} = \begin{bmatrix} 0.4 & 0 \\ 0 & 0 \end{bmatrix}, \quad H_{u2} = \begin{bmatrix} 0 & 5 \end{bmatrix}, \quad (5.44)$$

$$F_{u3} = \begin{bmatrix} -0.15 & -1 \\ 2 & 0 \end{bmatrix}, \quad G_{u3} = \begin{bmatrix} 0.2 & 0 \\ 0 & 0 \end{bmatrix}, \quad H_{u3} = \begin{bmatrix} 0 & 3 \end{bmatrix}, \quad (5.45)$$

$$F_{u4} = \begin{bmatrix} -0.1 & -1 \\ 2 & 0 \end{bmatrix}, \quad G_{u4} = \begin{bmatrix} 0.4 & 0 \\ 0 & 0 \end{bmatrix}, \quad H_{u4} = \begin{bmatrix} 0 & 5 \end{bmatrix}, \quad (5.46)$$
Figure 5.1: A connected and undirected path graph with 5 agents, where the dark colored agent on the middle is the leader agent and all other agents are follower agents.

\[
F_{u5} = \begin{bmatrix}
-0.15 & -1 \\
2 & 0
\end{bmatrix},
G_{u5} = \begin{bmatrix}
0.4 & 0 \\
0 & 0
\end{bmatrix},
H_{u5} = \begin{bmatrix}
0 & 5.5
\end{bmatrix},
\]

where the agent on the middle is the leader agent; see Figure 5.1. Here, we consider \( W_i = [2/1.5; 1/1.5] \) as the uncertain weights with \( \sigma_i(x_i(t)) = x_i(t) \), where the projection bounds are set to \( w^* = [-5; 5] \). In addition, we use \( K_1 = [5, 10] \) and \( K_2 = 3.5 \) as the control gain matrices, \( \alpha_i = 5 \) as the learning rates, and set the command only available to the leader agent to \( c = 1 \).

Figures 5.2 and 5.3 respectively show agent states along with reference states and control responses for the standard distributed adaptive control method in the presence of coupled dynamics. In this case, the agent-wise local stability conditions given by (5.26) for the standard distributed adaptive control method do not hold for any \( \beta_i \) value. Therefore, it is not surprising to see that the presence of coupled dynamics yield unbounded trajectories for agents 2 and 4.

Next, Figures 5.4 and 5.5 respectively show agent states along with reference states and control responses for the proposed distributed adaptive control method in the presence of coupled dynamics, where we use \( \beta_1 = 33, \beta_2 = 33, \beta_3 = 10, \beta_4 = 33, \text{ and } \beta_5 = 33 \). In this case, the agent-wise local stability conditions given by (5.34) hold for the considered \( \beta_i \) values. Therefore, overall system stability is ensured the presence of coupled dynamics.

5.6 Conclusions

In practical applications of multiagent systems, each agent can be subject to system uncertainties and coupled dynamics. While there exists notable research results on adaptively controlling uncertain multiagent systems, there is a scientific gap on developing stability conditions in the presence of coupled dynamics.
Figure 5.2: Responses for agent states and reference states for the distributed adaptive control method in Theorem 5.3.1.

The purpose of this paper was to make the first attempt in addressing this fundamental gap, where we studied distributed adaptive architectures for controlling uncertain multiagent systems with unmeasurable coupled dynamics. Specifically, in Theorem 5.3.1, which presented our first contribution, we analyzed a standard distributed adaptive control method with system uncertainties and coupled dynamics in a leader-follower setting and developed local stability conditions. In Theorem 5.4.1, which presented our second contribution, we then proposed an additional feedback term within the control signal of each agent in order to relax the aforementioned local stability conditions. An illustrative numerical result finally demonstrated our theoretical contributions. While a leader-follower setting was adopted to present our research results,
we note that these results can be readily applied to other classes of multiagent system problems including but not limited to consensus algorithms, formation algorithms, and containment algorithms.
Figure 5.4: Responses for agent states and reference states for the distributed adaptive control method in Theorem 5.4.1.
Figure 5.5: Responses for agent control inputs for the distributed adaptive control method in Theorem 5.4.1.
Chapter 6: Distributed Adaptive Control for Linear Multiagent Systems with Heterogeneous Actuator Dynamics and Uncertainties

This chapter provides control synthesis and stability verification framework for linear multiagent systems not only with heterogeneous system uncertainties but also with heterogeneous actuator dynamics. Specifically, two novel model reference adaptive control architectures are considered one for i) first-order multiagent systems and another for ii) high-order multiagent systems.

6.1 Distributed Adaptive Control of Networked Multiagent Systems with Heterogeneous Actuator Dynamics

Distributed adaptive control is a powerful framework to preserve stability of networked multiagent systems in the presence of uncertainties resulting from, for example, modeling errors, unknown control effectiveness, and perturbed information exchange. However, considering multiagent systems that consist of agents with heterogeneous actuator capabilities, implementation of distributed adaptive control approaches is not a trivial task. This is due to the fact that each agent in this case cannot identically execute given local control laws and this can lead to a poor networked multiagent system performance or even overall instability. To make the first attempt to this challenging problem, we consider a class of uncertain networked multiagent systems with single integrator dynamics in the context of a leader-follower problem and propose a novel distributed adaptive control design procedure for guaranteeing overall stability in the presence of agents having different actuator bandwidths. Specifically, a distributed adaptive control architecture is implemented for agent uncertainties and a hedging method, which modifies ideal reference models of each agent, is utilized to allow for correct adaptation that does not get affected due to the presence of actuator bandwidths. We then analyze the stability of the networked multiagent system and compute the actuator bandwidth limits of each agent using tools from Lyapunov stability and linear matrix inequalities. An illustrative numerical example is provided to demonstrate the efficacy of the proposed design procedure.

59This section is previously published in [120]. Permission is included in Appendix C.
6.1.1 Introduction

Networked multiagent systems consist of groups of agents that locally sense their environment, communicate with each other, and process distributed information in order to achieve a given set of system goals. Since these systems have widespread applications in not only military but also mixed civilian environments, it is not surprising that the last decade has witnessed an increased interest in networked multiagent systems (see, for example, [41–43], and references therein). Yet, due to the low-cost and small-size nature of agents, classical distributed control algorithms developed for networked multiagent systems often suffer from the presence of uncertainties resulting from modeling errors, unknown control effectiveness, and perturbed information exchange; to name but a few examples.

To this end, distributed adaptive control is a powerful emerging framework (see, for example, [11–16, 45–47, 121–123], and references therein) to preserve stability of networked multiagent systems in the presence of uncertainties. However, considering multiagent systems that consist of agents with heterogeneous actuator capabilities, implementation of distributed adaptive control approaches is not a trivial task. This is due to the fact that each agent in this case cannot identically execute given local control laws (e.g., one agent may be slow or fast in executing its control laws as compared to its neighbors) and this can lead to a poor networked multiagent system performance or even overall instability.

To address this challenging problem, this paper considers a class of uncertain networked multiagent systems with single integrator dynamics in the context of a leader-follower problem and proposes a novel distributed adaptive control design procedure for guaranteeing overall stability in the presence of agents having different actuator bandwidths. The proposed methodology is an extension of the adaptive control approach adopted in [35, 116, 124, 125] to networked multiagent systems and distributed adaptive control. Specifically, a distributed adaptive control architecture is implemented for agent uncertainties and a hedging method, which modifies ideal reference models of each agent, is utilized to allow for correct adaptation that does not get affected due to the presence of actuator bandwidths. We then analyze the stability of the networked multiagent system and compute the actuator bandwidth limits of each agent using tools from Lyapunov stability theory and linear matrix inequalities. An illustrative numerical example is provided to demonstrate the efficacy of the proposed design procedure.

Throughout this paper, \( \mathbb{R} \) denotes the set of real numbers, \( \mathbb{R}^n \) denotes the set of \( n \times 1 \) real column vectors, \( \mathbb{R}^{n\times m} \) denotes the set of \( n \times m \) real matrices, \( \mathbb{R}_+ \) (resp., \( \mathbb{R}_+^* \)) denotes the set of positive (resp.,
nonnegative) real numbers, \( \mathbb{R}^{n \times n}_+ \) (resp., \( \mathbb{R}^{n \times n}_+ \)) denotes the set of \( n \times n \) positive-definite (resp., nonnegative-definite) real matrices, \( \mathbb{S}^{n \times n}_+ \) denotes the set of \( n \times n \) symmetric nonnegative-definite real matrices, \( \mathbb{Z}_+ \) (resp., \( \mathbb{Z}_+ \)) denotes the set of positive integers (resp., nonnegative integers), \( \mathbb{D}^{n \times n} \) denotes the set of \( n \times n \) real matrices with diagonal scalar entries, and “\( \triangleq \)” denotes the equality by definition. In addition, we use \((\cdot)^T\) for the transpose operator, \((\cdot)^{-1}\) for the inverse operator, and \(\text{diag}(a)\) for the diagonal matrix with the vector \(a\) on its diagonal.

### 6.1.2 Mathematical Preliminaries

In this section, we introduce necessary mathematical preliminaries that are needed to develop the main results of this paper. We begin with the following definition.

**Definition 6.1.1** Let

\[
\Omega = \{ \theta \in \mathbb{R}^n : (\theta_{i}^{\min} \leq \theta_{i} \leq \theta_{i}^{\max})_{i=1,2,\ldots,n} \} \tag{6.1}
\]

be a convex hypercube in \( \mathbb{R}^n \), where \( (\theta_{i}^{\min}, \theta_{i}^{\max}) \) represent the minimum and maximum bounds for the \( i \)th component of the \( n \)-dimensional parameter vector \( \theta \). In addition, let

\[
\Omega_\varepsilon = \{ \theta \in \mathbb{R}^n : (\theta_{i}^{\min} + \varepsilon \leq \theta_{i} \leq \theta_{i}^{\max} - \varepsilon)_{i=1,2,\ldots,n} \} \tag{6.2}
\]

be a second hypercube for a sufficiently small positive constant \( \varepsilon \), where \( \Omega_\varepsilon \subset \Omega \). Then, the projection operator \( \text{Proj} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) is defined component-wise by

\[
\text{Proj}(\theta, y) \triangleq \begin{cases} 
\left( \frac{\theta_{i}^{\max} - \theta_{i}}{\varepsilon} \right) y_{i}, & \text{if } \theta_{i} > \theta_{i}^{\max} - \varepsilon \\
\text{and } y_{i} > 0 \\
\left( \frac{\theta_{i}^{\min} - \theta_{i}}{\varepsilon} \right) y_{i}, & \text{if } \theta_{i} < \theta_{i}^{\min} + \varepsilon \\
\text{and } y_{i} < 0 \\
y_{i}, & \text{otherwise}
\end{cases} \tag{6.3}
\]

where \( y \in \mathbb{R}^n \) [74].
Remark 6.1.1 It follows from the Definition 6.1.1 that

\[(\theta - \theta^*)^T(\text{Proj}(\theta, y) - y) \leq 0, \quad \theta^* \in \mathbb{R}^n, \tag{6.4}\]

holds [53, 74].

Next, we recall some basic notions from graph theory, where we refer to [43, 126] for further details. In the multiagent systems literature, graphs are broadly adopted to encode interactions in networked systems. An undirected graph \(G\) is defined by a set \(V_G = \{1, \ldots, n\}\) of nodes and a set \(E_G \subset V_G \times V_G\) of edges. If \((i, j) \in E_G\), then the nodes \(i\) and \(j\) are neighbors and the neighboring relation is indicated with \(i \sim j\).

The degree of a node is given by the number of its neighbors. Letting \(d_i\) be the degree of node \(i\), then the degree matrix of a graph \(G\), \(D(G) \in \mathbb{R}^{n \times n}\), is given by \(D(G) \triangleq \text{diag}(d)\), \(d = [d_1, \ldots, d_n]^T\). A path \(i_0i_1 \ldots i_L\) is a finite sequence of nodes such that \(i_{k-1} \sim i_k\), \(k = 1, \ldots, L\), and a graph \(G\) is connected if there is a path between any pair of distinct nodes. The adjacency matrix of a graph \(G\), \(A(G) \in \mathbb{R}^{n \times n}\), is given by

\[
[A(G)]_{ij} \triangleq \begin{cases} 
1, & \text{if } (i, j) \in E_G, \\
0, & \text{otherwise.}
\end{cases} \tag{6.5}
\]

The Laplacian matrix of a graph, \(L(G) \in \mathbb{S}^{n \times n}_+\), playing a central role in many graph theoretic treatments of multiagent systems is given by \(L(G) \triangleq D(G) - A(G)\), where the spectrum of the Laplacian for an undirected and connected graph \(G\) can be ordered as \(0 = \lambda_1(L(G)) < \lambda_2(L(G)) \leq \cdots \leq \lambda_n(L(G))\), with \(1_n\) as the eigenvector corresponding to the zero eigenvalue \(\lambda_1(L(G))\) and \(L(G)1_n = 0_n\) holds.

Throughout this paper, we model a given multiagent system by a connected, undirected graph \(G\), where nodes and edges represent agents and inter-agent communication links, respectively, and use the following lemma.

Lemma 6.1.1 [43]. Let \(\mathcal{K} = \text{diag}(k)\), \(k = [k_1, k_2, \ldots, k_n]^T\), \(k_i \in \mathbb{Z}_+, i = 1, \ldots, n\), and assume that at least one element of \(k\) is nonzero. Then, for the Laplacian of a connected, undirected graph, \(F(G) \triangleq L(G) + \mathcal{K} \in \mathbb{S}^{n \times n}_+\) and \(-F(G)\) is a Hurwitz matrix.

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6.1.3 Problem Formulation

Consider an uncertain networked multiagent system consisting of \( n \) agents that locally exchange information according to a connected, undirected graph \( G \). In addition, from the \( n \) agents, there are \( m \geq 1 \) agents receiving a command input, referred to as the leaders. The remaining agents not directly receiving a command input are then referred to as followers. Noting this, let \( x_i(t) \in \mathbb{R} \) denote the state of agent \( i \), \( i = 1, \ldots, n \), at time \( t \in \mathbb{R}_+ \), subject to the dynamics given by

\[
\dot{x}_i(t) = (1 + \delta_i) v_i(t) + w_i x_i(t), \quad x_i(0) = x_{i0}, \tag{6.6}
\]

for \( i = 1, \ldots, n \), where \( \delta_i > -1 \) is an uncertainty in the control effectiveness, \( w_i \in \mathbb{R} \) is an unknown constant weight, and \( v_i(t) \in \mathbb{R} \) is the actuator output of the actuator dynamics given by

\[
\dot{x}_{ci}(t) = -\lambda_i x_{ci}(t) + u_i(t), \quad x_{ci}(0) = x_{ci0}, \tag{6.7}
\]

\[
v_i(t) = \lambda_i x_{ci}(t), \tag{6.8}
\]

for \( i = 1, \ldots, n \), with \( x_{ci}(t) \in \mathbb{R} \) being the actuator state, \( \lambda_i \in \mathbb{R}_+ \) being the actuator bandwidth, and \( u_i(t) \in \mathbb{R} \) being the control input of agent \( i \).

The objective of this paper is to closely drive the states of all agents (leaders and followers) to an applied command.

**Remark 6.1.2** In the case there is no uncertainty (i.e. \( \delta_i = 0 \) and \( w_i = 0 \)) and there are no actuator dynamics, the agent state dynamics given by (6.6) can be written as

\[
\dot{x}_i(t) = u_i(t), \quad x_i(0) = x_{i0}. \tag{6.9}
\]

The objective of driving all the agent states to an applied command is then accomplished through the control law given by

\[
u_i(t) = -\sum_{i \sim j} (x_i(t) - x_j(t)) - k_i (x_i(t) - c), \tag{6.10}\]
where \( c \in \mathbb{R} \) is a constant command and \( k_i \in \{0, 1\} \) denotes whether the agent is a leader or follower (i.e. \( k_i = 1 \) for leaders and \( k_i = 0 \) for followers). To see this, we first note that the error between the agent state and the applied command is \( e_i(t) \triangleq x_i(t) - c \). The error dynamics then follow from using (6.9) and (6.10) as

\[
\dot{e}_i(t) = -\sum_{i \sim j} (x_i(t) - x_j(t)) - k_i(x_i(t) - c)
\]

\[
= -\sum_{i \sim j} (x_i(t) - c - (x_j(t) - c)) - k_i(x_i(t) - c)
\]

\[
= -\sum_{i \sim j} (e_i(t) - e_j(t)) - k_i e_i(t).
\]

(6.11)

Now, using Lemma 6.1.1 with \( e(t) \triangleq [e_1(t), \ldots, e_n(t)]^T \) and \( K \triangleq \text{diag}(k_1, \ldots, k_n) \) we can write (6.11) in compact form as

\[
\dot{e}(t) = -(\mathcal{L}(\mathcal{G}) + K)e(t)
\]

\[
= -\mathcal{F}(\mathcal{G})e(t).
\]

(6.12)

Since \(-\mathcal{F}(\mathcal{G})\) is Hurwitz, it follows that (6.12) is asymptotically stable, and hence, \( x_i(t) \to c \) as \( t \to \infty \).

**Remark 6.1.3** Now, we consider the case in which there exist system uncertainties (i.e. \( \delta_i \neq 0 \) and \( w_i \neq 0 \)) and the actuator dynamics are still negligible as in Remark 6.1.2. In this case, we can readily obtain

\[
\dot{x}_i(t) = (1 + \delta_i)u_i(t) + w_i x_i(t), \quad x_i(0) = x_{i0}.
\]

(6.13)

In the presence of uncertainty, it is desired to design a control law that adaptively estimates for the unknown parameters such that we achieve the “nominal performance” observed in Remark 6.1.2. To do this, we first design a reference model which captures the ideal (i.e., nominal), closed-loop dynamical system performance given by

\[
\dot{x}_r(t) = -\sum_{i \sim j} (x_r_i(t) - x_r_j(t)) - k_i(x_r_i(t) - c), \quad x_r_i(0) = x_{r0}.
\]

(6.14)

Now, the objective of driving all agent states to an applied command is accomplished by designing a control law such that the agents subject to the uncertain state dynamics given by (6.13) follow the reference model.
given by (6.14). For this purpose, we consider the adaptive control given by

\[ u_i(t) = -(1 + \hat{\delta}_i(t))^{-1}\left( \sum_{i \neq j} (x_i(t) - x_j(t)) - k_i(x_i(t) - c) - \dot{w}_i(t)x_i(t) \right), \quad (6.15) \]

where \( \hat{\delta}_i(t) > -1 \) is the estimate of \( \delta_i \) and \( \dot{w}_i(t) \in \mathbb{R} \) is an estimate of \( w_i \), respectively satisfying the update laws

\[ \hat{\delta}_i(t) = \gamma_i \text{Proj}_i[\hat{\delta}_i(t), \bar{x}_i(t)u_i(t)], \quad \hat{\delta}_i(0) = \hat{\delta}_{i0} > -1, \quad (6.16) \]
\[ \dot{w}_i(t) = \gamma_2 \text{Proj}_i[\dot{w}_i(t), x_i(t)\bar{x}_i(t)], \quad \dot{w}_i(0) = \dot{w}_{i0}, \quad (6.17) \]

where \( \gamma_{i1} \in \mathbb{R}^+ \) and \( \gamma_{i2} \in \mathbb{R}^+ \) are the learning rate gain, \( \bar{x}_i(t) \triangleq x_i(t) - x_i(t) \) is the agent system error state, and the projection bounds are defined such that \( |\hat{\delta}_i(t)| \leq \hat{\delta}_{i,max} < 1 \) and \( |\dot{w}_i(t)| \leq \dot{w}_{i,max} \). To see how the update laws (6.16) and (6.17) are derived, first note that the adaptive control law (6.15) can be equivalently written as

\[ u_i(t) = -\hat{\delta}_i(t)u_i(t) - \sum_{i \neq j} (x_i(t) - x_j(t)) - k_i(x_i(t) - c) - \dot{w}_i(t)x_i(t), \quad (6.18) \]

such that using (6.18) in (6.13) along with (6.14), the error dynamics follow as

\[ \dot{x}_i(t) = -\hat{\delta}_i(t)u_i(t) - \sum_{i \neq j} (\bar{x}_i(t) - \bar{x}_j(t)) - k_i\bar{x}_i(t) - \dot{w}_i(t)x_i(t). \quad (6.19) \]

where \( \bar{x}_i(t) \triangleq \bar{x}_i(t) - \delta_i \in \mathbb{R} \) and \( \bar{w}_i(t) \triangleq \dot{w}_i(t) - w_i \in \mathbb{R} \) are the update errors. Then consider the quadratic function given by

\[ V_i(\bar{x}_i, \bar{\delta}_i) = \frac{1}{2} \bar{x}_i^2 + \frac{1}{2\gamma_i} \bar{\delta}_i^2 + \frac{1}{2\gamma_{i2}} \bar{w}_i^2, \quad (6.20) \]

and note that \( V(0, 0, 0) = 0 \) and \( V_i(\bar{x}_i, \bar{\delta}_i, \bar{w}_i) > 0 \) for all \( (\bar{x}_i, \bar{\delta}_i, \bar{w}_i) \neq (0, 0, 0) \). Furthermore, \( V_i(\bar{x}_i, \bar{\delta}_i, \bar{w}_i) \) is radially unbounded. Now, differentiating (6.20) yields

\[ \dot{V}_i(\bar{x}_i(t), \bar{\delta}_i(t), \bar{w}_i(t)) = \bar{x}_i(t)\dot{x}_i(t) + \frac{1}{\gamma_i} \bar{\delta}_i(t)\dot{\bar{x}}_i(t) + \frac{1}{\gamma_{i2}} \bar{w}_i(t)\dot{\bar{w}}_i(t) \]
\[
\begin{align*}
\dot{x}_i(t) & = -\ddot{x}_i(t)\hat{\delta}_i(t) - \ddot{x}_i(t)\delta_i(t) + \frac{1}{\gamma_{1i}}\ddot{\delta}_i(t) + \frac{1}{\gamma_{2i}}\ddot{w}_i(t), \\
& \quad -k_i\ddot{x}_i^2(t) - \ddot{x}_i(t)\hat{\delta}_i(t)\delta_i(t) - k_i\ddot{x}_i^2(t),
\end{align*}
\] (6.21)

where using (6.16) and (6.17) in (6.21) results in

\[
\dot{V}_i(x_i(t), \delta_i(t)) \leq -\ddot{x}_i(t)\sum_{j \sim i}(\ddot{x}_i(t) - \ddot{x}_j(t)) - k_i\ddot{x}_i^2(t).
\] (6.22)

Now, consider the Lyapunov function candidate given by

\[
V(\cdot) = \sum_{i=1}^{n} V_i(\ddot{x}_i, \ddot{\delta}_i, \ddot{w}_i).
\] (6.23)

The time derivative of (6.23) follows using (6.22) as

\[
\dot{V}(\cdot) \leq \ddot{x}^T(t)(L(\mathcal{G}) + K)\ddot{x}(t)
\]
\[
= -\ddot{x}^T(t)F(\mathcal{G})\ddot{x}(t) \leq 0,
\] (6.24)

where \( \ddot{x}(t) = [\ddot{x}_1(t), \cdots, \ddot{x}_n(t)]^T \). Since \( \dot{V}(\cdot) \leq 0 \), it follows that the error states \( \ddot{x}_i(t) \) and the update errors \( \ddot{\delta}_i(t) \) and \( \ddot{w}_i(t) \) are Lyapunov stable, and therefore, bounded for all \( t \in \mathbb{R}_+ \). In addition, \( x_i(t) \) is bounded for all \( t \in \mathbb{R}_+ \), such that it follows from (6.19) that \( \ddot{x}_i(t) \) is bounded, and hence, \( \dot{V}(\cdot) \) is bounded for all \( t \in \mathbb{R}_+ \).

It then follows from Barbalat’s lemma that \( \lim_{t \to \infty} \dot{V}(\cdot) = 0 \), which consequently shows that \( \ddot{x}_i(t) \to 0 \) as \( t \to \infty \).

**Remark 6.1.4** Although this paper considers a specific leader-follower problem to focus on its main contribution of guaranteeing overall stability in the presence of uncertain agents having heterogeneous actuator bandwidths, which has not been addressed in the literature, the distributed adaptive control design to be presented in the next section can be easily extended or applied as is to other multiagent problems. In addition, note that we will consider generalizations to agents having high-order linear (or nonlinear) dynamics as a part of our future research studies.
6.1.4 Distributed Adaptive Control Based on Linear Matrix Inequalities

In the nontrivial case that there are system uncertainties and the actuator dynamics are not negligible, we would like to implement an adaptive control law such that the adaptation performance is not affected by the presence of the actuator dynamics. For this purpose, we begin by adding and subtracting \( u_i(t) \) to obtain the equivalent form of (6.6) given by

\[
\dot{x}_i(t) = u_i(t) + \delta_i v_i(t) + w_i x_i(t) + \left( v_i(t) - u_i(t) \right).
\] (6.25)

Using the adaptive control law given by

\[
u_i(t) = -\tilde{\delta}_i(t) v_i(t) - \sum_{i\sim j} (x_i(t) - x_j(t)) - k_i (x_i(t) - c) - \tilde{w}_i(t) x_i(t),
\] (6.26)

where \( \tilde{w}_i(t) \) satisfies the update law (6.17) and \( \tilde{\delta}_i(t) \) satisfies the update law

\[
\dot{\tilde{\delta}}_i(t) = \gamma_i \text{Proj} [\tilde{\delta}_i(t), \tilde{x}_i(t) v_i(t)], \quad \tilde{\delta}(0) = \tilde{\delta}_0 > -1,
\] (6.27)

the agent state dynamics, (6.25), can be written as

\[
\dot{x}_i(t) = -\sum_{i\sim j} (x_i(t) - x_j(t)) - k_i (x_i(t) - c) - \delta_i v_i(t) - \tilde{w}_i(t) x_i(t) + \left( v_i(t) - u_i(t) \right).
\] (6.28)

Motivated from the structure of (6.28), we use a hedging approach [113], to modify the ideal reference model given by (6.14) to the following

\[
\dot{x}_{m_i}(t) = -\sum_{i\sim j} (x_{m_i}(t) - x_{m_j}(t)) - k_i (x_{m_i}(t) - c) + \left( v_i(t) - u_i(t) \right), \quad x_{m_i}(0) = x_{m_0},
\] (6.29)

such that the agent state error dynamics can be given using (6.28) and (6.29) as

\[
\dot{\tilde{x}}_i(t) = -\sum_{i\sim j} (\tilde{x}_i(t) - \tilde{x}_j(t)) - k_i \tilde{x}_i(t) - \tilde{\delta}_i(t) v_i(t) - \tilde{w}_i(t) x_i(t).
\] (6.30)

Note that this modification to the ideal reference model does not prevent the agents from tracking the input commands as will be shown later in Corollary 6.1.1.
The following lemma is needed for the results of this section. For this purpose, let \( \Lambda \triangleq \text{diag}(\lambda_1, \ldots, \lambda_n) \), \( \hat{\Lambda}(t) \triangleq \text{diag}(\hat{\lambda}_1(t), \ldots, \hat{\lambda}_n(t)) \), \( \hat{W}(t) \triangleq \text{diag}(\hat{w}_1(t), \ldots, \hat{w}_n(t)) \), \( \bar{\omega} \in \mathbb{R}_+ \) be such that \( \max \{ \hat{\omega}_i \} \leq \bar{\omega} \), \( \bar{\zeta} \in \mathbb{R}_+ \) be such that \( \max \{ \hat{\delta}_i \} \leq \bar{\zeta} \) and \( \lambda \in \mathbb{R}_+ \) be such that \( \lambda \leq \min \{ \lambda_i \} \).

**Lemma 6.1.2** There exists a set \( \kappa \triangleq \{ \lambda : \lambda \leq \min \{ \lambda_i \} \} \cup \{ \bar{\omega}, \bar{\zeta} : \max \{ \hat{\omega}_i \} \leq \bar{\omega} \) and \( \max \{ \hat{\delta}_i \} \leq \bar{\zeta} \} \) such that if \( (\lambda, \bar{\omega}, \bar{\zeta}) \in \kappa \), then \( A(\hat{\Lambda}(t), \hat{W}(t), \Lambda) = \begin{bmatrix} \hat{W}(t) & (I + \hat{\Lambda}(t))\Lambda \\ -(\mathcal{F}(\mathcal{G}) - \hat{W}(t)) & -(I + \hat{\Lambda}(t))\Lambda \end{bmatrix} \) (6.31)

is quadratically stable.

**Proof.** We first show that there exists \( \lambda \) such that (6.31) is quadratically stable. For this purpose, consider the Lyapunov inequality given by

\[ A^T(\cdot)P + PA(\cdot) < 0, \quad P = P^T > 0, \] (6.32)

with

\[ P = \begin{bmatrix} P & P \\ P & P + \alpha I \end{bmatrix}, \] (6.33)

where \( P \in \mathbb{R}^{n \times n}_+ \) is a solution of the Lyapunov equation given by

\[ (-\mathcal{F}^T(\mathcal{G}))P + P(-\mathcal{F}(\mathcal{G}))+I = 0, \] (6.34)

and \( \alpha \in \mathbb{R}_+ \). Note that \( \mathcal{F}^T(\mathcal{G}) = \mathcal{F}(\mathcal{G}) \) for a connected, undirected graph \( \mathcal{G} \), and hence, (6.34) can be rewritten as

\[ \mathcal{F}(\mathcal{G})P + P\mathcal{F}(\mathcal{G}) = I. \] (6.35)

The positive-definiteness of (6.33) follows from the positive-definiteness of \( P \) and the positive-definiteness of the Schur complement of (6.33) given by
\[ S_1 = P + \alpha I - P(P)^{-1}P = \alpha I > 0. \]  

Next, note that

\[ Q = A^T(\hat{\Delta}(t), \hat{\omega}(t), \Lambda)P + P A(\hat{\Delta}(t), \hat{\omega}(t), \Lambda) \]
\[ = \begin{bmatrix} -I & Q_2(t) \\ Q_2^T(t) & -2\alpha(I + \hat{\Delta}(t))\Lambda \end{bmatrix}, \]  

where \( Q_2(t) \triangleq -\mathcal{F}(\mathcal{G})(P + \alpha I) - \alpha \hat{\omega}(t) \). Since \(-I\) is a negative-definite matrix, it follows from the Schur complement of (6.37)

\[ S_2 = -2\alpha(I + \hat{\Delta}(t))\Lambda + (\mathcal{F}(\mathcal{G})(P + \alpha I) - \alpha \hat{\omega}(t))^T(\mathcal{F}(\mathcal{G})(P + \alpha I) - \alpha \hat{\omega}(t)), \]  

that (6.38) is a negative-definite matrix when \( \lambda \) is sufficiently large, which yields to the quadratic stability of (6.31).

We next show that there exists \( \overline{\nu} \) and \( \overline{\zeta} \) such that (6.31) is quadratically stable. For this purpose, we note that when \( \overline{\nu} = 0 \) and \( \overline{\zeta} = 0 \), (6.31) can equivalently be written as

\[ \mathcal{A}(0, 0, \Lambda) = \begin{bmatrix} 0 & \Lambda \\ -\mathcal{F}(\mathcal{G}) & -\Lambda \end{bmatrix}. \]  

Then, using

\[ T \triangleq \begin{bmatrix} I & 0 \\ 0 & \Lambda^{-1} \end{bmatrix}, \]  

which is nonsingular, note that

\[ T^{-1}\mathcal{A}(0, 0, \Lambda)T = \begin{bmatrix} 0 & I \\ -\Lambda\mathcal{F}(\mathcal{G}) & -\Lambda \end{bmatrix}, \]
is Hurwitz by Fact 5.11.9 in [127]. It now follows from similarity that \(\mathcal{A}(0,0,\Lambda)\) is Hurwitz. Since (6.31) depends continuously to the variations in \(0 < \hat{\omega}(t) \leq \overline{\omega}\) and \(0 < \hat{\zeta}(t) \leq \overline{\zeta}\), the quadratic stability of (6.31) is assured when \(\overline{\omega}\) and \(\overline{\zeta}\) are sufficiently small.

Finally, since there exist a (sufficiently large) \(\lambda\) or a (sufficiently small) \(\overline{\omega}\) and \(\overline{\zeta}\) such that (6.31) is quadratically stable, the existence of set \(\kappa\) is immediate. \(\blacksquare\)

The next theorem presents the main results of this paper. For this purpose, note that \(x(t) = [x_1(t), \cdots, x_n(t)]^T, \tilde{x}(t) = [\tilde{x}_1(t), \cdots, \tilde{x}_n(t)]^T, \tilde{\delta}(t) = [\tilde{\delta}_1(t), \cdots, \tilde{\delta}_h(t)]^T, \tilde{w}(t) = [\tilde{w}_1(t), \cdots, \tilde{w}_h(t)]^T, x_m(t) = [x_{m1}(t), \cdots, x_{m_n}(t)]^T, x_c(t) = [x_{c1}(t), \cdots, x_{cn}(t)]^T,\) and \(v(t) = [v_1(t), \cdots, v_n(t)]^T.\)

**Theorem 6.1.1** Consider the uncertain networked multiagent system given by (6.6), the reference model given by (6.29), the actuator dynamics given by (6.7) and (6.8), and the adaptive control law given by (6.26) along with the update laws (6.17) and (6.27). If \((\lambda, \overline{\omega}, \overline{\zeta}) \in \kappa,\) then the solution \((\tilde{x}(t), \tilde{\delta}(t), \tilde{w}(t), x_m(t), v(t))\) of the closed-loop dynamical system is bounded and \(\lim_{t \to \infty} \tilde{x}(t) = 0.\)

**Proof.** To show Lyapunov stability and guarantee boundedness of the agent error state \(\tilde{x}(t)\) and the update errors \(\tilde{\delta}(t)\) and \(\tilde{w}(t)\), consider the Lyapunov function candidate

\[
V(\tilde{x}, \tilde{\delta}, \tilde{w}) = \sum_{i=1}^{n} \left( \frac{1}{2} \tilde{x}_i^2 + \frac{1}{2\gamma_i} \tilde{\delta}_i^2 + \frac{1}{2\gamma_{ii}} \tilde{w}_i^2 \right). \tag{6.42}
\]

Note that \(V(0,0,0) = 0\) and \(V(\tilde{x}, \tilde{\delta}, \tilde{w}) > 0\) for all \((\tilde{x}, \tilde{\delta}, \tilde{w}) \neq (0,0,0).\) Then, differentiating (6.42) yields \(\dot{V}(\tilde{x}(t), \tilde{\delta}(t), \tilde{w}(t)) \leq -\tilde{x}^T(t) \mathcal{F}(\mathcal{G}) \tilde{x}(t) \leq 0,\) which guarantees the Lyapunov stability, and hence, the boundedness of the solution \((\tilde{x}(t), \tilde{\delta}(t), \tilde{w}(t)).\)

To show the boundedness of \(x_c(t)\) and \(x_c(t)\) (and therefore \(v(t)\)), consider the reference system (6.29) and the actuator dynamics (6.7) and (6.8) subject to (6.26) in compact form as

\[
\begin{align*}
\dot{x}_m(t) &= \tilde{W}(t)x_m(t) + (I + \hat{\Delta}(t))\Lambda x_c(t) + \tilde{W}(t)\tilde{x}(t) + \mathcal{F}(\mathcal{G})\tilde{x}(t) \tag{6.43} \\
\dot{x}_c(t) &= -(I + \hat{\Delta}(t))\Lambda x_c(t) - \mathcal{F}(\mathcal{G})x_m(t) + K_c - \tilde{W}(t)x_m(t) - \mathcal{F}(\mathcal{G})\tilde{x}(t) - \tilde{W}(t)\tilde{x}(t), \tag{6.44}
\end{align*}
\]

where (6.43) and (6.44) can be rewritten as

\[
\begin{align*}
\dot{\xi}(t) &= \mathcal{A}(\hat{\Delta}(t), \tilde{W}(t), \Lambda)\xi(t) + \omega(\cdot), \tag{6.45}
\end{align*}
\]
with $\xi(t) = [x_m^T(t), x_c^T(t)]^T$ and

$$\omega(\cdot) = \begin{bmatrix} \hat{W}(t)\ddot{x}(t) + \mathcal{F}(G)\ddot{x}(t) \\ -\hat{W}(t)\ddot{x}(t) - \mathcal{F}(G)\ddot{x}(t) + Kc \end{bmatrix}. \quad (6.46)$$

Note that $\omega(\cdot)$ in (6.45) is a bounded perturbation as a result of Lyapunov stability of the triple $(\ddot{x}(t), \ddot{\delta}(t), \ddot{\nu}(t))$. Now, it follows that since $\omega(\cdot)$ is bounded and $\mathcal{A}(\hat{\Delta}(t), \hat{W}(t), \Lambda)$ is quadratically stable for $(\hat{\Delta}, \bar{\omega}, \bar{\xi}) \in \kappa$ by Lemma 6.1.2, then $x_m(t)$ and $x_c(t)$ are also bounded (see, for example, [57]). This further implies that the actuator output $v(t)$ is bounded.

To show $\lim_{t \to \infty} \ddot{x}(t) = 0$, note that $\ddot{x}(t)$ is bounded as a consequence of the boundedness of $\ddot{x}(t)$ and $x_m(t)$. It now follows from (6.30) that $\ddot{x}(t)$ is bounded, and hence, $\dot{V}(\ddot{x}(t), \ddot{\delta}(t), \ddot{\nu}(t))$ is bounded. As a consequence of the boundedness of $\dot{V}(\ddot{x}(t), \ddot{\delta}(t), \ddot{\nu}(t))$ and Barbalat’s lemma [57], $\lim_{t \to \infty} \dot{V}(\ddot{x}(t), \ddot{\delta}(t), \ddot{\nu}(t)) = 0$, and hence, $\lim_{t \to \infty} \ddot{x}(t) = 0$. ■

**Remark 6.1.5** We now utilize linear matrix inequalities to satisfy the quadratic stability of (6.31) for given projection bounds $\hat{\delta}_{1, \text{max}}$ and $\hat{\nu}_{1, \text{max}}$ for the diagonal elements of $\hat{\Delta}(t)$ and $\hat{W}(t)$ respectively, and the bandwidths of the actuator dynamics $\Lambda$. For this purpose, let $\bar{\Delta}_{j_1,...,j_l} \in \mathbb{D}^{n \times n}$ and $\bar{W}_{j_1,...,j_l} \in \mathbb{D}^{n \times n}$ be defined as

$$\bar{\Delta}_{j_1,...,j_l} = \text{diag}\left((-1)^{j_1} \hat{\delta}_{1, \text{max}}, (-1)^{j_2} \hat{\delta}_{2, \text{max}}, \cdots, (-1)^{j_n} \hat{\delta}_{n, \text{max}}\right), \quad (6.47)$$

$$\bar{W}_{j_1,...,j_l} = \text{diag}\left((-1)^{j_1} \hat{\nu}_{1, \text{max}}, (-1)^{j_2} \hat{\nu}_{2, \text{max}}, \cdots, (-1)^{j_n} \hat{\nu}_{n, \text{max}}\right), \quad (6.48)$$

where $j_l \in \{1,2\}$, $l \in \{1,...,n^2\}$, such that $\bar{\Delta}_{j_1,...,j_l}$ and $\bar{W}_{j_1,...,j_l}$ represent the corners of the hypercube defining the maximum variation of $\hat{\Delta}(t)$ and $\hat{W}(t)$. Following the results in [128] and [111], if

$$\mathcal{A}_{j_1,...,j_l} = \begin{bmatrix} \bar{W}_{j_1,...,j_l} & (I + \bar{\Delta}_{j_1,...,j_l})\Lambda \\ -\mathcal{F}(G) - \bar{W}_{j_1,...,j_l} & -(I + \bar{\Delta}_{j_1,...,j_l})\Lambda \end{bmatrix}, \quad (6.49)$$

satisfies the matrix inequality

$$\mathcal{A}_{j_1,...,j_l}^T \mathcal{P} + \mathcal{P} \mathcal{A}_{j_1,...,j_l} < 0, \quad \mathcal{P} = \mathcal{P}^T > 0, \quad (6.50)$$
for all permutations of $\tilde{\Lambda}_{j_1,\ldots,j_l}$ and $\tilde{W}_{j_1,\ldots,j_l}$, then (6.31) is quadratically stable. Since (6.31) is quadratically stable for large values of $\Lambda$ (see Lemma 6.1.2), we cast (6.50) as a convex optimization problem given by

$$\begin{align*}
\text{minimize} & \quad \Lambda, \\
\text{subject to} & \quad (6.50).
\end{align*}$$

Therefore, we can satisfy (6.50) by minimizing $\Lambda$ for given projection bounds.

It has been shown that the distance between the networked multiagent system given by (6.6) and the modified reference model given by (6.29) asymptotically vanishes. To analyze the convergence properties of the modified reference model to the ideal reference model capturing a given desired closed-loop adaptive multiagent system behavior, we provide the following corollary.

**Corollary 6.1.1** Consider the ideal reference model (6.14), the modified reference model (6.29), the actuator dynamics (6.7) and (6.8), and the adaptive control law (6.26). If $(\bar{\lambda}, \bar{\omega}, \bar{\zeta}) \in \kappa$, the modified reference model (6.29) will asymptotically converge to the ideal reference model (6.14). In addition, using the results from Theorem 6.1.1, it follows that $x_i(t) - x_{ri}(t) \to 0$ as $t \to \infty$, for all $i = 1, \ldots, n$.

**Proof.** The proof follows from the results presented in [116].

6.1.5 Illustrative Numerical Example

In order to illustrate the distributed adaptive control architecture for uncertain networked multiagent systems, we consider a group of 4 agents with the state dynamics given by (6.6) and the actuator dynamics given by (6.7) and (6.8). We consider the formation as depicted in Figure 6.1, with agent 1 being the leader, and resulting in the Laplacian and gain matrix as follows

$$L(G) = \begin{bmatrix}
2 & -1 & 0 & -1 \\
-1 & 2 & -1 & 0 \\
0 & -1 & 2 & -1 \\
-1 & 0 & -1 & 2
\end{bmatrix}, \quad (6.51)$$
$$K = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$  \hfill (6.52)

We assume that agents 1 and 2 have the same actuator bandwidths (i.e., $\lambda_1 = \lambda_2$) and likewise agents 3 and 4 have the same actuator bandwidths (i.e., $\lambda_3 = \lambda_4$). We next examine the effect of the actuator bandwidths of agents to the overall networked multiagent system stability in the presence of uncertainties.

For this numerical example, we assume $\delta_1 = \delta_3 = -0.6$ and $\delta_2 = \delta_4 = -0.2$ and $w_i = 0.5, i = 1, 2, 3, 4$, such that we set the projection bounds as $|\hat{\delta}_1(t)| = |\hat{\delta}_3(t)| \leq 0.65, |\hat{\delta}_2(t)| = |\hat{\delta}_4(t)| \leq 0.25$, and $|\hat{w}(t)_i| \leq 0.55$. Using these bounds in the linear matrix inequalities-based analysis highlighted in Section 6.1.4, the feasible region of allowable actuator bandwidth limits is calculated.

Figure 6.2 shows the feasible region of allowable actuator bandwidth values as well as the feasible limit provided by the simulation results. We select two points to simulate the proposed controller performance as seen in Figures 6.3 and 6.4. Since the feasible boundary corresponds to calculated minimum feasible values for the actuator bandwidths, it is expected that the system performances are guaranteed to be bounded for actuator dynamics at points greater than or equal to the calculated feasible boundary. This can be seen by the fact that all agents track the applied command of $c = 1$ in Figure 6.3 when the actuator bandwidths are at the point $(\lambda_1 = \lambda_2 = 4.0, \lambda_3 = \lambda_4 = 3.35)$, which is located on the feasible boundary. In Figure 6.4, we let the actuator bandwidth values be outside the calculated feasible region to show that the system remains bounded until the point $(\lambda_1 = \lambda_2 = 1.2, \lambda_3 = \lambda_4 = 1.2)$ is reached. Note that

![Figure 6.1: Formation of multiagent network, with $1^*$ being the leader (solid lines represent an undirected information exchange between agents).](image-url)
this is consistent with the proposed theory in that the linear matrix inequalities-based analysis provides a conservative limit. We also note from Figure 6.2 that agents 1 and 2 (respectively, agents 3 and 4) can have large $\lambda_i$ values (i.e., fast actuation capabilities) provided that $\lambda_i$ values of other agents are not smaller than...
the values shown in this figure. This shows the stability guarantee regarding when slow and fast agents can work together in coherence.

6.1.6 Conclusions

In this paper, a novel distributed adaptive control design procedure was proposed for guaranteeing overall networked multiagent system stability in the presence of agents having heterogeneous actuator dynamics. For this purpose, a distributed adaptive control architecture was implemented for agent uncertainties and a hedging method was utilized to allow for correct adaptation that does not get affected due to the presence of actuator bandwidths. Stability of the overall system was analyzed using tools from Lyapunov stability theory and linear matrix inequalities and an illustrative numerical example was further provided that shows the efficacy of the proposed design procedure. Although this paper focused on a class of uncertain networked multiagent systems with single integrator dynamics in the context of a leader-follower
problem, the presented design and analysis procedure can be generalized to agents having high-order linear (or nonlinear) dynamics and this will be done as a part of our future research studies.

6.2 Distributed Adaptive Control and Stability Verification for Linear Multiagent Systems with Heterogeneous Actuator Dynamics and System Uncertainties

The contribution of this paper is a control synthesis and stability verification framework for linear time-invariant multiagent systems with heterogeneous actuator dynamics and system uncertainties. In particular, we first propose a distributed adaptive control architecture in a leader-follower setting for this class of high-order multiagent systems. The proposed architecture uses a hedging method, which alters the ideal reference model dynamics of each agent in order to ensure correct adaptation in the presence of heterogeneous actuator dynamics of these agents. We then use Lyapunov stability theory and linear matrix inequalities to analyze the proposed architecture. This analysis reveals a stability condition, where evaluation of this condition with respect to a given graph topology allows stability verification of the controlled multiagent system. From a practical point of view, this condition also shows a fundamental tradeoff between heterogeneous agent actuation capabilities and unknown parameters in agent dynamics. Several illustrative numerical examples are also provided to demonstrate the efficacy of the proposed architecture.

6.2.1 Introduction

6.2.1.1 Literature Review

Multiagent systems generally consist of groups of agents (e.g., low-cost and small-size unmanned vehicles) that work cooperatively with each other in order to achieve a given set of system objectives. The last two decades have witnessed extensive research studies on these systems owing to their widespread applications in military and civilian environments (e.g., see [41–43, 129, 130]). One of the open problems in the control design for these systems, which is of paramount importance for their practical applications, is the ability of the controlled system to guarantee stability and performance with respect to often nonidentical agent actuation capabilities and unknown parameters in the agent dynamics.

In particular, if a multiagent system has agents with nonidentical actuation capabilities (e.g., unmanned vehicles working together with low and high bandwidth actuators), they cannot execute given

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60This section is previously published in [50]. Permission is included in Appendix C.
distributed control approaches identically. As shown by the authors of, for example, [120] in this context and by the authors of, for example, [51, 52] in the context of temporal heterogeneity, this then can lead to the instability of the controlled multiagent system. Therefore, it is of both theoretical and practical interest to determine the conditions when the controlled multiagent system remains stable in this case. It should be also noted here that these conditions not only depend on how different the actuation capabilities of these agents are, but also the number of, locations of, and communication links between these agents with nonidentical actuation capabilities (see Section 6.2.3.1 for a motivational example).

In addition to heterogeneous actuation between agents, distributed control approaches also suffer from the presence of unknown parameters in agent dynamics. For example, the authors of [45, 48, 131, 132] (see also references therein) propose intelligent (e.g., neural networks-based) adaptive control approaches and the authors of [12–16, 46, 47, 49, 121, 133] (see also references therein) propose direct adaptive control approaches to deal with unknown parameters resulting from the lack of excessive modeling efforts and environmental disturbances. Furthermore, the authors of, for example, [134] propose an adaptive control approach that not only deals with such unknown parameters but also considers the uncertainties resulting from gain variations of controllers. While the above adaptive control approaches can preserve the stability of multiagent systems in the presence of unknown parameters, they do not consider the presence of actuator dynamics under an assumption that such dynamics are sufficiently fast as compared with the controlled multiagent system.

To summarize, there is a fundamental gap in the design of distributed adaptive controllers for multiagent systems with heterogeneous actuator dynamics and system uncertainties, where stability verifications steps must be considered in order to rigorously determine the conditions on the stability of the controlled multiagent system. The research purpose of this paper explained in the next subsection is to close this fundamental gap, where we focus on the heterogeneous case (since this case is more general owing to the fact that the homogeneous case is a special case of the heterogeneous case).

6.2.1.2 Contribution

The main contribution of this paper is a control synthesis and stability verification framework for linear time-invariant multiagent systems with heterogeneous actuator dynamics and system uncertainties. In particular:
• We first propose a distributed adaptive control architecture in a leader-follower setting for this class of high-order multiagent systems. The proposed architecture uses a hedging method, which alters the ideal reference model dynamics of each agent in order to ensure correct adaptation in the presence of actuator dynamics of these agents.

• We then use Lyapunov stability theory and linear matrix inequalities to analyze the proposed architecture. Specifically, Lyapunov stability theory is utilized to show each agent error (i.e., the error between an agent dynamics and its hedged reference model dynamics) remains bounded. We then resort to linear matrix inequalities to show the boundedness of the hedged reference model dynamics of each agent, which in turn guarantees that the agent errors asymptotically vanish. It is important to note that our analysis procedure reveals a stability condition discussed next.

• Evaluation of the resulting condition with respect to a given graph topology allows stability verification of the controlled multiagent system. From a practical point of view, this condition also shows a fundamental tradeoff between heterogeneous agent actuation capabilities and unknown parameters in agent dynamics. In particular, one can use this condition to assess stability of the controlled multiagent system with respect to, for example, number of leaders and followers in a given multiagent system, location of these leaders and followers on a given graph, and communication links between leaders and followers (i.e., the edges on the given graph).

• Finally, illustrative numerical examples are provided to demonstrate the efficacy of the proposed architecture, where we show how the stability of the controlled multiagent system changes with respect to heterogeneity in the actuator dynamics of agents as well as the number of, locations of, and communication links between leaders and followers.

While our system-theoretical results are presented in a leader-follower setting, it should be noted here that they can be readily used for many other problems (e.g., consensus, formation, and containment) of multiagent systems. In particular, the proposed control synthesis and stability verification framework has broad applications involving groups of unmanned air, ground, surface, or underwater vehicles for scenarios including cooperative surveillance, reconnaissance, transportation, and data gathering to name but a few examples, where each possibly low-cost and small-size agent needs to operate in the presence of uncertainties as well as nonidentical actuation capabilities. Note that the proposed architecture of this paper can be viewed as a generalization of the results in [34, 35, 124, 125], where these results focus on adaptive control synthesis and stability verification for sole uncertain dynamical systems subject to actuator
dynamics (i.e., they are not related with multiagent systems). Finally, preliminary conference versions of this paper appeared in [120, 135], where the first one only considers scalar agent dynamics and the present paper considerably goes beyond the second one by providing detailed proofs of all the results with additional examples and motivation.

6.2.2 Notation and Definitions

The purpose of this section is to introduce the notation used throughout the paper (Section 6.2.2.1) and recall some basic definitions from graph theory (e.g., see books [43, 118] for further details), where we also overview necessary lemmas for our paper (Section 6.2.2.2).

6.2.2.1 Notation

A fairly standard notation is used in this paper. Specifically, \( \mathbb{R} \) denotes the set of real numbers, \( \mathbb{R}^n \) denotes the set of \( n \times 1 \) real column vectors, \( \mathbb{R}^{n \times m} \) denotes the set of \( n \times m \) real matrices, \( \mathbb{R}_+ \) denotes the set of positive real numbers, \( \mathbb{R}_+^{n \times n} \) (resp., \( \mathbb{R}_-^{n \times n} \)) denotes the set of \( n \times n \) positive-definite (resp., nonnegative-definite) real matrices, \( \mathbb{Z} \) denotes the set of integers, \( \mathbb{Z}_+ \) (resp., \( \mathbb{Z}_- \)) denotes the set of positive (resp., nonnegative) integers, \( 0_n \) denotes the \( n \times 1 \) vector of all zeros, \( 1_n \) denotes the \( n \times 1 \) vector of all ones, \( 0_{n \times n} \) denotes the \( n \times n \) zero matrix, and \( I_n \) denotes the \( n \times n \) identity matrix. In addition, we write \((\cdot)^T\) for transpose operator, \((\cdot)^{-1}\) for the inverse operator, and \(\text{Proj}(\cdot, \cdot)\) for the projection operator Chapter 11.4 of [74]. Finally, we write \(\lambda_i(A)\) for the \(i\)-th eigenvalue of \(A\) (\(A\) is symmetric and the eigenvalues are ordered from least to greatest value), \(\text{diag}(a)\) for the diagonal matrix with the vector \(a\) on its diagonal, \(\text{block-diag}(\cdot)\) for the block diagonal matrix, and \(\triangleq\) for the equality by definition.

6.2.2.2 Graph-Theoretical Definitions and Lemmas

Graphs are broadly adopted as a fundamental tool in the multiagent systems literature to encode agent-to-agent interactions. In particular, an undirected graph \(G\) is defined by a set \(V_G = \{1, \ldots, N\}\) of nodes and a set \(E_G \subset V_G \times V_G\) of edges. If \((i, j) \in E_G\), then the nodes \(i\) and \(j\) are neighbors and the neighboring relation is indicated with \(i \sim j\). The degree of a node is given by the number of its neighbors. Letting \(d_i\) be the degree of node \(i\), the degree matrix of a graph \(G\), \(D(G) \in \mathbb{R}^{N \times N}\), is defined by \(D(G) \triangleq \text{diag}(d)\), where \(d = [d_1, \ldots, d_N]^T\). A path \(i_0i_1 \ldots i_L\) is a finite sequence of nodes such that \(i_{k-1} \sim i_k\), \(k = 1, \ldots, L\), and a graph \(G\) is connected when there is a path between any pair of distinct nodes. The adjacency matrix of a
graph $G$, $A(G) \in \mathbb{R}^{N \times N}$, is defined by $[A(G)]_{ij} = 1$ if $(i, j) \in \mathcal{E}(G)$ and $[A(G)]_{ij} = 0$ otherwise. The Laplacian matrix of a graph, $L(G) \in \mathbb{R}^{N \times N}$, playing a central role in many graph theoretic treatments of multiagent systems, is defined by $L(G) \triangleq \mathcal{D}(G) - A(G)$. Throughout this paper, we model a given multiagent system by a connected, undirected graph $G$, where nodes and edges respectively represent agents and inter-agent communication links. In addition, we need the following lemmas respectively from [43], [119], and [127].

**Lemma 6.2.1** The spectrum of the Laplacian of a connected, undirected graph can be ordered as $0 = \lambda_1(L(G)) < \lambda_2(L(G)) \leq \ldots \leq \lambda_N(L(G))$ with $1_n$ as the eigenvector corresponding to the zero eigenvalue $\lambda_1(L(G))$ and $L(G)1_N = 0_N$.

**Lemma 6.2.2** Let $K = \text{diag}(k)$, $k = [k_1, k_2, \ldots, k_N]^T$, $k_i \in \mathbb{Z}_+$, $i = 1, \ldots, N$, and assume at least one element of $k$ is nonzero. Then, for the Laplacian of a connected, undirected graph, $F(G) \triangleq L(G) + K \in \mathbb{R}^{N \times N}$ is a positive-definite matrix and $-F(G)$ is Hurwitz.

**Lemma 6.2.3** Let $A_1 \in \mathbb{R}^{n \times n}$, $A_2 \in \mathbb{R}^{n \times m}$, $A_3 \in \mathbb{R}^{m \times n}$ and $A_4 \in \mathbb{R}^{m \times m}$. If $A_1$ and $A_4 - A_3A_1^{-1}A_2$ are nonsingular, then

$$
\begin{bmatrix}
A_1 & A_2 \\
A_3 & A_4
\end{bmatrix}^{-1} = \begin{bmatrix}
M_1 & M_2 \\
M_3 & M_4
\end{bmatrix},
$$

(6.53)

where $M_2 \triangleq -A_1^{-1}A_2(A_4 - A_3A_1^{-1}A_2)^{-1}$.

Note that the result of Lemma 6.2.2 implicitly depends on the result of Lemma 6.2.1, where the matrix $F(G)$ of Lemma 6.2.2 is utilized in the results of this paper. Moreover, the result of Lemma 6.2.3 is needed for a discussion in the following section.

### 6.2.3 Motivation, Problem Formulation, and Mathematical Preliminaries

#### 6.2.3.1 Motivational Example

To elucidate the stability tradeoff between heterogeneous agent actuation capabilities and existing parameters in the dynamics of agents, we first give an example in a simple setting with two scalar agent dynamics given by

$$
\dot{x}_1(t) = u_1(t) + w_1x_1(t), \quad x_1(0) = x_{10}, \quad (6.54)
$$

$$
\dot{x}_2(t) = u_2(t) + w_2x_2(t), \quad x_2(0) = x_{20}, \quad (6.55)
$$
where \( x_i(t) \in \mathbb{R} \), \( u_i(t) \in \mathbb{R} \), and \( w_i \in \mathbb{R} \) respectively denote the state, the control input, and a constant of agent \( i \), \( i = 1, 2 \). To simplify the control design for now, assume that \( w_i \) is known and let our objective be to achieve a leader-following behavior while removing the effect of the term \( w_i x_i(t) \) in (6.54) and (6.55). In the absence of actuator dynamics, this problem can be trivially solved with asymptotic stability for any \( w_i \) using 

\[
    u_1(t) = -\left( x_1(t) - x_2(t) \right) - (x_1(t) - c) - w_1 x_1(t) \quad \text{and} \quad u_2(t) = -\left( x_2(t) - x_1(t) \right) - w_2 x_2(t)
\]

(e.g., see [43, 120]), where first agent (6.54) acts as a leader since it has the knowledge of the bounded command \( c \in \mathbb{R} \) to be followed by the states of both agents.

In the presence of actuator dynamics, however, one often has the dynamics given by

\[
\begin{align*}
    \dot{x}_1(t) &= v_1(t) + w_1 x_1(t), \quad v_1(t) = -m_1 \left( v_1(t) - u_1(t) \right), \quad x_1(0) = x_{10}, \quad v_1(0) = v_{10}, \quad (6.56) \\
    \dot{x}_2(t) &= v_2(t) + w_2 x_2(t), \quad v_2(t) = -m_2 \left( v_2(t) - u_2(t) \right), \quad x_2(0) = x_{20}, \quad v_2(0) = v_{20}, \quad (6.57)
\end{align*}
\]

where \( v_i(t) \in \mathbb{R} \) and \( m_i \) denotes the actuator state and the actuator pole of agent \( i \), \( i = 1, 2 \). To further simplify the following discussion on the aforementioned tradeoff, set the command \( c \) to zero. Based on the control inputs given above, one can then write the resulting closed-loop dynamical system in the compact form given by

\[
\begin{bmatrix}
    \dot{x}_1(t) \\
    \dot{v}_1(t) \\
    \dot{x}_2(t) \\
    \dot{v}_2(t)
\end{bmatrix} =
\begin{bmatrix}
    w_1 & 1 & 0 & 0 \\
    -m_1(2 + w_1) & -m_1 & m_1 & 0 \\
    0 & 0 & w_2 & 1 \\
    m_2 & 0 & -m_2(1 + w_2) & -m_2
\end{bmatrix}
\begin{bmatrix}
    x_1(t) \\
    v_1(t) \\
    x_2(t) \\
    v_2(t)
\end{bmatrix}
\begin{bmatrix}
    x_{10} \\
    v_{10} \\
    x_{20} \\
    v_{20}
\end{bmatrix}, \quad (6.58)
\]

such that \( A \) must be Hurwitz for the asymptotic stability of the controlled system. In this simple yet illustrative setting, one can construct, for example, Figure 6.5 to understand the tradeoff between agent actuation capabilities and unknown parameters in the agent dynamics, where the pair \((m_i, w_i)\) changes in homogeneously in the left subfigure and heterogeneously in the right subfigure.

For the homogenous case in Figure 6.5, it is clear that the pole \( m_i \) of the agent actuators must be large enough for asymptotic stability (i.e., \( \max(\text{Re}(\lambda(A))) < 0 \)). For the heterogeneous case, however, we cannot make such a straightforward argument. From a practical standpoint, in addition, it is known that \( w_i \) can be uncertain, agents of a given multiagent system generally do not obey scalar dynamics, and the
number of agents can be finitely many. Thus, the following fundamental research question is immediate:

*For a given multiagent system having \( N \) high-order agents subject to actuator dynamics and system uncertainties, how can we develop a distributed control architecture and a method for performing stability verification of the resulting controlled system?*

In the remainder of this section, we first introduce the multiagent system setup considered throughout the paper (Section 6.2.3.2). We next concisely overview mathematical preliminaries with regard to basic distributed control methods (Sections 6.2.3.3.1 and 6.2.3.3.2) focusing on special cases of the presented multiagent system setup. Note that the main contribution of this paper presented in Section 6.2.4 builds on and significantly generalizes these basic distributed control methods to rigorously answer the question stated at the end of the previous paragraph.
6.2.3.2 Multiagent System Setup and Distributed Control Objective

Consider a multiagent system consisting of \(N\) linear time-invariant agents with heterogeneous actuator dynamics and system uncertainties. Specifically, the dynamics of agent \(i, i = 1, \ldots, N\), is given by

\[
\begin{align*}
\dot{x}_i(t) & = Ax_i(t) + B \left[ v_i(t) + W_i^T x_i(t) \right], \quad x_i(0) = x_{i0}, \quad (6.59) \\
y_i(t) & = C x_i(t), \quad (6.60)
\end{align*}
\]

where \(A \in \mathbb{R}^{n \times n}\) is a known system matrix, \(B \in \mathbb{R}^{n \times m}\) is a known control input matrix, \(C \in \mathbb{R}^{p \times n}\) is a known output matrix, and \(W_i \in \mathbb{R}^{n \times m}\) is an unknown weight matrix, which captures the uncertain parameters in the agent dynamics. In addition, \(x_i(t) \in \mathbb{R}^n\) denotes the state vector of agent \(i\), which is only available to this agent and not to its neighbors, \(y_i(t) \in \mathbb{R}^p\) denotes the output vector of agent \(i\), which is shared with its neighbors locally through a connected, undirected graph \(G\), and \(v_i(t) \in \mathbb{R}^m\) denotes the output vector of the actuator dynamics of agent \(i\) given by

\[
\begin{align*}
\dot{x}_{ci}(t) & = -M_i x_{ci}(t) + u_i(t), \quad x_{ci}(0) = x_{ci0}, \quad (6.61) \\
v_i(t) & = M_i x_{ci}(t), \quad (6.62)
\end{align*}
\]

with \(x_{ci}(t) \in \mathbb{R}^m\) being the actuator state vector, \(M_i = \text{diag}(\{\lambda_j\}) \in \mathbb{R}^{m \times m}\) being a matrix with diagonal entries \(\{\lambda_j\} > 0, j = 1, \ldots, m\), where these entries represent the actuator bandwidths of each control channel for this agent, and \(u_i(t) \in \mathbb{R}^m\) being the feedback control input vector. A graphical depiction of the open-loop dynamics (6.59), (6.60), (6.61), and (6.62) of agent \(i, i = 1, \ldots, N\), is given in Figure 6.6.

For the given multiagent system having \(N\) high-order linear time-invariant agents subject to heterogeneous actuator dynamics and system uncertainties, the overarching objective of this paper is to develop a distributed adaptive control architecture and a linear matrix inequalities-based method for performing stability verification such that the agents of the resulting controlled system work in coherence. To address this objective, we utilize a benchmark leader-follower setting in what follows, where the results of this paper can be readily used for many other problems of multiagent systems as discussed above. In this leader-following setting, as standard, we consider that there exists a subset of agents (at least one) that has the knowledge of a given command input to be tracked (i.e., leader agents), where the other agents do not have any knowledge with regard to this command input (i.e., follower agents). Finally, while we consider agents
6.2.3.3 Mathematical Preliminaries

In this section, we first present a review of distributed control design in the absence of agent actuator dynamics and uncertainties. In this case, (6.59), (6.60), (6.61), and (6.62) simplify to

\[ \dot{x}_i(t) = Ax_i(t) + Bu_i(t), \quad x_i(0) = x_{i0}, \]  
\[ y_i(t) = Cx_i(t), \]  

for \( i = 1, \ldots, N \). In order to drive the outputs of all agents to a given command input only available to the leader agents, one can consider the distributed control approach given by

\[ \dot{x}_i = Ax_i + Bu_i + W_i^T \delta_i, \]

\[ y_i = Cx_i, \]

having identical \( A, B, \) and \( C \) matrices in (6.59) and (6.60) for directly focusing on the main results of this paper, the presented contribution can be extended to the case where these matrices are different by resorting to the works in, for example, [49, 136].

In the remainder of this section, we overview necessary mathematical preliminaries with regard to basic distributed control methods (e.g., [15, 41, 42, 47, 49, 136–139]) focusing on special cases of the above multiagent system setup. Specifically, we first present a concise review of distributed control design in a leader-follower setting for the case when there does not exist actuator dynamics and unknown parameters in agent dynamics (i.e., \( v_i(t) = u_i(t) \) and \( W_i = 0 \) in (6.59)). We then present a similar concise review for the case when there does not exist actuator dynamics (i.e., \( v_i(t) = u_i(t) \) in (6.59)) but there are unknown parameters in the agent dynamics.

Figure 6.6: Open–loop dynamics of agent \( i, i = 1, \ldots, N \), given by (6.59)–(6.62).
\[ u_i(t) = -K_1 x_i(t) + K_2 z_i(t), \]  
(6.65)

\[ \dot{z}_i(t) = -\sum_{i \sim j} (y_i(t) - y_j(t)) - k_i (y_i(t) - c), \]  
(6.66)

where \( K_1 \in \mathbb{R}^{m \times n} \) and \( K_2 \in \mathbb{R}^{m \times p} \) are feedback controller gain matrices, \( z_i(t) \in \mathbb{R}^p \) is an integral state vector, \( c \in \mathbb{R}^p \) is a constant command input, and \( k_i \in \{0, 1\} \) denotes whether the agent is a leader or a follower (i.e., \( k_i = 0 \) for follower agents and \( k_i = 1 \) for leader agents). We refer to, for example, [41, 42, 136, 137, 139] for the distributed control approach given by (6.65) and (6.66) as well as its similar versions, where the following two assumptions are needed.

**Assumption 6.2.1** There exists feedback controller gain matrices \( K_1 \) and \( K_2 \) such that

\[ H_{ii} \triangleq \begin{bmatrix} A - BK_1 & BK_2 \\ -\rho_i C & 0_{p \times p} \end{bmatrix} \in \mathbb{R}^{(n+p) \times (n+p)}, \]  
(6.67)

is Hurwitz for all \( \rho_i, \ i = 1, \ldots, N \), where \( \rho_i \in \text{spec}(\mathcal{F}(\mathcal{G})) \).

**Assumption 6.2.2** There exists \( K_1 \) and \( K_2 \) such that

\[ J \triangleq C(A - BK_1)^{-1}BK_2, \]  
(6.68)

is invertible, where \( J \in \mathbb{R}^{p \times p} \).

With Assumptions 6.2.1 and 6.2.2, the distributed control approach given by (6.65) and (6.66) asymptotically drives the outputs of all agents to a given constant command input only available to leader agents (i.e., \( \lim_{t \to \infty} y_i(t) = c, \ i = 1, \ldots, N \)). To elucidate this point based on the available results in, for example, [136, 139], we now provide the following lemma.

**Lemma 6.2.4** Consider a multiagent system consisting of \( N \) linear time-invariant agents with the dynamics of agent \( i, \ i = 1, \ldots, N \), given by (6.63) and (6.64). In addition, consider the distributed adaptive control architecture in (6.65) and (6.66). If Assumptions 6.2.1 and 6.2.2 hold, then the distributed control approach given by (6.65) and (6.66) asymptotically drives the outputs of all agents to a given constant command input only available to leader agents (i.e., \( \lim_{t \to \infty} y_i(t) = c, \ i = 1, \ldots, N \)).

**Proof.** Based on the results in [136, 139], define \( x(t) \triangleq [x_1(t), x_2(t), \ldots, x_N(t)]^T \in \mathbb{R}^{Nn} \), \( z(t) \triangleq [z_1(t), z_2(t), \ldots, z_N(t)]^T \in \mathbb{R}^{Np} \), and \( y(t) \triangleq [y_1(t), y_2(t), \ldots, y_N(t)]^T \in \mathbb{R}^{Np} \), and \( p_c \triangleq 1_N \otimes c \in \mathbb{R}^{Np} \). Now,
one can write the dynamics given by

\[
\dot{x}(t) = \left( I_N \otimes (A - BK_1) \right) x(t) + \left( I_N \otimes BK_2 \right) z(t), \quad (6.69)
\]

\[
\dot{z}(t) = -\left( \mathcal{F}(G) \otimes C \right) x(t) + (K \otimes I_p) p_c, \quad (6.70)
\]

\[
y(t) = \left( I_N \otimes C \right) x(t). \quad (6.71)
\]

From Lemma 6.2.2, note that \( \mathcal{F}(G) \in \mathbb{R}^{N \times N} \) is a positive-definite matrix that includes the Laplacian matrix of the considered undirected, connected graph \( G \). In a compact form, (6.69) and (6.70) can be rewritten as

\[
\dot{q}(t) = \bar{A} q(t) + \bar{B} p_c, \quad (6.72)
\]

with \( q(t) \triangleq [x^T(t), z^T(t)]^T \in \mathbb{R}^{N(n+p)} \) and

\[
\bar{A} \triangleq \begin{bmatrix}
I_N \otimes (A - BK_1) & I_N \otimes BK_2 \\
-\mathcal{F}(G) \otimes C & 0_{p \times p}
\end{bmatrix} \in \mathbb{R}^{N(n+p) \times N(n+p)}, \quad (6.73)
\]

\[
\bar{B} \triangleq \begin{bmatrix}
0_{Nn \times Np} \\
K \otimes I_p
\end{bmatrix} \in \mathbb{R}^{N(n+p) \times Np}. \quad (6.74)
\]

From Assumption 6.2.1, note that (6.73) is Hurwitz, and hence, there exists a unique positive-definite matrix \( \bar{P} \in \mathbb{R}_{+}^{N(n+p) \times N(n+p)} \) such that \( 0 = \bar{A}^T \bar{P} + \bar{P} \bar{A} + I \) holds. Next, consider the Lyapunov function candidate given by \( V(q + \bar{A}^{-1} \bar{B} p_c) = (q + \bar{A}^{-1} \bar{B} p_c)^T \bar{P} (q + \bar{A}^{-1} \bar{B} p_c) \). Note that \( \bar{A} \) is invertible owing to the fact that it is Hurwitz, and hence, has a nonzero determinant, \( V(0) = 0, V(q + \bar{A}^{-1} \bar{B} p_c) > 0 \) for all \( q + \bar{A}^{-1} \bar{B} p_c \neq 0 \), and \( V(q + \bar{A}^{-1} \bar{B} p_c) \) is radially unbounded. The time-derivative of this Lyapunov function candidate is now given by \( \dot{V}(\cdot) = (q(t) + \bar{A}^{-1} \bar{B} p_c)^T \bar{A}^T \bar{P} + \bar{P} \bar{A}) (q(t) + \bar{A}^{-1} \bar{B} p_c) = -\left\| (q(t) + \bar{A}^{-1} \bar{B} p_c) \right\|_2^2 < 0 \). Thus, \( \lim_{t \to \infty} q(t) = -\bar{A}^{-1} \bar{B} p_c \). Letting \( \bar{C} \triangleq \begin{bmatrix} I_N \otimes C & 0_{Np \times Np} \end{bmatrix} \in \mathbb{R}^{Np \times N(n+p)}, q = -\bar{A}^{-1} \bar{B} p_c \) gives \( y = -\bar{C} \bar{A}^{-1} \bar{B} p_c \) at steady-state. Finally, let

\[
\bar{A}^{-1} = \begin{bmatrix} M_1 & M_2 \\
M_3 & M_4
\end{bmatrix}, \quad (6.75)
\]
where we have $M_2 = -\mathcal{F}(\mathcal{G})^{-1} \otimes (A - BK_1)^{-1} BK_2 \left( C(A - BK_1)^{-1} BK_2 \right)^{-1}$ from Lemma 6.2.3. Note that $M_2$ is well-defined owing to Assumption 6.2.2. From $y = -\bar{C}\bar{A}^{-1}\bar{B}_p c$, we can now write

$$y = -\begin{bmatrix} I_N \otimes C & 0_{Np \times Np} \end{bmatrix} \begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix} \begin{bmatrix} 0_{Np \times Np} \\ K \otimes I_p \end{bmatrix} p_c$$

$$= -(I_N \otimes C) M_2 (K \otimes I_p) p_c$$

$$= (I_N \otimes C) \left( \mathcal{F}(\mathcal{G})^{-1} \otimes (A - BK_1)^{-1} BK_2 \left( C(A - BK_1)^{-1} BK_2 \right)^{-1} \right) (K \otimes I_p) p_c$$

$$= (I_N \otimes I_p) p_c. \quad (6.76)$$

Therefore, it follows that $\lim_{t \to \infty} y_i(t) = c, \ i = 1, \ldots, N.$

**Remark 6.2.1** While our above discussions consider the case when the command input is constant, they can be readily extended to the case when the command input is time-varying with a bounded rate (e.g., see Theorem 1 and Corollary 1 of [136] and Section 3 of [139]). In this case, it is concluded that $y_i(t)$ converges to a neighborhood set of $c(t)$, where the size of this set depends on the command input rate as well as the bandwidth of the controlled system given by (6.72). Therefore, we consider a constant command input case in what follows for directly focusing on the main results of this paper without loss of much generality. Finally, since the content presented in this subsection based on Assumptions 6.2.1 and 6.2.2 is a building block for our main results, these two assumptions are implicitly assumed throughout this paper.

We next present a review of distributed control design in the absence of actuator dynamics and the presence of agent uncertainties. In this case, (6.59), (6.60), (6.61), and (6.62) simplify to

\[
\dot{x}_i(t) = Ax_i(t) + Bu_i(t) + W_i^T x_i(t), \quad x_i(0) = x_{i0}, \quad (6.77)
\]

\[
y_i(t) = Cx_i(t), \quad (6.78)
\]

for $i = 1, \ldots, N$. To suppress the effects of agent uncertainties in a model reference distributed adaptive control setting, we now consider the reference model for agent $i, i = 1, \ldots, N,$ given by

\[
\dot{x}_r_i(t) = A_r x_r_i(t) + B_r z_r_i(t), \quad x_r_i(0) = x_{r0}, \quad (6.79)
\]

\[
\dot{z}_r_i(t) = -\sum_{i\neq j} (y_r_i(t) - y_j(t)) - k_i(y_r_i(t) - c), \quad z_r(0) = z_{r0}, \quad (6.80)
\]

\[
y_r_i(t) = C_x_i(t), \quad (6.81)
\]
where $x_r(t) \in \mathbb{R}^n$ is the reference state vector, $z_r(t) \in \mathbb{R}^p$ is the reference integral state vector, $y_r(t) \in \mathbb{R}^p$ is the reference output vector, and $A_r \triangleq A - BK_1 \in \mathbb{R}^{n \times n}$ and $B_r \triangleq BK_2 \in \mathbb{R}^{n \times p}$. We refer to Remark 6.2.3 below for details on this reference model selection including why $y_j(t)$ is utilized in (6.80) instead of $y_r(t)$.

In order to drive the outputs of all agents to a given command input only available to the leader agents in the presence of agent uncertainties, one can now consider the distributed adaptive control approach given by

$$u_i(t) = -K_1 x_i(t) + K_2 z_i(t) - \hat{W}_i^T(t) x_i(t), \quad (6.82)$$

$$\dot{z}_i(t) = -\sum_{i \sim j} (y_i(t) - y_j(t)) - k_i (y_i(t) - c), \quad (6.83)$$

where $\hat{W}_i(t) \in \mathbb{R}^{n \times m}$ is an estimate of unknown parameters $W_i$ that satisfies the local (i.e., agent-wise) update law

$$\dot{\hat{W}}_i(t) = \alpha \text{Proj}[\hat{W}_i(t), x_i(t) \tilde{q}_i(t) P_i G], \quad (6.84)$$

with $G \triangleq [B^T, 0_{m \times p}]^T \in \mathbb{R}^{(n+p) \times m}$, $\alpha \in \mathbb{R}_+$ being the learning rate gain, and

$$\tilde{q}_i(t) \triangleq \begin{bmatrix} \tilde{x}_i(t) \\ \tilde{z}_i(t) \end{bmatrix} = \begin{bmatrix} x_i(t) \\ z_i(t) \end{bmatrix} - \begin{bmatrix} x_r(t) \\ z_r(t) \end{bmatrix} \in \mathbb{R}^{n+p}, \quad (6.85)$$

being the augmented agent error state. Considering (6.84), in addition, the projection bounds are defined such that $|\hat{W}_i(t)| \leq \hat{W}_{i,\text{max}}$. Moreover, $P_i \in \mathbb{R}_+^{(n+p) \times (n+p)}$ in (6.84) satisfies the local Lyapunov equation given by $0 = \mathcal{H}_2^T P_i + P_i \mathcal{H}_2 + I$ with

$$\mathcal{H}_2i \triangleq \begin{bmatrix} A - BK_1 & BK_2 \\ - (d_i + k_i)C & 0_{p \times p} \end{bmatrix} \in \mathbb{R}^{(n+p) \times (n+p)}, \quad (6.86)$$

when the following assumption holds.

**Assumption 6.2.3** There exists $K_1$ and $K_2$ such that $\mathcal{H}_2i$ is Hurwitz for $i = 1, \ldots, N$.

**Remark 6.2.2** Assumptions 6.2.1 and 6.2.3 include the same feedback controller gain matrices $K_1$ and $K_2$ and the same system matrices $A$, $B$, and $C$. Motivated from this fact, one can group these assumptions into...
a single matrix as

\[
H^*_i \triangleq \begin{bmatrix}
A-BK_1 & BK_2 \\
-\mu C & 0_{p\times p}
\end{bmatrix} \in \mathbb{R}^{(n+p)\times(n+p)},
\]

(6.87)

where \(\mu_i\) takes values from the set containing both \(\rho_i\) and \(d_i+k_i\); hence, there exists a positive lower bound \(\underline{\mu} \in \mathbb{R}_+\) and a positive upper bound \(\overline{\mu} \in \mathbb{R}_+\) such that \(0 < \underline{\mu} \leq \mu_i \leq \overline{\mu} < \infty\) since all \(\rho_i\) and \(d_i+k_i\) numbers are positive. That is, when \(K_1\) and \(K_2\) make \(H^*_i\) Hurwitz for all \(\mu_i\), then Assumptions 6.2.1 and 6.2.3 hold.

Here, one can resort to methods from robust control analysis or synthesis to respectively check or ensure \(H^*_i\) is Hurwitz for all \(\mu_i\) [41]. Specifically, if desired, robust control analysis methods (see, for example, Section 2 of [140]) can be utilized to check whether \(H^*_i\) is Hurwitz or not on the parameter set of \(\mu_i \in [\underline{\mu}, \overline{\mu}]\) with respect to given \(K_1\) and \(K_2\). If \(H^*_i\) is not Hurwitz for all \(\mu_i\), then one can judiciously redesign feedback controller gain matrices \(K_1\) and \(K_2\). To perform this design process more systematically, one can also consider adopting methods from robust control synthesis. To this end, several methods (see, for example, Section 4 of [140]) can be utilized to systematically choose the feedback controller gain matrices \(K_1\) and \(K_2\) which ensure \(H^*_i\) is Hurwitz on the parameter set of \(\mu_i \in [\underline{\mu}, \overline{\mu}]\).

We refer to, for example, [15, 47, 49, 138] for the distributed adaptive control approach given by (6.82), (6.83), and (6.84) as well as its similar versions, where the following result is based on these references.

**Lemma 6.2.5** Consider a multiagent system consisting of \(N\) linear time-invariant agents with the dynamics of agent \(i, i = 1, \ldots, N\), given by (6.77) and (6.78). In addition, consider the reference model given by (6.79), (6.80), and (6.81) along with the proposed distributed adaptive control architecture in (6.82), (6.83), and (6.84). If Assumptions 6.2.1, 6.2.2, and 6.2.3 hold, then the solution \((\bar{q}(t), \bar{W}(t))\) of the closed-loop dynamical system is bounded.

**Proof.** From (6.77), (6.79), (6.80), (6.82), and (6.83), we write the local error dynamics given by

\[
\dot{x}_i(t) = A_x \bar{x}_i(t) + B_i \bar{z}_i(t) - B \bar{W}_i^T(t) x_i(t),
\]

(6.88)

\[
\dot{\bar{z}}_i(t) = -(d_i+k_i)C \bar{x}_i(t),
\]

(6.89)
where \( \hat{W}_i(t) = \hat{W}_i(t) - W_i \in \mathbb{R}^{n \times m} \). Using (6.85), these local error dynamics can now be written in a compact form given by

\[
\tilde{q}_i(t) = H_{2i}q_i(t) - G\hat{W}_i^T(t)x_i(t), \tag{6.90}
\]

where notice that \( H_{2i} \) is Hurwitz by Assumption 6.2.3. Next, consider the quadratic function

\[
\mathcal{V}_i(\hat{q}_i, \hat{W}_i) = \hat{q}_i^TP_i\hat{q}_i + \alpha^{-1}\text{tr}\hat{W}_i^T\hat{W}_i, \tag{6.91}
\]

and note that \( \mathcal{V}(0,0) = 0, \mathcal{V}(\hat{q}_i, \hat{W}_i) > 0 \) for all \( (\hat{q}_i, \hat{W}_i) \neq (0,0) \), and \( \mathcal{V}(\hat{q}_i, \hat{W}_i) \) is radially unbounded. The time-derivative of (6.91) yields \( \dot{\mathcal{V}}(\hat{q}_i, \hat{W}_i) = 2\tilde{q}_i^TP_i[H_{2i}q_i(t) - G\hat{W}_i^T(t)x_i(t)] + 2\text{tr}\hat{W}_i^T(t)\text{Proj}[\hat{W}_i(t), x_i(t)\tilde{q}_i^T(t)PG] \). Using a projection operator property, \( \text{tr}\hat{W}_i^T(t)(\text{Proj}[\hat{W}_i(t), x_i(t)\tilde{q}_i^T(t)PG] - x_i(t)\tilde{q}_i^T(t)PG) \leq 0 \) Lemma 11.3 of [74]. \( \mathcal{V}(\hat{q}_i, \hat{W}_i) \) as follows. Based on (6.91), let the Lyapunov function candidate be given by \( \mathcal{V}(\hat{q}, \hat{W}) = \sum_{i=1}^N \mathcal{V}_i(\hat{q}_i, \hat{W}_i) \), which yields \( \dot{\mathcal{V}}(\cdot) \leq -q^T(t)(I_N \otimes I_{n+p})\hat{q}(t) \). Thus, it is immediate that the pair \( (\hat{q}_i(t), \hat{W}_i(t)) \) is bounded for all \( i = 1, \ldots, N \). Finally, let \( x_i(t) \triangleq [x_{r_1}(t), \ldots, x_{r_N}(t)]^T \in \mathbb{R}^{Nn}, z_i(t) \triangleq [z_{r_1}(t), \ldots, z_{r_N}(t)]^T \in \mathbb{R}^{Np}, \) and \( q_i \triangleq [x_i^T(t), z_i^T(t)]^T \in \mathbb{R}^{N(p+n)} \). Then, the reference models of all agents can be written in the compact form given by \( \dot{\tilde{x}}_i(t) = \tilde{A}_i(t) + \tilde{B}_iP \tilde{c} + \left[0_{Nn \times Nn}, (\tilde{A}(G) \otimes C)^T\right] \tilde{x}(t) \), where this expression gives bounded trajectories owing to the facts that \( \tilde{A} \) is Hurwitz by Assumption 6.2.1 and the term \([0_{Nn \times Nn}, (\tilde{A}(G) \otimes C)^T]^T \tilde{x}(t)\) is bounded from the above analysis. From Barbalat’s lemma [57], \( \dot{\mathcal{V}}(\cdot) \to 0 \) as \( t \to \infty \) is now evident. This result further implies \( x_i(t) \to x_i(t) \) as \( t \to \infty \), \( z_i(t) \to z_i(t) \) as \( t \to \infty \), and \( y_i(t) \to y_i(t) \) as \( t \to \infty \).

**Remark 6.2.3** We use \( y_j(t) \) in (6.80) of the reference model for agent \( i, i = 1, \ldots, N \), instead of \( y_j(t) \). This is motivated by our prior works in, for example, [15, 138], which yields to local update law given by (6.84) and the aforementioned local Lyapunov equation. In this setting, in addition, the reference model given by (6.79), (6.80), and (6.81) does not require any additional information from neighboring agents such as the output of their reference models \( y_i(t) \). Note that this does not prevent the outputs of all agents to converge to a given constant command input only available to leader agents. To elucidate this point, notice that since the term \([0_{Nn \times Nn}, (\tilde{A}(G) \otimes C)^T]^T \tilde{x}(t)\) asymptotically goes to zero in \( \dot{\tilde{q}}_i(t) = \tilde{A}_i(t) + \tilde{B}_iP \tilde{c} + \left[0_{Nn \times Nn}, (\tilde{A}(G) \otimes C)^T\right] \tilde{x}(t) \), one can still recover the ideal reference model dynamics given by \( \dot{\tilde{q}}(t) = \tilde{A}\tilde{q}(t) + \tilde{B}\tilde{c}, \) which can be given in the local form for \( i = 1, \ldots, N \) as
\begin{align}
\dot{x}_{\text{id}}^{i}(t) &= A_{\text{id}} x_{\text{id}}^{i}(t) + B_{\text{id}} z_{\text{id}}^{i}(t), \quad x_{\text{id}}^{i}(0) = x_{\text{id}}^{i0}, \\
\dot{z}_{\text{id}}^{i}(t) &= - \sum_{i \sim j} (y_{\text{id}}^{i}(t) - y_{\text{id}}^{j}(t)) - k_{i}(y_{\text{id}}^{i}(t) - c), \quad z_{\text{id}}^{i}(0) = z_{\text{id}}^{i0}, \\
\dot{y}_{\text{id}}^{i}(t) &= C x_{\text{id}}^{i}(t),
\end{align}

where $x_{\text{id}}^{i}(t) \in \mathbb{R}^{n}$ is the ideal reference state vector, $z_{\text{id}}^{i}(t) \in \mathbb{R}^{p}$ is the ideal reference integral state vector, $y_{\text{id}}^{i}(t) \in \mathbb{R}^{p}$ is the ideal reference output vector. Here, $\hat{q}(t) = [ (x_{\text{id}}^{i1}(t))^{T}, \ldots, (x_{\text{id}}^{iN}(t))^{T}, (z_{\text{id}}^{i1}(t))^{T}, \ldots, (z_{\text{id}}^{iN}(t))^{T}]^{T}$. Since $\lim_{t \to \infty} y_{\text{id}}^{i}(t) = c$ by Lemma 6.2.4 subject to Assumptions 6.2.1 and 6.2.2 (notice that (6.92), (6.93), and (6.94) are dynamically identical to (6.63) subject to (6.65), (6.66), and (6.64) of Section 6.2.3.3.1, respectively), it now follows from the above discussion that $\lim_{t \to \infty} y_{i}(t) = c$. Finally, similar to the last part of the discussion in Remark 6.2.1, since the contents presented in this subsection and Section 6.2.3.3.1 are based on Assumptions 6.2.1, 6.2.2, and 6.2.3 and since these contents are building bricks for our main results presented in the next section, we make these three assumptions implicitly throughout this paper.

6.2.4 Distributed Adaptive Control and Stability Verification

In this section, we present a distributed adaptive control architecture and a method for performing stability verification based on the multiagent system setup introduced in Section 6.2.3.2, where each agent can be subject to heterogeneous actuation capabilities and unknown parameters (see Figure 6.6). While the preliminary contents presented in Sections 6.2.3.3.1 and 6.2.3.3.2 are the building bricks of our theoretical development here, these results do not consider actuator dynamics. Thus, they can yield to unstable controlled system trajectories when they are directly considered for the multiagent system setup in Section 6.2.3.2 (e.g., see Figure 6.5 for a motivational simple example). To this end, we utilize linear matrix inequalities in this section in order to reveal the fundamental tradeoff between heterogeneous agent actuation capabilities and unknown parameters in agent dynamics, and hence, the proposed approach allows for stable distributed adaptation.
We start by arranging (6.59) as

\[
\dot{x}_i(t) = A x_i(t) + B \left[ v_i(t) + W^T_i x_i(t) \right]
\]

\[
= A x_i(t) + B \left[ u_i(t) + W^T_i x_i(t) \right] + B \left[ v_i(t) - u_i(t) \right].
\] (6.95)

Using the distributed adaptive control approach in (6.82), (6.83), and (6.84), we now rewrite (6.95) as

\[
\dot{x}_i(t) = A x_i(t) + B \left[ u_i(t) + (W^T_i x_i(t) - W^T_i x_i(t)) \right] + B \left[ v_i(t) - u_i(t) \right].
\] (6.96)

To achieve correct local adaptation that is not affected by the presence of heterogeneous agent actuator dynamics, we next consider a hedging method based reference model given by

\[
\dot{x}_{ri}(t) = A r x_{ri}(t) + B r z_{ri}(t) - B \tilde{W}^T_i x_i(t) + B \left[ v_i(t) - u_i(t) \right],
\] (6.97)

\[
\dot{z}_{ri}(t) = -\sum_{i=1}^{N} (y_{ri}(t) - y_{j}(t)) - k_i (y_{ri}(t) - c),
\] (6.98)

\[
y_{ri}(t) = C x_{ri}(t),
\] (6.99)

instead of the one introduced in (6.79), (6.80), and (6.81), where “B \left[ v_i(t) - u_i(t) \right]” in (6.97) is the hedging term. Specifically, since not all agent reference models can be followed in the presence of actuator dynamics, this method adds “B \left[ v_i(t) - u_i(t) \right]” to the reference models for \(i = 1, \ldots, N\) such that the resulting reference model trajectories can be followed by the agent dynamics in the presence of system uncertainties (we refer to [34, 113] for additional details).

Now, with the hedging term added to the reference model, one can write the same agent error dynamics in Section 6.2.3.3.1 as

\[
\dot{\tilde{x}}_i(t) = A \tilde{x}_i(t) + B \tilde{z}_i(t) - B \tilde{W}_i^T(t) x_i(t),
\] (6.100)

\[
\dot{\tilde{z}}_i(t) = -(d_i + k_i) C \tilde{x}_i(t),
\] (6.101)

which can be written in a compact form given by

\[
\dot{\tilde{q}}_i(t) = H \tilde{q}_i(t) - G \tilde{W}_i^T(t) x_i(t).
\] (6.102)
The following stability condition is needed for the results of this paper.

**Assumption 6.2.4** The square matrix given by

\[
\mathcal{A}(\hat{W}_i(t), M_i) = \begin{bmatrix}
I_N \otimes A + \phi_1 & 0_{Nn \times Np} & \bar{B} \\
-\mathcal{F}(G) \otimes C & 0_{Np \times Np} & 0_{Np \times Nm} \\
-I_N \otimes K_1 - \phi_2 & I_N \otimes K_2 & -\bar{M}
\end{bmatrix} \in \mathbb{R}^{N(n+p+m) \times N(n+p+m)},
\]  

(6.103)

is quadratically stable such that there exists a positive-definite matrix \( P \in \mathbb{R}^{N(n+p+m) \times N(n+p+m)} \) satisfying the Lyapunov inequality given by

\[
\mathcal{A}^T(\hat{W}_i(t), M_i)P + PA(\hat{W}_i(t), M_i) < 0,
\]  

(6.104)

where \( \phi_1 \triangleq \text{block-diag}(B\hat{W}_1^T, \ldots, B\hat{W}_N^T) \in \mathbb{R}^{Nn \times Nn}, \phi_2 \triangleq \text{block-diag}(\hat{W}_1^T, \ldots, \hat{W}_N^T) \in \mathbb{R}^{Nm \times Nn}, \bar{B} \triangleq \text{block-diag}(BM_1, \ldots, BM_N) \in \mathbb{R}^{Nn \times Nn}, \text{and } \bar{M} \triangleq \text{block-diag}(M_1, \ldots, M_N) \in \mathbb{R}^{Nm \times Nn}.

The following theorem, which presents the main result of this paper, addresses the distributed control objective highlighted in the second paragraph of Section 6.2.3.2.

**Theorem 6.2.1** Consider a multiagent system consisting of \( N \) linear time-invariant agents with the dynamics of agent \( i, i = 1, \ldots, N, \) given by (6.59) and (6.60) subject to the actuator dynamics in (6.61) and (6.62). In addition, consider the proposed hedging method based reference model given by (6.97), (6.98), and (6.99) along with the proposed distributed adaptive control architecture in (6.82), (6.83), and (6.84). If Assumptions 6.2.1, 6.2.2, 6.2.3, and 6.2.4 hold, then the solution \((\tilde{q}(t), \hat{W}(t), x_i(t), z_i(t), v(t))\) of the closed-loop dynamical system is bounded and \(\lim_{t \to \infty} \tilde{q}(t) = 0\).

**Proof.** We first show the boundedness of the pair \((\tilde{q}(t), \hat{W}(t))\) based on the results in Lemma 6.2.5. Specifically, considering the quadratic function given by (6.91), one can write \(\mathcal{V}_i(\tilde{q}_i(t), W_i(t)) \leq -\tilde{q}_i^T(t)\tilde{q}_i(t)\). Now, based on the Lyapunov function candidate given by \(\mathcal{V}(\tilde{q}, \hat{W}) = \sum_{i=1}^{N} \mathcal{V}_i(\tilde{q}_i, W_i)\), one can readily see that \(\dot{\mathcal{V}}(\cdot) \leq -\tilde{q}^T(t)(I_N \otimes I_{n+p})\tilde{q}(t)\) holds, and hence, this conclusion yields the boundedness of the pair \((\tilde{q}(t), \hat{W}(t))\). Unlike the results in Lemma 6.2.5, however, one needs to ensure the boundedness of the triple \((x_i(t), z_i(t), v(t))\) to utilize the Barbalat’s lemma here.

In particular, to show the boundedness of this triple, the dynamics of the agent reference models given by (6.97) and (6.98) and the dynamics of the agent actuators given by (6.61) and (6.62) can be
rearranged using the distributed adaptive control approach in (6.82) and (6.83) as

\[
\dot{x}_i(t) = A_1 x_i(t) + B_1 z_i(t) + B[M_1 x_c(t) + K_1 x_i(t) - K_2 z_i(t) + \hat{W}_i^T(t) x_i(t)] \\
= A x_i(t) + B M x_c(t) + B \hat{W}_i^T(t) x_i(t) + BK_1 \hat{x}_i(t) - BK_2 \hat{z}_i(t) + B \hat{W}_i^T(t) \hat{x}_i(t),
\]

(6.105)

\[
\dot{z}_i(t) = - \sum_{i \sim j} (y_i(t) - y_j(t)) - k_i(y_i(t) - c) \\
= - \sum_{i \sim j} (y_i(t) - y_j(t)) - k_i(y_i(t) - c) - \sum_{j \in N} (y_j(t) - y_j(t)) \\
= - \sum_{i \sim j} (y_i(t) - y_j(t)) - k_i(y_i(t) - c) - \sum_{j \in N} (C x_j),
\]

(6.106)

\[
\dot{x}_c(t) = - M_2 x_c(t) - K_1 x_i(t) + K_2 z_i(t) - \hat{W}_i^T(t) x_i(t) \\
= - M_2 x_c(t) - K_1 x_i(t) + K_2 z_i(t) - \hat{W}_i^T(t) x_i(t) - K_1 \hat{x}_i(t) + K_2 \hat{z}_i(t) - \hat{W}_i^T(t) \hat{x}_i(t).
\]

(6.107)

Now, let the aggregated vectors be given by \( \bar{x}(t) = [\bar{x}_1(t), \bar{x}_2(t), \ldots, \bar{x}_N(t)]^T \in \mathbb{R}^{Nn} \), \( \bar{z}(t) = [\bar{z}_1(t), \bar{z}_2(t), \ldots, \bar{z}_N(t)]^T \in \mathbb{R}^{Nn} \), \( z_c(t) = [z_{c1}(t), z_{c2}(t), \ldots, z_{cN}(t)]^T \in \mathbb{R}^{Nn} \), such that (6.105), (6.106), and (6.107) can be compactly written in the form given by

\[
\dot{x}(t) = (I_N \otimes A + \Phi_1) x(t) + (I_N \otimes B K_1) \bar{x}(t) - (I_N \otimes B K_2) \bar{z}(t) + \Phi_1 \bar{x}(t),
\]

(6.108)

\[
\dot{z}(t) = -(F(G) \otimes C) x(t) + (K \otimes I_p) z_c,
\]

(6.109)

\[
\dot{z}_c(t) = - M_2 x_c(t) - (I_N \otimes K_1) x_i(t) - \Phi_2 x(t) + (I_N \otimes K_2) z_i(t) - (I_N \otimes K_1) \bar{x}(t) - \Phi_2 \bar{x}(t) + (I_N \otimes K_2) \bar{z}(t).
\]

(6.110)

Next, defining \( \xi(t) \triangleq [x_i^T(t), z_i^T(t), x_c^T(t)]^T \), (6.108), (6.109), and (6.110) can be written as

\[
\dot{\xi}(t) = A(\hat{W}(t), M) \xi(t) + \omega(\cdot),
\]

(6.111)

where \( \omega(\cdot) \) satisfies

\[
\omega(\cdot) = \begin{bmatrix}
(I_N \otimes B K_1) \bar{x}(t) - (I_N \otimes B K_2) \bar{z}(t) + \Phi_1 \bar{x}(t) \\
(K \otimes I_p) z_c \\
-(I_N \otimes K_1) \bar{x}(t) - \Phi_2 \bar{x}(t) + (I_N \otimes K_2) \bar{z}(t)
\end{bmatrix}.
\]

(6.112)
Notice that $\phi(\cdot)$ is a bounded perturbation as a result of Lyapunov stability of $(\tilde{q}, \tilde{W})$. It now follows that since $\phi(\cdot)$ is bounded and $A(\tilde{W}(t), M)$ is quadratically stable by Assumption 6.2.4, then $x_i(t)$, $z_c(t)$, and $x_c(t)$ are bounded (see, for example, [57]). This further implies the boundedness of the actuator output $v(t)$, and hence, the boundedness of the triple $(x_i(t), z_c(t), v(t))$.

Finally, in order to conclude that $\lim_{t \to \infty} \tilde{q}(t) = 0$, note that $q(t)$ is bounded as a result of the boundedness of $\tilde{q}(t)$, $x_i(t)$, and $z_c$. It now follows from (6.102) that $\dot{\tilde{q}}(t)$ is bounded, and hence, $\tilde{V}(\cdot)$ is bounded for all $t \in \mathbb{R}_+$. It then follows from Barbalat’s lemma that $\lim_{t \to \infty} \tilde{V}(\cdot) = 0$, which consequently shows that $\tilde{q}(t) \to 0$ as $t \to \infty$. The proof is now complete.

The quadratic stability condition on the matrix $A(\tilde{W}_i(t), M_i)$ stated in Assumption 6.2.4 and utilized in Theorem 6.2.1 is important in order to reveal the fundamental tradeoff between heterogeneous agent actuation capabilities and unknown parameters in the agent dynamics (see also Section 6.2.5). Yet, we need the following remark that highlights how we can numerically evaluate this quadratic stability condition using linear matrix inequalities.

**Remark 6.2.4** We now utilize linear matrix inequalities to satisfy the quadratic stability of (6.103) for given projection bounds $\tilde{W}_{i, \max}$ and the bandwidths of the actuator dynamics $M_i$. For this purpose, let $\bar{W}_{i_{j_1 \ldots j_l}} \in \mathbb{R}^{n \times m}$ be defined as

$$
\bar{W}_{i_{j_1 \ldots j_l}} = \begin{bmatrix}
(-1)^{j_1} \bar{W}_{i_{max,1}} & (-1)^{j_2} \bar{W}_{i_{max,2+n}} & \cdots & (-1)^{j_1 + (m-1)n} \bar{W}_{i_{max,1+(m-1)n}} \\
(-1)^{j_2} \bar{W}_{i_{max,2}} & (-1)^{j_3} \bar{W}_{i_{max,2+2n}} & \cdots & (-1)^{j_2 + (m-1)n} \bar{W}_{i_{max,2+(m-1)n}} \\
\vdots & \vdots & \ddots & \vdots \\
(-1)^{j_m} \bar{W}_{i_{max,m}} & (-1)^{j_{m+1}} \bar{W}_{i_{max,2n}} & \cdots & (-1)^{j_m+n} \bar{W}_{i_{max,nn}}
\end{bmatrix},
$$

(6.113)

for agents $i = 1, \ldots, N$, where $j_l \in \{1, 2\}$, $l \in \{1, \ldots, 2^nn\}$ such that $\bar{W}_{i_{j_1 \ldots j_l}}$ represents the corners of the hypercube defining the maximum variation of $\bar{W}_i(t)$, for each agent $i = 1, \ldots, N$. Furthermore, since we define $\bar{\phi}_1 \triangleq \text{block-diag}(B\bar{W}_1^T, \ldots, B\bar{W}_N^T)$ and $\bar{\phi}_2 \triangleq \text{block-diag}(\tilde{W}_1^T, \ldots, \tilde{W}_N^T)$, let $\bar{\phi}_{1, r} \in \mathbb{R}^{Nn \times Nn}$ and $\bar{\phi}_{2, r} \in \mathbb{R}^{n \times n}$.
be defined respectively as

\[
\bar{\varphi}_{1,...,r} \triangleq \begin{bmatrix}
B \overline{W}_{1j_1...j_l}^T & 0_{n \times n} & \cdots & 0_{n \times n} \\
0_{n \times n} & B \overline{W}_{2j_1...j_l}^T & \cdots & \vdots \\
\vdots & \ddots & \ddots & 0_{n \times n} \\
0_{n \times n} & \cdots & 0_{n \times n} & B \overline{W}_{Nj_1...j_l}^T
\end{bmatrix},
\]

(6.114)

\[
\bar{\varphi}_{21,...,r} \triangleq \begin{bmatrix}
W_{1j_1...j_l}^T & 0_{m \times n} & \cdots & 0_{m \times n} \\
0_{m \times n} & W_{2j_1...j_l}^T & \cdots & \vdots \\
\vdots & \ddots & \ddots & 0_{m \times n} \\
0_{m \times n} & \cdots & 0_{m \times n} & W_{Nj_1...j_l}^T
\end{bmatrix},
\]

(6.115)

where \(r \in \{1, \ldots, (2^m)^N\}\). Following the results in [111, 128], if the square matrix given by

\[
A_{1,...,r} = \begin{bmatrix}
I_N \otimes A + \bar{\varphi}_{1,...,r} & 0_{Nn \times Np} & B \\
-\mathcal{F}(\mathcal{G}) \otimes C & 0_{Np \times Np} & 0_{Np \times Nm} \\
-I_N \otimes K_1 - \bar{\varphi}_{21,...,r} & I_N \otimes K_2 & -\mathcal{M}
\end{bmatrix},
\]

(6.116)

satisfies the linear matrix inequality

\[
A_{1,...,r}^T \mathcal{P} + \mathcal{P} A_{1,...,r} < 0, \quad \mathcal{P} > 0,
\]

(6.117)

for all permutations of \(\bar{\varphi}_{1,...,r}\) and \(\bar{\varphi}_{21,...,r}\), then (6.103) is quadratically stable. We can then cast (6.117) as a convex optimization problem and solve it using linear matrix inequalities for stability verification purposes.

As a result of Theorem 6.2.1 and the use of linear matrix inequalities highlighted in Remark 6.2.4, for a given multiagent system with heterogeneous actuator dynamics and internal system uncertainties, a feasible stability region can be obtained to determine appropriate operating actuator bandwidth values for each agent. In the illustrative numerical section, this is shown next for several different multiagent formations to investigate how the linear matrix inequalities calculated feasible stability region changes.
In order to illustrate the proposed distributed adaptive control architecture for linear time-invariant multiagent systems with heterogeneous actuator dynamics and system uncertainties, we consider a group of agents with the dynamics

\[
\dot{x}_i(t) = \begin{bmatrix} 0 & 1 \\ 0.5 & 0.5 \end{bmatrix} x_i(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (v_i(t) + W_i^T x_i(t)), \quad x_i(0) = [0, 0]^T, 
\]

(6.118)

\[
y_i(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} x_i(t). 
\]

(6.119)

For each agent, we choose the reference model given by (6.97) with the matrices

\[
A_r = \begin{bmatrix} 0 & 1 \\ -0.5 & -1 \end{bmatrix}, \quad B_r = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}, 
\]

(6.120)

which yields the nominal control gains \(K_1 = [1, 1.5]\) and \(K_2 = 0.5\). In the following examples, all agents have a single channel actuator for executing their control input such that \(M_i = \lambda_i, \lambda_i \in \mathbb{R}^+.\) In addition, we set \(W_i = [0.45; 0.45]\) and use \(|[\hat{W}_i(t)]_{1,1}| \leq 0.5\) and \(|[\hat{W}_i(t)]_{2,1}| \leq 0.5\) as the bounds of the (rectangular) projection operator.

In what follows, we consider nonidentical multiagent graph topologies to investigate the effect of leader location, additional information links, and ratio of leaders to followers. In each case, we consider the leaders (“dark grey” in respective figures) to have an actuator bandwidth \(\lambda_1\) and the followers (“light grey” in respective figures) to have an actuator bandwidth \(\lambda_2\). Using the projection bounds in the linear matrix inequality-based stability verification analysis outlined in Remark 6.2.4, the feasible region of allowable actuator bandwidth limits for \(\lambda_1\) and \(\lambda_2\) is calculated. As a result, appropriate operating actuator bandwidth values for the leader and follower agents are obtained.

**Example 6.2.1 Effect of Leader Location on a Star Graph:** In this example, we consider a star graph topology given in Figure 6.7. The effect of the leader location is investigated by comparing two cases. For the first case (Example 6.2.1.1), we place the leader at one of the corners of the star graph and in the second case (Example 6.2.1.2), we place the leader in the center of the star graph. An linear matrix inequality analysis is then conducted for both configurations of the leader location in the star graph. To this end,
Figure 6.7: Star graph for Example 6.2.1 (Example 6.2.1.1 on the left subfigure and Example 6.2.1.2 on the right subfigure) with “dark grey” color indicating the leader and “light grey” color indicating the followers.

Figure 6.8 provides a comparison of the resulting feasible stability regions of allowable actuator bandwidth values. Specifically, the top subplot on this figure shows the linear matrix inequality results such that when the pair \((\lambda_1, \lambda_2)\) is selected from the region staying above these lines, then the controlled multiagent system is guaranteed to be stable. On the same figure, the middle and bottom subplots show the minimum unstable values of the pair \((\lambda_1, \lambda_2)\) obtained numerically from the simulation with \(c = 1\) and \(c(t) = 0.5\sin(0.01\pi t)\), respectively\(^{61}\). Comparing the results of middle and bottom subplots with the one on top subplot, it can be readily seen that the proposed linear matrix inequality results provide less conservative conclusions on determining the feasible stability regions. Furthermore, it is clear that the change in the location of the leader significantly alters the resulting feasible stability regions.

To show the performance of the proposed distributed adaptive control design, three points are selected from Figure 6.8 for the case of Example 6.2.1, which are indicated by the black circles. The numerical simulation results are shown at these actuator bandwidth values in Figures 6.9 and 6.10 with the constant command \(c = 1\) and Figures 6.11 and 6.12 with the bounded time varying command \(c(t) = 0.5\sin(0.01\pi t)\), respectively. In particular, since the feasible linear matrix inequality boundary corresponds to calculated minimum feasible values for the actuator bandwidths, it is expected that the system performances are guaranteed to be stable for actuator dynamics at points greater than or equal to the calculated feasible values on the pair \((\lambda_1, \lambda_2)\) as can be seen in Figures 6.9 and 6.11. It is not until the actuator bandwidth values are decreased below the linear matrix inequalities calculated limit that the multiagent system reaches a point of instability due to the unbounded reference model trajectory as seen in Figures 6.10 and 6.12. Note that we choose to show only three points of only one configuration of the leader location to avoid

\^{61}\)Since the numerically obtained minimum unstable values of the pair \((\lambda_1, \lambda_2)\) do not significantly change in these middle and bottom subplots corresponding to time-invariant and time-varying command inputs, respectively, we only consider the case with the time-invariant command input \(c = 1\) in the following examples.
Figure 6.8: Feasible region of actuator bandwidth values for Example 6.2.1. The top subplot presents the linear matrix inequalities calculated region, the middle subplot shows the simulated points yielding unbounded reference model trajectories with $c = 1$ and the bottom subplot shows the simulated points yielding unbounded reference model trajectories with $c(t) = 0.5 \sin(0.01 \pi t)$.

repetitiveness as the other actuator bandwidth value pairs shown in Figure 6.8 would provide similar simulation plots as in Figures 6.9-6.12.

**Example 6.2.2 Effect of Leader Location on a Line Graph:** In this example, we consider a line graph topology with nonidentical leader locations given in Figure 6.13. The effect of the leader location is investigated by comparing three cases. For the first case (Example 6.2.2.1) we place the leader in the middle of the line graph and for the remaining two cases we move the leader toward the front of the line graph (Examples 6.2.2.2 and 6.2.2.3 respectively). We then conducted an linear matrix inequality analysis for these three configurations. Specifically, the top subplot of Figure 6.14 presents the resulting linear matrix inequality feasible stability regions of allowable actuator bandwidth values and the bottom subplot of the same figure shows the minimum unstable values obtained numerically from the simulation for a given
Figure 6.9: Proposed distributed adaptive control performance for Example 6.2.1.1 with actuator bandwidth values \((\lambda_1, \lambda_2) = (50, 27.02)\) and command \(c = 1\).

Figure 6.10: Proposed distributed adaptive control performance for Example 6.2.1.1 with actuator bandwidth values \((\lambda_1, \lambda_2) = (50, 17.46)\) and command \(c = 1\).
constant command $c = 1$. Consistent with the first example, it can be seen that the change in the location of the leader significantly alters the resulting feasible stability regions. Finally, to show the performance of the proposed distributed adaptive control design, two points are selected from Figure 6.14 for the case of Example 6.2.2.1, which are indicated by the black circles. The numerical simulation results are shown at these actuator bandwidth values in Figures 6.15 and 6.16 with the command $c(t) = 0.5 \sin(0.01\pi t)$.

**Example 6.2.3** Effect of Leader Location on a Circle Graph: In this example, we consider a circle graph topology with nonidentical leader locations given in Figure 6.17. The effect of leader location is investigated by comparing two cases. These cases now include two leaders. For the first case (Example 6.2.3.1) we place the leaders as separated by followers such that they do not directly exchange information with each other, and in the second case (Example 6.2.3.2) we place the leaders next to each other such that they do directly exchange information. The feasible region of allowable actuator bandwidth pairs is shown in Figure 6.18 for the two different leader locations in the circle graph. We then perform a linear matrix inequality analysis for these two configurations. In particular, the top subplot of Figure 6.18 presents the resulting
linear matrix inequality feasible stability regions of allowable actuator bandwidth values and the bottom subplot of the same figure shows the minimum unstable values obtained numerically from the simulation for a given constant command $c = 1$. Once again, the change in the locations of two leaders results in different feasible stability regions. Finally, to show the performance of the proposed distributed adaptive control design, two points are selected from Figure 6.18 for the case of Example 6.2.3.2, which are indicated by the black circles. The numerical simulation results are shown at these actuator bandwidth values in Figures 6.19 and 6.20 with command $c = 1$, where similar conclusions are observed as for the case in Examples 6.2.1 and 6.2.2.

**Example 6.2.4 Number of the Information Links on a Circle Graph:** In this example, we consider a circle graph topology with added information links between agents; see in Figure 6.21. The effect of the added information links is investigated by comparing three cases. For the first case (Example 6.2.4.1) the leader shares information with two followers, for the second case (Example 6.2.4.2) the leader shares information with three followers, and for the last case (Example 6.2.4.3) all the agents share information with each other. We then conducted an linear matrix inequality analysis for these three configurations. Specifically, the top

![Graphs showing proposed distributed adaptive control performance for Example 6.2.1.1](image)

Figure 6.12: Proposed distributed adaptive control performance for Example 6.2.1.1 with actuator bandwidth values $(\lambda_1, \lambda_2) = (50, 16.51)$ and command $c(t) = 0.5 \sin(0.01\pi t)$.  

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Figure 6.13: Line graphs for Example 6.2.2 (Example 6.2.2.1 on the top subfigure, Example 6.2.2.2 in the middle subfigure, and Example 6.2.2.3 on the bottom subfigure) with “dark grey” color indicating the leader and “light grey” color indicating the followers.

Figure 6.14: Feasible region of actuator bandwidth values for Example 6.2.2. The top subplot presents the linear matrix inequalities calculated region and the bottom subplot shows the simulated points yielding unbounded reference model trajectories with \( c = 1 \).

The subplot of Figure 6.22 presents the resulting linear matrix inequality feasible stability regions of allowable actuator bandwidth values and the bottom subplot of the same figure shows the minimum unstable values obtained numerically from the simulation for a given constant command \( c = 1 \). Comparing the result of bottom subplot with the top subplot, it is obvious that the proposed linear matrix inequality results provide
Figure 6.15: Proposed distributed adaptive control performance for Example 6.2.2.1 with actuator bandwidth values $(\lambda_1, \lambda_2) = (30, 15.24)$.

Figure 6.16: Proposed distributed adaptive control performance for Example 6.2.2.1 with actuator bandwidth values $(\lambda_1, \lambda_2) = (30, 9.1)$.
Figure 6.17: Circle graphs for Example 6.2.3 (Example 6.2.3.1 on the left subfigure and Example 6.2.3.2 on the right subfigure) with “dark grey” color indicating the leader and “light grey” color indicating the followers.

Figure 6.18: Feasible region of actuator bandwidth values for Example 6.2.3. The top subplot presents the linear matrix inequalities calculated region and the bottom subplot shows the simulated points yielding unbounded reference model trajectories with $c = 1$.

less conservative conclusions on determining the feasible stability regions. Furthermore, it is clear that the change in the number of the information links significantly alters the resulting feasible stability regions. In particular, one can conclude from Figure 6.22 that as information links are added, the feasible stability region shifts upwards; hence, it decreases with the added information links for the considered range of the pair $(\lambda_1, \lambda_2)$. Finally, to show the performance of the proposed distributed adaptive control design, two points are selected from Figure 6.22 for the case of Example 6.2.4.1, which are indicated by black circles.
Figure 6.19: Proposed distributed adaptive control performance for Example 6.2.3.2 with actuator bandwidth values \((\lambda_1, \lambda_2) = (35, 15.26)\).

Figure 6.20: Proposed distributed adaptive control performance for Example 6.2.3.2 with actuator bandwidth values \((\lambda_1, \lambda_2) = (35, 4.45)\).
The numerical simulation results are shown at these actuator bandwidth values in Figures 6.23 and 6.24 with command $c = 1$, where a similar conclusion is observed as for the case in the previous examples.

Figure 6.21: Circle graphs with additional links for Example 6.2.4 (Example 6.2.4.1 on the left subfigure, Example 6.2.4.2 in the middle subfigure, and Example 6.2.4.3 on the right subfigure) with “dark grey” color indicating the leader and “light grey” color indicating the followers.

Figure 6.22: Feasible region of actuator bandwidth values for Example 6.2.4. The top subplot presents the linear matrix inequalities calculated region and the bottom subplot shows the simulated points yielding unbounded reference model trajectories with $c = 1$.

Example 6.2.5 Effects of Multiple Leaders on a Circle Graph: Finally, we again consider a circle graph topology with nonidentical number of leaders given in Figure 6.25. The effect of added leaders is investigated by comparing four cases. For the first case (Example 6.2.5.1) we use one leader and for the remaining
Figure 6.23: Proposed distributed adaptive control performance for Examples 6.2.4.1 and 6.2.5.1 with actuator bandwidth values \((\lambda_1, \lambda_2) = (35, 17.62)\).

Figure 6.24: Proposed distributed adaptive control performance for Examples 6.2.4.1 and 6.2.5.1 with actuator bandwidth values \((\lambda_1, \lambda_2) = (35, 8.9)\).
Figure 6.25: Circle graphs with a nonidentical number of leaders for Example 6.2.5 (Example 6.2.5.1 on the top left subfigure, Example 6.2.5.2 on the top right subfigure, Example 6.2.5.3 on the bottom left subfigure, and Example 6.2.5.4 on the bottom right subfigure) with “dark grey” color indicating the leader and “light grey” color indicating the followers.

three cases we respectively use two leaders, three leaders, and four leaders (Example 6.2.5.2, Example 6.2.5.3 and Example 6.2.5.4, respectively). We then conducted an linear matrix inequality analysis for these four configurations. Specifically, the top subplot of Figure 6.26 presents the resulting linear matrix inequality feasible stability regions of allowable actuator bandwidth values and the bottom subplot of the same figure shows the minimum unstable values obtained numerically from the simulation for a given constant command $c = 1$. Comparing the result of bottom subplot with the top subplot, it can be readily seen that the proposed linear matrix inequality results provide less conservative conclusions on determining the feasible stability regions. Moreover, it is clear that the change in the number of leaders significantly alters resulting feasible regions. In particular, one can conclude from the Figure 6.26 that as number of the leaders increase, the feasible stability region increases. Finally, to show the performance of the proposed distributed adaptive control design, two points are selected from Figure 6.26 for the case of Example 6.2.5.1 which has the same graph topology as Example 6.2.4.1. Thus, the simulation results at these actuator bandwidth values are the same as in Figures 6.23 and 6.24 with command $c = 1$, where a similar conclusion is observed as for the case in the previous examples.
6.2.6 Conclusions

An open problem in distributed control design for multiagent systems is the ability of the controlled system to guarantee stability and performance with respect to often nonidentical (e.g., slow and fast) agent actuation capabilities and unknown parameters in the agent dynamics. To this end, we proposed a distributed adaptive control architecture based on a hedging method for linear time-invariant multiagent systems with heterogeneous actuator dynamics and system uncertainties. The stability of the controlled multiagent system was analyzed using Lyapunov stability theory. Linear matrix inequalities were then utilized to perform stability verification in terms of numerically computing the fundamental tradeoff between heterogeneous agent actuation capabilities and unknown parameters in agent dynamics. Several illustrative numerical examples thoroughly show the efficacy of the proposed design procedure by investigating the effect of different leader locations in different graph topologies, the effect of additional information links, and the effect of additional leaders. As a future research direction, one can consider extending the results of this paper to the case with actuator amplitude and rate saturation, where the approaches given in, for example, [141-150] can be used to this end.
Chapter 7: Concluding Remarks and Future Research

7.1 Concluding Remarks

In adaptive control of physical systems, it is well-known that the presence of actuator and/or unmodeled dynamics in feedback loops can yield to unstable closed-loop system trajectories. Motivated by this standpoint, the feedback architectures reported in this dissertation have investigated new model reference adaptive control tools and methods for uncertain sole and multiagent dynamical systems with unmodeled and/or actuator dynamics.

The challenges associated with the system uncertainties and unmodeled dynamics were first addressed using novel approaches that determine and relax the stability limits (e.g., conditions and trade-offs) as well as improve transient performance. The first approach was predicated on a novel model reference adaptive control architecture that was augmented with an adaptive robustifying term (Chapter 2). Unlike standard adaptive controllers, it was shown that the proposed architecture allows the closed-loop dynamical system to remain stable in the presence of large system uncertainties when the unmodeled system dynamics satisfy a set of conditions. The second, third, fourth, fifth, and sixth approaches of this dissertation are the generalizations of the first one. These approaches respectively considered an experimental verification, a theoretical extension to a class of nonlinear unmodeled dynamics, an architecture to achieve a guaranteed performance, a theoretical extension for dynamical systems with unstructured uncertainties, and an asymptotic decoupling approach for the problem of presence of unmodeled dynamics in the dynamical system.

In particular, the second approach presented an experimental result for the purpose of demonstrating the efficacy of the first approach, where a benchmark mechanical system setup was used involving an inverted pendulum on a cart coupled with another cart through a spring in the presence of unknown frictions (Section 3.1). The third approach presented an extension of the first approach to a wider class of unmodeled dynamics involving nonlinear functions (Section 3.2). Moreover, the fourth approach presented a direct uncertainty minimization framework with the added term in the control signal and the update law, which was developed through a gradient descent procedure with a new cost function involving a cost function gain, for minimizing
the effect of both system uncertainties and unmodeled dynamics on the closed-loop system response (Section 3.3). The fifth approach presented stability conditions of model reference adaptive control architectures in the presence of unstructured system uncertainties subject to (residual) approximating errors satisfying the linear growth inequality and unmodeled dynamics (Section 3.4). Note that the fourth and fifth approaches were also experimentally validated on the benchmark mechanical system setup. Finally, the last approach presented a new framework guaranteeing asymptotic convergence between the trajectories of an uncertain dynamical system and a given reference model without relying on any measurements from the coupled dynamics (Section 3.5). The challenges associated with the uncertain dynamical systems in the presence of both unmodeled and actuator dynamics were second addressed using novel approaches that helped to determine and relax the stability limits. In particular, model reference adaptive control architectures with standard, hedging-based (that alters the ideal reference model dynamics of each agent in order to ensure correct adaptation in the presence of actuator dynamics), and expanded reference models were analyzed for this class of uncertain dynamical systems and sufficient stability conditions were developed (Chapter 4). A robustifying term was then synthesized for the latter architecture and to analytically show that this term could allow for a relaxed sufficient stability condition. The challenges associated with the uncertain multiagent systems in the presence of unmeasurable unmodeled dynamics were third addressed using a novel distributed adaptive architecture. Specifically, standard distributed adaptive control method with system uncertainties and coupled dynamics in a leader-follower setting was analyzed, where local stability conditions were developed (Chapter 5). An additional feedback term within the control signal of each agent was also proposed for relaxing the local stability conditions. The challenges associated with the uncertain multiagent systems in the presence of heterogeneous actuator dynamics were finally addressed using novel distributed adaptive architectures. First, a distributed adaptive control architecture in a leader-follower setting for the class of both scalar and high-order multiagent systems was proposed (Section 6.1). The proposed architectures used a hedging method for ensuring correct adaptation in the presence of heterogeneous actuator dynamics of these agents. Second, sufficient stability conditions were showed, where evaluation of these conditions with respect to a given graph topology allows stability verification of the controlled multiagent system (Section 6.2).

To summarize, verifiable model reference adaptive control architectures for both sole and multiagent systems were developed and analyzed in this dissertation using Lyapunov stability theory, linear matrix
inequalities, and matrix mathematics. For bridging the gap between theory and practice, several simulation and experimental results were also presented.

### 7.2 Future Research Directions

There are several possible research directions that can be considered based on the presented results of this dissertation. In particular, more experimental results on different platforms can be shown to demonstrate the efficacy of the control architectures. One candidate for experimental result for the control architectures given in Chapters 2 and 3 can be utilizing one flexible/ high–aspect ratio aerial vehicle that aim to deliver a supply with a physical link between this supply and aerial vehicle (e.g., slung load system). Other candidate for experimental result for the control architecture given in Chapter 5 can be utilizing networked aerial vehicles aim to deliver multiple supplies in the presence of physical links between these supplies and aerial vehicles as well as communication data exchange between these vehicles (e.g., transportation system). In addition, another challenge to control uncertain networked multiagent systems given in Chapters 5 and 6 involve the presence of agents having both coupled dynamics and different actuator bandwidths. Therefore, another future research direction can be to address the presence of coupled and actuator dynamics together with all sufficient stability conditions for the networked multiagent systems over undirected, directed, and switching communication graph topologies. Finally, the proposed architectures on this dissertation assume fully measurable dynamical system states (not unmodeled dynamics states). Thus, another immediate future research direction can be extend to the case, where there is limited state information (i.e., output feedback adaptive control).
References


Appendix A: Projection Operator for Section 3.1

We use the following (rectangular) projection operator definition from Exercise 11.3 of [7] and [85] in this paper.

**Definition A.1** Consider a convex hypercube in the form $\Omega = \{ \theta \in \mathbb{R}^n : (\theta_i^{\min} \leq \theta_i \leq \theta_i^{\max})_{i=1,\ldots,n} \}$, where $\Omega \in \mathbb{R}^n$, and $\theta_i^{\min}$ and $\theta_i^{\max}$ respectively represent the minimum and maximum bounds for the $i$th component of the $n$-dimensional parameter vector $\theta$ (we set $\theta_i^{\min} = -\theta_i^{\max}$ for the results of this paper and without loss of generality). Furthermore, for a sufficiently small positive constant $\varepsilon_0$, consider another hypercube in the form $\Omega_{\varepsilon_0} = \{ \theta \in \mathbb{R}^n : (\theta_i^{\min} + \varepsilon_0 \leq \theta_i \leq \theta_i^{\max} - \varepsilon_0)_{i=1,\ldots,n} \}$, where $\Omega_{\varepsilon_0} \subset \Omega$. The projection operator $\text{Proj} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is then defined component-wise by $\text{Proj}(\theta, y) = (\theta_i^{\max} - \theta_i)y_i/\varepsilon_0$ if $\theta_i > \theta_i^{\max} - \varepsilon_0$ and $y_i > 0$, $\text{Proj}(\theta, y) = (\theta_i - \theta_i^{\min})y_i/\varepsilon_0$ if $\theta_i < \theta_i^{\min} + \varepsilon_0$ and $y_i < 0$, and $\text{Proj}(\theta, y) = y_i$ otherwise, where $y \in \mathbb{R}^n$.

Based on the above definition and $\theta^* \in \Omega_{\varepsilon_0}$, one can show that the inequality $(\theta - \theta^*)^T(\text{Proj}(\theta, y) - y) \leq 0$ holds for $\theta \in \Omega$ and $y \in \mathbb{R}^n$. Moreover, the above definition can be similarly generalized to matrices as $\text{Proj}_m(\Theta, Y) = (\text{Proj}(\text{col}_1(\Theta), \text{col}_1(Y)), \ldots, \text{Proj}(\text{col}_m(\Theta), \text{col}_m(Y)))$ with the inequality $\text{tr}[(\Theta - \Theta^*)^T(\text{Proj}_m(\Theta, Y) - Y)] = \sum_{i=1}^m[\text{col}_i(\Theta - \Theta^*)^T(\text{Proj}(\text{col}_i(\Theta), \text{col}_i(Y)) - \text{col}_i(Y))] \leq 0$ for $n \times m$ matrices $Y$, $\Theta$, and $\Theta^*$ (here, $\text{col}_i(\cdot)$ denotes $i$th column function).
Appendix B: Dead-zone Function for Section 3.1

The projection operator with a dead-zone function in the update laws is utilized in the experimental results of this paper. Specifically, the proposed model reference adaptive control architecture given in this paper consist of the dead-zone function in the form given by

\[
\dot{\hat{W}}(t) = \gamma \text{Proj}_{\infty} \left[ \hat{W}(t), \phi W N x(t) e^T(t) P \right], \quad \hat{W}(0) = \hat{W}_0. \tag{A.1}
\]

In (A.1), \( \phi_W \equiv 0 \) when \( e^T P e \leq E_0 \in \mathbb{R}_+ \) and \( \phi_W \equiv 1 \) otherwise. In addition, the robustifying term’s update law in the proposed model reference adaptive control architecture also consists of the dead-zone function in the form given by

\[
\dot{\hat{\mu}}(t) = \mu_0 \text{Proj} \left( \hat{\mu}(t), \phi_W \left[ \| B^T P e(t) \|_2^2 - \sigma \mu \hat{\mu}(t) \right] \right), \quad \hat{\mu}(0) = \hat{\mu}_0, \quad \hat{\mu}(0) \in \mathbb{R}_+. \tag{A.2}
\]

In (A.2), once again, \( \phi_W \equiv 0 \) when \( e^T P e \leq E_0 \in \mathbb{R}_+ \) and \( \phi_W \equiv 1 \) otherwise. This also motivate us to select Lyapunov function in Section 3.1.2.2 as

\[
V(e, \hat{W}, z, \hat{\mu}) = E_0 + \gamma^{-1} \text{tr} (\hat{W} A \hat{W}^T) + \alpha z^T S z + \mu_0^{-1} \hat{\mu}^2 \lambda_L, \tag{A.3}
\]

when \( \phi_W \equiv 0 \) and otherwise as given in [1]. Note that the first term in (A.3) when \( \phi_W \equiv 1 \) is inspired by the results of the seminar paper [151]. Specifically, if \( \phi_W \equiv 0 \) in both (A.1) and (A.2), then the time derivative of (A.3) results with \( \dot{V}(\cdot) \leq -\alpha \| z(t) \|_2^2 + 2\alpha \| z(t) \|_2 \| S G_2 \|_2 (e_0 + x_r^*) \), where \( e_0 \) and \( x_r^* \) satisfy the inequalities \( \| e(t) \|_2 \leq e_0 \) and \( \| x_r(t) \|_2 \leq x_r^* \). On the other hand, if \( \phi_W \equiv 1 \) in both (A.1) and (A.2), then \( \dot{V}(\cdot) \) is as given in [1] with \( G_1 = 0 \) and results with stability conditions given by (3.24) and (3.25). Once again, this is consistent with the analysis procedure presented in [151].
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Relaxing the stability limit of adaptive control systems in the presence of unmodelled dynamics

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Experimental Results of a Model Reference Adaptive Control Law on an Uncertain System with Unmodeled Dynamics

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I Introduction

Model reference adaptive control is a powerful theoretical method that has a natural capability to suppress the effect of system uncertainties for achieving a desired level of closed-loop system performance. For a wide array of applications including unmodeled dynamics however, the closed-loop system stability with model reference adaptive control laws can be challenged. These applications include coupled rigid body systems with flexible interconnection links, airplanes with high aspect ratio wings, and high speed vehicles with strong rigid body and flexible dynamics coupling, to name a few examples. The contribution of this paper is to present experimental results in order to complement these recent theoretical studies and demonstrate the efficacy of this proposed term.

1. Model reference adaptive control is a powerful tool that has a capability to suppress the effect of system uncertainties for achieving a desired level of closed-loop system performance. Yet, for a wide array of applications including unmodeled dynamics such as coupled rigid body systems with flexible interconnection links, airplanes with high aspect ratio wings, and high speed vehicles with strong rigid body and flexible dynamics coupling, the closed-loop system stability with model reference adaptive control laws can be challenged. To this end, the authors have recently studied the stability interplay between a class of unmodeled dynamics and system uncertainties for model reference adaptive control laws, and proposed a robustifying term to relax the resulting interplay. The contribution of this paper is to present experimental results for the purpose of demonstrating the efficacy of this proposed term, where we use a benchmark mechanical system setup involving an inverted pendulum on a cart coupled with another cart through a spring in the presence of unknown frictions.

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A Generalization of Fundamental Stability Limits of Model Reference Adaptive Controllers in the Presence of a Class of Nonlinear Unmodeled Dynamics

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A fundamental trade-off in the design of model reference adaptive controllers is to obtain a stable closed-loop dynamical system in the presence of not only system uncertainties but also unmodeled dynamics, which can be either linear or nonlinear. Specifically, there exist stability limits for these controllers, where the closed-loop dynamical system subject to an adaptive controller preserves stability either if there does not exist significant unmodeled dynamics or the effect of systems uncertainties is negligible. To address this problem we recently propose an adaptive control architecture to relax these stability limits for linear unmodeled system dynamics. Generalizations of the proposed adaptive control architecture are made in this paper to a broader class of unmodeled dynamics including nonlinear functions. Specifically, we utilize tools and methods from Lyapunov stability to relax stability limits for model reference adaptive controllers in the presence of nonlinear unmodeled dynamics. An illustrative numerical example is provided to demonstrate the efficacy of the proposed approach.

I Introduction

A fundamental trade-off in the design of model reference adaptive controllers is to obtain a stable closed-loop dynamical system in the presence of not only system uncertainties but also unmodeled dynamics (see the seminal paper [1]), which can be either linear or nonlinear. Specifically, there exist stability limits for these controllers, where the closed-loop dynamical system subject to an adaptive controller preserves stability either if there does not exist significant unmodeled dynamics or the effect of systems uncertainties

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The presence of unmodeled dynamics degrades the stability and performance of adaptive control architectures. While there are studies that focus on the stability aspects of adaptive control architectures for uncertain systems with unmodeled dynamics, methods that offer performance guarantees in the sense of predictably minimizing the difference between uncertain system trajectories and given reference model trajectories are not available in the literature. In this paper, we address this gap through a recently developed direct uncertainty minimization framework. Specifically, a model reference adaptive control architecture is proposed and mathematically analyzed for uncertain systems with unmodeled dynamics. The key feature of our architecture is the added term in the control signal and the update law, which is developed through a gradient descent procedure with a new cost function involving a cost function gain in order to minimize the effect of both system uncertainties and unmodeled dynamics on the closed-loop system response. Therefore, the proposed architecture can be effective in achieving performance guarantees, where an illustrative numerical example shows the offered predictable performance as a function of the cost function gain. Finally, we also provide an experimental study on a physical system involving two carts connected with each other through a spring in order to demonstrate the efficacy of the proposed architecture.

I Introduction

In the absence of unmodeled dynamics, adaptive control architectures guarantee Lyapunov stability of the closed-loop system and asymptotic convergence of uncertain system trajectories to given reference model trajectories (see, for example, [1–5]). However, the presence of unmodeled dynamics, which result from the coupling between rigid body and flexible appendages in applications such as airplanes with high aspect
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