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An Efficient Scheme for Numerical Solution of Burgers' Equation Using Quintic Hermite Interpolating Polynomials

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An efficient scheme for numerical solution of Burgers' equation using quintic Hermite interpolating polynomials

Abstract A numerical scheme combining the features of quintic Hermite interpolating polynomials and orthogonal collocation method has been presented to solve the well-known non-linear Burgers' equation. The quintic Hermite collocation method (QHCM) solves the non-linear Burgers' equation directly without converting it into linear form using Hopf–Cole transformation. Stability of the QHCM has been checked using Eucledian and Supremum norms. Numerical values obtained from QHCM are compared with the values obtained from other techniques such as orthogonal collocation method, orthogonal collocation on finite elements and *pdepe* solver. Numerical values have been plotted using plane and surface plots to demonstrate the results graphically.

Mathematics Subject Classification 35K05 · 35K10 · 35K57 · 65M60 · 65M70

الملخص

تم تقديم مخطط يجمع بين خصائص كثيرات حدود هرميت الاستكمالية من الدرجة الخامسة وطريقة الترتيب العمودي لحل معادلة بورغز غير الخطية المشهورة. تحل طريقة ترتيب هرميت الخماسية (QHCM) معادلة بورغز غير الخطية مباشرةً دون الحاجة إلى تحويلها إلى معادلة خطية باستخدام تحويل هوبف – كول. تم فحص استقرار QHCM باستخدام معيار إقليدس ومعيار أدنى حد علوي. تمت مقارنة نتائج عددية حصلنا عليها من QHCM مع نتائج تم الحصول عليها باستخدام تقنيات أخرى مثل طريقة الترتيب العمودي (OCM)، وطريقة الترتيب العمودي باستخدام عدد منته من العناصر (OCFE)، وحالّ *pdepe*. تم رسم القيم العددية على مستوى وروسم سطوح لتمثيل النتائج بيانيا.

1 Introduction

The majority of the problems arising in the field of physics, engineering, chemistry and biology, etc. are modelled using linear or non-linear partial differential equations. One such type of equation having numerous applications in physics and engineering is Burgers' equation. It is a well-known non-linear problem which gives an insight into the relation between convection and diffusion.

Consider the following Burgers' equation

$$\frac{\partial u}{\partial t} = \varepsilon \frac{\partial^2 u}{\partial x^2} - u \frac{\partial u}{\partial x}, \quad (x, t) \in \Omega \times (0, T). \tag{1}$$

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Initially,

$$u(x, 0) = u_0(x), \quad \forall x \in \Omega. \tag{2}$$

Boundary condition:

$$u(0, t) = g_1(t) \quad \text{and} \quad u(1, t) = g_2(t), \quad \forall t \in (0, T). \tag{3}$$

A variety of numerical methods have been developed to solve the Burgers' equation, such as finite difference scheme [12,14], finite element method [2], quadratic B-spline [13,16], cubic B-spline [5,9], automatic differentiation [6], and modified Adomain method [1]. In the present study, numerical solution of Burgers' equation has been shown by applying the quintic Hermite collocation method directly, without transforming the non-linear form into the linear form using Hopf–Cole transformation.

The paper is divided into six sections. Section 1 gives the introduction of Burgers' equation, whereas Sect. 2 discusses about QHCM and collocation points. In Sect. 3, application and implementation of QHCM are discussed. In Sect. 4, stability analysis is discussed and Sect. 5 gives the discussion of all the results obtained and finally in Sect. 6, the crust of the present study is concluded.

2 Quintic Hermite collocation method(QHCM)

Quintic Hermite collocation is one of the Hermite collocation method [10,17], where Hermite interpolating polynomials are used as base functions. The trial function is approximated by Hermite interpolating polynomials of the order $2k + 1 (k > 0)$. It is the generalization of the Lagrange interpolation with polynomials that not only interpolate function at each node but also its consecutive derivatives. In general for real numbers $x_1 < x_2 < x_3 < \cdots < x_k$ and all integers $m_1, m_2, m_3, \ldots, m_k$ greater than zero, there exists a unique polynomial of degree $m_1 + m_2 + m_3 + \cdots + m_k - 1$. In the present work, quintic Hermite interpolating polynomials which are of order 5 are used to approximate the trial function.

Quintic Hermite interpolating polynomials can be expressed in the following form [8]:

$$\sum_{j=1}^{6} p(x_j) P_j(x) + p'(x_j) \bar{P}_j(x) + p''(x_j) \bar{\bar{P}}_j(x), \tag{4}$$

where $P_j, \bar{P}_j, \bar{\bar{P}}_j$ can be expressed as: $P_j(x) = \begin{cases} 6\left(\frac{x_{i+1}-x}{x_{i+1}-x_i}\right)^5 - 15\left(\frac{x_{i+1}-x}{x_{i+1}-x_i}\right)^4 + 10\left(\frac{x_{i+1}-x}{x_{i+1}-x_i}\right)^3; & x_i \le x \le x_{i+1} \\ 6\left(\frac{x-x_{i-1}}{x_i-x_{i-1}}\right)^5 - 15\left(\frac{x-x_{i-1}}{x_i-x_{i-1}}\right)^4 + 10\left(\frac{x-x_{i-1}}{x_i-x_{i-1}}\right)^3; & x_{i-1} \le x \le x_i \\ 0; & \text{elsewhere} \end{cases}$

$$\bar{P}_j(x) = \begin{cases} 3\frac{(x_{i+1}-x)^5}{(x_{i+1}-x_i)^4} - 7\frac{(x_{i+1}-x)^4}{(x_{i+1}-x_i)^3} + 4\frac{(x_{i+1}-x)^3}{(x_{i+1}-x_i)^2}; & x_i \le x \le x_{i+1} \\ -3\frac{(x-x_{i-1})^5}{(x_i-x_{i-1})^4} + 7\frac{(x-x_{i-1})^4}{(x_i-x_{i-1})^3} - 4\frac{(x-x_{i-1})^3}{(x_i-x_{i-1})^2}; & x_{i-1} \le x \le x_i \\ 0; & \text{elsewhere} \end{cases}$$

$$\bar{\bar{P}}_j(x) = \begin{cases} 0.5\frac{(x_{i+1}-x)^5}{(x_{i+1}-x_i)^3} - \frac{(x_{i+1}-x)^4}{(x_{i+1}-x_i)^2} + 0.5\frac{(x_{i+1}-x)^3}{(x_{i+1}-x_i)}; & x_i \le x \le x_{i+1} \\ 0.5\frac{(x-x_{i-1})^5}{(x_i-x_{i-1})^3} - \frac{(x-x_{i-1})^4}{(x_i-x_{i-1})^2} + 0.5\frac{(x-x_{i-1})^3}{(x_i-x_{i-1})}; & x_{i-1} \le x \le x_i \\ 0; & \text{elsewhere} \end{cases}$$

where,

$P_j(x_i) = \delta_{ij}, \quad P'_j(x_i) = 0, \quad P''_j(x_i) = 0, \quad i, j = 1, 2, \ldots, 6$

$\bar{P}_j(x_i) = 0, \quad \bar{P}'_j(x_i) = \delta_{ij}, \quad \bar{P}''_j(x_i) = 0, \quad i, j = 1, 2, \ldots, 6$

$\bar{\bar{P}}_j(x_i) = 0, \quad \bar{\bar{P}}'_j(x_i) = 0, \quad \bar{\bar{P}}''_j(x_i) = \delta_{ij}, \quad i, j = 1, 2, \ldots, 6.$

Orthogonal collocation is applied within each subdomain by introducing a new variable $\xi = \frac{(x-x_\gamma)}{h}$, where $h = x_{\gamma+1} - x_\gamma, \xi = 0$ at $x = x_\gamma$ and $\xi = 1$ at $x = x_{\gamma+1}$.

After rearranging the terms, $P_j, \bar{P}_j, \bar{\bar{P}}_j$ can also be written in simplified form as:

$H_1 = (1 - 10\xi^3 + 15\xi^4 - 6\xi^5)$

$$H_2 = h(\xi - 6\xi^3 + 8\xi^4 - 3\xi^5)$$
$$H_3 = h^2(0.5\xi^2 - 1.5\xi^3 + 1.5\xi^4 - 0.5\xi^5)$$
$$H_4 = h^2(0.5\xi^3 - \xi^4 + 0.5\xi^5)$$
$$H_5 = (10\xi^3 - 15\xi^4 + 6\xi^5)$$
$$H_6 = h(-4\xi^3 + 7\xi^4 - 3\xi^5)$$

with $H_1(\xi) = H_5(1 - \xi)$, $H_2(\xi) = -H_6(1 - \xi)$, $H_3(\xi) = H_4(1 - \xi)$.

2.1 Collocation points

Choice of collocation points is an important characteristic to be well thought out while considering the technique of orthogonal collocation. Collocation points are taken to be the zeros of orthogonal polynomials. Basically, the discretization end points are taken to be 0 and 1. In QHCM, four interior collocation points are taken within each element to discretize the problem. These interior collocation points are the zeros of orthogonal polynomials such as Jacobi polynomials [3]. The basic recurrence formula for Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ is

$$2(n + 1)(\alpha + \beta + n + 1)(\alpha + \beta + 2n)P_{n+1}^{(\alpha,\beta)}(x)$$
$$= (\alpha + \beta + 2n + 1) \times [\alpha^2 - \beta^2 + x(\alpha + \beta + 2n + 2)(\alpha + \beta + 2n)]P_n^{(\alpha,\beta)}(x)$$
$$-2(\alpha + n)(\beta + n)(\alpha + \beta + 2n + 2)P_{n-1}^{(\alpha,\beta)}(x); \quad \text{for all } x \in [-1, 1] \tag{5}$$

with $P_0^{(\alpha,\beta)}(x) = 1$ and $P_1^{(\alpha,\beta)}(x) = 0.5(\alpha + 1)(\beta + 1)(x + 1) + 0.5(x - 1)$, $n \in \mathbb{N}$ is the degree of the polynomial with α and β as parameters. The zeros of $P_n^{(\alpha,\beta)}(x)$ are transformed from $[-1,1]$ onto $[0,1]$ using the following one to one relation:

$$f(x) = \frac{x + 1}{2}. \tag{6}$$

In the present, the zeros of Legendre polynomials have been taken as collocation points as a special case of Jacobi polynomials for $\alpha = \beta = 0$.

3 Application of QHCM

To apply QHCM on system of equations of non-linear Burgers' equation defined by (1)–(3), the approximating function is defined as:

$$u^\gamma(x, t) = \sum_{i=1}^{6} P_i(x)c_i(t) + \bar{P}_i(x)c_i(t) + \bar{\bar{P}}_i(x)c_i(t); \quad x \in [x_{i-1}x_{i+1}]$$

where $c_i's$ are continuous function of t. As defined earlier, to apply collocation, a new variable ξ is introduced within each of the subdomains such that the trial function takes the form:

$$u^\gamma(x, t) = \sum_{i=1}^{6} H_i(\xi)c_i^\gamma(\tau); \quad \xi \in [0, 1].$$

The boundary conditions are defined on $x = 0$ and $x = 1$. At jth collocation point, the system of equations (1)–(3) can be expressed as:

$$\sum_{i=1}^{6} \frac{dc_i^\gamma}{d\tau} H_i(\xi_j) = \frac{\varepsilon}{h^2} \sum_{i=1}^{6} c_i^\gamma H_i''(\xi_j) - \frac{1}{h} \sum_{i=1}^{6} c_i^\gamma H_i(\xi_j)$$
$$\times \sum_{i=1}^{6} c_i^\gamma H_i'(\xi_j); \quad j = 2, 3, 4 \text{ and } \gamma = 1, 2, \ldots, \text{ne} \tag{7}$$

Initially,

$$u(x_\gamma + \xi_j h, 0) = u_0(x_\gamma + \xi_j h) \tag{8}$$
$$u^1(\xi_1, \tau) = g_1(\tau) \text{ and } u^{ne}(\xi_6, \tau) = g_2(\tau) \tag{9}$$
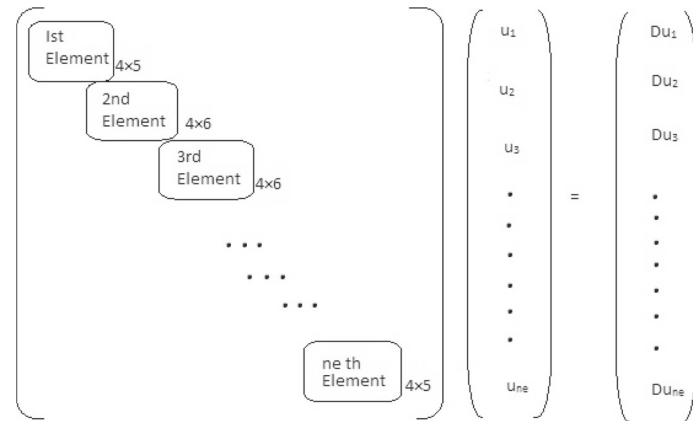
Fig. 1 Matrix structure for system of equations defined in Eq. (7)

After application of QHCM, the system of equations defined from (1) to (3) transforms into a set of ordinary differential equations (ODEs) with four ODEs within each subinterval $[x_{i-1}, x_i]$. As the approximating function consists of quintic Hermite interpolating polynomials which have the property to interpolate the first- and second-order derivatives at node points, due to which the additional condition of continuity waives off. It reduces the system of partial differential equations into a system of ODEs instead of the system of differential algebraic equations as in orthogonal collocation on finite elements (OCFE) [4,7]. After implementation of QHCM, the system of equations defined in Eqs. (7)–(9) can be written as:

$$\mathfrak{D}u = \mathfrak{M}u \tag{10}$$

where $\mathfrak{D}$ is the differential operator, u is the vector of collocation solutions $u_i's$ and $\mathfrak{M}$ is the coefficient matrix at jth collocation point. The Matrix structure for QHCM is shown in Fig. 1. $\mathfrak{M}$ is the square matrix of order $4ne \times 4ne$, $\mathfrak{D}u$ and u are vectors of order $4ne$. Bandwidth of each subinterval except first and last in matrix $\mathfrak{M}$ is of order 4×6 and bandwidth of first and last subinterval is of order 4×5 due to boundary conditions.

3.1 Implementation of QHCM

Problem 3.1 *First solve the Burgers' equation (1) with the initial condition $u(x, 0) = \sin \pi x$; the problem becomes*

$$\frac{\partial u}{\partial \tau} = \frac{\varepsilon}{h^2} \frac{\partial^2 u}{\partial \xi^2} - \frac{u}{h} \frac{\partial u}{\partial \xi}, \quad 0 < \xi < 1 \text{ and } \tau > 0. \tag{11}$$

Initially,

$$u(\xi, 0) = \sin(\pi \times (x_\gamma + \xi h)), \quad \forall \xi \in (0, 1). \tag{12}$$

Boundary conditions,

$$u(0, \tau) = 0 \text{ and } u(1, \tau) = 0, \quad \forall \tau > 0. \tag{13}$$

Problem 3.2 *In the second problem, the initial condition is taken to be $u(x, 0) = 4x(1 - x)$.*
The problem becomes

$$\frac{\partial u}{\partial \tau} = \frac{\varepsilon}{h^2} \frac{\partial^2 u}{\partial \xi^2} - \frac{u}{h} \frac{\partial u}{\partial \xi}, \quad 0 < \xi < 1 \text{ and } \tau > 0. \tag{14}$$

Initially,

$$u(\xi, 0) = 4(x_\gamma + \xi h) \times (1 - (x_\gamma + \xi h)), \quad \forall \xi \in (0, 1). \tag{15}$$

Boundary condition,

$$u(0, \tau) = 0 \text{ and } u(1, \tau) = 0, \quad \forall \, \tau > 0. \tag{16}$$

After the application of quintic Hermite collocation method to the above system of equations, $4ne$ number of equations appear, with ne as the number of elements. The resulting system of equation has been solved numerically using MATLAB with $ode15s$ system solver.

4 Stability analysis

In the present work, stability of the numerical method has been checked by Eucledian norm and supremum norm. Let $E = u - u^\gamma$, where E defines the pointwise rate of error, u being the exact solution and u^γ is the approximate solution. $\| U \|_2 = \sqrt{\sum_{\gamma=1}^{ne} h_\gamma \sum_{i=1}^{6} w_i E_i^2}$, where E_i is the pointwise error, $w_i's$ are the corresponding weight functions and h_γ is the length of the γth subdomain.

$$\| U \|_\infty = \max_{x_0, x_1, \ldots, x_n} | u - u^\gamma |$$

The order of convergence can be determined by the lemma given in [11] and is mentioned below:

Lemma 4.1 *Let H be the interpolating polynomial of function u defined on [a, b], such that H belongs to the space of quintic Hermite interpolating polynomials. Then the order of convergence of quintic Hermite interpolation technique is defined to be:*

$$\| u - H \| \leq C \| u^{(6)} \| h^6$$

where C is the generic constant.

Therefore, the order of convergence of quintic Hermite interpolation is of the order h^6. The stability analysis has been performed on the basis of maximum and Eucledian norm and is shown in Tables 1 and 2 for different values of ε. It has been observed that both the norms lie between 0 and 1.

5 Results and discussion

All the numerical findings obtained in this study have been adequately described in this section. Problem 3.1 has been solved numerically using different techniques such as QHCM, OCFE [4,7], *pdepe* solver, OCM

Table 1 $\|U\|_\infty$ and $\|U\|_2$ norm for different ε and $\tau = 0.1$ for Problem 3.1

ϵ	$\|U\|_\infty$ γ			$\|U\|_2$ γ		
	10	50	100	10	50	100
1	2.4900×10^{-3}	1.8000×10^{-4}	7.0000×10^{-5}	7.2897×10^{-4}	2.0668×10^{-5}	5.4536×10^{-6}
0.25	1.0053×10^{-2}	1.4930×10^{-3}	8.3100×10^{-4}	2.5428×10^{-3}	1.9750×10^{-4}	7.3101×10^{-5}
0.0625	1.6200×10^{-2}	3.2200×10^{-3}	1.6200×10^{-3}	4.7004×10^{-3}	3.9394×10^{-4}	1.3802×10^{-4}
0.015625	1.6165×10^{-1}	2.9040×10^{-2}	2.0760×10^{-2}	2.1831×10^{-2}	2.0577×10^{-3}	8.9404×10^{-4}

Table 2 $\|U\|_\infty$ and $\|U\|_2$ norm for different ε and $\tau = 0.1$ for Problem 3.2

ϵ	$\|U\|_\infty$ γ			$\|U\|_2$ γ		
	10	50	100	10	50	100
1	2.5600×10^{-3}	1.8000×10^{-4}	7.0000×10^{-5}	7.5291×10^{-4}	2.0936×10^{-5}	5.3571×10^{-6}
0.25	1.0031×10^{-2}	1.4510×10^{-3}	6.9100×10^{-4}	2.5093×10^{-3}	1.8355×10^{-4}	6.1722×10^{-5}
0.0625	1.6426×10^{-2}	3.1550×10^{-3}	1.6150×10^{-3}	4.5489×10^{-3}	3.7851×10^{-4}	1.2992×10^{-4}
0.015625	1.2971×10^{-1}	2.0304×10^{-2}	8.6860×10^{-3}	1.7878×10^{-2}	1.3429×10^{-3}	4.8217×10^{-4}

[15,18] for different values of ε and τ. In Tables 3, 4, 5, 6 and 7, the comparison between exact and the numerical solution with different techniques has been shown. It has been observed from these tables that the numerical values obtained from QHCM agree well with the exact ones and this fact authenticates that quintic Hermite collocation method is better than OCFE, *pdepe* solver and OCM. In Fig. 2, the graphical view of the numerical solution has been shown for different values of ε and τ in the form of 2D plots. In Fig. 3, surface plots have been shown for the graphical view of the solution for different values of ε and all τ. It has been observed that in all the figures values are symmetric and lie between 0 and 1. From these surface plots, the smoothness of the solution can also be observed easily.

Table 3 Comparison between exact and numerical values at $\tau = 0.1$ and $\varepsilon = 1$

ξ	QHCM	OCFE	*pdepe*	OCM	Exact solution
0.1	0.10952	0.10947	0.10960	0.11932	0.10954
0.2	0.20976	0.20892	0.20992	0.18236	0.20979
0.3	0.29185	0.29178	0.29207	0.30825	0.29190
0.4	0.34788	0.34751	0.34745	0.35214	0.34792
0.5	0.37155	0.37154	0.37157	0.37161	0.37158
0.6	0.35903	0.35901	0.35924	0.36221	0.35905
0.7	0.30990	0.30981	0.30893	0.32517	0.30990
0.8	0.22782	0.22763	0.22791	0.19908	0.22782
0.9	0.12069	0.12058	0.12073	0.13139	0.12069

Table 4 Comparison between exact and numerical values at $\tau = 0.1$ and $\varepsilon = 0.25$

ξ	QHCM	OCFE	*pdepe*	OCM	Exact solution
0.1	0.20217	0.20099	0.20252	0.22068	0.20241
0.2	0.39279	0.39827	0.39338	0.34005	0.39320
0.3	0.55961	0.56721	0.56025	0.59563	0.56007
0.4	0.68906	0.68887	0.68959	0.70053	0.68946
0.5	0.76605	0.76618	0.76634	0.76628	0.76625
0.6	0.77373	0.77418	0.77422	0.77737	0.77447
0.7	0.69999	0.69920	0.69995	0.72630	0.69991
0.8	0.53643	0.53957	0.53601	0.47294	0.53598
0.9	0.29251	0.29265	0.29220	0.31744	0.29219

Table 5 Comparison between exact and numerical values at $\tau = 0.1$ and $\varepsilon = 0.0625$

ξ	QHCM	OCFE	*pdepe*	OCM	Exact solution
0.1	0.22810	0.22913	0.22874	0.24941	0.22868
0.2	0.44558	0.44579	0.44657	0.38540	0.44643
0.3	0.64065	0.64151	0.64171	0.68411	0.64148
0.4	0.79959	0.80003	0.80054	0.81450	0.80024
0.5	0.90591	0.90581	0.90652	0.90629	0.90628
0.6	0.93943	0.93938	0.93946	0.94005	0.93940
0.7	0.87654	0.87601	0.87554	0.78907	0.87601
0.8	0.69519	0.69414	0.69361	0.61817	0.69414
0.9	0.39045	0.38936	0.38913	0.42209	0.38937

Table 6 Comparison between exact and numerical values at $\tau = 0.1$ and $\varepsilon = 0.015625$

ξ	QHCM	OCFE	*pdepe*	OCM	Exact solution
0.1	0.23421	0.23663	0.23529	0.25652	0.23517
0.2	0.45863	0.45664	0.45988	0.39665	0.45964
0.3	0.66100	0.66136	0.66222	0.70637	0.66188
0.4	0.82779	0.82866	0.82881	0.84360	0.82847
0.5	0.94258	0.94298	0.94304	0.94299	0.94298
0.6	0.98463	0.98466	0.98411	0.98441	0.98467
0.7	0.92823	0.92858	0.92904	0.95222	0.92852
0.8	0.76252	0.76944	0.76487	0.84168	0.76650
0.9	0.29116	0.29567	0.28148	0.26113	0.28470

Table 7 Comparison between exact and numerical values at different τ and $\varepsilon = 0.01$

| ξ	τ	QHCM	OCFE	*pdepe*	OCM	Exact solution
0.25	0.4	0.33686	0.34449	0.34206	0.33489	0.34191
	0.6	0.26501	0.29989	0.26906	0.27445	0.26896
	0.8	0.21826	0.26828	0.22155	0.22667	0.22148
	1	0.18548	0.23456	0.18821	0.18972	0.18819
	3	0.07408	0.14405	0.07507	0.07341	0.07511
0.5	0.4	0.66015	0.66182	0.66084	0.65585	0.66071
	0.6	0.52895	0.55618	0.52955	0.50368	0.52942
	0.8	0.43875	0.48441	0.43925	0.42531	0.43914
	1	0.37410	0.43071	0.37444	0.36914	0.37442
	3	0.15011	0.21979	0.15013	0.15079	0.15018
0.75	0.4	0.91228	0.91216	0.91009	0.92317	0.91026
	0.6	0.76973	0.77237	0.76726	0.79907	0.76724
	0.8	0.64969	0.66152	0.64747	0.67187	0.64740
	1	0.55809	0.58114	0.55608	0.57230	0.55605
	3	0.22568	0.28034	0.22477	0.22750	0.22481
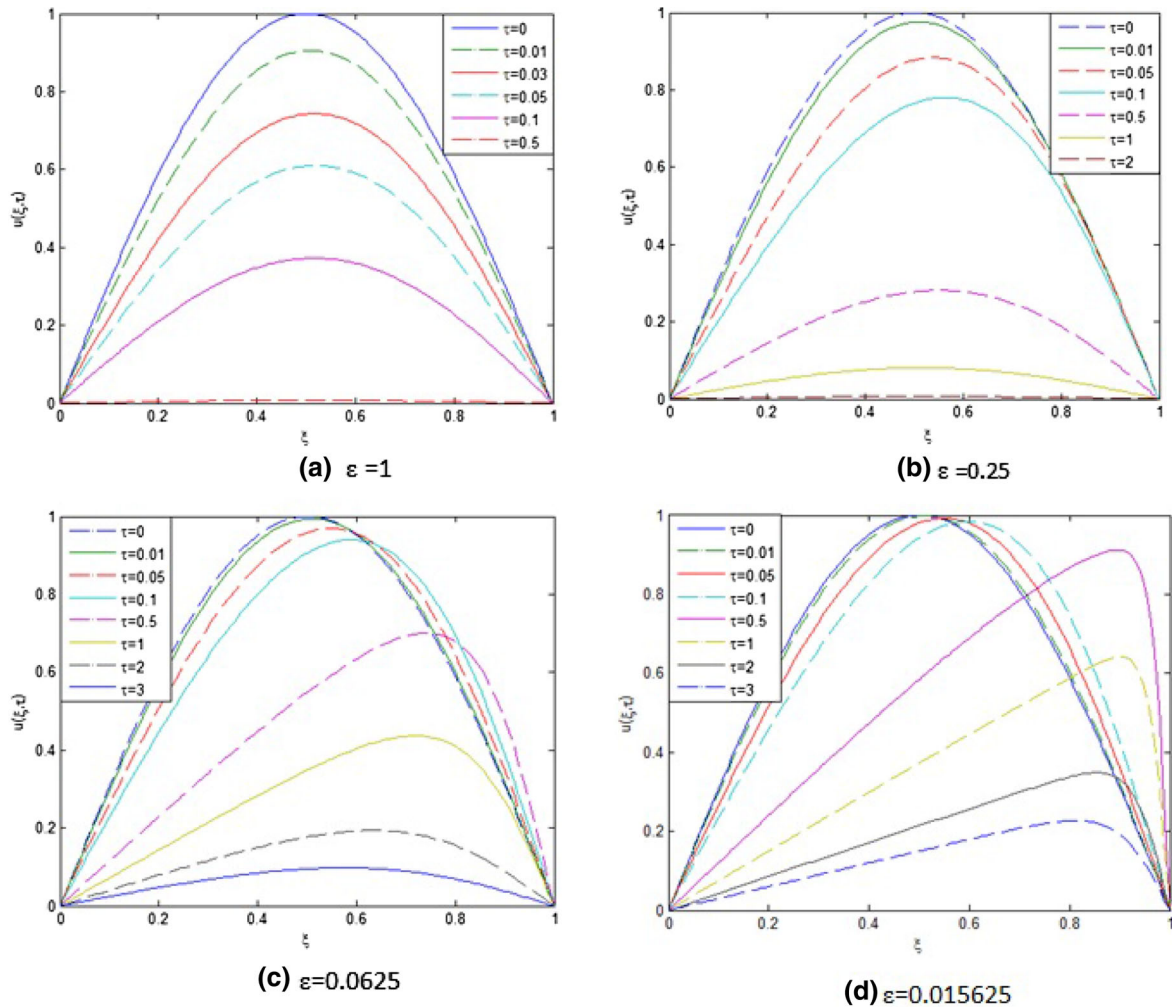
Fig. 2 Graphical view of the numerical solution for $\varepsilon = 1, 0.25, 0.0625, 0.015625$ and different τ in the form of 2D plots for Problem 3.1
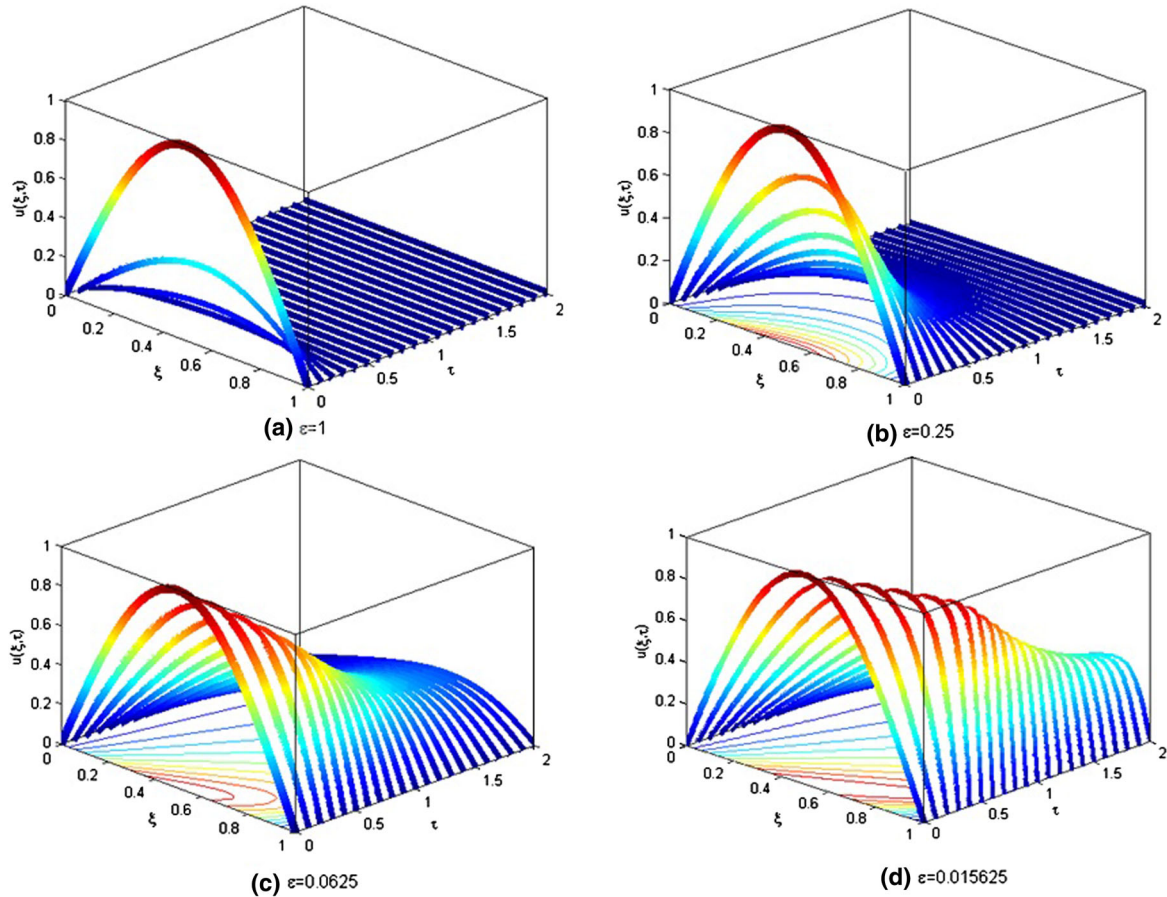
Fig. 3 Graphical view of the numerical solution for $\varepsilon = 1, 0.25, 0.0625, 0.015625$ using *pdepe* solver for Problem 3.1

Table 8 Comparison between exact and numerical values at $\tau = 0.1$ and $\varepsilon = 1$

ξ	QHCM	OCFE	*pdepe*	OCM	Exact solution
0.1	0.11287	0.11252	0.11292	0.12297	0.11289
0.2	0.21622	0.21676	0.21631	0.25604	0.21625
0.3	0.30092	0.30085	0.30102	0.31785	0.30097
0.4	0.35882	0.35880	0.35891	0.36323	0.35886
0.5	0.38339	0.38326	0.38344	0.38346	0.38342
0.6	0.37064	0.37060	0.37065	0.37391	0.37066
0.7	0.32006	0.32014	0.32005	0.33580	0.32007
0.8	0.23538	0.23537	0.23535	0.20569	0.23537
0.9	0.12472	0.12447	0.12470	0.13577	0.12472

Table 9 Comparison between exact and numerical values at $\tau = 0.1$ and $\varepsilon = 0.25$

ξ	QHCM	OCFE	*pdepe*	OCM	Exact solution
0.1	0.21109	0.21771	0.21127	0.23033	0.21131
0.2	0.40855	0.40208	0.40887	0.35412	0.40894
0.3	0.57905	0.57615	0.57943	0.61550	0.57950
0.4	0.70951	0.70980	0.70985	0.72098	0.70989
0.5	0.78654	0.78615	0.78675	0.78676	0.78673
0.6	0.79584	0.79521	0.79596	0.79854	0.79578
0.7	0.72294	0.72214	0.72269	0.74875	0.72264
0.8	0.55796	0.55800	0.55759	0.49307	0.55755
0.9	0.30638	0.30655	0.30610	0.33236	0.30608

Table 10 Comparison between exact and numerical values at $\tau = 0.1$ and $\varepsilon = 0.0625$

ξ	QHCM	OCFE	*pdepe*	OCM	Exact solution
0.1	0.24844	0.24586	0.24901	0.27129	0.24903
0.2	0.47740	0.47728	0.47823	0.41514	0.47826
0.3	0.67224	0.67339	0.67305	0.71414	0.67305
0.4	0.82268	0.82366	0.82331	0.83640	0.82328
0.5	0.91913	0.91936	0.91949	0.91947	0.91946
0.6	0.94944	0.94947	0.94946	0.94991	0.94949
0.7	0.89659	0.89656	0.89614	0.82010	0.89614
0.8	0.73469	0.73379	0.73370	0.66190	0.73373
0.9	0.43199	0.43191	0.43087	0.46524	0.43090

Table 11 Comparison between exact and numerical values at $\tau = 0.1$ and $\varepsilon = 0.015625$

ξ	QHCM	OCFE	*pdepe*	OCM	Exact solution
0.1	0.26281	0.26472	0.26377	0.28707	0.26384
0.2	0.49915	0.50892	0.50016	0.43564	0.50018
0.3	0.69761	0.69635	0.69850	0.74030	0.69845
0.4	0.85130	0.85119	0.85204	0.86541	0.85192
0.5	0.95177	0.95211	0.95223	0.95213	0.95211
0.6	0.98746	0.98745	0.98747	0.98722	0.98746
0.7	0.94192	0.94151	0.94138	0.96138	0.94127
0.8	0.77554	0.78198	0.78455	0.72043	0.78370
0.9	0.54206	0.54316	0.54494	0.52528	0.54869

Table 12 Comparison between exact and numerical values at different τ and $\varepsilon = 0.01$

ξ	τ	QHCM	OCFE	*pdepe*	OCM	Exact solution
0.25	0.4	0.36628	0.36210	0.36230	0.35715	0.36226
	0.6	0.28531	0.31174	0.28207	0.28985	0.28204
	0.8	0.23320	0.27690	0.23048	0.23697	0.23045
	1	0.19706	0.25109	0.19466	0.19702	0.19469
	3	0.07712	0.14563	0.07608	0.07444	0.07613
0.5	0.4	0.68313	0.68409	0.68364	0.67315	0.68368
	0.6	0.54784	0.57712	0.54632	0.52215	0.54832
	0.8	0.45332	0.50111	0.45314	0.43974	0.45371
	1	0.38536	0.44356	0.38561	0.38001	0.38568
	3	0.15212	0.22185	0.15211	0.15288	0.15218
0.75	0.4	0.92235	0.92059	0.92039	0.93946	0.92050
	0.6	0.78539	0.78708	0.78291	0.81563	0.78299
	0.8	0.66499	0.67826	0.66268	0.68849	0.66272
	1	0.57136	0.59724	0.56924	0.58677	0.56932
	3	0.22863	0.28391	0.22867	0.23059	0.22774

Problem 3.2 has also been solved numerically using different techniques as Problem 3.1 for different values of ε and τ. The results are presented in Tables 8, 9, 10, 11 and 12 and it can be easily observed that numerical values agree well with the exact ones for QHCM as compared to OCFE, *pdepe* solver and OCM. In Fig. 4, numerical solution has been shown in the form of 2D plots for different values of ε and τ. In Fig. 5, surface plots have been plotted for different values of ε and all τ. From both the problems, it has been observed that the values obtained from QHCM agree well with the exact solution as compared to the OCFE, OCM and *pdepe* solver. Since for OCFE one needs a large number of elements, *pdepe* solver gives oscillations for certain values of parameter and because of stiff system of equations OCM does not give good results.
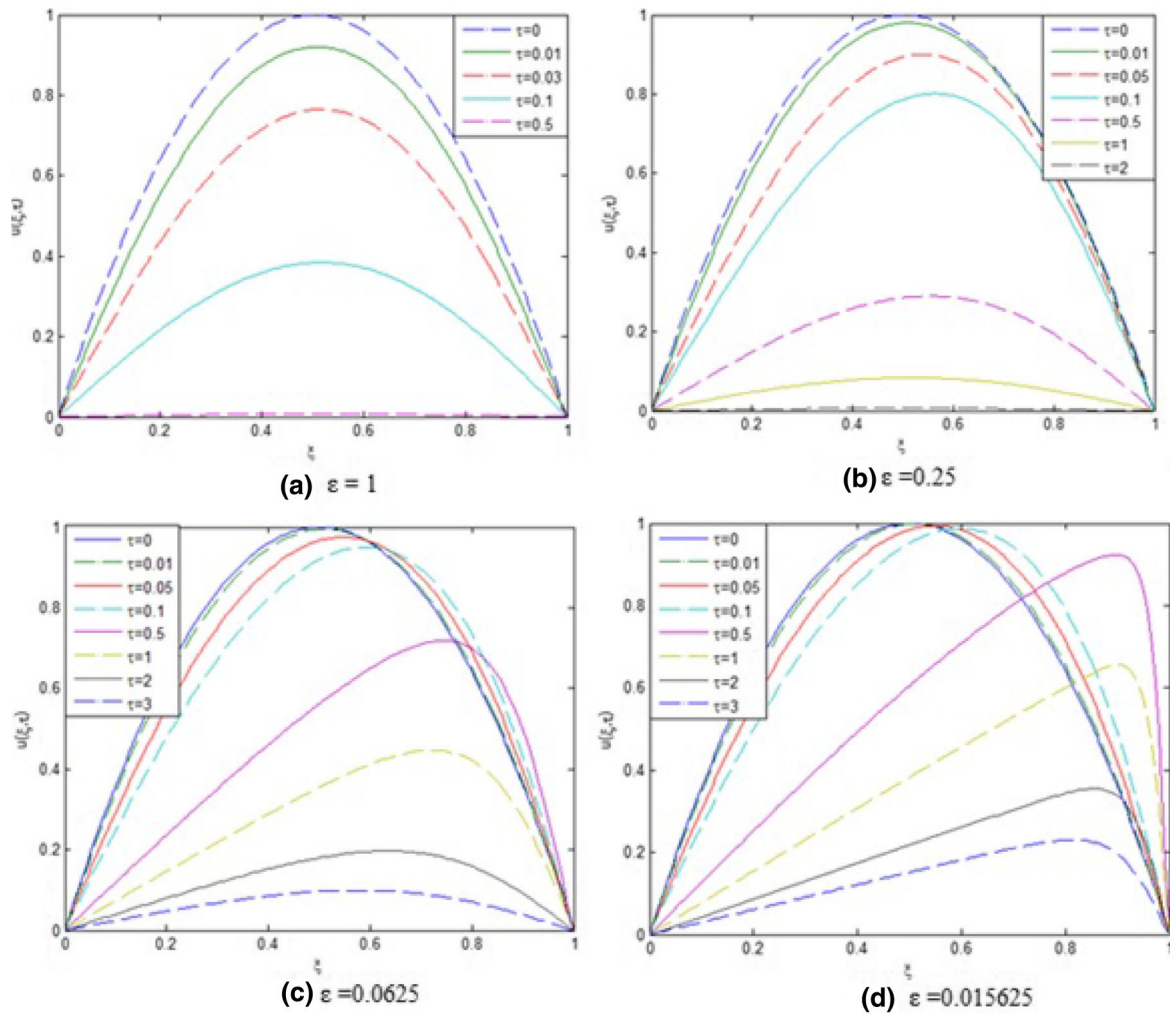
Fig. 4 Graphical view of the numerical solution for $\varepsilon = 1, 0.25, 0.0625, 0.015625$ and different τ in the form of 2D plots for Problem 3.2

Error analysis has been performed by calculating relative error (RE) given in the following formula:

$$\text{RE} = \left| \frac{u_{\text{exact}} - u_{\text{numerical}}}{u_{\text{exact}}} \right|$$

RE has been presented in Tables 13 and 14 for Problems 3.1 and 3.2, respectively. It has been observed from these tables that maximum R.E. for QHCM is of order 10^{-2}. This simply shows that the values obtained from QHCM agree quite well with the exact ones up to a certain degree of accuracy.
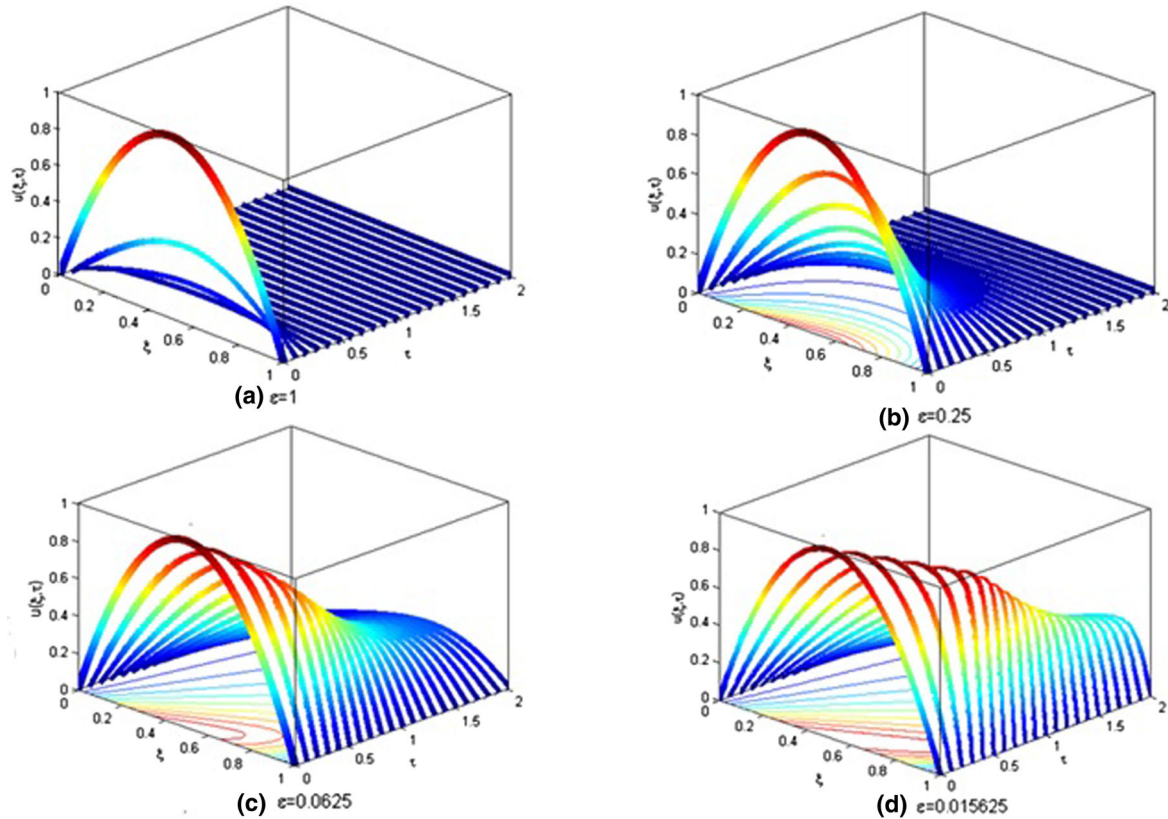
Fig. 5 Graphical view of the numerical solution for $\varepsilon = 1, 0.25, 0.0625, 0.015625$ using *pdepe* solver for Problem 3.2

Table 13 Relative error for different ε at $\tau = 0.1$ for Problem 3.1

ξ	$\varepsilon = 1$	$\varepsilon = 0.25$	$\varepsilon = 0.0625$	$\varepsilon = 0.015625$
0.1	1.8258×10^{-4}	1.1857×10^{-3}	2.5363×10^{-3}	4.0822×10^{-3}
0.2	1.4300×10^{-4}	1.0427×10^{-3}	1.9040×10^{-3}	2.1974×10^{-3}
0.3	1.7129×10^{-4}	8.2133×10^{-4}	1.2939×10^{-3}	1.3295×10^{-3}
0.4	1.1497×10^{-4}	5.8016×10^{-4}	8.1226×10^{-4}	8.2079×10^{-4}
0.5	8.0736×10^{-5}	2.6101×10^{-4}	4.0826×10^{-4}	4.2419×10^{-4}
0.6	5.5703×10^{-5}	9.5549×10^{-4}	3.1935×10^{-5}	4.0623×10^{-5}
0.7	0	1.1430×10^{-4}	6.0502×10^{-4}	3.1232×10^{-4}
0.8	0	8.3958×10^{-4}	1.5127×10^{-3}	5.1924×10^{-3}
0.9	0	1.0952×10^{-3}	2.7737×10^{-3}	2.2691×10^{-2}

Table 14 Relative error for different ε at $\tau = 0.1$ for Problem 3.2

ξ	$\varepsilon = 1$	$\varepsilon = 0.25$	$\varepsilon = 0.0625$	$\varepsilon = 0.015625$
0.1	1.7716×10^{-4}	1.0411×10^{-3}	2.3692×10^{-3}	3.9039×10^{-3}
0.2	1.3873×10^{-4}	9.5369×10^{-4}	1.7982×10^{-3}	2.0593×10^{-3}
0.3	1.6613×10^{-4}	7.7653×10^{-4}	1.2035×10^{-3}	1.2027×10^{-3}
0.4	1.1146×10^{-4}	5.3529×10^{-4}	7.2879×10^{-4}	7.2777×10^{-4}
0.5	7.8243×10^{-5}	2.4151×10^{-4}	3.5891×10^{-4}	3.5710×10^{-4}
0.6	5.3958×10^{-5}	7.5398×10^{-5}	5.2660×10^{-5}	0
0.7	3.1243×10^{-5}	4.1514×10^{-4}	5.0215×10^{-4}	6.9056×10^{-4}
0.8	4.2486×10^{-5}	7.3536×10^{-4}	1.3084×10^{-3}	1.0412×10^{-2}
0.9	0	9.8014×10^{-4}	2.5296×10^{-3}	1.2083×10^{-2}

6 Conclusions

The Hermite collocation method developed here solves the non-linear Burgers' equation directly without transforming it into the linear form using Hopf–Cole transformation. The numerical study on two problems shows the accuracy of QHCM by comparing the values with other techniques as OCFE, *pdepe* solver and OCM. Results obtained from both the problems agree fairly well with the exact ones for QHCM as compared to other techniques, which authenticate the applicability of QHCM in solving non-linear stiff system of partial differential equations. This method is efficient due to its simplicity and easily programmable nature. For stability reasons, $\|U\|_2$ and $\|U\|_\infty$ norms have been calculated.

References

1. Abbasbandy, S.; Darvishi, M.T.: A numerical solution of Burgers' equation by modified Adomain method. Appl. Math. Comput. **163**, 1265–1272 (2005)
2. Aksan, E.N.: A numerical solution of Burgers' equation by finite element method constructed on the method of discretization in time. Appl. Math. Comput. **170**, 895–904 (2005)
3. Andrews, L.C.: Special Functions for Engineers and Applied Mathematicians. Macmillan Publishing Company, A Divison of Macmillan, New York, Collier Macmillan Publishers, London (1984)
4. Arora, S.; Dhaliwal, S.S.; Kukreja, V.K.: Modelling of displacement washing of packed bed of fibers. Braz. J. Chem. Eng. **26**, 385–393 (2009)
5. Arora, G.; Singh, B.K.: Numerical solution of Burgers' equation with modified cubic B-spline differential quadrature method. Appl. Math. Comput. **224**, 166–177 (2013)
6. Asaithambi, A.: Numerical solution of the Burgers' equation by automatic differentiation. Appl. Math. Comput. **216**, 2700–2708 (2010)
7. Carey, G.F.; Finlayson, B.A.: Orthogonal collocation on finite elements. Chem. Eng. Sci. **30**, 587–596 (1975)
8. Chen, F.; Wong, P.J.Y.: Error estimates for discrete spline interpolation: quintic and biquintic splines. J. Comput. Appl. Math. **236**, 3835–3854 (2012)
9. Dağ, İ.; Irk, D.; Saka, B.: A numerical solution of the Burgers' equation using cubic B-splines. Appl. Math. Comput. **163**, 199–211 (2005)
10. Dyksen, W.R.; Lynch, R.E.: A new decoupling technique for the Hermite cubic collocation equations arising from boundary value problems. Math. Comput. Simul. **54**, 359–372 (2000)
11. Hall, C.: On Onerror bounds for spline interpolation. J. Approx. Theory. **1**, 209–218 (1968)
12. Kadalbajoo, M.K.; Awasthi, A.: A numerical method based on Crank-Nicolson scheme for Burgers' equation. Appl. Math. Comput. **182**, 1430–1442 (2006)
13. Kutluay, S.; Esen, A.; Dag, I.: Numerical solutions of the Burgers' equation by the least-squares quadratic B-spline finite element method. J. Comput. Appl. Math. **167**, 21–33 (2004)
14. Liao, W.: An implicit fourth-order compact finite difference scheme for one-dimensional Burgers' equation. Appl. Math. Comput. **206**, 755–764 (2008)
15. Michelsen, M.L.; Villadsen, J.: A convenient computational procedure for collocation constants. Chem. Eng. J. **4**, 64–68 (1971)
16. Özis, T.; Esen, A.; Kutluay, S.: Numerical solution of Burgers' equation by quadratic B-spline finite elements. Appl. Math. Comput. **165**, 237–249 (2005)
17. Ricciardi, K.L.; Brill, S.H.: Optimal Hermite collocation applied to a one-dimensional convection-diffusion equation using an adaptive hybrid optimization algorithm. Int. J. Numer. Methods Heat Fluid Flow. **19**, 874–893 (2009)
18. Villadsen, J.; Stewart, W.E.: Solution of boundary value problems by orthogonal collocation. Chem. Eng. Sci. **22**, 1483–1501 (1967)