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Student Solutions Manual for Elementary Differential Equations and Elementary Differential Equations with Boundary Value Problems

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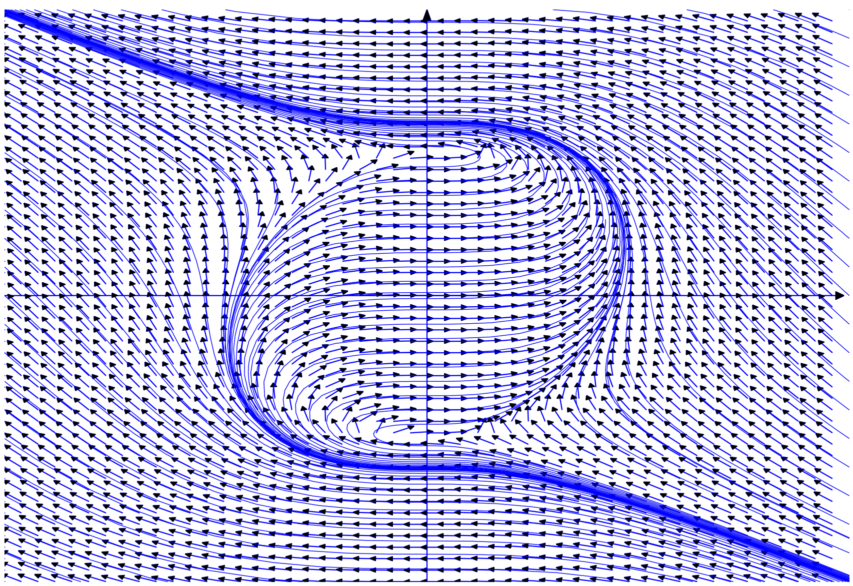
STUDENT SOLUTIONS MANUAL FOR

ELEMENTARY DIFFERENTIAL EQUATIONS

AND

ELEMENTARY DIFFERENTIAL EQUATIONS WITH BOUNDARY VALUE PROBLEMS



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TO BEVERLY

Contents

CHAPTER 1
Introduction

1.2 BASIC CONCEPTS

1.2.2. **(a)** If $y = ce^{2x}$, then $y' = 2ce^{2x} = 2y$.

(b) If $y = \dfrac{x^2}{3} + \dfrac{c}{x}$, then $y' = \dfrac{2x}{3} - \dfrac{c}{x^2}$, so $xy' + y = \dfrac{2x^2}{3} - \dfrac{c}{x} + \dfrac{x^2}{3} + \dfrac{c}{x} = x^2$.

(c) If

$$y = \frac{1}{2} + ce^{-x^2}, \quad \text{then} \quad y' = -2xce^{-x^2}$$

and

$$y' + 2xy = -2xce^{-x^2} + 2x\left(\frac{1}{2} + ce^{-x^2}\right) = -2xce^{-x^2} + x + 2cxe^{-x^2} = x.$$

(d) If

$$y = \frac{1 + ce^{-x^2/2}}{1 - ce^{-x^2/2}}$$

then

$$\begin{aligned} y' &= \frac{(1 - ce^{-x^2/2})(-cxe^{-x^2/2}) - (1 + ce^{-x^2/2})cxe^{-x^2/2}}{(1 - cxe^{-x^2/2})^2} \\ &= \frac{-2cxe^{-x^2/2}}{(1 - ce^{-x^2/2})^2} \end{aligned}$$

and

$$\begin{aligned} y^2 - 1 &= \left(\frac{1 + ce^{-x^2/2}}{1 - ce^{-x^2/2}}\right)^2 - 1 \\ &= \frac{(1 + ce^{-x^2/2})^2 - (1 - ce^{-x^2/2})^2}{(1 - ce^{-x^2/2})^2} \\ &= \frac{4ce^{-x^2/2}}{(1 - ce^{-x^2/2})^2}, \end{aligned}$$

so

$$2y' + x(y^2 - 1) = \frac{-4cx + 4cx}{(1 - ce^{-x^2/2})^2} = 0.$$

(e) If $y = \tan\left(\frac{x^3}{3} + c\right)$, then $y' = x^2 \sec^2\left(\frac{x^3}{3} + c\right) = x^2\left(1 + \tan^2\left(\frac{x^3}{3 + c}\right)\right) = x^2(1 + y^2)$.

(f) If

$$\begin{aligned} y &= (c_1 + c_2 x)e^x + \sin x + x^2, \quad \text{then} \\ y' &= (c_1 + 2c_2 x)e^x + \cos x + 2x, \\ y' &= (c_1 + 3c_2 x)e^x - \sin x + 2, \end{aligned}$$

and

$$\begin{aligned} y'' - 2y' + y &= c_1 e^x(1 - 2 + 1) + c_2 x e^x(3 - 4 + 1) \\ &\quad - \sin x - 2\cos x + \sin x + 2 - 4x + x^2 \\ &= -2\cos x + x^2 - 4x + 2. \end{aligned}$$

(g) If $y = c_1 e^x + c_2 x + \frac{2}{x}$, then $y' = c_1 e^x + c_2 - \frac{2}{x^2}$ and $y'' = c_1 e^x + \frac{4}{x^3}$, so $(1-x)y'' + xy' - y = c_1(1 - x + x - 1) + c_2(x - x) + \frac{4(1-x)}{x^3} - \frac{2}{x} - \frac{2}{x} = \frac{4(1 - x - x^2)}{x^3}$

(h) If $y = \frac{c_1 \sin x + c_2 \cos x}{x^{1/2}} + 4x + 8$ then $y' = \frac{c_1 \cos x - c_2 \sin x}{x^{1/2}} - \frac{c_1 \sin x + c_2 \cos x}{2x^{3/2}} + 4$ and $y'' = -\frac{c_1 \sin x + c_2 \cos x}{x^{1/2}} - \frac{c_1 \sin x - c_2 \cos x}{x^{3/2}} + \frac{3}{4}\frac{c_1 \sin x + c_2 \cos x}{x^{5/2}}$, so $x^2 y'' + xy' + \left(x^2 - \frac{1}{4}\right) y = c_1\left(-x^{-3/2}\sin x - x^{1/2}\cos x + \frac{3}{4}x^{-1/2}\sin x + x^{1/2}\cos x - \frac{1}{2}x^{-1/2}\sin x + x^{3/2}\sin x - \frac{1}{4}x^{-1/2}\sin x\right) + c_2\left(-x^{-3/2}\cos x + x^{1/2}\sin x + \frac{3}{4}x^{-1/2}\cos x - x^{1/2}\sin x - \frac{1}{2}x^{-1/2}\cos x + x^{3/2}\cos x - \frac{1}{4}x^{-1/2}\cos x\right) + 4x + \left(x^2 - \frac{1}{4}\right)(4x + 8) = 4x^3 + 8x^2 + 3x - 2$.

1.2.4. **(a)** If $y' = -xe^x$, then $y = -xe^x + \int e^x\,dx + c = (1 - x)e^x + c$, and $y(0) = 1 \Rightarrow 1 = 1 + c$, so $c = 0$ and $y = (1 - x)e^x$.

(b) If $y' = x \sin x^2$, then $y = -\frac{1}{2}\cos x^2 + c$; $y\left(\sqrt{\frac{\pi}{2}}\right) = 1 \Rightarrow 1 = 0 + c$, so $c = 1$ and $y = 1 - \frac{1}{2}\cos x^2$.

(c) Write $y' = \tan x = \frac{\sin x}{\cos x} = -\frac{1}{\cos x}\frac{d}{dx}(\cos x)$. Integrating this yields $y = -\ln|\cos x| + c$; $y(\pi/4) = 3 \Rightarrow 3 = -\ln(\cos(\pi/4)) + c$, or $3 = \ln\sqrt{2} + c$, so $c = 3 - \ln\sqrt{2}$, so $y = -\ln(|\cos x|) + 3 - \ln\sqrt{2} = 3 - \ln(\sqrt{2}|\cos x|)$.

(d) If $y'' = x^4$, then $y' = \frac{x^5}{5} + c_1$; $y'(2) = -1 \Rightarrow \frac{32}{5} + c_1 = -1 \Rightarrow c_1 = -\frac{37}{15}$, so $y' = \frac{x^5}{5} - \frac{37}{15}$. Therefore, $y = \frac{x^6}{30} - \frac{37}{15}(x - 2) + c_2$; $y(2) = -1 \Rightarrow \frac{64}{30} + c_2 = -1 \Rightarrow c_2 = -\frac{47}{15}$, so $y = -\frac{47}{15} - \frac{37}{5}(x - 2) + \frac{x^6}{30}$.

(e) (A) $\int xe^{2x}\,dx = \frac{xe^{2x}}{2} - \frac{1}{2}\int e^{2x}\,dx = \frac{xe^{2x}}{2} - \frac{e^{2x}}{4}$. Therefore, $y' = \frac{xe^{2x}}{2} - \frac{e^{2x}}{4} + c_1$; $y'(0) = 1 \Rightarrow -\frac{1}{4} + c_1 = \frac{5}{4} \Rightarrow c_1 = \frac{5}{4}$, so $y' = \frac{xe^{2x}}{2} - \frac{e^{2x}}{4} + \frac{5}{4}$; Using (A) again, $y = \frac{xe^{2x}}{4} - \frac{e^{2x}}{8} - \frac{e^{2x}}{8} + \frac{5}{4}x + c_2 = \frac{xe^{2x}}{4} - \frac{e^{2x}}{4} + \frac{5}{4}x + c_2$; $y(0) = 7 \Rightarrow -\frac{1}{4} + c_2 = 7 \Rightarrow c_2 = \frac{29}{4}$, so $y = \frac{xe^{2x}}{4} - \frac{e^{2x}}{4} + \frac{5}{4}x + \frac{29}{4}$.

(f) (A) $\int x \sin x\,dx = -x\cos x + \int \cos x\,dx = -x\cos x + \sin x$ and (B) $\int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x$. If $y'' = -x\sin x$, then (A) implies that $y' = x\cos x - \sin x + c_1$; $y'(0) = -3 \Rightarrow c = -3$, so $y' = x\cos x - \sin x - 3$. Now (B) implies that $y = x\sin x + \cos x + \cos x - 3x + c_2 = x\sin x + 2\cos x - 3x + c_2$; $y(0) = 1 \Rightarrow 2 + c_2 = 1 \Rightarrow c_2 = -1$, so $y = x\sin x + 2\cos x - 3x - 1$.

(g) If $y''' = x^2e^x$, then $y'' = \int x^2e^x\,dx = x^2e^x - 2\int xe^x\,dx = x^2e^x - 2xe^x + 2e^x + c_1$; $y''(0) = 3 \Rightarrow 2 + c_1 = 3 \Rightarrow c_1 = 1$, so (A) $y'' = (x^2 - 2x + 2)e^x + 1$. Since $\int (x^2 - 2x + 2)e^x\,dx = (x^2 - 2x + 2)e^x - \int (2x - 2)e^x\,dx = (x^2 - 2x + 2)e^x - (2x - 2)e^x + 2e^x = (x^2 - 4x + 6)e^x$, (A) implies that $y' = (x^2 - 4x + 6)e^x + x + c_2$; $y'(0) = -2 \Rightarrow 6 + c_2 = -2 \Rightarrow c_2 = -8$, so (B) $y' = (x^2 - 4x + 6)e^x + x - 8$; Since $\int (x^2 - 4x + 6)e^x\,dx = (x^2 - 4x + 6)e^x - \int (2x - 4)e^x\,dx = (x^2 - 4x + 6)e^x - (2x - 4)e^x + 2e^x = (x^2 - 6x + 12)e^x$, (B) implies that $y = (x^2 - 6x + 12)e^x + \dfrac{x^2}{2} - 8x + c_3$; $y(0) = 1 \Rightarrow 12 + c_3 = 1 \Rightarrow c_3 = -11$, so $y = (x^2 - 6x + 12)e^x + \dfrac{x^2}{2} - 8x - 11$.

(h) If $y''' = 2 + \sin 2x$, then $y'' = 2x - \dfrac{\cos 2x}{2} + c_1$; $y''(0) = 3 \Rightarrow -\dfrac{1}{2} + c_1 = 3 \Rightarrow c_1 = \dfrac{7}{2}$, so $y'' = 2x - \dfrac{\cos 2x}{2} + \dfrac{7}{2}$. Then $y' = x^2 - \dfrac{\sin 2x}{4} + \dfrac{7}{2}x + c_2$; $y'(0) = -6 \Rightarrow c_2 = -6$, so $y' = x^2 - \dfrac{\sin 2x}{4} + \dfrac{7}{2}x - 6$. Then $y = \dfrac{x^3}{3} + \dfrac{\cos 2x}{8} + \dfrac{7}{4}x^2 - 6x + c_3$; $y(0) = 1 \Rightarrow \dfrac{1}{8} + c_3 = 1 \Rightarrow c_3 = \dfrac{7}{8}$, so $y = \dfrac{x^3}{3} + \dfrac{\cos 2x}{8} + \dfrac{7}{4}x^2 - 6x + \dfrac{7}{8}$.

(i) If $y''' = 2x + 1$, then $y'' = x^2 + x + c_1$; $y''(2) = 7 \Rightarrow 6 + c_1 = 7 \Rightarrow c_1 = 1$; so $y'' = x^2 + x + 1$. Then $y' = \dfrac{x^3}{3} + \dfrac{x^2}{2} + (x - 2) + c_2$; $y'(2) = -4 \Rightarrow \dfrac{14}{3} + c_2 = -4 \Rightarrow c_2 = -\dfrac{26}{3}$, so $y' = \dfrac{x^3}{3} + \dfrac{x^2}{2} + (x - 2) - \dfrac{26}{3}$. Then $y = \dfrac{x^4}{12} + \dfrac{x^3}{6} + \dfrac{1}{2}(x - 2)^2 - \dfrac{26}{3}(x - 2) + c_3$; $y(2) = 1 \Rightarrow \dfrac{8}{3} + c_3 = 1 \Rightarrow c_3 = -\dfrac{5}{3}$, so $y = \dfrac{x^4}{12} + \dfrac{x^3}{6} + \dfrac{1}{2}(x - 2)^2 - \dfrac{26}{3}(x - 2) - \dfrac{5}{3}$.

1.2.6. **(a)** If $y = x^2(1 + \ln x)$, then $y(e) = e^2(1 + \ln e) = 2e^2$; $y' = 2x(1 + \ln x) + x = 3x + 2x\ln x$, so $y'(e) = 3e + 2e\ln e = 5e$; (A) $y'' = 3 + 2 + 2\ln x = 5 + 2\ln x$. Now, $3xy' - 4y = 3x(3x + 2x\ln x) - 4x^2(1 + \ln x) = 5x^2 + 2x^2\ln x = x^2y''$, from (A).

(b) If $y = \dfrac{x^2}{3} + x - 1$, then $y(1) = \dfrac{1}{3} + 1 - 1 = \dfrac{1}{3}$; $y' = \dfrac{2}{3}x + 1$, so $y'(1) = \dfrac{2}{3} + 1 = \dfrac{5}{3}$; (A) $y'' = \dfrac{2}{3}$. Now $x^2 - xy' + y + 1 = x^2 - x\left(\dfrac{2}{3}x + 1\right) + \dfrac{x^2}{3} + x - 1 + 1 = \dfrac{2}{3}x^2 = x^2y''$, from (A).

(c) If $y = (1 + x^2)^{-1/2}$, then $y(0) = (1 + 0^2)^{-1/2} = 1$; $y' = -x(1 + x^2)^{-3/2}$, so $y'(0) = 0$; (A) $y'' = (2x^2 - 1)(1 + x^2)^{-5/2}$. Now, $(x^2 - 1)y - x(x^2 + 1)y' = (x^2 - 1)(1 + x^2)^{-1/2} - x(x^2 + 1)(-x)(1 + x^2)^{-3/2} = (2x^2 - 1)(1 + x^2)^{-1/2} = y''(1 + x^2)^2$ from (A), so $y'' = \dfrac{(x^2 - 1)y - x(x^2 + 1)y'}{(x^2 + 1)^2}$.

(d) If $y = \dfrac{x^2}{1 - x}$, then $y(1/2) = \dfrac{1/4}{1 - 1/2} = \dfrac{1}{2}$; $y' = -\dfrac{x(x - 2)}{(1 - x)^2}$, so $y'(1/2) = \dfrac{(-1/2)(-3/2)}{(1 - 1/2)^2} = 3$; (A) $y'' = \dfrac{2}{(1 - x)^3}$. Now, (B) $x + y = x + \dfrac{x^2}{1 - x} = \dfrac{x}{1 - x}$ and (C) $xy' - y = -\dfrac{x^2(x - 2)}{(1 - x)^2} - \dfrac{x^2}{1 - x} = \dfrac{x^2}{(1 - x)^2}$. From (B) and (C), $(x + y)(xy' - y) = \dfrac{x^3}{(1 - x)^3} = \dfrac{x^3}{2}y''$, so $y'' = \dfrac{2(x + y)(xy' - y)}{x^3}$.

1.2.8. **(a)** $y = (x - c)^a$ is defined and $x - c = y^{1/a}$ on (c, ∞); moreover, $y' = a(x - c)^{a-1} = a\left(y^{1/a}\right)^{a-1} = ay^{(a-1)/a}$.

(b) if $a > 1$ or $a < 0$, then $y \equiv 0$ is a solution of (B) on $(-\infty, \infty)$.

1.2.10. **(a)** Since $y' = c$ we must show that the right side of (B) reduces to c for all values of x in some interval. If $y = c^2 + cx + 2c + 1$,

$$\begin{aligned} x^2 + 4x + 4y &= x^2 + 4x + 4c^2 + 4cx + 8c + 4 \\ &= x^2 + 4(1+c)x + 4(c^2 + 2c + 1) \\ &= x^2 + 4(1+c) + 2(c+1)^2 = (x + 2c + 2)^2. \end{aligned}$$

Therefore, $\sqrt{x^2 + 4x + 4y} = x + 2c + 2$ and the right side of (B) reduces to c if $x > -2c - 2$.

(b) If $y_1 = -\dfrac{x(x+4)}{4}$, then $y_1' = -\dfrac{x+2}{2}$ and $x^2 + 4x + 4y = 0$ for all x. Therefore, y_1 satisfies (A) on $(-\infty, \infty)$.

CHAPTER 2
First Order Equations

2.1 LINEAR FIRST ORDER EQUATIONS

2.1.2. $\dfrac{y'}{y} = -3x^2$; $\quad |\ln|y| = -x^3 + k$; $\quad y = ce^{-x^3}$. $y = ce^{-(\ln x)^2/2}$.

2.1.4. $\dfrac{y'}{y} = -\dfrac{3}{x}$; $\quad \ln|y| = -3\ln|x| + k = -\ln|x|^3 + k$; $\quad y = \dfrac{c}{x^3}$.

2.1.6. $\dfrac{y'}{y} = -\dfrac{1+x}{x} = -\dfrac{1}{x} - 1$; $\quad |\ln|y| = -\ln|x| - x + k$; $\quad y = \dfrac{ce^{-x}}{x}$; $\quad y(1) = 1 \Rightarrow c = e$; $y = \dfrac{e^{-(x-1)}}{x}$.

2.1.8. $\dfrac{y'}{y} = -\dfrac{1}{x} - \cot x$; $\quad |\ln|y| = -\ln|x| - \ln|\sin x| + k = -\ln|x\sin x| + k$; $\quad y = \dfrac{c}{x\sin x}$; $y(\pi/2) = 2 \Rightarrow c = \pi$; $\quad y = \dfrac{\pi}{x\sin x}$.

2.1.10. $\dfrac{y'}{y} = -\dfrac{k}{x}$; $\quad |\ln|y| = -k\ln|x| + k_1 = \ln|x^{-k}| + k_1$; $\quad y = c|x|^{-k}$; $\quad y(1) = 3 \Rightarrow c = 3$; $y = 3x^{-k}$.

2.1.12. $\dfrac{y_1'}{y_1} = -3$; $\quad \ln|y_1| = -3x$; $\quad y_1 = e^{-3x}$; $\quad y = ue^{-3x}$; $\quad u'e^{-3x} = 1$; $\quad u' = e^{3x}$; $\quad u = \dfrac{e^{3x}}{3} + c$; $\quad y = \dfrac{1}{3} + ce^{-3x}$.

2.1.14. $\dfrac{y_1'}{y_1} = -2x$; $\quad \ln|y_1| = -x^2$; $\quad y_1 = e^{-x^2}$; $\quad y = ue^{-x^2}$; $\quad u'e^{-x^2} = xe^{-x^2}$; $\quad u' = x$; $u = \dfrac{x^2}{2} + c$; $\quad y = e^{-x^2}\left(\dfrac{x^2}{2} + c\right)$.

2.1.16. $\dfrac{y_1'}{y_1} = -\dfrac{1}{x}$; $\quad \ln|y_1| = -\ln|x|$; $\quad y_1 = \dfrac{1}{x}$; $\quad y = \dfrac{u}{x}$; $\quad \dfrac{u'}{x} = \dfrac{7}{x^2} + 3$; $\quad u' = \dfrac{7}{x} + 3x$; $u = 7\ln|x| + \dfrac{3x^2}{2} + c$; $\quad y = \dfrac{7\ln|x|}{x} + \dfrac{3x}{2} + \dfrac{c}{x}$.

2.1.18. $\frac{y_1'}{y_1} = -\frac{1}{x} - 2x$; $\ln|y_1| = -\ln|x| - x^2$; $y_1 = \frac{e^{-x^2}}{x}$; $y = \frac{ue^{-x^2}}{x}$; $\frac{u'e^{-x^2}}{x} = x^2e^{-x^2}$; $u' = x^3$; $u = \frac{x^4}{4} + c$; $y = e^{-x^2}\left(\frac{x^3}{4} + \frac{c}{x}\right)$.

2.1.20. $\frac{y_1'}{y_1} = -\tan x$; $\ln|y_1| = \ln|\cos x|$; $y_1 = \cos x$; $y = u\cos x$; $u'\cos x = \cos x$; $u' = 1$; $u = x + c$; $y = (x+c)\cos x$.

2.1.22. $\frac{y_1'}{y_1} = \frac{4x-3}{(x-2)(x-1)} = \frac{5}{x-2} - \frac{1}{x-1}$; $\ln|y_1| = 5\ln|x-2| - \ln|x-1| = \ln\left|\frac{(x-2)^5}{x-1}\right|$; $y_1 = \frac{(x-2)^5}{x-1}$; $y = \frac{u(x-2)^5}{x-1}$; $\frac{u'(x-2)^5}{x-1} = \frac{(x-2)^2}{x-1}$; $u' = \frac{1}{(x-2)^3}$; $u = -\frac{1}{2}\frac{1}{(x-2)^2} + c$; $y = -\frac{1}{2}\frac{(x-2)^3}{(x-1)} + c\frac{(x-2)^5}{(x-1)}$.

2.1.24. $\frac{y_1'}{y_1} = -\frac{3}{x}$; $\ln|y_1| = -3\ln|x| = \ln|x|^{-3}$; $y_1 = \frac{1}{x^3}$; $y = \frac{u}{x^3}$; $\frac{u'}{x^3} = \frac{e^x}{x^2}$; $u' = xe^x$; $u = xe^x - e^x + c$; $y = \frac{e^x}{x^2} - \frac{e^x}{x^3} + \frac{c}{x^3}$.

2.1.26. $\frac{y_1'}{y_1} = -\frac{4x}{1+x^2}$; $\ln|y_1| = -2\ln(1+x^2) = \ln(1+x^2)^{-2}$; $y_1 = \frac{1}{(1+x^2)^2}$; $y = \frac{u}{(1+x^2)^2}$; $\frac{u'}{(1+x^2)^2} = \frac{2}{(1+x^2)^2}$; $u' = 2$; $u = 2x + c$; $y = \frac{2x+c}{(1+x^2)^2}$; $y(0) = 1 \Rightarrow c = 1$; $y = \frac{2x+1}{(1+x^2)^2}$.

2.1.28. $\frac{y_1'}{y_1} = -\cot x$; $\ln|y_1| = -\ln|\sin x|$; $y_1 = \frac{1}{\sin x}$; $y = \frac{u}{\sin x}$; $\frac{u'}{\sin x} = \cos x$; $u' = \sin x\cos x$; $u = \frac{\sin^2 x}{2} + c$; $y = \frac{\sin x}{2} + c\csc x$; $y(\pi/2) = 1 \Rightarrow c = \frac{1}{2}$; $y = \frac{1}{2}(\sin x + \csc x)$.

2.1.30. $\frac{y_1'}{y_1} = -\frac{3}{x-1}$; $\ln|y_1| = -3\ln|x-1| = \ln|x-1|^{-3}$; $y_1 = \frac{1}{(x-1)^3}$; $y = \frac{u}{(x-1)^3}$; $\frac{u'}{(x-1)^3} = \frac{1}{(x-1)^4} + \frac{\sin x}{(x-1)^3}$; $u' = \frac{1}{x-1} + \sin x$; $u = \ln|x-1| - \cos x + c$; $y = \frac{\ln|x-1| - \cos x + c}{(x-1)^3}$; $y(0) = 1 \Rightarrow c = 0$; $y = \frac{\ln|x-1| - \cos x}{(x-1)^3}$.

2.1.32. $\frac{y_1'}{y_1} = -\frac{2}{x}$; $\ln|y_1| = 2\ln|x| = \ln(x^2)$; $y_1 = x^2$; $y = ux^2$; $u'x^2 = -x$; $u' = -\frac{1}{x}$; $u = -\ln|x| + c$; $y = x^2(c - \ln|x|)$; $y(1) = 1 \Rightarrow c = 1$; $y = x^2(1 - \ln x)$.

2.1.34. $\frac{y_1'}{y_1} = -\frac{3}{x-1}$; $\ln|y_1| = -3\ln|x-1| = \ln|x-1|^{-3}$; $y_1 = \frac{1}{(x-1)^3}$; $y = \frac{u}{(x-1)^3}$; $\frac{u'}{(x-1)^3} = \frac{1 + (x-1)\sec^2 x}{(x-1)^4}$; $u' = \frac{1}{x-1} + \sec^2 x$; $u = \ln|x-1| + \tan x + c$; $y = \frac{\ln|x-1| + \tan x + c}{(x-1)^3}$; $y(0) = -1 \Rightarrow c = 1$; $y = \frac{\ln|x-1| + \tan x + 1}{(x-1)^3}$.

2.1.36. $\dfrac{y_1'}{y_1} = \dfrac{2x}{x^2-1}$; $\ln|y_1| = \ln|x^2-1|$; $y_1 = x^2-1$; $y = u(x^2-1)$; $u'(x^2-1) = x$; $u' = \dfrac{x}{x^2-1}$; $u = \dfrac{1}{2}\ln|x^2-1| + c$; $y = (x^2-1)\left(\dfrac{1}{2}\ln|x^2-1| + c\right)$; $y(0) = 4 \Rightarrow c = -4$; $y = (x^2-1)\left(\dfrac{1}{2}\ln|x^2-1| - 4\right)$.

2.1.38. $\dfrac{y_1'}{y_1} = -2x$; $\ln|y_1| = -x^2$; $y_1 = e^{-x^2}$; $y = ue^{-x^2}$; $u'e^{-x^2} = x^2$; $u' = x^2e^{x^2}$; $u = c + \displaystyle\int_0^x t^2e^{t^2}\,dt$; $y = e^{-x^2}\left(c + \displaystyle\int_0^x t^2e^{t^2}\,dt\right)$; $y(0) = 3 \Rightarrow c = 3$; $y = e^{-x^2}\left(3 + \displaystyle\int_0^x t^2e^{t^2}\,dt\right)$.

2.1.40. $\dfrac{y_1'}{y_1} = -1$; $\ln|y_1| = -x$; $y_1 = e^{-x}$; $y = ue^{-x}$; $u'e^{-x} = \dfrac{e^{-x}\tan x}{x}$; $u' = \dfrac{\tan x}{x}$; $u = c + \displaystyle\int_1^x \frac{\tan t}{t}\,dt$; $y = e^{-x}\left(c + \displaystyle\int_1^x \frac{\tan t}{t}\,dt\right)$; $y(1) = 0 \Rightarrow c = 0$; $y = e^{-x}\displaystyle\int_1^x \frac{\tan t}{t}\,dt$.

2.1.42. $\dfrac{y_1'}{y_1} = -1 - \dfrac{1}{x}$; $\ln|y_1| = -x - \ln|x|$; $y_1 = \dfrac{e^{-x}}{x}$; $y = \dfrac{ue^{-x}}{x}$; $\dfrac{u'e^{-x}}{x} = \dfrac{e^{x^2}}{x}$; $u' = e^xe^{x^2}$; $u = c + \displaystyle\int_1^x e^te^{t^2}\,dt$; $y = \dfrac{e^{-x}}{x}\left(c + \displaystyle\int_1^x e^te^{t^2}\,dt\right)$; $y(1) = 2 \Rightarrow c = 2e$; $y = \dfrac{1}{x}\left(2e^{-(x-1)} + e^{-x}\displaystyle\int_1^x e^te^{t^2}\,dt\right)$.

2.1.44. **(b)** Eqn. (A) is equivalent to

$$y' - \frac{2}{x} = -\frac{1}{x} \tag{B}$$

on $(-\infty, 0)$ and $(0, \infty)$. Here $\dfrac{y_1'}{y_1} = \dfrac{2}{x}$; $\ln|y_1| = 2\ln|x|$; $y_1 = x^2$; $y = ux^2$; $u'x^2 = -\dfrac{1}{x}$; $u' = -\dfrac{1}{x^3}$; $u = \dfrac{1}{2x^2} + c$, so $y = \dfrac{1}{2} + cx^2$ is the general solution of (A) on $(-\infty, 0)$ and $(0, \infty)$.

(c) From the proof of **(b)**, any solution of (A) must be of the form

$$y = \begin{cases} \dfrac{1}{2} + c_1x^2, & x \geq 0, \\ \dfrac{1}{2} + c_2x^2, & x < 0, \end{cases} \tag{C}$$

for $x \neq 0$, and any function of the form (C) satisfies (A) for $x \neq 0$. To complete the proof we must show that any function of the form (C) is differentiable and satisfies (A) at $x = 0$. By definition,

$$y'(0) = \lim_{x\to 0} \frac{y(x) - y(0)}{x - 0} = \lim_{x\to 0} \frac{y(x) - 1/2}{x}$$

if the limit exists. But

$$\frac{y(x) - 1/2}{x} = \begin{cases} c_1x, & x > 0 \\ c_2x, & x < 0, \end{cases}$$

so $y'(0) = 0$. Since $0y'(0) - 2y(0) = 0 \cdot 0 - 2(1/2) = -1$, any function of the form (C) satisfies (A) at $x = 0$.

(d) From **(b)** any solution y of (A) on $(-\infty, \infty)$ is of the form (C), so $y(0) = 1/2$.

(e) If $x_0 > 0$, then every function of the form (C) with $c_1 = \dfrac{y_0 - 1/2}{x_0^2}$ and c_2 arbitrary is a solution of the initial value problem on $(-\infty, \infty)$. Since these functions are all identical on $(0, \infty)$, this does not contradict Theorem 2.1.1, which implies that (B) (so (A)) has exactly one solution on $(0, \infty)$ such that $y(x_0) = y_0$. A similar argument applies if $x_0 < 0$.

2.1.46. **(a)** Let $y = c_1 y_1 + c_2 y_2$. Then

$$\begin{aligned} y' + p(x)y &= (c_1y_1 + c_2y_2)' + p(x)(c_1y_1 + c_2y_2) \\ &= c_1y_1' + c_2y_2' + c_1p(x)y_1 + c_2p(x)y_2 \\ &= c_1(y_1' + p(x)y_1) + c_2(y_2 + p(x)y_2) = c_1f_1(x) + c_2f_2(x). \end{aligned}$$

(b) Let $f_1 = f_2 = f$ and $c_1 = -c_2 = 1$.
(c) Let $f_1 = f$, $f_2 = 0$, and $c_1 = c_2 = 1$.

2.1.48. **(a)** If $z = \tan y$, then $z' = (\sec^2 y)y'$, so $z' - 3z = -1$; $z_1 = e^{3x}$; $z = ue^{3x}$; $u'e^{3x} = -1$; $u' = -e^{-3x}$; $u = \dfrac{e^{-3x}}{3} + c$; $z = \dfrac{1}{3} + ce^{3x} = \tan y$; $y = \tan^{-1}\left(\dfrac{1}{3} + ce^{3x}\right)$.

(b) If $z = e^{y^2}$, then $z' = 2yy'e^{y^2}$, so $z' + \dfrac{2}{x}z = \dfrac{1}{x^2}$; $z_1 = \dfrac{1}{x^2}$; $z = \dfrac{u}{x^2}$; $\dfrac{u'}{x^2} = \dfrac{1}{x^2}$; $u' = 1$; $u = x + c$; $z = \dfrac{1}{x} + \dfrac{c}{x^2} = e^{y^2}$; $y = \pm\left[\ln\left(\dfrac{1}{x} + \dfrac{c}{x^2}\right)\right]^{1/2}$.

(c) Rewrite the equation as $\dfrac{y'}{y} + \dfrac{2}{x}\ln y = 4x$. If $z = \ln y$, then $z' = \dfrac{y'}{y}$, so $z' + \dfrac{2}{x}z = 4x$; $z_1 = \dfrac{1}{x^2}$; $z = \dfrac{u}{x^2}$; $\dfrac{u'}{x^2} = 4x$; $u' = 4x^3$; $u = x^4 + c$; $z = x^2 + \dfrac{c}{x^2} = \ln y$; $y = \exp\left(x^2 + \dfrac{c}{x^2}\right)$.

(d) If $z = -\dfrac{1}{1+y}$, then $z' = \dfrac{y'}{(1+y)^2}$, so $z' + \dfrac{1}{x}z = -\dfrac{3}{x^2}$; $z_1 = \dfrac{1}{x}$; $z = \dfrac{u}{x}$; $\dfrac{u'}{x} = -\dfrac{3}{x^2}$; $u' = -\dfrac{3}{x}$; $u = -3\ln|x| - c$; $z = -\dfrac{3\ln|x| + c}{x} = -\dfrac{1}{1+y}$; $y = -1 + \dfrac{x}{3\ln|x| + c}$.

2.2 SEPARABLE EQUATIONS

2.2.2. By inspection, $y \equiv k\pi$ (k =integer) is a constant solution. Separate variables to find others: $\left(\dfrac{\cos y}{\sin y}\right)y' = -\sin x$; $\ln(|\sin y|) = \cos x + c$.

2.2.4. $y \equiv 0$ is a constant solution. Separate variables to find others: $\left(\dfrac{\ln y}{y}\right)y' = -x^2$; $\dfrac{(\ln y)^2}{2} = -\dfrac{x^3}{3} + c$.

2.2.6. $y \equiv 1$ and $y \equiv -1$ are constant solutions. For others, separate variables: $(y^2-1)^{-3/2}yy' = \dfrac{1}{x^2}$; $-(y^2-1)^{-1/2} = -\dfrac{1}{x} - c = -\left(\dfrac{1+cx}{x}\right)$; $(y^2-1)^{1/2} = \left(\dfrac{x}{1+cx}\right)$; $(y^2-1) = \left(\dfrac{x}{1+cx}\right)^2$; $y^2 = 1 + \left(\dfrac{x}{1+cx}\right)^2$; $y = \pm\left(1 + \left(\dfrac{x}{1+cx}\right)^2\right)^{1/2}$.

2.2.8. By inspection, $y \equiv 0$ is a constant solution. Separate variables to find others: $\dfrac{y'}{y} = -\dfrac{x}{1+x^2}$; $\ln|y| = -\dfrac{1}{2}\ln(1+x^2) + k$; $y = \dfrac{c}{\sqrt{1+x^2}}$, which includes the constant solution $y \equiv 0$.

2.2.10. $(y-1)^2 y' = 2x+3$; $\dfrac{(y-1)^3}{3} = x^2+3x+c$; $(y-1)^3 = 3x^2+9x+c$; $y = 1 + \left(3x^2+9x+c\right)^{1/3}$.

2.2.12. $\dfrac{y'}{y(y+1)} = -x$; $\left[\dfrac{1}{y} - \dfrac{1}{y+1}\right]y' = -x$; $\ln\left|\dfrac{y}{y+1}\right| = -\dfrac{x^2}{2} + k$; $\dfrac{y}{y+1} = ce^{-x^2/2}$; $y(2) = 1 \Rightarrow c = \dfrac{e^2}{2}$; $y = (y+1)ce^{-x^2/2}$; $y(1 - ce^{-x^2/2}) = ce^{-x^2/2}$; $y = \dfrac{ce^{-x^2/2}}{1 - ce^{-x^2}/2}$; setting $c = \dfrac{e^2}{2}$ yields $y = \dfrac{e^{-(x^2-4)/2}}{2 - e^{-(x^2-4)/2}}$.

2.2.14. $\dfrac{y'}{(y+1)(y-1)(y-2)} = -\dfrac{1}{x+1}$; $\left[\dfrac{1}{6}\dfrac{1}{y+1} - \dfrac{1}{2}\dfrac{1}{y-1} + \dfrac{1}{3}\dfrac{1}{y-2}\right]y' = -\dfrac{1}{x+1}$; $\left[\dfrac{1}{y+1} - \dfrac{3}{y-1} + \dfrac{2}{y-2}\right]y' = -\dfrac{6}{x+1}$; $\ln|y+1| - 3\ln|y-1| + 2\ln|y-2| = -6\ln|x+1| + k$; $\dfrac{(y+1)(y-2)^2}{(y-1)^3} = \dfrac{c}{(x+1)^6}$; $y(1) = 0 \Rightarrow c = -256$; $\dfrac{(y+1)(y-2)^2}{(y-1)^3} = -\dfrac{256}{(x+1)^6}$.

2.2.16. $\dfrac{y'}{y(1+y^2)} = 2x$; $\left[\dfrac{1}{y} - \dfrac{y}{y^2+1}\right]y' = 2x$; $\ln\left(\dfrac{|y|}{\sqrt{y^2+1}}\right) = x^2 + k$; $\dfrac{y}{\sqrt{y^2+1}} = ce^{x^2}$; $y(0) = 1 \Rightarrow c = \dfrac{1}{\sqrt{2}}$; $\dfrac{y}{\sqrt{y^2+1}} = \dfrac{e^{x^2}}{\sqrt{2}}$; $2y^2 = (y^2+1)e^{x^2}$; $y^2(2-e^{x^2}) = e^{2x^2}$; $y = \dfrac{1}{\sqrt{2e^{-2x^2} - 1}}$.

2.2.18. $\dfrac{y'}{(y-1)(y-2)} = -2x$; $\left[\dfrac{1}{y-2} - \dfrac{1}{y-1}\right]y' = -2x$; $\ln\left|\dfrac{y-2}{y-1}\right| = -x^2 + k$; $\dfrac{y-2}{y-1} = ce^{-x^2}$; $y(0) = 3 \Rightarrow c = \dfrac{1}{2}$; $\dfrac{y-2}{y-1} = \dfrac{e^{-x^2}}{2}$; $y-2 = \dfrac{e^{-x^2}}{2}(y-1)$; $y\left(1 - \dfrac{e^{-x^2}}{2}\right) = 2 - \dfrac{e^{-x^2}}{2}$; $y = \dfrac{4 - e^{-x^2}}{2 - e^{-x^2}}$. The interval of validity is $(-\infty, \infty)$.

2.2.20. $\dfrac{y'}{y(y-2)} = -1$; $\dfrac{1}{2}\left[\dfrac{1}{y-2} - \dfrac{1}{y}\right]y' = -1$; $\left[\dfrac{1}{y-2} - \dfrac{1}{y}\right]y' = -2$; $\ln\left|\dfrac{y-2}{y}\right| = -2x + k$; $\dfrac{y-2}{y} = ce^{-2x}$; $y(0) = 1 \Rightarrow c = -1$; $\dfrac{y-2}{y} = -e^{-2x}$; $y - 2 = -ye^{-2x}$; $y(1 + e^{-2x}) = 2$; $y = \dfrac{2}{1+e^{-2x}}$. The interval of validity is $(-\infty, \infty)$.

2.2.22. $y \equiv 2$ is a constant solution of the differential equation, and it satisfies the initial condition. Therefore, $y \equiv 2$ is a solution of the initial value problem. The interval of validity is $(-\infty, \infty)$.

2.2.24. $\dfrac{y'}{1+y^2} = \dfrac{1}{1+x^2}$; $\tan^{-1} y = \tan^{-1} x + k$; $y = \tan(\tan^{-1} x + k)$. Now use the identity $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$ with $A = \tan^{-1} x$ and $B = \tan^{-1} c$ to rewrite y as $y = \dfrac{x+c}{1-cx}$, where $c = \tan k$.

2.2.26. $(\sin y)y' = \cos x$; $-\cos y = \sin x + c$; $y(\pi) = \frac{\pi}{2} \Rightarrow c = 0$, so (A) $\cos y = -\sin x$. To obtain y explicity we note that $-\sin x = \cos(x + \pi/2)$, so (A) can be rewritten as $\cos y = \cos(x + \pi/2)$. This equation holds if an only if one of the following conditions holds for some integer k:

$$\text{(B) } y = x + \frac{\pi}{2} + 2k\pi; mbox(C)\ y = -x - \frac{\pi}{2} + 2k\pi.$$

Among these choices the only way to satisfy the initial condition is to let $k = 1$ in (C), so $y = -x + \frac{3\pi}{2}$.

2.2.28. Rewrite the equation as $P' = -a\alpha P(P - 1/\alpha)$. By inspection, $P \equiv 0$ and $P \equiv 1/\alpha$ are constant solutions. Separate variables to find others: $\frac{P'}{P(P-1/\alpha)} = -a\alpha$; $\left[\frac{1}{P-1/\alpha} - \frac{1}{P}\right]P' = -a$; $\ln\left|\frac{P-1/\alpha}{P}\right| = -at + k$; (A) $\frac{P-1/\alpha}{P} = ce^{-\alpha t}$; $P(1 - ce^{-\alpha t}) = 1/\alpha$; (B) $P = \frac{1}{\alpha(1-ce^{-\alpha t})}$. From (A), $P(0) = P_0 \Rightarrow c = \frac{P_0 - 1/\alpha}{P_0}$. Substituting this into (B) yields $P = \frac{P_0}{\alpha P_0 + (1-\alpha P_0)e^{-at}}$. From this $\lim_{t\to\infty} P(t) = 1/\alpha$.

2.2.30. If $q = rS$ the equation for I reduces to $I' = -rI^2$, so $\frac{I'}{I^2} = -r$; $-\frac{1}{I} = -rt - \frac{1}{I_0}$; so $I = \frac{I_0}{1 + rI_0 t}$ and $\lim_{t\to\infty} I(t) = 0$. If $q \neq rS$, then rewrite the equation for I as $I' = -rI(I - \alpha)$ with $\alpha = S - \frac{q}{r}$. Separating variables yields $\frac{I'}{I(I-\alpha)} = -r$; $\left[\frac{1}{I-\alpha} - \frac{1}{I}\right]I' = -r\alpha$; $\ln\left|\frac{I-\alpha}{I}\right| = -r\alpha t + k$; (A) $\frac{I-\alpha}{I} = ce^{-r\alpha t}$; $I(1 - ce^{-r\alpha t}) = \alpha$; (B) $I = \frac{\alpha}{1 - ce^{-r\alpha t}}$. From (A), $I(0) = I_0 \Rightarrow c = \frac{I_0 - \alpha}{I_0}$. Substituting this into (B) yields $I = \frac{\alpha I_0}{I_0 + (\alpha - I_0)e^{-r\alpha t}}$. If $q < rS$, then $\alpha > 0$ and $\lim_{t\to\infty} I(t) = \alpha = S - \frac{q}{r}$. If $q > rS$, then $\alpha < 0$ and $\lim_{t\to\infty} I(t) = 0$.

2.2.34. The given equation is separable if $f = ap$, where a is a constant. In this case the equation is

$$y' + p(x)y = ap(x). \tag{A}$$

Let P be an antiderivative of p; that is, $P' = p$.

SOLUTION BY SEPARATION OF VARIABLES. $y' = -p(x)(y - a)$; $\frac{y'}{y-a} = -p(x)$; $\ln|y - a| = -P(x) + k$; $y - a = ce^{-P(x)}$; $y = a + ce^{-P(x)}$.

SOLUTION BY VARIATION OF PARAMETERS. $y_1 = e^{-P(x)}$ is a solution of the complementary equation, so solutions of (A) are of the form $y = ue^{-P(x)}$ where $u'e^{-P(x)} = ap(x)$. Hence, $u' = ap(x)e^{P(x)}$; $u = ae^{P(x)} + c$; $y = a + ce^{-P(x)}$.

2.2.36. Rewrite the given equation as (A) $y' - \frac{2}{x}y = \frac{x^5}{y + x^2}$. $y_1 = x^2$ is a solution of $y' - \frac{2}{x}y = 0$. Look for solutions of (A) of the form $y = ux^2$. Then $u'x^2 = \frac{x^5}{(u+1)x^2} = \frac{x^3}{u+1}$; $u' = \frac{x}{u+1}$; $(u + 1)u' = x$; $\frac{(1+u)^2}{2} = \frac{x^2}{2} + \frac{c}{2}$; $u = -1 \pm \sqrt{x^2 + c}$; $y = x^2\left(-1 \pm \sqrt{x^2 + c}\right)$.

2.2.38. $y_1 = e^{2x}$ is a solution of $y' - 2y = 0$. Look for solutions of the nonlinear equation of the form $y = ue^{2x}$. Then $u'e^{2x} = \dfrac{xe^{2x}}{1-u}$; $u' = \dfrac{x}{1-u}$; $(1-u)u' = x$; $-\dfrac{(1-u)^2}{2} = \dfrac{1}{2}(x^2 - c)$; $u = 1 \pm \sqrt{c - x^2}$; $y = e^{2x}\left(1 \pm \sqrt{c - x^2}\right)$.

2.3 EXISTENCE AND UNIQUENESS OF SOLUTIONS OF NONLINEAR EQUATIONS

2.3.2. $f(x, y) = \dfrac{e^x + y}{x^2 + y^2}$ and $f_y(x, y) = \dfrac{1}{x^2 + y^2} - \dfrac{2y(e^x + y)}{(x^2 + y^2)^2}$ are both continuous at all $(x, y) \neq (0, 0)$. Hence, Theorem 2.3.1 implies that if $(x_0, y_0) \neq (0, 0)$, then the initial value problem has a a unique solution on some open interval containing x_0. Theorem 2.3.1 does not apply if $(x_0, y_0) = (0, 0)$.

2.3.4. $f(x, y) = \dfrac{x^2 + y^2}{\ln xy}$ and $f_y(x, y) = \dfrac{2y}{\ln xy} - \dfrac{x^2 + y^2}{x(\ln xy)^2}$ are both continuous at all (x, y) such that $xy > 0$ and $xy \neq 1$. Hence, Theorem 2.3.1 implies that if $x_0y_0 > 0$ and $x_0y_0 \neq 1$, then the initial value problem has unique solution on an open interval containing x_0. Theorem 2.3.1 does not apply if $x_0y_0 \leq 0$ or $x_0y_0 = 1$.

2.3.6. $f(x, y) = 2xy$ and $f_y(x, y) = 2x$ are both continuous at all (x, y). Hence, Theorem 2.3.1 implies that if (x_0, y_0) is arbitrary, then the initial value problem has a unique solution on some open interval containing x_0.

2.3.8. $f(x, y) = \dfrac{2x + 3y}{x - 4y}$ and $f_y(x, y) = \dfrac{3}{x - 4y} + 4\dfrac{2x + 3y}{(x - 4y)^2}$ are both continuous at all (x, y) such that $x \neq 4y$. Hence, Theorem 2.3.1 implies that if $x_0 \neq 4y_0$, then the initial value problem has a unique solution on some open interval containing x_0. Theorem 2.3.1 does not apply if $x_0 = 4y_0$.

2.3.10. $f(x, y) = x(y^2 - 1)^{2/3}$ is continuous at all (x, y), but $f_y(x, y) = \dfrac{4}{3}xy(y^2 - 1)^{1/3}$ is continuous at (x, y) if and only if $y \neq \pm 1$. Hence, Theorem 2.3.1 implies that if $y_0 \neq \pm 1$, then the initial value problem has a unique solution on some open interval containing x_0, while if $y_0 = \pm 1$, then the initial value problem has at least one solution (possibly not unique on any open interval containing x_0).

2.3.12. $f(x, y) = (x + y)^{1/2}$ and $f_y(x, y) = \dfrac{1}{2(x + y)^{1/2}}$ are both continuous at all (x, y) such that $x + y > 0$ Hence, Theorem 2.3.1 implies that if $x_0 + y_0 > 0$, then the initial value problem has a unique solution on some open interval containing x_0. Theorem 2.3.1 does not apply if $x_0 + y_0 \leq 0$.

2.3.14. To apply Theorem 2.3.1, rewrite the given initial value problem as (A) $y' = f(x, y)$, $y(x_0) = y_0$, where $f(x, y) = -p(x)y + q(x)$ and $f_y(x, y) = -p(x)$. If p and f are continuous on some open interval (a, b) containing x_0, then f and f_y are continuous on some open rectangle containing (x_0, y_0), so Theorem 2.3.1 implies that (A) has a unique solution on *some* open interval containing x_0. The conclusion of Theorem 2.1.2 is more specific: the solution of (A) exists and is unique on (a, b). For example, in the extreme case where $(a, b) = (-\infty, \infty)$, Theorem 2.3.1 still implies only existence and uniqueness on *some* open interval containing x_0, while Theorem 2.1.2 implies that the solution exists and is unique on $(-\infty, \infty)$.

2.3.16. First find solutions of (A) $y' = y^{2/5}$. Obviously $y \equiv 0$ is a solution. If $y \not\equiv 0$, then we can separate variables on any open interval where y has no zeros: $y^{-2/5}y' = 1$; $\dfrac{5}{3}y^{3/5} = x + c$; $y = \left(\dfrac{3}{5}(x + c)^{5/3}\right)$. (Note that this solution is also defined at $x = -c$, even though $y(-c) = 0$.

To satisfy the initial condition, let $c = 1$. Thus, $y = \left(\frac{3}{5}(x+1)^{5/3}\right)$ is a solution of the initial value problem on $(-\infty, \infty)$; moreover, since $f(x, y) = y^{2/5}$ and $f_y(x, y) = \frac{2}{5}y^{-3/5}$ are both continuous at all (x, y) such that $y \neq 0$, this is the only solution on $(-5/3, \infty)$, by an argument similar to that given in Example 2.3.7, the function

$$y = \begin{cases} 0, & -\infty < x \le -\frac{5}{3} \\ \left(\frac{3}{5}x+1\right)^{5/3}, & -\frac{5}{3} < x < \infty \end{cases}$$

(To see that y satisfies $y' = y^{2/5}$ at $x = -\frac{5}{3}$ use an argument similar to that of Discussion 2.3.15-2) For every $a \ge \frac{5}{3}$, the following function is also a solution:

$$y = \begin{cases} \left(\frac{3}{5}(x+a)\right)^{5/3}, & -\infty < x < -a, \\ 0, & -a \le x \le -\frac{5}{3} \\ \left(\frac{3}{5}x+1\right)^{5/3}, & -\frac{5}{3} < x < \infty. \end{cases}$$

2.3.18. Obviously, $y_1 \equiv 1$ is a solution. From Discussion 2.3.18 (taking $c = 0$ in the two families of solutions) yields $y_2 = 1 + |x|^3$ and $y_3 = 1 - |x|^3$. Other solutions are $y_4 = 1 + x^3$, $y_5 = 1 - x^3$,

$$y_6 = \begin{cases} 1 + x^3, & x \ge 0, \\ 1, & x < 0 \end{cases}; \quad y_7 = \begin{cases} 1 - x^3, & x \ge 0, \\ 1, & x < 0 \end{cases};$$

$$y_8 = \begin{cases} 1, & x \ge 0, \\ 1 + x^3, & x < 0 \end{cases}; \quad y_9 = \begin{cases} 1, & x \ge 0, \\ 1 - x^3, & x < 0 \end{cases}$$

It is straightforward to verify that all these functions satisfy $y' = 3x(y-1)^{1/3}$ for all $x \neq 0$. Moreover, $y_i'(0) = \lim_{x\to 0} \frac{y_i(x) - 1}{x} = 0$ for $1 \le i \le 9$, which implies that they also satisfy the equation at $x = 0$.

2.3.20. Let y be any solution of (A) $y' = 3x(y-1)^{1/3}$, $y(3) = -7$. By continuity, there is some open interval I containing $x_0 = 3$ on which $y(x) < 1$. From Discussion 2.3.18, $y = 1 + (x^2 + c)^{3/2}$ on I; $y(3) = -7 \Rightarrow c = -5$; (B) $y = 1 - (x^2 - 5)^{3/2}$. It now follows that every solution of (A) satisfies $y(x) < 1$ and is given by (B) on $(\sqrt{5}, \infty)$; that is, (B) is the unique solution of (A) on $(\sqrt{5}, \infty)$. This solution can be extended uniquely to $(0, \infty)$ as

$$y = \begin{cases} 1, & 0 < x \le \sqrt{5}, \\ 1 - (x^2 - 5)^{3/2}, & \sqrt{5} < x < \infty \end{cases}$$

It can be extended to $(-\infty, \infty)$ in infinitely many ways. Thus,

$$y = \begin{cases} 1, & -\infty < x \le \sqrt{5}, \\ 1 - (x^2 - 5)^{3/2}, & \sqrt{5} < x < \infty \end{cases}$$

is a solution of the initial value problem on $(-\infty, \infty)$. Moreover, if $\alpha \ge 0$, then

$$y = \begin{cases} 1 + (x^2 - \alpha^2)^{3/2}, & -\infty < x < -\alpha, \\ 1, & -\alpha \le x \le \sqrt{5}, \\ 1 - (x^2 - 5)^{3/2}, & \sqrt{5} < x < \infty, \end{cases}$$

and

$$y = \begin{cases} 1-(x^2-\alpha^2)^{3/2}, & -\infty < x < -\alpha, \\ 1, & -\alpha \le x \le \sqrt{5}, \\ 1-(x^2-5)^{3/2}, & \sqrt{5} < x < \infty, \end{cases}$$

are also solutions of the initial value problem on $(-\infty, \infty)$.

2.4 TRANSFORMATION OF NONLINEAR EQUATIONS INTO SEPARABLE EQUATIONS

2.4.2. Rewrite as $y' - \frac{2}{7x}y = -\frac{x}{7y^6}$. Then $\frac{y_1'}{y_1} = \frac{2}{7x}$; $\ln|y_1| = \frac{2}{7}\ln|x| = \ln|x|^{2/7}$; $y_1 = x^{2/7}$; $y = ux^{2/7}$; $u'x^{2/7} = -\frac{1}{7u^6x^{5/7}}$; $u^6u' = -\frac{1}{7x}$; $\frac{u^7}{7} = -\frac{1}{7}\ln|x| + \frac{c}{7}$; $u = (c - \ln|x|)^{1/7}$; $y = x^{2/7}(c - \ln|x|)^{1/7}$.

2.4.4. Rewrite as $y' + \frac{2x}{1+x^2}y = \frac{1}{(1+x^2)^2y}$. Then $\frac{y_1'}{y_1} = -\frac{2x}{1+x^2}$; $\ln|y_1| = -\ln(1+x^2)$; $y_1 = \frac{1}{1+x^2}$; $y = \frac{u}{1+x^2}$; $\frac{u'}{1+x^2} = \frac{1}{u(1+x^2)}$; $u'u = 1$; $\frac{u^2}{2} = x + \frac{c}{2}$; $u = \pm\sqrt{2x+c}$; $y = \pm\frac{\sqrt{2x+c}}{1+x^2}$.

2.4.6. $\frac{y_1'}{y_1} = \frac{1}{3}\left(\frac{1}{x}+1\right)$; $\ln|y_1| = \frac{1}{3}(\ln|x| + x)$; $y_1 = x^{1/3}e^{x/3}$; $y = ux^{1/3}e^{x/3}$; $u'x^{1/3}e^{x/3} = x^{4/3}e^{4x/3}u^4$; $\frac{u'}{u^4} = xe^x$; $-\frac{1}{3u^3} = (x-1)e^x - \frac{c}{3}$; $u = \frac{1}{[3(1-x)e^x + c]^{1/3}}$; $y = \left[\frac{x}{3(1-x)+ce^{-x}}\right]^{1/3}$.

2.4.8. $\frac{y_1'}{y_1} = x$; $\ln|y_1| = \frac{x^2}{2}$; $y_1 = e^{x^2/2}$; $y = ue^{x^2/2}$; $u'e^{x^2/2} = xu^{3/2}e^{3x^2/4}$; $\frac{u'}{u^{3/2}} = xe^{x^2/4}$; (A) $-\frac{2}{u^{1/2}} = 2e^{x^2/4} + 2c$; $u^{1/2} = -\frac{1}{c+e^{x^2/4}}$; $u = \frac{1}{(c+e^{x^2/4})^2}$; $y = \frac{1}{(1+ce^{-x^2/4})^2}$. Because of (A) we must choose c so that $y(1) = 4$ and $1 + ce^{-1/4} < 0$. This implies that $c = -3e^{1/4}$; $y = \left[1 - \frac{3}{2}e^{-(x^2-1)/4}\right]^{-2}$.

2.4.10. $\frac{y_1'}{y_1} = 2$; $\ln|y_1| = 2x$; $y_1 = e^{2x}$; $y = ue^{2x}$; $u'e^{2x} = 2u^{1/2}e^x$; $u^{-1/2}u' = 2e^{-x}$; $2u^{1/2} = -2e^{-x} + 2c$; $u^{1/2} = c - e^{-x} > 0$; $y(0) = 1 \Rightarrow u(0) = 1 \Rightarrow c = 2$; $u = (2-e^{-x})^2$; $y = (2e^x - 1)^2$.

2.4.12. Rewrite as $y' + \frac{2}{x}y = \frac{y^3}{x^2}$. Then $\frac{y_1'}{y_1} = -\frac{2}{x}$; $\ln|y_1| = -2\ln|x| = \ln x^{-2}$; $y_1 = \frac{1}{x^2}$; $y = \frac{u}{x^2}$; $\frac{u'}{x^2} = \frac{u^3}{x^8}$; $\frac{u'}{u^3} = \frac{1}{x^6}$; $-\frac{1}{2u^2} = -\frac{1}{5x^5} + c$; $y(1) = \frac{1}{\sqrt{2}} \Rightarrow u(1) = \frac{1}{\sqrt{2}} \Rightarrow c = -\frac{4}{5}$; $u = \left[\frac{5x^5}{2(1+4x^5)}\right]^{1/2}$; $y = \left[\frac{5x}{2(1+4x^5)}\right]^{1/2}$.

2.4.14. $P = ue^{at}$; $u'e^{at} = -a\alpha u^2e^{2at}$; $\frac{u'}{u^2} = -a\alpha e^{at}$; $-\frac{1}{u} = -a\int_0^t \alpha(\tau)e^{a\tau}\,d\tau - \frac{1}{P_0}$; $P = \frac{P_0e^{at}}{1 + aP_0\int_0^t \alpha(\tau)e^{a\tau}\,d\tau}$, which can also be written as $P = \frac{P_0}{e^{-at} + aP_0e^{-at}\int_0^t \alpha(\tau)e^{a\tau}\,d\tau}$. Therefore,

$\lim_{t\to\infty} P(t) = \begin{cases} \infty & \text{if } L = 0, \\ 0 & \text{if } L = \infty, \\ 1/aL & \text{if } 0 < L < \infty. \end{cases}$

2.4.16. $y = ux$; $u'x + u = u^2 + 2u$; (A) $u'x = u(u+1)$. Since $u \equiv 0$ and $u \equiv -1$ are constant solutions of (A), $y \equiv 0$ and $y = -x$ are solutions of the given equation. The nonconstant solutions of (A) satisfy $= \dfrac{u'}{u(u+1)} = \dfrac{1}{x}$; $\left[\dfrac{1}{u} - \dfrac{1}{u+1}\right]u' = \dfrac{1}{x}$; $\ln\left|\dfrac{u}{u+1}\right| = \ln|x| + k$; $\dfrac{u}{u+1} = cx$; $u = (u+1)cx$; $u(1-cx) = cx$; $u = \dfrac{cx}{1-cx}$; $y = \dfrac{cx^2}{1-cx}$.

2.4.18. $y = ux$; $u'x + u = u + \sec$; $u'x = \sec u$; $(\cos u)u' = \dfrac{1}{x}$; $\sin u = \ln|x| + c$; $u = \sin^{-1}(\ln|x| + c)$; $y = x\sin^{-1}(\ln|x| + c)$.

2.4.20. Rewrite the given equation as $y' = \dfrac{x^2 + 2y^2}{xy}$; $y = ux$; $u'x + u = \dfrac{1}{u} + 2u$; $u'x = \dfrac{1+u^2}{u}$; $\dfrac{uu'}{1+u^2} = \dfrac{1}{x}$; $\dfrac{1}{2}\ln(1+u^2) = \ln|x| + k$; $\ln\left(1 + \dfrac{y^2}{x^2}\right) = \ln x^2 + 2k$; $1 + \dfrac{y^2}{x^2} = cx^2$; $x^2 + y^2 = cx^4$; $y = \pm x\sqrt{cx^2 - 1}$.

2.4.22. $y = ux$; $u'x + u = u + u^2$; $u'x = u^2$; $\dfrac{u'}{u^2} = \dfrac{1}{x}$; $-\dfrac{1}{u} = \ln|x| + c$; $y(-1) = 2 \Rightarrow u(-1) = -2 \Rightarrow c = \dfrac{1}{2}$; $u = -\dfrac{2}{2\ln|x| + 1}$; $y = -\dfrac{2x}{2\ln|x| + 1}$.

2.4.24. Rewrite the given equation as $y' = -\dfrac{x^2 + y^2}{xy}$; $y = ux$; $u'x + u = -\dfrac{1}{u} - u$; $u'x = -\dfrac{1 + 2u^2}{u}$; $-\dfrac{uu'}{1+2u^2} = \dfrac{1}{x}$; $-\dfrac{1}{4}\ln(1+2u^2) = \ln|x| + k$; $x^4(1+2u^2) = c$; $y(1) = 2 \Rightarrow u(1) = 2 \Rightarrow c = 9$; $x^4(1+2u^2) = 9$; $u^2 = \dfrac{9 - x^4}{2x^4}$; $u = \dfrac{1}{x^2}\left(\dfrac{9 - x^4}{2}\right)^{1/2}$; $y = \dfrac{1}{x}\left(\dfrac{9 - x^4}{2}\right)^{1/2}$.

2.4.26. Rewrite the given equation as $y' = 2 + \dfrac{y^2}{x^2} + 4\dfrac{y}{x}$; $y = ux$; $u'x + u = 2 + u^2 + 4u$; $u'x = u^2 + 3u + 2 = (u+1)(u+2)$; $\dfrac{u'}{(u+1)(u+2)} = \dfrac{1}{x}$; $\left[\dfrac{1}{u+1} - \dfrac{1}{u+2}\right]u' = \dfrac{1}{x}$; $\ln\left|\dfrac{u+1}{u+2}\right| = \ln|x| + k$; $\dfrac{u+1}{u+2} = cx$; $y(1) = 1 \Rightarrow u(1) = 1 \Rightarrow c = \dfrac{2}{3}$; $\dfrac{u+1}{u+2} = \dfrac{2}{3}x$; $u + 1 = \dfrac{2}{3}x(u+2)$; $u\left(1 - \dfrac{2}{3}x\right) = -1 + \dfrac{4}{3}x$; $u = -\dfrac{4x - 3}{2x - 3}$; $y = -\dfrac{x(4x-3)}{2x - 3}$.

2.4.28. $y = ux$; $u'x + u = \dfrac{1+u}{1-u}$; $u'x = \dfrac{1+u^2}{1-u}$; $\dfrac{(1-u)u'}{1+u^2} = \dfrac{1}{x}$; $\tan^{-1} u - \dfrac{1}{2}\ln(1+u^2) = \ln|x| + c$; $\tan^{-1}\dfrac{y}{x} - \dfrac{1}{2}\ln\left(1 + \dfrac{y^2}{x^2}\right) = \ln|x| + c$; $\tan^{-1}\dfrac{y}{x} - \dfrac{1}{2}\ln(x^2 + y^2) = c$.

2.4.30. $y = ux$; $u'x + u = \dfrac{u^3 + 2u^2 + u + 1}{(u+1)^2} = \dfrac{u(u+1)^2 + 1}{(u+1)^2} = u + \dfrac{1}{(u+1)^2}$; $u'x = \dfrac{1}{(u+1)^2}$; $(u+1)^2u' = \dfrac{1}{x}$; $\dfrac{(u+1)^3}{3} = \ln|x| + c$; $(u+1)^3 = 3(\ln|x| + c)$; $\left(\dfrac{y}{x} + 1\right)^3 = 3(\ln|x| + c)$; $(y+x)^3 = 3x^3(\ln|x| + c)$.

2.4.32. $y = ux$; $u'x + u = \dfrac{u}{u-2}$; (A) $u'x = \dfrac{u(u-3)}{2-u}$; Since $u \equiv 0$ and $u \equiv 3$ are constant solutions of (A), $y \equiv 0$ and $y = 3x$ are solutions of the given equation. The nonconstant solutions of (A) satisfy $\dfrac{(2-u)u'}{u(u-3)} = \dfrac{1}{x}$; $\left[\dfrac{1}{u-3} + \dfrac{2}{u}\right]u' = -\dfrac{3}{x}$; $\ln|u-3| + 2\ln|u| = -3\ln|x| + k$; $u^2(u-3) = \dfrac{c}{x^3}$; $y^2(y-3x) = c$.

2.4.34. $y = ux$; $u'x + u = \dfrac{1 + u + 3u^3}{1 + 3u^2} = u + \dfrac{1}{1+3u^2}$; $(1 + 3u^2)u' = \dfrac{1}{x}$; $u + u^3 = \ln|x| + c$; $\dfrac{y}{x} + \dfrac{y^3}{x^3} = \ln|x| + c$.

2.4.36. Rewrite the given equation as $y' = \dfrac{x^2 - xy + y^2}{xy}$; $y = ux$; $u'x + u = \dfrac{1}{u} - 1 + u$; $u'x = \dfrac{1-u}{u}$; $\dfrac{uu'}{u-1} = -\dfrac{1}{x}$; $\left[1 + \dfrac{1}{u-1}\right]u' = -\dfrac{1}{x}$; $u + \ln|u-1| = -\ln|x| + k$; $e^u(u-1) = \dfrac{c}{x}$; $e^{y/x}(y-x) = c$.

2.4.38. $y = ux$; $u'x + u = 1 + \dfrac{1}{u} + u$; (A) $u'x = \dfrac{u+1}{u}$. Since (A) has the constant solution $u = -1$; $y = -x$ is a solution of the given equation. The nonconstant solutions of (A) satisfy $\dfrac{uu'}{u+1} = \dfrac{1}{x}$; $\left[1 - \dfrac{1}{u+1}\right]u' = \dfrac{1}{x}$; $u - \ln|u+1| = \ln|x| + c$; $\dfrac{y}{x} - \ln\left|\dfrac{y}{x} - 1\right| = \ln|x| + c$; $y - x\ln|y-x| = cy$.

2.4.40. If $x = X - X_0$ and $y = Y - Y_0$, then $\dfrac{dy}{dx} = \dfrac{dY}{dx} = \dfrac{dY}{dX}\dfrac{dX}{dx} = \dfrac{dY}{dX}$, so $y = y(x)$ satisfies the given equation if and only if $Y = Y(X)$ satisfies

$$\frac{dY}{dX} = \frac{a(X - X_0) + b(Y - Y_0) + \alpha}{c(X - X_0) + d(Y - Y_0) + \beta},$$

which reduces to the nonlinear homogeneous equation

$$\frac{dY}{dX} = \frac{aX + bY}{cX + dY}$$

if and only if

$$\begin{aligned} aX_0 + bY_0 &= \alpha \\ cX_0 + dY_0 &= \beta. \end{aligned} \tag{B}$$

We will now show that if $ad - bc \neq 0$, then it is possible (for any choice of α and β) to solve (B). Multiplying the first equation in (B) by d and the second by b yields

$$\begin{aligned} daX_0 + dbY_0 &= d\alpha \\ bcX_0 + bdY_0 &= b\beta. \end{aligned}$$

Subtracting the second of these equations from the first yields $(ad - bc)X_0 = \alpha d - \beta b$. Since $ad - bc \neq 0$, this implies that $X_0 = \dfrac{\alpha d - \beta b}{ad - bc}$. Multiplying the first equation in (B) by c and the second by a yields

$$\begin{aligned} caX_0 + cbY_0 &= c\alpha \\ acX_0 + adY_0 &= a\beta. \end{aligned}$$

Subtracting the first of these equation from the second yields $(ad-bc)Y_0 = \alpha c - \beta a$. Since $ad-bc \neq 0$ this implies that $Y_0 = \dfrac{\alpha c - \beta a}{ad-bc}$.

2.4.42. For the given equation, (B) of Exercise 2.4.40 is

$$\begin{aligned} 2X_0 + Y_0 &= 1 \\ X_0 + 2Y_0 &= -4. \end{aligned}$$

Solving this pair of equations yields $X_0 = 2$ and $Y_0 = -3$. The transformed differential equation is

$$\frac{dY}{dX} = \frac{2X+Y}{X+2Y}. \tag{A}$$

Let $Y = uX$; $u'X + U = \dfrac{2+u}{1+2u}$; (B) $u'X = -\dfrac{2(u-1)(u+1)}{2u+1}$. Since $u \equiv 1$ and $u \equiv -1$ satisfy (B), $Y = X$ and $Y = -X$ are solutions of (A). Since $X = x+2$ and $Y = y-3$, it follows that $y = x+5$ and $y = -x+1$ are solutions of the given equation. The nonconstant solutions of (B) satisfy $\dfrac{(2u+1)u'}{(u-1)(u+1)} = -\dfrac{2}{X}$; $\left[\dfrac{1}{u+1} + \dfrac{3}{u-1}\right]u' = -\dfrac{4}{X}$; $\ln|u+1| + 3\ln|u-1| = -4\ln|X| + k$; $(u+1)(u-1)^3 = \dfrac{c}{X^4}$; $(Y+X)(Y-X)^3 = c$; Setting $X = x+2$ and $Y = y-3$ yields $(y+x-1)(y-x-5)^3 = c$.

2.4.44. Rewrite the given equation as $y' = \dfrac{y^3+x}{3xy^2}$; $y = ux^{1/3}$; $u'x^{1/3} + \dfrac{1}{3x^{2/3}}u = \dfrac{u^3+1}{3u^2x^{2/3}}$; $u'x^{1/3} = \dfrac{1}{3x^{2/3}u^2}$; $u^2u' = \dfrac{1}{3x}$; $\dfrac{u^3}{3} = \dfrac{1}{3}(\ln|x|+c)$; $u = (\ln|x|+c)^{1/3}$; $y = x^{1/3}(\ln|x|+c)^{1/3}$.

2.4.46. Rewrite the given equation as $y' = \dfrac{2(y^2+x^2y-x^4)}{x^3}$; $y = ux^2$; $u'x^2 + 2xu = 2x(u^2+u-1)$; (A) $u'x^2 = 2x(u^2-1)$. Since $u \equiv 1$ and $u \equiv -1$ are constant solutions of (A), $y = x^2$ and $y = -x^2$ are solutions of the given equation. The nonconstant solutions of (A) satisfy $\dfrac{u'}{u^2-1} = \dfrac{2}{x}$; $\left[\dfrac{1}{u-1} - \dfrac{1}{u+1}\right]u' = \dfrac{4}{x}$; $\ln\left|\dfrac{u-1}{u+1}\right| = 4\ln|x| + k$; $\dfrac{u-1}{u+1} = cx^4$; $(u-1) = (u+1)cx^4$; $u(1-cx^4) = 1+cx^4$; $u = \dfrac{1+cx^4}{1-cx^4}$; $y = \dfrac{x^2(1+cx^4)}{1-cx^4}$.

2.4.48. $y = u\tan x$; $u'\tan x + u\sec^2 x = (u^2+u+1)\sec^2 x$; $u'\tan x = (u^2+1)\sec^2 x$; $\dfrac{u'}{u^2+1} = \sec^2 x \cot x = \cot x + \tan x$; $\tan^{-1} u = \ln|\sin x| - \ln|\cos x| + c = \ln|\tan x| + c$; $u = \tan(\ln|\tan x| + c)$; $y = \tan x\ \tan(\ln|\tan x| + c)$.

2.4.50. Rewrite the given equation as $y' = \dfrac{(y+\sqrt{x})^2}{2x(y+2\sqrt{x})}$; $y = ux^{1/2}$; $u'x^{1/2} + \dfrac{1}{2\sqrt{x}}u = \dfrac{(u+1)^2}{2\sqrt{x}\,(u+2)}$; $u'x^{1/2} = \dfrac{1}{2\sqrt{x}\,(u+2)}$; $(u+2)u' = \dfrac{1}{2x}$; $\dfrac{(u+2)^2}{2} = \dfrac{1}{2}(\ln|x|+c)$; $(u+2)^2 = \ln|x|+c$; $u = -2 \pm \sqrt{\ln|x|+c}$; $y = x^{1/2}(-2 \pm \sqrt{\ln|x|+c})$.

2.4.52. $y_1 = \dfrac{1}{x^2}$ is a solution of $y' + \dfrac{2}{x}y = 0$. Let $y = \dfrac{u}{x^2}$; then

$$\frac{u'}{x^2} = \frac{3x^2(u^2/x^4) + 6x(u/x^2) + 2}{x^2\,(2x(u/x^2)+3)} = \frac{3(u/x)^2 + 6(u/x) + 2}{x^2\,(2(u/x)+3)},$$

so (A) $u' = \dfrac{3(u/x)^2 + 6(u/x) + 2}{2(u/x) + 3}$. Since (A) is a homogeneous nonlinear equation, we now substitute $u = vx$ into (A). This yields $v'x + v = \dfrac{3v^2 + 6v + 2}{2v + 3}$; $v'x = \dfrac{(v+1)(v+2)}{2v+3}$; $\dfrac{(2v+3)v'}{(v+1)(v+2)} = \dfrac{1}{x}$; $\left[\dfrac{1}{v+1} + \dfrac{1}{v+2}\right] v' = \dfrac{1}{x}$; $\ln|(v+1)(v+2)| = \ln|x| + k$; (B) $(v+1)(v+2) = cx$. Since $y(2) = 2 \Rightarrow u(2) = 8 \Rightarrow v(2) = 4$, (B) implies that $c = 15$. $(v+1)(v+2) = 15x$; $v^2 + 3v + 2 - 15x = 0$. From the quadratic formula, $v = \dfrac{-3 + \sqrt{1 + 60x}}{2}$; $u = vx = \dfrac{x(-3 + \sqrt{1 + 60x})}{2}$; $y = \dfrac{u}{x^2} = \dfrac{-3 + \sqrt{1 + 60x}}{2x}$.

2.4.54. Differentiating (A) $y_1(x) = \dfrac{y(ax)}{a}$ yields (B) $y_1'(x) = \dfrac{1}{a} y'(ax) \cdot a = y'(ax)$. Since $y'(x) = q(y(x)/x)$ on some interval I, (C) $y'(ax) = q(y(ax)/ax)$ on some interval J. Substituting (A) and (B) into (C) yields $y_1'(x) = q(y_1(x)/x)$ on J.

2.4.56. If $y = z + 1$, then $z' + z = xz^2$; $z = ue^{-x}$; $u'e^{-x} = xu^2 e^{-2x}$; $\dfrac{u'}{u^2} = xe^{-x}$; $-\dfrac{1}{u} = -e^{-x}(x+1) - c$; $u = \dfrac{1}{e^{-x}(x+1) + c}$; $z = \dfrac{1}{x + 1 + ce^x}$; $y = 1 + \dfrac{1}{x + 1 + ce^x}$.

2.4.58. If $y = z + 1$, then $z' + \dfrac{2}{x} z = z^2$; $z_1 = \dfrac{1}{x^2}$; $z = \dfrac{u}{x^2}$; $\dfrac{u'}{x^2} = \dfrac{u^2}{x^4}$; $\dfrac{u'}{u^2} = \dfrac{1}{x^2}$; $-\dfrac{1}{u} = -\dfrac{1}{x} + c = -\dfrac{1 - cx}{x}$; $u = -\dfrac{x}{1 - cx}$; $z = -\dfrac{1}{x(1 - cx)}$; $y = 1 - \dfrac{1}{x(1 - cx)}$.

2.5 EXACT EQUATIONS

2.5.2. $M(x,y) = 3y\cos x + 4xe^x + 2x^2e^x$; $N(x,y) = 3\sin x + 3$; $M_y(x,y) = 3\cos x = N_x(x,y)$, so the equation is exact. We must find F such that (A) $F_x(x,y) = 3y\cos x + 4xe^x + 2x^2e^x$ and (B) $F_y(x,y) = 3\sin x + 3$. Integrating (B) with respect to y yields (C) $F(x,y) = 3y\sin x + 3y + \psi(x)$. Differentiating (C) with respect to x yields (D) $F_x(x,y) = 3y\cos x + \psi'(x)$. Comparing (D) with (A) shows that (E) $\psi'(x) = 4xe^x + 2x^2e^x$. Integration by parts yields $\int xe^x\,dx = xe^x - e^x$ and $\int x^2e^x\,dx = x^2e^x - 2xe^x + 2e^x$. Substituting from the last two equations into (E) yields $\psi(x) = 2x^2e^x$. Substituting this into (C) yields $F(x,y) = 3y\sin x + 3y + 2x^2e^x$. Therefore, $3y\sin x + 3y + 2x^2e^x = c$.

2.5.4. $M(x,y) = 2x - 2y^2$; $N(x,y) = 12y^2 - 4xy$; $M_y(x,y) = -4y = N_x(x,y)$, so the equation is exact. We must find F such that (A) $F_x(x,y) = 2x - 2y^2$ and (B) $F_y(x,y) = 12y^2 - 4xy$. Integrating (A) with respect to x yields (C) $F(x,y) = x^2 - 2xy^2 + \phi(y)$. Differentiating (C) with respect to y yields (D) $F_y(x,y) = -4xy + \phi'(y)$. Comparing (D) with (B) shows that $\phi'(y) = 12y^2$, so we take $\phi(y) = 4y^3$. Substituting this into (C) yields $F(x,y) = x^2 - 2xy^2 + 4y^3$. Therefore, $x^2 - 2xy^2 + 4y^3 = c$.

2.5.6. $M(x,y) = 4x + 7y$; $N(x,y) = 3x + 4y$; $M_y(x,y) = 7 \neq 3 = N_x(x,y)$, so the equation is not exact.

2.5.8. $M(x,y) = 2x + y$; $N(x,y) = 2y + 2x$; $M_y(x,y) = 1 \neq 2 = N_x(x,y)$, so the equation is not exact.

2.5.10. $M(x,y) = 2x^2 + 8xy + y^2$; $N(x,y) = 2x^2 + \frac{xy^3}{3}$; $M_y(x,y) = 8x + 2y \neq 4x + \frac{y^3}{3} = N_x(x,y)$, so the equation is not exact.

2.5.12. $M(x,y) = y \sin xy + xy^2 \cos xy$; $N(x,y) = x \sin xy + xy^2 \cos xy$; $M_y(x,y) = 3xy \cos xy + (1 - x^2y^2) \sin xy \neq (xy + y^2) \cos xy + (1 - xy^3) \sin xy = N_x(x,y)$, so the equation is not exact.

2.5.14. $M(x,y) = e^x(x^2y^2 + 2xy^2) + 6x$; $N(x,y) = 2x^2ye^x + 2$; $M_y(x,y) = 2xye^x(x+2) = N_x(x,y)$, so the equation is exact. We must find F such that (A) $F_x(x,y) = e^x(x^2y^2 + 2xy^2) + 6x$ and (B) $F_y(x,y) = 2x^2ye^x + 2$. Integrating (B) with respect to y yields (C) $F(x,y) = x^2y^2e^x + 2y + \psi(x)$. Differentiating (C) with respect to x yields (D) $F_x(x,y) = e^x(x^2y^2 + 2xy^2) + \psi'(x)$. Comparing (D) with (A) shows that $\psi'(x) = 6x$, so we take $\psi(x) = 3x^2$. Substituting this into (C) yields $F(x,y) = x^2y^2e^x + 2y + 3x^2$. Therefore, $x^2y^2e^x + 2y + 3x^2 = c$.

2.5.16. $M(x,y) = e^{xy}(x^4y + 4x^3) + 3y$; $N(x,y) = x^5e^{xy} + 3x$; $M_y(x,y) = x^4e^{xy}(xy+5) + 3 = N_x(x,y)$, so the equation is exact. We must find F such that (A) $F_x(x,y) = e^{xy}(x^4y + 4x^3) + 3y$ and (B) $F_y(x,y) = x^5e^{xy} + 3x$. Integrating (B) with respect to y yields (C) $F(x,y) = x^4e^{xy} + 3xy + \psi(x)$. Differentiating (C) with respect to x yields (D) $F_x(x,y) = e^{xy}(x^4y + 4x^3) + 3y + \psi'(x)$. Comparing (D) with (A) shows that $\psi'(x) = 0$, so we take $\psi(x) = 0$. Substituting this into (C) yields $F(x,y) = x^4e^{xy} + 3xy$. Therefore, $x^4e^{xy} + 3xy = c$.

2.5.18. $M(x,y) = 4x^3y^2 - 6x^2y - 2x - 3$; $N(x,y) = 2x^4y - 2x^3$; $M_y(x,y) = 8x^3y - 6x^2 = N_x(x,y)$, so the equation is exact. We must find F such that (A) $F_x(x,y) = 4x^3y^2 - 6x^2y - 2x - 3$ and (B) $F_y(x,y) = 2x^4y - 2x^3$. Integrating (A) with respect to x yields (C) $F(x,y) = x^4y^2 - 2x^3y - x^2 - 3x + \phi(y)$. Differentiating (C) with respect to y yields (D) $F_y(x,y) = 2x^4y - 2x^3 + \phi'(y)$. Comparing (D) with (B) shows that $\phi'(y) = 0$, so we take $\phi(y) = 0$. Substituting this into (C) yields $F(x,y) = x^4y^2 - 2x^3y - x^2 - 3x$. Therefore, $x^4y^2 - 2x^3y - x^2 - 3x = c$. Since $y(1) = 3 \Rightarrow c = -1$, $x^4y^2 - 2x^3y - x^2 - 3x + 1 = 0$ is an implicit solution of the initial value problem. Solving this for y by means of the quadratic formula yields $y = \frac{x + \sqrt{2x^2 + 3x - 1}}{x^2}$.

2.5.20. $M(x,y) = (y^3 - 1)e^x$; $N(x,y) = 3y^2(e^x + 1)$; $M_y(x,y) = 3y^2e^x = N_x(x,y)$, so the equation is exact. We must find F such that (A) $F_x(x,y) = (y^3 - 1)e^x$ and (B) $F_y(x,y) = 3y^2(e^x + 1)$. Integrating (A) with respect to x yields (C) $F(x,y) = (y^3 - 1)e^x + \phi(y)$. Differentiating (C) with respect to y yields (D) $F_y(x,y) = 3y^2e^x + \phi'(y)$. Comparing (D) with (B) shows that $\phi'(y) = 3y^2$, so we take $\phi(y) = y^3$. Substituting this into (C) yields $F(x,y) = (y^3 - 1)e^x + y^3$. Therefore, $(y^3 - 1)e^x + y^3 = c$. Since $y(0) = 0 \Rightarrow c = -1$, $(y^3 - 1)e^x + y^3 = -1$ is an implicit solution of the initial value problem. Therefore, $y^3(e^x + 1) = e^x - 1$, so $y = \left(\frac{e^x - 1}{e^x + 1}\right)^{1/3}$.

2.5.22. $M(x,y) = (2x-1)(y-1)$; $N(x,y) = (x+2)(x-3)$; $M_y(x,y) = 2x - 1 = N_x(x,y)$, so the equation is exact. We must find F such that (A) $F_x(x,y) = (2x-1)(y-1)$ and (B) $F_y(x,y) = (x+2)(x-3)$. Integrating (A) with respect to x yields (C) $F(x,y) = (x^2 - x)(y-1) + \phi(y)$. Differentiating (C) with respect to y yields (D) $F_y(x,y) = x^2 - x + \phi'(y)$. Comparing (D) with (B) shows that $\phi'(y) = -6$, so we take $\phi(y) = -6y$. Substituting this into (C) yields $F(x,y) = (x^2 - x)(y-1) - 6y$. Therefore, $(x^2 - x)(y-1) - 6y = c$. Since $y(1) = -1 \Rightarrow c = 6$, $(x^2 - x)(y-1) - 6y = 6$ is an implicit solution of the initial value problem. Therefore, $(x^2 - x - 6)y = x^2 - x + 6$, so $y = \frac{x^2 - x + 6}{(x-3)(x+2)}$.

2.5.24. $M(x,y) = e^x(x^4y^2 + 4x^3y^2 + 1)$; $N(x,y) = 2x^4ye^x + 2y$; $M_y(x,y) = 2x^3ye^x(x+4) = N_x(x,y)$, so the equation is exact. We must find F such that (A) $F_x(x,y) = e^x(x^4y^2 + 4x^3y^2 + 1)$

and (B) $F_y(x,y) = 2x^4ye^x + 2y$. Integrating (B) with respect to y yields (C) $F(x,y) = x^4y^2e^x + y^2 + \psi(x)$. Differentiating (C) with respect to x yields (D) $F_x(x,y) = e^xy^2(x^4+4x^3) + \psi'(x)$. Comparing (D) with (A) shows that $\psi'(x) = e^x$, so we take $\psi(x) = e^x$. Substituting this into (C) yields $F(x,y) = (x^4y^2+1)e^x + y^2$. Therefore, $(x^4y^2+1)e^x + y^2 = c$.

2.5.28. $M(x,y) = x^2+y^2$; $N(x,y) = 2xy$; $M_y(x,y) = 2y = N_x(x,y)$, so the equation is exact. We must find F such that (A) $F_x(x,y) = x^2+y^2$ and (B) $F_y(x,y) = 2xy$. Integrating (A) with respect to x yields (C) $F(x,y) = \dfrac{x^3}{3} + xy^2 + \phi(y)$. Differentiating (C) with respect to y yields (D) $F_y(x,y) = 2xy + \phi'(y)$. Comparing (D) with (B) shows that $\phi'(y) = 0$, so we take $\phi(y) = 0$. Substituting this into (C) yields $F(x,y) = \dfrac{x^3}{3} + xy^2$. Therefore, $\dfrac{x^3}{3} + xy^2 = c$.

2.5.30. **(a)** Exactness requires that $N_x(x,y) = M_y(x,y) = \dfrac{\partial}{\partial y}(x^3y^2+2xy+3y^2) = 2x^3y+2x+6y$. Hence, $N(x,y) = \dfrac{x^4y}{4} + x^2 + 6xy + g(x)$, where g is differentiable.

(b) Exactness requires that $N_x(x,y) = M_y(x,y) = \dfrac{\partial}{\partial y}(\ln xy + 2y\sin x) = \dfrac{1}{y} + 2\sin x$. Hence, $N(x,y) = \dfrac{x}{y} - 2\cos x + g(x)$, where g is differentiable.

(c) Exactness requires that $N_x(x,y) = M_y(x,y) = \dfrac{\partial}{\partial y}(x\sin x + y\sin y) = y\cos y + \sin y$. Hence, $N(x,y) = x(y\cos y + \sin y) + g(x)$, where g is differentiable.

2.5.32. The assumptions imply that $\dfrac{\partial M_1}{\partial y} = \dfrac{\partial N_1}{\partial x}$ and $\dfrac{\partial M_2}{\partial y} = \dfrac{\partial N_2}{\partial x}$. Therefore, $\dfrac{\partial}{\partial y}(M_1+M_2) = \dfrac{\partial M_1}{\partial y} + \dfrac{\partial M_2}{\partial y} = \dfrac{\partial N_1}{\partial x} + \dfrac{\partial N_2}{\partial x} = \dfrac{\partial}{\partial x}(N_1+N_2)$, which implies that $(M_1+M_2)\,dx + (N_1+N_2)\,dy = 0$ is exact on R.

2.5.34. Here $M(x,y) = Ax^2+Bxy+Cy^2$ and $N(x,y) = Dx^2+Exy+Fy^2$. Since $M_y = Bx+2Cy$ and $N_x = 2Dx+Ey$, the equation is exact if and only if $B = 2D$ and $E = 2C$.

2.5.36. Differentiating (A) $F(x,y) = \displaystyle\int_{y_0}^{y} N(x_0,s)\,ds + \int_{x_0}^{x} M(t,y)\,dt$ with respect to x yields $F_x(x,y) = M(x,y)$, since the first integral in (A) is independent of x and $M(t,y)$ is a continuous function of t for each fixed y. Differentiating (A) with respect to y and using the assumption that $M_y = N_x$ yields $F_y(x,y) = N(x_0,y) + \displaystyle\int_{x_0}^{x} \frac{\partial M}{\partial y}(t,y)\,dt = N(x_0,y) + \int_{x_0}^{x} \frac{\partial N}{\partial x}(t,y)\,dt = N(x_0,y) + N(x,y) - N(x_0,y) = N(x,y)$.

2.5.38. $y_1 = \dfrac{1}{x^2}$ is a solution of $y' + \dfrac{2}{x}y = 0$. Let $y = \dfrac{u}{x^2}$; then

$$\frac{u'}{x^2} = -\frac{2x(u/x^2)}{(x^2+2x^2(u/x^2)+1)} = -\frac{2xu}{x^2(x^2+2u+1)},$$

so $u' = -\dfrac{2xu}{x^2+2u+1}$, which can be rewritten as (A) $2xu\,dx + (x^2+2u+1)\,du = 0$. Since $\dfrac{\partial}{\partial u}(2xu) = \dfrac{\partial}{\partial x}(x^2+2u+1) = 2x$, (A) is exact. To solve (A) we must find F such that (A) $F_x(x,u) =$

$2xu$ and (B) $F_u(x,u) = x^2+2u+1$. Integrating (A) with respect to x yields (C) $F(x,u) = x^2u+\phi(u)$. Differentiating (C) with respect to u yields (D) $F_u(x,u) = x^2 + \phi'(u)$. Comparing (D) with (B) shows that $\phi'(u) = 2u + 1$, so we take $\phi(u) = u^2 + u$. Substituting this into (C) yields $F(x,u) = x^2u + u^2 + u = u(x^2 + u + 1)$. Therefore, $u(x^2 + u + 1) = c$. Since $y(1) = -2 \Rightarrow u(1) = -2, c = 0$. Therefore, $u(x^2 + u + 1) = 0$. Since $u \equiv 0$ does not satisfy $u(1) = -2$, it follows that $u = -x^2 - 1$ and $y = -1 - \dfrac{1}{x^2}$.

2.5.40. $y_1 = e^{-x^2}$ is a solution of $y' + 2xy = 0$. Let $y = ue^{-x^2}$; then $u'e^{-x^2} = -e^{-x^2}\left(\dfrac{3x+2u}{2x+3u}\right)$, so $u' = -\dfrac{3x+2u}{2x+3u}$, which can be rewritten as (A) $(3x+2u)\,dx+(2x+3u)\,du = 0$. Since $\dfrac{\partial}{\partial u}(3x+2u) = \dfrac{\partial}{\partial x}(2x + 3u) = 2$, (A) is exact. To solve (A) we must find F such that (A) $F_x(x,u) = 3x + 2u$ and (B) $F_u(x,u) = 2x + 3u$. Integrating (A) with respect to x yields (C) $F(x,u) = \dfrac{3x^2}{2} + 2xu + \phi(u)$. Differentiating (C) with respect to u yields (D) $F_u(x,u) = 2x + \phi'(u)$. Comparing (D) with (B) shows that $\phi'(u) = 3u$, so we take $\phi(u) = \dfrac{3u^2}{2}$. Substituting this into (C) yields $F(x,u) = \dfrac{3x^2}{2} + 2xu + \dfrac{3u^2}{2}$. Therefore, $\dfrac{3x^2}{2} + 2xu + \dfrac{3u^2}{2} = c$. Since $y(0) = -1 \Rightarrow u(0) = -1, c = \dfrac{3}{2}$. Therefore, $3x^2 + 4xu + 3u^2 = 3$ is an implicit solution of the initial value problem. Rewriting this as $3u^2 + 4xu + (3x^2 - 3) = 0$ and solving for u by means of the quadratic formula yields $u = -\left(\dfrac{2x+\sqrt{9-5x^2}}{3}\right)$, so

$$y = -e^{-x^2}\left(\frac{2x+\sqrt{9-5x^2}}{3}\right).$$

2.5.42. Since $M\,dx + N\,dy = 0$ is exact, (A) $M_y = N_x$. Since $-N\,dx + M\,dy = 0$ is exact, (B) $M_x = -N_y$. Differentiating (A) with respect to y and (B) with respect to x yields (C) $M_{yy} = N_{xy}$ and (D) $M_{xx} = -N_{yx}$. Since $N_{xy} = N_{yx}$, adding (C) and (D) yields $M_{xx} + M_{yy} = 0$. Differentiating (A) with respect to x and (B) with respect to y yields (E) $M_{yx} = N_{xx}$ and (F) $M_{xy} = -N_{yy}$. Since $M_{xy} = M_{yx}$, subtracting (F) from (E) yields $N_{xx} + N_{yy} = 0$.

2.5.44. **(a)** If $F(x,y) = x^2 - y^2$, then $F_x(x,y) = 2x$, $F_y(x,y) = -2y$, $F_{xx}(x,y) = 2$, and $F_{yy}(x,y) = -2$. Therefore, $F_{xx} + F_{yy} = 0$, and G must satisfy (A) $G_x(x,y) = 2y$ and (B) $G_y(x,y) = 2x$. Integrating (A) with respect to x yields (C) $G(x,y) = 2xy + \phi(y)$. Differentiating (C) with respect to y yields (D) $G_y(x,y) = 2x + \phi'(y)$. Comparing (D) with (B) shows that $\phi'(y) = 0$, so we take $\phi(y) = c$. Substituting this into (C) yields $G(x,y) = 2xy + c$.

(b) If $F(x,y) = e^x\cos y$, then $F_x(x,y) = e^x\cos y$, $F_y(x,y) = -e^x\sin y$, $F_{xx}(x,y) = e^x\cos y$, and $F_{yy}(x,y) = -e^x\cos y$. Therefore, $F_{xx} + F_{yy} = 0$, and G must satisfy (A) $G_x(x,y) = e^x\sin y$ and (B) $G_y(x,y) = e^x\cos y$. Integrating (A) with respect to x yields (C) $G(x,y) = e^x\sin y + \phi(y)$. Differentiating (C) with respect to y yields (D) $G_y(x,y) = e^x\cos y + \phi'(y)$. Comparing (D) with (B) shows that $\phi'(y) = 0$, so we take $\phi(y) = c$. Substituting this into (C) yields $G(x,y) = e^x\sin y + c$.

(c) If $F(x,y) = x^3 - 3xy^2$, then $F_x(x,y) = 3x^2 - 3y^2$, $F_y(x,y) = -6xy$, $F_{xx}(x,y) = 6x$, and $F_{yy}(x,y) = -6x$. Therefore, $F_{xx} + F_{yy} = 0$, and G must satisfy (A) $G_x(x,y) = 6xy$ and (B) $G_y(x,y) = 3x^2 - 3y^2$. Integrating (A) with respect to x yields (C) $G(x,y) = 3x^2y + \phi(y)$. Differentiating (C) with respect to y yields (D) $G_y(x,y) = 3x^2 + \phi'(y)$. Comparing (D) with (B) shows that $\phi'(y) = -3y^2$, so we take $\phi(y) = -y^3 + c$. Substituting this into (C) yields $G(x,y) = 3x^2y - y^3 + c$.

(d) If $F(x,y) = \cos x\cosh y$, then $F_x(x,y) = -\sin x\cosh y$, $F_y(x,y) = \cos x\sinh y$, $F_{xx}(x,y) =$

$-\cos x \cosh y$, and $F_{yy}(x, y) = \cos x \cosh y$. Therefore, $F_{xx} + F_{yy} = 0$, and G must satisfy (A) $G_x(x, y) = -\cos x \sinh y$ and (B) $G_y(x, y) = -\sin x \cosh y$. Integrating (A) with respect to x yields (C) $G(x, y) = -\sin x \sinh y + \phi(y)$. Differentiating (C) with respect to y yields (D) $G_y(x, y) = -\sin x \cosh y + \phi'(y)$. Comparing (D) with (B) shows that $\phi'(y) = 0$, so we take $\phi(y) = c$. Substituting this into (C) yields $G(x, y) = -\sin x \sinh y + c$.

(e) If $F(x, y) = \sin x \cosh y$, then $F_x(x, y) = \cos x \cosh y$, $F_y(x, y) = \sin x \sinh y$, $F_{xx}(x, y) = -\sin x \cosh y$, and $F_{yy}(x, y) = \sin x \cosh y$. Therefore, $F_{xx} + F_{yy} = 0$, and G must satisfy (A) $G_x(x, y) = -\sin x \sinh y$ and (B) $G_y(x, y) = \cos x \cosh y$. Integrating (A) with respect to x yields (C) $G(x, y) = \cos x \sinh y + \phi(y)$. Differentiating (C) with respect to y yields (D) $G_y(x, y) = \cos x \cosh y + \phi'(y)$. Comparing (D) with (B) shows that $\phi'(y) = 0$, so we take $\phi(y) = c$. Substituting this into (C) yields $G(x, y) = \cos x \sinh y + c$.

2.6 INTEGRATING FACTORS

2.6.2. **(a)** and **(b)**. To show that $\mu(x, y) = \dfrac{1}{(x-y)^2}$ is an integrating factor for (A) and that (B) is exact, it suffices to observe that $\dfrac{\partial}{\partial x}\left(\dfrac{xy}{x-y}\right) = -\dfrac{y^2}{(x-y)^2}$ and $\dfrac{\partial}{\partial y}\left(\dfrac{xy}{x-y}\right) = \dfrac{x^2}{(x-y)^2}$. By Theorem 2.5.1 this also shows that (C) is an implicit solution of (B). Since $\mu(x, y)$ is never zero, any solution of (B) is a solution of (A).

(c) If we interpret (A) as $-y^2 + x^2 y' = 0$, then substituting $y = x$ yields $-x^2 + x^2 \cdot 1 = 0$.

(NOTE: In Exercises 2.6.3–2.6.23, the given equation is multiplied by an integrating factor to produce an exact equation, and an implicit solution is found for the latter. For a complete analysis of the relationship between the sets of solutions of the two equations it is necessary to check for additional solutions of the given equation "along which" the integrating factor is undefined, or for solutions of the exact equation "along which" the integrating factor vanishes. In the interests of brevity we omit these tedious details except in cases where there actually is a difference between the sets of solutions of the two equations.)

2.6.4. $M(x, y) = 3x^2y$; $N(x, y) = 2x^3$; $M_y(x, y) - N_x(x, y) = 3x^2 - 6x^2 = -3x^2$; $p(x) = \dfrac{M_y(x, y) - N_x(x, y)}{N(x, y)} = -\dfrac{3x^2}{2x^3} = -\dfrac{3}{2x}$; $\int p(x)\,dx = -\dfrac{3}{2}\ln|x|$; $\mu(x) = P(x) = x^{-3/2}$; therefore $3x^{1/2}y\,dx + 2x^{3/2}\,dy = 0$ is exact. We must find F such that (A) $F_x(x, y) = 3x^{1/2}y$ and (B) $F_y(x, y) = 2x^{3/2}$. Integrating (A) with respect to x yields (C) $F(x, y) = 2x^{3/2}y + \phi(y)$. Differentiating (C) with respect to y yields (D) $F_y(x, y) = 2x^{3/2} + \phi'(y)$. Comparing (D) with (B) shows that $\phi'(y) = 0$, so we take $\phi(y) = 0$. Substituting this into (C) yields $F(x, y) = 2x^{3/2}y$, so $x^{3/2}y = c$.

2.6.6. $M(x, y) = 5xy + 2y + 5$; $N(x, y) = 2x$; $M_y(x, y) - N_x(x, y) = (5x + 2) - 2 = 5x$; $p(x) = \dfrac{M_y(x, y) - N_x(x, y)}{N(x, y)} = \dfrac{5x}{2x} = \dfrac{5}{2}$; $\int p(x)\,dx = \dfrac{5x}{2}$; $\mu(x) = P(x) = e^{5x/2}$; therefore $e^{5x/2}(5xy + 2y + 5)\,dx + 2xe^{5x/2}\,dy = 0$ is exact. We must find F such that (A) $F_x(x, y) = e^{5x/2}(5xy + 2y + 5)$ and (B) $F_y(x, y) = 2xe^{5x/2}$. Integrating (B) with respect to y yields (C) $F(x, y) = 2xye^{5x/2} + \psi(x)$. Differentiating (C) with respect to x yields (D) $F_x(x, y) = 5xye^{5x/2} + 2ye^{5x/2} + \psi'(x)$. Comparing (D) with (A) shows that $\psi'(x) = 5e^{5x/2}$, so we take $\psi(x) = 2e^{5x/2}$. Substituting this into (C) yields $F(x, y) = 2e^{5x/2}(xy + 1)$, so $e^{5x/2}(xy + 1) = c$.

2.6.8. $M(x, y) = 27xy^2 + 8y^3$; $N(x, y) = 18x^2y + 12xy^2$; $M_y(x, y) - N_x(x, y) = (54xy + 24y^2) - (36xy + 12y^2) = 18xy + 12y^2$; $p(x) = \dfrac{M_y(x, y) - N_x(x, y)}{N(x, y)} = \dfrac{18xy + 12y^2}{18x^2y + 12y^2x} = \dfrac{1}{x}$; $\int p(x)\,dx = \ln|x|$; $\mu(x) = P(x) = x$; therefore $(27x^2y^2 + 8xy^3)\,dx + (18x^3y + 12x^2y^2)\,dy = 0$ is exact. We must find F such that (A) $F_x(x, y) = 27x^2y^2 + 8xy^3$ and (B) $F_y(x, y) = 18x^3y + 12x^2y^2$. Integrating (A) with respect to x yields (C) $F(x, y) = 9x^3y^2 + 4x^2y^3 + \phi(y)$. Differentiating (C) with respect to y yields (D) $F_y(x,y) = 18x^3y + 12x^2y^2 + \phi'(y)$. Comparing (D) with (B) shows that $\phi'(y) = 0$, so we take $\phi(y) = 0$. Substituting this into (C) yields $F(x,y) = 9x^3y^2 + 4x^2y^3$, so $x^2y^2(9x+4y) = c$.

2.6.10. $M(x,y) = y^2$; $N(x,y) = \left(xy^2 + 3xy + \dfrac{1}{y}\right)$; $M_y(x,y) - N_x(x,y) = 2y - (y^2 + 3y) = -y(y+1)$; $q(y) = \dfrac{N_x(x,y) - M_y(x,y)}{M(x,y)} = \dfrac{y(y+1)}{y^2} = 1 + \dfrac{1}{y}$; $\int q(y)\,dy = y \ln|y|$; $\mu(y) = Q(y) = ye^y$; therefore $y^3e^y\,dx + e^y(xy^3 + 3xy^2 + 1)\,dy = 0$ is exact. We must find F such that (A) $F_x(x,y) = y^3e^y$ and (B) $F_y(x,y) = e^y(xy^3 + 3xy^2 + 1)$. Integrating (A) with respect to x yields (C) $F(x,y) = xy^3e^y + \phi(y)$. Differentiating (C) with respect to y yields (D) $F_y(x,y) = xy^3e^y + 3xy^2e^y + \phi'(y)$. Comparing (D) with (B) shows that $\phi'(y) = e^y$, so we take $\phi(y) = e^y$. Substituting this into (C) yields $F(x,y) = xy^3e^y + e^y$, so $e^y(xy^3+1) = c$.

2.6.12. $M(x,y) = x^2y + 4xy + 2y$; $N(x,y) = x^2 + x$; $M_y(x,y) - N_x(x,y) = (x^2 + 4x + 2) - (2x + 1) = x^2 + 2x + 1 = (x+1)^2$; $p(x) = \dfrac{M_y(x,y) - N_x(x,y)}{N(x,y)} = \dfrac{(x+1)^2}{x(x+1)} = 1 + \dfrac{1}{x}$; $\int p(x)\,dx = x + \ln|x|$; $\mu(x) = P(x) = xe^x$; therefore $e^x(x^3y + 4x^2y + 2xy)\,dx + e^x(x^3 + x^2)\,dy = 0$ is exact. We must find F such that (A) $F_x(x,y) = e^x(x^3y + 4x^2y + 2xy)$ and (B) $F_y(x,y) = e^x(x^3 + x^2)$. Integrating (B) with respect to y yields (C) $F(x,y) = y(x^3 + x^2)e^x + \psi(x)$. Differentiating (C) with respect to x yields (D) $F_x(x,y) = e^x(x^3y + 4x^2y + 2xy) + \psi'(x)$. Comparing (D) with (A) shows that $\psi'(x) = 0$, so we take $\psi(x) = 0$. Substituting this into (C) yields $F(x,y) = y(x^3 + x^2)e^x = x^2y(x+1)e^x$, so $x^2y(x+1)e^x = c$.

2.6.14. $M(x,y) = \cos x \cos y$; $N(x,y) = \sin x \cos y - \sin x \sin y + y$; $M_y(x,y) - N_x(x,y) = -\cos x \sin y - (\cos x \cos y - \cos x \sin y) = -\cos x \cos y$; $q(y) = \dfrac{N_x(x,y) - M_y(x,y)}{M(x,y)} = \dfrac{\cos x \cos y}{\cos x \cos y} = 1$; $\int q(y)\,dy = 1$; $\mu(y) = Q(y) = e^y$; therefore $e^y \cos x \cos y\,dx + e^y(\sin x \cos y - \sin x \sin y + y)\,dy = 0$ is exact. We must find F such that (A) $F_x(x,y) = e^y \cos x \cos y$ and (B) $F_y(x,y) = e^y(\sin x \cos y - \sin x \sin y + y)$. Integrating (A) with respect to x yields (C) $F(x,y) = e^y \sin x \cos y + \phi(y)$. Differentiating (C) with respect to y yields (D) $F_y(x,y) = e^y(\sin x \cos y - \sin x \sin y) + \phi'(y)$. Comparing (D) with (B) shows that $\phi'(y) = ye^y$, so we take $\phi(y) = e^y(y-1)$. Substituting this into (C) yields $F(x,y) = e^y(\sin x \cos y + y - 1)$, so $e^y(\sin x \cos y + y - 1) = c$.

2.6.16. $M(x,y) = y \sin y$; $N(x,y) = x(\sin y - y \cos y)$; $M_y(x,y) - N_x(x,y) = (y \cos y + \sin y) - (\sin y - y \cos y) = 2y \cos y$; $q(y) = \dfrac{N_x(x,y) - M_y(x,y)}{N(x,y)} = -\dfrac{2\cos y}{\sin y}$; $\int q(y)\,dy = -2\ln|\sin y|$; $\mu(y) = Q(y) = \dfrac{1}{\sin^2 y}$; therefore $\left(\dfrac{y}{\sin y}\right)dx + x\left(\dfrac{1}{\sin y} - \dfrac{y \cos y}{\sin^2 y}\right)dy = 0$ is exact. We must find F such that (A) $F_x(x,y) = \dfrac{y}{\sin y}$ and (B) $F_y(x,y) = x\left(\dfrac{1}{\sin y} - \dfrac{y\cos y}{\sin^2 y}\right)$. Integrating (A) with respect to x yields (C) $F(x,y) = \dfrac{xy}{\sin y} + \phi(y)$. Differentiating (C) with respect to y yields (D) $F_y(x,y) = x\left(\dfrac{1}{\sin y} - \dfrac{y\cos y}{\sin^2 y}\right) + \phi'(y)$. Comparing (D) with (B) shows that $\phi'(y) = 0$, so we take $\phi(y) = 0$. Substituting this into (C) yields $F(x,y) = \dfrac{xy}{\sin y}$, so $\dfrac{xy}{\sin y} = c$. In addition, the given equation has the constant solutions $y = k\pi$, where k is an integer.

2.6.18. $M(x,y) = \alpha y + \gamma xy$; $N(x,y) = \beta x + \delta xy$; $M_y(x,y) - N_x(x,y) = (\alpha + \gamma x) - (\beta + \delta y)$; and $p(x)N(x,y) - q(y)M(x,y) = p(x)x(\beta + \delta y) - q(y)y(\alpha + \gamma x)$. so exactness requires that

$(\alpha+\gamma x)-(\beta+\delta y) = p(x)x(\beta+\delta y)-q(y)y(\alpha+\gamma x)$, which holds if $p(x)x = -1$ and $q(y)y = -1$. Thus $p(x) = -\dfrac{1}{x}$; $q(y) = -\dfrac{1}{y}$; $\int p(x)\,dx = -\ln|x|$; $\int q(y)\,dy = -\ln|y|$; $P(x) = \dfrac{1}{x}$; $Q(y) = \dfrac{1}{y}$; $\mu(x,y) = \dfrac{1}{xy}$. Therefore, $\left(\dfrac{\alpha}{x}+\gamma\right)dx + \left(\dfrac{\beta}{y}+\delta\right)dy = 0$ is exact. We must find F such that (A) $F_x(x,y) = \dfrac{\alpha}{x}+\gamma$ and (B) $F_y(x,y) = \dfrac{\beta}{y}+\delta$. Integrating (A) with respect to x yields (C) $F(x,y) = \alpha\ln|x|+\gamma x+\phi(y)$. Differentiating (C) with respect to y yields (D) $F_y(x,y) = \phi'(y)$. Comparing (D) with (B) shows that $\phi'(y) = \dfrac{\beta}{y}+\delta$, so we take $\phi(y) = \beta\ln|y|+\delta y$. Substituting this into (C) yields $F(x,y) = \alpha\ln|x|+\gamma x+\beta\ln|y|+\delta y$, so $|x|^\alpha|y|^\beta e^{\gamma x}e^{\delta y} = c$. The given equation also has the solutions $x\equiv 0$ and $y\equiv 0$.

2.6.20. $M(x,y) = 2y$; $N(x,y) = 3(x^2+x^2y^3)$; $M_y(x,y)-N_x(x,y) = 2-(6x+6xy^3)$; and $p(x)N(x,y)-q(y)M(x,y) = 3p(x)(x^2+x^2y^3)-2q(y)y$. so exactness requires that (A) $2-6x-6xy^3 = 3p(x)x(x+xy^3)-2q(y)y$. To obtain similar terms on the two sides of (A) we let $p(x)x = a$ and $q(y)y = b$ where a and b are constants such that $2-6x-6xy^3 = 3a(x+xy^3)-2b$, which holds if $a = -2$ and $b = -1$. Thus, $p(x) = -\dfrac{2}{x}$; $q(y) = -\dfrac{1}{y}$; $\int p(x)\,dx = -2\ln|x|$; $\int q(y)\,dy = -\ln|y|$; $P(x) = \dfrac{1}{x^2}$; $Q(y) = \dfrac{1}{y}$; $\mu(x,y) = \dfrac{1}{x^2y}$. Therefore, $\dfrac{2}{x^2}\,dx + 3\left(\dfrac{1}{y}+y^2\right)dy = 0$ is exact. We must find F such that (B) $F_x(x,y) = \dfrac{2}{x^2}$ and (C) $F_y(x,y) = 3\left(\dfrac{1}{y}+y^2\right)$. Integrating (B) with respect to x yields (D) $F(x,y) = -\dfrac{2}{x}+\phi(y)$. Differentiating (D) with respect to y yields (E) $F_y(x,y) = \phi'(y)$. Comparing (E) with (C) shows that $\phi'(y) = 3\left(\dfrac{1}{y}+y^2\right)$, so we take $\phi(y) = y^3+3\ln|y|$. Substituting this into (D) yields $F(x,y) = -\dfrac{2}{x}+y^3+3\ln|y|$, so $-\dfrac{2}{x}+y^3+3\ln|y| = c$. The given equation also has the solutions $x\equiv 0$ and $y\equiv 0$.

2.6.22. $M(x,y) = x^4y^4$; $N(x,y) = x^5y^3$; $M_y(x,y)-N_x(x,y) = 4x^4y^3-5x^4y^3 = -x^4y^3$; and $p(x)N(x,y)-q(y)M(x,y) = p(x)x^5y^3-q(y)x^4y^4$. so exactness requires that $-x^4y^3 = p(x)x^5y^3-q(y)x^4y^4$, which is equivalent to $p(x)x-q(y)y = -1$. This holds if $p(x)x = a$ and $q(y)y = a+1$ where a is an arbitrary real number. Thus, $p(x) = \dfrac{a}{x}$; $q(y) = \dfrac{a+1}{y}$; $\int p(x)\,dx = a\ln|x|$; $\int q(y)\,dy = (a+1)\ln|y|$; $P(x) = |x|^a$; $Q(y) = |y|^{a+1}$; $\mu(x,y) = |x^a||y|^{a+1}$. Therefore, $|x|^a|y|^{a+1}\left(x^4y^4\,dx+x^5y^3\,dy\right) = 0$ is exact for any choice of a. For simplicity we let $a = -4$, so (A) is equivalent to $y\,dx+x\,dy = 0$. We must find F such that (B) $F_x(x,y) = y$ and (C) $F_y(x,y) = x$. Integrating (B) with respect to x yields (D) $F(x,y) = xy+\phi(y)$. Differentiating (D) with respect to y yields (E) $F_y(x,y) = x+\phi'(y)$. Comparing (E) with (C) shows that $\phi'(y) = 0$, so we take $\phi(y) = 0$. Substituting this into (D) yields $F(x,y) = xy$, so $xy = c$.

2.6.24. $M(x,y) = x^4y^3+y$; $N(x,y) = x^5y^2-x$; $M_y(x,y)-N_x(x,y) = (3x^4y^2+1)-(5x^4y^2-1) = -2x^4y^2+2$; $p(x) = \dfrac{M_y(x,y)-N_x(x,y)}{N(x,y)} = -\dfrac{2x^4y^2-2}{x^5y^2-x} = -\dfrac{2}{x}$; $\int p(x)\,dx = -2\ln|x|$; $\mu(x) = P(x) = \dfrac{1}{x^2}$; therefore $\left(x^2y^3+\dfrac{y}{x^2}\right)dx + \left(x^3y^2-\dfrac{1}{x}\right)dy = 0$ is exact. We must find F such that (A) $F_x(x,y) = \left(x^2y^3+\dfrac{y}{x^2}\right)$ and (B) $F_y(x,y) = \left(x^3y^2-\dfrac{1}{x}\right)$. Integrating (A) with respect to x yields (C) $F(x,y) = \dfrac{x^3y^3}{3} - \dfrac{y}{x} + \phi(y)$. Differentiating (C) with respect to y yields (D) $F_y(x,y) = x^3y^2 - \dfrac{1}{x} + \phi'(y)$. Comparing (D) with (B) shows that $\phi'(y) = 0$, so we take $\phi(y) = 0$. Substituting this into (C) yields $F(x,y) = \dfrac{x^3y^3}{3} - \dfrac{y}{x}$, so $\dfrac{x^3y^3}{3} - \dfrac{y}{x} = c$.

2.6.26. $M(x,y) = 12xy + 6y^3$; $N(x,y) = 9x^2 + 10xy^2$; $M_y(x,y) - N_x(x,y) = (12x + 18y^2) - (18x+10y^2) = -6x+8y^2$; and $p(x)N(x,y)-q(y)M(x,y) = p(x)x(9x+10y^2)-q(y)y(12x+6y^2)$, so exactness requires that (A) $-6x + 8y^2 = p(x)x(9x + 10y^2) - q(y)y(12x + 6y^2)$. To obtain similar terms on the two sides of (A) we let $p(x)x = a$ and $q(y)y = b$ where a and b are constants such that $-6x + 8y^2 = a(9x + 10y^2) - b(12x + 6y^2)$, which holds if $9a - 12b = -6$, $10a - 6b = 8$; that is, $a = b = 2$. Thus $p(x) = \dfrac{2}{x}$; $q(y) = \dfrac{2}{y}$; $\int p(x)\,dx = 2\ln|x|$; $\int q(y)\,dy = 2\ln|y|$; $P(x) = x^2$; $Q(y) = y^2$; $\mu(x,y) = x^2y^2$. Therefore, $(12x^3y^3 + 6x^2y^5)\,dx + (9x^4y^2 + 10x^3y^4)\,dy = 0$ is exact. We must find F such that (B) $F_x(x,y) = 12x^3y^3 + 6x^2y^5$ and (C) $F_y(x,y) = 9x^4y^2 + 10x^3y^4$. Integrating (B) with respect to x yields (D) $F(x,y) = 3x^4y^3 + 2x^3y^5 + \phi(y)$. Differentiating (D) with respect to y yields (E) $F_y(x,y) = 9x^4y^2 + 10x^3y^4 + \phi'(y)$. Comparing (E) with (C) shows that $\phi'(y) = 0$, so we take $\phi(y) = 0$. Substituting this into (D) yields $F(x,y) = 3x^4y^3 + 2x^3y^5$, so $x^3y^3(3x + 2y^2) = c$.

2.6.28. $M(x,y) = ax^my+by^{n+1}$; $N(x,y) = cx^{m+1}+dxy^n$; $M_y(x,y)-N_x(x,y) = \left[ax^{m+1} + (n+1)by^n\right] - [(m+1)cx^m + dy^n]$; $p(x)N(x,y) - q(y)M(x,y) = xp(x)(cx^m + dy^n) - yp(y)(ax^m + by^n)$. Let (A) $xp(x) = \alpha$ and (B) $yp(y) = \beta$, where α and β are to be chosen so that $\left[ax^{m+1} + (n+1)by^n\right] - [(m+1)cx^m + dy^n] = \alpha(cx^m + dy^n) - \beta(ax^m + by^n)$, which will hold if

$$\begin{array}{rclcl} c\alpha - a\beta &=& a-(m+1)c &=_{\rm df}& A \\ d\alpha - b\beta &=& -d + (n+1)b &=_{\rm df}& B. \end{array} \qquad \text{(C)}$$

Since $ad - bc \neq 0$ it can be verified that $\alpha = \dfrac{aB - bA}{ad - bc}$ and $\beta = \dfrac{cB - dA}{ad - bc}$ satisfy (C). From (A) and (B), $p(x) = \dfrac{\alpha}{x}$ and $q(y) = \dfrac{\beta}{y}$, so $\mu(x,y) = x^\alpha y^\beta$ is an integrating factor for the given equation.

2.6.30. **(a)** Since $M(x,y) = p(x)y - f(x)$ and $N(x,y) = 1$, $\dfrac{M_y(x,y) - N_x(x,y)}{N(x,y)} = p(x)$ and Theorem 2.6.1 implies that $\mu(x) \pm e^{\int p(x)\,dx}$ is an integrating factor for (C).

(b) Multiplying (A) through $\mu = \pm e^{\int p(x)\,dx}$ yields (D) $\mu(x)y' + \mu'(x)y = \mu(x)f(x)$, which is equivalent to $(\mu(x)y)' = \mu(x)f(x)$. Integrating this yields $\mu(x)y = c + \displaystyle\int \mu(x)f(x)\,dx$, so $y = \dfrac{1}{\mu(x)}\left(c + \displaystyle\int \mu(x)f(x)\,dx\right)$, which is equivalent to (B) since $y_1 = \dfrac{1}{\mu}$ is a nontrivial solution of $y' + p(x)y = 0$.

CHAPTER 3
Numerical Methods

3.1 EULER'S METHOD

3.1.2. $y_1 = 1.200000000,\ y_2 = 1.440415946,\ y_3 = 1.729880994$

3.1.4. $y_1 = 2.962500000,\ y_2 = 2.922635828,\ y_3 = 2.880205639$

3.1.6.

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	Exact
0.0	2.000000000	2.000000000	2.000000000	2.000000000
0.1	2.100000000	2.169990965	2.202114518	2.232642918
0.2	2.514277288	2.649377900	2.713011720	2.774352565
0.3	3.317872752	3.527672599	3.628465025	3.726686582
0.4	4.646592772	4.955798226	5.106379369	5.254226636
0.5	6.719737638	7.171467977	7.393322991	7.612186259
0.6	9.876155616	10.538384528	10.865186799	11.188475269
0.7	14.629532397	15.605686107	16.088630652	16.567103199
0.8	21.751925418	23.197328550	23.913328531	24.623248150
0.9	32.399118931	34.545932627	35.610005377	36.665439956
1.0	48.298147362	51.492825643	53.076673685	54.647937102

3.1.8.

x	$h = 0.05$	$h = 0.025$	$h = 0.0125$	Exact
1.00	2.000000000	2.000000000	2.000000000	2.000000000
1.05	2.250000000	2.259280190	2.264490570	2.270158103
1.10	2.536734694	2.559724746	2.572794280	2.587150838
1.15	2.867950854	2.910936426	2.935723355	2.963263785
1.20	3.253613825	3.325627715	3.367843117	3.415384615
1.25	3.706750613	3.820981064	3.889251900	3.967391304
1.30	4.244700641	4.420781829	4.528471927	4.654198473
1.35	4.891020001	5.158883503	5.327348558	5.528980892
1.40	5.678467290	6.085075790	6.349785943	6.676923077
1.45	6.653845988	7.275522641	7.698316221	8.243593315
1.50	7.886170437	8.852463793	9.548039907	10.500000000

3.1.10.

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	$h = 0.1$	$h = 0.05$	$h = 0.025$
1.0	1.000000000	1.000000000	1.000000000	0.0000	0.0000	0.0000
1.1	0.920000000	0.921898275	0.922822717	-0.0384	-0.0189	-0.0094
1.2	0.847469326	0.851018464	0.852746371	-0.0745	-0.0368	-0.0183
1.3	0.781779403	0.786770087	0.789197876	-0.1092	-0.0540	-0.0268
1.4	0.722453556	0.728682209	0.731709712	-0.1428	-0.0707	-0.0351
1.5	0.669037867	0.676299618	0.679827306	-0.1752	-0.0868	-0.0432
1.6	0.621054176	0.629148585	0.633080163	-0.2062	-0.1023	-0.0509
1.7	0.578000416	0.586740390	0.590986601	-0.2356	-0.1170	-0.0583
1.8	0.539370187	0.548588902	0.553070392	-0.2631	-0.1310	-0.0653
1.9	0.504674296	0.514228603	0.518877246	-0.2889	-0.1441	-0.0719
2.0	0.473456737	0.483227470	0.487986391	-0.3129	-0.1563	-0.0781
	Approximate Solutions			Residuals		

3.1.12.

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
1.0	0.000000000	0.000000000	0.000000000	0.000000000
1.1	-0.100000000	-0.099875000	-0.099780455	-0.099664000
1.2	-0.199000000	-0.198243434	-0.197800853	-0.197315517
1.3	-0.294996246	-0.293129862	-0.292110713	-0.291036003
1.4	-0.386095345	-0.382748403	-0.380986158	-0.379168221
1.5	-0.470695388	-0.465664569	-0.463078857	-0.460450590
1.6	-0.547627491	-0.540901018	-0.537503081	-0.534085626
1.7	-0.616227665	-0.607969574	-0.603849795	-0.599737720
1.8	-0.676329533	-0.666833345	-0.662136956	-0.657473792
1.9	-0.728190908	-0.717819639	-0.712718751	-0.707670533
2.0	-0.772381768	-0.761510960	-0.756179726	-0.750912371

3.1.14.

Euler's method				
x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
2.0	2.000000000	2.000000000	2.000000000	2.000000000
2.1	2.420000000	2.440610764	2.451962006	2.464119569
2.2	2.922484288	2.972198224	2.999753046	3.029403212
2.3	3.524104434	3.614025082	3.664184099	3.718409925
2.4	4.244823572	4.389380160	4.470531822	4.558673929
2.5	5.108581185	5.326426396	5.449503467	5.583808754
2.6	6.144090526	6.459226591	6.638409411	6.834855438
2.7	7.385795229	7.828984275	8.082588076	8.361928926
2.8	8.875017001	9.485544888	9.837137672	10.226228709
2.9	10.661332618	11.489211987	11.969020902	12.502494409
3.0	12.804226135	13.912944662	14.559623055	15.282004826

Euler semilinear method				
x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
2.0	2.000000000	2.000000000	2.000000000	2.000000000
2.1	2.467233571	2.465641081	2.464871435	2.464119569
2.2	3.036062650	3.032657307	3.031011316	3.029403212
2.3	3.729169725	3.723668026	3.721008466	3.718409925
2.4	4.574236356	4.566279470	4.562432696	4.558673929
2.5	5.605052990	5.594191643	5.588940276	5.583808754
2.6	6.862874116	6.848549921	6.841623814	6.834855438
2.7	8.398073101	8.379595572	8.370660695	8.361928926
2.8	10.272163096	10.248681420	10.237326199	10.226228709
2.9	12.560265110	12.530733531	12.516452106	12.502494409
3.0	15.354122287	15.317257705	15.299429421	15.282004826

3.1.16.

Euler's method				
x	$h = 0.2$	$h = 0.1$	$h = 0.05$	"Exact"
1.0	2.000000000	2.000000000	2.000000000	2.000000000
1.2	1.768294197	1.786514499	1.794412375	1.801636774
1.4	1.603028371	1.628427487	1.639678822	1.650102616
1.6	1.474580412	1.502563111	1.515157063	1.526935885
1.8	1.368349549	1.396853671	1.409839229	1.422074283
2.0	1.276424761	1.304504818	1.317421794	1.329664953
2.2	1.194247156	1.221490111	1.234122458	1.246155344
2.4	1.119088175	1.145348276	1.157607418	1.169334346
2.6	1.049284410	1.074553688	1.086419453	1.097812069
2.8	0.983821745	1.008162993	1.019652023	1.030719114
3.0	0.922094379	0.945604800	0.956752868	0.967523153

Euler semilinear method				
x	$h = 0.2$	$h = 0.1$	$h = 0.05$	"Exact"
1.0	2.000000000	2.000000000	2.000000000	2.000000000
1.2	1.806911831	1.804304958	1.802978526	1.801636774
1.4	1.659738603	1.654968381	1.652547436	1.650102616
1.6	1.540257861	1.533652916	1.530308405	1.526935885
1.8	1.438532932	1.430361800	1.426232584	1.422074283
2.0	1.348782285	1.339279577	1.334486249	1.329664953
2.2	1.267497415	1.256876924	1.251528766	1.246155344
2.4	1.192497494	1.180958765	1.175157264	1.169334346
2.6	1.122416379	1.110147777	1.103988310	1.097812069
2.8	1.056405906	1.043585743	1.037158237	1.030719114
3.0	0.993954754	0.980751307	0.974140320	0.967523153

3.1.18.

Euler's method				
x	$h = 0.2$	$h = 0.1$	$h = 0.05$	"Exact"
0.0	1.000000000	1.000000000	1.000000000	1.000000000
0.2	1.200000000	1.186557290	1.179206574	1.171515153
0.4	1.333543409	1.298441890	1.280865289	1.263370891
0.6	1.371340142	1.319698328	1.295082088	1.271251278
0.8	1.326367357	1.270160237	1.243958980	1.218901287
1.0	1.233056306	1.181845667	1.158064902	1.135362070
1.2	1.122359136	1.080477477	1.060871608	1.042062625
1.4	1.013100262	0.981124989	0.965917496	0.951192532
1.6	0.914000211	0.890759107	0.879460404	0.868381328
1.8	0.827848558	0.811673612	0.803582000	0.795518627
2.0	0.754572560	0.743869878	0.738303914	0.732638628

Euler semilinear method				
x	$h = 0.2$	$h = 0.1$	$h = 0.05$	"Exact"
0.0	1.000000000	1.000000000	1.000000000	1.000000000
0.2	1.153846154	1.162906599	1.167266650	1.171515153
0.4	1.236969953	1.250608357	1.257097924	1.263370891
0.6	1.244188456	1.258241892	1.264875987	1.271251278
0.8	1.195155456	1.207524076	1.213335781	1.218901287
1.0	1.115731189	1.125966437	1.130768614	1.135362070
1.2	1.025938754	1.034336918	1.038283392	1.042062625
1.4	0.937645707	0.944681597	0.948002346	0.951192532
1.6	0.856581823	0.862684171	0.865583126	0.868381328
1.8	0.784832910	0.790331183	0.792963532	0.795518627
2.0	0.722610454	0.727742966	0.730220211	0.732638628

3.1.20.

Euler's method				
x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
0.0	1.000000000	1.000000000	1.000000000	1.000000000
0.1	0.700000000	0.725841563	0.736671690	0.746418339
0.2	0.498330000	0.532982493	0.547988831	0.561742917
0.3	0.356272689	0.392592562	0.408724303	0.423724207
0.4	0.254555443	0.289040639	0.304708942	0.319467408
0.5	0.181440541	0.212387189	0.226758594	0.240464879
0.6	0.128953069	0.155687255	0.168375130	0.180626161
0.7	0.091393543	0.113851516	0.124744976	0.135394692
0.8	0.064613612	0.083076641	0.092230966	0.101293057
0.9	0.045585102	0.060505907	0.068068776	0.075650324
1.0	0.032105117	0.043997045	0.050159310	0.056415515

Euler semilinear method				
x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
0.0	1.000000000	1.000000000	1.000000000	1.000000000
0.1	0.740818221	0.743784320	0.745143557	0.746418339
0.2	0.555889275	0.558989106	0.560410719	0.561742917
0.3	0.418936461	0.421482025	0.422642541	0.423724207
0.4	0.315890439	0.317804400	0.318668549	0.319467408
0.5	0.237908421	0.239287095	0.239902094	0.240464879
0.6	0.178842206	0.179812811	0.180239888	0.180626161
0.7	0.134165506	0.134840668	0.135133367	0.135394692
0.8	0.100450939	0.100918118	0.101117514	0.101293057
0.9	0.075073968	0.075396974	0.075532643	0.075650324
1.0	0.056020154	0.056243980	0.056336491	0.056415515

3.1.22.

Euler's method				
x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
2.0	1.000000000	1.000000000	1.000000000	1.000000000
2.1	1.000000000	1.005062500	1.007100815	1.008899988
2.2	1.020500000	1.026752091	1.029367367	1.031723469
2.3	1.053489840	1.059067423	1.061510137	1.063764243
2.4	1.093521685	1.097780573	1.099748225	1.101614730
2.5	1.137137554	1.140059654	1.141496651	1.142903776
2.6	1.182269005	1.184090031	1.185056276	1.186038851
2.7	1.227745005	1.228755801	1.229350441	1.229985178
2.8	1.272940309	1.273399187	1.273721920	1.274092525
2.9	1.317545833	1.317651554	1.317786528	1.317967533
3.0	1.361427907	1.361320824	1.361332589	1.361383810

Euler semilinear method				
x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
2.0	1.000000000	1.000000000	1.000000000	1.000000000
2.1	0.982476904	0.996114142	1.002608435	1.008899988
2.2	0.988105346	1.010577663	1.021306044	1.031723469
2.3	1.009495813	1.037358814	1.050731634	1.063764243
2.4	1.041012955	1.071994816	1.086964414	1.101614730
2.5	1.078631301	1.111346365	1.127262285	1.142903776
2.6	1.119632590	1.153300133	1.169781376	1.186038851
2.7	1.162270287	1.196488725	1.213325613	1.229985178
2.8	1.205472927	1.240060456	1.257146091	1.274092525
2.9	1.248613584	1.283506001	1.300791772	1.317967533
3.0	1.291345518	1.326535737	1.344004102	1.361383810

3.2 THE IMPROVED EULER METHOD AND RELATED METHODS

3.2.2. $y_1 = 1.220207973,\ y_2 = 1.489578775\ y_3 = 1.819337186$

3.2.4. $y_1 = 2.961317914;\ y_2 = 2.920132727;\ y_3 = 2.876213748.$

3.2.6.

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	Exact
0.0	2.000000000	2.000000000	2.000000000	2.000000000
0.1	2.257138644	2.238455342	2.234055168	2.232642918
0.2	2.826004666	2.786634110	2.777340360	2.774352565
0.3	3.812671926	3.747167263	3.731674025	3.726686582
0.4	5.387430580	5.285996803	5.261969043	5.254226636
0.5	7.813298361	7.660199197	7.623893064	7.612186259
0.6	11.489337756	11.260349005	11.206005869	11.188475269
0.7	17.015861211	16.674352914	16.593267820	16.567103199
0.8	25.292140630	24.783149862	24.662262731	24.623248150
0.9	37.662496723	36.903828191	36.723608928	36.665439956
1.0	56.134480009	55.003390448	54.734674836	54.647937102

3.2.8.

x	$h = 0.05$	$h = 0.025$	$h = 0.0125$	Exact
1.00	2.000000000	2.000000000	2.000000000	2.000000000
1.05	2.268367347	2.269670336	2.270030868	2.270158103
1.10	2.582607299	2.585911295	2.586827341	2.587150838
1.15	2.954510022	2.960870733	2.962638822	2.963263785
1.20	3.400161788	3.411212150	3.414293964	3.415384615
1.25	3.942097142	3.960434900	3.965570792	3.967391304
1.30	4.612879780	4.642784826	4.651206769	4.654198473
1.35	5.461348619	5.510188575	5.524044591	5.528980892
1.40	6.564150753	6.645334756	6.668600859	6.676923077
1.45	8.048579617	8.188335998	8.228972215	8.243593315
1.50	10.141969585	10.396770409	10.472502111	10.500000000

3.2.10.

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	$h = 0.1$	$h = 0.05$	$h = 0.025$
1.0	1.000000000	1.000000000	1.000000000	0.00000	0.000000	0.000000
1.1	0.923734663	0.923730743	0.923730591	0.00004	0.000001	-0.000001
1.2	0.854475600	0.854449616	0.854444697	0.00035	0.000068	0.000015
1.3	0.791650344	0.791596016	0.791584634	0.00078	0.000167	0.000039
1.4	0.734785779	0.734703826	0.734686010	0.00125	0.000277	0.000065
1.5	0.683424095	0.683318666	0.683295308	0.00171	0.000384	0.000091
1.6	0.637097057	0.636973423	0.636945710	0.00213	0.000483	0.000115
1.7	0.595330359	0.595193634	0.595162740	0.00250	0.000572	0.000137
1.8	0.557658422	0.557513000	0.557479947	0.00283	0.000650	0.000156
1.9	0.523638939	0.523488343	0.523453958	0.00311	0.000718	0.000173
2.0	0.492862999	0.492709931	0.492674855	0.00335	0.000777	0.000187
	Approximate Solutions			Residuals		

3.2.12.

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
1.0	0.000000000	0.000000000	0.000000000	0.000000000
1.1	-0.099500000	-0.099623114	-0.099653809	-0.099664000
1.2	-0.196990313	-0.197235180	-0.197295585	-0.197315517
1.3	-0.290552949	-0.290917718	-0.291006784	-0.291036003
1.4	-0.378532718	-0.379013852	-0.379130237	-0.379168221
1.5	-0.459672297	-0.460262848	-0.460404546	-0.460450590
1.6	-0.533180153	-0.533868468	-0.534032512	-0.534085626
1.7	-0.598726853	-0.599496413	-0.599678824	-0.599737720
1.8	-0.656384109	-0.657214624	-0.657410640	-0.657473792
1.9	-0.706530934	-0.707400266	-0.707604759	-0.707670533
2.0	-0.749751364	-0.750637632	-0.750845571	-0.750912371

3.2.14.

Improved Euler method				
x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
2.0	2.000000000	2.000000000	2.000000000	2.000000000
2.1	2.461242144	2.463344439	2.463918368	2.464119569
2.2	3.022367633	3.027507237	3.028911026	3.029403212
2.3	3.705511610	3.714932709	3.717507170	3.718409925
2.4	4.537659565	4.553006531	4.557202414	4.558673929
2.5	5.551716960	5.575150456	5.581560437	5.583808754
2.6	6.787813853	6.822158665	6.831558101	6.834855438
2.7	8.294896222	8.343829180	8.357227947	8.361928926
2.8	10.132667135	10.200955596	10.219663917	10.226228709
2.9	12.373954732	12.467758807	12.493470722	12.502494409
3.0	15.107600968	15.234856000	15.269755072	15.282004826

Improved Euler semilinear method				
x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
2.0	2.000000000	2.000000000	2.000000000	2.000000000
2.1	2.464261688	2.464155139	2.464128464	2.464119569
2.2	3.029706047	3.029479005	3.029422165	3.029403212
2.3	3.718897663	3.718531995	3.718440451	3.718409925
2.4	4.559377397	4.558849990	4.558717956	4.558673929
2.5	5.584766724	5.584048510	5.583868709	5.583808754
2.6	6.836116246	6.835170986	6.834934347	6.834855438
2.7	8.363552464	8.362335253	8.362030535	8.361928926
2.8	10.228288880	10.226744312	10.226357645	10.226228709
2.9	12.505082132	12.503142042	12.502656361	12.502494409
3.0	15.285231726	15.282812424	15.282206780	15.282004826

3.2.16.

Improved Euler method				
x	$h = 0.2$	$h = 0.1$	$h = 0.05$	"Exact"
1.0	2.000000000	2.000000000	2.000000000	2.000000000
1.2	1.801514185	1.801606135	1.801629115	1.801636774
1.4	1.649911580	1.650054870	1.650090680	1.650102616
1.6	1.526711768	1.526879870	1.526921882	1.526935885
1.8	1.421841570	1.422016119	1.422059743	1.422074283
2.0	1.329441172	1.329609020	1.329650971	1.329664953
2.2	1.245953205	1.246104819	1.246142713	1.246155344
2.4	1.169162994	1.169291515	1.169323639	1.169334346
2.6	1.097677870	1.097778523	1.097803683	1.097812069
2.8	1.030626179	1.030695880	1.030713305	1.030719114
3.0	0.967473721	0.967510790	0.967520062	0.967523153

Improved Euler semilinear method				
x	$h = 0.2$	$h = 0.1$	$h = 0.05$	"Exact"
1.0	2.000000000	2.000000000	2.000000000	2.000000000
1.2	1.801514185	1.801606135	1.801629115	1.801636774
1.4	1.649911580	1.650054870	1.650090680	1.650102616
1.6	1.526711768	1.526879870	1.526921882	1.526935885
1.8	1.421841570	1.422016119	1.422059743	1.422074283
2.0	1.329441172	1.329609020	1.329650971	1.329664953
2.2	1.245953205	1.246104819	1.246142713	1.246155344
2.4	1.169162994	1.169291515	1.169323639	1.169334346
2.6	1.097677870	1.097778523	1.097803683	1.097812069
2.8	1.030626179	1.030695880	1.030713305	1.030719114
3.0	0.967473721	0.967510790	0.967520062	0.967523153

3.2.18.

Improved Euler method				
x	$h = 0.2$	$h = 0.1$	$h = 0.05$	"Exact"
0.0	1.000000000	1.000000000	1.000000000	1.000000000
0.2	1.166771705	1.170394902	1.171244037	1.171515153
0.4	1.255835116	1.261642355	1.262958788	1.263370891
0.6	1.263517157	1.269528214	1.270846761	1.271251278
0.8	1.212551997	1.217531648	1.218585457	1.218901287
1.0	1.130812573	1.134420589	1.135150284	1.135362070
1.2	1.039104333	1.041487727	1.041938536	1.042062625
1.4	0.949440052	0.950888923	0.951132561	0.951192532
1.6	0.867475787	0.868263999	0.868364849	0.868381328
1.8	0.795183973	0.795523696	0.795530315	0.795518627
2.0	0.732679223	0.732721613	0.732667905	0.732638628

Improved Euler semilinear method				
x	$h = 0.2$	$h = 0.1$	$h = 0.05$	"Exact"
0.0	1.000000000	1.000000000	1.000000000	1.000000000
0.2	1.170617859	1.171292452	1.171459576	1.171515153
0.4	1.261629934	1.262938347	1.263262919	1.263370891
0.6	1.269173253	1.270734290	1.271122186	1.271251278
0.8	1.216926014	1.218409355	1.218778420	1.218901287
1.0	1.133688235	1.134944960	1.135257876	1.135362070
1.2	1.040721691	1.041728386	1.041979126	1.042062625
1.4	0.950145706	0.950931597	0.951127345	0.951192532
1.6	0.867573431	0.868179975	0.868331028	0.868381328
1.8	0.794899034	0.795364245	0.795480063	0.795518627
2.0	0.732166678	0.732521078	0.732609267	0.732638628

3.2.20.

Improved Euler method				
x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
0.0	1.000000000	1.000000000	1.000000000	1.000000000
0.1	0.749165000	0.747022742	0.746561141	0.746418339
0.2	0.565942699	0.562667885	0.561961242	0.561742917
0.3	0.428618351	0.424803657	0.423978964	0.423724207
0.4	0.324556426	0.320590918	0.319732571	0.319467408
0.5	0.245417735	0.241558658	0.240723019	0.240464879
0.6	0.185235654	0.181643813	0.180866303	0.180626161
0.7	0.139546094	0.136310496	0.135610749	0.135394692
0.8	0.104938506	0.102096319	0.101482503	0.101293057
0.9	0.078787731	0.076340645	0.075813072	0.075650324
1.0	0.059071894	0.056999028	0.056553023	0.056415515

Improved Euler semilinear method				
x	$h=0.1$	$h=0.05$	$h=0.025$	"Exact"
0.0	1.000000000	1.000000000	1.000000000	1.000000000
0.1	0.745595127	0.746215164	0.746368056	0.746418339
0.2	0.560827568	0.561515647	0.561686492	0.561742917
0.3	0.422922083	0.423524585	0.423674586	0.423724207
0.4	0.318820339	0.319306259	0.319427337	0.319467408
0.5	0.239962317	0.240339716	0.240433757	0.240464879
0.6	0.180243441	0.180530866	0.180602470	0.180626161
0.7	0.135106416	0.135322934	0.135376855	0.135394692
0.8	0.101077312	0.101239368	0.101279714	0.101293057
0.9	0.075489492	0.075610310	0.075640381	0.075650324
1.0	0.056295914	0.056385765	0.056408124	0.056415515

3.2.22.

Improved Euler method				
x	$h=0.1$	$h=0.05$	$h=0.025$	"Exact"
2.0	1.000000000	1.000000000	1.000000000	1.000000000
2.1	1.010250000	1.009185754	1.008965733	1.008899988
2.2	1.033547273	1.032105322	1.031811002	1.031723469
2.3	1.065562151	1.064135919	1.063849094	1.063764243
2.4	1.103145347	1.101926450	1.101685553	1.101614730
2.5	1.144085693	1.143140125	1.142957158	1.142903776
2.6	1.186878796	1.186202854	1.186075600	1.186038851
2.7	1.230530804	1.230088035	1.230007943	1.229985178
2.8	1.274404357	1.274147657	1.274104430	1.274092525
2.9	1.318104153	1.317987551	1.317971490	1.317967533
3.0	1.361395309	1.361379259	1.361382239	1.361383810

Improved Euler semilinear method				
x	$h=0.1$	$h=0.05$	$h=0.025$	"Exact"
2.0	1.000000000	1.000000000	1.000000000	1.000000000
2.1	1.012802674	1.009822081	1.009124116	1.008899988
2.2	1.038431870	1.033307426	1.032108359	1.031723469
2.3	1.072484834	1.065821457	1.064263950	1.063764243
2.4	1.111794329	1.104013534	1.102197168	1.101614730
2.5	1.154168041	1.145554968	1.143547198	1.142903776
2.6	1.198140189	1.188883373	1.186728849	1.186038851
2.7	1.242762459	1.232984559	1.230712361	1.229985178
2.8	1.287441845	1.277221941	1.274850828	1.274092525
2.9	1.331821976	1.321210992	1.318753047	1.317967533
3.0	1.375699933	1.364730937	1.362193997	1.361383810

3.2.24.

x	$h=0.1$	$h=0.05$	$h=0.025$	Exact
1.0	1.000000000	1.000000000	1.000000000	1.000000000
1.1	1.151019287	1.153270661	1.153777957	1.153937085
1.2	1.238798618	1.241884421	1.242580821	1.242799540
1.3	1.289296258	1.292573128	1.293313355	1.293546032
1.4	1.317686801	1.320866599	1.321585242	1.321811247
1.5	1.333073855	1.336036248	1.336705820	1.336916440
1.6	1.341027170	1.343732006	1.344343232	1.344535503
1.7	1.345001345	1.347446389	1.347998652	1.348172348
1.8	1.347155352	1.349355473	1.349852082	1.350008229
1.9	1.348839325	1.350816158	1.351261995	1.351402121
2.0	1.350890736	1.352667599	1.353067951	1.353193719

3.2.26.

x	$h = 0.05$	$h = 0.025$	$h = 0.0125$	Exact
1.00	2.000000000	2.000000000	2.000000000	2.000000000
1.05	2.268496358	2.269703943	2.270043628	2.270158103
1.10	2.582897367	2.585985695	2.586855275	2.587150838
1.15	2.954995034	2.960992388	2.962683751	2.963263785
1.20	3.400872342	3.411384294	3.414355862	3.415384615
1.25	3.943047906	3.960651794	3.965644965	3.967391304
1.30	4.614039436	4.643018510	4.651277424	4.654198473
1.35	5.462568051	5.510357362	5.524069547	5.528980892
1.40	6.564985580	6.645224236	6.668472955	6.676923077
1.45	8.047824947	8.187384679	8.228413044	8.243593315
1.50	10.136329642	10.393419681	10.470731411	10.500000000

3.2.28.

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
0.0	1.000000000	1.000000000	1.000000000	1.000000000
0.1	0.984142840	0.984133302	0.984130961	0.984130189
0.2	0.965066124	0.965044455	0.965039117	0.965037353
0.3	0.942648578	0.942611457	0.942602279	0.942599241
0.4	0.916705578	0.916648569	0.916634423	0.916629732
0.5	0.886970525	0.886887464	0.886866778	0.886859904
0.6	0.853066054	0.852948011	0.852918497	0.852908668
0.7	0.814458249	0.814291679	0.814249848	0.814235883
0.8	0.770380571	0.770143777	0.770083998	0.770063987
0.9	0.719699643	0.719355385	0.719267905	0.719238519
1.0	0.660658411	0.660136630	0.660002840	0.659957689

3.2.30.

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
1.0	0.000000000	0.000000000	0.000000000	0.000000000
1.1	-0.099666667	-0.099665005	-0.099664307	-0.099674132
1.2	-0.197322275	-0.197317894	-0.197316222	-0.197355914
1.3	-0.291033227	-0.291036361	-0.291036258	-0.291123993
1.4	-0.379131069	-0.379160444	-0.379166504	-0.379315647
1.5	-0.460350276	-0.460427667	-0.460445166	-0.460662347
1.6	-0.533897316	-0.534041581	-0.534075026	-0.534359685
1.7	-0.599446325	-0.599668984	-0.599721072	-0.600066382
1.8	-0.657076288	-0.657379719	-0.657450947	-0.657845646
1.9	-0.707175010	-0.707553135	-0.707641993	-0.708072516
2.0	-0.750335016	-0.750775571	-0.750879100	-0.751331499

3.2.32. **(a)** Let $x_i = a + ih$, $i = 0, 1, \dots, n$. If y is the solution of the initial value problem $y' = f(x)$, $y(a) = 0$, then $y(b) = \int_a^b f(x)\,dx$. The improved Euler method yields $y_{i+1} = y_i + .5h\,(f(a+ih) + f(a+(i+1)h))$, $i = 0, 1, \dots, n-1$, where $y_0 = a$ and y_n is an approximation to $\displaystyle\int_a^b f(x)\,dx$. But

$$y_n = \sum_{i=0}^{n-1}(y_{i+1} - y_i) = .5h\,(f(a) + f(b)) + h\sum_{i=1}^{n-1} f(a+ih).$$

(c) The local truncation error is a multiple of $y'''(\tilde{x}_i) = f''(\tilde{x}_i)$, where $x_i < \tilde{x}_i < x_{i+1}$. Therefore, the quadrature formula is exact if f is a polynomial of degree < 2.

(d) Let $E(f) = \displaystyle\int_a^b f(x)\,dx - y_n$. Note that E is linear. If f is a polynomial of degree 2, then

$f(x) = f_0(x) + K(x-a)^2$ where $\deg(f_0) \le 1$. Since $E(f_0) = 0$ from **(c)** and

$$\begin{aligned} E((x-a)^2) &= \frac{(b-a)^3}{3} - \frac{(b-a)^2 h}{2} - h^3 \sum_{i=1}^{n-1} i^2 \\ &= h^3 \left[\frac{n^3}{3} - \frac{n^2}{2} - \frac{n(n-1)(2n-1)}{6} \right] = -\frac{nh^3}{6} = -\frac{(b-a)h^2}{6}, \end{aligned}$$

$E(f) = -\dfrac{K(b-a)h^2}{6}$; therefore the error is proportional to h^2.

3.3 THE RUNGE–KUTTA METHOD

3.3.2. $y_1 = 1.221551366,\ y_2 = 1.492920208$

3.3.4. $y_1 = 2.961316248$; $y_2 = 2.920128958$.

3.3.6.

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	Exact
0.0	2.000000000	2.000000000	2.000000000	2.000000000
0.1	2.232752507	2.232649573	2.232643327	2.232642918
0.2	2.774582759	2.774366625	2.774353431	2.774352565
0.3	3.727068686	3.726710028	3.726688030	3.726686582
0.4	5.254817388	5.254263005	5.254228886	5.254226636
0.5	7.613077020	7.612241222	7.612189662	7.612186259
0.6	11.189806778	11.188557546	11.188480365	11.188475269
0.7	16.569088310	16.567225975	16.567110808	16.567103199
0.8	24.626206255	24.623431201	24.623259496	24.623248150
0.9	36.669848687	36.665712858	36.665456874	36.665439956
1.0	54.654509699	54.648344019	54.647962328	54.647937102

3.3.8.

x	$h = 0.05$	$h = 0.025$	$h = 0.0125$	Exact
1.00	2.000000000	2.000000000	2.000000000	2.000000000
1.05	2.270153785	2.270157806	2.270158083	2.270158103
1.10	2.587139846	2.587150083	2.587150789	2.587150838
1.15	2.963242415	2.963262317	2.963263689	2.963263785
1.20	3.415346864	3.415382020	3.415384445	3.415384615
1.25	3.967327077	3.967386886	3.967391015	3.967391304
1.30	4.654089950	4.654191000	4.654197983	4.654198473
1.35	5.528794615	5.528968045	5.528980049	5.528980892
1.40	6.676590929	6.676900116	6.676921569	6.676923077
1.45	8.242960669	8.243549415	8.243590428	8.243593315
1.50	10.498658198	10.499906266	10.499993820	10.500000000

3.3.10.

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	$h = 0.1$	$h = 0.05$	$h = 0.025$
1.0	1.000000000	1.000000000	1.000000000	0.000000000	0.0000000000	0.00000000000
1.1	0.923730622	0.923730677	0.923730681	-0.000000608	-0.0000000389	-0.00000000245
1.2	0.854443253	0.854443324	0.854443328	-0.000000819	-0.0000000529	-0.00000000335
1.3	0.791581155	0.791581218	0.791581222	-0.000000753	-0.0000000495	-0.00000000316
1.4	0.734680497	0.734680538	0.734680541	-0.000000523	-0.0000000359	-0.00000000233
1.5	0.683288034	0.683288051	0.683288052	-0.000000224	-0.0000000178	-0.00000000122
1.6	0.636937046	0.636937040	0.636937040	0.000000079	0.0000000006	-0.00000000009
1.7	0.595153053	0.595153029	0.595153028	0.000000351	0.0000000171	0.00000000093
1.8	0.557469558	0.557469522	0.557469520	0.000000578	0.0000000309	0.00000000179
1.9	0.523443129	0.523443084	0.523443081	0.000000760	0.0000000421	0.00000000248
2.0	0.492663789	0.492663738	0.492663736	0.000000902	0.0000000508	0.00000000302
	Approximate Solutions			Residuals		

3.3.12.

x	$h=0.1$	$h=0.05$	$h=0.025$	"Exact"
1.0	0.000000000	0.000000000	0.000000000	0.000000000
1.1	-0.099663901	-0.099663994	-0.099664000	-0.099664000
1.2	-0.197315322	-0.197315504	-0.197315516	-0.197315517
1.3	-0.291035700	-0.291035983	-0.291036001	-0.291036003
1.4	-0.379167790	-0.379168194	-0.379168220	-0.379168221
1.5	-0.460450005	-0.460450552	-0.460450587	-0.460450590
1.6	-0.534084875	-0.534085579	-0.534085623	-0.534085626
1.7	-0.599736802	-0.599737663	-0.599737717	-0.599737720
1.8	-0.657472724	-0.657473726	-0.657473788	-0.657473792
1.9	-0.707669346	-0.707670460	-0.707670529	-0.707670533
2.0	-0.750911103	-0.750912294	-0.750912367	-0.750912371

3.3.14.

Runge–Kutta method				
x	$h=0.1$	$h=0.05$	$h=0.025$	"Exact"
2.0	2.000000000	2.000000000	2.000000000	2.000000000
2.1	2.464113907	2.464119185	2.464119544	2.464119569
2.2	3.029389360	3.029402271	3.029403150	3.029403212
2.3	3.718384519	3.718408199	3.718409812	3.718409925
2.4	4.558632516	4.558671116	4.558673746	4.558673929
2.5	5.583745479	5.583804456	5.583808474	5.583808754
2.6	6.834762639	6.834849135	6.834855028	6.834855438
2.7	8.361796619	8.361919939	8.361928340	8.361928926
2.8	10.226043942	10.226216159	10.226227891	10.226228709
2.9	12.502240429	12.502477158	12.502493285	12.502494409
3.0	15.281660036	15.281981407	15.282003300	15.282004826

Runge–Kutta semilinear method				
x	$h=0.1$	$h=0.05$	$h=0.025$	"Exact"
2.0	2.000000000	2.000000000	2.000000000	2.000000000
2.1	2.464119623	2.464119573	2.464119570	2.464119569
2.2	3.029403325	3.029403219	3.029403212	3.029403212
2.3	3.718410105	3.718409936	3.718409925	3.718409925
2.4	4.558674188	4.558673945	4.558673930	4.558673929
2.5	5.583809105	5.583808776	5.583808755	5.583808754
2.6	6.834855899	6.834855467	6.834855440	6.834855438
2.7	8.361929516	8.361928963	8.361928928	8.361928926
2.8	10.226229456	10.226228756	10.226228712	10.226228709
2.9	12.502495345	12.502494468	12.502494413	12.502494409
3.0	15.282005990	15.282004899	15.282004831	15.282004826

3.3.16.

Runge–Kutta method				
x	$h=0.2$	$h=0.1$	$h=0.05$	"Exact"
1.0	2.000000000	2.000000000	2.000000000	2.000000000
1.2	1.801636785	1.801636775	1.801636774	1.801636774
1.4	1.650102633	1.650102617	1.650102616	1.650102616
1.6	1.526935904	1.526935886	1.526935885	1.526935885
1.8	1.422074302	1.422074284	1.422074283	1.422074283
2.0	1.329664970	1.329664954	1.329664953	1.329664953
2.2	1.246155357	1.246155345	1.246155344	1.246155344
2.4	1.169334355	1.169334347	1.169334346	1.169334346
2.6	1.097812074	1.097812070	1.097812069	1.097812069
2.8	1.030719113	1.030719114	1.030719114	1.030719114
3.0	0.967523147	0.967523152	0.967523153	0.967523153

Runge–Kutta semilinear method				
x	$h=0.2$	$h=0.1$	$h=0.05$	"Exact"
1.0	2.000000000	2.000000000	2.000000000	2.000000000
1.2	1.801636785	1.801636775	1.801636774	1.801636774
1.4	1.650102633	1.650102617	1.650102616	1.650102616
1.6	1.526935904	1.526935886	1.526935885	1.526935885
1.8	1.422074302	1.422074284	1.422074283	1.422074283
2.0	1.329664970	1.329664954	1.329664953	1.329664953
2.2	1.246155357	1.246155345	1.246155344	1.246155344
2.4	1.169334355	1.169334347	1.169334346	1.169334346
2.6	1.097812074	1.097812070	1.097812069	1.097812069
2.8	1.030719113	1.030719114	1.030719114	1.030719114
3.0	0.967523147	0.967523152	0.967523153	0.967523153

3.3.18.

Runge–Kutta method				
x	$h=0.2$	$h=0.1$	$h=0.05$	"Exact"
0.0	1.000000000	1.000000000	1.000000000	1.000000000
0.2	1.171515610	1.171515156	1.171515152	1.171515153
0.4	1.263365845	1.263370556	1.263370869	1.263370891
0.6	1.271238957	1.271250529	1.271251232	1.271251278
0.8	1.218885528	1.218900353	1.218901230	1.218901287
1.0	1.135346772	1.135361174	1.135362016	1.135362070
1.2	1.042049558	1.042061864	1.042062579	1.042062625
1.4	0.951181964	0.951191920	0.951192495	0.951192532
1.6	0.868372923	0.868380842	0.868381298	0.868381328
1.8	0.795511927	0.795518241	0.795518603	0.795518627
2.0	0.732633229	0.732638318	0.732638609	0.732638628

Runge–Kutta semilinear method				
x	$h=0.2$	$h=0.1$	$h=0.05$	"Exact"
0.0	1.000000000	1.000000000	1.000000000	1.000000000
0.2	1.171517316	1.171515284	1.171515161	1.171515153
0.4	1.263374485	1.263371110	1.263370904	1.263370891
0.6	1.271254636	1.271251485	1.271251291	1.271251278
0.8	1.218903802	1.218901442	1.218901297	1.218901287
1.0	1.135363869	1.135362181	1.135362077	1.135362070
1.2	1.042063952	1.042062706	1.042062630	1.042062625
1.4	0.951193560	0.951192595	0.951192536	0.951192532
1.6	0.868382157	0.868381378	0.868381331	0.868381328
1.8	0.795519315	0.795518669	0.795518629	0.795518627
2.0	0.732639212	0.732638663	0.732638630	0.732638628

3.3.20.

Runge–Kutta method				
x	$h=0.1$	$h=0.05$	$h=0.025$	"Exact"
0.0	1.000000000	1.000000000	1.000000000	1.000000000
0.1	0.746430962	0.746418992	0.746418376	0.746418339
0.2	0.561761987	0.561743921	0.561742975	0.561742917
0.3	0.423746057	0.423725371	0.423724274	0.423724207
0.4	0.319489811	0.319468612	0.319467478	0.319467408
0.5	0.240486460	0.240466046	0.240464947	0.240464879
0.6	0.180646105	0.180627244	0.180626225	0.180626161
0.7	0.135412569	0.135395665	0.135394749	0.135394692
0.8	0.101308709	0.101293911	0.101293107	0.101293057
0.9	0.075663769	0.075651059	0.075650367	0.075650324
1.0	0.056426886	0.056416137	0.056415552	0.056415515

Runge–Kutta semilinear method				
x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
0.0	1.000000000	1.000000000	1.000000000	1.000000000
0.1	0.746416306	0.746418217	0.746418332	0.746418339
0.2	0.561740647	0.561742780	0.561742908	0.561742917
0.3	0.423722193	0.423724084	0.423724199	0.423724207
0.4	0.319465760	0.319467308	0.319467402	0.319467408
0.5	0.240463579	0.240464800	0.240464874	0.240464879
0.6	0.180625156	0.180626100	0.180626158	0.180626161
0.7	0.135393924	0.135394645	0.135394689	0.135394692
0.8	0.101292474	0.101293021	0.101293055	0.101293057
0.9	0.075649884	0.075650297	0.075650322	0.075650324
1.0	0.056415185	0.056415495	0.056415514	0.056415515

3.3.22.

Runge–Kutta method				
x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
2.0	1.000000000	1.000000000	1.000000000	1.000000000
2.1	1.008912398	1.008900636	1.008900025	1.008899988
2.2	1.031740789	1.031724368	1.031723520	1.031723469
2.3	1.063781819	1.063765150	1.063764295	1.063764243
2.4	1.101630085	1.101615517	1.101614774	1.101614730
2.5	1.142915917	1.142904393	1.142903811	1.142903776
2.6	1.186047678	1.186039295	1.186038876	1.186038851
2.7	1.229991054	1.229985469	1.229985194	1.229985178
2.8	1.274095992	1.274092692	1.274092535	1.274092525
2.9	1.317969153	1.317967605	1.317967537	1.317967533
3.0	1.361384082	1.361383812	1.361383809	1.361383810

Runge–Kutta semilinear method				
x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
2.0	1.000000000	1.000000000	1.000000000	1.000000000
2.1	1.008913934	1.008900843	1.008900041	1.008899988
2.2	1.031748526	1.031725001	1.031723564	1.031723469
2.3	1.063798300	1.063766321	1.063764371	1.063764243
2.4	1.101656264	1.101617259	1.101614886	1.101614730
2.5	1.142951721	1.142906691	1.142903955	1.142903776
2.6	1.186092475	1.186042105	1.186039051	1.186038851
2.7	1.230043983	1.229988742	1.229985397	1.229985178
2.8	1.274156172	1.274096377	1.274092762	1.274092525
2.9	1.318035787	1.317971658	1.317967787	1.317967533
3.0	1.361456502	1.361388196	1.361384079	1.361383810

3.3.24.

x	$h = .1$	$h = .05$	$h = .025$	Exact
1.00	0.142854841	0.142857001	0.142857134	0.142857143
1.10	0.053340745	0.053341989	0.053342066	0.053342071
1.20	-0.046154629	-0.046153895	-0.046153849	-0.046153846
1.30	-0.153363206	-0.153362764	-0.153362736	-0.153362734
1.40	-0.266397049	-0.266396779	-0.266396762	-0.266396761
1.50	-0.383721107	-0.383720941	-0.383720931	-0.383720930
1.60	-0.504109696	-0.504109596	-0.504109589	-0.504109589
1.70	-0.626598326	-0.626598268	-0.626598264	-0.626598264
1.80	-0.750437351	-0.750437320	-0.750437318	-0.750437318
1.90	-0.875050587	-0.875050574	-0.875050573	-0.875050573
2.00	-1.000000000	-1.000000000	-1.000000000	-1.000000000

3.3.26.

x	$h = .1$	$h = .05$	$h = .025$	Exact
0.50	-8.954103230	-8.954063245	-8.954060698	-8.954060528
0.60	-5.059648314	-5.059633293	-5.059632341	-5.059632277
0.70	-2.516755942	-2.516749850	-2.516749465	-2.516749439
0.80	-0.752508672	-0.752506238	-0.752506084	-0.752506074
0.90	0.530528482	0.530529270	0.530529319	0.530529323
1.00	1.500000000	1.500000000	1.500000000	1.500000000
1.10	2.256519743	2.256519352	2.256519328	2.256519326
1.20	2.863543039	2.863542454	2.863542417	2.863542415
1.30	3.362731379	3.362730700	3.362730658	3.362730655
1.40	3.782361948	3.782361231	3.782361186	3.782361183
1.50	4.142171279	4.142170553	4.142170508	4.142170505

3.3.28. **(a)** Let $x_i = a + ih$, $i = 0, 1, \dots, n$. If y is the solution of the initial value problem $y' = f(x)$, $y(a) = 0$, then $y(b) = \int_a^b f(x)\,dx$. The Runge-Kutta method yields $y_{i+1} = y_i + \frac{h}{6}(f(a+ih) + 4f(a+(2i+1)h/2) + f(a+(i+1)h))$, $i = 0, 1, \dots, n-1$, where $y_0 = a$ and y_n is an approximation to $\displaystyle\int_a^b f(x)\,dx$. But

$$y_n = \sum_{i=0}^{n-1}(y_{i+1} - y_i) = \frac{h}{6}(f(a) + f(b)) + \frac{h}{3}\sum_{i=1}^{n-1} f(a+ih) + \frac{2h}{3}\sum_{i=1}^{n} f\left(a + (2i-1)h/2\right).$$

(c) The local truncation error is a multiple of $y^{(5)}(\tilde{x}_i) = f^{(4)}(\tilde{x}_i)$, where $x_i < \tilde{x}_i < x_{i+1}$. Therefore, the quadrature formula is exact if f is a polynomial of degree < 4.

(d) Let $E(f) = \displaystyle\int_a^b f(x)\,dx - y_n$. Note that E is linear. If f is a polynomial of degree 4, then $f(x) = f_0(x) + K(x-a)^4$ where $\deg(f_0) \le 3$ and K is constant. Since $E(f_0) = 0$ from **(c)** and

$$\begin{aligned} E((x-a)^4) &= \frac{(b-a)^5}{5} - \frac{(b-a)^4 h}{6} - \frac{h^5}{3}\sum_{i=1}^{n-1} i^4 - \frac{2h^5}{3}\sum_{i=1}^{n}(i-1/2)^4 \\ &= h^5\left[\frac{n^5}{5} - \frac{n^4}{6} - \left(\frac{n^5}{15} - \frac{n^4}{6} + \frac{n^3}{9} - \frac{n}{90}\right) - \left(\frac{2n^5}{15} - \frac{n^3}{9} + \frac{7n}{360}\right)\right] \\ &= -\frac{nh^5}{120} = -\frac{(b-a)h^4}{120}, \end{aligned}$$

$E(f) = -\dfrac{(b-a)h^4}{120}$; thus, the error is proportional to h^4.

CHAPTER 4
Applications of First Order Equations

4.1 GROWTH AND DECAY

4.1.2. $k\tau = \ln 2$ and $\tau = 2 \Rightarrow k = \dfrac{\ln 2}{2}$; $Q(t) = Q_0 e^{-t\ln 2/2}$; if $Q(T) = \dfrac{Q_0}{10}$, then $\dfrac{Q_0}{10} = Q_0 e^{-T\ln 2/2}$; $\ln 10 = \dfrac{T\ln 2}{2}$; $T = \dfrac{2\ln 10}{\ln 2}$ days.

4.1.4. Let t_1 be the elapsed time since the tree died. Since $p(t) = e^{-(t\ln 2)\tau}$, it follows that $p_1 = p_0 e^{-(t_1\ln 2)/\tau}$, so $\ln\left(\dfrac{p_1}{p_0}\right) = -\dfrac{t_1}{\tau}\ln 2$ and $t_1 = \tau\dfrac{\ln(p_0/p_1)}{\ln 2}$.

4.1.6. $Q = Q_0 e^{-kt}$; $Q_1 = Q_0 e^{-kt_1}$; $Q_2 = Q_0 e^{-kt_2}$; $\dfrac{Q_2}{Q_1} = e^{-k(t_2-t_1)}$; $\ln\left(\dfrac{Q_1}{Q_2}\right) = k(t_2 - t_1)$; $k = \dfrac{1}{t_2 - t_1}\ln\left(\dfrac{Q_1}{Q_2}\right)$.

4.1.8. $Q' = .06Q,\ Q(0) = Q_0$; $Q = Q_0 e^{.06t}$. We must find τ such that $Q(\tau) = 2Q_0$; that is, $Q_0 e^{.06\tau} = 2Q_0$, so $.06\tau = \ln 2$ and $\tau = \dfrac{\ln 2}{.06} = \dfrac{50\ln 2}{3}$ yr.

4.1.10. **(a)** If T is the time to triple the value, then $Q(T) = Q_0 e^{.05T} = 3Q_0$, so $e^{.05T} = 3$. Therefore, $.05T = \ln 3$ and $T = 20\ln 3$.

(b) If $Q(10) = 100000$, then $Q_0 e^{.5} = 100000$, so $Q_0 = 100000e^{-.5}$

4.1.12. $Q' = -\dfrac{Q^2}{2},\ Q(0) = 50$; $\dfrac{Q'}{Q^2} = -\dfrac{1}{2}; -\dfrac{1}{Q} = -\dfrac{t}{2} + c$; $Q(0) = 50 \Rightarrow c = -\dfrac{1}{50}$; $\dfrac{1}{Q} = \dfrac{t}{2} + \dfrac{1}{50} = \dfrac{1+25t}{50}$; $Q = \dfrac{50}{1+25t}$. Now $Q(T) = 25 \Rightarrow 1 + 25T = 2 \Rightarrow 25T = 1 \Rightarrow T = \dfrac{1}{25}$ years.

4.1.14. Since $\tau = 1500, k = \dfrac{\ln 2}{1500}$; hence $Q = Q_0 e^{-(t\ln 2)/1500}$. If $Q(t_1) = \dfrac{3Q_0}{4}$, then $e^{-(t_1\ln 2)/1500} = \dfrac{3}{4}$; $-t_1\dfrac{\ln 2}{1500} = \ln\left(\dfrac{3}{4}\right) = -\ln\left(\dfrac{4}{3}\right)$; $t_1 = 1500\dfrac{\ln\left(\frac{4}{3}\right)}{\ln 2}$. Finally, $Q(2000) = Q_0 e^{-\frac{4}{3}\ln 2} = 2^{-4/3}Q_0$.

4.1.16. (A) $S' = 1 - \frac{S}{10}$, $S(0) = 20$. Rewrite the differential equation in (A) as (B) $S' + \frac{S}{10} = 1$. Since $S_1 = e^{-t/10}$ is a solution of the complementary equation, the solutions of (B) are given by $S = ue^{-t/10}$, where $u'e^{-t/10} = 1$. Therefore, $u' = e^{t/10}$; $u = 10e^{t/10} + c$; $S = 10 + ce^{-t/10}$. Now $S(0) = 20 \Rightarrow c = 10$, so $S = 10 + 10e^{-t/10}$ and $\lim_{t\to\infty} S(t) = 10$ g.

4.1.18. (A) $V' = -750 + \frac{V}{20}$, $V(0) = 25000$. Rewrite the differential equation in (A) as (B) $V' - \frac{V}{20} = -750$. Since $V_1 = e^{t/20}$ is a solution of the complementary equation, the solutions of (B) are given by $V = ue^{t/20}$, where $u'e^{t/20} = -750$. Therefore, $u' = -750e^{-t/20}$; $u = 15000e^{-t/20} + c$; $V = 15000 + ce^{t/20}$; $V(0) = 25000 \Rightarrow c = 10000$. Therefore, $V = 15000 + 10000e^{t/20}$.

4.1.20. $p' = \frac{p}{2} - \frac{p^2}{8} = -\frac{1}{8}p(p-4)$; $\frac{p'}{p(p-4)} = -\frac{1}{8}$; $\frac{1}{4}\left[\frac{1}{p-4} - \frac{1}{p}\right]p' = -\frac{1}{8}$; $\left[\frac{1}{p-4} - \frac{1}{p}\right]p' = -\frac{1}{2}$; $\left|\frac{p-4}{p}\right] = -\frac{t}{2} + k$; $\frac{p-4}{p} = ce^{-t/2}$; $p(0) = 100 \Rightarrow c = \frac{24}{25}$; $\frac{p-4}{p} = \frac{24}{25}e^{-t/2}$; $p - 4 = \frac{24}{25}pe^{-t/2}$; $p\left(1 - \frac{24}{25}e^{-t/2}\right) = 4$; $p = \frac{4}{1 - \frac{24}{25}e^{-t/2}} = \frac{100}{24 - 24e^{-t/2}}$.

4.1.22. **(a)** $P' = rP - 12M$.

(b) $P = ue^{rt}$; $u'e^{rt} = -12M$; $u' = -12Me^{rt}$; $u = \frac{12M}{r}e^{-rt} + c$; $P = \frac{12M}{r} + ce^{rt}$; $P(0) = P_0 \Rightarrow c = P_0 - \frac{12M}{r}$; $P = \frac{12M}{r}(1 - e^{rt}) + P_0e^{rt}$.

(c) Since $P(N) = 0$, the answer to **(b)** implies that $M = \frac{rP_0}{12(1 - e^{-rN})}$

4.1.24. The researcher's salary is the solution of the initial value problem $S' = aS$, $S(0) = S_0$. Therefore, $S = S_0e^{at}$. If $P = P(t)$ is the value of the trust fund, then $P' = -S_0e^{at} + rP$, or $P' - rP = -S_0e^{at}$. Therefore, (A) $P = ue^{rt}$, where $u'e^{rt} = -S_0e^{at}$, so (B) $u' = -S_0e^{(a-r)t}$. If $a \neq r$, then (B) implies that $u = \frac{S_0}{r-a}e^{(a-r)t} + c$, so (A) implies that $P = \frac{S_0}{r-a}e^{at} + ce^{rt}$. Now $P(0) = P_0 \Rightarrow c = P_0 - \frac{S_0}{r-a}$; therefore $P = \frac{S_0}{r-a}e^{at} + \left(P_0 - \frac{S_0}{r-a}\right)e^{rt}$. We must choose P_0 so that $P(T) = 0$; that is, $P = \frac{S_0}{r-a}e^{aT} + \left(P_0 - \frac{S_0}{r-a}\right)e^{rT} = 0$. Solving this for P_0 yields $P_0 = \frac{S_0(1 - e^{(a-r)T})}{r-a}$. If $a = r$, then (B) becomes $u' = -S_0$, so $u = -S_0t + c$ and (A) implies that $P = (-S_0t + c)e^{rt}$. Now $P(0) = P_0 \Rightarrow c = P_0$; therefore $P = (-S_0t + P_0)e^{rt}$. To make $P(T) = 0$ we must take $P_0 = S_0T$.

4.1.26. $Q' = \frac{at}{1 + btQ^2} - kQ$; $\lim_{t\to\infty} Q(t) = (a/bk)^{1/3}$.

4.2 COOLING AND MIXING

4.2.2. Since $T_0 = 100$ and $T_M = -10$, $T = -10 + 110e^{-kt}$. Now $T(1) = 80 \Rightarrow 80 = -10 + 110e^{-k}$, so $e^{-k} = \frac{9}{11}$ and $k = \ln\frac{11}{9}$. Therefore, $T = -10 + 110e^{-t\ln\frac{11}{9}}$.

4.2.4. Let T be the thermometer reading. Since $T_0 = 212$ and $T_M = 70$, $T = 70 + 142e^{-kt}$. Now $T(2) = 125 \Rightarrow 125 = 70 + 142e^{-2k}$, so $e^{-2k} = \frac{55}{142}$ and $k = \frac{1}{2}\ln\frac{142}{55}$. Therefore, (A) $T =$

$70 + 142e^{-\frac{t}{2}\ln\frac{142}{55}}$.

(a) $T(2) = 70 + 142e^{-2\ln\frac{142}{55}} = 70 + 142\left(\dfrac{55}{142}\right)^2 \approx 91.30°\text{F}$.

(b) Let τ be the time when $T(\tau) = 72$, so $72 = 70 + 142e^{-\frac{\tau}{2}\ln\frac{142}{55}}$, or $e^{-\frac{\tau}{2}\ln\frac{142}{55}} = \dfrac{1}{71}$. Therefore, $\tau = 2\dfrac{\ln 71}{\ln\frac{142}{55}} \approx 8.99$ min.

(c) Since (A) implies that $T > 70$ for all $t > 0$, the thermometer will never read 69°F.

4.2.6. Since $T_M = 20$, $T = 20 + (T_0 - 20)e^{-kt}$. Now $T_0 - 5 = 20 + (T_0 - 20)e^{-4k}$ and $T_0 - 7 = 20 + (T_0 - 20)e^{-8k}$. Therefore, $\dfrac{T_0 - 25}{T_0 - 20} = e^{-4k}$ and $\left(\dfrac{T_0 - 27}{T_0 - 20}\right) = e^{-8k}$, so $\dfrac{T_0 - 27}{T_0 - 20} = \left(\dfrac{T_0 - 25}{T_0 - 20}\right)^2$, which implies that $(T_0 - 20)(T_0 - 27) = (T_0 - 25)^2$, or $T_0^2 - 47T_0 + 540 = T_0^2 - 50T_0 + 625$; hence $3T_0 = 85$ and $T_0 = (85/3)°C$.

4.2.8. $Q' = 3 - \dfrac{3}{40}Q$, $\quad Q(0) = 0$. Rewrite the differential equation as (A) $Q' + \dfrac{3}{40}Q = 3$. Since $Q_1 = e^{-3t/40}$ is a solution of the complementary equation, the solutions of (A) are given by $Q = ue^{-3t/40}$ where $u'e^{-3t/40} = 3$. Therefore,$u' = 3e^{3t/40}$, $u = 40e^{3t/40} + c$, and $Q = 40 + ce^{-3t/40}$. Now $Q(0) = 0 \Rightarrow c = -40$, so $Q = 40(1 - e^{-3t/40})$.

4.2.10. $Q' = \dfrac{3}{2} - \dfrac{Q}{20}$, $\quad Q(0) = 10$. Rewrite the differential equation as (A) $Q' + \dfrac{Q}{20} = \dfrac{3}{2}$. Since $Q_1 = e^{-t/20}$ is a solution of the complementary equation, the solutions of (A) are given by $Q = ue^{-t/20}$ where $u'e^{-t/20} = \dfrac{3}{2}$. Therefore,$u' = \dfrac{3}{2}e^{t/20}$, $u = 30e^{t/20} + c$, and $Q = 30 + ce^{-t/20}$. Now $Q(0) = 10 \Rightarrow c = -20$, so $Q = 30 - 20e^{-t/20}$ and $K = \dfrac{Q}{100} = .3 - .2e^{-t/20}$.

4.2.12. $Q' = 10 - \dfrac{Q}{5}$, or (A) $Q' + \dfrac{Q}{5} = 10$. Since $Q_1 = e^{-t/5}$ is a solution of the complementary equation, the solutions of (A) are given by $Q = ue^{-t/5}$ where $u'e^{-t/5} = 10$. Therefore,$u' = 10e^{t/5}$, $u = 50e^{t/10} + c$, and $Q = 50 + ce^{-t/5}$. Since $\lim_{t\to\infty} Q(t) = 50$, the mininum capacity is 50 gallons.

4.2.14. Since there are $2t + 600$ gallons of mixture in the tank at time t and mixture is being drained at 4 gallons/min, $Q' = 3 - \dfrac{2}{t + 300}Q$, $\quad Q(0) = 40$. Rewrite the differential equation as (A) $Q' + \dfrac{2}{t + 300}Q = 3$. Since $Q_1 = \dfrac{1}{(t + 300)^2}$ is a solution of the complementary equation, the solutions of (A) are given by $Q = \dfrac{u}{(t + 300)^2}$ where $\dfrac{u'}{(t + 300)^2} = 3$. Therefore, $u' = 3(t + 300)^2$, $u = (t + 300)^3 + c$, and $Q = t + 300 + \dfrac{c}{(t + 300)^2}$. Now $Q(0) = 40 \Rightarrow c = -234 \times 10^5$, so $Q = t + 300 - \dfrac{234 \times 10^5}{(t + 300)^2}$, $0 \le t \le 300$.

4.2.16. **(a)** $S' = -k_m(S - T_m)$, $S(0) = 0$, so (A) $S = T_m + (S_0 - T_m)e^{-k_m t}$. $T' = -k(T - S) = -k\left(T - T_m - (S_0 - T_m)e^{-k_m t}\right)$, from (A). Therefore,$T' + kT = kT_m + k(S_0 - T_m)e^{-k_m t}$; $T = ue^{-kt}$; (B) $u' = kT_m e^{kt} + k(S_0 - T_m)e^{(k-k_m)t}$; $u = T_m e^{kt} + \dfrac{k}{k - k_m}(S_0 - T_m)e^{(k-k_m)t} + c$; $T(0) = T_0 \Rightarrow c = T_0 - T_m - \dfrac{k}{k - k_m}(S_0 - T_m)$; $u = T_m e^{kt} + \dfrac{k}{k - k_m}(S_0 - T_m)e^{(k-k_m)t} + T_0 - T_m - \dfrac{k}{k - k_m}(S_0 - T_m)$;

$T = T_m + (T_0 - T_m)e^{-kt} + \dfrac{k(S_0 - T_m)}{(k - k_m)}\left(e^{-k_m t} - e^{-kt}\right).$

(b) If $k = k_m$ (B) becomes (B) $u' = kT_m e^{kt} + k(S_0 - T_m)$; $u = T_m e^{kt} + k(S_0 - T_m)t + c$; $T(0) = T_0 \Rightarrow c = T_0 - T_m$; $u = T_m e^{kt} + k(S_0 - T_m)t + (T_0 - T_m)$; $T = T_m + k(S_0 - T_m)te^{-kt} + (T_0 - T_m)e^{-kt}$.

(c) $\lim_{t\to\infty} T(t) = \lim_{t\to\infty} S(t) = T_m$ in either case.

4.2.18. $V' = aV - bV^2 = -bV(V - b/a)$; $\dfrac{V'}{V(V - a/b)} = -b$; $\left[\dfrac{1}{V - a/b} - \dfrac{1}{V}\right]V' = -a$; $\ln\left|\dfrac{V - a/b}{V}\right| = -at + k$; (A) $\dfrac{V - a/b}{V} = ce^{-at}$; (B) $V = \dfrac{a}{b}\dfrac{1}{1 - ce^{-at}}$. Since $V(0) = V_0$, (A) $\Rightarrow c = \dfrac{V_0 - a/b}{V_0}$. Substituting this into (B) yields $V = \dfrac{a}{b}\dfrac{V_0}{V_0 - (V_0 - a/b)\,e^{-at}}$ so $\lim_{t\to\infty} V(t) = a/b$

4.2.20. If $Q_n(t)$ is the number of pounds of salt in T_n at time t, then $Q'_{n+1} + \dfrac{r}{W}Q_{n+1} = rc_n(t)$, $n = 0, 1, \dots,$ where $c_0(t) \equiv c$. Therefore,$Q_{n+1} = u_{n+1}e^{-rt/w}$; (A) $u'_{n+1} = re^{rt/W}c_n(t)$. In particular, with $n = 0$, $u_1 = cW(e^{rt/W} - 1)$, so $Q_1 = cW(1 - e^{-rt/W})$ and $c_1 = c(1 - e^{-rt/W})$. We will shown by induction that $c_n = c\left(1 - e^{-rt/W}\displaystyle\sum_{j=0}^{n-1}\frac{1}{j!}\left(\frac{rt}{W}\right)^j\right)$. This is true for $n = 1$; if it is true for a given n, then, from (A),

$$u'_{n+1} = cre^{rt/W}\left(1 - e^{-rt/W}\sum_{j=0}^{n-1}\frac{1}{j!}\left(\frac{rt}{W}\right)^j\right) = cre^{rt/W} - cr\sum_{j=0}^{n-1}\frac{1}{j!}\left(\frac{rt}{W}\right)^j,$$

so (since $Q_{n+1}(0) = 0$),

$$u_{n+1} = cW(e^{rt/W} - 1) - c\sum_{j=0}^{n-1}\frac{1}{(j+1)!}\frac{r^{j+1}}{W^j}t^{j+1}.$$

Therefore,

$$c_{n+1} = \frac{1}{W}u_{n+1}e^{-rt/W} = c\left(1 - e^{-rt/W}\sum_{j=0}^{n}\frac{1}{j!}\left(\frac{rt}{W}\right)^j\right),$$

which completes the induction. From this, $\lim_{t\to\infty} c_n(t) = c$.

4.2.22. Since the incoming solution contains 1/2 lb of salt per gallon and there are always 600 gallons in the tank, we conclude intuitively that $\lim_{t\to\infty} Q(t) = 300$. To verify this rigorously, note that $Q_1(t) = \exp\left(-\dfrac{1}{150}\displaystyle\int_0^t a(\tau)\,d\tau\right)$ is a solution of the complementary equation, (A) $Q_1(0) = 1$, and (B) $\lim_{t\to\infty} Q_1(t) = 0$ (since $\lim_{t\to\infty} a(t) = 1$). Therefore,$Q = Q_1 u$; $Q_1 u' = 2$; $u' = \dfrac{2}{Q_1}$; $u = Q_0 + 2\displaystyle\int_0^t \frac{d\tau}{Q_1(\tau)}$ (see (A)), and $Q(t) = Q_0 Q_1(t) + 2Q_1(t)\displaystyle\int_0^t \frac{d\tau}{Q_1(\tau)}$. From (B), $\displaystyle\lim_{t\to\infty} Q(t) = 2\lim_{t\to\infty} Q_1(t)\int_0^t \frac{d\tau}{Q_1(\tau)}$, a $0\cdot\infty$ indeterminate form. By L'Hospital's rule, $\displaystyle\lim_{t\to\infty} Q(t) = 2\lim_{t\to\infty}\frac{1}{Q_1(t)}\Big/\left(\frac{-Q_1'(t)}{Q_1^2(t)}\right) = -2\lim_{t\to\infty}\frac{Q_1(t)}{Q_1'(t)} = 300.$

vspace*10pt

4.3 ELEMENTARY MECHANICS

4.3.2. The firefighter's mass is $m = \dfrac{192}{32} = 6$ sl, so $6v' = -192 - kv$, or (A) $v' + \dfrac{k}{6}v = -32$. Since $v_1 = e^{-kt/6}$ is a solution of the complementary equation, the solutions of (A) are $v = ue^{-kt/6}$ where $u'e^{-kt/6} = -32$. Therefore,$u' = -32e^{kt/6}$; $u = -\dfrac{192}{k}e^{kt/6} + c$; $v = -\dfrac{192}{k} + ce^{-kt/6}$. Now $v(0) = 0 \Rightarrow c = \dfrac{192}{k}$. Therefore,$v = -\dfrac{192}{k}(1 - e^{-kt/6})$ and $\lim_{t\to\infty} v(t) = -\dfrac{192}{k} = -16$ ft/s, so $k = 12$ lb-s/ft and $v = -16(1 - e^{-2t})$.

4.3.3. $m = \dfrac{64000}{32} = 2000$, so $2000v' = 50000 - 2000v$, or (A) $v' + v = 25$. Since $v_1 = e^{-t}$ is a solution of the complementary equation, the solutions of (A) are $v = ue^{-t}$ where $u'e^{-t} = 25$. Therefore,$u' = 25e^t$; $u = 25e^t + c$; $v = 25 + ce^{-t}$. Now $v(0) = 0 \Rightarrow c = -25$. Therefore,$v = 25(1 - e^{-t})$ and $\lim_{t\to\infty} v(t) = 25$ ft/s.

4.3.4. $20v' = 10 - \dfrac{1}{2}v$, or (A) $v' + \dfrac{1}{20}v = \dfrac{1}{2}$. Since $v_1 = e^{-t/40}$ is a solution of the complementary equation, the solutions of (A) are $v = ue^{-t/40}$ where $u'e^{-t/40} = \dfrac{1}{2}$. Therefore,$u' = \dfrac{e^{t/40}}{2}$; $u = 20e^{t/40} + c$; $v = 20 + ce^{-t/40}$. Now $v(0) = -7 \Rightarrow c = -27$. Therefore,$v = 20 - 27e^{-t/40}$.

4.3.6. $m = \dfrac{3200}{32} = 100$ sl. The component of the gravitational force in the direction of motion is $-3200\cos(\pi/3) = -1600$ lb. Therefore, $100v' = -1600 + v^2$. Separating variables yields $\dfrac{v'}{(v-40)(v+40)} = \dfrac{1}{100}$, or $\left[\dfrac{1}{v-40} - \dfrac{1}{v+40}\right] = \dfrac{4}{5}$. Therefore, $\ln\left|\dfrac{v-40}{v+40}\right| = \dfrac{4t}{5} + k$ and $\dfrac{v-40}{v+40} = ce^{4t/5}$. Now $v(0) = -64 \Rightarrow c = \dfrac{13}{3}$; therefore $\dfrac{v-40}{v+40} = \dfrac{13e^{4t/5}}{3}$, so $v = \dfrac{40(3 + 13e^{4t/5})}{3 - 13e^{4t/5}}$, or $v = -\dfrac{40(13 + 3e^{-4t/5})}{13 - 3e^{-4t/5}}$.

4.3.8. From Example 4.3.1, (A) $v = -\dfrac{mg}{k} + \left(v_0 + \dfrac{mg}{k}\right)e^{-kt/m}$. Integrating this yields (B) $y = -\dfrac{mgt}{k} - \dfrac{m}{k}\left(v_0 + \dfrac{mg}{k}\right)e^{-kt/m} + c$. Now $y(0) = y_0 \Rightarrow c = y_0 + \dfrac{m}{k}\left(v_0 + \dfrac{mg}{k}\right)$. Substituting this into (B) yields

$$\begin{aligned} y &= -\frac{mgt}{k} - \frac{m}{k}\left(v_0 + \frac{mg}{k}\right)e^{-kt/m} + y_0 + \frac{m}{k}\left(v_0 + \frac{mg}{k}\right) \\ &= y_0 + \frac{m}{k}\left(v_0 - gt + \frac{mg}{k} - \left(v_0 + \frac{mg}{k}\right)e^{-kt/m}\right) \\ &= y_0 + \frac{m}{k}(v_0 - v - gt) \end{aligned}$$

where the last equality follows from (A).

4.3.10. $m = \dfrac{256}{32} = 8$ sl. Since the resisting force is 1 lb when $|v| = 4$ ft/s, $k = \dfrac{1}{16}$. Therefore, $8v' = -256 + \dfrac{1}{16}v^2 = \dfrac{1}{16}\left(v^2 - (64)^2\right)$. Separating variables yields $\dfrac{v'}{(v-64)(v+64)} = \dfrac{1}{128}$, or $\left[\dfrac{1}{v-64} - \dfrac{1}{v+64}\right]v' = 1$. Therefore,$\ln\left|\dfrac{v-64}{v+64}\right| = t + k$ and $\dfrac{v-64}{v+64} = ce^t$. Now $v(0) = 0 \Rightarrow c = -1$; therefore $\dfrac{v-64}{v+64} = -e^t$, so $v = \dfrac{64(1-e^t)}{1+e^t}$, or $v = -\dfrac{64(1-e^{-t})}{1+e^{-t}}$. Therefore, $\lim_{t\to\infty} v(t) = -64$.

4.3.12. **(a)** $mv' = -mg - kv^2 = -mg(1+\gamma^2v^2)$, where $\gamma = \sqrt{\frac{k}{mg}}$. Therefore,(A) $\frac{v'}{1+\gamma^2v^2} = -g$. With the substitution $u = \gamma v$, $\int \frac{dv}{1+\gamma^2v^2} = \frac{1}{\gamma}\int \frac{du}{1+u^2} = \frac{1}{\gamma}\tan^{-1}u = \frac{1}{\gamma}\tan^{-1}(\gamma v)$. Therefore, $\frac{1}{\gamma}\tan^{-1}(\gamma v) = -gt + c$. Now $v(0) = v_0 \Rightarrow c = \frac{1}{\gamma}\tan^{-1}(\gamma v_0)$, so $\frac{1}{\gamma}\tan^{-1}(\gamma v) = -gt + \frac{1}{\gamma}\tan^{-1}(\gamma v_0)$. Since $v(T) = 0$, it follows that $T = \frac{1}{\gamma g}\tan^{-1}\gamma v_0 = \sqrt{\frac{m}{kg}}\tan^{-1}\left(v_0\sqrt{\frac{k}{mg}}\right)$.

(b) Replacing t by $t-T$ and setting $v_0 = 0$in the answer to the previous exercise yields $v = -\sqrt{\frac{mg}{k}}\,\frac{1-e^{-2\sqrt{\frac{gk}{m}}(t-T)}}{1+e^{-2\sqrt{\frac{gk}{m}}(t-T)}}$.

4.3.14. **(a)** $mv' = -mg + f(|v|)$; since $s = |v| = -v$, (A) $ms' = mg - f(s)$.

(b) Since f is increasing and $\lim_{t\to\infty} f(s) \le mg$, $mg - f(s) > 0$ for all s. This and (A) imply that s is an increasing function of t, so either (B) $\lim_{t\to\infty} s(t) = \infty$ or (C) $\lim_{t\to\infty} s(t) = \overline{s} < \infty$. However, (A) and (C) imply that $s'(t) > K = g - f(\overline{s})/m$ for all $t > 0$. Consequently, $s(t) > s_0 + Kt$ for all $t > 0$, which contradicts (C) because $K > 0$.

(c) There is a unique positive number s_T such that $f(s_T) = mg$, and $s \equiv s_T$ is a constant solution of (A). Now suppose that $s(0) < s_T$. Then Theorem 2.3.1 implies that (D) $s(t) < s_T$ for all $t > 0$, so (A) implies that s is strictly increasing. This and (D) imply that $\lim_{t\to\infty} s(t) = \overline{s} \le s_T$. If $\overline{s} < s_T$ then (A) implies that $s'(t) > K = g - f(\overline{s})/m$. Consequently, $s(t) > s(0) + Kt$, which contradicts (D) because $K > 0$. Therefore, $s(0) < s_T \Rightarrow \lim_{t\to\infty} s(t) = s_T$. A similar proof with inequlities reversed shows that $s(0) > s_T \Rightarrow \lim_{t\to\infty} s(t) = s_T$.

4.3.16. **(a)** (A) $mv' = -mg + k\sqrt{|v|}$; since the magnitude of the resistance is 64 lb when $v = 16$ ft/s, $4k = 64$, so $k = 16$ lb$\cdot$ s$^{1/2}$/ft$^{1/2}$. Since $m = 2$ and $g = 32$, (A) becomes $2v' = -64 + 16\sqrt{|v|}$, or $v' = -32 + 8\sqrt{|v|}$.

(b) From Exercise 4.3.14**(c)**, v_T is the negative number such that $-32 + 8\sqrt{|v_T|} = 0$; thus, $v_T = -16$ ft/s.

4.3.18. With $h = 0$, $v_e = \sqrt{2gR}$, where R is the radius of the moon and g is the acceleration due to gravity at the moon's surface. With length in miles, $g = \frac{5.31}{5280}$ mi/s^2, so $v_e = \sqrt{\frac{2\cdot 5.31\cdot 1080}{5280}} \approx 1.47$ miles/s.

4.3.20. Suppose that there is a number y_m such that $y(t) \le y_m$ for all $t \ge 0$ and let $\alpha = \frac{gR^2}{(y_m+R)^2}$. Then $\frac{d^2y}{dt^2} \le -\alpha$ for all $t \ge 0$. Integrating this inequality from $t = 0$ to $t = T > 0$ yields $v(T) - v_0 \le -\alpha T$, or $v(T) \le v_0 - \alpha T$, so $v(T) < 0$ for $T > \frac{v_0}{\alpha}$. This implies that the vehicle must eventually fall back to Earth, which contradicts the assumption that it continues to climb forever.

4.4 AUTONOMOUS SECOND ORDER EQUATIONS

4.4.1. $\overline{y} = 0$ is a stable equilibrium. The phase plane equivalent is $v\frac{dv}{dy} + y^3 = 0$, so the trajectories are $v^2 + \frac{y^4}{4} = c$.

4.4.2. $\overline{y} = 0$ is an unstable equilibrium. The phase plane equivalent is $v\dfrac{dv}{dy} + y^2 = 0$, so the trajectories are $v^2 + \dfrac{2y^3}{3} = c$.

4.4.4. $\overline{y} = 0$ is a stable equilibrium. The phase plane equivalent is $v\dfrac{dv}{dy} + ye^{-y} = 0$, so the trajectories are $v^2 - e^{-y}(y+1) = c$.

4.4.6. $p(y) = y^3 - 4y = (y+2)y(y-2)$, so the equilibria are $-2, 0, 2$. Since

$$y(y-2)(y+2) \quad \begin{array}{ll} < 0 & \text{if } y < -2 \text{ or } 0 < y < 2, \\ > 0 & \text{if } -2 < y < 0 \text{ or } y > 2, \end{array}$$

0 is unstable and $-2, 2$ are stable. The phase plane equivalent is $v\dfrac{dv}{dy} + y^3 - 4y = 0$, so the trajectories are $2v^2 + y^4 - 8y^2 = c$. Setting $(y, v) = (0, 0)$ yields $c = 0$, so the equation of the separatrix is $2v^2 - y^4 + 8y^2 = 0$.

4.4.8. $p(y) = y(y-2)(y-1)(y+2)$, so the equilibria are $-2, 0, 1, 2$. Since

$$y(y-2)(y-1)(y+2) \quad \begin{array}{ll} > 0 & \text{if } y < -2 \text{ or } 0 < y < 1 \text{ or } y > 2, \\ < 0 & \text{if } -2 < y < 0 \text{ or } 1 < y < 2, \end{array}$$

$0, 2$ are stable and $-2, 1$ are unstable. The phase plane equivalent is $v\dfrac{dv}{dy} + y(y-2)(y-1)(y+2) = 0$, so the trajectories are $30v^2 + y^2(12y^3 - 15y^2 - 80y + 120) = c$. Setting $(y, v) = (-2, 0)$ and $(y, v) = (1, 0)$ yields $c = 496$ and $c = 37$ respectively, so the equations of the separatrices are $30v^2 + y^2(12y^3 - 15y^2 - 80y + 120) = 496$ and $30v^2 + y^2(12y^3 - 15y^2 - 80y + 120) = 37$.

4.4.10. $p(y) = y^3 - ay$. If $a \le 0$, then $p(0) = 0$, $p(y) > 0$ if $y > 0$, and $p(y) < 0$ if $y < 0$, so 0 is stable. If $a > 0$, then

$$y^3 - ay = y(y - \sqrt{a})(y + \sqrt{a}) \quad \begin{array}{ll} > & 0 \text{ if } -\sqrt{a} < y < 0 \text{ or } y > \sqrt{a}, \\ < & 0 \text{ if } y < -\sqrt{a} \text{ or } 0 < y < \sqrt{a}, \end{array}$$

so $-\sqrt{a}$ and $\sqrt{a}$ are stable and 0 is unstable. We say that $a = 0$ is a critical value because it separates the two cases.

4.4.12. $p(y) = y - ay^3$. If $a \le 0$, then $p(0) = 0$, $p(y) > 0$ if $y > 0$, and $p(y) < 0$ if $y < 0$, so 0 is stable. If $a > 0$, then

$$y - ay^3 = -ay(y - 1/\sqrt{a})(y + 1/\sqrt{a}) \quad \begin{array}{ll} > & 0 \text{ if } y < -1/\sqrt{a} < y < 0 \text{ or } 0 < y < 1/\sqrt{a} \\ < & 0 \text{ if } -1/\sqrt{a} < y < 0 \text{ or } y > 1/\sqrt{a}, \end{array}$$

so $-\sqrt{a}$ and $\sqrt{a}$ are unstable and 0 is stable. We say that $a = 0$ is a critical value because it separates the two cases.

4.4.24. **(a)** Since $v' = -p(y) \ge k$ and $v(0) = 0$, $v \ge kt$ and therefore $y \ge y_0 + kt^2/2$ for $0 \le t < T$.
(b) Let $0 < \epsilon < \rho$. Suppose that y is the solution of the initial value problem (A) $y'' + p(y) = 0$, $y(0) = y_0$, $y'(0) = 0$, where $\overline{y} < y_0 < \overline{y} + \epsilon$. Now let $Y = y - \overline{y}$ and $P(Y) = p(Y + \overline{y})$. Then $P(0) = 0$ and $P(Y) < 0$ if $0 < Y \le \rho$. Morover, Y is the solution of $Y'' + p(Y) =$

$0, \quad Y(0) = Y_0, \quad Y''(0) = 0$, where $Y_0 = y_0 - \overline{y}$, so $0 < Y_0 < \epsilon$. From **(a)**, $Y(t) \geq \epsilon$ for some $t > 0$. Therefore,$y(t) > \overline{y} + \epsilon$ for some $t > 0$, so $\overline{y}$ is an unstable equilibrium of $y'' + p(y) = 0$.

4.5 APPLICATIONS TO CURVES

4.5.2. Differentiating (A) $e^{xy} = cy$ yields (B) $(xy' + y)e^{xy} = cy'$. From (A), $c = \dfrac{e^{xy}}{y}$. Substituting this into (B) and cancelling e^{xy} yields $xy' + y = \dfrac{y'}{y}$, so $y' = -\dfrac{y^2}{(xy-1)}$.

4.5.4. Solving $y = x^{1/2} + cx$ for c yields $c = \dfrac{y}{x} - x^{-1/2}$, and differentiating yields $0 = \dfrac{y'}{x} - \dfrac{y}{x^2} + \dfrac{x^{-3/2}}{2}$, or $xy' - y = -\dfrac{x^{1/2}}{2}$.

4.5.6. Rewriting $y = x^3 + \dfrac{c}{x}$ as $xy = x^4 + c$ and differentiating yields $xy' + y = 4x^3$.

4.5.8. Rewriting $y = e^x + c(1 + x^2)$ as $\dfrac{y}{1+x^2} = \dfrac{e^x}{1+x^2} + c$ and differentiating yields $\dfrac{y'}{1+x^2} - \dfrac{2xy}{(1+x^2)^2} = \dfrac{e^x}{1+x^2} - \dfrac{2xe^x}{(1+x^2)^2}$, so $(1+x^2)y' - 2xy = (1-x)^2 e^x$.

4.5.10. If (A) $y = f + cg$, then (B) $y' = f' + cg'$. Mutiplying (A) by g' and (B) by g yields (C) $yg' = fg' + cgg'$ and (D) $y'g = f'g + cg'g$, and subtracting (C) from (D) yields $y'g - yg' = f'g - fg'$.

4.5.12. Let (x_0, y_0) be the center and r be the radius of a circle in the family. Since $(-1, 0)$ and $(1, 0)$ are on the circle, $(x_0 + 1)^2 + y_0^2 = (x_0 - 1)^2 + y_0^2$, which implies that $x_0 = 0$. Therefore,the equation of the circle is (A) $x^2 + (y - y_0)^2 = r^2$. Since $(1, 0)$ is on the circle, $r^2 = 1 + y_0^2$. Substituting this into (A) shows that the equation of the circle is $x^2 + y^2 - 2yy_0 = 1$, so $2y_0 = \dfrac{x^2 + y^2 - 1}{y}$. Differentiating $y(2x + 2yy') - y'(x^2 + y^2 - 1) = 0$, so $y'(y^2 - x^2 + 1) + 2xy = 0$.

4.5.14. From Example 4.5.6 the equation of the line tangent to the parabola at (x_0, x_0^2) is (A) $y = -x_0^2 + 2x_0 x$.

(a) From (A), $(x, y) = (5, 9)$ is on the tangent line through (x_0, x_0^2) if and only if $9 = -x_0^2 + 10x_0$, or $x_0^2 - 10x_0 + 9 = (x_0 - 1)(x_0 - 9) = 0$. Letting $x_0 = 1$ in (A) yields the line $y = -1 + 2x$, tangent to the parabola at $(x_0, x_0^2) = (1, 1)$. Letting $x_0 = 9$ in (A) yields the line $y = -81 + 18x$, tangent to the parabola at $(x_0, x_0^2) = (9, 81)$.

(b) From (A), $(x, y) = (6, 11)$ is on the tangent line through (x_0, x_0^2) if and only if $11 = -x_0^2 + 12x_0$, or $x_0^2 - 12x_0 + 11 = (x_0 - 1)(x_0 - 11) = 0$. Letting $x_0 = 1$ in (A) yields the line $y = -1 + 2x$, tangent to the parabola at $(x_0, x_0^2) = (1, 1)$. Letting $x_0 = 11$ in (A) yields the line $y = -121 + 22x$, tangent to the parabola at $(x_0, x_0^2) = (11, 21)$.

(c) From (A), $(x, y) = (-6, 20)$ is on the tangent line through (x_0, x_0^2) if and only if $20 = -x_0^2 - 12x_0$, or $x_0^2 + 12x_0 + 20 = (x_0 + 2)(x_0 + 10) = 0$. Letting $x_0 = -2$ in (A) yields the line $y = -4 - 4x$, tangent to the parabola at $(x_0, x_0^2) = (-2, 4)$. Letting $x_0 = -10$ in (A) yields the line $y = -100 - 20x$, tangent to the parabola at $(x_0, x_0^2) = (-10, 100)$.

(d) From (A), $(x, y) = (-3, 5)$ is on the tangent line through (x_0, x_0^2) if and only if $5 = -x_0^2 - 6x_0$, or $x_0^2 + 6x_0 + 5 = (x_0 + 1)(x_0 + 5) = 0$. Letting $x_0 = -1$ in (A) yields the line $y = -1 - 2x$, tangent to the parabola at $(x_0, x_0^2) = (-1, 1)$. Letting $x_0 = -5$ in (A) yields the line $y = -25 - 10x$, tangent to the parabola at $(x_0, x_0^2) = (-5, 25)$.

4.5.15. **(a)** If (x_0, y_0) is any point on the circle such that $x_0 \neq \pm 1$ (and therefore $y_0 \neq 0$), then differentiating (A) yields $2x_0 + 2y_0y_0' = 0$, so $y_0' = -\dfrac{x_0}{y_0}$. Therefore,the equation of the tangent line is $y = y_0 - \dfrac{x_0}{y_0}(x - x_0)$. Since $x_0^2 + y_0^2 = 1$, this is equivalent to (B).

(b) Since $y' = -\dfrac{x_0}{y_0}$ on the tangent line, we can rewrite (B) as $y - xy' = \dfrac{1}{y_0}$. Hence (F) $\dfrac{1}{(y - xy')^2} = y_0^2$ and (G) $x_0^2 = 1 - y_0^2 = \dfrac{(y - xy')^2 - 1}{(y - xy')^2}$. Since $(y')^2 = \dfrac{x_0^2}{y_0^2}$, (F) and (G) imply that $(y')^2 = (y - xy')^2 - 1$, which implies (C).

(c) Using the quadratic formula to solve (C) for y' yields

$$y' = \frac{xy \pm \sqrt{x^2 + y^2 - 1}}{x^2 - 1} \tag{H}$$

if (x, y) is on a tangent line with slope y'. If $y = \dfrac{1 - x_0x}{y_0}$, then $x^2 + y^2 - 1 = x^2 + \left(\dfrac{1 - x_0x}{y_0}\right)^2 - 1 = \left(\dfrac{x - x_0}{y_0}\right)^2$ (since $x_0^2 + y_0^2 = 1$). Since $y' = -\dfrac{x_0}{y_0}$, this implies that (H) is equivalent to $-\dfrac{x_0}{y_0} = \dfrac{1}{x^2 - 1}\left[\dfrac{x(1 - x_0x)}{y_0} \pm \left|\dfrac{x - x_0}{y_0}\right|\right]$, which holds if and only if we choose the “$\pm$" so that $\pm\left|\dfrac{x - x_0}{y_0}\right| = -\left(\dfrac{x - x_0}{y_0}\right)$. Therefore,we must choose $\pm = -$ if $\dfrac{x - x_0}{y_0} > 0$, so (H) reduces to (D), or $\pm = +$ if $\dfrac{x - x_0}{y_0} < 0$, so (H) reduces to (E).

(d) Differentiating (A) yields $2x + 2yy' = 0$, so $y' = -\dfrac{x}{y}$ on either semicircle. Since (D) and (E) both reduce to $y' = \dfrac{xy}{1 - x^2} = -\dfrac{x}{y}$ (since $x^2 + y^2 = 1$) on both semicircles, the conclusion follows.

(e) From (D) and (E) the slopes of tangent lines from (5,5) tangent to the circle are $y' = \dfrac{25 \pm \sqrt{49}}{24} = \dfrac{3}{4}, \dfrac{4}{3}$. Therefore, tangent lines are $y = 5 + \dfrac{3}{4}(x - 5) = \dfrac{1 + 3x/5}{4/5}$ and $y = 5 + \dfrac{4}{3}(x - 5) = \dfrac{1 - 4x/5}{-3/5}$, which intersect the circle at $(-3/5, 4/5)$ $(4/5, -3/5)$, respectively. (See (B)).

4.5.16. **(a)** If (x_0, y_0) is any point on the parabola such that $x_0 > 0$ (and therefore $y_0 \neq 0$), then differentiating (A) yields $1 = 2y_0y_0'$, so $y_0' = \dfrac{1}{2y_0}$. Therefore,the equation of the tangent line is $y = y_0 + \dfrac{1}{2y_0}(x - x_0)$. Since $x_0 = y_0^2$, this is equivalent to (B).

(b) Since $y' = \dfrac{1}{2y_0}$ on the tangent line, we can rewrite (B) as $\dfrac{y_0}{2} = y - xy'$. Substituting this into (B) yields $y = (y - xy') + \dfrac{x}{4(y - xy')}$, which implies (C).

(c) Using the quadratic formula to solve (C) for y' yields

$$y' = \frac{y \pm \sqrt{y^2 - x}}{2x} \tag{F}$$

if (x, y) is on a tangent line with slope y'. If $y = \dfrac{y_0}{2} + \dfrac{x}{2y_0}$, then $y^2 - x = \dfrac{1}{4}\left(y_0 - \dfrac{x}{y_0}\right)^2$ so (F) is equivalent to $\dfrac{1}{2y_0} = \dfrac{y_0 + \dfrac{x}{y_0} \pm \left|y_0 - \dfrac{x}{y_0}\right|}{4x}$ which holds if and only if we choose the "$\pm$" so that $\pm\left|y_0 - \dfrac{x}{y_0}\right| = -\left(y_0 - \dfrac{x}{y_0}\right)$. Therefore,we must choose $\pm = +$ if $x > y_0^2 = x_0$, so (F) reduces to (D), or $\pm = -$ if $x < y_0^2 = x_0$, so (F) reduces to (E).

(d) Differentiating (A) yields $1 = 2yy'$, so $y' = \dfrac{1}{2y}$ on either half of the parabola. Since (D) and (E) both reduce to this if $x = y^2$, the conclusion follows.

4.5.18. The equation of the line tangent to the curve at $(x_0, y(x_0))$ is $y = y(x_0) + y'(x_0)(x - x_0)$. Since $y(x_0/2) = 0$, $y(x_0) - \dfrac{y'(x_0)x_0}{2} = 0$. Since x_0 is arbitrary, it follows that $y' = \dfrac{2y}{x}$, so $\dfrac{y'}{y} = \dfrac{2}{x}$, $\ln|y| = 2\ln|x| + k$, and $y = cx^2$. Since $(1, 2)$ is on the curve, $c = 2$. Therefore,$y = 2x^2$.

4.5.20. The equation of the line tangent to the curve at $(x_0, y(x_0))$ is $y = y(x_0) + y'(x_0)(x - x_0)$. Since (x_1, y_1) is on the line, $y(x_0) + y'(x_0)(x_1 - x_0) = y_1$. Since x_0 is arbitrary, it follows that $y + y'(x_1 - x) = y_1$, so $\dfrac{y'}{y - y_1} = \dfrac{1}{x - x_1}$, $\ln|y - y_1| = \ln|x - x_1| + k$, and $y - y_1 = c(x - x_1)$.

4.5.22. The equation of the line tangent to the curve at $(x_0, y(x_0))$ is $y = y(x_0) + y'(x_0)(x - x_0)$. Since $y(0) = x_0$, $x_0 = y(x_0) - y'(x_0)x_0$. Since x_0 is arbitrary, it follows that $x = y - xy'$, so (A) $y' - \dfrac{y}{x} = -1$. The solutions of (A) are of the form $y = ux$, where $u'x = -1$, so $u' = -\dfrac{1}{x}$. Therefore,$u = -\ln|x| + c$ and $y = -x\ln|x| + cx$.

4.5.24. The equation of the line normal to the curve at (x_0, y_0) is $y = y(x_0) - \dfrac{x - x_0}{y'(x_0)}$. Since $y(0) = 2y(x_0)$, $y(x_0) + \dfrac{x_0}{y'(x_0)} = 2y(x_0)$. Since x_0 is arbitrary, it follows that $y'y = x$, so (A) $\dfrac{y^2}{2} = \dfrac{x^2}{2} + \dfrac{c}{2}$ and $y^2 = x^2 + c$. Now $y(2) = 1 \leftrightarrow c = -3$. Therefore,$y = \sqrt{x^2 - 3}$.

4.5.26. Differentiating the given equation yields $2x + 4y + 4xy' + 2yy' = 0$, so $y' = -\dfrac{x + 2y}{2x + y}$ is a differential equation for the given family, and (A) $y' = \dfrac{2x + y}{x + 2y}$ is a differential equation for the orthogonal trajectories. Substituting $y = ux$ in (A) yields $u'x + u = \dfrac{2 + u}{1 + 2u}$, so $u'x = -\dfrac{2(u^2 - 1)}{1 + 2u}$ and $\dfrac{1 + 2u}{(u - 1)(u + 1)}u' = -\dfrac{2}{x}$, or $\left[\dfrac{3}{u - 1} + \dfrac{1}{u + 1}\right]u' = -\dfrac{4}{x}$. Therefore, $3\ln|u-1| + \ln|u+1| = -4\ln|x| + K$, so $(u-1)^3(u+1) = \dfrac{k}{x^4}$. Substituting $u = \dfrac{y}{x}$ yields the orthogonal trajectories $(y-x)^3(y+x) = k$.

4.5.28. Differentiating yields $ye^{x^2}(1+2x^2) + xe^{x^2}y' = 0$, so $y' = \dfrac{y(1 + 2x^2)}{x}$ is a differential equation for the given family. Therefore,(A) $y' = -\dfrac{x}{y(1 + 2x^2)}$ is a differential equation for the orthogonal trajectories. From (A), $yy' = -\dfrac{x}{1 + 2x^2}$, so $\dfrac{y^2}{2} = -\dfrac{1}{4}\ln(1 + 2x^2) + \dfrac{k}{2}$, and the orthogonal trajectories are given by $y^2 = -\dfrac{1}{2}\ln(1 + 2x^2) + k$.

4.5.30. Differentiating (A) $y = 1 + cx^2$ yields (B) $y' = 2cx$. From (C), $c = \dfrac{y-1}{x^2}$. Substituting this into (B) yields the differential equation $y' = \dfrac{2(y-1)}{x}$ for the given family of parabolas. Therefore,$y' = -\dfrac{x}{2(y-1)}$ is a differential equation for the orthogonal trajectories. Separating variables yields $2(y-1)y' = -x$, so $(y-1)^2 = -\dfrac{x^2}{2} + k$. Now $y(-1) = 3 \leftrightarrow k = \dfrac{9}{2}$, so $(y-1)^2 = -\dfrac{x^2}{2} + \dfrac{9}{2}$. Therefore,(D) $y = 1 + \sqrt{\dfrac{9-x^2}{2}}$. This curve interesects the parabola (A) if and only if the equation (C) $cx^2 = \sqrt{\dfrac{9-x^2}{2}}$ has a solution x^2 in $(0, 9)$. Therefore,$c > 0$ is a necessaary condition for intersection. We will show that it is also sufficient. Squaring both sides of (C) and simplifying yields $2c^2x^4 + x^2 - 9 = 0$. Using the quadratic formula to solve this for x^2 yields $x^2 = \dfrac{-1 + \sqrt{1 + 72c^2}}{4c^2}$. The condition $x^2 < 9$ holds if and only if $-1 + \sqrt{1 + 72c^2} < 36c^4$, which is equivalent to $1 + 72c^2 < (1 + 36c^2)^2 = 1 + 72c^2 + 1296c^4$, which holds for all $c > 0$.

4.5.32. The angles θ and θ_1 from the x-axis to the tangents to C and C_1 satisfy $\tan\theta = f(x_0, y_0)$ and $\tan\theta_1 = \dfrac{f(x_0, y_0) + \tan\alpha}{1 - f(x_0, y_0)\tan\alpha} = \dfrac{\tan\theta + \tan\alpha}{1 - \tan\theta\tan\alpha} = \tan(\theta + \alpha)$. Therefore, assuming θ and θ_1 are both in $[0, 2\pi)$, $\theta_1 = \theta + \alpha$.

4.5.34. Circles centered at the origin are given by $x^2 + y^2 = r^2$. Differentiating yields $2x + 2yy' = 0$, so $y' = -\dfrac{x}{y}$ is a differential equation for the given family, and $y' = \dfrac{-(x/y) + \tan\alpha}{1 + (x/y)\tan\alpha}$ is a differential equation for the desired family. Substituting $y = ux$ yields $u'x + u = \dfrac{-1/u + \tan\alpha}{1 + (1/u)\tan\alpha} = \dfrac{-1 + u\tan\alpha}{u + \tan\alpha}$. Therefore,$u'x = -\dfrac{1+u^2}{u + \tan\alpha}$, $\dfrac{u + \tan\alpha}{1 + u^2}u' = -\dfrac{1}{x}$ and $\dfrac{1}{2}\ln(1 + u^2) + \tan\alpha\tan^{-1}u = -\ln|x| + k$. Substituting $u = \dfrac{y}{x}$ yields $\dfrac{1}{2}\ln(x^2 + y^2) + (\tan\alpha)\tan^{-1}\dfrac{y}{x} = k$.

CHAPTER 5
Linear Second Order Equations

5.1 HOMOGENEOUS LINEAR EQUATIONS

5.1.2. **(a)** If $y_1 = e^x \cos x$, then $y_1' = e^x(\cos x - \sin x)$ and $y_1'' = e^x(\cos x - \sin x - \sin x - \cos x) = -2e^x \sin x$, so $y_1'' - 2y_1' + 2y_1 = e^x(-2\sin x - 2\cos x + 2\sin x + 2\cos x) = 0$. If $y_2 = e^x \sin x$, then $y_2' = e^x(\sin x + \cos x)$ and $y_2'' = e^x(\sin x + \cos x + \cos x - \sin x) = 2e^x \cos x$, so $y_2'' - 2y_2' + 2y_2 = e^x(2\cos x - 2\sin x - 2\cos x + 2\sin x) = 0$.

(b) If (B) $y = e^x(c_1 \cos x + c_2 \sin x)$, then

$$y' = e^x\left(c_1(\cos x - \sin x) + c_2(\sin x + \cos x)\right) \tag{C}$$

and

$$\begin{aligned} y'' &= c_1 e^x(\cos x - \sin x - \sin x - \cos x) \\ &\quad + c_2 e^x(\sin x + \cos x + \cos x - \sin x) \\ &= 2e^x(-c_1 \sin x + c_2 \cos x), \end{aligned}$$

so

$$\begin{aligned} y'' - 2y' + 2y &= c_1 e^x(-2\sin x - 2\cos x + 2\sin x + 2\cos x) \\ &\quad + c_2 e^x(2\cos x - 2\sin x - 2\cos x + 2\sin x) = 0. \end{aligned}$$

(c) We must choose c_1 and c_2 in (B) so that $y(0) = 3$ and $y'(0) = -2$. Setting $x = 0$ in (B) and (C) shows that $c_1 = 3$ and $c_1 + c_2 = -2$, so $c_2 = -5$. Therefore, $y = e^x(3\cos x - 5\sin x)$.

(d) We must choose c_1 and c_2 in (B) so that $y(0) = k_0$ and $y'(0) = k_1$. Setting $x = 0$ in (B) and (C) shows that $c_1 = k_0$ and $c_1 + c_2 = k_1$, so $c_2 = k_1 - k_0$. Therefore, $y = e^x\left(k_0 \cos x + (k_1 - k_0)\sin x\right)$.

5.1.4. **(a)** If $y_1 = \dfrac{1}{x-1}$, then $y_1' = -\dfrac{1}{(x-1)^2}$ and $y_1'' = \dfrac{2}{(x-1)^3}$, so

$$\begin{aligned} (x^2-1)y_1'' + 4xy_1' + 2y_1 &= \frac{2(x^2-1)}{(x-1)^3} - \frac{4x}{(x-1)^2} + \frac{2}{x-1} \\ &= \frac{2(x+1) - 4x + 2(x-1)}{(x-1)^2} = 0. \end{aligned}$$

Similar manipulations show that $(x^2-1)y_2'' + 4xy_2' + 2y_2 = 0$. The general solution on each of the intervals $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$ is (B) $y = \dfrac{c_1}{x-1} + \dfrac{c_2}{x+1}$.

(b) Differentiating (B) yields (C) $y' = -\frac{c_1}{(x-1)^2} - \frac{c_2}{(x+1)^2}$. We must choose c_1 and c_2 in (B) so that $y(0) = -5$ and $y'(0) = 1$. Setting $x = 0$ in (B) and (C) shows that $-c_1 + c_2 = -5$, $-c_1 - c_2 = 1$. Therefore,$c_1 = 2$ and $c_2 = -3$, so $y = \frac{2}{x-1} - \frac{3}{x+1}$ on $(-1, 1)$.

(d) The Wronskian of $\{y_1, y_2\}$ is

$$W(x) = \begin{vmatrix} \dfrac{1}{x-1} & \dfrac{1}{x+1} \\ -\dfrac{1}{(x-1)^2} & -\dfrac{1}{(x+1)^2} \end{vmatrix} = \frac{2}{(x^2-1)^2}, \tag{D}$$

so $W(0) = 2$. Since $p(x) = \frac{4x}{x^2-1}$, so $\int_0^x p(t)\,dt = \int_0^x \frac{4t}{t^2-1}\,dt = \ln(x^2-1)^2$, Abel's formula implies that $W(x) = W(0)e^{-\ln(x^2-1)^2} = \frac{2}{(x^2-1)^2}$, consistent with (D).

5.1.6. From Abel's formula, $W(x) = W(\pi)e^{-3\int_\pi^x (t^2+1)\,dt} = 0 \cdot e^{-3\int_\pi^x (t^2+1)\,dt} = 0$.

5.1.8. $p(x) = \frac{1}{x}$; therefore $\int_1^x p(t)\,dt = \int_1^x \frac{dt}{t} = \ln x$, so Abel's formula yields $W(x) = W(1)e^{-\ln x} = \frac{1}{x}$.

5.1.10. $p(x) = -2$; $P(x) = -2x$; $y_2 = uy_1 = ue^{3x}$; $u' = \frac{Ke^{-P(x)}}{y_1^2(x)} = \frac{Ke^{2x}}{e^{6x}} = Ke^{-4x}$; $u = -\frac{K}{4}e^{-4x}$. Choose $K = -4$; then $y_2 = e^{-4x}e^{3x} = e^{-x}$.

5.1.12. $p(x) = -2a$; $P(x) = -2ax$; $y_2 = uy_1 = ue^{ax}$; $u' = \frac{Ke^{-P(x)}}{y_1^2(x)} = \frac{Ke^{2ax}}{e^{2ax}} = K$; $u = Kx$. Choose $K = 1$; then $y_2 = xe^{ax}$.

5.1.14. $p(x) = -\frac{1}{x}$; $P(x) = -\ln x$; $y_2 = uy_1 = ux$; $u' = \frac{Ke^{-P(x)}}{y_1^2(x)} = \frac{Kx}{x^2} = \frac{K}{x}$; $u = K\ln x$. Choose $K = 1$; then $y_2 = x\ln x$.

5.1.16. $p(x) = -\frac{1}{x}$; $P(x) = -\ln|x|$; $y_2 = uy_1 = ux^{1/2}e^{2x}$; $u' = \frac{Ke^{-P(x)}}{y_1^2(x)} = \frac{Kx}{xe^{4x}} = e^{-4x}$; $u = -\frac{Ke^{-4x}}{4}$. Choose $K = -4$; then $y_2 = e^{-4x}(x^{1/2}e^{2x}) = x^{1/2}e^{-2x}$.

5.1.18. $p(x) = -\frac{2}{x}$; $P(x) = -2\ln|x|$; $y_2 = uy_1 = ux\cos x$; $u' = \frac{Ke^{-P(x)}}{y_1^2(x)} = \frac{Kx^2}{x^2\cos^2 x} = K\sec^2 x$; $u = K\tan x$. Choose $K = 1$; then $y_2 = \tan x(x\cos x) = x\sin x$.

5.1.20. $p(x) = -\frac{3x+2}{3x-1} = -1 - \frac{3}{3x-1}$; $P(x) = -x - \ln|3x-1|$; $y_2 = uy_1 = ue^{2x}$; $u' = \frac{Ke^{-P(x)}}{y_1^2(x)} = \frac{K(3x-1)e^x}{e^{4x}} = K(3x-1)e^{-3x}$; $u = -Kxe^{-3x}$. Choose $K = -1$; then $y_2 = xe^{-3x}e^{2x} = xe^{-x}$.

5.1.22. $p(x) = -\dfrac{2(2x^2-1)}{x(2x+1)} = -2 - \dfrac{2}{2x+1} + \dfrac{2}{x}$; $P(x) = -2x - \ln|2x+1| + 2\ln|x|$; $y_2 = uy_1 = \dfrac{u}{x}$; $u' = \dfrac{Ke^{-P(x)}}{y_1^2(x)} = \dfrac{K(2x+1)e^{2x}}{x^2}x^2 = K(2x+1)e^{2x}$; $u = Kxe^{2x}$. Choose $K = 1$; then $y_2 = \dfrac{xe^{2x}}{x} = e^{2x}$.

5.1.24. Suppose that $y \equiv 0$ on (a,b). Then $y' \equiv 0$ and $y'' \equiv 0$ on (a,b), so y is a solution of (A) $y'' + p(x)y' + q(x)y = 0,\ y(x_0) = 0,\ y'(x_0) = 0$ on (a,b). Since Theorem 5.1.1 implies that (A) has only one solution on (a,b), the conclusion follows.

5.1.26. If $\{z_1, z_2\}$ is a fundamental set of solutions of (A) on (a,b), then every solution y of (A) on (a,b) is a linear combination of $\{z_1, z_2\}$; that is, $y = c_1z_1 + c_2z_2 = c_1(\alpha y_1 + \beta y_2) + c_2(\gamma y_1 + \delta y_2) = (c_1\alpha + c_2\gamma)y_1 + (c_1\beta + c_2\delta)y_2$, which shows that every solution of (A) on (a,b) can be written as a linear combination of $\{y_1, y_2\}$. Therefore,$\{y_1, y_2\}$ is a fundamental set of solutions of (A) on (a,b).

5.1.28. The Wronskian of $\{y_1, y_2\}$ is

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} y_1 & ky_1 \\ y_1' & ky_1' \end{vmatrix} = k(y_1y_1' - y_1'y_1) = 0.$$

nor y_2 can be a solution of $y'' + p(x)y' + q(x)y = 0$ on (a,b).

5.1.30. $W(x_0) = \big(y_1(x_0)y_2'(x_0) - y_1'(x_0)y_2(x_0)\big) = 0$ if either $y_1(x_0) = y_2(x_0) = 0$ or $y_1'(x_0) = y_2'(x_0) = 0$, and Theorem 5.1.6 implies that $\{y_1, y_2\}$ is linearly dependent on (a,b).

5.1.32. Let x_0 be an arbitrary point in (a,b). By the motivating argument preceding Theorem 5.1.4, (B) $W(x_0) = y_1(x_0)y_2'(x_0) - y_1'(x_0)y_2(x_0) \neq 0$. Now let y be the solution of $y'' + p(x)y' + q(x)y = 0,\ y(x_0) = y_1(x_0),\ y'(x_0) = y_1'(x_0)$. By assumption, y is a linear combination of $\{y_1, y_2\}$ on (a,b); that is, $y = c_1y_1 + c_2y_2$, where

$$\begin{aligned} c_1y_1(x_0) + c_2y_2(x_0) &= y_1(x_0) \\ c_1y_1'(x_0) + c_2y_2'(x_0) &= y_1'(x_0). \end{aligned}$$

Solving this system by Cramers' rule yields

$$c_1 = \frac{1}{W(x_0)}\begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} = 1 \text{ and } c_2 = \frac{1}{W(x_0)}\begin{vmatrix} y_1(x_0) & y_1(x_0) \\ y_1'(x_0) & y_1(x_0) \end{vmatrix} = 0.$$

Therefore,$y = y_1$, which shows that y_1 is a solution of (A). A similar argument shows that y_2 is a solution of (A).

5.1.34. Expanding the determinant by cofactors of its first column shows that the first equation in the exercise can be written as

$$\frac{y}{W}\begin{vmatrix} y_1' & y_2' \\ y_1'' & y_2'' \end{vmatrix} - \frac{y'}{W}\begin{vmatrix} y_1 & y_2 \\ y_1'' & y_2'' \end{vmatrix} + \frac{y''}{W}\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = 0,$$

which is of the form (A) with

$$p = -\frac{1}{W}\begin{vmatrix} y_1 & y_2 \\ y_1'' & y_2'' \end{vmatrix} \quad \text{and} \quad q = \frac{1}{W}\begin{vmatrix} y_1' & y_2' \\ y_1'' & y_2'' \end{vmatrix}.$$

5.1.36. Theorem 5.1.6 implies that that there are constants c_1 and c_2 such that (B) $y = c_1y_1 + c_2y_2$ on (a, b). To see that c_1 and c_2 are unique, assume that (B) holds, and let x_0 be a point in (a, b). Then (C) $y' = c_1y_1' + c_2y_2'$. Setting $x = x_0$ in (B) and (C) yields

$$\begin{aligned} c_1y_1(x_0) + c_2y_2(x_0) &= y(x_0) \\ c_1y_1'(x_0) + c_2y_2'(x_0) &= y'(x_0). \end{aligned}$$

Since Theorem 5.1.6 implies that $y_1(x_0)y_2'(x_0) - y_1'(x_0)y_2(x_0) \neq 0$, the argument preceding Theorem 5.1.4 implies that c_1 and c_2 are given uniquely by

$$c_1 = \frac{y_2'(x_0)y(x_0) - y_2(x_0)y'(x_0)}{y_1(x_0)y_2'(x_0) - y_1'(x_0)y_2(x_0)} \quad c_2 = \frac{y_1(x_0)y'(x_0) - y_1'(x_0)y(x_0)}{y_1(x_0)y_2'(x_0) - y_1'(x_0)y_2(x_0)}.$$

5.1.38. The general solution of $y'' = 0$ is $y = c_1 + c_2x$, so $y' = c_2$. Imposing the stated initial conditions on $y_1 = c_1 + c_2x$ yields $c_1 + c_2x_0 = 1$ and $c_2 = 0$; therefore $c_1 = 1$, so $y_1 = 1$. Imposing the stated initial conditions on $y_2 = c_1 + c_2x$ yields $c_1 + c_2x_0 = 0$ and $c_2 = 1$; therefore $c_1 = -x_0$, so $y_2 = x - x_0$. The solution of the general initial value problem is $y = k_0 + k_1(x - x_0)$.

5.1.40. Let $y_1 = a_1 \cos \omega x + a_2 \sin \omega x$ and $y_2 = b_1 \cos \omega x + b_2 \sin \omega x$. Then

$$\begin{aligned} a_1 \cos \omega x_0 + a_2 \sin \omega x_0 &= 1 \\ \omega(-a_1 \sin \omega x_0 + a_2 \cos \omega x_0) &= 0 \end{aligned}$$

and

$$\begin{aligned} b_1 \cos \omega x_0 + b_2 \sin \omega x_0 &= 0 \\ \omega(-b_1 \sin \omega x_0 + b_2 \cos \omega x_0) &= 1. \end{aligned}$$

Solving these systems yields $a_1 = \cos \omega x_0$, $a_2 = \sin \omega x_0$, $b_1 = -\dfrac{\sin \omega x_0}{\omega}$, and $b_2 = \dfrac{\cos \omega x_0}{\omega}$. Therefore, $y_1 = \cos \omega x_0 \cos \omega x + \sin \omega x_0 \sin \omega x = \cos \omega(x - x_0)$ and $y_2 = \dfrac{1}{\omega}(-\sin \omega x_0 \cos \omega x + \cos \omega x_0 \sin \omega x) = \dfrac{1}{\omega} \sin \omega(x - x_0)$. The solution of the general initial value problem is $y = k_0 \cos \omega(x - x_0) + \dfrac{k_1}{\omega} \sin \omega(x - x_0)$.

5.1.42. **(a)** If $y_1 = x^2$, then $y_1' = 2x$ and $y_1'' = 2$, so $x^2y_1'' - 4xy_1' + 6y_1 = x^2(2) - 4x(2x) + 6x^2 = 0$ for x in $(-\infty, \infty)$. If $y_2 = x^3$, then $y_2' = 3x^2$ and $y_2'' = 6x$, so $x^2y_2'' - 4xy_2' + 6y_2 = x^2(6x) - 4x(3x^2) + 6x^3 = 0$ for x in $(-\infty, \infty)$. If $x \neq 0$, then $y_2(x)/y_1(x) = x$, which is nonconstant on $(-\infty, 0)$ and $(0, \infty)$, so Theorem 5.1.6 implies that $\{y_1, y_2\}$ is a fundamental set of solutions of (A) on each of these intervals.

(b) Theorem 5.1.6 and **(a)** imply that y satisfies (A) on $(-\infty, 0)$ and on $(0, \infty)$ if and only if $y = \begin{cases} a_1x^2 + a_2x^3, & x > 0, \\ b_1x^2 + b_2x^3, & x < 0. \end{cases}$ Since $y(0) = 0$ we can complete the proof that y is a solution of (A) on $(-\infty, \infty)$ by showing that $y'(0)$ and $y''(0)$ both exist if and only if $a_1 = b_1$. Since

$$\frac{y(x) - y(0)}{x - 0} = \begin{cases} a_1x + a_2x^2, & \text{if } x > 0, \\ b_1x + b_2x^2, & \text{if } x < 0, \end{cases}$$

it follows that $y'(0) = \displaystyle\lim_{x \to 0} \frac{y(x) - y(0)}{x - 0} = 0$. Therefore, $y' = \begin{cases} 2a_1x + 3a_2x^2, & x \geq 0, \\ 2b_1x + 3b_2x^2, & x < 0. \end{cases}$ Since $\dfrac{y'(x) - y'(0)}{x - 0} = \begin{cases} 2a_1 + 3a_2x, & \text{if } x > 0, \\ 2b_1 + 3b_2x, & \text{if } x < 0, \end{cases}$ it follows that $y''(0) = \displaystyle\lim_{x \to 0} \frac{y'(x) - y'(0)}{x - 0}$ exists if and only if $a_1 = b_1$. By renaming $a_1 = b_1 = c_1$, $a_2 = c_2$, and $b_2 = c_3$ we see that y is a solution of (A) on $(-\infty, \infty)$ if and only if $y = \begin{cases} c_1x^2 + c_2x^3, & x \geq 0, \\ c_1x^2 + c_3x^3, & x < 0. \end{cases}$

(c) We have shown that $y(0) = y'(0) = 0$ for any choice of c_1 and c_2 in (C). Therefore,the given initial value problem has a solution if and only if $k_0 = k_1 = 0$, in which case every function of the form (C) is a solution.

(d) If $x_0 > 0$, then c_1 and c_2 in (C) are uniquely determined by k_0 and k_1, but c_3 can be chosen arbitrarily. Therefore,(B) has a unique solution on $(0, \infty)$, but infinitely many solutions on $(-\infty, \infty)$. If $x_0 < 0$, then c_1 and c_3 in (C) are uniquely determined by k_0 and k_1, but c_2 can be chosen arbitrarily. Therefore,(B) has a unique solution on $(-\infty, 0)$, but infinitely many solutions on $(-\infty, \infty)$.

5.1.44. **(a)** If $y_1 = x^3$, then $y_1' = 3x^2$ and $y_1'' = 6x$, so $x^2y_1'' - 6xy_1' + 12y_1 = x^2(6x) - 6x(3x^2) + 12x^3 = 0$ for x in $(-\infty, \infty)$. If $y_2 = x^4$, then $y_2' = 4x^3$ and $y_2'' = 12x^2$, so $x^2y_2'' - 6xy_2' + 12y_2 = x^2(12x^2) - 6x(4x^3) + 12x^4 = 0$ for x in $(-\infty, \infty)$. If $x \neq 0$, then $y_2(x)/y_1(x) = x$, which is nonconstant on $(-\infty, 0)$ and $(0, \infty)$, so Theorem 5.1.6 implies that $\{y_1, y_2\}$ is a fundamental set of solutions of (A) on each of these intervals.

(b) Theorem 5.1.2 and **(a)** imply that y satisfies (A) on $(-\infty, 0)$ and on $(0, \infty)$ if and only if (C) $y = \begin{cases} a_1x^3 + a_2x^4, & x > 0, \\ b_1x^3 + b_2x^4, & x < 0. \end{cases}$ Since $y(0) = 0$ we can complete the proof that y is a solution of (A) on $(-\infty, \infty)$ by showing that $y'(0)$ and $y''(0)$ both exist for any choice of a_1, a_2, b_1, and b_2. Since $\dfrac{y(x) - y(0)}{x - 0} = \begin{cases} a_1x^2 + a_2x^3, & \text{if } x > 0, \\ b_1x^2 + b_2x^3, & \text{if } x < 0, \end{cases}$ it follows that $y'(0) = \lim_{x \to 0} \dfrac{y(x) - y(0)}{x - 0} = 0$. Therefore, $y' = \begin{cases} 3a_1x^2 + 4a_2x^3, & x \geq 0, \\ 3b_1x^2 + 4b_2x^3, & x < 0. \end{cases}$ Since $\dfrac{y'(x) - y'(0)}{x - 0} = \begin{cases} 3a_1x + 4a_2x^2, & \text{if } x > 0, \\ 3b_1x + 4b_2x^2, & \text{if } x < 0, \end{cases}$ it follows that $y''(0) = \lim_{x \to 0} \dfrac{y'(x) - y'(0)}{x - 0} = 0$. Therefore,(B) is a solution of (A) on $(-\infty, \infty)$.

(c) We have shown that $y(0) = y'(0) = 0$ for any choice of a_1, a_2, b_1, and b_2 in (B). Therefore,the given initial value problem has a solution if and only if $k_0 = k_1 = 0$, in which case every function of the form (B) is a solution.

(d) If $x_0 > 0$, then a_1 and a_2 in (B) are uniquely determined by k_0 and k_1, but b_1 and b_2 can be chosen arbitrarily. Therefore,(C) has a unique solution on $(0, \infty)$, but infinitely many solutions on $(-\infty, \infty)$. If $x_0 < 0$, then b_1 and b_2 in (B) are uniquely determined by k_0 and k_1, but a_1 and a_2 can be chosen arbitrarily. Therefore,(C) has a unique solution on $(-\infty, 0)$, but infinitely many solutions on $(-\infty, \infty)$.

5.2 CONSTANT COEFFICIENT HOMOGENEOUS EQUATIONS

5.2.2. $p(r) = r^2 - 4r + 5 = (r - 2)^2 + 1$; $y = e^{2x}(c_1 \cos x + c_2 \sin x)$.

5.2.4. $p(r) = r^2 - 4r + 4 = (r - 2)^2$; $y = e^{2x}(c_1 + c_2x)$.

5.2.6. $p(r) = r^2 + 6r + 10 = (r + 3)^2 + 1$; $y = e^{-3x}(c_1 \cos x + c_2 \sin x)$.

5.2.8. $p(r) = r^2 + r = r(r + 1)$; $y = c_1 + c_2e^{-x}$.

5.2.10. $p(r) = r^2 + 6r + 13y = (r + 3)^2 + 4$; $y = e^{-3x}(c_1 \cos 2x + c_2 \sin 2x)$.

5.2.12. $p(r) = 10r^2 - 3r - 1 = (2r - 1)(5r + 1) = 10(r - 1/2)(r + 1/5)$; $y = c_1e^{-x/5} + c_2e^{x/2}$.

5.2.14. $p(r) = 6r^2 - r - 1 = (2r - 1)(3r + 1) = 6(r - 1/2)(r + 1/3)$; $y = c_1e^{-x/3} + c_2e^{x/2}$; $y' = -\dfrac{c_1}{3}e^{-x/3} + \dfrac{c_2}{2}e^{x/2}$; $y(0) = 10 \Rightarrow c_1 + c_2 = 10$, $y'(0) = 0 \Rightarrow -\dfrac{c_1}{3} + \dfrac{c_2}{2} = 0$; $c_1 = 6, c_2 = 4$; $y = 4e^{x/2} + 6e^{-x/3}$.

5.2.16. $p(r) = 4r^2 - 4r - 3 = (2r-3)(2r+1) = 4(r-3/2)(r+1/2)$; $y = c_1e^{-x/2} + c_2e^{3x/2}$; $y' = -\frac{c_1}{2}e^{-x/2} + \frac{3c_2}{2}e^{3x/2}$; $y(0) = \frac{13}{12} \Rightarrow c_1 + c_2 = \frac{13}{12}$, $y'(0) = \frac{23}{24} \Rightarrow -\frac{c_1}{2} + \frac{3c_2}{2} = \frac{23}{24}$; $c_1 = \frac{1}{3}, c_2 = \frac{3}{4}$; $y = \frac{e^{-x/2}}{3} + \frac{3e^{3x/2}}{4}$.

5.2.18. $p(r) = r^2 + 7r + 12 = (r+3)(r+4)$; $y = c_1e^{-4x} + c_2e^{-3x}$; $y' = -4c_1e^{-4x} - 3c_2e^{-3x}$; $y(0) = -1 \Rightarrow c_1 + c_2 = -1$, $y'(0) = 0 \Rightarrow -4c_1 - 3c_2 = 0$; $c_1 = 3, c_2 = -4$; $y = 3e^{-4x} - 4e^{-3x}$.

5.2.20. $p(r) = 36r^2 - 12r + 1 = (6r-1)^2 = 36(r-1/6)^2$; $y = e^{x/6}(c_1 + c_2x)$; $y' = \frac{e^{x/6}}{6}(c_1 + c_2x) + c_2e^{x/6}$; $y(0) = 3 \Rightarrow c_1 = 3$, $y'(0) = \frac{5}{2} \Rightarrow \frac{c_1}{6} + c_2 = \frac{5}{2} \Rightarrow c_2 = 2$; $y = e^{x/6}(3 + 2x)$.

5.2.22. **(a)** From (A), $ay''(x) + by'(x) + cy(x) = 0$ for all x. Replacing x by $x - x_0$ yields (C) $ay''(x-x_0) + by'(x-x_0) + cy(x-x_0) = 0$. If $z(x) = y(x-x_0)$, then the chain rule implies that $z'(x) = y'(x-x_0)$ and $z''(x) = y''(x-x_0)$, so (C) is equivalent to $az'' + bz' + cz = 0$.

(b) If $\{y_1, y_2\}$ is a fundamental set of solutions of (A) then Theorem 5.1.6 implies that y_2/y_1 is nonconstant. Therefore, $\frac{z_2(x)}{z_1(x)} = \frac{y_2(x-x_0)}{y_1(x-x_0)}$ is also nonconstant, so Theorem 5.1.6 implies that $\{z_1, z_2\}$ is a fundamental set of solutions of (A).

(c) *Let $p(r) = ar^2 + br + c$ be the characteristic polynomial of* (A). *Then:*

- *If $p(r) = 0$ has distinct real roots r_1 and r_2, then the general solution of* (A) *is*

$$y = c_1e^{r_1(x-x_0)} + c_2e^{r_2(x-x_0)}.$$

- *If $p(r) = 0$ has a repeated root r_1, then the general solution of* (A) *is*

$$y = e^{r_1(x-x_0)}(c_1 + c_2(x-x_0)).$$

- *If $p(r) = 0$ has complex conjugate roots $r_1 = \lambda + i\omega$ and $r_2 = \lambda - i\omega$ (where $\omega > 0$), then the general solution of* (A) *is*

$$y = e^{\lambda(x-x_0)}(c_1\cos\omega(x-x_0) + c_2\sin\omega(x-x_0)).$$

5.2.24. $p(r) = r^2 - 6r - 7 = (r-7)(r+1)$;

$$\begin{aligned} y &= c_1e^{-(x-2)} + c_2e^{7(x-2)}; \\ y' &= -c_1e^{-(x-2)} + 7c_2e^{7(x-2)}; \end{aligned}$$

$y(2) = -\frac{1}{3} \Rightarrow c_1 + c_2 = -\frac{1}{3}$, $y'(2) = -5 \Rightarrow -c_1 + 7c_2 = -5$; $c_1 = \frac{1}{3}, c_2 = -\frac{2}{3}$; $y = \frac{1}{3}e^{-(x-2)} - \frac{2}{3}e^{7(x-2)}$.

5.2.26. $p(r) = 9r^2 + 6r + 1 = (3r+1)^2 = 9(r+1/3)^2$;

$$\begin{aligned} y &= e^{-(x-2)/3}\left(c_1 + c_2(x-2)\right); \\ y' &= -\frac{1}{3}e^{-(x-2)/3}\left(c_1 + c_2(x-2)\right) + c_2e^{-(x-2)/3}; \end{aligned}$$

$y(2) = 2 \Rightarrow c_1 = 2,\ y'(2) = -\frac{14}{3} \Rightarrow -\frac{c_1}{3} + c_2 = -\frac{14}{3} \Rightarrow c_2 = -4;\ \ y = e^{-(x-2)/3}\,(2 - 4(x-2))$.

5.2.28. $p(r) = r^2 + 3$;

$$\begin{aligned} y &= c_1 \cos\sqrt{3}\left(x - \frac{\pi}{3}\right) + c_2 \sin\sqrt{3}\left(x - \frac{\pi}{3}\right); \\ y' &= -\sqrt{3}c_1 \sin\sqrt{3}\left(x - \frac{\pi}{3}\right) + \sqrt{3}c_2 \cos\sqrt{3}\left(x - \frac{\pi}{3}\right); \end{aligned}$$

$y(\pi/3) = 2 \Rightarrow c_1 = 2,\ y'(\pi/3) = -1 \Rightarrow c_2 = -\frac{1}{\sqrt{3}}$;

$$y = 2\cos\sqrt{3}\left(x - \frac{\pi}{3}\right) - \frac{1}{\sqrt{3}}\sin\sqrt{3}\left(x - \frac{\pi}{3}\right).$$

5.2.30. y is a solution of $ay'' + by' + cy = 0$ if and only if

$$\begin{aligned} y &= c_1 e^{r_1(x-x_0)} + \quad e^{r_2(x-x_0)} \\ y' &= r_1 c_1 e^{r_1(x-x_0)} + r_2 e^{r_2}(x - x_0). \end{aligned}$$

Now $y_1(x_0) = k_0$ and $y_1'(x_0) = k_1 \Rightarrow c_1 + c_2 = k_0, r_1c_1 + r_2c_2 = k_1$. Therefore,$c_1 = \frac{r_2k_0 - k_1}{r_2 - r_1}$ and $c_2 = \frac{k_1 - r_1k_0}{r_2 - r_1}$. Substituting c_1 and c_2 into the above equations for y and y' yields

$$\begin{aligned} y &= \frac{r_2k_0 - k_1}{r_2 - r_1} e^{r_1(x-x_0)} + \frac{k_1 - r_1k_0}{r_2 - r_1} e^{r_2(x-x_0)} \\ &= \frac{k_0}{r_2 - r_1}\left(r_2 e^{r_1(x-x_0)} - r_1 e^{r_2(x-x_0)}\right) + \frac{k_1}{r_2 - r_1}\left(e^{r_2(x-x_0)} - e^{r_1(x-x_0)}\right). \end{aligned}$$

5.2.32. y is a solution of $ay'' + by' + cy = 0$ if and only if

$$y = e^{\lambda(x-x_0)}\left(c_1 \cos\omega(x - x_0) + c_2 \sin\omega(x - x_0)\right) \tag{A}$$

and

$$\begin{aligned} y' &= \lambda e^{\lambda(x-x_0)}\left(c_1 \cos\omega(x - x_0) + c_2 \sin\omega(x - x_0)\right) \\ &\quad + \omega e^{\lambda(x-x_0)}\left(-c_1 \sin\omega(x - x_0) + c_2 \cos\omega(x - x_0)\right). \end{aligned}$$

Now $y_1(x_0) = k_0 \Rightarrow c_1 = k_0$ and $y_1'(x_0) = k_1 \Rightarrow \lambda c_1 + \omega c_2 = k_1$, so $c_2 = \frac{k_1 - \lambda k_0}{\omega}$. Substituting c_1 and c_2 into (A) yields

$$y = e^{\lambda(x-x_0)}\left[k_0 \cos\omega(x - x_0) + \left(\frac{k_1 - \lambda k_0}{\omega}\right)\sin\omega(x - x_0)\right].$$

5.2.34. (b)

$$\begin{aligned} e^{i\theta_1}e^{i\theta_2} &= (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2) \\ &= \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2) = e^{i(\theta_1+\theta_2)}. \end{aligned}$$

(c)

$$
\begin{aligned}
e^{z_1+z_2} &= e^{(\alpha_1+i\beta_1)+(\alpha_2+i\beta_2)} = e^{(\alpha_1+\alpha_2)+i(\beta_1+\beta_2)} \\
&= e^{(\alpha_1+\alpha_2)}e^{i(\beta_1+\beta_2)} \text{ (from (F) with } \alpha=\alpha_1+\alpha_2 \text{ and } \beta=\beta_1+\beta_2) \\
&= e^{\alpha_1}e^{\alpha_2}e^{i(\beta_1+\beta_2)} \text{ (property of the real–valued exponential function)} \\
&= e^{\alpha_1}e^{\alpha_2}e^{i\beta_1}e^{i\beta_2} \text{ (from } \mathbf{(b)}) \\
&= e^{\alpha_1}e^{i\beta_1}e^{\alpha_2}e^{i\beta_2} = e^{\alpha_1+i\beta_1}e^{\alpha_2+i\beta_2} = e^{z_1}e^{z_2}.
\end{aligned}
$$

(d) The real and imaginary parts of $z_1 = e^{(\lambda+i\omega)x}$ are $u_1 = e^{\lambda x}\cos\omega x$ and $v_1 = e^{\lambda x}\sin\omega x$, which are both solutions of $ay''+by'+cy=0$, by Theorem 5.2.1(c). Similarly, the real and imaginary parts of $z_2 = e^{(\lambda-i\omega)x}$ are $u_2 = e^{\lambda x}\cos(-\omega x) = e^{\lambda x}\cos\omega x$ and $v_1 = e^{\lambda x}\sin(-\omega x) = -e^{\lambda x}\sin\omega x$, which are both solutions of $ay''+by'+cy=0$, by Theorem 5.2.1,(c).

5.3 NONHOMOGENEOUS LINEAR EQUATIONS

5.3.2. The characteristic polynomial of the complementary equation is $p(r) = r^2-4r+5 = (r-2)^2+1$, so $\{e^{2x}\cos x, e^{2x}\sin x\}$ is a fundamental set of solutions for the complementary equation. Let $y_p = A+Bx$; then $y_p''-4y_p'+5y_p = -4B+5(A+Bx) = 1+5x$. Therefore,$5B=5$, $-4B+5A=1$, so $B=1$, $A=1$. Therefore,$y_p = 1+x$ is a particular solution and $y = 1+x+e^{2x}(c_1\cos x+c_2\sin x)$ is the general solution.

5.3.4. The characteristic polynomial of the complementary equation is $p(r) = r^2-4r+4 = (r-2)^2$, so $\{e^{2x}, xe^{2x}\}$ is a fundamental set of solutions for the complementary equation. Let $y_p = A+Bx+Cx^2$; then $y_p''-4y_p'+4y_p = 2C-4(B+2Cx)+4(A+Bx+Cx^2) = (2C-4B+4A)+(-8C+4B)x+4Cx^2 = 2+8x-4x^2$. Therefore,$4C=-4$, $-8C+4B=8$, $2C-4B+4A=2$, so $C=-1$, $B=0$, and $A=1$. Therefore,$y_p = 1-x^2$ is a particular solution and $y = 1-x^2+e^{2x}(c_1+c_2x)$ is the general solution.

5.3.6. The characteristic polynomial of the complementary equation is $p(r) = r^2+6r+10 = (r+3)^2+1$, so $\{e^{-3x}\cos x, e^{-3x}\sin x\}$ is a fundamental set of solutions for the complementary equation. Let $y_p = A+Bx$; then $y_p''+6y_p'+10y_p = 6B+10(A+Bx) = 22+20x$. Therefore,$10B=20$, $6B+10A=22$, so $B=2$, $A=1$. Therefore,$y_p = 1+2x$ is a particular solution and (A) $y = 1+2x+e^{-3x}(c_1\cos x+c_2\sin x)$ is the general solution. Now $y(0)=2 \Rightarrow 2 = 1+c_1 \Rightarrow c_1=1$. Differentiating (A) yields $y' = 2-3e^{-3x}(c_1\cos x+c_2\sin x)+e^{-3x}(-c_1\sin x+c_2\cos x)$, so $y'(0) = -2 \Rightarrow -2 = 2-3c_1+c_2 \Rightarrow c_2=-1$. $y = 1+2x+e^{-3x}(\cos x-\sin x)$ is the solution of the initial value problem.

5.3.8. If $y_p = \dfrac{A}{x}$, then $x^2y_p''+7xy_p'+8y_p = A\left(x^2\left(\dfrac{2}{x^3}\right)+7x\left(\dfrac{-1}{x^2}\right)+\left(\dfrac{8}{x}\right)\right) = \dfrac{3A}{x} = \dfrac{6}{x}$ if $A=2$. Therefore,$y_p = \dfrac{2}{x}$ is a particular solution.

5.3.10. If $y_p = Ax^3$, then $x^2y_p''-xy_p'+y_p = A\left(x^2(6x)-x(3x^2)+x^3\right) = 4Ax^3 = 2x^3$ if $A=\dfrac{1}{2}$. Therefore,$y_p = \dfrac{x^3}{2}$ is a particular solution.

5.3.12. If $y_p = Ax^{1/3}$, then $x^2y_p''+xy_p'+y_p = A\left(x^2\left(\dfrac{-2x^{-5/3}}{9}\right)+x\left(\dfrac{x^{-2/3}}{3}\right)+x^{1/3}\right) = \dfrac{10A}{9}x^{1/3} = 10x^{1/3}$ if $A=9$. Therefore,$y_p = 9x^{1/3}$ is a particular solution.

5.3.14. If $y_p = \dfrac{A}{x^3}$, then $x^2y_p'' + 3xy_p' - 3y_p = A\left(x^2\left(\dfrac{12}{x^5}\right) + 3x\left(\dfrac{-3}{x^4}\right) + \dfrac{3}{x^3}\right) = 0$. Therefore,$y_p$ is not a solution of the given equation for any choice of A.

5.3.16. The characteristic polynomial of the complementary equation is $p(r) = r^2+5r-6 = (r+6)(r-1)$, so $\{e^{-6x}, e^x\}$ is a fundamental set of solutions for the complementary equation. Let $y_p = Ae^{3x}$; then $y_p'' + 5y_p' - 6y_p = p(3)Ae^{3x} = 18Ae^{3x} = 6e^{3x}$ if $A = \dfrac{1}{3}$. Therefore,$y_p = \dfrac{e^{3x}}{3}$ is a particular solution and $y = \dfrac{e^{3x}}{3} + c_1e^{-6x} + c_2e^x$ is the general solution.

5.3.18. The characteristic polynomial of the complementary equation is $p(r) = r^2+8r+7 = (r+1)(r+7)$, so $\{e^{-7x}, e^{-x}\}$ is a fundamental set of solutions for the complementary equation. Let $y_p = Ae^{-2x}$; then $y_p'' + 8y_p' + 7y_p = p(-2)Ae^{-2x} = -5Ae^{-2x} = 10e^{-2x}$ if $A = -2$. Therefore,$y_p = -2e^{-2x}$ is a particular solution and (A) $y = -2e^{-2x} + c_1e^{-7x} + c_2e^{-x}$ is the general solution. Differentiating (A) yields $y' = 4e^{-2x} - 7c_1e^{-7x} - c_2e^{-x}$. Now $y(0) = -2 \Rightarrow -2 = -2 + c_1 + c_2$ and $y'(0) = 10 \Rightarrow 10 = 4 - 7c_1 - c_2$. Therefore,$c_1 = -1$ and $c_2 = 1$, so $y = -2e^{-2x} - e^{-7x} + e^{-x}$ is the solution of the initial value problem.

5.3.20. The characteristic polynomial of the complementary equation is $p(r) = r^2 + 2r + 10 = (r+1)^2 + 9$, so $\{e^{-x}\cos 3x, e^{-x}\sin 3x\}$ is a fundamental set of solutions for the complementary equation. If $y_p = Ae^{x/2}$, then $y_p'' + 2y_p' + 10y_p = p(1/2)Ae^{x/2} = \dfrac{45}{4}Ae^{x/2} = e^{x/2}$ if $A = \dfrac{4}{45}$. Therefore, $y_p = \dfrac{4}{45}e^{x/2}$ is a particular solution and $y = \dfrac{4}{45}e^{x/2} + e^{-x}(c_1\cos 3x + c_2\sin 3x)$ is the general solution.

5.3.22. The characteristic polynomial of the complementary equation is $p(r) = r^2 - 7r + 12 = (r-4)(r-3)$. If $y_p = Ae^{4x}$, then $y_p'' - 7y_p' + 12y_p = p(4)Ae^{4x} = 0\cdot e^{4x} = 0$, so $y_p'' - 7y_p' + 12y_p \neq 5e^{4x}$ for any choice of A.

5.3.24. The characteristic polynomial of the complementary equation is $p(r) = r^2 - 8r + 16 = (r-4)^2$, so $\{e^{4x}, xe^{4x}\}$ is a fundamental set of solutions for the complementary equation. If $y_p = A\cos x + B\sin x$, then $y_p'' - 8y_p' + 16y_p = -(A\cos x + B\sin x) - 8(-A\sin x + B\cos x) + 16(A\cos x + B\sin x) = (15A - 8B)\cos x + (8A + 15B)\sin x$, so $15A - 8B = 23,\ 8A + 15B = -7$, which implies that $A = 1$ and $B = -1$. Hence $y_p = \cos x - \sin x$ and $y = \cos x - \sin x + e^{4x}(c_1 + c_2x)$ is the general solution.

5.3.26. The characteristic polynomial of the complementary equation is $p(r) = r^2 - 2r + 3 = (r-1)^2 + 2$, so $\{e^x\cos\sqrt{2}x, e^x\sin\sqrt{2}x\}$ is a fundamental set of solutions for the complementary equation. If $y_p = A\cos 3x + B\sin 3x$, then $y_p'' - 2y_p' + 3y_p = -9(A\cos 3x + B\sin 3x) - 6(-A\sin 3x + B\cos 3x) + 3(A\cos 3x + B\sin 3x) = -(6A + 6B)\cos 3x + (6A - 6B)\sin 3x$, so $-6A - 6B = -6,\ 6A - 6B = 6$, which implies that $A = 1$ and $B = 0$. Hence $y_p = \cos 3x$ is a particular solution and $y = \cos 3x + e^x(c_1\cos\sqrt{2}x + c_2\sin\sqrt{2}x)$ is the general solution.

5.3.28. The characteristic polynomial of the complementary equation is $p(r) = r^2 + 7r + 12 = (r+3)(r+4)$, so $\{e^{-4x}, e^{-3x}\}$ is a fundamental set of solutions for the complementary equation. If $y_p = A\cos 2x + B\sin 2x$, then $y_p'' + 7y_p' + 12y_p = -4(A\cos 2x + B\sin 2x) + 14(-A\sin 2x + B\cos 2x) + 12(A\cos x + B\sin x) = (8A + 14B)\cos 2x + (8B - 14a)\sin 2x$, so $8A + 14B = -2,\ -14A + 8B = 36$, which implies that $A = -2$ and $B = 1$. Hence $y_p = -2\cos 2x + \sin 2x$ is a particular solution and (A) $y = -2\cos 2x + \sin 2x + c_1e^{-4x} + c_2e^{-3x}$ is the general solution. Differentiating (A) yields $y' = 2\sin 2x + 2\cos 2x - 4c_1e^{-4x} - 3c_2e^{-3x}$. Now $y(0) = -3 \Rightarrow -3 = -2 + c_1 + c_2$ and $y'(0) = 3 \Rightarrow 3 = 2 - 4c_1 - 3c_2$. Therefore, $c_1 = 2$ and $c_2 = -3$, so $y = -2\cos 2x + \sin 2x + 2e^{-4x} - 3e^{-3x}$ is the solution of the initial value problem.

5.3.30. $\{\cos\omega_0 x, \sin\omega_0 x\}$ is a fundamental set of solutions of the complementary equation. If $y_p = A\cos\omega x + B\sin\omega x$, then $y_p'' + \omega_0^2 y_p = -\omega^2(A\cos\omega x + B\sin\omega x) + \omega_0^2(A\cos\omega x + B\sin\omega x) = (\omega_0^2 - \omega^2)(A\cos\omega x + B\sin\omega x) = M\cos\omega x + N\sin\omega x$ if $A = \dfrac{M}{\omega_0^2-\omega^2}$ and $B = \dfrac{N}{\omega_0^2-\omega^2}$. Therefore,

$$y_p = \frac{1}{\omega_0^2-\omega^2}(M\cos\omega x + N\sin\omega x)$$

is a particular solution of the given equation and

$$y = \frac{1}{\omega_0^2-\omega^2}(M\cos\omega x + N\sin\omega x) + c_1\cos\omega_0 x + c_2\sin\omega_0 x$$

is the general solution.

5.3.32. If $y_p = A\cos\omega x + B\sin\omega x$, then $ay_p'' + by_p' + cy_p = -a\omega^2(A\cos\omega x + B\sin\omega x) + b\omega(-A\sin\omega x + B\cos\omega x) + c(A\cos\omega x + B\sin\omega x) = \left[(c-a\omega^2)A + b\omega B\right]\cos\omega x + \left[-b\omega A + (c-a\omega^2)B\right]\sin\omega x$. Therefore, y_p is a solution of (A) if and only if the set of equations (B) $(c-a\omega^2)A + b\omega B = M$, $-b\omega A + (c-a\omega^2)B = N$ has a solution. If $(c-a\omega^2)^2 + (b\omega)^2 \ne 0$, then (B) has the solution $A = \dfrac{(c-a\omega^2)M - b\omega N}{(c-a\omega^2)^2+(b\omega)^2}$, $B = \dfrac{(c-a\omega^2)N + b\omega M}{(c-a\omega^2)^2+(b\omega)^2}$, and $y_p = A\cos\omega x + B\sin\omega x$ is a solution of (A). If $(c-a\omega^2)^2 + (b\omega)^2 = 0$ (which is true if and only if the left side of (A) is of the form $a(y'' + \omega^2 y)$, then the coefficients of A and B in (B) are all zero, so (B) does not have a solution, so (A) does not have a solution of the form $y_p = A\cos\omega x + B\sin\omega x$.

5.3.34. From Exercises 5.3.2 and 5.3.17, $y_{p_1} = 1 + x$ and $y_{p_2} = e^{2x}$ are particular solutions of $y'' - 4y' + 5y = 1 + 5x$ and $y'' - 4y' + 5y = e^{2x}$ respectively, and $\{e^{2x}\cos x, e^{2x}\sin x\}$ is a fundamental set of solutions of the complementary equation. Therefore,$y_p = y_{p_1} + y_{p_2} = 1 + x + e^{2x}$ is a particular solution of the given equation, and $y = 1 + x + e^{2x}(1 + c_1\cos x + c_2\sin x)$ is the general solution.

5.3.36. From Exercises 5.3.4 and 5.3.19, $y_{p_1} = 1 - x^2$ and $y_{p_2} = e^x$ are particular solutions of $y'' - 4y' + 4y = 2 + 8x - 4x^2$ and $y'' - 4y' + 4y = e^x$ respectively, and $\{e^{2x}, xe^{2x}\}$ is a fundamental set of solutions of the complementary equation. Therefore,$y_p = y_{p_1} + y_{p_2} = 1 - x^2 + e^x$ is a particular solution of the given equation, and $y = 1 - x^2 + e^x + e^{2x}(c_1 + c_2 x)$ is the general solution.

5.3.38. From Exercises 5.3.6 and 5.3.21, $y_{p_1} = 1 + 2x$ and $y_{p_2} = e^{-3x}$ are particular solutions of $y'' + 6y' + 10y = 22 + 20x$ and $y'' + 6y' + 10y = e^{-3x}$ respectively, and $\{e^{-3x}\cos x, e^{-3x}\sin x\}$ is a fundamental set of solutions of the complementary equation. Therefore,$y_p = y_{p_1} + y_{p_2} = 1 + 2x + e^{-3x}$ is a particular solution of the given equation, and $y = 1 + 2x + e^{-3x}(1 + c_1\cos x + c_2\sin x)$ is the general solution.

5.3.40. Letting $c_1 = c_2 = 0$ shows that (A) $y_p'' + p(x)y_p' + q(x)y_p = f$. Letting $c_1 = 1$ and $c_2 = 0$ shows that (B) $(y_1 + y_p)'' + p(x)(y_1 + y_p)' + q(x)(y_1 + y_p) = f$. Now subtract (A) from (B) to see that $y_1'' + p(x)y_1' + q(x)y_1 = 0$. Letting $c_1 = 0$ and $c_2 = 1$ shows that (C) $(y_2 + y_p)'' + p(x)(y_2 + y_p)' + q(x)(y_2 + y_p) = f$. Now subtract (A) from (C) to see that $y_2'' + p(x)y_2' + q(x)y_2 = 0$.

5.4 THE METHOD OF UNDETERMINED COEFFICIENTS I

5.4.2. If $y = ue^{-3x}$, then $y'' - 6y' + 5y = e^{-3x}\left[(u'' - 6u' + 9u) - 6(u' - 3u) + 5u\right] = e^{-3x}(35 - 8x)$, so $u'' - 12u' + 32u = 35 - 8x$ and $u_p = A + Bx$, where $-12B + 32(A + Bx) = 35 - 8x$. Therefore,$32B = -8$, $32A - 12B = 35$, so $B = -\dfrac{1}{4}$, $A = 1$, and $u_p = 1 - \dfrac{x}{4}$. Therefore, $y_p = e^{-3x}\left(1 - \dfrac{x}{4}\right)$.

5.4.4. If $y = ue^{2x}$, then $y'' + 2y' + y = e^{2x}\left[(u'' + 4u' + 4u) + 2(u' + 2u) + u\right] = e^{2x}(-7 - 15x + 9x^2)$ so $u'' + 6u' + 9u = -7 - 15x + 9x^2$ and $u_p = A + Bx + Cx^2$, where $2C + 6(B + 2Cx) + 9(A + Bx + Cx^2) = -7 - 15x + 9x^2$. Therefore,$9C = 9$, $9B + 12C = -15$, $9A + 6B + 2C = -7$, so $C = 1$, $B = -3$, $A = 1$, and $u_p = 1 - 3x + x^2$. Therefore,$y_p = e^{2x}(1 - 3x + x^2)$.

5.4.6. If $y = ue^{x}$, then $y'' - y' - 2y = e^{x}\left[(u'' + 2u' + u) - (u' + u) - 2u\right] = e^{x}(9 + 2x - 4x^2)$ so $u'' + u' - 2u = 9 + 2x - 4x^2$, and $u_p = A + Bx + Cx^2$, where $2C + (B + 2Cx) - 2(A + Bx + Cx^2) = 9 + 2x - 4x^2$. Therefore,$-2C = -4$, $-2B + 2C = 2$, $-2A + B + 2C = 9$, so $C = 2$, $B = 1$, $A = -2$, and $u_p = -2 + x + 2x^2$. Therefore, $y_p = e^{x}(-2 + x + 2x^2)$.

5.4.8. If $y = ue^{x}$, then $y'' - 3y' + 2y = e^{x}\left[(u'' + 2u' + u) - 3(u' + u) + 2u\right] = e^{x}(3 - 4x)$, so $u'' - u' = 3 - 4x$ and $u_p = Ax + Bx^2$, where $2B - (A + 2Bx) = 3 - 4x$. Therefore,$-2B = -4$, $-A + 2B = 3$, so $B = 2$, $A = 1$, and $u_p = x(1 + 2x)$. Therefore,$y_p = xe^{x}(1 + 2x)$.

5.4.10. If $y = ue^{2x}$, then $2y'' - 3y' - 2y = e^{2x}\left[2(u'' + 4u' + 4u) - 3(u' + 2u) - 2u\right] = e^{2x}(-6 + 10x)$, so $2u'' + 5u' = -6 + 10x$ and $u_p = Ax + Bx^2$, where $2(2B) + 5(A + 2Bx) = -6 + 10x$. Therefore,$10B = 10$, $5A + 4B = -6$, so $B = 1$, $A = -2$, and $u_p = x(-2 + x)$. Therefore,$y_p = xe^{2x}(-2 + x)$.

5.4.12. If $y = ue^{x}$, then $y'' - 2y' + y = e^{x}\left[(u'' + 2u' + u) - 2(u' + u) + u\right] = e^{x}(1 - 6x)$, so $u'' = 1 - 6x$ Integrating twice and taking the constants of integration to be zero yields $u_p = x^2\left(\frac{1}{2} - x\right)$. Therefore, $y_p = x^2e^{x}\left(\frac{1}{2} - x\right)$.

5.4.14. If $y = ue^{-x/3}$, then $9y'' + 6y' + y = e^{-x/3}\left[9\left(u'' - \frac{2u'}{3} + \frac{u}{9}\right) + 6\left(u' - \frac{u}{3}\right) + u\right] = e^{-x/3}(2 - 4x + 4x^2)$, so $9u'' = 2 - 4x + 4x^2$, or $u'' = \frac{1}{9}(2 - 4x + 4x^2)$. Integrating twice and taking the constants of integration to be zero yields $u_p = \frac{x^2}{27}(3 - 2x + x^2)$. Therefore, $y_p = \frac{x^2e^{-x/3}}{27}(3 - 2x + x^2)$.

5.4.16. If $y = ue^{x}$, then $y'' - 6y' + 8y = e^{x}\left[(u'' + 2u' + u) - 6(u' + u) + 8u\right] = e^{x}(11 - 6x)$, so $u'' - 4u' + 3u = 11 - 6x$ and $u_p = A + Bx$, where $-4B + 3(A + Bx) = 11 - 6x$. Therefore,$3B = -6$, $3A - 4B = 11$, so $B = -2$, $A = 1$ and $u_p = 1 - 2x$. Therefore,$y_p = e^{x}(1 - 2x)$. The characteristic polynomial of the complementary equation is $p(r) = r^2 - 6r + 8 = (r - 2)(r - 4)$, so $\{e^{2x}, e^{4x}\}$ is a fundamental set of solutions of the complementary equation. Therefore, $y = e^{x}(1 - 2x) + c_1e^{2x} + c_2e^{4x}$ is the general solution of the nonhomogeneous equation.

5.4.18. If $y = ue^{x}$, then $y'' + 2y' - 3y = e^{x}\left[(u'' + 2u' + u) + 2(u' + u) - 3u\right] = -16xe^{x}$, so $u'' + 4u' = -16x$ and $u_p = Ax + Bx^2$, where $2B + 4(A + 2Bx) = -16x$. Therefore,$8B = -16$, $4A + 2B = 0$, so $B = -2$, $A = 1$, and $u_p = x(1 - 2x)$. Therefore,$y_p = xe^{x}(1 - 2x)$. The characteristic polynomial of the complementary equation is $p(r) = r^2 + 2r - 3 = (r + 3)(r - 1)$, so $\{e^{x}, e^{-3x}\}$ is a fundamental set of solutions of the complementary equation. Therefore, $y = xe^{x}(1 - 2x) + c_1e^{x} + c_2e^{-3x}$ is the general solution of the nonhomogeneous equation.

5.4.20. If $y = ue^{2x}$, then $y'' - 4y' - 5y = e^{2x}\left[(u'' + 4u' + 4u) - 4(u' + 2u) - 5u\right] = 9e^{2x}(1 + x)$, so $u'' - 9u = 9 + 9x$ and $u_p = A + Bx$, where $-9(A + Bx) = 9 + 9x$. Therefore,$-9B = -9$, $-9A = 9$, so $B = -1$, $A = -1$, and $u_p = -1 - x$. Therefore,$y_p = -e^{2x}(1 + x)$. The characteristic polynomial of the complementary equation is $p(r) = r^2 - 4r - 5 = (r - 5)(r + 1)$, so $\{e^{-x}, e^{5x}\}$ is a fundamental set of solutions of the complementary equation. Therefore,(A) $y = -e^{2x}(1+x) + c_1e^{-x} + c_2e^{5x}$ is the general solution of the nonhomogeneous equation. Differentiating (A) yields $y' = -2e^{2x}(1+x) - e^{2x} - c_1e^{-x} + 5c_2e^{5x}$. Now $y(0) = 0,\ y'(0) = -10 \Rightarrow 0 = -1 + c_1 + c_2,\ -10 = -3 - c_1 + 5c_2$, so $c_1 = 2, c_2 = -1$. Therefore,$y = -e^{2x}(1+x) + 2e^{-x} - e^{5x}$ is the solution of the initial value problem.

5.4.22. If $y = ue^{-x}$, then $y'' + 4y' + 3y = e^{-x}\left[(u'' - 2u' + u) + 4(u' - u) + 3u\right] = -e^{-x}(2+8x)$, so $u'' + 2u' = -2 - 8x$ and $u_p = Ax + Bx^2$, where $2B + 2(A + 2Bx) = -2 - 8x$. Therefore,$4B = -8$, $2A + 2B = -2$, so $B = -2$, $A = 1$, and $u_p = x(1-2x)$. Therefore,$y_p = xe^{-x}(1-2x)$. The characteristic polynomial of the complementary equation is $p(r) = r^2 + 4r + 3 = (r+3)(r+1)$, so $\{e^{-x}, e^{-3x}\}$ is a fundamental set of solutions of the complementary equation. Therefore,(A) $y = xe^{-x}(1-2x) + c_1e^{-x} + c_2e^{-3x}$ is the general solution of the nonhomogeneous equation. Differentiating (A) yields $y' = -xe^{-x}(1-2x) + e^{-x}(1-4x) - c_1e^{-x} - 3c_2e^{-3x}$. Now $y(0) = 1,\ y'(0) = 2 \Rightarrow 1 = c_1 + c_2,\ 2 = 1 - c_1 - 3c_2$, so $c_1 = 2, c_2 = -1$. Therefore,$y = e^{-x}(2 + x - 2x^2) - e^{-3x}$ is the solution of the initial value problem.

5.4.24. We must find particular solutions y_{p_1} and y_{p_2} of (A) $y'' + y' + y = xe^x$ and (B) $y'' + y' + y = e^{-x}(1+2x)$, respectively. To find a particular solution of (A) we write $y = ue^x$. Then $y'' + y' + y = e^x\left[(u'' + 2u' + u) + (u' + u) + u\right] = xe^x$ so $u'' + 3u' + 3u = x$ and $u_p = A + Bx$, where $3B + 3(A + Bx) = x$. Therefore,$3B = 1$, $3A + 3B = 0$, so $B = \frac{1}{3}$, $A = -\frac{1}{3}$, and $u_p = -\frac{1}{3}(1-x)$, so $y_{p_1} = -\frac{e^x}{3}(1-x)$. To find a particular solution of (B) we write $y = ue^{-x}$. Then $y'' + y' + y = e^{-x}\left[(u'' - 2u' + u) + (u' - u) + u\right] = e^{-x}(1+2x)$, so $u'' - u' + u = 1 + 2x$ and $u_p = A + Bx$, where $-B + (A + Bx) = 1 + 2x$. Therefore, $B = 2$, $A - B = 1$, so $A = 3$, and $u_p = 2 + 3x$, so $y_{p_2} = e^{-x}(3+2x)$. Now $y_p = y_{p_1} + y_{p_2} = -\frac{e^x}{3}(1-x) + e^{-x}(3+2x)$.

5.4.26. We must find particular solutions y_{p_1} and y_{p_2} of (A) $y'' - 8y' + 16y = 6xe^{4x}$ and (B) $y'' - 8y' + 16y = 2 + 16x + 16x^2$, respectively. To find a particular solution of (A) we write $y = ue^{4x}$. Then $y'' - 8y' + 16y = e^x\left[(u'' + 8u' + 16u) - 8(u' + 4u) + 16u\right] = 6xe^{4x}$, so $u'' = 6x$, $u_p = x^3$. and $y_{p_1} = x^3e^{4x}$. To find a particular solution of (B) we write $y_p = A + Bx + Cx^2$. Then $y_p'' - 8y_p' + 16y_p = 2C - 8(B + 2Cx) + 16(A + Bx + Cx^2) = (16A - 8B + 2C) + (16B - 16C)x + 16Cx^2 = 2 + 16x + 16x^2$ if $16C = 16$, $16B - 16C = 16$, $16A - 8B + 2C = 2$. Therefore,$C = 1$, $B = 2$, $A = 1$, and $y_{p_2} = 1 + 2x + x^2$. Now $y_p = y_{p_1} + y_{p_2} = x^3e^{4x} + 1 + 2x + x^2$.

5.4.28. We must find particular solutions y_{p_1} and y_{p_2} of (A) $y'' - 2y' + 2y = e^x(1+x)$ and (B) $y'' - 2y' + 2y = e^{-x}(2 - 8x + 5x^2)$, respectively. To find a particular solution of (A) we write $y = ue^x$. Then $y'' - 2y' + 2y = e^x\left[(u'' + 2u' + u) - 2(u' + u) + 2u\right] = e^x(1+x)$, so $u'' + u = 1 + x$ and $u_p = 1 + x$, so $y_{p_1} = e^x(1+x)$. To find a particular solution of (B) we write $y = ue^{-x}$. Then $y'' - 2y' + 2y = e^{-x}\left[(u'' - 2u' + u) - 2(u' - u) + 2u\right] = e^{-x}(2 - 8x + 5x^2)$, so $u'' - 4u' + 5u = 2 - 8x + 5x^2$ and $u_p = A + Bx + Cx^2$, where $2C - 4(B + 2Cx) + 5(A + Bx + Cx^2) = 2 - 8x + 5x^2$. Therefore,$5C = 5$, $5B - 8C = -8$, $5A - 4B + 2C = 2$, so $C = 1$, $B = 0$, $A = 0$, and $u_p = x^2$. Therefore,$y_{p_2} = x^2e^{-x}$. Now $y_p = y_{p_1} + y_{p_2} = e^x(1+x) + x^2e^{-x}$.

5.4.30. **(a)** If $y = ue^{\alpha x}$, then $ay'' + by' + cy = e^{\alpha x}\left[a(u'' + 2\alpha u' + \alpha^2 u) + b(u' + \alpha u) + cu\right] = e^{\alpha x}\left[au'' + (2a\alpha + b))u' + (a\alpha^2 + b\alpha + c)u\right] = e^{\alpha x}(au'' + p'(\alpha)u' + p(\alpha)u)$. Therefore,$ay'' + by' + cy = e^{\alpha x}G(x)$ if and only if $au'' + p'(\alpha)u' + p(\alpha)u = G(x)$.

(b) Substituting $u_p = A + Bx + Cx^2 + Dx^3$ into (B) yields

$$
\begin{aligned}
&a(2C + 6Dx) + p'(\alpha)(B + 2Cx + 3Dx^2) + p(\alpha)(A + Bx + Cx^2 + Dx^3) \\
&= [p(\alpha)A + p'(\alpha)B + 2aC] + [p(\alpha)B + 2p'(\alpha)C + 6aD]x \\
&\quad + [p(\alpha)C + 3p'(\alpha)D]x^2 + p(\alpha)Dx^3 = g_0 + g_1x + g_2x^2 + g_3x^3
\end{aligned}
$$

if

$$\begin{array}{rcl} p(\alpha)D &=& g_3 \\ p(\alpha)C+3p'(\alpha)D &=& g_2 \\ p(\alpha)B+2p'(\alpha)C+6aD &=& g_1 \\ p(\alpha)A+p'(\alpha)B+2aC &=& g_0. \end{array} \qquad \text{(C)}$$

Since $e^{\alpha x}$ is not a solution of the complementary equation, $p(\alpha)\neq 0$. Therefore,the triangular system (C) can be solved successively for D, C, B and A.

(c) Since $e^{\alpha x}$ is a solution of the complementary equation while $xe^{\alpha x}$ is not, $p(\alpha)=0$ and $p'(\alpha)\neq 0$. Therefore, (B) reduces to (D) $au''+p'(\alpha)u=G(x)$. Substituting $u_p=Ax+Bx^2+Cx^3+Dx^4$ into (D) yields

$$\begin{aligned} & a(2B+6Cx+12Dx^2)+p'(\alpha)(A+2Bx+3Cx^2+4Dx^3) \\ &= (p'(\alpha)A+2aB)+(2p'(\alpha)B+6aC)x+(3p'(\alpha)C+12aD)x^2 \\ &\quad +4p'(\alpha)Dx^3 = g_0+g_1x+g_2x^2+g_3x^3 \end{aligned}$$

if

$$\begin{array}{rcl} 4p'(\alpha)D &=& g_3 \\ 3p'(\alpha)C+12aD &=& g_2 \\ 2p'(\alpha)B+6aC &=& g_1 \\ p'(\alpha)A+2aB &=& g_0. \end{array}$$

Since $p'(\alpha)\neq 0$ this triangular system can be solved successively for D, C, B and A.

(d) Since $e^{\alpha x}$ and $xe^{\alpha x}$ are solutions of the complementary equation, $p(\alpha)=0$ and $p'(\alpha)=0$. Therefore, (B) reduces to (D) $au''=G(x)$, so $u''=\dfrac{G(x)}{a}$. Integrating this twice and taking the constants of integration yields the particular solution $u_p=x^2\left(\dfrac{g_0}{2}+\dfrac{g_1}{6}x+\dfrac{g_2}{12}x^2+\dfrac{g_3}{20}x^3\right)$.

5.4.32. If $y_p=Axe^{4x}$, then $y_p''-7y_p'+12y_p=[(8+16x)-7(1+4x)+12x]Ae^{4x}=Ae^{4x}=5e^{4x}$ if $A=1$, so $y_p=5xe^{4x}$.

5.4.34. If $y_p=e^{3x}(A+Bx+Cx^2)$, then

$$\begin{aligned} y_p''-3y_p'+2y_p &= e^{3x}[(9A+6B+2C)+(9B+12C)x+9Cx^2] \\ &\quad -3e^{3x}[(3A+B)+(3B+2C)x+3Cx^2] \\ &\quad +2e^{3x}(A+Bx+Cx^2) \\ &= e^{3x}[(2A+3B+2C)+(2B+6C)x+2Cx^2] \\ &= e^{3x}(-1+2x+x^2) \end{aligned}$$

if $2C=1,\ 2B+6C=2,\ 2A+3B+2C=-1$. Therefore,$C=\dfrac{1}{2}$, $B=-\dfrac{1}{2}$, $A=-\dfrac{1}{4}$, and $y_p=-\dfrac{e^{3x}}{4}(1+2x-2x^2)$.

5.4.36. If $y_p=e^{-x/2}(Ax^2+Bx^3+Cx^4)$, then

$$\begin{aligned} 4y_p''+4y_p'+y_p &= e^{-x/2}[8A-(8A-24B)x+(A-12B+48C)x^2] \\ &\quad +e^{-x/2}[(B-16C)x^3+Cx^4] \\ &\quad +e^{-x/2}[8Ax-(2A-12B)x^2-(2B-16C)x^3-2Cx^4] \\ &\quad +e^{-x/2}(Ax^2+Bx^3+Cx^4) \\ &= e^{-x/2}(8A+24Bx+48Cx^2)=e^{-x/2}(-8+48x+144x^2) \end{aligned}$$

if $48C = 144$, $24B = 48$, and $8A = -8$. Therefore,$C = 3$, $B = 2$, $A = -1$, and $y_p = x^2e^{-x/2}(-1 + 2x + 3x^2)$.

5.4.38. If $y = \int e^{\alpha x}P(x)\,dx$, then $y' = e^{\alpha x}P(x)$. Let $y = ue^{\alpha x}$; then $(u' + \alpha u)e^{\alpha x} = e^{\alpha x}P(x)$, which implies (A). We must show that it is possible to choose $A_0, \dots, A_k$ so that (B) $(A_0 + A_1x \cdots + A_kx^k)' + \alpha(A_0 + A_1x \cdots + A_kx^k) = p_0 + p_1x + \cdots + p_kx^k$. By equating the coefficients of $x^k, x^{k-1}, \dots, 1$ (in that order) on the two sides of (B), we see that (B) holds if and only if $\alpha A_k = p_k$ and $(k - j + 1)A_{k-j+1} + \alpha A_k = p_{k-j}$, $1 \le j \le k$.

5.4.40. If $y = \int x^ke^{\alpha x}\,dx$, then $y' = x^ke^{\alpha x}$. Let $y = ue^{\alpha x}$; then $(u' + \alpha u)e^{\alpha x} = x^ke^{\alpha x}$, so $u' + \alpha u = x^k$. This equation has a particular solution $u_p = A_0 + A_1x \cdots + A_kx^k$, where (A) $(A_0 + A_1x \cdots + A_kx^k)' + \alpha(A_0 + A_1x \cdots + A_kx^k) = x^k$. By equating the coefficients of $x^k, x^{k-1}, \dots, 1$ on the two sides of (A), we see that (A) holds if and only if $\alpha A_k = 1$ and $(k - j + 1)A_{k-j+1} + \alpha A_k - j = 0$, $1 \le j \le k$. Therefore, $A_k = \dfrac{1}{\alpha}$, $A_{k-1} = -\dfrac{k}{\alpha^2}$, $A_{k-2} = \dfrac{k(k-1)}{\alpha^3}$, and, in general, $A_{k-j} = (-1)^j\dfrac{k(k-1)\cdots(k-j+1)}{\alpha^{j+1}} = \dfrac{(-1)^jk!}{\alpha^{j+1}(k-j)!}$, $1 \le j \le k$. By introducing the index $r = k - j$ we can rewrite this as $A_r = \dfrac{(-1)^{k-r}k!}{\alpha^{k-r+1}r!}$, $0 \le r \le k$. Therefore, $u_p = \dfrac{(-1)^kk!}{\alpha^{k+1}}\displaystyle\sum_{r=0}^{k}\frac{(-\alpha x)^r}{r!}$ and $y = \dfrac{(-1)^kk!e^{\alpha x}}{\alpha^{k+1}}\displaystyle\sum_{r=0}^{k}\frac{(-\alpha x)^r}{r!} + c$.

5.5 THE METHOD OF UNDETERMINED COEFFICIENTS II

5.5.2. Let

$$\begin{aligned} y_p &= (A_0 + A_1x)\cos x + (B_0 + B_1x)\sin x;\ \text{then} \\ y_p' &= (A_1 + B_0 + B_1x)\cos x + (B_1 - A_0 - A_1x)\sin x \\ y_p'' &= (2B_1 - A_0 - A_1x)\cos x - (2A_1 + B_0 + B_1x)\sin x,\ \text{so} \end{aligned}$$

$$\begin{aligned} y_p'' + 3y_p' + y_p &= (3A_1 + 3B_0 + 2B_1 + 3B_1x)\cos x \\ &\quad +(3B_1 - 3A_0 - 2A_1 - 3A_1x)\sin x \\ &= (2 - 6x)\cos x - 9\sin x \end{aligned}$$

if $3B_1 = -6$, $-3A_1 = 0$, $3B_0 + 3A_1 + 2B_1 = 2$, $-3A_0 + 3B_1 + 2A_1 = -9$. Therefore,$A_1 = 0$, $B_1 = -2$, $A_0 = 1$, $B_0 = 2$, and $y_p = \cos x + (2 - 2x)\sin x$.

5.5.4. Let $y = ue^{2x}$. Then

$$\begin{aligned} y'' + 3y' - 2y &= e^{2x}\left[(u'' + 4u' + 4u) + 3(u' + 2u) - 2u\right] \\ &= e^{2x}(u'' + 7u' + 8u) = -e^{2x}(5\cos 2x + 9\sin 2x) \end{aligned}$$

if $u'' + 7u' + 8u = -5\cos 2x - 9\sin 2x$. Now let $u_p = A\cos 2x + B\sin 2x$. Then

$$\begin{aligned} u_p'' + 7u_p' + 8u_p &= -4(A\cos 2x + B\sin 2x) + 14(-A\sin 2x + B\cos 2x) \\ &\quad +8(A\cos 2x + B\sin 2x) \\ &= (4A + 14B)\cos 2x - (14A - 4B)\sin 2x \\ &= -5\cos 2x - 9\sin 2x \end{aligned}$$

if $4A + 14B = -5$, $-14A + 4B = -9$. Therefore, $A = \frac{1}{2}$, $B = -\frac{1}{2}$,

$$u_p = \frac{1}{2}(\cos 2x - \sin 2x), \text{ and } y_p = \frac{e^{2x}}{2}(\cos 2x - \sin 2x).$$

5.5.6. Let $y = ue^{-2x}$. Then

$$\begin{aligned} y'' + 3y' - 2y &= e^{-2x}\left[(u'' - 4u' + 4u) + 3(u' - 2u) - 2u\right] \\ &= e^{-2x}(u'' - u' - 4u) \\ &= e^{-2x}\left[(4 + 20x)\cos 3x + (26 - 32x)\sin 3x\right] \end{aligned}$$

if $u'' - u' - 4u = (4 + 20x)\cos 3x + (26 - 32x)\sin 3x$. Let

$$\begin{aligned} u_p &= (A_0 + A_1x)\cos 3x + (B_0 + B_1x)\sin 3x; \text{ then} \\ u_p' &= (A_1 + 3B_0 + 3B_1x)\cos 3x + (B_1 - 3A_0 - 3A_1x)\sin 3x \\ u_p'' &= (6B_1 - 9A_0 - 9A_1x)\cos 3x - (2A_1 + 9B_0 + 9B_1x)\sin 3x, \text{ so} \end{aligned}$$

$$\begin{aligned} u_p'' - u_p' - 4u_p &= -\left[13A_0 + A_1 + 3B_0 - 6B_1 + (13A_1 + 3B_1)x\right]\cos 3x \\ &\quad -\left[13B_0 + B_1 - 3A_0 + 6A_1 + (13B_1 - 3A_1)x\right]\sin 3x \\ &= (4 + 20x)\cos 3x + (26 - 32x)\sin 3x \text{ if} \end{aligned}$$

$$\begin{aligned} -13A_1 - 3B_1 &= 20 \\ 3A_1 - 13B_1 &= -32 \end{aligned} \quad \text{and} \quad \begin{aligned} -13A_0 - 3B_0 - A_1 + 6B_1 &= 4 \\ 3A_0 - 13B_0 - 6A_1 - B_1 &= 26. \end{aligned}$$

From the first two equations, $A_1 = -2$, $B_1 = 2$. Substituting these in the last two equations yields $-13A_0 - 3B_0 = -10$, $3A_0 - 13B_0 = 16$. Solving this pair yields $A_0 = 1$, $B_0 = -1$. Therefore,

$$u_p = (1 - 2x)(\cos 3x - \sin 3x) \text{ and } y_p = e^{-2x}(1 - 2x)(\cos 3x - \sin 3x).$$

5.5.8. Let

$$\begin{aligned} y_p &= (A_0x + A_1x^2)\cos x + (B_0x + B_1x^2)\sin x; \text{ then} \\ y_p' &= \left[A_0 + (2A_1 + B_0)x + B_1x^2\right]\cos x + \left[B_0 + (2B_1 - A_0)x - B_1x^2\right]\sin x \\ y_p'' &= \left[2A_1 + 2B_0 - (A_0 - 4B_1)x - A_1x^2\right]\cos x \\ &\quad + \left[2B_1 - 2A_0 - (B_0 + 4A_1)x - B_1x^2\right]\sin x, \text{ so} \end{aligned}$$

$$\begin{aligned} y_p'' + y_p &= (2A_1 + 2B_0 + 4B_1x)\cos x + (2B_1 - 2A_0 - 4A_1x)\sin x \\ &= (-4 + 8x)\cos x + (8 - 4x)\sin x \end{aligned}$$

if $4B_1 = 8$, $-4A_1 = -4$, $2B_0 + 2A_1 = -4$, $-2A_0 + 2B_1 = 8$. Therefore, $A_1 = 1$, $B_1 = 2$, $A_0 = -2$, $B_0 = -3$, and $y_p = -x\left[(2 - x)\cos x + (3 - 2x)\sin x\right]$.

5.5.10. Let $y = ue^{-x}$. Then

$$\begin{aligned} y'' + 2y' + 2y &= e^{-x}\left[(u'' - 2u' + u) + 2(u' - u) + 2u\right] \\ &= e^{-x}(u'' + u) = e^{-x}(8\cos x - 6\sin x) \end{aligned}$$

if $u'' + u = 8\cos x - 6\sin x$. Now let

$$\begin{array}{rcl} u_p &=& Ax\cos x + Bx\sin x;\ \text{then} \\ u_p' &=& (A + Bx)\cos x + (B - Ax)\sin x \\ u_p'' &=& (2B - Ax)\cos x - (2A + Bx)\sin x,\ \text{so} \end{array}$$

$$u_p'' + u_p = 2B\cos x - 2A\sin x = 8\cos x - 6\sin x$$

if $2B = 8$, $-2A = -6$. Therefore,$A = 3$, $B = 4$, $u_p = x(3\cos x + 4\sin x)$, and $y_p = xe^{-x}(3\cos x + 4\sin x)$.

5.5.12. Let

$$\begin{array}{rcl} y_p &=& (A_0 + A_1x + A_2x^2)\cos x + (B_0 + B_1x + B_2x^2)\sin x;\ \text{then} \\ y_p' &=& \left[A_1 + B_0 + (2A_2 + B_1)x + B_2x^2\right]\cos x \\ && +\left[B_1 - A_0 + (2B_2 - A_1)x - A_2x^2\right]\sin x, \\ y_p'' &=& \left[-A_0 + 2A_2 + 2B_1 - (A_1 - 4B_2)x - A_2x^2\right]\cos x \\ && +\left[-B_0 + 2B_2 - 2A_1 - (B_1 + 4A_2)x - B_2x^2\right]\sin x,\ \text{so} \end{array}$$

$$\begin{array}{rcl} y_p'' + 2y_p' + y_p &=& 2\left[A_1 + A_2 + B_0 + B_1 + (2A_2 + B_1 + 2B_2)x + B_2x^2\right]\cos x \\ && +2\left[B_1 + B_2 - A_0 - A_1 + (2B_2 - A_1 - 2A_2)x - A_2x^2\right]\sin x \\ &=& 8x^2\cos x - 4x\sin x\ \text{if} \end{array}$$

$$\text{(i)}\ \begin{array}{rcl} 2B_2 &=& 8 \\ -2A_2 &=& 0 \end{array},\ \text{(ii)}\ \begin{array}{rcl} 2B_1 + 4A_2 + 4B_2 &=& 0 \\ -2A_1 - 4A_2 + 4B_2 &=& -4 \end{array},$$

$$\text{(iii)}\ \begin{array}{rcl} 2B_0 + 2A_1 + 2B_1 + 2A_2 &=& 0 \\ -2A_0 - 2A_1 + 2B_1 + 2B_2 &=& 0 \end{array}.$$

From (i), $A_2 = 0$, $B_2 = 4$. Substituting these into (ii) and solving for A_1 and B_1 yields $A_1 = 10$, $B_1 = -8$. Substituting the known coefficients into (iii) and solving for A_0 and B_0 yields $A_0 = -14$, $B_0 = -2$. Therefore,$y_p = -(14 - 10x)\cos x - (2 + 8x - 4x^2)\sin x$.

5.5.14. Let

$$\begin{array}{rcl} y_p &=& (A_0 + A_1x + A_2x^2)\cos 2x + (B_0 + B_1x + B_2x^2)\sin 2x;\ \text{then} \\ y_p' &=& \left[A_1 + 2B_0 + (2A_2 + 2B_1)x + 2B_2x^2\right]\cos 2x \\ && +\left[B_1 - 2A_0 + (2B_2 - 2A_1)x - 2A_2x^2\right]\sin 2x \\ y_p'' &=& \left[-4A_0 + 2A_2 + 4B_1 - (4A_1 - 8B_2)x - 4A_2x^2\right]\cos 2x \\ && +\left[-4B_0 - 2B_2 - 4A_1 - (4B_1 + 8A_2)x - 4B_2x^2\right]\sin 2x,\ \text{so} \end{array}$$

$$\begin{array}{rcl} y_p'' + 3y_p' + 2y_p &=& [-2A_0 + 3A_1 + 4A_2 + 6B_0 + 4B_1 \\ && -(2A_1 - 6A_2 - 6B_1 - 8B_2)x - (2A_2 - 6B_2)x^2\big]\cos 2x \\ && +[-2B_0 + 3B_1 + 4B_2 - 6A_0 - 4A_1 \\ && -(2B_1 - 6B_2 + 6A_1 + 8A_2)x - (2B_2 + 6A_2)x^2\big]\sin 2x \\ &=& (1 - x - 4x^2)\cos 2x - (1 + 7x + 2x^2)\sin 2x\ \text{if} \end{array}$$

$$\text{(i)}\quad \begin{array}{rcr} -2A_2+6B_2 &=& -4 \\ -6A_2-2B_2 &=& -2 \end{array},\quad \text{(ii)}\quad \begin{array}{rcr} -2A_1+6B_1+6A_2+8B_2 &=& -1 \\ -6A_1-2B_1-8A_2+6B_2 &=& -7 \end{array},$$

$$\text{(iii)}\quad \begin{array}{rcr} -2A_0+6B_0+3A_1+4B_1+2A_2 &=& 1 \\ -6A_0-2B_0-4A_0+3B_1+2B_2 &=& -1 \end{array}.$$

From (i), $A_2=\frac{1}{2}$, $B_2=-\frac{1}{2}$. Substituting these into (ii) and solving for A_1 and B_1 yields $A_1=0$, $B_1=0$. Substituting the known coefficients into (iii) and solving for A_0 and B_0 yields $A_0=0$, $B_0=0$. Therefore, $y_p=\frac{x^2}{2}(\cos 2x-\sin 2x)$.

5.5.16. Let $y=ue^x$. Then

$$\begin{aligned} y''-2y'+y &= e^x\left[(u''+2u'+u)-2(u'+u)+u\right]=e^xu'' \\ &= -e^x\left[(3+4x-x^2)\cos x+(3-4x-x^2)\sin x\right] \end{aligned}$$

if $u''=-(3+4x-x^2)\cos x-(3-4x-x^2)\sin x$. Now let

$$\begin{aligned} u_p &= (A_0+A_1x+A_2x^2)\cos x+(B_0+B_1x+B_2x^2)\sin x;\ \text{then} \\ u_p' &= \left[A_1+B_0+(2A_2+B_1)x+B_2x^2\right]\cos x \\ &\quad +\left[B_1-A_0+(2B_2-A_1)x-A_2x^2\right]\sin x, \\ u_p'' &= \left[-A_0+2A_2+2B_1-(A_1-4B_2)x-A_2x^2\right]\cos x \\ &\quad +\left[-B_0+2B_2-2A_1-(B_1+4A_2)x-B_2x^2\right]\sin x \\ &= -(3+4x-x^2)\cos x-(3-4x-x^2)\sin x\ \text{if} \end{aligned}$$

$$\text{(i)}\quad \begin{array}{rcr} -A_2 &=& 1 \\ -B_2 &=& 1 \end{array},\quad \text{(ii)}\quad \begin{array}{rcr} -A_1+4B_2 &=& -4 \\ -B_1-4A_2 &=& 4 \end{array},$$

$$\text{(iii)}\quad \begin{array}{rcr} -A_0+2B_1+2A_2 &=& -3 \\ -B_0-2A_1+2B_2 &=& -3 \end{array}.$$

From (i), $A_2=-1$, $B_2=-1$. Substituting these into (ii) and solving for A_1 and B_1 yields $A_1=0$, $B_1=0$. Substituting the known coefficients into (iii) and solving for A_0 and B_0 yields $A_0=1$, $B_0=1$. Therefore,$u_p=(1-x^2)(\cos x+\sin x)$ and $y_p=e^x(1-x^2)(\cos x+\sin x)$.

5.5.18. Let $y=ue^{-x}$. Then

$$\begin{aligned} y''+2y'+y &= e^{-x}\left[(u''-2u'+u)+2(u'-u)+u\right] \\ &= e^{-x}u''=e^{-x}\left[(5-2x)\cos x-(3+3x)\sin x\right] \end{aligned}$$

if $u''=(5-2x)\cos x-(3+3x)\sin x$. Let

$$\begin{aligned} u_p &= (A_0+A_1x)\cos x+(B_0+B_1x)\sin x;\ \text{then} \\ u_p' &= (A_1+B_0+B_1x)\cos x+(B_1-A_0-A_1x)\sin x \\ u_p'' &= (2B_1-A_0-A_1x)\cos x-(2A_1+B_0+B_1x)\sin x \\ &= (5-2x)\cos x-(3+3x)\sin x \end{aligned}$$

if $-A_1=-2$, $-B_1=-3$, $-A_0+2B_1=5$, $-B_0-2A_1$=-3. Therefore, $A_1=2$, $B_1=3$, $A_0=1$, $B_0=-1$, $u_p=e^{-x}\left[(1+2x)\cos x-(1-3x)\sin x\right]$, and $y_p=e^{-x}\left[(1+2x)\cos x-(1-3x)\sin x\right]$.

5.5.20. Let

$$\begin{aligned}
y_p &= (A_0x + A_1x^2 + A_2x^3)\cos x + (B_0x + B_1x^2 + B_2x^3)\sin x; \text{ then} \\
y_p' &= \left[A_0 + (2A_1 + B_0)x + (3A_2 + B_1)x^2 + B_2x^3\right]\cos x \\
&\quad + \left[B_0 + (2B_1 - A_0)x + (3B_2 - A_1)x^2 - A_2x^3\right]\sin x \\
y_p'' &= \left[2A_1 + 2B_0 - (A_0 - 6A_2 - 4B_1)x - (A_1 - 6B_2)x^2 - A_2x^3\right]\cos x \\
&\quad + \left[2B_1 - 2A_0 - (B_0 + 6B_2 + 4A_1)x - (B_1 + 6A_2)x^2 - B_2x^3\right]\sin x, \text{ so}
\end{aligned}$$

$$\begin{aligned}
y_p'' + y_p &= \left[2A_1 + 2B_0 + (6A_2 + 4B_1)x + 6B_2x^2\right]\cos x \\
&\quad + \left[2B_1 - 2A_0 + (6B_2 - 4A_1)x - 6A_2x^2\right]\sin x \\
&= (2 + 2x)\cos x + (4 + 6x^2)\sin x \text{ if}
\end{aligned}$$

$$\text{(i)} \begin{array}{rcl} 6B_2 &=& 0 \\ -6A_2 &=& 6 \end{array}, \text{ (ii)} \begin{array}{rcl} 4B_1 + 6A_2 &=& 2 \\ -4A_1 + 6B_2 &=& 0 \end{array}, \text{ (iii)} \begin{array}{rcl} 2B_0 + 2A_1 &=& 2 \\ -2A_0 + 2B_1 &=& 4 \end{array}.$$

From (i), $A_2 = -1$, $B_2 = 0$. Substituting these into (ii) and solving for A_1 and B_1 yields $A_1 = 0$, $B_1 = 2$. Substituting the known coefficients into (iii) and solving for A_0 and B_0 yields $A_0 = 0$, $B_0 = 2$. Therefore,$y_p = -x^3\cos x + (x + 2x^2)\sin x$.

5.5.22. Let $y = ue^x$. Then

$$\begin{aligned}
y'' - 7y' + 6y &= e^x\left[(u'' + 2u' + u) - 7(u' + u) + 6u\right] \\
&= e^x(u'' - 5u') = -e^x(17\cos x - 7\sin x)
\end{aligned}$$

if $u'' - 5u' = -17\cos x + 7\sin x$. Now let $u_p = A\cos x + B\sin x$. Then

$$\begin{aligned}
u_p'' - 5u_p' &= -(A\cos x + B\sin x) - 5(-A\sin x + B\cos x) \\
&= (-A - 5B)\cos x - (B - 5A)\sin x = -17\cos x + 7\sin x
\end{aligned}$$

if $-A - 5B = -17$, $5A - B = 7$. Therefore,$A = 2$, $B = 3$, $u_p = 2\cos x + 3\sin x$, and $y_p = e^x(2\cos x + 3\sin x)$. The characteristic polynomial of the complementary equation is $p(r) = r^2 - 7r + 6 = (r-1)(r-6)$, so $\{e^x, e^{6x}\}$ is a fundamental set of solutions of the complementary equation. Therefore, (A) $y = e^x(2\cos x + 3\sin x) + c_1e^x + c_2e^{6x}$ is the general solution of the nonhomogeneous equation. Differentiating (A) yields $y' = e^x(2\cos x + 3\sin x) + e^x(-2\sin x + 3\cos x) + c_1e^x + 6c_2e^{6x}$, so $y(0) = 4,\ y'(0) = 2 \Rightarrow 4 = 2 + c_1 + c_2,\ 2 = 2 + 3 + c_1 + 6c_2 \Rightarrow c_1 + c_2 = 2,\ c_1 + 6c_2 = -3$, so $c_1 = 3$, $c_2 = -1$, and $y = e^x(2\cos x + 3\sin x) + 3e^x - e^{6x}$.

5.5.24. Let $y = ue^x$. Then

$$\begin{aligned}
y'' + 6y' + 10y &= e^x\left[(u'' + 2u' + u) + 6(u' + u) + 10u\right] \\
&= e^x(u'' + 8u' + 17u) = -40e^x\sin x
\end{aligned}$$

if $u'' + 8u' + 17u = -40\sin x$. Let $u_p = A\cos x + B\sin x$. Then

$$\begin{aligned}
u_p'' + 6u_p' + 17u_p &= -(A\cos x + B\sin x) + 8(-A\sin x + B\cos x) \\
&\quad + 17(A\cos x + B\sin x) \\
&= (16A + 8B)\cos x - (8A - 16B)\sin x = -40\sin x
\end{aligned}$$

if $16A + 8B = 0$, $-8A + 16B = -40$. Therefore,$A = 1$, $B = -2$, and $y_p = e^x(\cos x - 2\sin x)$. The characteristic polynomial of the complementary equation is $p(r) = r^2 + 6r + 10 = (r+3)^2 + 1$, so $\{e^{-3x}\cos x, e^{-3x}\sin x\}$ is a fundamental set of solutions of the complementary equation, and (A) $y = e^x(\cos x - 2\sin x) + e^{-3x}(c_1\cos x + c_2\sin x)$ is the general solution of the nonhomogeneous equation. Therefore,$y(0) = 2 \Rightarrow 2 = 1 + c_1$, so $c_1 = 1$. Differentiating (A) yields $y' = e^x(\cos x - 2\sin x) - e^x(\sin x + 2\cos x) - 3e^{-3x}(c_1\cos x + c_2\sin x) + e^{-3x}(-c_1\sin x + c_2\cos x)$. Therefore,$y'(0) = -3 \Rightarrow -3 = 1 - 2 - 3c_1 + c_2$, so $c_2 = 1$, and $y = e^x(\cos x - 2\sin x) + e^{-3x}(\cos x + \sin x)$.

5.5.26. Let $y = ue^{3x}$. Then

$$\begin{aligned} y'' - 3y' + 2y &= e^{3x}\left[(u'' + 6u' + 9u) - 3(u' + 3u) + 2u\right] \\ &= e^{3x}(u'' + 3u' + 2u) = e^{3x}\left[21\cos x - (11 + 10x)\sin x\right] \end{aligned}$$

if $u'' + 3u' + 2u = 21\cos x - (11 + 10x)\sin x$. Now let

$$\begin{aligned} u_p &= (A_0 + A_1x)\cos x + (B_0 + B_1x)\sin x; \text{ then} \\ u_p' &= (A_1 + B_0 + B_1x)\cos x + (B_1 - A_0 - A_1x)\sin x \\ u_p'' &= (2B_1 - A_0 - A_1x)\cos x - (2A_1 + B_0 + B_1x)\sin x, \text{ so} \end{aligned}$$

$$\begin{aligned} u'' + 3u' + 2u &= [A_0 + 3A_1 + 3B_0 + 2B_1 + (A_1 + 3B_1)x]\cos x \\ &\quad + [B_0 + 3B_1 - 3A_0 - 2A_1 + (B_1 - 3A_1)x]\sin x \\ &= 21\cos x - (11 + 10x)\sin x \text{ if} \end{aligned}$$

$$\begin{array}{rcr} A_1 + 3B_1 &=& 0 \\ -3A_1 + B_1 &=& -10 \end{array} \text{ and } \begin{array}{rcr} A_0 + 3B_0 + 3A_1 + 2B_1 &=& 21 \\ -3A_0 + B_0 - 2A_1 + 3B_1 &=& -11 \end{array}.$$

From the first two equations $A_1 = 3$, $B_1 = -1$. Substituting these in last two equations yields and solving for A_0 and B_0 yields $A_0 = 2$, $B_0 = 4$. Therefore, $u_p = (2 + 3x)\cos x + (4 - x)\sin x$ and $y_p = e^{3x}\left[(2 + 3x)\cos x + (4 - x)\sin x\right]$. The characteristic polynomial of the complementary equation is $p(r) = r^2 - 3r + 2 = (r-1)(r-2)$, so $\{e^x, e^{2x}\}$ is a fundamental set of solutions of the complementary equation, and (A) $y = e^{3x}\left[(2 + 3x)\cos x + (4 - x)\sin x\right] + c_1e^x + c_2e^{2x}$ is the general solution of the nonhomogeneous equation. Differentiating (A) yields

$$\begin{aligned} y' &= 3e^{3x}\left[(2 + 3x)\cos x + (4 - x)\sin x\right] \\ &\quad + e^{3x}\left[(7 - x)\cos x - (3 + 3x)\sin x\right] + c_1e^x + 2c_2e^{2x}. \end{aligned}$$

Therefore,$y(0) = 0$, $y'(0) = 6 \Rightarrow 0 = 2 + c_1 + c_2$, $6 = 6 + 7 + c_1 + 2c_2$, so $c_1 + c_2 = -2$, $c_1 + 2c_2 = -7$. Therefore, $c_1 = 3$, $c_2 = -5$, and $y = e^{3x}\left[(2 + 3x)\cos x + (4 - x)\sin x\right] + 3e^x - 5e^{2x}$.

5.5.28. We must find particular solutions y_{p_1}, y_{p_2}, and y_{p_3} of (A) $y'' + y = 4\cos x - 2\sin x$ and (B) $y'' + y = xe^x$, and (C) $y'' + y = e^{-x}$, respectively. To find a particular solution of (A) we write

$$\begin{aligned} y_{p_1} &= Ax\cos x + Bx\sin x; \text{ then} \\ y_{p_1}' &= (A + Bx)\cos x + (B - Ax)\sin x \\ y_{p_1}'' &= (2B - Ax)\cos x - (2A + Bx)\sin x, \text{ so} \end{aligned}$$

$y_{p_1}'' + y_{p_1} = 2B\cos x - 2A\sin x = 4\cos x - 2\sin x$ if $2B = 4$, $-2A = -2$. Therefore, $A = 1$, $B = 2$, and $y_{p_1} = x(\cos x + 2\sin x)$. To find a particular solution of (B) we write $y = ue^x$. Then

$$\begin{aligned} y'' + y &= e^x\left[(u'' + 2u' + u) + u\right] \\ &= e^x(u'' + 2u' + 2u) = xe^x \end{aligned}$$

if $u'' + 2u' + 2u = x$. Now $u_p = A + Bx$, where $2B + 2(A + Bx) = x$. Therefore, $2B = 1$, $2A + 2B = 0$, so $B = \frac{1}{2}$, $A = -\frac{1}{2}$, $u_p = -\frac{1}{2}(1-x)$, and $y_{p2} = -\frac{e^x}{2}(1-x)$. To find a particular solution of (C) we write $y_{p3} = Ae^{-x}$. Then $y''_{p3} + y_{p3} = 2Ae^{-x} = e^{-x}$ if $2A = 1$, so $A = \frac{1}{2}$ and $y_{p3} = \frac{e^{-x}}{2}$ Now $y_p = y_{p_1} + y_{p2} + y_{p3} = x(\cos x + 2\sin x) - \frac{e^x}{2}(1-x) + \frac{e^{-x}}{2}$.

5.5.30. We must find particular solutions y_{p_1}, y_{p2} and y_{p3} of (A) $y'' - 2y' + 2y = 4xe^x\cos x$, (B) $y'' - 2y' + 2y = xe^{-x}$, and (C) $y'' - 2y' + 2y = 1 + x^2$, respectively. To find a particular solution of (A) we write $y = ue^x$. Then $y'' - 2y' + 2y = e^x\left[(u'' + 2u' + u) - 2(u' + u) + 2u\right] = e^x(u'' + u) = 4xe^x\cos x$ if $u'' + u = 4x\cos x$. Now let

$$\begin{aligned}
u_p &= (A_0x + A_1x^2)\cos x + (B_0x + B_1x^2)\sin x;\ \text{then}\\
u'_p &= \left[A_0 + (2A_1 + B_0)x + B_1x^2\right]\cos x + \left[B_0 + (2B_1 - A_0)x - B_1x^2\right]\sin x\\
u''_p &= \left[2A_1 + 2B_0 - (A_0 - 4B_1)x - A_1x^2\right]\cos x\\
&\quad + \left[2B_1 - 2A_0 - (B_0 + 4A_1)x - B_1x^2\right]\sin x,\ \text{so}
\end{aligned}$$

$$\begin{aligned}
u''_p + u_p &= (2A_1 + 2B_0 + 4B_1x)\cos x + (2B_1 - 2A_0 - 4A_1x)\sin x\\
&= 4x\cos x
\end{aligned}$$

if $4B_1 = 4$, $-4A_1 = 0$, $2B_0 + 2A_1 = 0$, $-2A_0 + 2B_1 = 0$. Therefore, $A_1 = 0$, $B_1 = 1$, $A_0 = 1$, $B_0 = 0$, $u_p = x(\cos x + x\sin x)$, and $y_{p_1} = xe^x(\cos x + x\sin x)$. To find a particular solution of (B) we write $y = ue^{-x}$. Then

$$\begin{aligned}
y'' - 2y' + 2y &= e^{-x}\left[(u'' - 2u' + u) - 2(u' - u) + 2u\right]\\
&= e^{-x}(u'' - 4u' + 5u) = xe^{-x}
\end{aligned}$$

if $u'' - 4u' + 5u = x$. Now $u_p = A + Bx$ where $-4B + 5(A + Bx) = x$. Therefore, $5B = 1$, $5A - 4B = 0$, $B = \frac{1}{5}$, $A = \frac{4}{25}$, $u_p = \frac{1}{25}(4 + 5x)$, and $y_{p2} = \frac{e^{-x}}{25}(4 + 5x)$. To find a particular solution of (C) we write $y_{p3} = A + Bx + Cx^2$. Then

$$\begin{aligned}
y''_{p3} - 2y'_{p3} + 2y_{p3} &= 2C - 2(B + 2Cx) + 2(A + Bx + Cx^2)\\
&= (2A - 2B + 2C) + (2B - 4C)x + 2Cx^2 = 1 + x^2
\end{aligned}$$

if $2A - 2B + 2C = 1$, $2B - 4C = 0$, $2C = 1$. Therefore,$C = \frac{1}{2}$, $B = 1$, $A = 1$, and $y_{p3} = 1 + x + \frac{x^2}{2}$. Now $y_p = y_{p_1} + y_{p2} + y_{p3} = xe^x(\cos x + x\sin x) + \frac{e^{-x}}{25}(4 + 5x) + 1 + x + \frac{x^2}{2}$.

5.5.32. We must find particular solutions y_{p_1} and y_{p2} of (A) $y'' - 4y' + 4y = 6e^{2x}$ and (B) $y'' - 4y' + 4y = 25\sin x$, respectively. To find a particular solution of (A), let $y = ue^{2x}$. Then

$$\begin{aligned}
y'' - 4y' + 4y &= e^{2x}\left[(u'' + 4u' + 4u) - 4(u' + 2u) + 4u\right]\\
&= e^{2x}u'' = 6e^{2x}
\end{aligned}$$

if $u'' = 6$. Integrating twice and taking the constants of integration to be zero yields $u_p = 3x^2$, so $y_{p_1} = 3x^2e^{2x}$. To find a particular solution of (B), let $y_{p2} = A\cos x + B\sin x$. Then

$$\begin{aligned}
y''_{p2} - 4y'_{p2} + 4y_{p2} &= -(A\cos x + B\sin x) - 4(-A\sin x + B\cos x)\\
&\quad +4(A\cos x + B\sin x)\\
&= (3A - 4B)\cos x + (4A + 3B)\sin x = 25\sin x
\end{aligned}$$

if $3A - 4B = 0$, $4A + 3B = 25$. Therefore,$A = 4$, $B = 3$, and $y_{p_2} = 4\cos x + 3\sin x$. Now $y_p = y_{p_1} + y_{p_2} = 3x^2e^{2x} + 4\cos x + 3\sin x$. The characteristic polynomial of the complementary equation is $p(r) = r^2 - 4r + 4 = (r-2)^2$, so $\{e^{2x}, xe^{2x}\}$ is a fundamental set of solutions of the complementary equation. Therefore,(C) $y = 3x^2e^{2x} + 4\cos x + 3\sin x + e^{2x}(c_1 + c_2x)$ is the general solution of the nonhomogeneous equation. Now $y(0) = 5 \Rightarrow 5 = 4 + c_1$, so $c_1 = 1$. Differentiating (C) yields $y' = 6e^{2x}(x + x^2) - 4\sin x + 3\cos x + 2e^{2x}(c_1 + c_2x) + c_2e^{2x}$, so $y'(0) = 3 \Rightarrow 3 = 3 + 2 + c_2$. Therefore,$c_2 = -2$, and $y = (1 - 2x + 3x^2)e^{2x} + 4\cos x + 3\sin x$.

5.5.34. We must find particular solutions y_{p_1} and y_{p_2} of (A) $y'' + 4y' + 4y = 2\cos 2x + 3\sin 2x$ and (B) $y'' + 4y' + 4y = e^{-x}$, respectively. To find a particular solution of (A) we write $y_{p_1} = A\cos 2x + B\sin 2x$. Then

$$\begin{aligned}
y''_{p_1} + 4y'_{p_1} + 4y_{p_1} &= -4(A\cos 2x + B\sin 2x) + 8(-A\sin 2x + B\cos 2x) \\
&\quad +4(A\cos 2x + B\sin 2x) = -8A\sin 2x + 8B\cos 2x \\
&= 2\cos 2x + 3\sin 2x
\end{aligned}$$

if $8B = 2$, $-8A = 3$. Therefore,$A = -\frac{3}{8}$, $B = \frac{1}{4}$, and $y_{p_1} = -\frac{3}{8}\cos 2x + \frac{1}{4}\sin 2x$. To find a particular solution of (B) we write $y_{p_2} = Ae^{-x}$. Then $y''_{p_2} + 4y'_{p_2} + 4y_{p_2} = A(1 - 4 + 4)e^{-x} = Ae^{-x} = e^{-x}$ if $A = 1$. Therefore,$y_{p_2} = e^{-x}$. Now $y_p = y_{p_1} + y_{p_2} = -\frac{3}{8}\cos 2x + \frac{1}{4}\sin 2x + e^{-x}$. The characteristic polynomial of the complementary equation is $p(r) = r^2 + 4r + 4 = (r-2)^2$, so $\{e^{2x}, xe^{2x}\}$ is a fundamental set of solutions of the complementary equation. Therefore,(C) $y = -\frac{3}{8}\cos 2x + \frac{1}{4}\sin 2x + e^{-x} + e^{-2x}(c_1 + c_2x)$ is the general solution of the nonhomogeneous equation. Now $y(0) = -1 \Rightarrow -1 = -\frac{3}{8} + 1 + c_1$, so $c_1 = -\frac{13}{8}$. Differentiating (C) yields $y' = \frac{3}{4}\sin 2x + \frac{1}{2}\cos 2x - e^{-x} - 2e^{-2x}(c_1 + c_2x) + c_2e^{-2x}$, so $y'(0) = 2 \Rightarrow 2 = \frac{1}{2} - 1 - 2c_1 + c_2$. Therefore,$c_2 = -\frac{3}{4}$, and $y = -\frac{3}{8}\cos 2x + \frac{1}{4}\sin 2x + e^{-x} - \frac{13}{8}e^{-2x} - \frac{3}{4}xe^{-2x}$.

5.5.36. **(a)**, **(b)**, and **(c)** require only routine manipulations. **(d)** The coefficients of $\sin\omega x$ in y'_p, y''_p, $ay''_p + by'_p + cy_p$, and $y''_p + \omega^2 y_p$ can be obtained by replacing A by B and B by $-A$ in the corresponding coefficients of $\cos\omega x$.

5.5.38. Let $y = ue^{\lambda x}$. Then

$$\begin{aligned}
ay'' + by' + cy &= e^{\lambda x}\left[a(u'' + 2\lambda u' + \lambda^2 u) + b(u' + \lambda u) + cu\right] \\
&= e^{\lambda x}\left[au'' + (2a\lambda + b)u' + (a\lambda^2 + b\lambda + c)u\right] \\
&= e^{\lambda x}\left[au'' + p'(\lambda)u' + p(\lambda)u\right] \\
&= e^{\lambda x}\left(P(x)\cos\omega x + Q(x)\sin\omega x\right) \text{ if}
\end{aligned}$$

(A) $au'' + p'(\lambda)u' + p(\lambda)uP(x)\cos\omega x + Q(x)\sin\omega x$, where $p(r) = ar^{+}br + c$ is that characteristic polynomial of the complementary equation (B) $ay'' + by' + cy = 0$. If $e^{\lambda x}\cos\omega x$ and $e^{\lambda x}\sin\omega x$ are not solutions of (B), then $\cos\omega x$ and $\sin\omega x$ are not solutions of the complementary equation for (A). Then Theorem 5.5.1 implies that (A) has a particular solution

$$u_p = (A_0 + A_1x + \cdots + A_kx^k)\cos\omega x + (B_0 + B_1x + \cdots + B_kx^k)\sin\omega x,$$

and $y_p = u_pe^{\lambda x}$ is a particular solution of the stated form for the given equation. If $e^{\lambda x}\cos\omega x$ and $e^{\lambda x}\sin\omega x$ are solutions of (B), then $\cos\omega x$ and $\sin\omega x$ are solutions of the complementary equation for

(A). Then Theorem 5.5.1 implies that (A) has a particular solution

$$u_p = (A_0 x + A_1 x^2 + \cdots + A_k x^{k+1})\cos\omega x + (B_0 x + B_1 x^2 + \cdots + B_k x^{k+1})\sin\omega x,$$

and $y_p = u_p e^{\lambda x}$ is a particular solution of the stated form for the given equation.

5.5.40. **(a)** Let $y = \int x^2 \cos x\,dx$; then $y' = x^2\cos x$ Now let

$$\begin{aligned} y_p &= (A_0 + A_1 x + A_2 x^2)\cos x + (B_0 + B_1 x + B_2 x^2)\sin x; \text{ then} \\ y_p' &= \left[A_1 + B_0 + (2A_2 + B_1)x + B_2 x^2\right]\cos x \\ &\quad + \left[B_1 - A_0 + (2B_2 - A_1)x - A_2 x^2\right]\sin x = x^2\cos x \text{ if} \end{aligned}$$

$$\text{(i)}\ \begin{array}{rcl} B_2 &=& 1 \\ -A_2 &=& 0 \end{array},\ \text{(ii)}\ \begin{array}{rcl} B_1 + 2A_2 &=& 0 \\ -A_1 + 2B_2 &=& 0 \end{array},\ \text{(iii)}\ \begin{array}{rcl} B_0 + A_1 &=& 0 \\ -A_0 + B_1 &=& 0 \end{array}.$$

Solving these equations yields $A_2 = 0$, $B_2 = 1$, $A_1 = 2$, $B_1 = 0$, $A_0 = 0$, $B_0 = -2$. Therefore, $y_p = 2x\cos x - (2 - x^2)\sin x$ and $y = 2x\cos x - (2 - x^2)\sin x + c$.

(b) Let $y = \int x^2 e^x \cos x\,dx = ue^x$; then $y' = (u' + u)e^x = x^2 e^x\cos x$ if $u' + u = x^2\cos x$. Now let

$$\begin{aligned} u_p &= (A_0 + A_1 x + A_2 x^2)\cos x + (B_0 + B_1 x + B_2 x^2)\sin x; \text{ then} \\ u_p' &= \left[A_1 + B_0 + (2A_2 + B_1)x + B_2 x^2\right]\cos x \\ &\quad + \left[B_1 - A_0 + (2B_2 - A_1)x - A_2 x^2\right]\sin x, \text{ so} \end{aligned}$$

$$\begin{aligned} u_p'' + u_p &= \left[A_0 + A_1 + B_0 + (A_1 + 2A_2 + B_1)x + (A_2 + B_2)x^2\right]\cos x \\ &\quad + \left[B_0 + B_1 - A_0 + (B_1 + 2B_2 - A_1)x + (B_2 - A_2)x^2\right]\sin x \\ &= x^2\cos x \text{ if} \end{aligned}$$

$$\text{(i)}\ \begin{array}{rcl} A_2 + B_2 &=& 1 \\ -A_2 + B_2 &=& 0 \end{array},\ \text{(ii)}\ \begin{array}{rcl} A_1 + B_1 + 2A_2 &=& 0 \\ -A_1 + B_1 + 2B_2 &=& 0 \end{array},$$

$$\text{(iii)}\ \begin{array}{rcl} A_0 + B_0 + A_1 &=& 0 \\ -A_0 + B_0 + B_1 &=& 0 \end{array}.$$

From (i), $A_2 = \frac{1}{2}$, $B_2 = \frac{1}{2}$. Substituting these into (ii) and solving for A_1 and B_1 yields $A_1 = 0$, $B_1 = -1$. Substituting these into (iii) and solving for A_0 and B_0 yields $A_0 = -\frac{1}{2}$, $B_0 = \frac{1}{2}$. Therefore, $u_p = -\frac{1}{2}\left[(1 - x^2)\cos x - (1 - x)^2\sin x\right]$ and $y = -\frac{e^x}{2}\left[(1 - x^2)\cos x - (1 - x)^2\sin x\right]$.

(c) Let $y = \int xe^{-x}\sin 2x\,dx = ue^{-x}$; then $y' = (u' - u)e^{-x} = xe^{-x}\sin 2x$ if $u' - u = x\sin 2x$. Now let

$$\begin{aligned} u_p &= (A_0 + A_1 x)\cos 2x + (B_0 + B_1 x)\sin 2x; \text{ then} \\ u_p' &= [(A_1 + 2B_0) + 2B_1 x]\cos 2x + [(B_1 - 2A_0) - 2A_1 x]\sin 2x, \text{ so} \end{aligned}$$

$$\begin{aligned} u_p'' - u_p &= [-A_0 + A_1 + 2B_0 - (A_1 - 2B_1)x]\cos 2x \\ &\quad + [-B_0 + B_1 - 2A_0 - (B_1 + 2A_1)x]\sin 2x = x\sin 2x \text{ if} \end{aligned}$$

$$\text{(i)}\quad \begin{array}{rcl} -A_1+2B_1 &=& 0\\ -2A_1-\ B_1 &=& 1 \end{array},\quad \text{(ii)}\quad \begin{array}{rcl} -\ A_0+2B_0+A_1 &=& 0\\ -2A_0-\ B_0+B_1 &=& 0 \end{array}.$$

From (i), $A_1=-\frac{2}{5}$, $B_1=-\frac{1}{5}$. Substituting these into (ii) and solving for A_0 and B_0 yields $A_0=-\frac{4}{25}$, $B_0=\frac{3}{25}$. Therefore,

$$u_p=-\frac{1}{25}\left[(4+10x)\cos 2x-(3-5x)\sin 2x\right]+c \text{ and}$$

$$y_p=-\frac{e^{-x}}{25}\left[(4+10x)\cos 2x-(3-5x)\sin 2x\right]+c.$$

(d) Let $y=\int x^2e^{-x}\sin x\,dx=ue^{-x}$; then $y'=(u'-u)e^{-x}=x^2e^{-x}\sin x$ if $u'-u=x^2\sin x$. Now let

$$\begin{array}{rcl} u_p &=& (A_0+A_1x+A_2x^2)\cos x+(B_0+B_1x+B_2x^2)\sin x;\ \text{then}\\ u_p' &=& \left[A_1+B_0+(2A_2+B_1)x+B_2x^2\right]\cos x\\ && +\left[B_1-A_0+(2B_2-A_1)x-A_2x^2\right]\sin x,\ \text{so} \end{array}$$

$$\begin{array}{rcl} u_p''-u_p &=& \left[-A_0+A_1+B_0-(A_1-2A_2-B_1)x-(A_2-B_2)x^2\right]\cos x\\ && +\left[-B_0+B_1-A_0-(B_1-2B_2+A_1)x-(B_2+A_2)x^2\right]\sin x\\ &=& x^2\sin x \text{ if} \end{array}$$

$$\text{(i)}\quad \begin{array}{rcl} -A_2+B_2 &=& 0\\ -A_2-B_2 &=& 1 \end{array},\quad \text{(ii)}\quad \begin{array}{rcl} -A_1+B_1+2A_2 &=& 0\\ -A_1-B_1+2B_2 &=& 0 \end{array},$$

$$\text{(iii)}\quad \begin{array}{rcl} -A_0+B_0+A_1 &=& 0\\ -A_0-B_0+B_1 &=& 0 \end{array}.$$

From (i), $A_2=-\frac{1}{2}$, $B_2=-\frac{1}{2}$. Substituting these into (ii) and solving for A_1 and B_1 yields $A_1=-1$, $B_1=0$. Substituting these into (iii) and solving for A_0 and B_0 yields $A_0=-\frac{1}{2}$, $B_0=\frac{1}{2}$. Therefore,

$$u_p=-\frac{e^{-x}}{2}\left[(1+x)^2\cos x-(1-x^2)\sin x\right] \text{ and}$$

$$y=-\frac{e^{-x}}{2}\left[(1+x)^2\cos x-(1-x^2)\sin x\right]+c.$$

(e) Let $y=\int x^3e^x\sin x\,dx=ue^x$; then $y'=(u'+u)e^x=x^3e^x\sin x$ if $u'+u=x^3\sin x$. Now let

$$\begin{array}{rcl} u_p &=& (A_0+A_1x+A_2x^2+A_3x^3)\cos x+(B_0+B_1x+B_2x^2+B_3x^3)\sin x;\ \text{then}\\ u_p' &=& \left[A_1+B_0+(2A_2+B_1)x+(3A_3+B_2)x^2+B_3x^3\right]\cos x\\ && +\left[B_1-A_0+(2B_2-A_1)x+(3B_3-A_2)x^2-A_3x^3\right]\sin x,\ \text{so} \end{array}$$

$$\begin{array}{rcl} u_p''+u_p &=& [A_0+A_1+B_0+(A_1+2A_2+B_1)x\\ && +(A_2+3A_3+B_2)x^2+(A_3+B_3)x^3\big]\cos x\\ && +[B_0+B_1-A_0+(B_1+2B_2-A_1)x\\ && +(B_2+3B_3-A_2)x^2+(B_3-A_3)x^3\big]\sin x=x^3\sin x \text{ if} \end{array}$$

$$\text{(i)} \begin{array}{rcl} A_3 + B_3 &=& 0 \\ -A_3 + B_3 &=& 1 \end{array}, \quad \text{(ii)} \begin{array}{rcl} A_2 + B_2 + 3A_3 &=& 0 \\ -A_2 + B_2 + 3B_3 &=& 0 \end{array},$$

$$\text{(iii)} \begin{array}{rcl} A_1 + B_1 + 2A_2 &=& 0 \\ -A_1 + B_1 + 2B_2 &=& 0 \end{array}, \quad \text{(iv)} \begin{array}{rcl} A_0 + B_0 + A_1 &=& 0 \\ -A_0 + B_0 + B_1 &=& 0 \end{array}.$$

From (i), $A_3 = -\frac{1}{2}$, $B_3 = \frac{1}{2}$. Substituting these into (ii) and solving for A_2 and B_2 yields $A_2 = \frac{3}{2}$, $B_2 = 0$. Substituting these into (iii) and solving for A_1 and B_1 yields $A_1 = -\frac{3}{2}$, $B_1 = -\frac{3}{2}$. Substituting these into (iv) and solving for A_0 and B_0 yields $A_0 = 0$, $B_0 = \frac{3}{2}$. Therefore,

$$u_p = -\frac{1}{2}\left[x(3 - 3x + x^2)\cos x - (3 - 3x + x^3)\sin x\right] \text{ and}$$

$$y = -\frac{e^x}{2}\left[x(3 - 3x + x^2)\cos x - (3 - 3x + x^3)\sin x\right] + c.$$

(f) Let $y = \int e^x\left[x\cos x - (1 + 3x)\sin x\right]\,dx = ue^x$; then $y' = (u'+u)e^x = e^x\left[x\cos x - (1 + 3x)\sin x\right]$ if $u' + u = x\cos x - (1 + 3x)\sin x$. Now let

$$\begin{array}{rcl} u_p &=& (A_0 + A_1x)\cos x + (B_0 + B_1x)\sin x;\ \text{then} \\ u_p' &=& [A_1 + B_0 + B_1x]\cos x + [B_1 - A_0 - A_1x]\sin x,\ \text{so} \end{array}$$

$$\begin{array}{rcl} u_p'' + u_p &=& [A_0 + A_1 + B_0 + (A_1 + B_1)x]\cos x \\ && + [B_0 + B_1 - A_0 + (B_1 - A_1)x]\sin x \\ &=& x\cos x - (1 + 3x)\sin x \text{ if} \end{array}$$

$$\text{(i)} \begin{array}{rcl} A_1 + B_1 &=& 1 \\ -A_1 + B_1 &=& -3 \end{array}, \quad \text{(ii)} \begin{array}{rcl} A_0 + B_0 + A_1 &=& 0 \\ -A_0 + B_0 + B_1 &=& -1 \end{array}.$$

From (i), $A_1 = 2$, $B_1 = -1$. Substituting these into (ii) and solving for A_0 and B_0 yields $A_0 = -1$, $B_0 = -1$. Therefore, $u_p = -\left[(1 - 2x)\cos x + (1 + x)\sin x\right]$ and $y = -e^x\left[(1 - 2x)\cos x + (1 + x)\sin x\right] + c$.

(g) Let $y = \int e^{-x}\left[(1 + x^2)\cos x + (1 - x^2)\sin x\right]\,dx = ue^{-x}$; then

$$y' = (u' - u)e^{-x} = e^{-x}\left[(1 + x^2)\cos x + (1 - x^2)\sin x\right]$$

if $u' - u = (1 + x^2)\cos x + (1 - x^2)\sin x$. Now let

$$\begin{array}{rcl} u_p &=& (A_0 + A_1x + A_2x^2)\cos x + (B_0 + B_1x + B_2x^2)\sin x;\ \text{then} \\ u_p' &=& \left[A_1 + B_0 + (2A_2 + B_1)x + B_2x^2\right]\cos x \\ && + \left[B_1 - A_0 + (2B_2 - A_1)x - A_2x^2\right]\sin x,\ \text{so} \end{array}$$

$$\begin{array}{rcl} u_p'' - u_p &=& \left[-A_0 + A_1 + B_0 - (A_1 - 2A_2 - B_1)x - (A_2 - B_2)x^2\right]\cos x \\ && + \left[-B_0 + B_1 - A_0 - (B_1 - 2B_2 + A_1)x - (B_2 + A_2)x^2\right]\sin x \\ &=& (1 + x^2)\cos x + (1 - x^2)\sin x \text{ if} \end{array}$$

$$\text{(i)} \begin{array}{rcl} -A_2 + B_2 &=& 1 \\ -A_2 - B_2 &=& -1 \end{array}, \quad \text{(ii)} \begin{array}{rcl} -A_1 + B_1 + 2A_2 &=& 0 \\ -A_1 - B_1 + 2B_2 &=& 0 \end{array},$$

$$\text{(iii)} \quad \begin{array}{rcl} -A_0 + B_0 + A_1 &=& 1 \\ -A_0 - B_0 + B_1 &=& 1 \end{array}.$$

From (i), $A_2 = 0$, $B_2 = 1$. Substituting these into (ii) and solving for A_1 and B_1 yields $A_1 = 1$, $B_1 = 1$. Substituting these into (iii) and solving for A_0 and B_0 yields $A_0 = 0$, $B_0 = 0$. Therefore,$u_p = x\cos x + x(1+x)\sin x$ and $y = e^{-x}\left[x\cos x + x(1+x)\sin x\right] + c$.

5.6 REDUCTION OF ORDER

(NOTE: The term uy_1'' is indicated by "$\cdots$" in some of the following solutions, where y_1'' is complicated. Since this term always drops out of the differential equation for u, it is not necessary to include it.)

5.6.2. If $y = ux$, then $y' = u'x + u$ and $y'' = u''x + 2u'$, so $x^2y'' + xy' - y = x^3u'' + 3x^2u' = \dfrac{4}{x^2}$ if $u' = z$, where (A) $z' + \dfrac{3}{x}z = \dfrac{4}{x^5}$. Since $\displaystyle\int \frac{3}{x}\,dx = 3\ln|x|$, $z_1 = \dfrac{1}{x^3}$ is a solution of the complementary equation for (A). Therefore,the solutions of (A) are of the form (B) $z = \dfrac{v}{x^3}$, where $\dfrac{v'}{x^3} = \dfrac{4}{x^5}$, so $v' = \dfrac{4}{x^2}$. Hence, $v = -\dfrac{4}{x} + C_1$; $u' = z = -\dfrac{4}{x^4} + \dfrac{C_1}{x^3}$ (see (B)); $u = \dfrac{4}{3x^3} - \dfrac{C_1}{2x^2} + C_2$; $y = ux = \dfrac{4}{3x^2} - \dfrac{C_1}{2x} + C_2x$, or $y = \dfrac{4}{3x^2} + c_1x + \dfrac{c_2}{x}$. As a byproduct, $\{x, 1/x\}$ is a fundamental set of solutions of the complementary equation.

5.6.4. If $y = ue^{2x}$, then $y' = (u' + 2u)e^{2x}$ and $y'' = (u'' + 4u' + 4u)e^{2x}$, so $y'' - 3y' + 2y = (u'' + u')e^{2x} = \dfrac{1}{1+e^{-x}}$ if $u' = z$, where (A) $z' + z = \dfrac{e^{-2x}}{1+e^{-x}}$. Since $z_1 = e^{-x}$ is a solution of the complementary equation for (A), the solutions of (A) are of the form (B) $z = ve^{-x}$, where $v'e^{-x} = \dfrac{e^{-2x}}{1+e^{-x}}$, so $v' = \dfrac{e^{-x}}{1+e^{-x}}$. Hence, $v = -\ln(1+e^{-x}) + C_1$; $u' = z = -e^{-x}\ln(1+e^{-x}) + C_1e^{-x}$ (see (B)); $u = (1+e^{-x})\ln(1+e^{-x}) - 1 - e^{-x} - C_1e^{-x} + C_2$; $y = ue^{2x} = (e^{2x}+e^x)\ln(1+e^{-x}) - (C_1+1)e^x + (C_2-1)e^{2x}$, or $y = (e^{2x}+e^x)\ln(1+e^{-x}) + c_1e^{2x} + c_2e^x$. As a byproduct, $\{e^{2x}, e^x\}$ is a fundamental set of solutions of the complementary equation.

5.6.6. If $y = ux^{1/2}e^x$, then $y' = u'x^{1/2}e^x + u\left(x^{1/2} + \dfrac{x^{-1/2}}{2}\right)e^x$ and $y'' = u''x^{1/2}e^x + 2u'\left(x^{1/2} + \dfrac{x^{-1/2}}{2}\right)e^x + \cdots$ so $4x^2y'' + (4x - 8x^2)y' + (4x^2 - 4x - 1)y = e^x(4x^{5/2}u'' + 8x^{3/2}u') = 4x^{1/2}e^x(1+4x)$ if $u' = z$, where (A) $z' + \dfrac{2}{x}z = \dfrac{1+4x}{x^2}$. Since $\displaystyle\int \frac{2}{x}\,dx = 2\ln|x|$, $z_1 = \dfrac{2}{x^2}$ is a solution of the complementary equation for (A). Therefore,the solutions of (A) are of the form (B) $z = \dfrac{v}{x^2}$, where $\dfrac{v'}{x^2} = \dfrac{1+4x}{x^2}$, so $v' = 1 + 4x$. Hence, $v = x + 2x^2 + C_1$; $u' = z = \dfrac{1}{x} + 2 + \dfrac{C_1}{x^2}$ (see (B)); $u = \ln x + 2x - \dfrac{C_1}{x} + C_2$; $y = ux^{1/2}e^x = e^x(2x^{3/2} + x^{1/2}\ln x - C_1x^{-1/2} + C_2x^{1/2})$, or $y = e^x(2x^{3/2} + x^{1/2}\ln x + c_1x^{1/2} + c_2x^{-1/2})$. As a byproduct, $\{x^{1/2}e^x, x^{-1/2}e^{-x}\}$ is a fundamental set of solutions of the complementary equation.

5.6.8. If $y = ue^{-x^2}$, then $y' = u'e^{-x^2} - 2xue^{-x^2}$ and $y'' = u''e^{-x^2} - 4xu'e^{-x^2} + \cdots$, so $y'' + 4xy' + (4x^2+2)y = u''e^{-x^2} = 8e^{-x(x+2)} = 8e^{-x^2}e^{-2x}$ if $u'' = 8e^{-2x}$. Therefore,$u' = -4e^{-2x} + C_1$; $u = 2e^{-2x} + C_1x + C_2$, and $y = ue^{-x^2} = e^{-x^2}(2e^{-2x} + C_1x + C_2)$, or $y = e^{-x^2}(2e^{-2x} + c_1 + c_2x)$. As a byproduct, $\{e^{-x^2}, xe^{-x^2}\}$ is a fundamental set of solutions of the complementary equation.

5.6.10. If $y = uxe^{-x}$, then $y' = u'xe^{-x} - ue^{-x}(x-1)$ and $y'' = u''xe^{-x} - 2u'e^{-x}(x-1) + \cdots$, so $x^2y'' + 2x(x-1)y' + (x^2-2x+2)y = x^3u'' = x^3e^{2x}$ if $u'' = e^{3x}$. Therefore,$u' = \dfrac{e^{3x}}{3} + C_1$; $u = \dfrac{e^{3x}}{9} + C_1x + C_2$, and $y = uxe^{-x} = \dfrac{xe^{2x}}{9} + xe^{-x}(C_1x + C_2)$, or $y = \dfrac{xe^{2x}}{9} + xe^{-x}(c_1 + c_2x)$. As a byproduct, $\{xe^{-x}, x^2e^{-x}\}$ is a fundamental set of solutions of the complementary equation.

5.6.12. If $y = ue^x$, then $y' = (u'+u)e^x$ and $y'' = (u''+2u'+u)e^x$, so $(1-2x)y'' + 2y' + (2x-3)y = e^x\left[(1-2x)u'' + (4-4x)u'\right] = (1-4x+4x^2)e^x$ if $u' = z$, where (A) $z' + \dfrac{4-4x}{1-2x}z = 1-2x$. Since $\displaystyle\int \frac{4-4x}{1-2x}\,dx = \int\left(2 + \frac{2}{1-2x}\right)dx = 2x - \ln|1-2x|$, $z_1 = (1-2x)e^{-2x}$ is a solution of the complementary equation for (A). Therefore,the solutions of (A) are of the form (B) $z = v(1-2x)e^{-2x}$, where $v'(1-2x)e^{-2x} = (1-2x)$, so $v' = e^{2x}$. Hence, $v = \dfrac{e^{2x}}{2} + C_1$; $u' = z = \left(\dfrac{1}{2} + C_1e^{-2x}\right)(1-2x)$ (see (B)); $u = -\dfrac{(2x-1)^2}{8} + C_1xe^{-2x} + C_2$; $y = ue^x = -\dfrac{(2x-1)^2e^x}{8} + C_1xe^{-x} + C_2e^x$, or $y = -\dfrac{(2x-1)^2e^x}{8} + c_1e^x + c_2xe^{-x}$. As a byproduct, $\{e^x, xe^{-x}\}$ is a fundamental set of solutions of the complementary equation.

5.6.14. If $y = ue^{-x}$, then $y' = (u'-u)e^{-x}$ and $y'' = (u''-2u'+u)e^{-x}$, so $2xy'' + (4x+1)y' + (2x+1)y = e^{-x}(2xu'' + u') = 3x^{1/2}e^{-x}$ if $u' = z$, where (A) $z' + \dfrac{1}{2x}z = \dfrac{3}{2}x^{-1/2}$. Since $\displaystyle\int \frac{1}{2x}\,dx = \frac{1}{2}\ln|x|$, $z_1 = x^{-1/2}$ is a solution of the complementary equation for (A). Therefore,the solutions of (A) are of the form (B) $z = vx^{-1/2}$, where $v'x^{-1/2} = \dfrac{3}{2}x^{-1/2}$, so $v' = \dfrac{3}{2}$. Hence, $v = \dfrac{3x}{2} + C_1$; $u' = z = \dfrac{3}{2}x^{1/2} + C_1x^{-1/2}$ (see (B)); $u = x^{3/2} + 2C_1x^{1/2} + C_2$; $y = ue^{-x} = e^{-x}(x^{3/2} + 2C_1x^{1/2} + C_2)$, or $y = e^{-x}(x^{3/2} + c_1 + c_2x^{1/2})$ As a byproduct, is a $\{e^{-x}, x^{1/2}e^{-x}\}$ fundamental set of solutions of the complementary equation.

5.6.16. If $y = ux^{1/2}$, then $y' = u'x^{1/2} + \dfrac{u}{2x^{1/2}}$ and $y'' = u''x^{1/2} + \dfrac{u'}{x^{1/2}} + \cdots$ so $4x^2y'' - 4x(x+1)y' + (2x+3)y = 4x^{5/2}(u''-u') = 4x^{5/2}e^{2x}$ if $u' = z$, where (A) $z' - z = e^{2x}$. Since $z_1 = e^x$ is a solution of the complementary equation for (A), the solutions of (A) are of the form (B) $z = ve^x$, where $v'e^x = e^{2x}$, so $v' = e^x$. Hence, $v = e^x + C_1$; $u' = z = e^{2x} + C_1e^x$ (see (B)); $u = \dfrac{e^{2x}}{2} + C_1e^x + C_2$; $y = ux^{1/2} = x^{1/2}\left(\dfrac{e^{2x}}{2} + C_1e^x + C_2\right)$, or $y = x^{1/2}\left(\dfrac{e^{2x}}{2} + c_1 + c_2e^x\right)$. As a byproduct, $\{x^{1/2}, x^{1/2}e^x\}$ is a fundamental set of solutions of the complementary equation.

5.6.18. If $y = ue^x$, then $y' = (u'+u)e^x$ and $y'' = (u''+2u'+u)e^x$, so $xy'' + (2-2x)y' + (x-2)y = e^x(xu'' + 2u') = 0$ if $\dfrac{u''}{u'} = -\dfrac{2}{x}$; $\ln|u'| = -2\ln|x| + k$; $u' = \dfrac{C_1}{x^2}$; $u = -\dfrac{C_1}{x} + C_2$. Therefore,$y = ue^x = e^x\left(-\dfrac{C_1}{x} + C_2\right)$ is the general solution, and $\{e^x, e^x/x\}$ is a fundamental set of solutions.

5.6.20. If $y = u\ln|x|$, then $y' = u'\ln|x| + \dfrac{u}{x}$ and $y'' = u''\ln|x| + \dfrac{2u'}{x}\cdots$, so $x^2(\ln|x|)^2y'' - (2x\ln|x|)y' + (2+\ln|x|)y = x^2(\ln|x|)^3u'' = 0$ if $u'' = 0$; $u' = C_1$; $u = C_1x + C_2$. Therefore,$y = u\ln|x| = (C_1x + C_2)\ln|x|$ is the general solution, and $\{\ln|x|, x\ln|x|\}$ is a fundamental set of solutions.

5.6.22. If $y = ue^x$, then $y' = u'e^x + ue^x$ and $y'' = u''e^x + 2u'e^x + ue^x$, so $xy'' - (2x+2)y' + (x+2)y = e^x(xu'' - 2u') = 0$ if $\frac{u''}{u'} = \frac{2}{x}$; $\ln|u'| = 2\ln|x| + k$; $u' = C_1x^2$; $u = \frac{C_1x^3}{3} + C_2$. Therefore, $y = ue^x = \left(\frac{C_1x^3}{3} + C_2\right)e^x$ is the general solution, and $\{e^x, x^3e^x\}$ is a fundamental set of solutions.

5.6.24. If $y = ux\sin x$, then $y' = u'x\sin x + u(x\cos x + \sin x)$ and $y'' = u''x\sin x + 2u'(x\cos x + \sin x) + \cdots$, so $x^2y'' - 2xy' + (x^2+2)y = (x^3\sin x)u'' + 2(x^3\cos x)u' = 0$ if $\frac{u''}{u'} = -\frac{2\cos x}{\sin x}$; $\ln|u'| = -2\ln|\sin x| + k$; $u' = \frac{C_1}{\sin^2 x}$; $u = -C_1\cot x + C_2$. Therefore, $y = ux\sin x = x(-C_1\cos x + C_2\sin x)$ is the general solution, and $\{x\sin x, x\cos x\}$ is a fundamental set of solutions.

5.6.26. If $y = ux^{1/2}$, then $y' = u'x^{1/2} + \frac{u}{2x^{1/2}}$ and $y'' = u''x^{1/2} + \frac{u'}{x^{1/2}} + \cdots$ so $4x^2(\sin x)y'' - 4x(x\cos x + \sin x)y' + (2x\cos x + 3\sin x)y = 4x^{5/2}(u''\sin x - u'\cos x) = 0$ if $\frac{u''}{u'} = \frac{\cos x}{\sin x}$; $\ln|u'| = \ln|\sin x| + k$; $u' = C_1\sin x$; $u = -C_1\cos x + C_2$. Therefore,$y = ux^{1/2} = (-C_1\cos x + C_2)x^{1/2}$ is the general solution, and $\{x^{1/2}, x^{1/2}\cos x\}$ is a fundamental set of solutions.

5.6.28. If $y = \frac{u}{x}$, then $y' = \frac{u'}{x} - \frac{u}{x^2}$ and $y'' = \frac{u''}{x} - \frac{2u'}{x^2} + \cdots$, so $(2x+1)xy'' - 2(2x^2-1)y' - 4(x+1)y = (2x+1)u'' - (4x+4)u' = 0$ if $\frac{u''}{u'} = \frac{4x+4}{2x+1} = 2 + \frac{2}{2x+1}$; $\ln|u'| = 2x + \ln|2x+1| + k$; $u' = C_1(2x+1)e^{2x}$; $u = C_1xe^{2x} + C_2$. Therefore,$y = \frac{u}{x} = C_1e^{2x} + \frac{C_2}{x}$ is the general solution, and $\{1/x, e^{2x}\}$ is a fundamental set of solutions.

5.6.30. If $y = ue^{2x}$, then $y' = (u' + 2u)e^{2x}$ and $y'' = (u'' + 4u' + 4u)e^{2x}$, so $xy'' - (4x+1)y' + (4x+2)y = e^{2x}(xu'' - u') = 0$ if $\frac{u''}{u'} = \frac{1}{x}$; $\ln|u'| = \ln|x| + k$; $u' = C_1x$; $u = \frac{C_1x^2}{2} + C_2$. Therefore, $y = ue^{2x} = e^{2x}\left(\frac{C_1x^2}{2} + C_2\right)$ is the general solution, and $\{e^{2x}, x^2e^{2x}\}$ is a fundamental set of solutions.

5.6.32. If $y = ue^{2x}$, then $y' = (u' + 2u)e^{2x}$ and $y'' = (u'' + 4u' + 4u)e^{2x}$, so $(3x-1)y'' - (3x+2)y' - (6x-8)y = e^{2x}[(3x-1)u'' + (9x-6)u'] = 0$ if $\frac{u''}{u'} = -\frac{9x-6}{3x-1} = -3 + \frac{3}{3x-1}$. Therefore,$\ln|u'| = -3x + \ln|3x-1| + k$, so $u' = C_1(3x-1)e^{-3x}$, $u = -C_1xe^{-3x} + C_2$. Therefore,the general solution is $y = ue^{2x} = -C_1xe^{-x} + C_2e^{2x}$, or (A) $y = c_1e^{2x} + c_2xe^{-x}$. Now $y(0) = 2 \Rightarrow c_1 = 2$. Differentiating (A) yields $y' = 2c_1e^{2x} + c_2(e^{-x} - xe^{-x})$. Now $y'(0) = 3 \Rightarrow 3 = 2c_1 + c_2$, so $c_2 = -1$ and $y = 2e^{2x} - xe^{-x}$.

5.6.34. If $y = ux$, then $y' = u'x + u$ and $y'' = u''x + 2u'$, so $x^2y'' + 2xy' - 2y = x^3u'' + 4x^2u' = x^2$ if $u' = z$, where (A) $z' + \frac{4}{x}z = \frac{1}{x}$. Since $\int \frac{4}{x}\,dx = 4\ln|x|$, $z_1 = \frac{1}{x^4}$ is a solution of the complementary equation for (A). Therefore,the solutions of (A) are of the form (B) $z = \frac{v}{x^4}$, where $\frac{v'}{x^4} = \frac{1}{x}$, so $v' = x^3$. Hence, $v = \frac{x^4}{4} + C_1$; $u' = z = \frac{1}{4} + \frac{C_1}{x^4}$ (see (B)); $u = \frac{x}{4} - \frac{C_1}{3x^3} + C_2$. Therefore,the general solution is $y = ux = \frac{x^2}{4} - \frac{C_1}{3x^2} + C_2x$, or (C) $y = \frac{x^2}{4} + c_1x + \frac{c_2}{x^2}$. Differentiating (C) yields $y' = \frac{x}{2} + c_1 - 2\frac{c_2}{x^3}$.

Now $y(1) = \frac{5}{4}$, $y'(1) = \frac{3}{2} \Rightarrow c_1 + c_2 = 1$, $c_1 - 2c_2 = 1$, so $c_1 = 1$, $c_2 = 0$ and $y = \frac{x^2}{4} + x$.

5.6.36. If $y = uy_1$, then $y' = u'y_1 + uy_1'$ and $y'' = u''y_1 + 2u'y_1' + uy_1''$, so $y'' + p_1(x)y' + p_2(x)y = y_1u'' + (2y_1' + p_1y_1)u' = 0$ if u is any function such that (B) $\frac{u''}{u'} = -2\frac{y_1'}{y_1} - p_1$. If $\ln|u'(x)| = -2\ln|y_1(x)| - \int_{x_0}^{x} p_1(t)\,dt$, then u satisfies (B); therefore, if (C) $u'(x) = \frac{1}{y_1^2(x)}\exp\left(-\int_{x_0}^{x} p_1(s)\,ds\right)$, then u satisfies (B). Since $u(x) = \int_{x_0}^{x} \frac{1}{y_1^2(t)}\exp\left(-\int_{x_0}^{t} p_1(s)ds\right)$ satisfies (C), $y_2 = uy_1$ is a solution of (A) on (a, b). Since $\frac{y_2}{y_1} = u$ is nonconstant, Theorem 5.1.6 implies that $\{y_1, y_2\}$ is a fundamental set of solutions of (A) on (a, b).

5.6.38. **(a)** The associated linear equation is (A) $z'' + k^2z = 0$, with characteristic polynomial $p(r) = r^2 + k^2$. The general solution of (A) is $z = c_1\cos kx + c_2\sin kx$. Since $z' = -kc_1\sin kx + kc_2\cos kx$, $y = \frac{z'}{z} = \frac{-kc_1\sin kx + kc_2\cos kx}{c_1\cos kx + c_2\sin kx}$.

(b) The associated linear equation is (A) $z'' - 3z' + 2z = 0$, with characteristic polynomial $p(r) = r^2 - 3r + 2 = (r-1)(r-2)$. The general solution of (A) is $z = c_1e^x + c_2e^{2x}$. Since $z' = c_1e^x + c_2e^{2x}$, $y = \frac{z'}{z} = \frac{c_1 + 2c_2e^x}{c_1 + c_2e^x}$.

(c) The associated linear equation is (A) $z'' + 5z' - 6z = 0$, with characteristic polynomial $p(r) = r^2 + 5r - 6 = (r+6)(r-1)$. The general solution of (A) is $z = c_1e^{-6x} + c_2e^x$. Since $z' = -6c_1e^{-6x} + c_2e^x$, $y = \frac{z'}{z} = \frac{-6c_1 + c_2e^{7x}}{c_1 + c_2e^{7x}}$.

(d) The associated linear equation is (A) $z'' + 8z' + 7z = 0$, with characteristic polynomial $p(r) = r^2 + 8r + 7 = (r+7)(r+1)$. The general solution of (A) is $z = c_1e^{-7x} + c_2e^{-x}$. Since $z' = -7c_1e^{-7x} - 2c_2e^{-x}$, $y = \frac{z'}{z} = -\frac{7c_1 + c_2e^{6x}}{c_1 + c_2e^{6x}}$.

(e) The associated linear equation is (A) $z'' + 14z' + 50z = 0$, with characteristic polynomial $p(r) = r^2 + 14r + 50 = (r+7)^2 + 1$. The general solution of (A) is $z = e^{-7x}(c_1\cos x + c_2\sin x)$. Since $z' = -7e^{-7x}(c_1\cos x + c_2\sin x) + e^{-7x}(-c_1\sin x + c_2\cos x) = -(7c_1 - c_2)\cos x - (c_1 + 7c_2)\sin x$, $y = \frac{z'}{z} = -\frac{(7c_1 - c_2)\cos x + (c_1 + 7c_2)\sin x}{c_1\cos x + c_2\sin x}$.

(f) The given equation is equivalent to (A) $y' + y^2 - \frac{1}{6}y - \frac{1}{6} = 0$. The associated linear equation is (B) $z'' - \frac{1}{6}z' - \frac{1}{6}z = 0$, with characteristic polynomial $p(r) = r^2 - \frac{1}{6}r - \frac{1}{6} = \left(r + \frac{1}{3}\right)\left(r - \frac{1}{2}\right)$. The general solution of (B) is $z = c_1e^{-x/3} + c_2e^{x/2}$. Since $z' = -\frac{c_1}{3}e^{-x/3} + \frac{c_2}{2}e^{x/2}$, $y = \frac{z'}{z} = \frac{-2c_1 + 3c_2e^{5x/6}}{6(c_1 + c_2e^{5x/6})}$.

(g) The given equation is equivalent to (A) $y' + y^2 - \frac{1}{3}y + \frac{1}{36} = 0$. The associated linear equation is (B) $z'' - \frac{1}{3}z' + \frac{1}{36}z = 0$, with characteristic polynomial $p(r) = r^2 - \frac{1}{3}r + \frac{1}{36} = \left(r - \frac{1}{6}\right)^2$. The general solution of (B) is $z = e^{x/6}(c_1 + c_2x)$. Since $z' = \frac{e^{x/6}}{6}(c_1 + c_2x) + c_2e^{x/6} = \frac{e^{x/6}}{6}(c_1 + c_2(x + 6))$,

$y = \frac{z'}{z} = \frac{c_1 + c_2(x+6)}{6(c_1 + c_2 x)}$.

5.6.40. **(a)** Suppose that z is a solution of (B) and let $y = \frac{z'}{rz}$. Then (D) $\frac{z''}{rz} + \left[p(x) - \frac{r'(x)}{r(x)}\right] y + q(x) = 0$ and $y' = \frac{z''}{rz} - \frac{1}{r}\left(\frac{z'}{z}\right)^2 - \frac{r'z'}{r^2 z} = \frac{z''}{rz} - ry^2 - \frac{r'}{r}y$, so $\frac{z''}{rz} = y' + ry^2 + \frac{r'}{r}y$. Therefore, (D) implies that y satisfies (A). Now suppose that y is a solution of (A) and let z be any function such that $z' = ryz$. Then $z'' = r'yz + ry'z + ryz' = \frac{r'}{r}z' + (y' + ry^2)rz = \frac{r'}{r}z' - (p(x)y + q(x))rz$, so $z'' - \frac{r'}{r}z' + p(x)ryz + q(x)rz = 0$, which implies that z satisfies (B), since $ryz = z'$.

(b) If $\{z_1, z_2\}$ is a fundamental set of solutions of (B) on (a, b), then $z = c_1 z_1 + c_2 z_2$ is the general solution of (B) on (a, b). This and **(a)** imply that (C) is the general solution of (A) on (a, b).

5.7 VARIATION OF PARAMETERS

5.7.2. (A) $y_p = u_1 \cos 2x + u_2 \sin 2x$;

$$\begin{aligned} u_1' \cos 2x + u_2' \sin 2x &= 0 \qquad \text{(B)} \\ -2u_1' \sin 2x + 2u_2' \cos 2x &= \sin 2x \sec^2 x. \qquad \text{(C)} \end{aligned}$$

Multiplying (B) by $2\sin 2x$ and (C) by $\cos 2x$ and adding the resulting equations yields $2u_2' = \tan 2x$, so $u_2' = \frac{\tan 2x}{2}$. Then (B) implies that $u_1' = -u_2' \tan(2x) = -\frac{\tan^2 2x}{2} = \frac{1 - \sec^2 2x}{2}$. Therefore,$u_1 = \frac{x}{2} - \frac{\tan 2x}{4}$ and $u_2 = -\frac{\ln|\cos 2x|}{4}$. Now (A) yields $y_p = -\frac{\sin 2x \ln|\cos 2x|}{4} + \frac{x\cos 2x}{2} - \frac{\sin 2x}{4}$. Since $\sin 2x$ satisfies the complementary equation we redefine $y_p = -\frac{\sin 2x \ln|\cos 2x|}{4} + \frac{x \cos 2x}{2}$.

5.7.4. (A) $y_p = u_1 e^x \cos x + u_2 e^x \sin x$;

$$\begin{aligned} u_1' e^x \cos x + u_2' e^x \sin x &= 0 \qquad \text{(B)} \\ u_1'(e^x \cos x - e^x \sin x) + u_2'(e^x \sin x + e^x \cos x) &= 3e^x \sec x. \qquad \text{(C)} \end{aligned}$$

Subtracting (B) from (C) and cancelling e^x from the resulting equations yields

$$\begin{aligned} u_1' \cos x + u_2' \sin x &= 0 \qquad \text{(D)} \\ -u_1' \sin x + u_2' \cos x &= 3 \sec x. \qquad \text{(E)} \end{aligned}$$

Multiplying (D) by $\sin x$ and (E) by $\cos x$ and adding the results yields $u_2' = 3$. From (D), $u_1' = -u_2' \tan x = -3\tan x$. Therefore $u_1 = 3\ln|\cos x|$, $u_2 = 3x$. Now (A) yields $y_p = 3e^x(\cos x \ln|\cos x| + x \sin x)$.

5.7.6. (A) $y_p = u_1 e^x + u_2 e^{-x}$;

$$\begin{aligned} u_1' e^x + u_2' e^{-x} &= 0 \qquad \text{(B)} \\ u_1' e^x - u_2' e^{-x} &= \frac{4e^{-x}}{1 + e^{-2x}}. \qquad \text{(C)} \end{aligned}$$

Adding (B) to (C) yields $2u_1' e^x = \frac{4e^{-x}}{1 + e^{-2x}}$, so $u_1' = \frac{2e^{-2x}}{1 - e^{-2x}}$. From (B), $u_2' = -e^{2x} u_1' = -\frac{2}{1 - e^{-2x}} = \frac{2e^{2x}}{1 - e^{2x}}$. Using the substitution $v = e^{-2x}$ we integrate u_1' to obtain $u_1 = \ln(1 - e^{-2x})$.

Using the substitution $v = e^{2x}$ we integrate u_2' to obtain $u_1 = \ln(1-e^{2x})$. Now (A) yields $y_p = e^x \ln(1-e^{-2x}) - e^{-x}\ln(e^{2x}-1)$.

5.7.8. (A) $y_p = u_1 e^x + u_2 \dfrac{e^x}{x}$;

$$\begin{array}{rcll} u_1' e^x + u_2' \dfrac{e^x}{x} &=& 0 & \text{(B)} \\ u_1' e^x + u_2' \left(\dfrac{e^x}{x} - \dfrac{e^x}{x^2}\right) &=& \dfrac{e^{2x}}{x}. & \text{(C)} \end{array}$$

Subtracting (B) from (C) yields $-\dfrac{u_2' e^x}{x^2} = \dfrac{e^{2x}}{x}$, so $u_2' = -xe^x$. From (B), $u_1' = \dfrac{u_2'}{x} = e^x$. Therefore $u_1 = e^x$, $u_2 = -xe^x + e^x$. Now (A) yields $y_p = \dfrac{e^{2x}}{x}$.

5.7.10. (A) $y_p = u_1 e^{-x^2} + u_2 x e^{-x^2}$;

$$\begin{array}{rcll} u_1' e^{-x^2} + u_2' x e^{-x^2} &=& 0 & \text{(B)} \\ -2xu_1' e^{-x^2} + u_2'(e^{-x^2} - 2x^2 e^{-x^2}) &=& 4e^{-x(x+2)}. & \text{(C)} \end{array}$$

Multiplying (B) by $2x$ and adding the result to (C) yields $u_2' e^{-x^2} = 4e^{-x(x+2)}$, so $u_2' = 4e^{-2x}$. From (B), $u_1' = -u_2' x = -4xe^{-2x}$. Therefore $u_1 = (2x+1)e^{-2x}$, $u_2 = -2e^{-2x}$. Now (A) yields $y_p = e^{-x(x+2)}$.

5.7.12. (A) $y_p = u_1 x + u_2 x^3$;

$$\begin{array}{rcll} u_1' x + \ u_2' x^3 &=& 0 & \text{(B)} \\ u_1' \ + 3u_2' x^2 &=& \dfrac{2x^4 \sin x}{x^2} = 2x^2 \sin x & \text{(C).} \end{array}$$

Multiplying (B) by $\dfrac{1}{x}$ and subtracting the result from (C) yields $2x^2 u_2' = 2x^2 \sin x$, so $u_2' = \sin x$. From (B), $u_1' = -u_2' x^2 = -x^2 \sin x$. Therefore $u_1 = (x^2-2)\cos x - 2x \sin x$, $u_2 = -\cos x$. Now (A) yields $y_p = -2x^2 \sin x - 2x \cos x$.

5.7.14. (A) $y_p = u_1 \cos\sqrt{x} + u_2 \sin\sqrt{x}$;

$$\begin{array}{rcll} u_1' \cos\sqrt{x} + u_2' \sin\sqrt{x} &=& 0 & \text{(B)} \\ -u_1' \dfrac{\sin\sqrt{x}}{2\sqrt{x}} + u_2' \dfrac{\cos\sqrt{x}}{2\sqrt{x}} &=& \dfrac{\sin\sqrt{x}}{4x} & \text{(C).} \end{array}$$

Multiplying (B) by $\dfrac{\sin\sqrt{x}}{2\sqrt{x}}$ and (C) by $\cos\sqrt{x}$ and adding the resulting equations yields $\dfrac{u_2'}{2\sqrt{x}} = \dfrac{\sin\sqrt{x}\cos\sqrt{x}}{4x}$, so $u_2' = \dfrac{\sin\sqrt{x}\cos\sqrt{x}}{2\sqrt{x}}$. From (B), $u_1' = -u_2' \tan\sqrt{x} = -\dfrac{\sin^2\sqrt{x}}{2\sqrt{x}}$. Therefore, $u_1 = \dfrac{\sin\sqrt{x}\cos\sqrt{x}}{2} - \dfrac{\sqrt{x}}{2}$, $u_2 = \dfrac{\sin^2\sqrt{x}}{2}$. Now (A) yields $y_p = \dfrac{\sin\sqrt{x}}{2} - \dfrac{\sqrt{x}\cos\sqrt{x}}{2}$. Since $\sin\sqrt{x}$ satisfies the complementary equation we redefine $y_p = -\dfrac{\sqrt{x}\cos\sqrt{x}}{2}$.

5.7.16. (A) $y_p = u_1 x^a + u_2 x^a \ln x$;

$$\begin{aligned} u_1' x^a + u_2' x^a \ln x &= 0 \qquad \text{(B)} \\ a u_1' x^{a-1} + u_2'(a x^{a-1} \ln x + x^{a-1}) &= \frac{x^{a+1}}{x^2} = x^{a-1} \qquad \text{(C)}. \end{aligned}$$

Multiplying (B) by $\frac{a}{x}$ and subtracting the result from (C) yields $u_2' x^{a-1} = x^{a-1}$, so $u_2' = 1$. From (B), $u_1' = -u_2' \ln x = -\ln x$. Therefore, $u_1 = x - \ln x$, $u_2 = x$. Now (A) yields $y_p = x^{a+1}$.

5.7.18. $y_p = u_1 e^{x^2} + u_2 e^{-x^2}$;

$$\begin{aligned} u_1' e^{x^2} + u_2' e^{-x^2} &= 0 \qquad \text{(B)} \\ 2u_1' x e^{x^2} - 2u_2' x e^{-x^2} &= \frac{8x^5}{x} = 8x^4. \qquad \text{(B)} \end{aligned}$$

Multiplying (B) by $2x$ and adding the result to (C) yields $4u_1' x e^{x^2} = 8x^4$, so $u_1' = 2x^3 e^{-x^2}$. From (B), $u_2' = -u_1' e^{2x^2} = -2x^3 e^{x^2}$. Therefore $u_1 = -e^{-x^2}(x^2+1)$, $u_2 = -e^{x^2}(x^2-1)$. Now (A) yields $y_p = -2x^2$.

5.7.20. (A) $y_p = u_1 \sqrt{x} e^{2x} + u_2 \sqrt{x} e^{-2x}$;

$$\begin{aligned} u_1' \sqrt{x} e^{2x} + u_2' \sqrt{x} e^{-2x} &= 0 \qquad \text{(B)} \\ u_1' e^{2x} \left(2\sqrt{x} + \frac{1}{2\sqrt{x}}\right) - u_2' e^{-2x} \left(2\sqrt{x} - \frac{1}{2\sqrt{x}}\right) &= \frac{8x^{5/2}}{4x^2} = 2\sqrt{x} \qquad \text{(C)}. \end{aligned}$$

Multiplying (B) by $\frac{1}{2x}$, subtracting the result from (C), and cancelling common factors from the resulting equations yields

$$\begin{aligned} u_1' e^{2x} + u_2' e^{-2x} &= 0 \qquad \text{(D)} \\ u_1' e^{2x} - u_2' e^{-2x} &= 1. \qquad \text{(E)} \end{aligned}$$

Adding (D) to (E) yields $2u_1' e^{2x} = 1$, so $u_1' = \frac{e^{-2x}}{2}$. From (D), $u_2' = -u_1' e^{4x} = -\frac{e^{2x}}{2}$. Therefore, $u_1 = -\frac{e^{-2x}}{4}$, $u_2 = -\frac{e^{2x}}{4}$. Now (A) yields $y_p = -\frac{\sqrt{x}}{2}$.

5.7.22. (A)$y_p = u_1 x e^x + u_2 x e^{-x}$;

$$\begin{aligned} u_1' x e^x + u_2' x e^{-x} &= 0 \qquad \text{(B)} \\ u_1'(x+1)e^x - u_2'(x-1)e^{-x} &= \frac{3x^4}{x^2} = 3x^2 \qquad \text{(C)}. \end{aligned}$$

Multiplying (B) by $\frac{1}{x}$, subtracting the resulting equation from (C), and cancelling common factors yields

$$\begin{aligned} u_1' e^x + u_2' e^{-x} &= 0 \qquad \text{(D)} \\ u_1' e^x - u_2' e^{-x} &= 3x. \qquad \text{(E)} \end{aligned}$$

Adding (D) to (E) yields $2u_1' e^x = 3x$, so $u_1' = \frac{3xe^{-x}}{2}$. From (D), $u_2' = -u_1' e^{2x} = -\frac{3xe^x}{2}$. Therefore $u_1 = -\frac{3e^x(x+1)}{2}$, $u_2 = -\frac{3e^x(x-1)}{2}$. Now (A) yields $y_p = -3x^2$.

5.7.24. (A) $y_p = \dfrac{u_1}{x} + u_2x^3$;

$$\begin{aligned} \frac{u_1'}{x} + u_2'x^3 &= 0 \qquad \text{(B)} \\ -\frac{u_1'}{x^2} + 3u_2'x^2 &= \frac{x^{3/2}}{x^2} = x^{-1/2}. \qquad \text{(C)} \end{aligned}$$

Multiplying (B) by $\dfrac{1}{x}$ and adding the result to (C) yields $4u_2'x^2 = x^{-1/2}$, so $u_2' = \dfrac{x^{-5/2}}{4}$. From (B), $u_1' = -u_2'x^4 = -\dfrac{x^{3/2}}{4}$. Therefore $u_1 = -\dfrac{x^{5/2}}{10}$, $u_2 = -\dfrac{x^{-3/2}}{6}$. Now (A) yields $y_p = -\dfrac{4x^{3/2}}{15}$.

5.7.26. (A) $y_p = u_1x^2e^x + u_2x^3e^x$;

$$\begin{aligned} u_1'x^2e^x + u_2'x^3e^x &= 0 \qquad \text{(B)} \\ u_1'(x^2e^x + 2xe^x) + u_2'(x^3e^x + 3x^2e^x) &= \frac{2xe^x}{x^2} = \frac{2e^x}{x}. \qquad \text{(C)} \end{aligned}$$

Subtracting (B) from (C) and cancelling common factors in the resulting equations yields

$$\begin{aligned} u_1' + u_2'x &= 0 \qquad \text{(D)} \\ 2u_1'x + 3u_2'x^2 &= \frac{2}{x}. \qquad \text{(E)} \end{aligned}$$

Multiplying (D) by $2x$ and subtracting the result from (E) yields $x^2u_2' = \dfrac{2}{x}$, so $u_2' = \dfrac{2}{x^3}$. From (D), $u_1' = -u_2'x = -\dfrac{2}{x^2}$. Therefore $u_1 = \dfrac{2}{x}$, $u_2 = -\dfrac{1}{x^2}$. Now (A) yields $y_p = xe^x$.

5.7.28. (A) $y_p = u_1x + u_2e^x$;

$$\begin{aligned} u_1'x + u_2'e^x &= 0 \qquad \text{(B)} \\ u_1' + u_2'e^x &= \frac{2(x-1)^2e^x}{x-1} = 2(x-1)e^x. \qquad \text{(C)} \end{aligned}$$

Subtracting (B) from (C) yields $u_1'(1-x) = 2(x-1)e^x$, so $u_1' = -2e^x$. From (B), $u_2' = -u_1'xe^{-x} = 2x$. Therefore, $u_1 = -2e^x$, $u_2 = x^2$. Now (A) yields $y_p = xe^x(x-2)$.

5.7.30. (A) $y_p = u_1e^{2x} + u_2xe^{-x}$;

$$\begin{aligned} u_1'e^{2x} + u_2'xe^{-x} &= 0 \qquad \text{(B)} \\ 2u_1'e^{2x} + u_2'(e^{-x} - xe^{-x}) &= \frac{(3x-1)^2e^{2x}}{3x-1} = (3x-1)e^{2x}. \qquad \text{(C)} \end{aligned}$$

Multiplying (B) by 2 and subtracting the result from (C) yields $u_2'(1-3x)e^{-x} = (3x-1)e^{2x}$, so $u_2' = -e^{3x}$. From (B), $u_1' = -u_2'xe^{-3x} = x$. Therefore $u_1 = \dfrac{x^2}{2}$, $u_2 = -\dfrac{e^{3x}}{3}$. Now (A) yields $y_p = \dfrac{xe^{2x}(3x-2)}{3}$. The general solution of the given equation is $y = \dfrac{xe^{2x}(3x-2)}{3} + c_1e^{2x} + c_2xe^{-x}$. Differentiating this yields $y' = \dfrac{e^{2x}(3x^2+x-1)}{3} + 2c_1e^{2x} + c_2(1-x)e^{-x}$. Now $y(0) = 1,\ y'(0) = 2 \Rightarrow c_1 = 1,\ 2 = -\dfrac{1}{3} + 2c_1 + c_2$, so $c_2 = \dfrac{1}{3}$, and $y = \dfrac{e^{2x}(3x^2-2x+6)}{6} + \dfrac{xe^{-x}}{3}$.

5.7.32. (A) $y_p = u_1(x-1)e^x + u_2(x-1)$;

$$\begin{aligned} u_1'(x-1)e^x + u_2'(x-1) &= 0 \qquad \text{(B)} \\ u_1'xe^x + u_2' &= \frac{(x-1)^3e^x}{(x-1)^2} = (x-1)e^x. \qquad \text{(C)} \end{aligned}$$

From (B), $u_1' = -u_2'e^{-x}$. Substituting this into (C) yields $-u_2'(x-1) = (x-1)e^x$, so $u_2' = -e^x$, $u_1' = 1$. Therefore $u_1 = x$, $u_2 = e^x$. Now (A) yields $y_p = e^x(x-1)^2$. The general solution of the given equation is $y = (x-1)^2e^x + c_1(x-1)e^x + c_2(x-1)$. Differentiating this yields $y' = (x^2-1)e^x + c_1xe^x + c_2$. Now $y(0) = 4$, $y'(0) = -6 \Rightarrow 4 = 1 - c_1 - c_2$, $-6 = -1 + c_2$, so $c_1 = 2, c_2 = -5$ and $y = (x^2-1)e^x - 5(x-1)$.

5.7.34. (A) $y_p = u_1x + \dfrac{u_2}{x^2}$;

$$\begin{aligned} u_1'x + \frac{u_2'}{x^2} &= 0 \qquad \text{(B)} \\ u_1' - \frac{2u_2'}{x^3} &= -\frac{2x^2}{x^2} = -2. \qquad \text{(C)} \end{aligned}$$

Multiplying (B) by $\dfrac{2}{x}$ and adding the result to (C) yields $3u_1' = -2$, so $u_1' = -\dfrac{2}{3}$. From (B), $u_2' = -u_1'x^3 = \dfrac{2x^3}{3}$. Therefore $u_1 = -\dfrac{2x}{3}$, $u_2 = \dfrac{x^4}{6}$. Now (A) yields $y_p = -\dfrac{x^2}{2}$. The general solution of the given equation is $y = -\dfrac{x^2}{2} + c_1x + \dfrac{c_2}{x^2}$. Differentiating this yields $y' = -x + c_1 - \dfrac{2c_2}{x^3}$. Now $y(1) = 1$, $y'(1) = -1 \Rightarrow 1 = -\dfrac{1}{2} + c_1 + c_2$, $-1 = -1 + c_1 - 2c_2$, so $c_1 = 1, c_2 = \dfrac{1}{2}$, and $y = -\dfrac{x^2}{2} + x + \dfrac{1}{2x^2}$.

5.7.36. Since $\overline{y} = y_p - a_1y_1 - a_2y_2$,

$$\begin{aligned} P_0(x)\overline{y}'' + P_1(x)\overline{y}' + P_2(x)\overline{y} &= P_0(x)(y_p - a_1y_1 - a_2y_2)'' \\ &\quad + P_1(x)(y_p - a_1y_1 - a_2y_2)' \\ &\quad + P_2(x)(y_p - a_1y_1 - a_2y_2) \\ &= (P_0(x)y_p'' + P_1(x)y_p' + P_2(x)y_p) \\ &\quad - a_1\left[P_0(x)y_1'' + P_1(x)y_1' + P_2(x)y_1\right] \\ &\quad - a_2\left[P_0(x)y_2'' + P_1(x)y_2' + P_2(x)y_2\right] \\ &= F(x) - a_1\cdot 0 - a_2\cdot 0 = F(x); \end{aligned}$$

hence $\overline{y}$ is a particular solution of (A).

5.7.38. **(a)** $y_p = u_1e^x + u_2e^{-x}$ is a solution of (A) on (a,∞) if $u_1'e^x + u_2'e^{-x} = 0$ and $u_1'e^x - u_2'e^{-x} = f(x)$. Solving these two equations yields $u_1' = \dfrac{e^{-x}f}{2}$, $u_2' = -\dfrac{e^xf}{2}$. The functions $u_1(x) = \dfrac{1}{2}\displaystyle\int_0^x e^{-t}f(t)\,dt$ and $u_2(x) = -\dfrac{1}{2}\displaystyle\int_0^x e^tf(t)\,dt$ satisfy these conditions. Therefore,

$$\begin{aligned} y_p(x) &= \frac{e^x}{2}\int_0^x e^{-t}f(t)\,dt - \frac{e^{-x}}{2}\int_0^x e^tf(t)\,dt \\ &= \frac{1}{2}\int_0^x f(t)\left(e^{(x-t) - e^{-(x-t)}}\right)dt = \int_0^x f(t)\sinh(x-t)\,dt. \end{aligned}$$

is a particular solution of $y'' - y = f(x)$. Differentiating y_p yields

$$\begin{aligned} y_p'(x) &= \frac{e^x}{2}\int_0^x e^{-t} f(t)\,dt + \frac{e^x}{2}e^{-x} + \frac{e^{-x}}{2}\int_0^x e^t f(t)\,dt - \frac{e^{-x}}{2}e^x \\ &= \frac{e^x}{2}\int_0^x e^{-t} f(t)\,dt + \frac{e^{-x}}{2}\int_0^x e^t f(t)\,dt \\ &= \frac{1}{2}\int_0^x f(t)\left(e^{(x-t)} + e^{-(x-t)}\right)\,dt = \int_0^x f(t)\cosh(x-t)\,dt. \end{aligned}$$

Since $y_p(x_0) = y_p'(x_0) = 0$, the solution of the initial value problem is

$$\begin{aligned} y &= y_p + k_0\cosh x + k_1\sinh x \\ &= k_0\cosh x + k_1\sinh x + \int_0^x \sinh(x-t) f(t)\,dt. \end{aligned}$$

The derivative of the solution is

$$\begin{aligned} y' &= y_p' + k_0\sinh x + k_1\cosh x \\ &= k_0\sinh x + k_1\cosh x + \int_0^x \cosh(x-t) f(t)\,dt. \end{aligned}$$

CHAPTER 6
Applications of Linear Second Order Equations

6.1 SPRING PROBLEMS I

6.1.2. Since $\dfrac{k}{m} = \dfrac{g}{\Delta l} = \dfrac{32}{.1} = 320$ the equation of motion is (A) $y'' + 320y = 0$. The general solution of (A) is $y = c_1 \cos 8\sqrt{5}t + c_2 \sin 8\sqrt{5}t$, so $y' = 8\sqrt{5}(-c_1 \sin 8\sqrt{5}t + c_2 \cos 8\sqrt{5}t)$. Now $y(0) = -\dfrac{1}{4} \Rightarrow c_1 = -\dfrac{1}{4}$ and $y'(0) = -2 \Rightarrow c_2 = -\dfrac{1}{4\sqrt{5}}$, so $y = -\dfrac{1}{4}\cos 8\sqrt{5}t - \dfrac{1}{4\sqrt{5}} \sin 8\sqrt{5}t$ ft.

6.1.4. Since $\dfrac{k}{m} = \dfrac{g}{\Delta l} = \dfrac{32}{.5} = 64$ the equation of motion is (A) $y'' + 64y = 0$. The general solution of (A) is $y = c_1 \cos 8t + c_2 \sin 8t$, so $y' = 8(-c_1 \sin 8t + c_2 \cos 8t)$. Now $y(0) = \dfrac{1}{4} \Rightarrow c_1 = \dfrac{1}{4}$ and $y'(0) = -\dfrac{1}{2} \Rightarrow c_2 = -\dfrac{1}{16}$, so $y = \dfrac{1}{4}\cos 8t - \dfrac{1}{16}\sin 8t$ ft; $R = \dfrac{\sqrt{17}}{16}$ ft; $\omega_0 = 8$ rad/s; $T = \pi/4$ s; $\phi \approx -.245$ rad $\approx -14.04°$.

6.1.6. Since $k = \dfrac{mg}{\Delta l} = \dfrac{(9.8)10}{.7} = 140$, the equation of motion of the 2 kg mass is (A) $y'' + 70y = 0$. The general solution of (A) is $y = c_1 \cos \sqrt{70}t + c_2 \sin \sqrt{70}t$, so $y' = \sqrt{70}(-c_1 \sin \sqrt{70}t + c_2 \cos \sqrt{70}t)$. Now $y(0) = -\dfrac{1}{4} \Rightarrow c_1 - \dfrac{1}{4} =$ and $y'(0) = 2 \Rightarrow c_2 = \dfrac{2}{\sqrt{70}}$, so $y = -\dfrac{1}{4}\cos \sqrt{70}\, t + \dfrac{2}{\sqrt{70}} \sin \sqrt{70}\, t$ m; $R = \dfrac{1}{4}\sqrt{\dfrac{67}{35}}$ m; $\omega_0 = \sqrt{70}$ rad/s; $T = 2\pi/\sqrt{70}$ s; $\phi \approx 2.38$ rad $\approx 136.28°$.

6.1.8. Since $\dfrac{k}{m} = \dfrac{g}{\Delta l} = \dfrac{32}{1/2} = 64$ the equation of motion is (A) $y'' + 64y = 0$. The general solution of (A) is $y = c_1 \cos 8t + c_2 \sin 8t$, so $y' = 8(-c_1 \sin 8t + c_2 \cos 8t)$. Now $y(0) = \dfrac{1}{2} \Rightarrow c_1 = \dfrac{1}{2}$ and $y'(0) = -3 \Rightarrow c_2 = -\dfrac{3}{8}$, so $y = \dfrac{1}{2}\cos 8t - \dfrac{3}{8}\sin 8t$ ft.

6.1.10. $m = \dfrac{64}{32} = 2$, so the equation of motion is $2y'' + 8y = 2\sin t$, or (A) $y'' + 4y = \sin t$. Let $y_p = A\cos t + B\sin t$; then $y_p'' = -A\cos t - B\sin t$, so $y_p'' + 4y_p = 3A\cos t + 3B\sin t = \sin t$ if $3A = 0, 3B = 1$. Therefore, $A = 0$, $B = \dfrac{1}{3}$, and $y_p = \dfrac{1}{3}\sin t$. The general solution of

(A) is (B) $y = \frac{1}{3}\sin t + c_1\cos 2t + c_2\sin 2t$, so $y(0) = \frac{1}{2} \Rightarrow c_1 = \frac{1}{2}$. Differentiating (B) yields $y' = \frac{1}{3}\cos t - 2c_1\sin 2t + 2c_2\cos 2t$, so $y'(0) = 2 \Rightarrow 2 = \frac{1}{3} + 2c_2$, so $c_2 = \frac{5}{6}$. Therefore, $y = \frac{1}{3}\sin t + \frac{1}{2}\cos 2t + \frac{5}{6}\sin 2t$ ft.

6.1.12. $m = \frac{4}{32} = \frac{1}{8}$ and $k = \frac{mg}{\Delta l} = 4$, so the equation of motion is $\frac{1}{8}y'' + 4y = \frac{1}{4}\sin 8t$, or (A) $y'' + 32y = 2\sin 8t$. Let $y_p = A\cos 8t + B\sin 8t$; then $y_p'' = -64A\cos t - 64B\sin 8t$, so $y_p'' + 32y_p = -32A\cos 8t - 32B\sin 8t = 2\sin 8t$ if $-32A = 0$, $-32B = 2$. Therefore,$A = 0$, $B = -\frac{1}{16}$, and $y_p = -\frac{1}{16}\sin 8t$. The general solution of (A) is (B) $y = -\frac{1}{16}\sin 8t + c_1\cos 4\sqrt{2}t + c_2\sin 4\sqrt{2}t$, so $y(0) = \frac{1}{3} \Rightarrow c_1 = \frac{1}{3}$. Differentiating (B) yields $y' = -\frac{1}{2}\cos 8t + 4\sqrt{2}(-c_1\sin 4\sqrt{2}t + c_2\cos 4\sqrt{2}t)$, so $y'(0) = -1 \Rightarrow -1 = -\frac{1}{2} + 4\sqrt{2}c_2$, so $c_2 = -\frac{1}{8\sqrt{2}}$. Therefore,$y = -\frac{1}{16}\sin 8t + \frac{1}{3}\cos 4\sqrt{2}t - \frac{1}{8\sqrt{2}}\sin 4\sqrt{2}t$ ft.

6.1.14. Since $T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{k}}$ the period is proportional to the square root of the mass. Therefore, doubling the mass mutiplies the period by $\sqrt{2}$; hence the period of the system with the 20 gm mass is $T = 4\sqrt{2}$ s.

6.1.16. $m = \frac{6}{32} = \frac{3}{16}$ and $k = \frac{mg}{\Delta l} = \frac{6}{1/3} = 18$ so the equation of motion is $\frac{3}{16}y'' + 18y = 4\sin\omega t - 6\cos\omega t$, or (A) $y'' + 96y = \frac{64}{3}\sin\omega t - 32\cos\omega t$. The displacement will be unbounded if $\omega = \sqrt{96} = 4\sqrt{6}$, in which case (A) becomes (B) $y'' + 96y = \frac{64}{3}\sin 4\sqrt{6}t - 32\cos 4\sqrt{6}t$. Let

$$
\begin{aligned}
y_p &= At\cos 4\sqrt{6}t + Bt\sin 4\sqrt{6}t; \text{ then} \\
y_p' &= (A + 4\sqrt{6}Bt)\cos 4\sqrt{6}t + (B - 4\sqrt{6}At)\sin 4\sqrt{6}t \\
y_p'' &= (8\sqrt{6}B - 96At)\cos 4\sqrt{6}t - (8\sqrt{6}A + 96Bt)\sin 4\sqrt{6}t, \text{ so}
\end{aligned}
$$

$$y_p'' + 96y_p = 8\sqrt{6}B\cos 4\sqrt{6}t - 8\sqrt{6}A\sin 4\sqrt{6}t = \frac{64}{3}\sin 4\sqrt{6}t - 32\cos 4\sqrt{6}t$$

if $8\sqrt{6}B = -32, -8\sqrt{6}A = \frac{64}{3}$. Therefore, $A = -\frac{8}{3\sqrt{6}}$, $B = -\frac{4}{\sqrt{6}}$, and $y_p = -\frac{t}{\sqrt{6}}\left(\frac{8}{3}\cos 4\sqrt{6}t + 4\sin 4\sqrt{6}t\right)$. The general solution of (B) is

$$y = -\frac{t}{\sqrt{6}}\left(\frac{8}{3}\cos 4\sqrt{6}t + 4\sin 4\sqrt{6}t\right) + c_1\cos 4\sqrt{6}t + c_2\sin 4\sqrt{6}t, \tag{C}$$

so $y(0) = 0 \Rightarrow c_1 = 0$. Differentiating (C) yields

$$
\begin{aligned}
y' &= -\left(\frac{8}{3\sqrt{6}}\cos 4\sqrt{6}t + \frac{4}{\sqrt{6}}\sin 4\sqrt{6}t\right) - 4t\left(-\frac{8}{3}\sin 4\sqrt{6}t + 4\cos 4\sqrt{6}t\right) \\
&\quad + 4\sqrt{6}(-c_1\sin 4\sqrt{6}t + c_2\cos\sqrt{6}t),
\end{aligned}
$$

so $y'(0) = 0 \Rightarrow 0 = -\frac{8}{3\sqrt{6}} + 4\sqrt{6}c_2$, and $c_2 = \frac{1}{9}$. Therefore,

$$y = -\frac{t}{\sqrt{6}}\left(\frac{8}{3}\cos 4\sqrt{6}t + 4\sin 4\sqrt{6}t\right) + \frac{1}{9}\sin 4\sqrt{6}t \text{ ft.}$$

6.1.18. The equation of motion is (A) $y'' + \omega_0^2 y = 0$. The general solution of (A) is $y = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t$. Now $y(0) = y_0 \Rightarrow c_1 = y_0$. Since $y' = \omega_0(-c_1 \sin \omega_0 t + c_2 \cos \omega_0 t)$, $y'(0) = v_0 \Rightarrow c_2 = \frac{v_0}{\omega_0}$. Therefore, $y = y_0 \cos \omega_0 t + \frac{v_0}{\omega_0} \sin \omega_0 t$;

$$R = \frac{1}{\omega_0} \sqrt{(\omega_0 y_0)^2 + (v_0)^2}; \ \cos \phi = \frac{y_0 \omega_0}{\sqrt{(\omega_0 y_0)^2 + (v_0)^2}}; \ \sin \phi = \frac{v_0}{\sqrt{(\omega_0 y_0)^2 + (v_0)^2}}.$$

Discussion 6.1.1 In Exercises 19, 20, and 21 we use the fact that in a spring–mass system with mass m and spring constant k the period of the motion is $T = 2\pi \sqrt{\frac{m}{k}}$. Therefore, if we have two systems with masses m_1 and m_2 and spring constants k_1 and k_2, then the periods are related by $\frac{T_2}{T_1} = \sqrt{\frac{m_2 k_1}{m_1 k_2}}$. We will use this formula in the solutions of these exercises.

6.1.20. Let $m_2 = 2m_1$. Since $k_1 = k_2$, $\frac{T_2}{T_1} = \sqrt{\frac{2m_1}{m_1}} = \sqrt{2}$, so $T_2 = \sqrt{2}T_1$.

6.1.21. Suppose that $T_2 = 3T_1$. Since $m_1 = m_2$, $\sqrt{\frac{k_1}{k_2}} = 3$, $k_1 = 9k_2$.

6.2 SPRING PROBLEMS II

6.2.2. Since $k = \frac{mg}{\Delta l} = \frac{16}{3.2} = 5$ the equation of motion is $\frac{1}{2}y'' + y' + 5y = 0$, or (A) $y'' + 2y' + 10y = 0$. The characteristic polynomial of (A) is $p(r) = r^2 + 2r + 10 = (r+1)^2 + 9$. Therefore,the general solution of (A) is $y = e^{-t}(c_1 \cos 3t + c_2 \sin 3t)$, so $y' = -y + 3e^{-t}(-c_1 \sin 3t + c_2 \cos 3t)$. Now $y(0) = -3$ and $y'(0) = 2 \Rightarrow c_1 = -3$ and $2 = 3 + 3c_2$, or $c_2 = -\frac{1}{3}$. Therefore,$y = -e^{-t}\left(3 \cos 3t + \frac{1}{3} \sin 3t\right)$ ft. The time–varying amplitude is $\frac{\sqrt{82}}{3} e^{-t}$ ft.

6.2.4. Since $k = \frac{mg}{\Delta l} = \frac{96}{3.2} = 30$ the equation of motion is $3y'' + 18y' + 30y = 0$, or (A) $y'' + 6y' + 10y = 0$. The characteristic polynomial of (A) is $p(r) = r^2 + 6r + 10 = (r+3)^2 + 1$. Therefore,the general solution of (A) is $y = e^{-3t}(c_1 \cos t + c_2 \sin t)$, so $y' = -3y + e^{-3t}(-c_1 \sin t + c_2 \cos t)$. Now $y(0) = -\frac{5}{4}$ and $y'(0) = -12 \Rightarrow c_1 = -\frac{5}{4}$ and $-12 = \frac{15}{4} + c_2$, or $c_2 = -\frac{63}{4}$. Therefore, $y = -\frac{e^{-3t}}{4}(5 \cos t + 63 \sin t)$ ft.

6.2.6. Since $k = \frac{mg}{\Delta l} = \frac{8}{.32} = 25$ the equation of motion is $\frac{1}{4}y'' + \frac{3}{2}y' + 25y = 0$, or (A) $y'' + 6y' + 100y = 0$. The characteristic polynomial of (A) is $p(r) = r^2 + 6r + 100 = (r+3)^2 + 91$. Therefore,the general solution of (A) is $y = e^{-3t}(c_1 \cos \sqrt{91}t + c_2 \sin \sqrt{91}t)$, so $y' = -3y + \sqrt{91}e^{-3t}(-c_1 \sin \sqrt{91}t + c_2 \cos \sqrt{91}t)$. Now $y(0) = \frac{1}{2}$ and $y'(0) = 4 \Rightarrow c_1 = \frac{1}{2}$ and $4 = -\frac{3}{2} + \sqrt{91}c_2$, or $c_2 = \frac{11}{2\sqrt{91}}$. Therefore,$y = \frac{1}{2}e^{-3t}\left(\cos \sqrt{91}t + \frac{11}{\sqrt{91}} \sin \sqrt{91}t\right)$ ft.

6.2.8. Since $k = \frac{mg}{\Delta l} = \frac{20 \cdot 980}{5} = 3920$ the equation of motion is $20y'' + 400y' + 3920y = 0$, or (A) $y'' + 20y' + 196y = 0$. The characteristic polynomial of (A) is $p(r) = r^2 + 20r + 196 =$

$(r+10)^2+96$. Therefore,the general solution of (A) is $y = e^{-10t}(c_1\cos 4\sqrt{6}t + c_2\sin 4\sqrt{6}t)$, so $y' = -10y + 4\sqrt{6}e^{-10t}(-c_1\sin 4\sqrt{6}t + c_2\cos 4\sqrt{6}t)$. Now $y(0)=9$ and $y'(0)=0 \Rightarrow c_1 = 9$ and $0 = -90 + 4\sqrt{6}c_2$, or $c_2 = \dfrac{45}{2\sqrt{6}}$. Therefore, $y = e^{-10t}\left(9\cos 4\sqrt{6}t + \dfrac{45}{2\sqrt{6}}\sin 4\sqrt{6}t\right)$ cm.

6.2.10. Since $k = \dfrac{mg}{\Delta l} = \dfrac{32}{1} = 32$ the equation of motion is (A) $y''+3y'+32y=0$. The characteristic polynomial of (A) is $p(r) = r^2+3r+32 = \left(r+\dfrac{3}{2}\right)^2 + \dfrac{119}{4}$. Therefore,the general solution of (A) is $y = e^{-3t/2}\left(c_1\cos\dfrac{\sqrt{119}}{2}t + c_2\sin\dfrac{\sqrt{119}}{2}t\right)$, so $y' = -\dfrac{3}{2}y + \dfrac{\sqrt{119}}{2}e^{-3t/2}\left(-c_1\sin\dfrac{\sqrt{119}}{2}t + c_2\cos\dfrac{\sqrt{119}}{2}t\right)$. Now $y(0) = \dfrac{1}{2}$ and $y'(0) = -3 \Rightarrow c_1 = \dfrac{1}{2}$ and $-3 = -\dfrac{3}{4} + \dfrac{\sqrt{119}}{2}c_2$, or $c_2 = -\dfrac{9}{2\sqrt{119}}$. Therefore, $y = e^{-\frac{3}{2}t}\left(\dfrac{1}{2}\cos\dfrac{\sqrt{119}}{2}t - \dfrac{9}{2\sqrt{119}}\sin\dfrac{\sqrt{119}}{2}t\right)$ ft.

6.2.12. Since $k = \dfrac{mg}{\Delta l} = \dfrac{2}{.32} = \dfrac{25}{4}$ the equation of motion is $\dfrac{1}{16}y'' + \dfrac{1}{8}y' + \dfrac{25}{4}y = 0$, or (A) $y''+2y'+100y=0$. The characteristic polynomial of (A) is $p(r) = r^2+2r+100 = (r+1)^2+99$. Therefore,the general solution of (A) is $y = e^{-t}(c_1\cos 3\sqrt{11}t + c_2\sin 3\sqrt{11}t)$, so $y' = -y + 3\sqrt{11}e^{-t}(-c_1\sin 3\sqrt{11}t + c_2\cos 3\sqrt{11}t)$. Now $y(0) = -\dfrac{1}{3}$ and $y'(0) = 5 \Rightarrow c_1 = -\dfrac{1}{3}$ and $5 = \dfrac{1}{3} + 3\sqrt{11}c_2$, or $c_2 = \dfrac{14}{9\sqrt{11}}$. Therefore,$y = e^{-t}\left(-\dfrac{1}{3}\cos 3\sqrt{11}t + \dfrac{14}{9\sqrt{11}}\sin 3\sqrt{11}t\right)$ ft.

6.2.14. Since $k = \dfrac{mg}{\Delta l} = 32$ the equation of motion is (A) $y''+12y'+32y=0$. The characteristic polynomial of (A) is $p(r) = r^2+12r+32 = (r+8)(r+4)$. Therefore, the general solution of (A) is $y = c_1e^{-8t}+c_2e^{-4t}$, so $y' = -8c_1e^{-8t} - 4c_2e^{-4t}$. Now $y(0) = -\dfrac{2}{3}$ and $y'(0) = 0 \Rightarrow c_1+c_2 = \dfrac{2}{3}$,- $8c_1 - 4c_2 = 0$, so $c_1 = -\dfrac{2}{3}$, $c_2 = \dfrac{4}{3}$, and $y = -\dfrac{2}{3}(e^{-8t} - 2e^{-4t})$.

6.2.16. Since $k = \dfrac{mg}{\Delta l} = \dfrac{100\cdot 980}{98} = 100$ the equation of motion is $100y''+600y'+1000y=0$, or (A) $y''+6y'+10y=0$. The characteristic polynomial of (A) is $p(r) = r^2+6r+10 = (r+3)^2+1$. Therefore, the general solution of (A) is $y = e^{-3t}(c_1\cos t + c_2\sin t)$, so $y' = -3y + e^{-t}(-c_1\sin t + c_2\cos t)$. Now $y(0) = 10$ and $y'(0) = -100 \Rightarrow c_1 = 10$ and $-100 = -30 + c_2$, or $c_2 = -70$. Therefore, $y = e^{-3t}(10\cos t - 70\sin t)$ cm.

6.2.18. The equation of motion is (A) $2y''+4y'+20y = 3\cos 4t - 5\sin 4t$. The steady state component of the solution of (A) is of the form $y_p = A\cos 4t + B\sin 4t$; therefore $y_p' = -4A\sin 4t + 4B\cos 4t$ and $y_p'' = -16A\cos 4t - 16B\sin 4t$, so $2y_p''+4y_p'+20y_p = (-12A+16B)\cos 4t - (16A+12B)\sin 4t = 3\cos 4t - 5\sin 4t$ if $-12A+16B = 3$, $-16A-12B = -5$; therefore $A = \dfrac{11}{100}$, $B = \dfrac{27}{100}$, and $y_p = \dfrac{11}{100}\cos 4t + \dfrac{27}{100}\sin 4t$ cm.

6.2.20. Since $k = \dfrac{mg}{\Delta l} = \dfrac{9.8}{.49} = 20$ the equation of motion is (A) $y''+4y'+20y = 8\sin 2t - 6\cos 2t$. The steady state component of the solution of (A) is of the form $y_p = A\cos 2t + B\sin 2t$; therefore

$y_p' = -2A\sin 2t + 2B\cos 2t$ and $y_p'' = -4A\cos 2t - 4B\sin 2t$, so $y_p'' + 4y_p' + 20y_p = (16A + 8B)\cos 2t - (8A - 16B)\sin 2t = 8\sin 2t - 6\cos 2t$ if $16A + 8B = -6$, $-8A + 16B = 8$; therefore $A = -\frac{1}{2}$, $B = \frac{1}{4}$, and $y = -\frac{1}{2}\cos 2t + \frac{1}{4}\sin 2t$ m.

6.2.22. If $e^{r_1 t}(c_1 + c_2 t) = 0$, then (A) $c_1 + c_2 t = 0$. If $c_2 = 0$, then $c_1 \neq 0$ (by assumption), so (A) is impossible. If $c_1 \neq 0$, then the left side of (A) is strictly monotonic and therefore cannot have the same value for two distinct values of t.

6.2.24. If $y = e^{-ct/2m}(c_1\cos\omega_1 t + c_2\sin\omega_1 t)$, then $y' = -\frac{c}{2m}y + \omega_1 e^{-ct/2m}(-c_1\sin\omega_1 t + c_2\cos\omega_1 t)$, so $y(0) = y_0$ and $y'(0) = v_0 \Rightarrow c_1 = y_0$ and $v_0 = -\frac{cy_0}{2m} + c_2\omega_1$, so $c_2 = \frac{1}{\omega_1}\left(v_0 + \frac{cy_0}{2m}t\right)$ and $y = e^{-ct/2m}\left(y_0\cos\omega_1 t + \frac{1}{\omega_1}\left(v_0 + \frac{cy_0}{2m}\right)\sin\omega_1 t\right)$.

6.2.26. If $y = e^{r_1 t}(c_1 + c_2 t)$, then $y' = r_1 y + c_2 e^{r_1 t}$, so $y(0) = y_0$ and $y'(0) = v_0 \Rightarrow c_1 = y_0$ and $v_0 = r_1 y_0 + c_2$, so $c_2 = v_0 - r_1 y_0$. Therefore, $y = e^{r_1 t}(y_0 + (v_0 - r_1 y_0)t)$.

6.3 THE RLC CIRCUIT

6.3.2. $\frac{1}{20}Q'' + 2Q' + 100Q = 0$; $Q'' + 40Q' + 2000Q = 0$; $r^2 + 40r + 2000 = (r+20)^2 + 1600 = 0$; $r = -20 \pm 40i$; $Q = e^{-20t}(2\cos 40t + c_2\sin 40t)$ (since $Q_0 = 2$); $I = Q' = e^{-20t}\left((40c_2 - 40)\cos 40t - (20c_2 + 80)\sin 40t\right)$; $I_0 = 2 \Rightarrow 40c_2 - 40 = 2 \Rightarrow c_2 = \frac{21}{20}$, so $20c_2 + 80 = 101$; $I = e^{-20t}(2\cos 40t - 101\sin 40t)$.

6.3.4. $\frac{1}{10}Q'' + 6Q' + 250Q = 0$; $Q'' + 60Q' + 2500Q = 0$; $r^2 + 60r + 2500 = (r+30)^2 + 1600 = 0$; $r = -30 \pm 40i$; $Q = e^{-30t}(3\cos 40t + c_2\sin 40t)$ (since $Q_0 = 3$); $I = Q' = e^{-30t}\left((40c_2 - 90)\cos 40t - (30c_2 + 120)\sin 40t\right)$; $I_0 = -10 \Rightarrow 40c_2 - 90 = -10 \Rightarrow c_2 = 2$, so $-30c_2 - 120 = -180$; $I = -10e^{-30t}(\cos 40t + 18\sin 40t)$.

6.3.6. $Q_p = A\cos 10t + B\sin 10t$; $Q_p' = 10B\cos 10t - 10A\sin 10t$; $Q_p'' = -100A\cos 10t - 100B\sin 10t$; $\frac{1}{10}Q_p'' + 3Q_p' + 100Q_9 = (90A + 30B)\cos 10t - (30A - 90B)\sin 10t = 5\cos 10t - 5\sin 10t$, so $90A + 30B = 5$, $-30A + 90B = -5$. Therefore, $A = 1/15$, $B = -1/30$, $Q_p = \frac{\cos 10t}{15} - \frac{\sin 10t}{30}$, and $I_p = -\frac{1}{3}(\cos 10t + 2\sin 10t)$.

6.3.8. $Q_p = A\cos 50t + B\sin 50t$; $Q_p' = 50B\cos 50t - 50A\sin 50t$; $Q_p'' = -2500A\cos 50t - 2500B\sin 50t$; $\frac{1}{10}Q_p'' + 2Q_p' + 100Q_p(-150A + 100B)\cos 50t - (100A + 150B)\sin 50t = 3\cos 50t - 6\sin 50t$, so $-150A + 100B = 3$, $-100A + 1500B = -6$. Therefore,$A = 3/650$, $B = 12/325$, $Q_p = \frac{3}{650}(\cos 50t + 8\sin 50t)$, and $I_p = \frac{3}{13}(8\cos 50t - \sin 50t)$.

6.3.10. $Q_p = A\cos 30t + B\sin 30t$; $Q_p' = 30B\cos 30t - 30A\sin 30t$; $Q_p'' = -900A\cos 30t - 900B\sin 30t$; $\frac{1}{20}Q_p'' + 4Q_p' + 125Q_p = (80A + 120B)\cos 30t - (120A - 80B)\sin 30t = 15\cos 30t - 30\sin 30t$, so $80A + 120B = 15$, $-120A + 80B = -30$, $A = 3/13$, $B = -3/104$, $Q_p = \frac{3}{104}(8\cos 30t - \sin 30t)$, and $I_p = -\frac{45}{52}(\cos 30t + 8\sin 30t)$.

6.3.12. Let $\sigma = \sigma(\omega)$ be the amplitude of I_p. From the solution of Exercise 6.3.11, $Q_p = A\cos\omega t + B\cos\omega t$, where $A = \dfrac{(1/C - L\omega^2)U - R\omega V}{\Delta}$, $B = \dfrac{R\omega U + (1/C - L\omega^2)V}{\Delta}$, and $\Delta = (1/C - L\omega^2)^2 + R^2\omega^2$. Since $I_p = Q_p' = \omega(-A\sin\omega t + B\cos\omega t)$, it follows that $\sigma^2(\omega) = \omega^2(A^2+B^2) = \dfrac{U^2+V^2}{\rho(\omega)}$, with $\rho(\omega) = \dfrac{\Delta}{\omega^2} = (1/C\omega - L\omega)^2 + R^2$, which attains it mininmum value R^2 when $\omega = \omega_0 = \dfrac{1}{\sqrt{LC}}$. The maximum amplitude of I_p is $\sigma(\omega) = \dfrac{\sqrt{U^2+V^2}}{R}$.

6.4 MOTION UNDER A CENTRAL FORCE

6.4.2. Let $h = r_0^2\theta_0'$; then $\rho = \dfrac{h^2}{k}$. Since $r = \dfrac{\rho}{1 + e\cos(\theta-\phi)}$, it follows that (A) $e\cos(\theta-\phi) = \dfrac{\rho}{r} - 1$. Differentiating this with respect to t yields $-e\sin(\theta-\phi)\theta' = -\dfrac{\rho r'}{r^2}$, so (B) $e\sin(\theta-\phi) = \dfrac{\rho r'}{h}$, since $r^2\theta' \equiv h$. Squaring and adding (A) and (B) and setting $t = 0$ in the result yields $e = \left[\left(\dfrac{\rho}{r_0} - 1\right)^2 + \left(\dfrac{\rho r_0'}{h}\right)^2\right]^{1/2}$. If $e = 0$, then θ_0 is undefined, but also irrelevant; if $e \neq 0$, then set $t = 0$ in (A) and (B) to see that $\phi = \theta_0 - \alpha$, where $-\pi \le \alpha < \pi$, $\cos\alpha = \dfrac{1}{e}\left(\dfrac{\rho}{r_0} - 1\right)$ and $\sin\alpha = \dfrac{\rho r_0'}{eh}$

6.4.4. Recall that (A) $\dfrac{d^2u}{d}\theta^2 = -\dfrac{1}{mh^2u^2}f(1/u)$. Let $u = \dfrac{1}{r} = \dfrac{1}{c\theta^2}$; then $\dfrac{d^2u}{d\theta^2} = \dfrac{6}{c\theta^4} = 6cu^2$. $6cu^2 + u = -\dfrac{1}{mh^2u^2}f(1/u)$, so $f(1/u) = -mh^2(6cu^4 + u^3)$ and $f(r) = -mh^2\left(\dfrac{6c}{r^4} + \dfrac{1}{r^3}\right)$.

6.4.6. **(a)** With $f(r) = -\dfrac{mk}{r^3}$, Eqn. 6.4.11 becomes (A) $\dfrac{d^2u}{d\theta^2} + \left(1 - \dfrac{k}{h^2}\right)u = 0$. The initial conditions imply that $u(\theta_0) = \dfrac{1}{r_0}$ and $\dfrac{du}{d\theta}(\theta_0) = -\dfrac{r_0'}{h}$ (see Eqn. (6.4.)

(b) Let $\gamma = \left|1 - \dfrac{k}{h^2}\right|^{1/2}$. (i) If $h^2 < k$, then (A) becomes $\dfrac{d^2u}{d\theta^2} - \gamma^2u = 0$, and the solution of the initial value problem for u is $u = \dfrac{1}{r_0}\cosh\gamma(\theta-\theta_0) - \dfrac{r_0'}{\gamma h}\sinh\gamma(\theta-\theta_0)$; therefore $r = r_0\left(\cosh\gamma(\theta-\theta_0) - \dfrac{r_0r_0'}{\gamma h}\sinh\gamma(\theta-\theta_0)\right)^{-1}$. (ii) If $h^2 = k$, then (A) becomes $\dfrac{d^2u}{d\theta^2} = 0$, and the solution of the initial value problem for u is $u = \dfrac{1}{r_0} - \dfrac{r_0'}{h}(\theta-\theta_0)$; therefore $r = r_0\left(1 - \dfrac{r_0r_0'}{h}(\theta-\theta_0)\right)^{-1}$. (iii) If $h^2 > k$, then (A) becomes $\dfrac{d^2u}{d\theta^2} + \gamma^2u = 0$, and the solution of the initial value problem for u is $u = \dfrac{1}{r_0}\cos\gamma(\theta-\theta_0) - \dfrac{r_0'}{\gamma h}\sin\gamma(\theta-\theta_0)$; therefore $r = r_0\left(\cos\gamma(\theta-\theta_0) - \dfrac{r_0r_0'}{\gamma h}\sin\gamma(\theta-\theta_0)\right)^{-1}$.

CHAPTER 7
Series Solutions of Linear Second Equations

7.1 REVIEW OF POWER SERIES

7.1.2. From Theorem 7.1.3, $\sum_{m=0}^{\infty} b_m z^m$ converges if $|z| < 1/L$ and diverges if $|z| > 1/L$. Therefore, $\sum_{m=0}^{\infty} b_m (x-x_0)^2$ converges if $|x-x_0| < 1/\sqrt{L}$ and diverges if $|x-x_0| > 1/\sqrt{L}$.

7.1.4. From Theorem 7.1.3, $\sum_{m=0}^{\infty} b_m z^m$ converges if $|z| < 1/L$ and diverges if $|z| > 1/L$. Therefore, $\sum_{m=0}^{\infty} b_m (x-x_0)^{km}$ converges if $|x-x_0| < 1/\sqrt[k]{L}$ and diverges if $|x-x_0| > 1/\sqrt[k]{L}$.

7.1.12. $(1+3x^2)y'' + 3x^2y' - 2y = \displaystyle\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + 3\sum_{n=2}^{\infty} n(n-1)a_n x^n + 3\sum_{n=1}^{\infty} n a_n x^{n+1} - 2\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + 3\sum_{n=1}^{\infty} n(n-1)n a_n x^n + 3\sum_{n=1}^{\infty}(n-1)a_{n-1}x^n - 2\sum_{n=0}^{\infty} a_n x^n = 2a_2 - 2a_0 + \sum_{n=1}^{\infty}[(n+2)(n+1)a_{n+2} + (3n(n-1)-2)a_n + 3(n-1)a_{n-1}]x^n$.

7.1.13. $(1+2x^2)y'' + (2-3x)y' + 4y = \displaystyle\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + 2\sum_{n=2}^{\infty} n(n-1)a_n x^n + 2\sum_{n=1}^{\infty} n a_n x^{n-1} - 2\sum_{n=0}^{\infty} a_n x^n + 4\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + 2\sum_{n=0}^{\infty} n(n-1)a_n x^n + 2\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n - 3\sum_{n=0}^{\infty} a_n x^n + 4\sum_{n=0}^{\infty} a_n x^n \sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2} + 2(n+1)a_{n+1} + (2n^2-5n+4)a_n\right]x^n$.

7.1.14. $(1+x^2)y'' + (2-x)y' + 3y = \displaystyle\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=2}^{\infty} n(n-1)a_n x^n + 2\sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=1}^{\infty} n a_n x^n + 3\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$

$$+\sum_{n=0}^{\infty} n(n-1)a_n x^n + 2\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n - \sum_{n=0}^{\infty} n a_n x^n + 3\sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2} + 2(n+1)a_{n+1} + (n^2-2n+3)a_n\right]x^n.$$

7.1.16. Let $t = x+1$; then $xy''+(4+2x)y'+(2+x)y = (-1+t)y''+(2+2t)y'+(1+t)y = -\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} + \sum_{n=2}^{\infty} n(n-1)a_n t^{n-1} + 2\sum_{n=1}^{\infty} na_n t^{n-1} + 2\sum_{n=1}^{\infty} na_n t^n + \sum_{n=0}^{\infty} a_n t^n + \sum_{n=0}^{\infty} a_n t^{n+1} = -\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}t^n + \sum_{n=0}^{\infty}(n+1)na_{n+1}t^n + 2\sum_{n=0}^{\infty}(n+1)a_{n+1}t^n + 2\sum_{n=0}^{\infty} na_n t^n + \sum_{n=0}^{\infty} a_n t^n + \sum_{n=1}^{\infty} a_{n-1}t^n = (-2a_2+2a_1+a_0)+\sum_{n=1}^{\infty}\left[-(n+2)(n+1)a_{n+2} + (n+1)(n+2)a_{n+1} + (2n+1)a_n + a_{n-1}\right](x+2)^n$.

7.1.20. $y'(x) = x^r \sum_{n=0}^{\infty} na_n x^{n-1} + rx^{r-1}\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty}(n+r)x^{n+r-1}$

$y'' = \frac{d}{dx}y'(x) = \frac{d}{dx}\left[x^{r-1}\sum_{n=0}^{\infty}(n+r)a_n x^n\right] = x^{r-1}\sum_{n=0}^{\infty}(n+r)na_n x^{n-1} + (r-1)x^{r-2}\sum_{n=0}^{\infty}(n+r)a_n x^n = \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$.

7.1.22. $x^2(1+x)y'' + x(1+2x)y' - (4+6x)y = (x^2y''+xy'-4y) + x(x^2y''+2xy'-6y) = \sum_{n=0}^{\infty}[(n+r)(n+r-1)+(n+r)-4]a_n x^{n+r} + \sum_{n=0}^{\infty}[(n+r)(n+r-1)+2(n+r)-6]a_n x^{n+r+1} = \sum_{n=0}^{\infty}(n+r-2)(n+r+2)a_n x^{n+r} + \sum_{n=0}^{\infty}(n+r+3)(n+r-2)a_n x^{n+r+1} = \sum_{n=0}^{\infty}(n+r-2)(n+r+2)a_n x^{n+r} + \sum_{n=1}^{\infty}(n+r+2)(n+r-3)a_{n-1}x^{n+r} = x^r\sum_{n=0}^{\infty} b_n x^n$ with $b_0 = (r-2)(r+2)a_0$ and $b_n = (n+r-2)(n+r+2)a_n + (n+r+2)(n+r-3)a_{n-1}$, $n \geq 1$.

7.1.24. $x^2(1+3x)y'' + x(2+12x+x^2)y' + 2x(3+x)y = (x^2y''+2xy') + x(3x^2y''+12xy'+6y) + x^2(xy'+2y) = \sum_{n=0}^{\infty}[(n+r)(n+r-1)+2(n+r)]a_n x^{n+r} + \sum_{n=0}^{\infty}[3(n+r)(n+r-1)+12(n+r)+6]a_n x^{n+r+1} + \sum_{n=0}^{\infty}[(n+r)+2]a_n x^{n+r+2} = \sum_{n=0}^{\infty}(n+r)(n+r+1)a_n x^{n+r} + 3\sum_{n=0}^{\infty}(n+r+1)(n+r+2)a_n x^{n+r+1} + \sum_{n=0}^{\infty}(n+r+2)a_n x^{n+r+2} = \sum_{n=0}^{\infty}(n+r)(n+r+1)a_n x^{n+r} + 3\sum_{n=1}^{\infty}(n+r)(n+r+1)a_{n-1}x^{n+r} + \sum_{n=2}^{\infty}(n+r)a_{n-2}x^{n+r} = x^r\sum_{n=0}^{\infty} b_n x^n$ with $b_0 = r(r+1)a_0$, $b_1 = (r+1)(r+2)a_1 + 3(r+1)(r+2)a_0$, $b_n = (n+r)(n+r+1)a_n + 3(n+r)(n+r+1)a_{n-1} + (n+r)a_{n-2}$, $n \geq 2$.

7.1.26. $x^2(2+x^2)y'' + 2x(5+x^2)y' + 2(3-x^2)y = (2x^2y''+10xy'+6y) + x^2(x^2y''+2xy'-2y) = \sum_{n=0}^{\infty}[2(n+r)(n+r-1)+10(n+r)+6]a_n x^{n+r} + \sum_{n=0}^{\infty}[(n+r)(n+r-1)+2(n+r)-2]a_n x^{n+r+2} = 2\sum_{n=0}^{\infty}(n+r+1)(n+r+3)a_n x^{n+r} + \sum_{n=0}^{\infty}(n+r-1)(n+r+2)a_n x^{n+r+2} = 2\sum_{n=0}^{\infty}(n+r+1)(n+$

$r+3)a_nx^{n+r}+\sum_{n=2}^{\infty}(n+r-3)(n+r)a_{n-2}x^{n+r} = x^r\sum_{n=0}^{\infty}b_nx^n$ with $b_0 = 2(r+1)(r+3)a_0$, $b_1 = 2(r+2)(r+4)a_1$ $b_n = 2(n+r+1)(n+r+3)a_n+(n+r-3)(n+r)a_{n-2}, n \ge 2$.

7.2 SERIES SOLUTIONS NEAR AN ORDINARY POINT I

7.2.2. $p(n) = n(n-1)+2n-2 = (n+2)(n-1)$; $a_{n+2} = -\frac{n-1}{n+1}a_n$; $a_{2m+2} = -\dfrac{2m-1}{2m+1}a_{2m}$, so $a_{2m} = \dfrac{(-1)^m}{2m-1}a_0$; $a_{2m+3} = -\dfrac{m}{m+2}a_{2m+1} = 0$ if $m \ge 0$; $y = a_0\sum_{m=0}^{\infty}(-1)^{m+1}\dfrac{x^{2m}}{2m-1}+a_1x$.

7.2.4. $p(n) = -n(n-1)-8n-12 = -(n+3)(n+4)$; $a_{n+2} = -\frac{(n+3)(n+4)}{(n+2)(n+1)}a_n$; $a_{2m+2} = -\dfrac{(m+2)(2m+3)}{(m+1)(2m+1)}a_{2m}$, so $a_{2m} = (m+1)(2m+1)a_0$; $a_{2m+3} = \dfrac{(m+2)(2m+5)}{(m+1)(2m+3)}a_{2m+1}$ so $a_{2m+1} = \dfrac{(m+1)(2m+3)}{3}a_1$; $y = a_0\sum_{m=0}^{\infty}(m+1)(2m+1)x^{2m}+\dfrac{a_1}{3}\sum_{m=0}^{\infty}(m+1)(2m+3)x^{2m+1}$.

7.2.6. $p(n) = n(n-1)+2n+\dfrac{1}{4} = \dfrac{(2n+1)^2}{4}$; $a_{n+2} = -\frac{(2n+1)^2}{4(n+2)(n+1)}a_n$; $a_{2m+2} = -\dfrac{(4m+1)^2}{8(m+1)(2m+1)}a_{2m}$, so $a_{2m} = (-1)^m\left[\prod_{j=0}^{m-1}\dfrac{(4j+1)^2}{2j+1}\right]\dfrac{1}{8^mm!}a_0$; $a_{2m+3} = -\dfrac{(4m+3)^2}{8(2m+3)(m+1)}a_{2m+1}$ so $a_{2m+1} = (-1)^m\left[\prod_{j=0}^{m-1}\dfrac{(4j+3)^2}{2j+3}\right]\dfrac{1}{8^mm!}a_1$; $y = a_0\sum_{m=0}^{\infty}(-1)^m\left[\prod_{j=0}^{m-1}\dfrac{(4j+1)^2}{2j+1}\right]\dfrac{x^{2m}}{8^mm!}+a_1\sum_{m=0}^{\infty}(-1)^m\left[\prod_{j=0}^{m-1}\dfrac{(4j+3)^2}{2j+3}\right]\dfrac{x^{2m+1}}{8^mm!}$

7.2.8. $p(n) = n(n-1)-10n+28 = (n-7)(n-4)$; $a_{n+2} = -\frac{(n-7)(n-4)}{(n+2)(n+1)}a_n$; $a_{2m+2} = -\dfrac{2(2m-7)(m-2)}{2(m+1)(2m+1)}a_{2m}$, so $a_2 = -14a_0$, $a_4 = -\dfrac{5}{6}a_2 = \dfrac{35}{3}a_0$, $a_{2m} = 0$ if $m \ge 3$; $a_{2m+3} = -\dfrac{(m-3)(2m-3)}{(2m+3)(m+1)}a_{2m+1}$, so $a_3 = -3a_1$, $a_5 = -\dfrac{1}{5}a_3 = \dfrac{3}{5}a_1$, $a_7 = \dfrac{1}{21}a_5 = \dfrac{1}{35}a_1$;
$y = a_0\left(1-14x^2+\dfrac{35}{3}x^4\right)+a_1\left(x-3x^3+\dfrac{3}{5}x^5+\dfrac{1}{35}x^7\right)$.

7.2.10. $p(n) = 2n+3$; $a_{n+2} = -\dfrac{2n+3}{(n+2)(n+1)}a_n$; $a_{2m+2} = -\dfrac{4m+3}{2(m+1)(2m+1)}a_{2m}$, so $a_{2m} = \left[\prod_{j=0}^{m-1}\dfrac{4j+3}{2j+1}\right]\dfrac{(-1)^m}{2^mm!}a_0$; $a_{2m+3} = -\dfrac{4m+5}{2(2m+3)(m+1)}a_{2m+1}$ so $a_{2m+1} = \left[\prod_{j=0}^{m-1}\dfrac{4j+5}{2j+3}\right]\dfrac{(-1)^m}{2^mm!}a_1$;
$y = a_0\sum_{m=0}^{\infty}(-1)^m\left[\prod_{j=0}^{m-1}\dfrac{4j+3}{2j+1}\right]\dfrac{x^{2m}}{2^mm!}+a_1\sum_{m=0}^{\infty}(-1)^m\left[\prod_{j=0}^{m-1}\dfrac{4j+5}{2j+3}\right]\dfrac{x^{2m+1}}{2^mm!}$

7.2.12. $p(n) = 2n(n-1)-9n-6 = (n-6)(2n+1)$; $a_{n+2} = -\dfrac{(n-6)(2n+1)}{(n+2)(n+1)}a_n$; $a_0 = y(0) = 1$; $a_1 = y'(0) = -1$.

7.2.13. $p(n) = 8n(n-1)+2 = 2(2n-1)^2$; $a_{n+2} = -\dfrac{2(2n-1)^2}{(n+2)(n+1)}a_n$; $a_0 = y(0) = 2$; $a_1 = y'(0) = -1$.

7.2.16. $p(n) = -1$; $a_{n+2} = \dfrac{1}{(n+2)(n+1)}a_n$; $a_{2m+2} = \dfrac{1}{(2m+2)(2m+1)}a_{2m}$, so $a_{2m} = \dfrac{1}{(2m)!}a_0$; $a_{2m+3} = \dfrac{1}{(2m+3)(2m+1)}a_{2m+1}$, so $a_{2m+1} = \dfrac{1}{(2m+1)!}a_1$; $y = a_0\displaystyle\sum_{m=0}^{\infty}\frac{(x-3)^{2m}}{(2m)!} + a_1\sum_{m=0}^{\infty}\frac{(x-3)^{2m+1}}{(2m+1)!}$.

7.2.18. Let $t = x-1$; then $(1-2t^2)y'' - 10ty' - 6y = 0$; $p(n) = -2n(n-1) - 10n - 6 = -2(n+1)(n+3)$; $a_{n+2} = \dfrac{2(n+3)}{n+2}a_n$; $a_{2m+2} = \dfrac{2m+3}{m+1}a_{2m}$, so $a_{2m} = \dfrac{1}{m!}\left[\displaystyle\prod_{j=0}^{m-1}(2j+3)\right]a_0$; $a_{2m+3} = \dfrac{4(m+2)}{2m+3}a_{2m+1}$, so $a_{2m+1} = \dfrac{4^m(m+1)!}{\prod_{j=0}^{m-1}(2j+3)}a_1$; $y = a_0\displaystyle\sum_{m=0}^{\infty}\left[\prod_{j=0}^{m-1}(2j+3)\right]\frac{(x-1)^{2m}}{m!}$ $+ a_1\displaystyle\sum_{m=0}^{\infty}\frac{4^m(m+1)!}{\prod_{j=0}^{m-1}(2j+3)}(x-1)^{2m+1}$.

7.2.20. Let $t = x+1$; then $\left(1+\dfrac{3t^2}{3}\right)y'' - \dfrac{9t}{2}y' + \dfrac{3}{2}y = 0$; $p(n) = \dfrac{3}{2}n(n-1) + \dfrac{9}{2}n + \dfrac{3}{2} = \dfrac{3}{2}(n+1)^2$; $a_{n+2} = -\dfrac{3(n+1)}{2(n+2)}a_n$; $a_{2m+2} = -\dfrac{3(2m+1)}{4(m+1)}a_{2m}$, so $a_{2m} = (-1)^m\left[\displaystyle\prod_{j=0}^{m-1}(2j+1)\right]\dfrac{3^m}{4^m m!}a_0$; $a_{2m+3} = -\dfrac{3(m+1)}{2m+3}a_{2m+1}$, so $a_{2m+1} = (-1)^m\dfrac{3^m m!}{\prod_{j=0}^{m-1}(2j+3)}a_1$; $y = a_0\displaystyle\sum_{m=0}^{\infty}(-1)^m\left[\prod_{j=0}^{m-1}(2j+1)\right]\frac{3^m}{4^m m!}(x+1)^{2m} + a_1\sum_{m=0}^{\infty}(-1)^m\frac{3^m m!}{\prod_{j=0}^{m-1}(2j+3)}(x+1)^{2m+1}$.

7.2.22. $p(n) = n+3$; $a_{n+2} = -\dfrac{n+3}{(n+2)(n+1)}a_n$; $a_0 = y(3) = -2$; $a_1 = y'(3) = 3$.

7.2.24. Let $t = x-3$; $(1+4t^2)y'' + y = 0$; $p(n) = (4n(n-1)+1 = 2n-1)^2$; $a_{n+2} = -\dfrac{(2n-1)^2}{(n+2)(n+1)}a_n$; $a_0 = y(3) = 4$; $a_1 = y'(3) = -6$.

7.2.26. Let $t = x+1$; $\left(1+\dfrac{2t^2}{3}\right)y'' - \dfrac{20}{3}ty' + 20y = 0$; $p(n) = \dfrac{2}{3}n(n-1) - \dfrac{20}{3}n + 20 = \dfrac{2(n-6)(n-5)}{3}$; $a_{n+2} = -\dfrac{2(n-6)(n-5)}{3(n+2)(n+1)}a_n$; $a_0 = y(-1) = 3$; $a_1 = y'(-1) = -3$.

7.2.28. From Theorem 7.2.2, $a_{n+2} = -\dfrac{p(n)}{(n+2)(n+1)}a_n$; $a_{2m+2} = -\dfrac{p(2m)}{(2m+2)(2m+1)}a_{2m}$, so $a_{2m} = \left[\displaystyle\prod_{j=0}^{m-1}p(2j)\right]\dfrac{(-1)^m}{(2m)!}a_0$; $a_{2m+3} = -\dfrac{p(2m+1)}{(2m+3)(2m+2)}a_{2m}$, so $a_{2m+1} = \left[\displaystyle\prod_{j=0}^{m-1}p(2j+1)\right]\dfrac{(-1)^m}{(2m+1)!}a_1$.

7.2.30. **(a)** Here $p(n) = -[n(n-1) + 2bn - \alpha(\alpha+2b-1)] = -(n-\alpha)(n+\alpha+2b-1)$, so Exercise 7.2.28 implies that y_1 and y_2 have the stated forms. If $\alpha = 2k$, then

$$y_1 = \sum_{m=0}^{\infty}\left[\prod_{j=0}^{m-1}(2j-2k)(2j+2k+2b-1)\right]\frac{x^{2m}}{(2m)!} \qquad \text{(C)}.$$

If $\alpha = 2k + 1$, then

$$y_2 = \sum_{m=0}^{\infty} \left[\prod_{j=0}^{m-1} (2j - 2k)(2j + 2k + 2b) \right] \frac{x^{2m+1}}{(2m+1)!}. \tag{D}$$

Since $2b$ is not a negative integer and $\prod_{j=0}^{m-1}(2j - 2k) = 0$ if $m > k$, y_1 in (C) and y_2 in (D) have the stated properties. This implies the conclusions regarding P_n.

(b) Multiplying (A) through by $(1 - x^2)^{b-1}$ yields

$$[(1 - x^2)^b P_n']' = -n(n + 2b - 1)(1 - x^2)^{b-1} P_n. \tag{E}$$

(c) Therefore,

$$[(1 - x^2)^b P_m']' = -m(m + 2b - 1)(1 - x^2)^{b-1} P_m. \tag{F}$$

Subtract P_n times (F) from P_m times (E) to obtain (B).

(d) Integrating the left side of (B) by parts over $[-1, 1]$ yields zero, which implies the conclusion.

7.2.32. **(a)** Let $Ly = (1 + \alpha x^3)y'' + \beta x^2 y' + \gamma x y$. If $y = \sum_{n=0}^{\infty} a_n x^n$, then $Ly = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} p(n) a_n x^{n+1} = 2a_2 + \sum_{n=0}^{\infty} [(n+3)(n+2)a_{n+3} + p(n)a_n] x^{n+1} = 0$ if and only if $a_2 = 0$ and $a_{n+3} = -\dfrac{p(n)}{(n+3)(n+2)} a_n$ for $n \ge 0$.

7.2.34. $p(r) = -2r(r-1) - 10r - 8 = -2(r+2)^2$; (A) $\prod_{j=0}^{m-1} \dfrac{p(3j)}{3j+2} = \prod_{j=0}^{m-1} \dfrac{(-2)(3j+2)^2}{3j+2} = (-1)^m 2^m \prod_{j=0}^{m-1} (3j+2)$; (B) $\prod_{j=0}^{m-1} \dfrac{p(3j+1)}{3j+4} = \prod_{j=0}^{m-1} \dfrac{(-2)(3j+3)^2}{3j+4} = \dfrac{(-1)^m 2^m (m!)^2}{\prod_{j=0}^{m-1}(3j+4)}$. Substituting (A) and (B) into the result of Exercise 7.2.32**(c)** yields

$$y = a_0 \sum_{m=0}^{\infty} \left(\frac{2}{3}\right)^m \left[\prod_{j=0}^{m-1} (3j+2) \right] \frac{x^{3m}}{m!} + a_1 \sum_{m=0}^{\infty} \frac{6^m m!}{\prod_{j=0}^{m-1}(3j+4)} x^{3m+1}.$$

7.2.36. $p(r) = -2r(r-1) + 6r + 24 = -2(r-6)(r+2)$; (A) $\prod_{j=0}^{m-1} \dfrac{p(3j)}{3j+2} = \prod_{j=0}^{m-1} (-6)(j-2)$.

(B) $\prod_{j=0}^{m-1} \dfrac{p(3j+1)}{3j+4} = \prod_{j=0}^{m-1} \dfrac{(-6)(j+1)(3j-5)}{3j+4} = (-1)^m 6^m m \prod_{j=0}^{m-1} \dfrac{3j-5}{3j+4}$. Substituting (A) and (B) into the result of Exercise 7.2.32**(c)** yields

$$y = a_0(1 - 4x^3 + 4x^6) + a_1 \sum_{m=0}^{\infty} 2^m \left[\prod_{j=0}^{m-1} \frac{3j-5}{3j+4} \right] x^{3m+1}.$$

7.2.38. **(a)** Let $Ly = (1 + \alpha x^{k+2})y'' + \beta x^{k+1} y' + \gamma x^k y$. If $y = \sum_{n=0}^{\infty} a_n x^n$, then $Ly = \sum_{n=2}^{\infty} n(n - 1)a_n x^{n-2} + \sum_{n=0}^{\infty} p(n) a_n x^{n+k} = \sum_{n=-k}^{-1} (n+k+2)(n+k-1) a_{n+k+2} x^{n+k} + \sum_{n=0}^{\infty} [(n+k+2)(n +$

$k+1)a_{n+k+2}+p(n)a_n]x^{n+k} = 0$ if and only if $a_k = 0$ for $2 \le n \le k+1$ and (A) $a_{n+k+1} = -\dfrac{p(n)}{(n+k+2)(n+k+1)}a_n$ for $n \ge 0$.
(b) If $a_n = 0$ the $a_{n+(k+2)m} = 0$ for all $m \ge 0$, from (A).

7.2.40. $k = 2$ and $p(r) = 1$; (A) $\displaystyle\prod_{j=0}^{m-1}\frac{p(4j)}{4j+3} = \frac{1}{\prod_{j=0}^{m-1}(4j+3)}$; (B) $\displaystyle\prod_{j=0}^{m-1}\frac{p(4j+1)}{(4j+5)} = \frac{1}{\prod_{j=0}^{m-1}4j+5}$.
Substituting (A) and (B) into the result of Exercise t.2.38(**c**) yields

$$y = a_0\sum_{m=0}^{\infty}(-1)^m\frac{x^{4m}}{4^m m!\prod_{j=0}^{m-1}(4j+3)} + a_1\sum_{m=0}^{\infty}(-1)^m\frac{x^{4m+1}}{4^m m!\prod_{j=0}^{m-1}(4j+5)}.$$

7.2.42. $k = 6$ and $p(r) = r(r-1)-16r+72 = (r-9)(r-8)$; (A) $\displaystyle\prod_{j=0}^{m-1}\frac{p(8j)}{8j+7} = \prod_{j=0}^{m-1}\frac{8(j-1)(8j-9)}{8j+7}$;
(B) $\displaystyle\prod_{j=0}^{m-1}\frac{p(8j+1)}{(8j+9)} = \prod_{j=0}^{m-1}\frac{8(j-1)(8j-7)}{8j+9}$;
Substituting (A) and (B) into the result of Exercise 7.2.38(**c**) yields

$$y = a_0\left(1-\frac{9}{7}x^8\right)+a_1\left(x-\frac{7}{9}x^9\right).$$

7.2.44. $k = 4$ and $p(r) = r+6$; (A) $\displaystyle\prod_{j=0}^{m-1}\frac{p(6j)}{6j+5} = \prod_{j=0}^{m-1}\frac{6(j+1)}{6j+5} = \frac{6^m m!}{\prod_{j=0}^{m-1}(6j+5)}$;
(B) $\displaystyle\prod_{j=0}^{m-1}\frac{p(6j+1)}{(6j+7)} = 1$;
Substituting (A) and (B) into the result of Exercise 7.2.38(**c**) yields

$$y = a_0\sum_{m=0}^{\infty}(-1)^m\frac{x^{6m}}{\prod_{j=0}^{m-1}(6j+5)} + a_1\sum_{m=0}^{\infty}(-1)^m\frac{x^{6m+1}}{6^m m!}.$$

7.3 SERIES SOLUTIONS NEAR AN ORDINARY POINT II

7.3.2. If $y = \sum_{n=0}^{\infty}a_nx^n$, then $(1+x+2x^2)y''+(2+8x)y'+4y = \sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}+\sum_{n=2}^{\infty}n(n-1)a_nx^{n-1}+2\sum_{n=2}^{\infty}n(n-1)a_nx^n+2\sum_{n=1}^{\infty}na_nx^{n-1}+8\sum_{n=1}^{\infty}na_nx^n+4\sum_{n=0}^{\infty}a_nx^n = \sum_{n=0}^{\infty}(n+2)(n+1)(a_{n+2}+a_{n+1}+2a_n)x^n = 0$ if $a_{n+2} = -a_{n+1}-2a_n$, $a_n \ge 0$. Starting with $a_0 = -1$ and $a_1 = 2$ yields $y = -1+2x-4x^3+4x^4+4x^5-12x^6+4x^7+\cdots$.

7.3.4. If $y = \sum_{n=0}^{\infty}a_nx^n$, then $(1+x+3x^2)y''+(2+15x)y'+12y = \sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}+\sum_{n=2}^{\infty}n(n-1)a_nx^{n-1}+3\sum_{n=2}^{\infty}n(n-1)a_nx^n+2\sum_{n=1}^{\infty}na_nx^{n-1}+15\sum_{n=1}^{\infty}na_nx^n+12\sum_{n=0}^{\infty}a_nx^n = \sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2}+(n+1)(n+2)a_{n+1}+3(n+2)^2a_n]x^n = 0$ if $a_{n+2} = -a_{n+1}-\dfrac{3(n+2)}{n+1}a_n$, $a_n \ge 0$. Starting with $a_0 = 0$ and $a_1 = 1$ yields $y = x-x^2-\dfrac{7}{2}x^3+\dfrac{15}{2}x^4+\dfrac{45}{8}x^5-\dfrac{261}{8}x^6+\dfrac{207}{16}x^7+\cdots$.

7.3.6. If $y = \sum_{n=0}^{\infty} a_n x^n$, then $(3+3x+x^2)y'' + (6+4x)y' + 2y = 3\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + 3\sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} + \sum_{n=2}^{\infty} n(n-1)a_n x^n + 6\sum_{n=1}^{\infty} n a_n x^{n-1} + 4\sum_{n=1}^{\infty} n a_n x^n + 2\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} (n+2)(n+1)[3a_{n+2} + 3a_{n+1} + a_n]x^n = 0$ if $a_{n+2} = -a_{n+1} - a_n/3$, $a_n \geq 0$. Starting with $a_0 = 7$ and $a_1 = 3$ yields $y = 7 + 3x - \frac{16}{3}x^2 + \frac{13}{3}x^3 - \frac{23}{9}x^4 + \frac{10}{9}x^5 - \frac{7}{27}x^6 - \frac{1}{9}x^7 + \cdots$.

7.3.8. The equation is equivalent to $(1+t+2t^2)y'' + (2+6t)y' + 2y = 0$ with $t = x-1$. If $y = \sum_{n=0}^{\infty} a_n t^n$, then $(1+t+2t^2)y'' + (2+6t)y' + 2y = \sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} + \sum_{n=2}^{\infty} n(n-1)a_n t^{n-1} + 2\sum_{n=2}^{\infty} n(n-1)a_n t^n + 2\sum_{n=1}^{\infty} n a_n t^{n-1} + 6\sum_{n=1}^{\infty} n a_n t^n + 2\sum_{n=0}^{\infty} a_n t^n = \sum_{n=0}^{\infty} (n+1)[(n+2)a_{n+2} + (n+2)a_{n+1} + 2(n+1)a_n]t^n = 0$ if $a_{n+2} = -a_{n+1} - \frac{2(n+1)}{n+2}a_n$, $a_n \geq 0$. Starting with $a_0 = 1$ and $a_1 = -1$ yields $y = 1 - (x-1) + \frac{4}{3}(x-1)^3 - \frac{4}{3}(x-1)^4 - \frac{4}{5}(x-1)^5 + \frac{136}{45}(x-1)^6 - \frac{104}{63}(x-1)^7 + \cdots$.

7.3.10. The equation is equivalent to $(1+t+t^2)y'' + (3+4t)y' + 2y = 0$ with $t = x-1$. If $y = \sum_{n=0}^{\infty} a_n t^n$, then $(1+t+t^2)y'' + (3+4t)y' + 2y = \sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} + \sum_{n=2}^{\infty} n(n-1)a_n t^{n-1} + \sum_{n=2}^{\infty} n(n-1)a_n t^n + 3\sum_{n=1}^{\infty} n a_n t^{n-1} + 4\sum_{n=1}^{\infty} n a_n t^n + 2\sum_{n=0}^{\infty} a_n t^n = \sum_{n=0}^{\infty} (n+1)[(n+2)a_{n+2} + (n+3)a_{n+1} + (n+2)a_n]t^n = 0$ if $a_{n+2} = -\frac{n+3}{n+2}a_{n+1} - a_n$, $a_n \geq 0$. Starting with $a_0 = 2$ and $a_1 = -1$ yields $y = 2 - (x-1) - \frac{1}{2}(x-1)^2 + \frac{5}{3}(x-1)^3 - \frac{19}{12}(x-1)^4 + \frac{7}{30}(x-1)^5 + \frac{59}{45}(x-1)^6 - \frac{1091}{630}(x-1)^7 + \cdots$

7.3.12. The equation is equivalent to $(1+2t+t^2)y'' + (1+7t)y' + 8y = 0$ with $t = x-1$. If $y = \sum_{n=0}^{\infty} a_n t^n$, then $(1+2t+t^2)y'' + (1+7t)y' + 8y = \sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} + 2\sum_{n=2}^{\infty} n(n-1)a_n t^{n-1} + \sum_{n=2}^{\infty} n(n-1)a_n t^n + \sum_{n=1}^{\infty} n a_n t^{n-1} + 7\sum_{n=1}^{\infty} n a_n t^n + 8\sum_{n=0}^{\infty} a_n t^n = \sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} + (n+1)(2n+1)a_{n+1} + (n+2)(n+4)a_n]t^n = 0$ if $a_{n+2} = -\frac{2n+1}{n+2}a_{n+1} - \frac{n+4}{n+1}a_n$, $a_n \geq 0$. Starting with $a_0 = 1$ and $a_1 = -2$ yields $y = 1 - 2(x-1) - 3(x-1)^2 + 8(x-1)^3 - 4(x-1)^4 - \frac{42}{5}(x-1)^5 + 19(x-1)^6 - \frac{604}{35}(x-1)^7 + \cdots$

7.3.16. If $y = \sum_{n=0}^{\infty} a_n x^n$, then $(1-x)y'' - (2-x)y' + y = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} - 2\sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} - (n+2)(n+1)a_{n+1} + (n+1)a_n]x^n = 0$ if $a_{n+2} = a_{n+1} - \frac{a_n}{n+2}$, $a_n \geq 0$.

7.3.18. If $y = \sum_{n=0}^{\infty} a_n x^n$, then $(1+x^2)y'' + y' + 2y = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=1}^{\infty} na_n x^{n-1} + 2\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2} + (n+1)a_{n+1} + (n^2-n+2)a_n]x^n = 0$ if $a_{n+2} = -\frac{1}{n+2}a_{n+1} - \frac{n^2-n+2}{(n+2)(n+1)}a_n$.

7.3.20. The equation is equivalent to $(3+2t)y'' + (1+2t)y' - (1-2t)y = 0$ with $t = x-1$. If $y = \sum_{n=0}^{\infty} a_n t^n$, then $(3+2t)y'' + (1+2t)y' - (1-2t)y = 3\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} + 2\sum_{n=2}^{\infty} n(n-1)a_n t^{n-1} + \sum_{n=1}^{\infty} na_n t^{n-1} + 2\sum_{n=1}^{\infty} na_n t^n - \sum_{n=0}^{\infty} a_n t^n + 2\sum_{n=0}^{\infty} a_n t^{n+1} = (6a_2 + a_1 - a_0) + \sum_{n=1}^{\infty}[3(n+2)(n+1)a_{n+2} + (n+1)(2n+1)a_{n+1} + (2n-1)a_n + 2a_{n-1}]t^n = 0$ if $a_2 = -\frac{a_1 - a_0}{6}$ and $a_{n+2} = -\frac{2n+1}{3(n+2)}a_{n+1} - \frac{2n-1}{3(n+2)(n+1)}a_n - \frac{2}{3(n+2)(n+1)}a_{n-1}$, $n \geq 1$. Starting with $a_0 = 1$ and $a_1 = -2$ yields $y = 1 - 2(x-1) + \frac{1}{2}(x-1)^2 - \frac{1}{6}(x-1)^3 + \frac{5}{36}(x-1)^4 - \frac{73}{1080}(x-1)^5 + \cdots$.

7.3.22. The equation is equivalent to $(1+t)y'' + (2-2t)y' + (3+t)y = 0$ with $t = x+3$. If $y = \sum_{n=0}^{\infty} a_n t^n$, then $(1+t)y'' + (2-2t)y' + (3+t)y = \sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} + \sum_{n=2}^{\infty} n(n-1)a_n t^{n-1} + 2\sum_{n=1}^{\infty} na_n t^{n-1} - 2\sum_{n=1}^{\infty} na_n t^n + 3\sum_{n=0}^{\infty} a_n t^n + \sum_{n=0}^{\infty} a_n t^{n+1} = (2a_2 + 2a_1 + 3a_0) + \sum_{n=1}^{\infty}[(n+2)(n+1)a_{n+2} + (n+2)(n+1)a_{n+1} - (2n-3)a_n + a_{n-1}]t^n = 0$ if $a_2 = -\frac{2a_1 + 3a_0}{2}$ and $a_{n+2} = -a_{n+1} + \frac{(2n-3)a_n - a_{n-1}}{(n+2)(n+1)}$, $n \geq 1$. Starting with $a_0 = 2$ and $a_1 = -2$ yields

$$y = 2 - 2(x+3) - (x+3)^2 + (x+3)^3 - \frac{11}{12}(x+3)^4 + \frac{67}{60}(x+3)^5 + \cdots.$$

7.3.24. The equation is equivalent to $(1+2t)y'' + 3y' + (1-t)y = 0$ with $t = x+1$. If $y = \sum_{n=0}^{\infty} a_n t^n$, then $(1+2t)y'' + 3y' + (1-t)y = \sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} + 2\sum_{n=2}^{\infty} n(n-1)a_n t^{n-1} + 3\sum_{n=1}^{\infty} na_n t^{n-1} + \sum_{n=0}^{\infty} a_n t^n - \sum_{n=0}^{\infty} a_n t^{n+1} = (2a_2 + 3a_1 + a_0) + \sum_{n=1}^{\infty}[(n+2)(n+1)a_{n+2} + (2n+3)(n+1)a_{n+1} + a_n - a_{n-1}]t^n = 0$ if $a_2 = -\frac{3a_1 + a_0}{2}$ and $a_{n+2} = -\frac{2n+3}{n+2}a_{n+1} - \frac{a_n - a_{n-1}}{(n+2)(n+1)}$, $n \geq 1$. Starting with $a_0 = 2$ and $a_1 = -3$ yields $y = 2 - 3(x+1) + \frac{7}{2}(x+1)^2 - 5(x+1)^3 + \frac{197}{24}(x+1)^4 - \frac{287}{20}(x+1)^5 + \cdots$.

7.3.26. The equation is equivalent to $(6-2t)y'' + (3+t)y = 0$ with $t = x-2$. If $y = \sum_{n=0}^{\infty} a_n t^n$, then

$(6-2t)y''+(3+t)y = 6\sum_{n=2}^{\infty} n(n-1)a_nt^{n-2} - 2\sum_{n=2}^{\infty} n(n-1)a_nt^{n-1} + 3\sum_{n=0}^{\infty} a_nt^n + \sum_{n=0}^{\infty} a_nt^{n+1} = (12a_2+3a_0) + \sum_{n=1}^{\infty}[6(n+2)(n+1)a_{n+2} - 2(n+1)na_{n+1} + 3a_n + a_{n-1}]t^n = 0$ if $a_2 = -\frac{a_0}{4}$ and $a_{n+2} = \frac{n}{3(n+2)}a_{n+1} - \frac{3a_n+a_{n-1}}{6(n+2)(n+1)}$, $n \geq 1$. Starting with $a_0 = 2$ and $a_1 = -4$ yields $y = 2-4(x-2)-\frac{1}{2}(x-2)^2+\frac{2}{9}(x-2)^3+\frac{49}{432}(x-2)^4+\frac{23}{1080}(x-2)^5+\cdots$.

7.3.28. The equation is equivalent to $(2+4t)y''-(1-2t)y=0$ with $t=x+4$. If $y=\sum_{n=0}^{\infty} a_nt^n$, then $(2+4t)y''-(1-2t)y = 2\sum_{n=2}^{\infty} n(n-1)a_nt^{n-2} + 4\sum_{n=2}^{\infty} n(n-1)a_nt^{n-1} - \sum_{n=0}^{\infty} a_nt^n + 2\sum_{n=0}^{\infty} a_nt^{n+1} = (4a_2-a_0) + \sum_{n=1}^{\infty}[2(n+2)(n+1)a_{n+2} + 4(n+1)na_{n+1} - a_n + 2a_{n-1}]t^n = 0$ if $a_2 = \frac{a_0}{4}$ and $a_{n+2} = -\frac{2n}{n+2}a_{n+1} + \frac{a_n-2a_{n-1}}{2(n+2)(n+1)}$, $n \geq 1$. Starting with $a_0=-1$ and $a_1=2$ yields $y = -1+2(x+1)-\frac{1}{4}(x+1)^2+\frac{1}{2}(x+1)^3-\frac{65}{96}(x+1)^4+\frac{67}{80}(x+1)^5+\cdots$.

N=5; b=zeros(N,1); b(1)=-1;b(2)=2; b(3)=b(1)/4; for n=1:N-2 b(n+3)=-2*n*b(n+2)/(n+2)+(b(n+1)-2*b(n))/(2*(n+2)*(n+1)); end

7.3.29. Let $Ly = (1+\alpha x+\beta x^2)y''+(\gamma+\delta x)y'+\epsilon y$. If $y=\sum_{n=0}^{\infty} a_nx^n$, then $Ly = \sum_{n=2}^{\infty} n(n-1)a_nx^{n-2} + \alpha\sum_{n=2}^{\infty} n(n-1)a_nx^{n-1} + \beta\sum_{n=2}^{\infty} n(n-1)a_nx^n + \gamma\sum_{n=1}^{\infty} na_nx^{n-1} + \delta\sum_{n=1}^{\infty} na_nx^n + \epsilon\sum_{n=0}^{\infty} a_nx^n = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \alpha\sum_{n=0}^{\infty}(n+1)na_{n+1}x^n + \beta\sum_{n=0}^{\infty} n(n-1)a_nx^n + \gamma\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + \delta\sum_{n=0}^{\infty} na_nx^n + \epsilon\sum_{n=0}^{\infty} a_nx^n = \sum_{n=0}^{\infty} b_nx^n$, where $b_n = (n+1)(n+2)a_{n+2} + (n+1)(\alpha n+\gamma)a_{n+1} + [\beta n(n-1)+\delta n+\epsilon]a_n$, which implies the conclusion.

7.3.30. **(a)** Let $\gamma=2\alpha$, $\delta=4\beta$, and $\epsilon=2\beta$ in Exercise 7.3.29 to obtain (B).

(b) If $a_n = c_1r_1^n+c_2r_2^n$, then $a_{n+2}+\alpha a_{n+1}+\beta a_n = c_1r_1^n(r_1^2+\alpha r+\beta)+c_2r_2^n(r_2^2+\alpha r_2+\beta) = c_1r_1^nP_0(r_1)+c_2r_2^nP_0(r_2) = 0$, so $\{a_n\}$ satisfies (B). Since $1/r_1$ and $1/r_2$ are the zeros of P_0, Theorem 7.2.1 implies that $\sum_{n=0}^{\infty}(c_1r_1^n+c_2r_2^n)x^n$ is a solution of (A) on $(-\rho,\rho)$.

(c) If $|x|<\rho$, then $|r_1x|<\rho$ and $|r_2x|<1$, so $\sum_{n=0}^{\infty} r_i^nx^n = \frac{1}{1-r_ix} = y_i$, $i=1,2$. Therefore, **(b)** implies that $\{y_1,y_2\}$ is a fundamental set of solutions of (A) on $(-\rho,\rho)$.

(d) (A) can written as $P_0y''+2P_0'y'+P_0''y = (P_0y)'' = 0$. Therefore, $P_0y = a+bx$ where a and b are arbitrary constants, and a partial fraction expansion shows that the general solution of (A) on any interval not containing $1/r_1$ or $1/r_2$ is $y = \frac{a+bx}{P_0(x)} = \frac{c_1}{1-r_1x}+\frac{c_2}{1-r_2x} = c_1y_1+c_2y_2$.

(e) If $a_n = c_1r_1^n+c_2r_2^n$, then $a_{n+2}+\alpha a_{n+1}+\beta a_n = c_1r_1^n(r_1^2+\alpha r+\beta)+c_2r_1^n[(n+2)r_1^2+\alpha(n+1)r_2+\beta n) = (c_1+nc_2)r_1^nP_0(r_1)+c_2r_1^nP_0'(r_1) = 0$, so $\{a_n\}$ satisfies (B). Since $1/r_1$ is the only zero of P_0, Theorem 7.2.1 implies that $\sum_{n=0}^{\infty}(c_1 + c_2 n)r_1^n)x^n$ is a solution of (A) on $(-\rho, \rho)$.

(f) If $|x| < \rho$, then $|r_1 x| < \rho$, so $\displaystyle\sum_{n=0}^{\infty} r_1^n x^n = \frac{1}{1 - r_1 x} = y_1$. Differentiating this and multiplying the result by x shows that $\displaystyle\sum_{n=0}^{\infty} n r_1^n x^n = \frac{r_1 x}{(1 - r_1 x)^2} = r_1 y_2$. Therefore, **(e)** implies that $\{y_1, y_2\}$ is a fundamental set of solutions of (A) on $(-\rho, \rho)$.

(g) The argument is the same as in **(c)**, but now the partial fraction expansion can be written as $\displaystyle y = \frac{a + bx}{P_0(x)} = \frac{c_1}{1 - r_1 x} + \frac{c_2 x}{(1 - r_2 x)^2} = c_1 y_1 + c_2 y_2$.

7.3.32. If $\displaystyle y = \sum_{n=0}^{\infty} a_n x^n$, then $\displaystyle y'' + 2xy' + (3 + 2x^2)y = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + 2\sum_{n=1}^{\infty} n a_n x^n + 3\sum_{n=0}^{\infty} a_n x^n + 2\sum_{n=0}^{\infty} a_n x^{n+2} = (2a_2 + 3a_0) + (6a_3 + 5a_1)x + \sum_{n=2}^{\infty}[(n+2)(n+1)a_{n+2} + (2n+3)a_n + 2a_{n-2}]x^n = 0$ if $a_2 = -3a_0/2$, $a_3 = -5a_1/6$, and $\displaystyle a_{n+2} = -\frac{(2n+3)a_n + 2a_{n-2}}{(n+2)(n+1)}$, $n \geq 2$. Starting with $a_0 = 1$ and $a_1 = -2$ yields $\displaystyle y = 1 - 2x - \frac{3}{2}x^2 + \frac{5}{3}x^3 + \frac{17}{24}x^4 - \frac{11}{20}x^5 + \cdots$.

7.3.34. If $\displaystyle y = \sum_{n=0}^{\infty} a_n x^n$, then $\displaystyle y'' + 5xy' - (3 - x^2)y = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + 5\sum_{n=1}^{\infty} n a_n x^n - 3\sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} a_n x^{n+2} = (2a_2 - 3a_0) + (6a_3 + 2a_1)x + \sum_{n=2}^{\infty}[(n+2)(n+1)a_{n+2} + (5n-3)a_n + a_{n-2}]x^n = 0$ if $a_2 = 3a_0/2$, $a_3 = -a_1/3$, and $\displaystyle a_{n+2} = -\frac{(5n-3)a_n + a_{n-2}}{(n+2)(n+1)}$, $n \geq 2$. Starting with $a_0 = 6$ and $a_1 = -2$ yields $\displaystyle y = 6 - 2x + 9x^2 + \frac{2}{3}x^3 - \frac{23}{4}x^4 - \frac{3}{10}x^5 + \cdots$.

7.3.36. If $\displaystyle y = \sum_{n=0}^{\infty} a_n x^n$, then $\displaystyle y'' - 3xy' + (2 + 4x^2)y = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - 3\sum_{n=1}^{\infty} n a_n x^n + 2\sum_{n=0}^{\infty} a_n x^n + 4\sum_{n=0}^{\infty} a_n x^{n+2} = (2a_2 + 2a_0) + (6a_3 - a_1)x + \sum_{n=2}^{\infty}[(n+2)(n+1)a_{n+2} - (3n-2)a_n + 4a_{n-2}]x^n = 0$ if $a_2 = -a_0$, $a_3 = a_1/6$, and $\displaystyle a_{n+2} = \frac{(3n-2)a_n - 4a_{n-2}}{(n+2)(n+1)}$, $n \geq 2$. Starting with $a_0 = 3$ and $a_1 = 6$ yields $\displaystyle y = 3 + 6x - 3x^2 + x^3 - 2x^4 - \frac{17}{20}x^5 + \cdots$.

7.3.38. If $\displaystyle y = \sum_{n=0}^{\infty} a_n x^n$, then $\displaystyle 3y'' + 2xy' + (4 - x^2)y = 3\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + 2\sum_{n=1}^{\infty} n a_n x^n + 4\sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{\infty} a_n x^{n+2} = (6a_2 + 4a_0) + (18a_3 + 6a_1)x + \sum_{n=2}^{\infty}[3(n+2)(n+1)a_{n+2} + (2n+4)a_n - a_{n-2}]x^n = 0$ if $a_2 = -2a_0/3$, $a_3 = -a_1/3$, and $\displaystyle a_{n+2} = -\frac{(2n+4)a_n - a_{n-2}}{3(n+2)(n+1)}$, $n \geq 2$. Starting with $a_0 = -2$ and $a_1 = 3$ yields $\displaystyle y = -2 + 3x + \frac{4}{3}x^2 - x^3 - \frac{19}{54}x^4 + \frac{13}{60}x^5 + \cdots$.

7.3.40. If $y = \sum_{n=0}^{\infty} a_n x^n$, then $(1+x)y'' + x^2y' + (1+2x)y = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} + \sum_{n=1}^{\infty} n a_n x^{n+1} + \sum_{n=0}^{\infty} a_n x^n + 2\sum_{n=0}^{\infty} a_n x^{n+1} = (2a_2 + a_0) + \sum_{n=1}^{\infty}[(n+2)(n+1)a_{n+2} + (n+1)n a_{n+1} + a_n + (n+1)a_{n-1}]x^n = 0$ if $a_2 = -a_0/2$ and $a_{n+2} = -\dfrac{(n+1)na_{n+1} + a_n + (n+1)a_{n-1}}{(n+2)(n+1)}$, $n \geq 1$. Starting with $a_0 = -2$ and $a_1 = 3$ yields $y = -2 + 3x + x^2 - \dfrac{1}{6}x^3 - \dfrac{3}{4}x^4 + \dfrac{31}{120}x^5 + \cdots$.

7.3.42. If $y = \sum_{n=0}^{\infty} a_n x^n$, then $(1+x^2)y'' + (2+x^2)y' + xy = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=2}^{\infty} n(n-1)a_n x^n + 2\sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=1}^{\infty} n a_n x^{n+1} + \sum_{n=1}^{\infty} a_n x^{n+1} = (2a_2 + 2a_1) + \sum_{n=1}^{\infty}[(n+2)(n+1)a_{n+2} + 2(n+1)a_{n+1} + n(n-1)a_n + na_{n-1}]x^n = 0$ if $a_2 = -a_1$ and $a_{n+2} = -\dfrac{[2(n+1)a_{n+1} + n(n-1)a_n + na_{n-1}]}{(n+2)(n+1)}$, $n \geq 1$. Starting with $a_0 = -3$ and $a_1 = 5$ yields $y = -3 + 5x - 5x^2 + \dfrac{23}{6}x^3 - \dfrac{23}{12}x^4 + \dfrac{11}{30}x^5 + \cdots$.

7.3.44. The equation is equivalent to $y'' + (1+3t^2)y' + (1+2t)y = 0$ with $t = x-2$. If $y = \sum_{n=0}^{\infty} a_n t^n$, then $y'' + (1+3t^2)y' + (1+2t)y = \sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} + \sum_{n=1}^{\infty} n a_n t^{n-1} + 3\sum_{n=1}^{\infty} n a_n t^{n+1} + \sum_{n=0}^{\infty} a_n t^n + 2\sum_{n=0}^{\infty} a_n t^{n+1} = (2a_2 + a_1 + a_0) + \sum_{n=1}^{\infty}[(n+2)(n+1)a_{n+2} + (n+1)a_{n+1} + a_n + (3n-1)a_{n-1}]t^n = 0$ if $a_2 = -(a_1 + a_0)/2$ and $a_{n+2} = -\dfrac{[(n+1)a_{n+1} + a_n + (3n-1)a_{n-1}]}{(n+2)(n+1)}$, $n \geq 1$. Starting with $a_0 = 2$ and $a_1 = -3$ yields

$$y = 2 - 3(x+2) + \frac{1}{2}(x+2)^2 - \frac{1}{3}(x+2)^3 + \frac{31}{24}(x+2)^4 - \frac{53}{120}(x+2)^5 + \cdots.$$

7.3.46. The equation is equivalent to $(1-t^2)y'' - (7-8t+t^2)y' + ty = 0$ with $t = x+2$. If $y = \sum_{n=0}^{\infty} a_n t^n$, then $(1-t^2)y'' - (7-8t+t^2)y' + ty = \sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n t^n - 7\sum_{n=1}^{\infty} n a_n t^{n-1} + 8\sum_{n=1}^{\infty} n a_n t^n - \sum_{n=1}^{\infty} n a_n t^{n+1} + \sum_{n=0}^{\infty} a_n t^{n+1} = (2a_2 - 7a_1) + \sum_{n=1}^{\infty}[(n+2)(n+1)a_{n+2} - 7(n+1)a_{n+1} - n(n-9)a_n - (n-2)a_{n-1}]t^n = 0$ if $a_2 = 7a_1/2$ and $a_{n+2} = \dfrac{[7(n+1)a_{n+1} + n(n-9)a_n + (n-2)a_{n-1}]}{(n+2)(n+1)}$, $n \geq 1$. Starting with $a_0 = 2$ and $a_1 = -1$ yields

$$y = 2 - (x+2) - \frac{7}{2}(x+2)^2 - \frac{43}{6}(x+2)^3 - \frac{203}{24}(x+2)^4 - \frac{167}{30}(x+2)^5 + \cdots.$$

7.3.48. The equation is equivalent to $(1+3t+2t^2)y'' - (3+t-t^2)y' - (3+t)y = 0$ with $t = x-1$. If $y = \sum_{n=0}^{\infty} a_n t^n$, then $(1+3t+2t^2)y'' - (3+t-t^2)y' - (3+t)y = \sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} + 3\sum_{n=2}^{\infty} n(n-$

$1)a_nt^{n-1}+2\sum_{n=2}^{\infty}n(n-1)a_nt^n-3\sum_{n=1}^{\infty}na_nt^{n-1}-\sum_{n=1}^{\infty}na_nt^n+\sum_{n=1}^{\infty}na_nt^{n+1}-3\sum_{n=0}^{\infty}a_nt^n-\sum_{n=0}^{\infty}a_nt^{n+1}=$ $(2a_2-3a_1-3a_0)+\sum_{n=1}^{\infty}[(n+2)(n+1)a_{n+2}+3(n^2-1)a_{n+1}+(2n^2-3n-3)(n+1)a_n+(n-2)a_{n-1}]t^n=0$ if $a_2=3(a_1+a_0)/2$ and $a_{n+2}=-\dfrac{[3(n^2-1)a_{n+1}+(2n^2-3n-3)(n+1)a_n+(n-2)a_{n-1}]}{(n+2)(n+1)}$, $n\geq 1$. Starting with $a_0=1$ and $a_1=0$ yields $y=1+\dfrac{3}{2}(x-1)^2+\dfrac{1}{6}(x-1)^3-\dfrac{1}{8}(x-1)^5+\cdots$.

7.4 REGULAR SINGULAR POINTS; EULER EQUATIONS

7.4.2. $p(r)=r(r-1)-7r+7=(r-7)(r-1)$; $y=c_1x+c_2x^7$.

7.4.4. $p(r)=r(r-1)+5r+4=(r+2)^2$;$y=x^{-2}(c_1+c_2\ln x)$

7.4.6. $p(r)=r(r-1)-3r+13=(r-2)^2+9$; $y=x^2[c_1\cos(3\ln x)+c_2\sin(3\ln x)]$.

7.4.8. $p(r)=12r(r-1)-5r+6=(3r-2)(4r-3)$; $y=c_1x^{2/3}+c_2x^{3/4}$.

7.4.10. $p(r)=3r(r-1)-r+1=(r-1)(3r-1)$; $y=c_1x+c_2x^{1/3}$.

7.4.12. $p(r)=r(r-1)+3r+5=(r+1)^2+4$; $y=\dfrac{1}{x}[c_1\cos(2\ln x)+c_2\sin(2\ln x]$

7.4.14. $p(r)=r(r-1)-r+10=(r-1)^2+9$; $y=x\,[c_1\cos(3\ln x)+c_2\sin(3\ln x)]$.

7.4.16. $p(r)=2r(r-1)+3r-1=(r+1)(2r-1)$; $y=\dfrac{c_1}{x}+c_2x^{1/2}$.

7.4.18. $p(r)=2r(r-1)+10r+9=2(r+2)^2+1$; $y=\dfrac{1}{x^2}\left[c_1\cos\left(\dfrac{1}{\sqrt{2}}\ln x\right)+c_2\sin\left(\dfrac{1}{\sqrt{2}}\ln x\right)\right]$.

7.4.20. If $p(r)=ar(r-1)+br+c=a(r-r_1)^2$, then (A) $p(r_1)=p'(r_1)=0$. If $y=ux^{r_1}$, then $y'=u'x^{r_1}+r_1ux^{r_1-1}$ and $y''=u''x^{r_1}+2r_1u'x_1^{r_1-1}+r_1(r_1-1)x_1^{r_1-2}$, so

$$\begin{aligned}ax^2y''+bxy'+cy &= ax^{r_1+2}u''+(2ar_1+b)x^{r_1+1}u'+(ar_1(r_1-1)+br_1+c)\,x^{r_1}u\\ &= ax^{r_1+2}u''+p'(r_1)x_1^{r_1+1}u'+p(r)x^{r_1}u=ax^{r_1+2}u'',\end{aligned}$$

from (A). Therefore,$u''=0$, so $u=c_1+c_2x$ and $y=x^{r_1}(c_1+c_2x)$.

7.4.22. **(a)** If $t=x-1$ and $Y(t)=y(t+1)=y(x)$, then $(1-x^2)y''-2xy'+\alpha(\alpha+1)y=-t(2+t)\dfrac{d^2Y}{dt^2}-2(1+t)\dfrac{dY}{dt}+\alpha(\alpha+1)Y=0$, so y satisfies Legendre's equation if and only if Y satisfies (A) $t(2+t)\dfrac{d^2Y}{dt^2}+2(1+t)\dfrac{dY}{dt}-\alpha(\alpha+1)Y=0$. Since (A) can be rewritten as $t^2(2+t)\dfrac{d^2Y}{dt^2}+2t(1+t)\dfrac{dY}{dt}-\alpha(\alpha+1)tY=0$, (A) has a regular singular point at $t=0_0$.

(b) If $t=x+1$ and $Y(t)=y(t-1)=y(x)$, then $(1-x^2)y''-2xy'+\alpha(\alpha+1)y=t(2-t)\dfrac{d^2Y}{dt^2}+2(1-t)\dfrac{dY}{dt}+\alpha(\alpha+1)Y$, so y satisfies Legendre's equation if and only if Y satisfies (B) $t(2-t)\dfrac{d^2Y}{dt^2}+$

$2(1-t)\dfrac{dY}{dt}+\alpha(\alpha+1)Y$, Since (B) can be rewritten as (B) $t^2(2-t)\dfrac{d^2Y}{dt^2}+2t(1-t)\dfrac{dY}{dt}+\alpha(\alpha+1)tY$, (B) has a regular singular point at $t=0_0$.

7.5 The Method of Frobenius I

7.5.2. $p_0(r)=r(3r-1)$; $p_1(r)=2(r+1)$; $p_2(r)=-4(r+2)$.

$a_1(r)=-\dfrac{2}{3r+2}$; $a_n(r)=-\dfrac{2a_{n-1}(r)-4a_{n-2}(r)}{3n+3r-1}$, $n\ge 1$.

$r_1=1/3$; $a_1(1/3)=-2/3$; $a_n(1/3)=-\dfrac{2a_{n-1}(1/3)-4a_{n-2}(1/3)}{3n}$, $n\ge1$;

$y_1=x^{1/3}\left(1-\dfrac{2}{3}x+\dfrac{8}{9}x^2-\dfrac{40}{81}x^3+\cdots\right)$.

$r_2=0$; $a_1(0)=-1$; $a_n(0)=-\dfrac{2a_{n-1}(0)-4a_{n-2}(0)}{3n-1}$, $n\ge 1$;

$y_2=1-x+\dfrac{6}{5}x^2-\dfrac{4}{5}x^3+\cdots$.

7.5.4. $p_0(r)=(r+1)(4r-1)$; $p_1(r)=2(r+2)$; $p_2(r)=4r+7$.

$a_1(r)=-\dfrac{2}{4r+3}$; $a_n(r)=-\dfrac{2}{4n+4r-1}a_{n-1}(r)-\dfrac{1}{n+r+1}a_{n-2}(r)$, $n\ge 1$.

$r_1=1/4$; $a_1(1/4)=-1/2$; $a_n(1/4)=-\dfrac{1}{2n}a_{n-1}(1/4)-\dfrac{4}{4n+5}a_{n-2}(1/4)$, $n\ge1$;

$y_1=x^{1/4}\left(1-\dfrac{1}{2}x-\dfrac{19}{104}x^2+\dfrac{1571}{10608}x^3+\cdots\right)$.

$r_2=-1$; $a_1(-1)=2$; $a_n(-1)=-\dfrac{2}{4n-5}a_{n-1}(-1)-\dfrac{1}{n}a_{n-2}(-1)$, $n\ge 1$;

$y_2=x^{-1}\left(1+2x-\dfrac{11}{6}x^2-\dfrac{1}{7}x^3+\cdots\right)$.

7.5.6. $p_0(r)=r(5r-1)$; $p_1(r)=(r+1)^2$; $p_2(r)=2(r+2)(5r+9)$.

$a_1(r)=-\dfrac{r+1}{5r+4}$; $a_n(r)=-\dfrac{n+r}{5n+5r-1}a_{n-1}(r)-2a_{n-2}(r)$, $n\ge 1$.

$r_1=1/5$; $a_1(1/5)=-6/25$; $a_n(1/5)=-\dfrac{5n+1}{25n}a_{n-1}(1/5)-2a_{n-2}(1/5)$, $n\ge1$;

$y_1=x^{1/5}\left(1-\dfrac{6}{25}x-\dfrac{1217}{625}x^2+\dfrac{41972}{46875}x^3+\cdots\right)$.

$r_2=0$; $a_1(0)=-1/4$; $a_n(0)=-\dfrac{n}{5n-1}a_{n-1}(0)-2a_{n-2}(0)$, $n\ge 1$;

$y_2=x-\dfrac{1}{4}x^2-\dfrac{35}{18}x^3+\dfrac{11}{12}x^4+\cdots$.

7.5.8. $p_0(r)=(3r-1)(6r+1)$; $p_1(r)=(3r+2)(6r+1)$; $p_2(r)=3r+5$.

$a_1(r)=-\dfrac{6r+1}{6r+7}$; $a_n(r)=-\dfrac{6n+6r-5}{6n+6r+1}a_{n-1}(r)-\dfrac{1}{6n+6r+1}a_{n-2}(r)$, $n\ge 1$.

$r_1=1/3$; $a_1(1/3)=-1/3$; $a_n(1/3)=-\dfrac{2n-1}{2n+1}a_{n-1}(1/3)-\dfrac{1}{6n+3}a_{n-2}(1/3)$, $n\ge1$;

$y_1=x^{1/3}\left(1-\dfrac{1}{3}x+\dfrac{2}{15}x^2-\dfrac{5}{63}x^3+\cdots\right)$.

$r_2=-1/6$; $a_1(-1/6)=0$; $a_n(-1/6)=-\dfrac{n-1}{n}a_{n-1}(-1/6)-\dfrac{1}{6n}a_{n-2}(-1/6)$, $n\ge 1$;

$y_2=x^{-1/6}\left(1-\dfrac{1}{12}x^2+\dfrac{1}{18}x^3+\cdots\right)$.

7.5.10. $p_0(r) = (2r+1)(5r-1)$; $p_1(r) = (2r-1)(5r+4)$; $p_2(r) = 2(2r+5)(5r-1)$.

$a_1(r) = -\dfrac{2r-1}{2r+3}$; $a_n(r) = -\dfrac{2n+2r-3}{2n+2r+1}a_{n-1}(r) - \dfrac{10n+10r-22}{5n+5r-1}a_{n-2}(r)$, $n \geq 1$.

$r_1 = 1/5$; $a_1(1/5) = 3/17$; $a_n(1/5) = -\dfrac{10n-13}{10n+7}a_{n-1}(1/5) - \dfrac{2n-4}{n}a_{n-2}(1/5)$, $n \geq 1$;

$y_1 = x^{1/5}\left(1 + \dfrac{3}{17}x - \dfrac{7}{153}x^2 - \dfrac{547}{5661}x^3 + \cdots\right)$.

$r_2 = -1/2$; $a_1(-1/2) = 1$; $a_n(-1/2) = -\dfrac{n-2}{n}a_{n-1}(-1/2) - \dfrac{20n-54}{10n-7}a_{n-2}(-1/2)$, $n \geq 1$;

$y_2 = x^{-1/2}\left(1 + x + \dfrac{14}{13}x^2 - \dfrac{556}{897}x^3 + \cdots\right)$.

7.5.14. $p_0(r) = (r+1)(2r-1)$; $p_1(r) = 2r+1$; $a_n(r) = -\dfrac{1}{n+r+1}a_{n-1}(r)$.

$r_1 = 1/2$; $a_n(1/2) = -\dfrac{2}{2n+3}a_{n-1}(1/2)$; $y_1 = x^{1/2}\displaystyle\sum_{n=0}^{\infty}\frac{(-2)^n}{\prod_{j=1}^n(2j+3)}x^n$.

$r_2 = -1$; $a_n(-1) = -\dfrac{1}{n}a_{n-1}(-1)$; $y_2 = x^{-1}\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}x^n$.

7.5.16. $p_0(r) = (r+2)(2r-1)$; $p_1(r) = r+3$; $a_n(r) = -\dfrac{1}{2n+2r-1}a_{n-1}(r)$.

$r_1 = 1/2$; $a_n(1/2) = -\dfrac{1}{2n}a_{n-1}(1/2)$; $y_1 = x^{1/2}\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n n!}x^n$.

$r_2 = -2$; $a_n(-2) = -\dfrac{1}{2n-5}a_{n-1}(-2)$; $y_2 = \dfrac{1}{x^2}\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{\prod_{j=1}^n(2j-5)}x^n$.

7.5.18. $p_0(r) = (r-1)(2r-1)$; $p_1(r) = -2$; $a_n(r) = \dfrac{2}{(n+r-1)(2n+2r-1)}a_{n-1}(r)$.

$r_1 = 1$; $a_n(1) = \dfrac{2}{n(2n+1}) a_{n-1}(1)$; $y_1 = x\displaystyle\sum_{n=0}^{\infty}\frac{2^n}{n!\prod_{j=1}^n(2j+1)}x^n$.

$r_2 = 1/2$; $a_n(1/2) = \dfrac{2}{n(2n-1}) a_{n-1}(1/2)$; $y_2 = x^{1/2}\displaystyle\sum_{n=0}^{\infty}\frac{2^n}{n!\prod_{j=1}^n(2j-1)}x^n$.

7.5.20. $p_0(r) = (r-1)(3r+1)$; $p_1(r) = r-3$; $a_n(r) = -\dfrac{n+r-4}{(n+r-1)(3n+3r+1)}a_{n-1}(r)$.

$r_1 =$; $a_n(1) = -\dfrac{n-3}{n(3n+4)}a_{n-1}(1)$; $y_1 = x\left(1 + \dfrac{2}{7}x + \dfrac{1}{70}x^2\right)$.

$r_2 = -1/3$; $a_n(-1/3) = -\dfrac{3n-13}{3n(3n-4)}a_{n-1}(-1/3)$; $y_2 = x^{-1/3}\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{3^n n!}\left(\prod_{j=1}^n\frac{3j-13}{3j-4}\right)x^n$.

7.5.22. $p_0(r) = (r-1)(4r-1)$; $p_1(r) = r(r+2)$; $a_n(r) = -\dfrac{n+r+1}{4n+4r-1}a_{n-1}(r)$.

$r_1 = 1$; $a_n(1) = -\dfrac{n+2}{4n+3}a_{n-1}(1)$; $y_1 = x\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n(n+2)!}{2\prod_{j=1}^n(4j+3)}x^n$.

$r_2 = 1/4$; $a_n(1/4) = -\dfrac{4n+5}{16n}a_{n-1}(1/4)$; $y_2 = x^{1/4}\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{16^n n!}\prod_{j=1}^n(4j+5)x^n$

7.5.24. $p_0(r) = (r+1)(3r-1)$; $p_1(r) = 2(r+2)(2r+3)$; $a_n(r) = -2\dfrac{2n+2r+1}{3n+3r-1}a_{n-1}(r)$.

$r_1 = 1/3$; $a_n(1/3) = -2\dfrac{6n+5}{9n}a_{n-1}(1/3)$; $y_1 = x^{1/3}\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left(\frac{2}{9}\right)^n\left(\prod_{j=1}^{n}(6j+5)\right)x^n$;

$r_2 = -1$; $a_n(-1) = -2\dfrac{2n-1}{3n-4}a_{n-1}(-1)$; $y_2 = x^{-1}\displaystyle\sum_{n=0}^{\infty}(-1)^n 2^n\left(\prod_{j=1}^{n}\frac{2j-1}{3j-4}\right)x^n$

7.5.28. $p_0(r) = (2r-1)(4r-1)$; $p_1(r) = (r+1)^2$; $a_n(r) = -\dfrac{(n+r)^2}{(2n+2r-1)(4n+4r-1)}a_{n-1}(r)$.

$r_1 = 1/2$; $a_n(1/2) = -\dfrac{4n^2+4n+1}{8n(4n+1)}a_{n-1}(1/2)$; $y_1 = x^{1/2}\left(1 - \dfrac{9}{40}x + \dfrac{5}{128}x^2 - \dfrac{245}{39936}x^3 + \cdots\right)$.

$r_2 = 1/4$; $a_n(1/4) = -\dfrac{16n^2+8n+1}{32n(4n-1)}a_{n-1}(1/4)$; $y_2 = x^{1/4}\left(1 - \dfrac{25}{96}x + \dfrac{675}{14336}x^2 - \dfrac{38025}{5046272}x^3 + \cdots\right)$.

7.5.30. $p_0(r) = (2r-1)(2r+1)$; $p_1(r) = (2r+1)(3r+1)$; $a_n(r) = -\dfrac{(3n+3r-2)}{(2n+2r+1)}a_n(r)$.

$r_1 = 1/2$; $a_n(1/2) = -\dfrac{6n-1}{4(n+1)}a_{n-1}(1/2)$; $y_1 = x^{1/2}\left(1 - \dfrac{5}{8}x + \dfrac{55}{96}x^2 - \dfrac{935}{1536}x^3 + \cdots\right)$.

$r_2 = -1/2$; $a_n(-1/2) = -\dfrac{6n-7}{4n}a_{n-1}(-1/2)$; $y_2 = x^{-1/2}\left(1 + \dfrac{1}{4}x - \dfrac{5}{32}x^2 - \dfrac{55}{384}x^3 + \cdots\right)$.

7.5.32. $p_0(r) = (2r+1)(3r+1)$; $p_1(r) = (r+1)(r+2)$; $a_n(r) = \dfrac{(n+r)(n+r+1)}{(2n+2r+1)(3n+3r+1)}a_n(r)$.

$r_1 = -1/3$; $a_n(-1/3) = -\dfrac{(3n-1)(3n+2)}{9n(6n+1)}a_{n-1}(-1/3)$; $y_1 = x^{-1/3}\left(1 - \dfrac{10}{63}x + \dfrac{200}{7371}x^2 - \dfrac{17600}{3781323}x^3 + \cdots\right)$.

$r_2 = -1/2$; $a_n(-1/2) = -\dfrac{(2n-1)(2n+1)}{4n(6n-1)}a_{n-1}(-1/2)$; $y_2 = x^{-1/2}\left(1 - \dfrac{3}{20}x + \dfrac{9}{352}x^2 - \dfrac{105}{23936}x^3 + \cdots\right)$.

7.5.34. $p_0(r) = (2r-1)(4r-1)$; $p_2(r) = -(2r+3)(4r+3)$; $a_{2m}(r) = \dfrac{8m+4r-5}{8m+4r-1}a_{2m-2}(r)$.

$r_1 = 1/2$; $a_{2m}(1/2) = \dfrac{8m-3}{8m+1}a_{2m-2}(1/2)$; $y_1 = x^{1/2}\displaystyle\sum_{m=0}^{\infty}\left(\prod_{j=1}^{m}\frac{8j-3}{8j+1}\right)x^{2m}$.

$r_2 = 1/4$; $a_{2m}(1/4) = \dfrac{2m-1}{2m}a_{2m-2}(1/4)$; $y_2 = x^{1/4}\displaystyle\sum_{m=0}^{\infty}\frac{1}{2^m m!}\left(\prod_{j=1}^{m}(2j-1)\right)x^{2m}$

7.5.36. $p_0(r) = r(3r-1)$; $p_2(r) = (r-4)(r+2)$; $a_{2m}(r) = -\dfrac{2m+r-6}{6m+3r-1}a_{2m-2}(r)$.

$r_1 = 1/3$; $a_{2m}(1/3) = -\dfrac{6m-17}{18m}a_{2m-2}(1/3)$; $y_1 = x^{1/3}\displaystyle\sum_{m=0}^{\infty}\frac{(-1)^m}{18^m m!}\left(\prod_{j=1}^{m}(6j-17)\right)x^{2m}$.

$r_2 = 0$; $a_{2m}(0) = -\dfrac{2m-6}{6m-1}a_{2m-2}(0)$; $y_2 = 1 + \dfrac{4}{5}x^2 + \dfrac{8}{55}x^4$

7.5.38. $p_0(r) = (2r-1)(3r-1)$; $p_2(r) = -(r+1)(3r+5)$; $a_{2m}(r) = \dfrac{2m+r-1}{4m+2r-1}a_{2m-2}(r)$.

$r_1 = 1/2; a_{2m}(1/2) = \dfrac{4m-1}{8m}a_{2m-2}(1/2)$; $y_1 = x^{1/2}\displaystyle\sum_{m=0}^{\infty}\frac{1}{8^m m!}\left(\prod_{j=1}^{m}(4j-1)\right)x^{2m}$.

$r_2 = 1/3; a_{2m}(1/3) = \dfrac{6m-2}{12m-1}a_{2m-2}(1/3)$; $y_2 = x^{1/3}\displaystyle\sum_{m=0}^{\infty}2^m\left(\prod_{j=1}^{m}\frac{3j-1}{12j-1}\right)x^{2m}$.

7.5.40. $p_0(r) = (2r-1)(2r+1)$; $p_1(r) = (r+1)(2r+3)$; $a_{2m}(r) = -\dfrac{2m+r-1}{4m+2r+1}a_{2m-2}(r)$.

$r_1 = 1/2; a_{2m}(1/2) = -\dfrac{4m-1}{4(2m+1)}a_{2m-2}(1/2)$; $y_1 = x^{1/2}\displaystyle\sum_{m=0}^{\infty}\frac{(-1)^m}{4^m}\left(\prod_{j=1}^{m}\frac{4j-1}{2j+1}\right)x^{2m}$.

$r_2 = -1/2; a_{2m}(-1/2) = -\dfrac{4m-3}{8m}a_{2m-2}(-1/2)$; $y_2 = x^{-1/2}\displaystyle\sum_{m=0}^{\infty}\frac{(-1)^m}{8^m m!}\left(\prod_{j=1}^{m}(4j-3)\right)x^{2m}$

7.5.42. $p_0(r) = (r+1)(3r-1)$; $p_1(r) = (r-1)(3r+5)$; $a_{2m}(r) = -\dfrac{2m+r-3}{2m+r+1}a_{2m-2}(r)$.

$r_1 = 1/3; a_{2m}(1/3) = -\dfrac{3m-4}{3m+2}a_{2m-2}(1/3)$; $y_1 = x^{1/3}\displaystyle\sum_{m=0}^{\infty}(-1)^m\left(\prod_{j=1}^{m}\frac{3j-4}{3j+2}\right)x^{2m}$.

$r_2 = -1; a_{2m}(-1) = -\dfrac{m-2}{m}a_{2m-2}(-1)$; $y_2 = x^{-1}(1+x^2)$

7.5.44. $p_0(r) = (r+1)(2r-1)$; $p_1(r) = r^2$; $a_{2m}(r) = -\dfrac{(2m+r-2)^2}{(2m+r+1)(4m+2r-1)}a_{2m-2}(r)$.

$r_1 = 1/2; a_{2m}(1/2) = -\dfrac{(4m-3)^2}{8m(4m+3)}a_{2m-2}(1/2)$; $y_1 = x^{1/2}\displaystyle\sum_{m=0}^{\infty}\frac{(-1)^m}{8^m m!}\left(\prod_{j=1}^{m}\frac{(4j-3)^2}{4j+3}\right)x^{2m}$.

$r_2 = -1; a_{2m}(-1) = -\dfrac{(2m-3)^2}{2m(4m-3)}a_{2m-2}(-1)$; $y_2 = x^{-1}\displaystyle\sum_{m=0}^{\infty}\frac{(-1)^m}{2^m m!}\left(\prod_{j=1}^{m}\frac{(2j-3)^2}{4j-3}\right)x^{2m}$.

7.5.46. $p_0(r) = (3r-1)(3r+1)$; $p_1(r) = 3r+5$; $a_{2m}(r) = -\dfrac{1}{6m+3r+1}a_{2m-2}(r)$.

$r_1 = 1/3; a_{2m}(1/3) = -\dfrac{1}{2(3m+1)}a_{2m-2}(1/3)$; $y_1 = x^{1/3}\displaystyle\sum_{m=0}^{\infty}\frac{(-1)^m}{2^m\prod_{j=1}^{m}(3j+1)}x^{2m}$.

$r_2 = -1/3; a_{2m}(-1/3) = -\dfrac{1}{6m}a_{2m-2}(-1/3)$; $y_2 = x^{-1/3}\displaystyle\sum_{m=0}^{\infty}\frac{(-1)^m}{6^m m!}x^{2m}$

7.5.48. $p_0(r) = 2(r+1)(4r-1)$; $p_2(r) = (r+3)^2$; $a_{2m}(r) = -\dfrac{2m+r+1}{2(8m+4r-1)}a_{2m-2}(r)$.

$r_1 = 1/4; a_{2m}(1/4) = -\dfrac{8m+5}{64m}a_{2m-2}(1/4)$; $y_1 = x^{1/4}\left(1-\dfrac{13}{64}x^2+\dfrac{273}{8192}x^4-\dfrac{2639}{524288}x^6+\cdots\right)$.

$r_2 = -1; a_{2m}(-1) = -\dfrac{m}{8m-5}a_{2m-2}(-1)$; $y_2 = x^{-1}\left(1-\dfrac{1}{3}x^2+\dfrac{2}{33}x^4-\dfrac{2}{209}x^6+\cdots\right)$.

7.5.50. $p_0(r) = (2r-1)(2r+1)$; $p_2(r) = (2r+5)^2$; $a_{2m}(r) = -\dfrac{4m+2r+1}{4m+2r-1}a_{2m-2}(r)$.

$$r_1 = 1/2; a_{2m}(1/2) = -\frac{2m+1}{2m}a_{2m-2}(1/2);\ y_1 = x^{1/2}\left(1 - \frac{3}{2}x^2 + \frac{15}{8}x^4 - \frac{35}{16}x^6 + \cdots\right).$$
$$r_2 = -1/2; a_{2m}(-1/2) = -\frac{2m}{2m-1}a_{2m-2}(-1/2);\ y_2 = x^{-1/2}\left(1 - 2x^2 + \frac{8}{3}x^4 - \frac{16}{5}x^6 + \cdots\right).$$

7.5.52. **(a)** Multiplying (A) $c_1y_1 + c_2y_2 \equiv 0$ by x^{-r_2} yields $c_1x^{r_1-r_2}\sum_{n=0}^{\infty} a_nx^n + c_2\sum_{n=0}^{\infty} b_nx^n = 0$, $0 < x < \rho$. Letting $x \to 0+$ shows that $c_2 = 0$, since $b_0 = 1$. Now (A) reduces to $c_1y_1 \equiv 0$, so $c_1 = 0$. Therefore,y_1 and y_2 are linearly independent on $(0, \rho)$.

(b) Since $y_1 = \sum_{n=0}^{\infty} a_n(r_1)x^n$ and $y_2 = \sum_{n=0}^{\infty} a_n(r_2)x^n$ are linearly independent solutions of $Ly = 0$ $(0, \rho)$, $\{y_1, y_2\}$ is a fundamental set of solutions of $Ly = 0$ on $(0, \rho)$, by Theorem 5.1.6.

7.5.54. **(a)** If $x > 0$, then $|x|^r x^n = x^{n+r}$, so the assertions are obvious. If $x < 0$, then $|x|^r = (-x)^r$, so $\frac{d}{dx}|x|^r = -r(-x)^{r-1} = \frac{r(-x)^r}{x} = \frac{r|x|^r}{x}$. Therefore,(A) $\frac{d}{dx}(|x|^r x^n) = \frac{r|x|^r}{x}x^n + |x|^r(nx^{n-1}) = (n+r)|x|^r x^{n-1}$ and $\frac{d^2}{dx^2}(|x|^r x^n) = (n+r)\frac{d}{dx}(|x|^r x^{n-1}) = (n+r)(n+r-1)|x|^r x^{n-2}$, from (A) with n replaced by $n-1$.

7.5.56. **(a)** Here $p_1 \equiv 0$, so Eqn. (7.5.12) reduces to $a_0(r) = 1$, $a_1(r) = 0$, $a_n(r) = -\frac{p_2(n+r-2)}{p_0(n+r)}a_{n-2}(r)$, $r \geq 0$, which implies that $a_{2m+1}(r) = 0$ for $m = 1, 2, 3, \ldots$. Therefore,Eqn. (7.5.12) actually reduces to $a_0(r) = 1$, $a_{2m}(r) = -\frac{p_2(2m+r-2)}{p_0(2m+r)}$, which holds because of condition (A).

(b) Similar to the proof of Exercise 7.5.55**(a)**.

(c) $p_0(2m + r_1) = 2m\alpha_0(2m + r_1 - r_2)$, which is nonzero if $m > 0$, since $r_1 - r_2 \geq 0$. Therefore, the assumptions of Theorem 7.5.2 hold with $r = r_1$, and $Ly_1 = p_0(r_1)x^{r_1} = 0$. If $r_1 - r_2$ is not an even integer, then $p_0(2m + r_2) = 2m\alpha_0(2m - r_1 + r_2) \neq 0$, $m = 1, 2, \cdots$. Hence, the assumptions of Theorem 7.5.2 hold with $r = r_2$ and $Ly_2 = p_0(r_2)x^{r_2} = 0$. From Exercise 7.5.52, $\{y_1, y_2\}$ is a fundamental set of solutions.

(d) Similar to the proof of Exercise 7.5.55**(c)**.

7.5.58. **(a)** From Exercise 7.5.57, $b_n = 0$ for $n \geq 1$.

7.5.60. **(a)** $(\alpha_0 + \alpha_1 x + \alpha_2 x^2)\sum_{n=0}^{\infty} a_nx^n = \alpha_0 a_0 + (\alpha_0 a_1 + \alpha_1 a_0)x + \sum_{n=2}^{\infty}(\alpha_0 a_n + \alpha_1 a_{n-1} + \alpha_2 a_{n-2})x^n = 1$, so $\sum_{n=0}^{\infty} a_nx^n = \frac{\alpha_0 a_0}{\alpha_0 + \alpha_1 x + \alpha_2 x^2}$.

(b) If $\frac{p_1(r-1)}{p_0(r)} = \frac{\alpha_1}{\alpha_0}$ and $\frac{p_2(r-2)}{p_0(r)} = \frac{\alpha_2}{\alpha_0}$, then Eqn. (7.5.12) is equivalent to $a_0(r) = 1$, $\alpha_0 a_1(r) + \alpha_1 a_0(r) = 0$, $\alpha_0 a_n(r) + \alpha_1 a_{n-1}(r) + \alpha_2 a_{n-2}(r) = 0$, $\quad n \geq 2$. Therefore,Theorem 7.5.2 implies the conclusion.

7.5.62. $p_0(r) = (2r-1)(3r-1)$; $p_1(r) = 0$; $p_2(r) = 2(2r+3)(3r+5)$; $\frac{p_1(r-1)}{p_0(r)} = 0 = \frac{\alpha_1}{\alpha_0}$; $\frac{p_2(r-2)}{p_0(r)} = 2 = \frac{\alpha_2}{\alpha_0}$; $y_1 = \frac{x^{1/3}}{1+2x^2}$; $y_2 = \frac{x^{1/2}}{1+2x^2}$.

7.5.64. $p_0(r) = 5(3r-1)(3r+1)$; $p_1(r) = (3r+2)(3r+4)$; $p_2(r) = 0$; $\frac{p_1(r-1)}{p_0(r)} = \frac{1}{5} = \frac{\alpha_1}{\alpha_0}$; $\frac{p_2(r-2)}{p_0(r)} = 0 = \frac{\alpha_2}{\alpha_0}$; $y_1 = \frac{x^{1/3}}{5+x}$; $y_2 = \frac{x^{-1/3}}{5+x}$.

7.5.66. $p_0(r) = (2r-3)(2r-1)$; $p_1(r) = 3(2r-1)(2r+1)$; $p_2(r) = (2r+1)(2r+3)$; $\dfrac{p_1(r-1)}{p_0(r)} = 3 = \dfrac{\alpha_1}{\alpha_0}$; $\dfrac{p_2(r-2)}{p_0(r)} = 1 = \dfrac{\alpha_2}{\alpha_0}$; $y_1 = \dfrac{x^{1/2}}{1+3x+x^2}$; $y_2 = \dfrac{x^{3/2}}{1+3x+x^2}$.

7.5.68. $p_0(r) = 3(r-1)(4r-1)$; $p_1(r) = 2r(4r+3)$; $p_2(r) = (r+1)(4r+7)$; $\dfrac{p_1(r-1)}{p_0(r)} = \dfrac{2}{3} = \dfrac{\alpha_1}{\alpha_0}$; $\dfrac{p_2(r-2)}{p_0(r)} = \dfrac{1}{3} = \dfrac{\alpha_2}{\alpha_0}$; $y_1 = \dfrac{x}{3+2x+x^2}$; $y_2 = \dfrac{x^{1/4}}{3+2x+x^2}$.

7.6 THE METHOD OF FROBENIUS II

7.6.2. $p_0(r) = (r+1)^2$; $p_1(r) = (r+2)(r+3)$; $p_2(r) = (r+3)(2r-1)$;

$a_1(r) = -\dfrac{r+3}{r+2}$; $a_n(r) = -\dfrac{n+r+2}{n+r+1}a_{n-1}(r) - \dfrac{2n+2r-5}{n+r+1}a_{n-2}(r)$, $n \ge 2$.

$a_1'(r) = \dfrac{1}{(r+2)^2}$; $a_n'(r) = -\dfrac{n+r+2}{n+r+1}a_{n-1}'(r) - \dfrac{2n+2r-5}{n+r+1}a_{n-2}'(r) + \dfrac{1}{(n+r+1)^2}a_{n-1}(r) - \dfrac{7}{(n+r+1)^2}a_{n-2}(r)$, $n \ge 2$.

$r_1 = -1$; $a_1(-1) = -2$; $a_n(-1) = -\dfrac{n+1}{n}a_{n-1}(-1) - \dfrac{2n-7}{n}a_{n-2}(-1)$, $n \ge 2$;

$y_1 = x^{-1}\left(1 - 2x + \dfrac{9}{2}x^2 - \dfrac{20}{3}x^3 + \cdots\right)$;

$a_1'(-1) = 1$; $a_n'(-1) = -\dfrac{n+1}{n}a_{n-1}(-1) - \dfrac{2n-7}{n}a_{n-2}(-1) + \dfrac{1}{n^2}a_{n-1}(-1) - \dfrac{7}{n^2}a_{n-2}(-1)$, $n \ge 2$;

$y_2 = y_1 \ln x + 1 - \dfrac{15}{4}x + \dfrac{133}{18}x^2 + \cdots$.

7.6.4. $p_0(r) = (2r-1)^2$; $p_1(r) = (2r+1)(2r+3)$; $p_2(r) = (2r+1)(2r+3)$;

$a_1(r) = -\dfrac{2r+3}{2r+1}$; $a_n(r) = -\dfrac{(2n+2r+1)a_{n-1}(r) - (2n+2r-3)a_{n-2}(r)}{2n+2r-1}$, $n \ge 2$.

$a_1'(r) = \dfrac{4}{(2r+1)^2}$; $a_n'(r) = -\dfrac{(2n+2r+1)a_{n-1}'(r) - (2n+2r-3)a_{n-2}'(r)}{2n+2r-1} + \dfrac{4(a_{n-1}(r) - a_{n-2}(r))}{(2n+2r-1)^2}$; $n \ge 2$.

$r_1 = 1/2$; $a_1(1/2) = -2$; $a_n(1/2) = -\dfrac{(n+1)a_{n-1}(1/2) + (n-1)a_{n-2}(1/2)}{n}$; $n \ge 2$;

$y_1 = x^{1/2}\left(1 - 2x + \dfrac{5}{2}x^2 - 2x^3 + \cdots\right)$;

$a_1'(1/2) = 1$; $a_n'(1/2) = -\dfrac{(n+1)a_{n-1}'(1/2) + (n-1)a_{n-2}'(1/2)}{n} + \dfrac{a_{n-1}(1/2) - a_{n-2}(1/2)}{n^2}$, $n \ge 2$;

$y_2 = y_1 \ln x + x^{3/2}\left(1 - \dfrac{9}{4}x + \dfrac{17}{6}x^2 + \cdots\right)$.

7.6.6. $p_0(r) = (3r+1)^2$; $p_1(r) = 3(3r+4)$; $p_2(r) = -2(3r+7)$;

$a_1(r) = -\dfrac{3}{3r+4}$; $a_n(r) = \dfrac{-3a_{n-1}(r) + 2a_{n-2}(r)}{3n+3r+1}$; $n \ge 2$;

$a_1'(r) = \dfrac{9}{(3r+4)^2}$; $a_n'(r) = \dfrac{-3a_{n-1}'(r) + 2a_{n-2}'(r)}{3n+3r+1} + \dfrac{9a_{n-1}(r) - 6a_{n-2}(r)}{(3n+3r+1)^2}$; $n \ge 2$.

$r_1 = -1/3$; $a_1(-1/3) = -1$; $a_n(-1/3) = \dfrac{-3a_{n-1}(-1/3) + 2a_{n-2}(-1/3)}{3n}$, $n \ge 2$;

$y_1 = x^{-1/3}\left(1 - x + \dfrac{5}{6}x^2 - \dfrac{1}{2}x^3 + \cdots\right)$;

$a_1'(-1/3) = 1$; $a_n'(-1/3) = \dfrac{-3a_{n-1}'(r) + 2a_{n-2}'(r)}{3n} + \dfrac{3a_{n-1}(r) - 2a_{n-2}(r)}{3n^2}$; $n \geq 2$;
$y_2 = y_1 \ln x + x^{2/3}\left(1 - \dfrac{11}{12}x + \dfrac{25}{36}x^2 + \cdots\right)$.

7.6.8. $p_0(r) = (r+2)^2$; $p_1(r) = 2(r+3)^2$; $p_2(r) = 3(r+4)$;
$a_1(r) = -2$; $a_n(r) = -2a_{n-1}(r) - \dfrac{3a_{n-2}(r)}{n+r+2}$; $n \geq 2$;
$a_1'(r) = 0$; $a_n'(r) = -2a_{n-1}'(r) - \dfrac{3a_{n-2}'(r)}{n+r+2} + \dfrac{3a_{n-2}(r)}{(n+r+2)^2}$, $n \geq 2$.
$r_1 = -2$; $a_1(-2) = -2$; $a_n(-2) = -2a_{n-1}(-2) - \dfrac{3a_{n-2}(-2)}{n}$, $n \geq 2$;
$y_1 = x^{-2}\left(1 - 2x + \dfrac{5}{2}x^2 - 3x^3 + \cdots\right)$;
$a_1'(-2) = 0$; $a_n'(-2) = -2a_{n-1}(-2) - \dfrac{3a_{n-2}(-2)}{n} + \dfrac{3a_{n-2}(-2)}{n^2}$; $n \geq 2$;
$y_2 = y_1 \ln x + \dfrac{3}{4} - \dfrac{13}{6}x + \cdots$.

7.6.10. $p_0(r) = (4r+1)^2$; $p_1(r) = 4r+5$; $p_2(r) = 2(4r+9)$;
$a_1(r) = -\dfrac{1}{4r+5}$; $a_n(r) = -\dfrac{a_{n-1}(r) + 2a_{n-2}(r)}{4n+4r+1}$; $n \geq 2$;
$a_1'(r) = \dfrac{4}{(4r+5)^2}$; $a_n'(r) = -\dfrac{a_{n-1}'(r) + 2a_{n-2}'(r)}{4n+4r+1} + \dfrac{4a_{n-1}(r) + 8a_{n-2}(r)}{(4n+4r+1)^2}$; $n \geq 2$.
$r_1 = -1/4$; $a_1(-1/4) = -1/4$; $a_n(-1/4) = -\dfrac{a_{n-1}(-1/4) + 2a_{n-2}(-1/4)}{4n}$; $n \geq 2$;
$y_1 = x^{-1/4}\left(1 - \dfrac{1}{4}x - \dfrac{7}{32}x^2 + \dfrac{23}{384}x^3 + \cdots\right)$;
$a_1'(-1/4) = 1/4$; $a_n'(-1/4) = -\dfrac{a_{n-1}'(-1/4) + 2a_{n-2}'(-1/4)}{4n} + \dfrac{a_{n-1}(-1/4) + 2a_{n-2}(-1/4)}{4n^2}$; $n \geq 2$;
$y_2 = y_1 \ln x + x^{3/4}\left(\dfrac{1}{4} + \dfrac{5}{64}x - \dfrac{157}{2304}x^2 + \cdots\right)$.

7.6.12. $p_0(r) = (2r-1)^2$; $p_1(r) = 4$;
$a_n(r) = -\dfrac{4}{(2n+2r-1)^2}a_{n-1}(r)$;
$a_n(r) = \dfrac{(-4)^n}{\prod_{j=1}^n (2j+2r-1)}$.
By logarithmic differentiation, $a_n'(r) = a_n(r)\displaystyle\sum_{j=1}^n \dfrac{2}{2j+2r-1}$;
$r_1 = 1/2$; $a_n(1/2) = \dfrac{(-1)^n}{(n!)^2}$;
$a_n'(1/2) = a_n(1/2)\left(-2\displaystyle\sum_{j=1}^n \dfrac{1}{j}\right)$;
$y_1 = x^{1/2}\displaystyle\sum_{n=0}^\infty \dfrac{(-1)^n}{(n!)^2}x^n$;

$$y_2 = y_1 \ln x - 2x^{1/2} \sum_{n=1}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\sum_{j=1}^{n} \frac{1}{j} \right) x^n;$$

7.6.14. $p_0(r) = (r-2)^2$; $p_1(r) = r^2$; $a_n(r) = -\dfrac{(n+r-1)^2}{(n+r-2)^2} a_{n-1}(r)$; $a_n(r) = (-1)^n \dfrac{(n+r-1)^2}{(r-1)^2}$; $a_n'(r) = (-1)^{n+1} \dfrac{2n(r+n-1)}{(r-1)^3}$; $r_1 = 2$; $a_n(2) = (-1)^n (n+1)^2$; $a_n'(2) = (-1)^{n+1} 2n(n+1)$; $y_1 = x^2 \displaystyle\sum_{n=0}^{\infty} (-1)^n (n+1)^2 x^n$; $y_2 = y_1 \ln x - 2x^2 \displaystyle\sum_{n=1}^{\infty} (-1)^n n(n+1) x^n$.

7.6.16. $p_0(r) = (5r-1)^2$; $p_1(r) = r+1$;

$$a_n(r) = -\frac{(n+r)}{(5n+5r-1)^2} a_{n-1}(r);$$

$$a_n(r) = (-1)^n \prod_{j=1}^{n} \frac{(j+r)}{(5j+5r-1)^2};$$

By logarithmic differentiation,

$$a_n'(r) = -a_n(r) \sum_{j=1}^{n} \frac{(5j+5r+1)}{(j+r)(5j+5r-1)};$$

$$r_1 = 1/5; a_n(1/5) = (-1)^n \prod_{j=1}^{n} \frac{(5j+1)}{125^n (n!)^2};$$

$$a_n'(1/5) = a_n(1/5) \sum_{j=1}^{n} \frac{5j+2}{j(5j+1)};$$

$$y_1 = x^{1/5} \sum_{n=0}^{\infty} \frac{(-1)^n \prod_{j=1}^{n} (5j+1)}{125^n (n!)^2} x^n;$$

$$y_2 = y_1 \ln x - x^{1/5} \sum_{n=1}^{\infty} \frac{(-1)^n \prod_{j=1}^{n} (5j+1)}{125^n (n!)^2} \left(\sum_{j=1}^{n} \frac{5j+2}{j(5j+1)} \right) x^n.$$

7.6.18. $p_0(r) = (3r-1)^2$; $p_1(r) = (2r-1)^2$;

$$a_n(r) = -\frac{(2n+2r-3)^2}{(3n+3r-1)^2} a_{n-1}(r);$$

$$a_n(r) = (-1)^n \prod_{j=1}^{n} \frac{(2j+2r-3)^2}{(3j+3r-1)^2};$$

By logarithmic differentiation,

$$a_n'(r) = 14 a_n(r) \sum_{j=1}^{n} \frac{1}{(2j+2r-3)(3j+3r-1)};$$

$$r_1 = 1/3; a_n(1/3) = \frac{(-1)^n \prod_{j=1}^{n} (6j-7)^2}{81^n (n!)^2};$$

$$a_n'(1/3) = 14 a_n(1/3) \sum_{j=1}^{n} \frac{1}{j(6j-7)});$$

$$y_1 = x^{1/3} \sum_{n=0}^{\infty} \frac{(-1)^n \prod_{j=1}^{n} (6j-7)^2}{81^n (n!)^2} x^n;$$

$$y_2 = y_1 \ln x + 14x^{1/3} \sum_{n=1}^{\infty} \frac{(-1)^n \prod_{j=1}^{n}(6j-7)^2}{81^n (n!)^2} \left(\sum_{j=1}^{n} \frac{1}{j(6j-7)} \right) x^n.$$

7.6.20. $p_0(r) = (r+1)^2$; $p_1(r) = -2(r+2)(2r+3)$;

$$a_n(r) = \frac{2(2n+2r+1)}{n+r+1} a_{n-1}(r),\ n \ge 1;\ a_n(r) = 2^n \prod_{j=1}^{n} \frac{2j+2r+1}{j+r+1};$$

By logarithmic differentiation,

$$a_n'(r) = a_n(r) \sum_{j=1}^{n} \frac{1}{(j+r+1)(2j+2r+1)};$$

$$r_1 = -1;\ a_n(-1) = \frac{2^n \prod_{j=1}^{n}(2j-1)}{n!};$$

$$a_n'(-1) = a_n(-1) \sum_{j=1}^{n} \frac{1}{j(2j-1)};$$

$$y_1 = \frac{1}{x} \sum_{n=0}^{\infty} \frac{2^n \prod_{j=1}^{n}(2j-1)}{n!} x^n;$$

$$y_2 = y_1 \ln x + \frac{1}{x} \sum_{n=1}^{\infty} \frac{2^n \prod_{j=1}^{n}(2j-1)}{n!} \left(\sum_{j=1}^{n} \frac{1}{j(2j-1)} \right) x^n.$$

7.6.22. $p_0(r) = 2(r-2)^2$; $p_1(r) = (r-1)(2r+1)$;

$$a_n(r) = -\frac{2n+2r-1}{2(n+r-2)} a_{n-1}(r);$$

$$a_n(r) = \frac{(-1)^n}{2^n} \prod_{j=1}^{n} \frac{2j+2r-1}{j+r-2};$$

By logarithmic differentiation,

$$a_n'(r) = -3a_n(r) \sum_{j=1}^{n} \frac{1}{(j+r-2)(2j+2r-1)};$$

$$r_1 = 2;\ a_n(2) = \frac{(-1)^n \prod_{j=1}^{n}(2j+3)}{2^n n!};$$

$$a_n'(2) = -3a_n(2) \sum_{j=1}^{n} \frac{1}{j(2j+3)};$$

$$y_1 = x^2 \sum_{n=0}^{\infty} \frac{(-1)^n \prod_{j=1}^{n}(2j+3)}{2^n n!} x^n;$$

$$y_2 = y_1 \ln x - 3x^2 \sum_{n=0}^{\infty} \frac{(-1)^n \prod_{j=1}^{n}(2j+3)}{2^n n!} \left(\sum_{j=1}^{n} \frac{1}{j(2j+3)} \right) x^n.$$

7.6.24. $p_0(r) = (r-3)^2$; $p_1(r) = -2(r-1)(r+2)$;

$$a_n(r) = \frac{2(n+r-2)(n+r+1)}{(n+r-3)^2} a_{n-1}(r);$$

$$a_n'(r) = \frac{2(n+r-2)(n+r+1)}{(n+r-3)^2} a_{n-1}'(r) - \frac{2(5n+5r-7)}{(n+r-3)^3} a_{n-1}(r);$$

$$r_1 = 3;\ a_n(3) = \frac{2(n+1)(n+4)}{n^2} a_{n-1}(3);$$

$$y_1 = x^3(1 + 20x + 180x^2 + 1120x^3 + \cdots;$$
$$a_n'(3) = \frac{2(n+1)(n+4)}{n^2}a_{n-1}'(3) - \frac{2(5n+8)}{n^3}a_{n-1}(3);$$
$$y_2 = y_1 \ln x - x^4\left(26 + 324x + \frac{6968}{3}x^2 + \cdots\right)$$

7.6.26. $p_0(r) = r^2$; $p_1(r) = r^2 + r + 1$;
$$a_n(r) = -\frac{(n^2 + n(2r-1) + r^2 - r + 1)}{(n+r)^2}a_{n-1}(r);$$
$$a_n'(r) = -\frac{(n^2 + n(2r-1) + r^2 - r + 1)}{(n+r)^2}a_{n-1}'(r) - \frac{(n+r-2)}{(n+r)^3}a_{n-1}(r);$$
$$r_1 = 0;\ a_n(0) = -\frac{(n^2 - n + 1)}{n^2}a_{n-1}(0);$$
$$y_1 = 1 - x + \frac{3}{4}x^2 - \frac{7}{12}x^3 + \cdots;$$
$$a_n'(0) = -\frac{(n^2 - n + 1)}{n^2}a_{n-1}'(0) - \frac{(n-2)}{n^3}a_{n-1}(0);$$
$$y_2 = y_1 \ln x + x\left(1 - \frac{3}{4}x + \frac{5}{9}x^2 + \cdots\right).$$

7.6.28. $p_0(r) = (r-1)^2$; $p_2(r) = r + 1$;
$$a_{2m}(r) = -\frac{1}{2m + r - 1}a_{2m-2}(r),\ n \ge 1;\ a_{2m}(r) = \frac{(-1)^m}{\prod_{j=1}^m (2j + r - 1)};$$
By logarithmic differentiation,
$$a_{2m}'(r) = a_{2m}(r)\sum_{j=1}^m;$$
$$r_1 = 1;\ a_{2m}(1) = \frac{(-1)^m}{2^m m!};$$
$$a_{2m}'(1) = -\frac{1}{2}a_{2m}(1)\sum_{j=1}^m \frac{1}{j};$$
$$y_1 = x\sum_{m=0}^\infty \frac{(-1)^m}{2^m m!}x^{2m};$$
$$y_2 = y_1 \ln x - \frac{x}{2}\sum_{m=1}^\infty \frac{(-1)^m}{2^m m!}\left(\sum_{j=1}^m \frac{1}{j}\right)x^{2m}.$$

7.6.30. $p_0(r) = (2r-1)^2$; $p_2(r) = 2r + 3$;
$$a_{2m}(r) = -\frac{1}{4m + 2r - 1}a_{2m-2}(r);$$
$$a_{2m}(r) = \frac{(-1)^m}{\prod_{j=1}^m (4j + 2r - 1)};$$
By logarithmic differentiation,
$$a_{2m}'(r) = -2a_{2m}(r)\sum_{j=1}^m \frac{1}{4j + 2r - 1};$$
$$r_1 = 1/2;\ a_{2m}(1/2) = \frac{(-1)^m}{4^m m!};$$

$$a'_{2m}(1/2) = -\frac{1}{2}a_{2m}(1/2)\sum_{j=1}^{m}\frac{1}{j};$$

$$y_1 = x^{1/2}\sum_{m=0}^{\infty}\frac{(-1)^m}{4^m m!}x^{2m};$$

$$y_2 = y_1 \ln x - \frac{x^{1/2}}{2}\sum_{m=1}^{\infty}\frac{(-1)^m}{4^m m!}\left(\sum_{j=1}^{m}\frac{1}{j}\right)x^{2m}.$$

7.6.32. $p_0(r) = (2r-1)^2$; $p_2(r) = (r+1)(2r+3)$; $a_{2m}(r) = -\dfrac{2m+r-1}{4m+2r-1}a_{2m-2}(r)$;

$$a_{2m}(r) = (-1)^m \prod_{j=1}^{m}\frac{2j+r-1}{4j+2r-1};$$

By logarithmic differentiation,

$$a'_{2m}(r) = a_{2m}(r)\sum_{j=1}^{m}\frac{1}{(2j+r-1)(4j+2r-1)};$$

$$r_1 = 1/2; a_{2m}(1/2) = \frac{(-1)^m \prod_{j=1}^{m}(4j-1)}{8^m m!};$$

$$a'_{2m}(1/2) = a_{2m}(1/2)\sum_{j=1}^{m}\frac{1}{2j(4j-1)};$$

$$y_1 = x^{1/2}\sum_{m=0}^{\infty}\frac{(-1)^m \prod_{j=1}^{m}(4j-1)}{8^m m!}x^{2m};$$

$$y_2 = y_1 \ln x + \frac{x^{1/2}}{2}\sum_{m=1}^{\infty}\frac{(-1)^m \prod_{j=1}^{m}(4j-1)}{8^m m!}\left(\sum_{j=1}^{m}\frac{1}{j(4j-1)}\right)x^{2m}.$$

7.6.34. $p_0(r) = (4r+1)^2$; $p_2(r) = (r-1)(4r+9)$;

$$a_{2m}(r) = -\frac{2m+r-3}{8m+4r+1}a_{2m-2}(r);$$

$$a_{2m}(r) = (-1)^m \prod_{j=1}^{m}\frac{2j+r-3}{8j+4r+1};$$

By logarithmic differentiation,

$$a'_{2m}(r) = a_{2m}(r)\sum_{j=1}^{m}\frac{13}{(2j+r-3)(8j+4r+1)};$$

$$r_1 = -1/4; a_{2m}(-1/4) = \frac{(-1)^m \prod_{j=1}^{m}(8j-13)}{(32)^m m!};$$

$$a'_{2m}(-1/4) = a_{2m}(-1/4)\sum_{j=1}^{m}\frac{13}{2j(8j-13)};$$

$$y_1 = x^{-1/4}\sum_{m=0}^{\infty}\frac{(-1)^m \prod_{j=1}^{m}(8j-13)}{(32)^m m!}x^{2m};$$

$$y_2 = y_1 \ln x + \frac{13}{2}x^{-1/4}\sum_{m=1}^{\infty}\frac{(-1)^m \prod_{j=1}^{m}(8j-13)}{(32)^m m!}\left(\sum_{j=1}^{m}\frac{1}{j(8j-13)}\right)x^{2m}.$$

7.6.36. $p_0(r) = (2r-1)^2$; $p_2(r) = 16r(r+1)$;

$$a_{2m}(r) = -\frac{16(2m+r-2)(2m+r-1)}{(4m+2r-1)^2} a_{2m-2}(r);$$

$$a_{2m}(r) = (-16)^m \prod_{j=1}^{m} \frac{(2j+r-2)(2j+r-1)}{(4j+2r-1)^2};$$

By logarithmic differentiation,

$$a'_{2m}(r) = a_{2m}(r) \sum_{j=1}^{m} \frac{8j+4r-5}{(2j+r-2)(2j+r-1)(4j+2r-1)};$$

$$r_1 = 1/2; a_{2m}(1/2) = \frac{(-1)^m \prod_{j=1}^{m}(4j-3)(4j-1)}{4^m (m!)^2};$$

$$a'_{2m}(1/2) = a_{2m}(1/2) \sum_{j=1}^{m} \frac{8j-3}{j(4j-3)(4j-1)};$$

$$y_1 = x^{1/2} \sum_{m=0}^{\infty} \frac{(-1)^m \prod_{j=1}^{m}(4j-3)(4j-1)}{4^m (m!)^2} x^{2m};$$

$$y_2 = y_1 \ln x + x^{1/2} \sum_{m=1}^{\infty} \frac{(-1)^m \prod_{j=1}^{m}(4j-3)(4j-1)}{4^m (m!)^2} \left(\sum_{j=1}^{m} \frac{8j-3}{j(4j-3)(4j-1)} \right) x^{2m}.$$

7.6.38. $p_0(r) = (r+1)^2$; $p_2(r) = (r+3)(2r-1)$;

$$a_{2m}(r) = -\frac{4m+2r-5}{2m+r+1} a_{2m-2}(r);$$

$$a_{2m}(r) = (-1)^m \prod_{j=1}^{m} \frac{4j+2r-5}{2j+r+1};$$

By logarithmic differentiation,

$$a'_{2m}(r) = a_{2m}(r) \sum_{j=1}^{m} \frac{7}{(2j+r+1)(4j+2r-5)};$$

$$r_1 = -1; a_{2m}(-1) = \frac{(-1)^m \prod_{j=1}^{m}(4j-7)}{2^m m!};$$

$$a'_{2m}(-1) = a_{2m}(-1) \sum_{j=1}^{m} \frac{7}{2j(4j-7)};$$

$$y_1 = \frac{1}{x} \sum_{m=0}^{\infty} \frac{(-1)^m \prod_{j=1}^{m}(4j-7)}{2^m m!} x^{2m};$$

$$y_2 = y_1 \ln x + \frac{7}{2x} \sum_{m=1}^{\infty} \frac{(-1)^m \prod_{j=1}^{m}(4j-7)}{2^m m!} \left(\sum_{j=1}^{m} \frac{1}{j(4j-7)} \right) x^{2m}.$$

7.6.40. $p_0(r) = (r-1)^2$; $p_2(r) = r+1$;

$$a_{2m}(r) = -\frac{1}{2m+r-1} a_{2m-2}(r);$$

$$a'_{2m}(r) = -\frac{1}{2m+r-1} a'_{2m-2}(r) + \frac{1}{(2m+r-1)^2} a_{2m-2}(r);$$

$$r_1 = 1; a_{2m}(1) = -\frac{1}{2m} a_{2m-2}(1);$$

$$y_1 = x \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{1}{48}x^6 + \cdots \right);$$

$$a'_{2m}(1) = -\frac{1}{2m}a'_{2m-2}(1) + \frac{1}{4m^2}a_{2m-2}(1),\ m \ge 1;$$
$$y_2 = y_1 \ln x + x^3\left(\frac{1}{4} - \frac{3}{32}x^2 + \frac{11}{576}x^4 + \cdots\right).$$

7.6.42. $p_0(r) = 2(r+3)^2;\ p_2(r) = r^2 - 2r + 2;$

$$a_{2m}(r) = -\frac{4m^2 + 4m(r-3) + r^2 - 6r + 10}{2(2m+r+3)^2}a_{2m-2}(r);$$
$$a'_{2m}(r) = -\frac{4m^2 + 4m(r-3) + r^2 - 6r + 10}{2(2m+r+3)^2}a'_{2m-2}(r) - \frac{12m+6r-19}{(2m+r+3)^3}a_{2m-2}(r);$$
$$r_1 = -3;\ a_{2m}(-3) = -\frac{4m^2 - 24m + 37}{8m^2}a_{2m-2}(-3);$$
$$y_1 = x^{-3}\left(1 - \frac{17}{8}x^2 + \frac{85}{256}x^4 - \frac{85}{18432}x^6 + \cdots\right);$$
$$a'_{2m}(-3) = -\frac{4m^2 - 24m + 37}{8m^2}a'_{2m-2}(-3) + \frac{37 - 12m}{8m^3}a_{2m-2}(-3),\ m \ge 1;$$
$$y_2 = y_1 \ln x + x^{-1}\left(\frac{25}{8} - \frac{471}{512}x^2 + \frac{1583}{110592}x^4 + \cdots\right).$$

7.6.44. $p_0(r) = (r+1)^2;\ p_1(r) = 2(2-r)(r+1);\ r_1 = -1.$

$a_n(r) = \dfrac{2(n+r)(n+r-3)}{(n+r+1)^2}a_{n-1}(r);\ a_n(r) = 2^n\displaystyle\prod_{j=1}^{n}\frac{(j+r)(j+r-3)}{(j+r+1)^2},\ n \ge 0$. Therefore,$a_n(-1) = 0$ if $n \ge 1$ and $y_1 = 1/x$. If $n \ge 4$, then $a_n(r) = (r+1)^2 b_n(r)$, where $b'_n(-1)$ exists; therefore $a'_n(-1) = 0$ if $n \ge 4$. For $r = 1, 2, 3$, $a_n(r) = (r+1)c_n(r)$, where $c_1(r) = \dfrac{2(r-2)}{(r+2)^2}$, $c_2(r) = \dfrac{4(r-2)(r-1)}{(r+2)(r+3)^2}$, $c_3(r) = \dfrac{8r(r-2)(r-1)}{(r+2)(r+3)(r+4)^2}$. Hence, $a'_1(-1) = c_1(-1) = -6$, $a'_2(-1) = c_2(-1) = 6$, $a'_3(-1) = c_3(-1) = -8/3$, and $y_2 = y_1 \ln x - 6 + 6x - \dfrac{8}{3}x^2$.

7.6.46. $p_0(r) = (r+1)^2;\ p_1(r) = -(r-1)(r+2);\ r_1 = -1.$

$a_n(r) = \dfrac{n+r-2}{n+r+1}a_{n-1}(r);\ a_n(r) = \displaystyle\prod_{j=1}^{m}\frac{j+r-2}{j+r+1},\ n \ge 0$. Therefore,$a_1(-1) = -2$, $a_2(-1) = 1$, and $a_n(-1) = 0$ if $n \ge 3$, so $y_1 = \dfrac{(x-1)^2}{x}$.

$a_1(r) = \dfrac{r-1}{r+2},\ a'_1(r) = \dfrac{3}{(r+2)^2},\ a'_1(-1) = 3;\ a_2(r) = \dfrac{r(r-1)}{(r+2)(r+3)},\ a'_2(r) = \dfrac{6(r^2+2r-1)}{(r+2)^2(r+3)^2}$, $a'_2(-1) = -3$; if $n \ge 3$ $a_n(r) = (r+1)c_n(r)$ where $c_n(r) = \dfrac{r(r-1)}{(n+r)(n+r-1)(n+r+1)}$, so $a'_n(-1) = c_n(-1) = \dfrac{2}{n(n-2)(n-1)}$ and $y_2 = y_1 \ln x + 3 - 3x + 2\displaystyle\sum_{n=2}^{\infty}\frac{1}{n(n^2-1)}x^n$.

7.6.48. $p_0(r) = (r-2)^2;\ p_1(r) = -(r-5)(r-1);\ r_1 = 2.$

$$a_n(r) = \frac{n+r-6}{n+r-2}a_{n-1}(r);$$

$a_n(r) = \displaystyle\prod_{j=1}^{m}\frac{j+r-6}{j+r-2},\ n \ge 0$. Therefore,$a_1(2) = -3$, $a_2(2) = 3$, $a_3(2) = -1$, and $a_n(2) = 0$ if $n \ge 4$, so $y_1 = x^2(1-x)^3$.

$a_1(r) = \dfrac{r-5}{r-1}$, $a_1'(r) = \dfrac{4}{(r-1)^2}$, $a_1'(2) = 4$;

$a_2(r) = \dfrac{(r-5)(r-4)}{r(r-1)}$, $a_2'(r) = \dfrac{4(2r^2-10r+5)}{r^2(r-1)^2}$, $a_2'(2) = -7$;

$a_3(r) = \dfrac{(r-5)(r-4)(r-3)}{r(r-1)(r+1)}$, $a_3'(r) = \dfrac{12(r^4-8r^3+16r^2-5)}{r^2(r-1)^2(r+1)^2}$, $a_3'(2) = 11/3$; if $n \geq 4$, then $a_n(r) = (r-2)c_n(r)$ where $c_n(r) = \dfrac{(r-5)(r-4)(r-3)}{(n+r-5)(n+r-4)(n+r-3)(n+r-2)}$, so $a_n'(2) = c_n(2) = -\dfrac{6}{n(n-2)(n^2-1)}$ and

$$y_2 = y_1 \ln x + x^3\left(4 - 7x + \frac{11}{3}x^2 - 6\sum_{n=3}^{\infty} \frac{1}{n(n-2)(n^2-1)}x^n\right).$$

7.6.50. $p_0(r) = (3r-1)^2$; $p_2(r) = 7-3r$; $r_1 = 1/3$.

$a_{2m}(r) = \dfrac{6m+3r-13}{(6m+3r-1)^2}a_{2m-2}(r)$;

$a_{2m}(r) = \displaystyle\prod_{j=1}^{m} \frac{6j+3r-13}{(6j+3r-1)^2}$, $m \geq 0$. Therefore,$a_2(1/3) = 1/6$ and $a_{2m}(1/3) = 0$ if $m \geq 2$, so $y_1 = x^{1/3}\left(1 - \dfrac{1}{6}x^2\right)$.

$a_2(r) = \dfrac{3r-7}{(3r+5)^2}$; $a_2'(r) = \dfrac{3(19-3r)}{(3r+5)^3}$; $a_2'(1/3) = 1/4$. If $m \geq 2$, then $a_{2m}(r) = (r-1/3)c_{2m}(r)$ where $c_{2m}(r) = \dfrac{3(3r-7)}{(6m+3r-7)(6m+3r-1)\prod_{j=1}^{m}(6j+3r-1)}$, so $a_{2m}'(1/3) = c_{2m}(1/3) = -\dfrac{1}{12}\dfrac{1}{6^{m-1}(m-1)m\,m!}$, and

$$y_2 = y_1 \ln x + x^{7/3}\left(\frac{1}{4} - \frac{1}{12}\sum_{m=1}^{\infty} \frac{1}{6^m m(m+1)(m+1)!}x^{2m}\right).$$

7.6.52. $p_0(r) = (2r+1)^2$; $p_2(r) = 7-2r$; $r_1 = -1/2$.

$a_{2m}(r) = \dfrac{4m+2r-11}{(4m+2r+1)^2}a_{2m-2}(r)$;

$a_{2m}(r) = \displaystyle\prod_{j=1}^{m} \frac{4j+2r-11}{(4j+2r+1)^2}$, $m \geq 0$. Therefore,$a_2(-1/2) = -1/2$, $a_4(-1/2) = 1/32$, and $a_{2m}(-1/2) = 0$ if $m \geq 3$, so $y_1 = x^{-1/2}\left(1 - \dfrac{1}{2}x^2 + \dfrac{1}{32}x^4\right)$.

$a_2(r) = \dfrac{2r-7}{(2r+5)^2}$, $a_2'(r) = \dfrac{2(19-2r)}{(2r+5)^3}$, $a_2'(-1/2) = 5/8$,

$a_4(r) = \dfrac{(2r-7)(2r-3)}{(2r+5)^2(2r+9)^2}$, $a_4'(r) = -\dfrac{4(8r^3-60r^2-146r+519)}{(2r+5)^3(2r+9)^3}$, $a_4'(-1/2) = -9/128$; if $m \geq 3$, then $a_{2m}(r) = (r+1/2)c_{2m}(r)$ where

$c_{2m}(r) = \dfrac{2(2r-7)(2r-3)}{(4m+2r-7)(4m+2r-3)(4m+2r+1)\prod_{j=1}^{m}(4j+2r+1)}$, so $a_{2m}'(-1/2) = c_{2m}(-1/2) = \dfrac{1}{4^m(m-2)(m-1)m\,m!}$, and

$$y_2 = y_1 \ln x + x^{3/2}\left(\frac{5}{8} - \frac{9}{128}x^2 + \sum_{m=2}^{\infty} \frac{1}{4^{m+1}(m-1)m(m+1)(m+1)!}x^{2m}\right).$$

7.6.54. **(a)** If $p_0(r) = \alpha_0(r-r_1)^2$, then (A) $a_n(r) = \dfrac{(-1)^n}{\alpha_0^n}\displaystyle\prod_{j=1}^{n} \frac{p_1(j+r-1)}{(j+r-r_1)^2}$. Therefore, $a_n(r_1) = \dfrac{(-1)^n}{\alpha_0^n(n!)^2}\displaystyle\prod_{j=1}^{n} p_1(j+r_1-1)$. Theorem 7.6.2 implies $Ly_1 = 0$.

(b) From (A), $\ln|a_n(r)| = -n\ln|\alpha_0| + \displaystyle\sum_{j=1}^{n}(\ln|p_1(j+r-1)| - 2\ln|j+r-r_1|)$, so $a_n'(r) = a_n(r)\displaystyle\sum_{j=1}^{n}\left(\frac{p_1'(j+r-1)}{p_1(j+r-1)} - \frac{2}{j+r-r_1}\right)$ and $a_n'(r_1) = a_n(r_1)\displaystyle\sum_{j=1}^{n}\left(\frac{p_1'(j+r_1-1)}{p_1(j+r_1-1)} - \frac{2}{j}\right)$. Theorem 7.6.2 implies that $Ly_2 = 0$.

(c) Since $p_1(r) = \gamma_1$, y_1 and y_2 reduce to the stated forms. If $\gamma_1 = 0$, then $y_1 = x^{r_1}$ and $y_2 = x^{r_1}\ln x$, which are solutions of the Euler equation $\alpha_0 x^2 y'' + \beta_0 x y' + \gamma_0 y$.

7.6.54. **(a)** $Ly_1 = p_0(r_1)x^{r_1} = 0$. Now use the fact that $p_0(j+r_1) = \alpha_0 j^2$, so $\prod_{j=1}^{n} p_0(j+r_1) = \alpha_0^n(n!)^2$.

(b) From Theorem 7.6.2, $y_2 = y_1\ln x + x^{r_1}\displaystyle\sum_{n=1}^{\infty} a_n'(r_1)x^n$ is a second solution of $Ly = 0$. Since $a_n(r) = \dfrac{(-1)^n}{\alpha_0^n}\displaystyle\prod_{j=1}^{n}\frac{p_1(j+r-1)}{(j+r-r_1)^2}$, (A) $\ln|a_n(r)| = -n\ln|\alpha_0| + \displaystyle\sum_{j=1}^{n}\ln|p_1(j+r-1)| - 2\sum_{j=1}^{n}\ln|j+r-r_1|$, provided that $p_1(j+r-1)$ and $j+r-r_1$ are nonzero for all positive integers j. Differentiating (A) and then setting $r = r_1$ yields $\dfrac{a_n'(r_1)}{a_n(r_1)} = \displaystyle\sum_{j=1}^{n}\frac{p_1'(j+r_1-1)}{p_2(j+r_1-1)} - 2\sum_{j=1}^{n}\frac{1}{j}$, which implies the conclusion.

(c) In this case $p_1(r) = \gamma_1$ and $p_1'(r) = 0$, so $a_n(r_1) = \dfrac{(-1)^n}{(n!)^2}\left(\dfrac{\gamma_1}{\alpha_0}\right)^n$ and $J_n = -2\displaystyle\sum_{j=1}^{n}\frac{1}{j}$. If $\gamma_1 = 0$, then $y_1 = x^{r_1}$ and $y_2 = x^{r_1}\ln x$, while the differential equation is an Euler equation with indicial polynomial $\alpha_0(r-r_1^2)$. See Theorem 7.4.3.

7.6.56. $p_0(r) = r^2$; $p_1(r) = 1$; $r_1 = 0$. $a_{2m}(r) = -\dfrac{a_{2m-1}(r)}{(2m+r)^2}$, $m \geq 1$; $a_{2m}(r) = \dfrac{(-1)^m}{\prod_{j=1}^{m}(2j+r)^2}$, $m \geq 0$. Therefore,$a_{2m}(0) = \dfrac{(-1)^m}{4^m(m!)^2}$, so $y_1 = \displaystyle\sum_{m=0}^{\infty}\frac{(-1)^m}{4^m(m!)^2}x^{2m}$.

By logarithmic differentiation, $a_{2m}'(r) = -2a_{2m}(r)\displaystyle\sum_{j=1}^{m}\frac{1}{2m+r}$, so $a_{2m}'(0) = -a_{2m}(0)\displaystyle\sum_{j=1}^{m}\frac{1}{j}$ and

$$y_2 = y_1\ln x - \sum_{m=1}^{\infty}\frac{(-1)^m}{4^m(m!)^2}\left(\sum_{j=1}^{m}\frac{1}{j}\right)x^{2m}.$$

7.6.58. $p_0(r) = (2r-1)^2$; $p_1(r) = (2r+1)^2$; $p_2(r) = 0$; $\dfrac{p_1(r-1)}{p_0(r)} = 1 = \dfrac{\alpha_1}{\alpha_0}$; $\dfrac{p_2(r-2)}{p_0(r)} = 0 = \dfrac{\alpha_2}{\alpha_0}$; $y_1 = \dfrac{x^{1/2}}{1+x}$; $y_2 = \dfrac{x^{1/2}\ln x}{1+x}$.

7.6.60. $p_0(r) = 2(r-1)^2$; $p_1(r) = 0$; $p_2(r) = -(r+1)^2$; $\dfrac{p_1(r-1)}{p_0(r)} = 0 = \dfrac{\alpha_1}{\alpha_0}$; $\dfrac{p_2(r-2)}{p_0(r)} = -1/2 = \dfrac{\alpha_2}{\alpha_0}$; $y_1 = \dfrac{x}{2-x^2}$; $y_2 = \dfrac{x\ln x}{2-x^2}$.

7.6.62. $p_0(r) = 4(r-1)^2$; $p_1(r) = 3r^2$; $p_2(r) = 0$; $\dfrac{p_1(r-1)}{p_0(r)} = 3/4 = \dfrac{\alpha_1}{\alpha_0}$; $\dfrac{p_2(r-2)}{p_0(r)} = 0 = \dfrac{\alpha_2}{\alpha_0}$; $y_1 = \dfrac{x}{4+3x}$; $y_2 = \dfrac{x\ln x}{4+3x}$.

7.6.64. $p_0(r) = (r-1)^2$; $p_1(r) = -2r^2$; $p_2(r) = (r+1)^2$; $\dfrac{p_1(r-1)}{p_0(r)} = -2 = \dfrac{\alpha_1}{\alpha_0}$; $\dfrac{p_2(r-2)}{p_0(r)} = 1 = \dfrac{\alpha_2}{\alpha_0}$; $y_1 = \dfrac{x}{(1-x)^2}$; $y_2 = \dfrac{x\ln x}{(1-x)^2}$.

7.6.66. See the proofs of Theorems 7.6.1 and 7.6.2.

7.7 THE METHOD OF FROBENIUS III

7.7.2. $p_0(r) = r(r-1)$; $p_1(r) = 1$; $r_1 = 1$; $r_2 = 0$; $k = r_1 - r_2 = 1$;

$$a_n(r) = -\frac{1}{(n+r)(n+r-1)}a_{n-1}(r);$$

$$a_n(r) = \frac{(-1)^n}{\prod_{j=1}^n (j+r)(j+r-1)};$$

$$a_n(1) = \frac{(-1)^n}{n!(n+1)!};$$

$$y_1 = x\sum_{n=0}^\infty \frac{(-1)^n}{n!(n+1)!}x^n;$$

$z = 1$; $C = -p_1(0)a_0(0) = -1$.

By logarithmic differentiation,

$$a_n'(r) = -a_n(r)\sum_{j=1}^n \frac{2n+2r-1}{(n+r)(n+r-1)};$$

$$a_n'(1) = -a_n(1)\sum_{j=1}^n \frac{2j+1}{j(j+1)};$$

$$y_2 = 1 - y_1\ln x + x\sum_{n=1}^\infty \frac{(-1)^n}{n!(n+1)!}\left(\sum_{j=1}^n \frac{2j+1}{j(j+1)}\right)x^n.$$

7.7.4. $p_0(r) = r(r-1)$; $p_1(r) = r+1$; $r_1 = 1$; $r_2 = 0$; $k = r_1 - r_2 = 1$; $a_n(r) = -\dfrac{a_{n-1}(r)}{n+r-1}$; $a_n(r) = \dfrac{(-1)^n}{\prod_{j=1}^n (j+r-1)}$; $a_n(1) = \dfrac{(-1)^n}{n!}$; $y_1 = x\displaystyle\sum_{n=0}^\infty \frac{(-1)^n}{n!}x^n = xe^{-x}$; $z = 1$; $C = -p_1(0)a_0(0) = -1$. By logarithmic differentiation, $a_n'(r) = -a_n(r)\displaystyle\sum_{j=1}^n \frac{1}{j+r-1}$; $a_n'(1) = -a_n(1)\displaystyle\sum_{j=1}^n \frac{1}{j}$; $y_2 = 1 - y_1\ln x + x\displaystyle\sum_{n=1}^\infty \frac{(-1)^n}{n!}\left(\sum_{j=1}^n \frac{1}{j}\right)$

7.7.6. $p_0(r) = (r-1)(r+2)$; $p_1(r) = r+3$; $r_1 = 1$; $r_2 = -2$; $k = r_1 - r_2 = 3$. $a_n(r) = -\dfrac{1}{n+r-1}a_{n-1}(r)$; $a_n(r) = \dfrac{(-1)^n}{\prod_{j=1}^n (j+r-1)}$; $a_n(1) = \dfrac{(-1)^n}{n!}$; $y_1 = x\displaystyle\sum_{n=0}^\infty \frac{(-1)^n}{n!}x^n = xe^{-x}$;

$z = x^{-2}\left(1+\frac{1}{2}x+\frac{1}{2}x^2\right)$; $C = -\frac{p_1(0)}{3}a_2(-2) = -1/2$. By logarithmic differentiation, $a_n'(r) = -a_n(r)\sum_{j=1}^{n}\frac{1}{j+r-1}$; $a_n'(1) = a_n(1)\sum_{j=1}^{n}\frac{1}{j}$; $y_2 = x^{-2}\left(1+\frac{1}{2}x+\frac{1}{2}x^2\right) - \frac{1}{2}\left(y_1 \ln x - x\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\left(\sum_{j=1}^{n}\frac{1}{j}\right)x^n\right)$;

7.7.8. $p_0(r) = (r+2)(r+7)$; $p_1(r) = 1$; $r_1 = -2$; $r_2 = -7$; $k = r_1 - r_2 = 5$; $a_n(r) = -\frac{a_{n-1}(r)}{(n+r+2)(n+r+7)}$; $a_n(r) = \frac{(-1)^n}{\prod_{j=1}^{n}(j+r+2)(j+r+7)}$; $a_n(-2) = 120\prod_{j=1}^{n}\frac{(-1)^n}{n!(n+5)!}$;

$y_1 = \frac{120}{x^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(n+5)!}x^n$;

$z = x^{-7}\left(1+\frac{1}{4}x+\frac{1}{24}x^2+\frac{1}{144}x^3+\frac{1}{576}x^4\right)$; $C = -\frac{p_1(-3)}{5}a_4(-7) = -1/2880$. By logarithmic differentiation, $a_n'(r) = -a_n(r)\sum_{j=1}^{n}-\frac{2j+2r+9}{(j+r+2)(j+r+7)}$; $a_n'(-2) = -a_n(-2)\sum_{j=1}^{n}\frac{2j+5}{j(j+5)}$;

$$y_2 = x^{-7}\left(1+\frac{1}{4}x+\frac{1}{24}x^2+\frac{1}{144}x^3+\frac{1}{576}x^4\right) - \frac{1}{2880}\left(y_1 \ln x - \frac{120}{x^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n!(n+5)!}\left(\sum_{j=1}^{n}\frac{2j+5}{j(j+5)}\right)x^n\right).$$

7.7.10. $p_0(r) = r(r-4)$; $p_1(r) = (r-6)(r-5)$; $r_1 = 4$; $r_2 = 0$; $k = r_1 - r_2 = 4$; $a_n(r) = -\frac{(n+r-7)(n+r-6)}{(n+r)(n+r-4)}a_{n-1}(r)$; $a_n(r) = (-1)^n\prod_{j=1}^{n}\frac{(j+r-7)(j+r-6)}{(j+r)(j+r-4)}$. Setting $r = 4$ yields $y_1 = x^4\left(1-\frac{2}{5}x\right)$. $z = 1+10x+50x^2+200x^3$; $C = -\frac{p_1(3)}{4}a_3(0) = -300$. $a_1(r) = -\frac{(r-6)(r-5)}{(r-3)(r+1)}$; $a_1'(r) = -\frac{3(3r^2-22r+31)}{(r-3)^2(r+1)^2}$; $a_1'(4) = 27/25$. $a_2(r) = (r-4)c_2(r)$, with $c_2(r) = \frac{(r-6)(r-5)^2}{(r-3)(r-2)(r+1)(r+2)}$, so $a_2'(4) = c_2(4) = -1/30$. If $n \ge 3$, then $a_n(r) = (r-4)^2 b_n(r)$ where $b_n'(4)$ exists, so $a_n'(4) = 0$ and $y_2 = 1+10x+50x^2+200x^3 - 300\left(y_1 \ln x + \frac{27}{25}x^5 - \frac{1}{30}x^6\right)$.

7.7.12. $p_0(r) = (r-2)(r+2)$; $p_1(r) = -2r-1$; $r_1 = 2$; $r_2 = -2$; $k = r_1 - r_2 = 4$; $a_n(r) = \frac{2j+2r-1}{(j+r-2)(j+r+2)}a_{n-1}(r)$; $a_n(r) = \prod_{j=1}^{n}\frac{2n+2r-1}{(n+r-2)(n+r+2)}$; $a_n(2) = \frac{1}{n!}\left(\prod_{j=1}^{n}\frac{2j+3}{j+4}\right)$;

$y_1 = x^2\sum_{n=0}^{\infty}\frac{1}{n!}\left(\prod_{j=1}^{n}\frac{2j+3}{j+4}\right)x^n$; $z = x^{-2}\left(1+x+\frac{1}{4}x^2-\frac{1}{12}x^3\right)$; $C = -\frac{p_1(1)}{4}a_3(-2) = -1/16$.

By logarithmic differentiation,

$$a_n'(r) = -2a_n(r)\sum_{j=1}^{n}\frac{j^2+j(2r-1)+r^2-r+4}{(j+r-2)(j+r+2)(2j+2r-1)};\ a_n'(2) = -2a_n(2)\sum_{j=1}^{n}\frac{(j^2+3j+6)}{j(j+4)(2j+3)};$$

$$y_2 = x^{-2}\left(1+x+\frac{1}{4}x^2-\frac{1}{12}x^3\right) - \frac{1}{16}y_1\ln x + \frac{x^2}{8}\sum_{n=1}^{\infty}\frac{1}{n!}\left(\prod_{j=1}^{n}\frac{2j+3}{j+4}\right)\left(\sum_{j=1}^{n}\frac{(j^2+3j+6)}{j(j+4)(2j+3)}\right)x^n.$$

7.7.14. $p_0(r) = (r+1)(r+7)$; $p_1(r) = (r+5)(2r+1)$; $r_1 = -1$; $r_2 = -7$; $k = r_1 - r_2 = 6$; $a_n(r) = -\frac{(n+r+4)(2n+2r-1)}{(n+r+1)(n+r+7)}a_{n-1}(r)$; $a_n(r) = (-1)^n\prod_{j=1}^{n}\frac{(j+r+4)(2j+2r-1)}{(j+r+1)(j+r+7)}$;

$a_n(-1) = \frac{(-1)^n}{n!}\left(\prod_{j=1}^{n}\frac{(j+3)(2j-3)}{j+6}\right)$; $y_1 = \frac{1}{x}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left(\prod_{j=1}^{n}\frac{(j+3)(2j-3)}{j+6}\right)x^n$; $z = x^{-7}\left(1+\frac{26}{5}x+\frac{143}{20}x^2\right)$; $C = -\frac{p_1(-2)}{6}a_5(-7) = 0$; $y_2 = x^{-7}\left(1+\frac{26}{5}x+\frac{143}{20}x^2\right)$.

7.7.16. $p_0(r) = (3r-10)(3r+2)$; $p_1(r) = r(3r-4)$; $r_1 = 10/3$; $r_2 = -2/3$; $k = r_1 - r_2 = 4$; $a_n(r) = -\frac{(n+r-1)(3n+3r-7)}{(3n+3r-10)(3n+3r+2)}a_{n-1}(r)$; $a_n(r) = (-1)^n\prod_{j=1}^{n}\frac{(j+r-1)(3j+3r-7)}{(3j+3r-10)(3j+3r+2)}$; $a_n(10/3) = \frac{(-1)^n(n+1)}{9^n}\left(\prod_{j=1}^{n}\frac{3j+7}{j+4}\right)$; $y_1 = x^{10/3}\sum_{n=0}^{\infty}\frac{(-1)^n(n+1)}{9^n}\left(\prod_{j=1}^{n}\frac{3j+7}{j+4}\right)x^n$; $z = x^{-2/3}\left(1+\frac{4}{27}x-\frac{1}{243}x^2\right)$; $C = -\frac{p_1(7/3)}{36}a_3(-2/3) = 0$; $y_2 = x^{-2/3}\left(1+\frac{4}{27}x-\frac{1}{243}x^2\right)$.

7.7.18. $p_0(r) = (r-3)(r+2)$; $p_1(r) = (r+1)^2$; $r_1 = 3$; $r_2 = -2$; $k = r_1 - r_2 = 5$; $a_n(r) = -\frac{(n+r)^2}{(n+r-3)(n+r+2)}a_{n-1}(r)$; $a_n(r) = (-1)^n\prod_{j=1}^{n}\frac{(j+r)^2}{(j+r-3)(j+r+2)}$; $a_n(3) = \frac{(-1)^n}{n!}\left(\prod_{j=1}^{n}\frac{(j+3)^2}{j+5}\right)$; $y_1 = x^3\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left(\prod_{j=1}^{n}\frac{(j+3)^2}{j+5}\right)x^n$; $z = x^{-2}\left(1+\frac{1}{4}x\right)$; $C = -\frac{p_1(2)}{5}a_4(-2) = 0$; $y_2 = x^{-2}\left(1+\frac{1}{4}x\right)$.

7.7.20. $p_0(r) = (r-6)(r-1)$; $p_1(r) = (r-8)(r-4)$; $r_1 = 6$; $r_2 = 1$; $k = r_1 - r_2 = 5$; $a_n(r) = -\frac{(n+r-9)(n+r-5)}{(n+r-6)(n+r-1)}a_{n-1}(r)$; $y_1 = x^6\left(1+\frac{2}{3}x+\frac{1}{7}x^2\right)$; $z = x\left(1+\frac{21}{4}x+\frac{21}{2}x^2+\frac{35}{4}x^3\right)$; $C = -\frac{p_1(5)}{6}a_5(1) = 0$; $y_2 = x\left(1+\frac{21}{4}x+\frac{21}{2}x^2+\frac{35}{4}x^3\right)$.

7.7.22. $p_0(r) = r(r-10)$; $p_1(r) = 2(r-6)(r+1)$; $r_1 = 10$; $r_2 = 0$; $k = r_1 - r_2 = 10$; $a_n(r) = -\frac{2(n+r-7)}{n+r-10}a_{n-1}(r)$; $a_n(r) = (-2)^n\frac{(n+r-9)(n+r-8)(n+r-7)}{(r-9)(r-8)(r-7)}$; $a_n(10) = \frac{(-1)^n2^n(n+1)(n+2)(n+3)}{6}$; $y_1 = \frac{x^{10}}{6}\sum_{n=0}^{\infty}(-1)^n2^n(n+1)(n+2)(n+3)x^n$; $z = \left(1-\frac{4}{3}x+\frac{5}{3}x^2-\frac{40}{21}x^3+\frac{40}{21}x^4-\frac{32}{21}x^5+\frac{16}{21}x^6\right)$; $C = -\frac{p_1(9)}{10}a_9(0) = 0$; $y_2 = \left(1-\frac{4}{3}x+\frac{5}{3}x^2-\frac{40}{21}x^3+\frac{40}{21}x^4-\frac{32}{21}x^5+\frac{16}{21}x^6\right)$.

Note: in the solutions to Exercises 7.7.23–7.7.40, $z = x^{r_2}\sum_{m=0}^{k-1}a_{2m}(r_2)x^{2m}$.

7.7.24. $p_0(r) = (r-6)(r-2)$; $p_2(r) = r$; $r_1 = 6$; $r_2 = 2$; $k = (r_1 - r_2)/2 = 2$; $a_{2m}(r) = -\frac{a_{2m-2}(r)}{2m+r-6}$; $a_{2m}(r) = \frac{(-1)^m}{\prod_{j=1}^{m}(2j+r-6)}$; $a_{2m}(6) = \frac{(-1)^m}{2^m m!}$; $y_1 = x^6\sum_{m=0}^{\infty}\frac{(-1)^m}{2^m m!}x^{2m} = x^6e^{-x^2/2}$; $z = x^2\left(1+\frac{1}{2}x^2\right)$; $C = -\frac{p_2(4)}{4}a_2(2) = -1/2$. By logarithmic differentiation, $a'_{2m}(r) = -a_{2m}(r)\sum_{j=1}^{m}\frac{1}{2j+r-6}$; $a'_{2m}(6) = -a_{2m}(6)\sum_{j=1}^{m}\frac{1}{2j}$; $y_2 = x^2\left(1+\frac{1}{2}x^2\right) - \frac{1}{2}y_1\ln x + \frac{x^6}{4}\sum_{m=1}^{\infty}\frac{(-1)^m}{2^m m!}\left(\sum_{j=1}^{m}\frac{1}{j}\right)x^{2m}$.

7.7.26. $p_0(r) = (r-1)(r+1)$; $p_2(r) = 2r+10$; $r_1 = 1$; $r_2 = -1$; $k = (r_1 - r_2)/2 = 1$; $a_{2m}(r) = -\dfrac{2(2m+r+3)}{(2m+r-1)(2m+r+1)}a_{2m-2}(r)$; $a_{2m}(r) = (-2)^m \displaystyle\prod_{j=1}^{m} \dfrac{2j+r+3}{(2j+r-1)(2j+r+1)}$; $a_{2m}(1) = \dfrac{(-1)^m(m+2)}{2\,m!}$; $y_1 = \dfrac{x}{2}\displaystyle\sum_{m=0}^{\infty}\dfrac{(-1)^m(m+2)}{m!}x^{2m}$; $z = x^{-1}$; $C = -\dfrac{p_2(-1)}{2}a_0(-1) = -4$.
By logarithmic differentiation,

$$a'_{2m}(r) = -a_{2m}(r)\sum_{j=1}^{m}\frac{(4j^2+4j(r+3)+r^2+6r+1)}{(2j+r-1)(2j+r+1)(2j+r+3)};$$

$$a'_{2m}(1) = -a_{2m}(1)\sum_{j=1}^{m}\frac{j^2+4j+2}{2j(j+1)(j+2)};$$

$$y_2 = x^{-1} - 4y_1\ln x + x\sum_{m=1}^{\infty}\frac{(-1)^m(m+2)}{m!}\left(\sum_{j=1}^{m}\frac{j^2+4j+2}{j(j+1)(j+2)}\right)x^{2m}.$$

7.7.28. $p_0(r) = (2r+1)(2r+5)$; $p_2(r) = 2r+3$; $r_1 = -1/2$; $r_2 = -5/2$; $k = (r_1 - r_2)/2 = 1$; $a_{2m}(r) = -\dfrac{(4m+2r-1)}{(4m+2r+1)(4m+2r+5)}a_{2m-2}(r)$; $a_{2m}(r) = (-1)^m\displaystyle\prod_{j=1}^{m}\dfrac{(4j+2r-1)}{(4j+2r+1)(4j+2r+5)}$; $a_{2m}(-1/2) = \dfrac{(-1)^m\prod_{j=1}^{m}(2j-1)}{8^m m!(m+1)!}$; $y_1 = x^{-1/2}\displaystyle\sum_{m=0}^{\infty}\dfrac{(-1)^m\prod_{j=1}^{m}(2j-1)}{8^m m!(m+1)!}x^{2m}$; $z = x^{-5/2}$; $C = -\dfrac{p_2(-5/2)}{8}a_0(-5/2) = 1/4$. By logarithmic differentiation,

$$a'_{2m}(r) = -2a_{2m}(r)\sum_{j=1}^{m}\frac{(16j^2+8j(2r-1)+4r^2-4r-11)}{(4j+2r-1)(4j+2r+1)(4j+2r+5)};\ a'_{2m}(-1/2) = -a_{2m}(-1/2)\sum_{j=1}^{m}\frac{2j^2-2j-1}{2j(j+1)(2j-1)};$$

$$y_2 = x^{-5/2} + \frac{1}{4}y_1\ln x - x^{-1/2}\sum_{m=1}^{\infty}\frac{(-1)^m\prod_{j=1}^{m}(2j-1)}{8^{m+1}m!(m+1)!}\left(\sum_{j=1}^{m}\frac{2j^2-2j-1}{j(j+1)(2j-1)}\right)x^{2m}.$$

7.7.30. $p_0(r) = (r-2)(r+2)$; $p_2(r) = -2(r+4)$; $r_1 = 2$; $r_2 = -2$; $k = (r_1 - r_2)/2 = 2$; $a_{2m}(r) = \dfrac{2}{2m+r-2}a_{2m-2}(r)$; $a_{2m}(r) = \dfrac{2^m}{\prod_{j=1}^{m}(2j+r-2)}$; $a_{2m}(2) = \dfrac{1}{m!}$; $y_1 = x^2\displaystyle\sum_{m=0}^{\infty}\dfrac{1}{m!}x^{2m} = x^2e^{x^2}$; $z = x^{-2}(1-x^2)$; $C = -\dfrac{p_2(0)}{4}a_2(-2) = -2$. By logarithmic differentiation, $a'_{2m}(r) = -a_{2m}(r)\displaystyle\sum_{j=1}^{m}\dfrac{1}{2j+r-2}$;

$$a'_{2m}(2) = -a_{2m}(2)\sum_{j=1}^{m}\frac{1}{2j};\ y_2 = x^{-2}(1-x^2) - 2y_1\ln x + x^2\sum_{m=1}^{\infty}\frac{1}{m!}\left(\sum_{j=1}^{m}\frac{1}{j}\right)x^{2m}.$$

7.7.32. $p_0(r) = (3r-13)(3r-1)$; $p_2(r) = 2(5-3r)$; $r_1 = 13/3$; $r_2 = 1/3$; $k = (r_1 - r_2)/2 = 2$; $a_{2m}(r) = \dfrac{2(6m+3r-11)}{(6m+3r-13)(6m+3r-1)}a_{2m-2}(r)$; $a_{2m}(r) = 2^m\displaystyle\prod_{j=1}^{m}\dfrac{(6j+3r-11)}{(6j+3r-13)(6j+3r-1)}$; $a_{2m}(13/3) = \dfrac{\prod_{j=1}^{m}(3j+1)}{9^m m!(m+2)!}$; $y_1 = 2x^{13/3}\displaystyle\sum_{m=0}^{\infty}\dfrac{\prod_{j=1}^{m}(3j+1)}{9^m m!(m+2)!}x^{2m}$; $z = x^{1/3}\left(1+\dfrac{2}{9}x^2\right)$; $C = -\dfrac{p_2(7/3)}{36}a_2(1/3) = 2/81$. By logarithmic differentiation, $a'_{2m}(r) = -9a_{2m}(r)\displaystyle\sum_{j=1}^{m}\dfrac{(12j^2+4j(3r-11)+3r^2-22r+47)}{(6j+3r-13)(6j+3r-11)(6j+3r-1)}$

$a'_{2m}(13/3) = -a_{2m}(13/3)\sum_{j=1}^{m}\frac{3j^2+2j+2}{2j(j+2)(3j+1)}$;

$$y_2 = x^{1/3}\left(1+\frac{2}{9}x^2\right)+\frac{2}{81}\left(y_1\ln x - x^{13/3}\sum_{m=0}^{\infty}\frac{\prod_{j=1}^{m}(3j+1)}{9^m m!(m+2)!}\left(\sum_{j=1}^{m}\frac{3j^2+2j+2}{j(j+2)(3j+1)}\right)x^{2m}\right).$$

7.7.34. $p_0(r) = (r-2)(r+2)$; $p_2(r) = -3(r-4)$; $r_1 = 2$; $r_2 = -2$; $k = (r_1-r_2)/2 = 2$; $a_{2m}(r) = \frac{3(2m+r-6)}{(2m+r-2)(2m+r+2)}a_{2m-2}(r)$; $y_1 = x^2\left(1-\frac{1}{2}x^2\right)$; $z = x^{-2}\left(1+\frac{9}{2}x^2\right)$; $C = -\frac{p_2(0)}{4}a_2(-2) = -27/2$; $a_2(r) = \frac{3(r-4)}{r(r+4)}$, $a'_2(r) = -\frac{3(r^2-8r-16)}{r^2(r+4)^2}$, $a'_2(2) = 7/12$. If $m \ge 2$, then $a_{2m}(r) = (r-2)c_{2m}(r)$ where $c_{2m}(r) = \frac{3^m(r-4)}{(2m+r-4)(2m+r-2)\prod_{j=1}^{m}(2j+r+2)}$, so $a'_{2m}(2) = c_{2m}(2) = -\frac{\left(\frac{3}{2}\right)^m}{m(m-1)(m+2)!}$; $y_2 = x^{-2}\left(1+\frac{9}{2}x^2\right)-\frac{27}{2}\left(y_1\ln x+\frac{7}{12}x^4-x^2\sum_{m=2}^{\infty}\frac{\left(\frac{3}{2}\right)^m}{m(m-1)(m+2)!}x^{2m}\right)$.

7.7.36. $p_0(r) = (2r-5)(2r+7)$; $p_2(r) = (2r-1)^2$; $r_1 = 5/2$; $r_2 = -7/2$; $k = (r_1-r_2)/2 = 3$; $a_{2m}(r) = -\frac{4m+2r-5}{4m+2r+7}a_{2m-2}(r)$; $a_{2m}(r) = \frac{(2r-1)(2r+3)(2r+7)}{(4m+2r-1)(4m+2r+3)(4m+2r+7)}$; $a_{2m}(5/2) = \frac{(-1)^m}{(m+1)(m+2)(m+3)}$; $y_1 = x^{5/2}\sum_{m=0}^{\infty}\frac{(-1)^m}{(m+1)(m+2)(m+3)}x^{2m}$; $z = x^{-7/2}(1+x^2)^2$ $C = -\frac{p_2(1/2)}{24}a_4(-7/2) = 0$; $y_2 = x^{-7/2}(1+x^2)^2$.

7.7.38. $p_0(r) = (r-3)(r+7)$; $p_2(r) = r(r+1)$; $r_1 = 3$; $r_2 = -7$; $k = (r_1-r_2)/2 = 5$; $a_{2m}(r) = -\frac{(2m+r-2)(2m+r-1)}{(2m+r-3)(2m+r+7)}a_{2m-2}(r)$; $a_{2m}(r) = (-1)^m\prod_{j=1}^{m}\frac{(2j+r-2)(2j+r-1)}{(2j+r-3)(2j+r+7)}$; $a_{2m}(3) = (-1)^m\frac{m+1}{2^m}\left(\prod_{j=1}^{m}\frac{2j+1}{j+5}\right)$; $y_1 = x^3\sum_{m=0}^{\infty}(-1)^m\frac{m+1}{2^m}\left(\prod_{j=1}^{m}\frac{2j+1}{j+5}\right)x^{2m}$; $z = x^{-7}\left(1+\frac{21}{8}x^2+\frac{35}{16}x^4+\frac{35}{64}x^6\right)$ $C = -\frac{p_2(1)}{10}a_8(-7) = 0$; $y_2 = x^{-7}\left(1+\frac{21}{8}x^2+\frac{35}{16}x^4+\frac{35}{64}x^6\right)$.

7.7.40. $p_0(r) = (2r-3)(2r+5)$; $p_2(r) = (2r-1)(2r+1)$; $r_1 = 3/2$; $r_2 = -5/2$; $k = (r_1-r_2)/2 = 2$; $a_{2m}(r) = -\frac{4m+2r-5}{4m+2r+5}a_{2m-2}(r)$; $a_{2m}(r) = (-1)^m\prod_{j=1}^{m}\frac{4j+2r-5}{4j+2r+5}$; $a_{2m}(3/2) = \frac{(-1)^m\prod_{j=1}^{m}(2j-1)}{2^{m-1}(m+2)!}$; $y_1 = x^{3/2}\sum_{m=0}^{\infty}\frac{(-1)^m\prod_{j=1}^{m}(2j-1)}{2^{m-1}(m+2)!}x^{2m}$; $z = x^{-5/2}\left(1+\frac{3}{2}x^2\right)$ $C = -\frac{p_2(-1/2)}{16}a_2(-5/2) = 0$; $y_2 = x^{-5/2}\left(1+\frac{3}{2}x^2\right)$.

7.7.42. $p_0(r) = r^2-\nu^2$; $p_2(r) = 1$; $r_1 = \nu$; $r_2 = -\nu$; $k = (r_1-r_2)/2 = \nu$; $a_{2m}(r) = -\frac{a_{2m-2}(r)}{(2m+r+\nu)(2m+r-\nu)}$; $a_{2m}(r) = \frac{(-1)^m}{\prod_{j=1}^{m}(2j+r+\nu)(2j+r-\nu)}$; $a_{2m}(\nu) = \frac{(-1)^m}{4^m m!\prod_{j=1}^{m}(j+\nu)}$; $a_{2m}(-\nu) = \frac{(-1)^m}{4^m m!\prod_{j=1}^{m}(j-\nu)}$, $j = 0,\dots,\nu-1$; $y_1 = x^\nu\sum_{m=0}^{\infty}\frac{(-1)^m}{4^m m!\prod_{j=1}^{m}(j+\nu)}x^{2m}$; $z =$

$x^{-\nu}\sum_{m=0}^{\nu-1}\frac{(-1)^m}{4^m m!\prod_{j=1}^m (j-\nu)}x^{2m}$; $C=-\frac{p_2(\nu-2)}{2\nu}a_{2\nu-2}(-\nu)=-\frac{a_{2\nu-2}(-\nu)}{2\nu}=-\frac{2}{4^\nu \nu!(\nu-1)!}$. By logarithmic differentiation, $a_{2m}'(r)=-2a_{2m}(r)\sum_{j=1}^{m}\frac{2j+\nu}{(2j+r+\nu)(2j+r-\nu)}$; $a_{2m}'(\nu)=-a_{2m}(\nu)\sum_{j=1}^{m}\frac{2j+\nu}{2j(j+\nu)}$;

$$y_2=x^{-\nu}\sum_{m=0}^{\nu-1}\frac{(-1)^m}{4^m m!\prod_{j=1}^m (j-\nu)}x^{2m}-\frac{2}{4^\nu \nu!(\nu-1)!}\left(y_1\ln x-\frac{x^\nu}{2}\sum_{m=1}^{\infty}\frac{(-1)^m}{4^m m!\prod_{j=1}^m (j+\nu)}\left(\sum_{j=1}^{m}\frac{2j+\nu}{j(j+\nu)}\right)x^{2m}\right).$$

7.7.44. Since $a_n(r_2)=-\frac{p_1(n+r_2-1)}{p_0(n+r_2)}a_{n-1}(r_2)$, $1\le n\le k-1$, $a_{k-1}(r_2)=(-1)^{k-1}\prod_{j=1}^{k-1}\frac{p_1(r_2+j-1)}{p_0(r_2+j)}$. But $C=-\frac{p_1(r_1-1)}{k\alpha_0}a_{k-1}(r_2)=-\frac{p_1(r_2+k-1)}{k\alpha_0}a_{k-1}(r_2)=(-)^k\frac{\prod_{j=1}^k p_1(r_2+j-1)}{k\alpha_0\prod_{j=1}^{k-1}p_0(r_2+j)}=0$ if and only if $\prod_{j=1}^{k}p_1(r_2+j-1)=0$.

7.7.46. Since $p_1(r)=\gamma_1$, $a_n(r)=-\frac{\gamma_1}{\alpha_0(n+r-r_1)(n+r-r_2)}a_{n-1}(r)$ and (A) $a_n(r)=(-1)^n\left(\frac{\gamma_1}{\alpha_0}\right)^n\frac{1}{\prod_{j=1}^n (j+r-r_1)(j+$ Therefore, $a_n(r_1)=\frac{(-1)^n}{n!}\left(\frac{\gamma_1}{\alpha_0}\right)^n\frac{1}{\prod_{j=1}^n (j+k)}$ for $n\ge 0$ (so $Ly_1=0$) and $a_n(r_2)=\frac{(-1)^n}{n!}\left(\frac{\gamma_1}{\alpha_0}\right)^n\frac{1}{\prod_{j=1}^n (j-k)}$ for $n=0,\dots,k-1$. $Ly_2=0$ if $y_2=x^{r_2}\sum_{n=0}^{k-1}a_n(r_2)x^n+C\left(y_1\ln x+x^{r_1}\sum_{n=1}^{\infty}a_n'(r_1)x^n\right)$ if $C=-\frac{\gamma_1}{k\alpha_0}a_{k-1}(r_2)=-\frac{\gamma_1}{k\alpha_0}\frac{(-1)^{k-1}}{(k-1)!}\left(\frac{\gamma_1}{\alpha_0}\right)^{k-1}\frac{(-1)^{k-1}}{(k-1)!}=-\frac{1}{k!(k-1)!}\left(\frac{\gamma_1}{\alpha_0}\right)^k$. From (A), $\ln|a_n(r)|=-n\ln\left|\frac{\gamma_1}{\alpha_0}\right|-\sum_{j=1}^{n}((\ln|j+r-r_1|+\ln|j+r-r_2|)$, so $a_n'(r)=-a_n(r)\sum_{j=1}^{n}\left(\frac{1}{j+r-r_1}+\frac{1}{j+r-r_2}\right)$ and $a_n'(r_1)=-a_n(r_1)\sum_{j=1}^{n}\frac{2j+k}{j(j+k)}$.

7.7.48. **(a)** From Exercise 7.6.66**(a)** of Section 7.6, $L\left(\frac{\partial y}{\partial r}(x,r)\right)=p_0'(r)x^r+x^r p_0(r)\ln x$. Setting $r=r_1$ yields $L\left(y_1\ln x+x^{r_1}\sum_{n=1}^{\infty}a_n'(r_1)\right)=p_0'(r_1)x^{r_1}$. Since $p_0'(r)=\alpha_0(2r-r_1-r_2)$, $p_0'(r_1)=k\alpha_0$.

(b) From Exercise 7.5.57 of Section 7.5, $L\left(x^{r_2}\sum_{n=0}^{\infty}a_n(r_2)x^n\right)=x^{r_2}\sum_{n=0}^{\infty}b_nx^n$, where $b_0=p_0(r_2)=0$ and $b_n=\sum_{j=0}^{n}p_j(n+r_2-j)a_{n-j}(r_2)$ if $n\ge 1$. From the definition of $\{a_n(r_2)\}$, $b_n=0$ if $n\ne k$, while $b_k=\sum_{j=0}^{k}p_k(k+r_2-j)a_{k-j}(r_2)=\sum_{j=1}^{k}p_j(r_1-j)a_{k-j}(r_2)$.

(d) Let $\{\tilde a_n(r_2)\}$ be the coefficients that would obtained if $\tilde a_k(r_2)=0$. Then $a_n(r_2)=\tilde a_n(r_2)$ if $n=$

$0, \dots, k-1$, and (A) $a_n(r_2) - \tilde{a}_n(r_2) = -\dfrac{1}{p_0(n+r_2)} \displaystyle\sum_{j=0}^{n-k} p_j(n+r_2-j)(a_{n-j}(r_2) - \tilde{a}_{n-j} r_2)$ if $n > k$. Now let $c_m = a_{k+m}(r_2) - \tilde{a}_{k+m}(r_2)$. Setting $n = m+k$ in (A) and recalling the $k + r_2 = r_1$ yields (B) $c_m = -\dfrac{1}{p_0(m+r_1)} \displaystyle\sum_{j=0}^{m} p_j(m+r_1-j)c_{m-j}$. Since $c_k = a_k(r_2)$, (B) implies that $c_m = a_k(r_2)a_m(r_1)$ for all $m \geq 0$, which implies the conclusion.

CHAPTER 8
Laplace Transforms

8.1 INTRODUCTION TO THE LAPLACE TRANSFORM

8.1.2. **(a)** $\cosh t \sin t = \dfrac{1}{2}\left(e^t \sin t + e^{-t}\sin t\right) \leftrightarrow \dfrac{1}{2}\left[\dfrac{1}{(s-1)^2+1} + \dfrac{1}{(s+1)^2+1}\right] = \dfrac{s^2+2}{[(s-1)^2+1][(s+1)^2+1]}$.

(b) $\sin^2 t = \dfrac{1-\cos 2t}{2} \leftrightarrow \dfrac{1}{2}\left[\dfrac{1}{s} - \dfrac{s}{(s^2+4)}\right] = \dfrac{2}{s(s^2+4)}$.

(c) $\cos^2 2t = \dfrac{1}{2}\left[\dfrac{1}{s} + \dfrac{s}{s^2+16}\right] = \dfrac{s^2+8}{s(s^2+16)}$.

(d) $\cosh^2 t = \dfrac{(e^t+e^{-t})^2)}{4} = \dfrac{(e^{2t}+2+e^{-2t})}{4} \leftrightarrow \dfrac{1}{4}\left(\dfrac{1}{s-2} + \dfrac{2}{s} + \dfrac{1}{s+2}\right) = \dfrac{s^2-2}{s(s^2-4)}$.

(e) $t \sinh 2t = \dfrac{te^{2t} - te^{-2t}}{2} \leftrightarrow \dfrac{1}{2}\left(\dfrac{1}{(s-2)^2} - \dfrac{1}{(s+2)^2}\right) = \dfrac{4s}{(s^2-4)^2}$.

(f) $\sin t \cos t = \dfrac{\sin 2t}{2} \leftrightarrow \dfrac{1}{s^2+4}$.

(g) $\sin(t+\pi/4) = \sin t \cos(\pi/4) + \cos t \cos(\pi/4) \leftrightarrow \dfrac{1}{\sqrt{2}}\dfrac{s+1}{s^2+1}$.

(h) $\cos 2t - \cos 3t \leftrightarrow \dfrac{s}{s^2+4} - \dfrac{s}{s^2+9} = \dfrac{5s}{(s^2+4)(s^2+9)}$.

(i) $\sin 2t + \cos 4t \leftrightarrow \dfrac{2}{s^2+4} + \dfrac{s}{s^2+16} = \dfrac{s^3+2s^2+4s+32}{(s^2+4)(s^2+16)}$.

8.1.6. If $F(s) = \displaystyle\int_0^\infty e^{-st} f(t)\,dt$, then $F'(s) = \displaystyle\int_0^\infty (-te^{-st}) f(t)\,dt = -\int_0^\infty e^{-st}(tf(t))\,dt$. Applying this argument repeatedly yields the assertion.

8.1.8. Let $f(t) = 1$ and $F(s) = 1/s$. From Exercise 8.1.6, $t^n \leftrightarrow (-1)^n F^{(n)}(s) = n!/s^{n+1}$.

8.1.10. If $|f(t)| \le Me^{s_0 t}$ for $t \ge t_0$, then $|f(t)e^{-st}| \le Me^{-(s-s_0)t}$ for $t \ge t_0$. Let $g(t) = e^{-st} f(t)$, $w(t) = Me^{-(s-s_0)t}$, and $\tau = t_0$. Since $\int_{t_0}^\infty w(t)\,dt$ converges if $s > s_0$, $F(s)$ is defined for $s > s_0$.

8.1.12. $\displaystyle\int_0^T e^{-st}\left(\int_0^t f(\tau)\,d\tau\right)dt = -\frac{e^{-st}}{s}\int_0^t f(\tau)\,d\tau\Big|_0^T + \frac{1}{s}\int_0^T e^{-st} f(t)\,dt = -\frac{e^{-sT}}{s}\int_0^T f(\tau)\,d\tau +$ $\displaystyle\frac{1}{s}\int_0^T e^{-st} f(t)\,dt$. Since f is of exponential order s_0, the second integral on the right converges to

$\frac{1}{s}L(f)$ as $T \to \infty$ (Exercise 8.1.10). Now it suffices to show that (A) $\lim_{T\to\infty} e^{-sT}\int_0^T f(\tau)\,d\tau = 0$ if $s > s_0$. Suppose that $|f(t)| \le Me^{s_0t}$ if $t \ge t_0$ and $|f(t)| \le K$ if $0 \le t \le t_0$, and let $T > t_0$. Then $\left|\int_0^T f(\tau)\,d\tau\right| \le \left|\int_0^{t_0} f(\tau)\,d\tau\right| + \left|\int_{t_0}^T f(\tau)\,d\tau\right| < Kt_0 + M\int_{t_0}^T e^{s_0\tau}\,d\tau < Kt_0 + \frac{Me^{s_0T}}{s_0}$, which proves (A).

8.1.14. **(a)** If $T > 0$, then $\int_0^T e^{-st}f(t)\,dt = \int_0^T e^{-(s-s_0)t}(e^{-s_0t}f(t))\,dt$. Use integration by parts with $u = e^{-(s-s_0)t}$, $dv = e^{-s_0t}f(t)\,dt$, $du = -(s-s_0)e^{-(s-s_0)t}$, and $v = g$ t obtain $\int_0^T e^{-st}f(t)\,dt = e^{-(s-s_0)t}g(t)\Big|_0^T + (s-s_0)\int_0^T e^{-(s-s_0)t}g(t)\,dt$. Since $g(0) = 0$ this reduces to $\int_0^T e^{-st}f(t)\,dt = e^{-(s-s_0)T}g(T) + (s-s_0)\int_0^T e^{-(s-s_0)t}g(t)\,dt$. Since $|g(t)| \le M$ for all $t \ge 0$, we can let $t \to \infty$ to conclude that $\int_0^\infty e^{-st}f(t)\,dt = (s-s_0)\int_0^\infty e^{-(s-s_0)t}g(t)\,dt$ if $s > s_0$.

(b) If $F(s_0)$ exists, then $g(t)$ is bounded on $[0,\infty)$. Now apply **(a)**.

(c) Since $f(t) = \frac{1}{2}\frac{d}{dt}\sin(e^{t^2})$, $\left|\int_0^t f(\tau)\,d\tau\right| = \frac{|\sin(e^{t^2}) - \sin(1)|}{2} \le 1$ for all $t \ge 0$. Now apply **(a)** with $s_0 = 0$.

8.1.16. **(a)** $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\,dx = \frac{x^\alpha e^{-x}}{\alpha}\Big|_0^\infty + \frac{1}{\alpha}\int_0^\infty x^\alpha e^{-x}\,dx = \frac{\Gamma(\alpha+1)}{\alpha}$.

(b) Use induction. $\Gamma(1) = \int_0^\infty e^{-x}\,dx = 1$. If (A) $\Gamma(n+1) = n!$, then $\Gamma(n+2) = (n+1)\Gamma(n+1)$ (from **(a)**) $= (n+1)n!$ (from (A)) $= (n+1)!$.

(c) $\Gamma(\alpha+1) = \int_0^\infty x^\alpha e^{-x}\,dx$. Let $x = st$. Then $\Gamma(\alpha+1) = \int_0^\infty (st)^\alpha e^{-st}s\,dt$, so $\int_0^\infty e^{-st}t^\alpha\,dt = \frac{\Gamma(\alpha+1)}{\alpha}$.

8.1.18. **(a)** $\int_0^2 e^{-st}f(t)\,dt = \int_0^1 e^{-st}t\,dt + \int_1^2 e^{-st}(2-t)\,dt = \left(\frac{1}{s^2} - \frac{e^{-s}(s+1)}{s^2}\right) + \left(\frac{e^{-s}(s-1)}{s^2} + \frac{e^{-2s}}{s^2}\right) = -\frac{2e^{-s}}{s^2} + \frac{e^{-2s}}{s^2} + \frac{1}{s^2} = \frac{(1-e^{-s})^2}{s^2}$. Therefore, $F(s) = \frac{(1-e^{-s})^2}{s^2(1-e^{-2s})} = \frac{1-e^{-s}}{s^2(1+e^{-s})} = \frac{1}{s^2}\tanh\frac{s}{2}$.

(b) $\int_0^1 e^{-st}f(t)\,dt = \int_0^{1/2} e^{-st}\,dt - \int_{1/2}^1 e^{-st}\,dt = \frac{1}{s} - \frac{e^{-s/2}}{s} + \frac{e^{-s}}{s} - \frac{e^{-s/2}}{s} = -\frac{2e^{-s/2}}{s} + \frac{e^{-s}}{s} + \frac{1}{s} = \frac{(1-e^{-s/2})^2}{s}$. Therefore, $F(s) = \frac{(1-e^{-s/2})^2}{s(1-e^{-s})} = \frac{1-e^{-s/2}}{s(1+e^{-s/2})} = \frac{1}{s}\tanh\frac{s}{4}$.

(c) $\int_0^\pi e^{-st}f(t)\,dt = \int_0^\pi e^{-st}\sin t\,dt = \frac{1+e^{-\pi s}}{(s^2+1)}$. Therefore, $F(s) = \frac{1+e^{-\pi s}}{(s^2+1)(1-e^{-\pi s})}\frac{1}{s^2+1}\coth\frac{\pi s}{2}$.

(d) $\int_0^{2\pi} e^{-st}f(t)\,dt = \int_0^\pi e^{-st}\sin t\,dt = \frac{1+e^{-\pi s}}{(s^2+1)}$. Therefore, $F(s) = \frac{1+e^{-\pi s}}{(s^2+1)1+e^{-2\pi s}} = \frac{1}{(s^2+1)(1-e^{-\pi s})}$.

8.2 THE INVERSE LAPLACE TRANSFORM

8.2.2. **(a)** $\dfrac{2s+3}{(s-7)^4} = \dfrac{2(s-7)+17}{(s-7)^4} = \dfrac{2}{(s-7)^3} + \dfrac{17}{(s-7)^4} = \dfrac{2!}{(s-7)^3} + \dfrac{17}{6}\dfrac{3!}{(s-7)^4} \leftrightarrow e^{7t}\left(t^2 + \dfrac{17}{6}t^3\right)$.

(b) $\dfrac{s^2-1}{(s-2)^6} = \dfrac{[(s-2)+2]^2-1}{(s-2)^6} = \dfrac{(s-2)^2+4(s-2)+3}{(s-2)^6} = \dfrac{1}{(s-2)^4} + \dfrac{4}{(s-2)^5} + \dfrac{3}{(s-2)^6} =$ $\dfrac{1}{6}\dfrac{3!}{(s-2)^4} + \dfrac{1}{6}\dfrac{4!}{(s-2)^5} + \dfrac{1}{40}\dfrac{5!}{(s-2)^6} \leftrightarrow \left(\dfrac{1}{6}t^3 + \dfrac{1}{6}t^4 + \dfrac{1}{40}t^5\right)e^{2t}$.

(c) $\dfrac{s+5}{s^2+6s+18} = \dfrac{(s+3)}{(s+3)^2+9} + \dfrac{2}{3}\dfrac{3}{(s+3)^2+9} \leftrightarrow e^{-3t}\left(\cos 3t + \dfrac{2}{3}\sin 3t\right)$.

(d) $\dfrac{2s+1}{s^2+9} = 2\dfrac{s}{s^2+9} + \dfrac{1}{3}\dfrac{3}{s^2+9} \leftrightarrow 2\cos 3t + \dfrac{1}{3}\sin 3t$.

(e) $\dfrac{s}{s^2+2s+1} = \dfrac{(s+1)-1}{(s+1)^2} = \dfrac{1}{s+1} - \dfrac{1}{(s+1)^2} \leftrightarrow (1-t)e^{-t}$.

(f) $\dfrac{s+1}{s^2-9} = \dfrac{s}{s^2-9} + \dfrac{1}{3}\dfrac{3}{s^2-9} \leftrightarrow \cosh 3t + \dfrac{1}{3}\sinh 3t$.

(g) Expand the numerator in powers of $s+1$: $s^3+2s^2-s-3 = [(s+1)-1]^3 + 2[(s+1)-1]^2 - [(s+1)-1]-3 = (s+1)^3-(s+1)^2-2(s+1)-1$; therefore $\dfrac{s^3+2s^2-s-3}{(s+1)^4} = \dfrac{1}{s+1} - \dfrac{1}{(s+1)^2} - \dfrac{2}{(s+1)^3} - \dfrac{1}{6}\dfrac{6}{(s+1)^4} \leftrightarrow \left(1-t-t^2-\dfrac{1}{6}t^3\right)e^{-t}$.

(h) $\dfrac{2s+3}{(s-1)^2+4} = 2\dfrac{(s-1)}{(s-1)^2+4} + \dfrac{5}{2}\dfrac{2}{(s-1)^2+4} \leftrightarrow e^t\left(2\cos 2t + \dfrac{5}{2}\sin 2t\right)$.

(i) $\dfrac{1}{s} - \dfrac{s}{s^2+1} \leftrightarrow 1-\cos t$.

(j) $\dfrac{3s+4}{s^2-1} = \dfrac{3s}{s^2-1} + \dfrac{4}{s^2-1} \leftrightarrow 3\cosh t + 4\sinh t$. Alternatively, $\dfrac{3s+4}{s^2-1} = \dfrac{3s+4}{(s-1)(s+1)} =$ $\dfrac{1}{2}\left[\dfrac{7}{s-1} - \dfrac{1}{s+1}\right] \leftrightarrow \dfrac{7e^t-e^{-t}}{2}$.

(k) $\dfrac{3}{s-1} + \dfrac{4s+1}{s^2+9} = 3\dfrac{1}{s-1} + 4\dfrac{s}{s^2+9} + \dfrac{1}{3}\dfrac{3}{s^2+9} \leftrightarrow 3e^t + 4\cos 3t + \dfrac{1}{3}\sin 3t$.

(l) $\dfrac{3}{(s+2)^2} - \dfrac{2s+6}{s^2+4} = 3\dfrac{1}{(s+2)^2} - 2\dfrac{s}{s^2+4} - 3\dfrac{2}{s^2+4} \leftrightarrow 3te^{-2t} - 2\cos 2t - 3\sin 2t$.

8.2.4. **(a)**

$$\frac{2+3s}{(s^2+1)(s+2)(s+1)} = \frac{A}{s+2} + \frac{B}{s+1} + \frac{Cs+D}{s^2+1},$$

where

$$A(s^2+1)(s+1) + B(s^2+1)(s+2) + (Cs+D)(s+2)(s+1) = 2+3s.$$

$$\begin{array}{rcll} -5A &=& -4 & (\text{set } s=-2); \\ 2B &=& -1 & (\text{set } s=-1); \\ A+2B+2D &=& 2 & (\text{set } s=0); \\ A+B+C &=& 0 & (\text{equate coefficients of } s^3). \end{array}$$

Solving this system yields $A = \dfrac{4}{5}$, $B = -\dfrac{1}{2}$, $C = -\dfrac{3}{10}$, $D = \dfrac{11}{10}$. Therefore,

$$\begin{aligned} \frac{2+3s}{(s^2+1)(s+2)(s+1)} &= \frac{4}{5}\frac{1}{s+2} - \frac{1}{2}\frac{1}{s+1} - \frac{1}{10}\frac{3s-11}{s^2+1} \\ &\leftrightarrow \frac{4}{5}e^{-2t} - \frac{1}{2}e^{-t} - \frac{3}{10}\cos t + \frac{11}{10}\sin t. \end{aligned}$$

(b)

$$\frac{3s^2+2s+1}{(s^2+1)(s^2+2s+2)} = \frac{As+B}{s^2+1} + \frac{C(s+1)+D}{(s+1)^2+1},$$

where

$$(As+B)((s+1)^2+1) + (C(s+1)+D)(s^2+1) = 3s^2+2s+1.$$

$$\begin{array}{rcll} 2B+C+D &=& 1 & \text{(set } s=0\text{);} \\ -A+B+2D &=& 2 & \text{(set } s=-1\text{);} \\ 2B+C+D &=& 1 & \text{(set } s=0\text{);} \\ A+C &=& 0 & \text{(equate coefficients of } s^3\text{).} \end{array}$$

Solving this system yields $A = 6/5$, $B = 2/5$, $C = -6/5$, $D = 7/5$. Therefore,

$$\begin{aligned} \frac{3s^2+2s+1}{(s^2+1)(s^2+2s+2)} &= \frac{1}{5}\left[\frac{6s+2}{s^2+1} - \frac{6(s+1)-7}{(s+1)^2+1}\right] \\ &\leftrightarrow \frac{6}{5}\cos t + \frac{2}{5}\sin t - \frac{6}{5}e^{-t}\cos t + \frac{7}{5}e^{-t}\sin t. \end{aligned}$$

(c) $s^2+2s+5 = (s+1)^2+4$;

$$\frac{3s+2}{(s-2)((s+1)^2+4)} = \frac{A}{s-2} + \frac{B(s+1)+C}{(s+1)^2+4},$$

where

$$A\left((s+1)^2)+4\right) + (B(s+1)+C)\,(s-2) = 3s+2.$$

$$\begin{array}{rcll} 13A &=& 8 & \text{(set } s=2\text{);} \\ 4A-3C &=& -1 & \text{(set } s=-1\text{);} \\ A+B &=& 0 & \text{(equate coefficients of } s^2\text{).} \end{array}$$

Solving this system yields $A = \frac{8}{13}$, $B = -\frac{8}{13}$, $C = \frac{15}{13}$. Therefore,

$$\begin{aligned} \frac{3s+2}{(s-2)((s+1)^2+4)} &= \frac{1}{13}\left[\frac{8}{s-2} - \frac{8(s-1)-15}{(s+1)^2+4}\right] \\ &\leftrightarrow \frac{8}{13}e^{2t} - \frac{8}{13}e^{-t}\cos 2t + \frac{15}{26}e^{-t}\sin 2t. \end{aligned}$$

(d)

$$\frac{3s^2+2s+1}{(s-1)^2(s+2)(s+3)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s+2} + \frac{D}{s+3},$$

where

$$(A(s-1)+B)(s+2)(s+3) + (C(s+3)+D(s+2))(s-1)^2 = 3s^2+2s+1.$$

$$\begin{array}{rcll} 12B &=& 6 & \text{(set } s=1\text{);} \\ 9C &=& 9 & \text{(set } s=-2\text{);} \\ -16D &=& 22 & \text{(set } s=-3\text{);} \\ A+C+D &=& 0 & \text{(equate coefficients of } s^3\text{).} \end{array}$$

Solving this system yields $A = 3/8$, $B = 1/2$, $C = 1$, $D = -11/8$. Therefore,

$$\begin{aligned} \frac{3s^2+2s+1}{(s-1)^2(s+2)(s+3)} &= \frac{3}{8}\frac{1}{s-1} + \frac{1}{2}\frac{1}{(s-1)^2} + \frac{1}{s+2} - \frac{11}{8}\frac{1}{s+3} \\ &\leftrightarrow \frac{3}{8}e^t + \frac{1}{2}te^t + e^{-2t} - \frac{11}{8}e^{-3t}. \end{aligned}$$

(e)

$$\frac{2s^2+s+3}{(s-1)^2(s+2)^2} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s+2} + \frac{D}{(s+2)^2},$$

where

$$(A(s-1)+B)(s+2)^2 + (C(s+2)+D)(s-1)^2 = 2s^2+s+3.$$

$$\begin{aligned} 9B &= 6 \quad \text{(set } s=1\text{);} \\ 9D &= 9 \quad \text{(set } s=-2\text{);} \\ -4A+4B+2C+D &= 3 \quad \text{(set } s=0\text{);} \\ A+C &= 0 \quad \text{(equate coefficients of } s^3\text{).} \end{aligned}$$

Solving this system yields $A=1/9$, $B=2/3$, $C=-1/9$, $D=1$. Therefore,

$$\begin{aligned} \frac{2s^2+s+3}{(s-1)^2(s+2)^2} &= \frac{1}{9}\frac{1}{s-1} + \frac{2}{3}\frac{1}{(s-1)^2} - \frac{1}{9}\frac{1}{s+2} + \frac{1}{(s+2)^2} \\ &\leftrightarrow \frac{1}{9}e^t + \frac{2}{3}te^t - \frac{1}{9}e^{-2t} + te^{-2t}. \end{aligned}$$

(f)

$$\frac{3s+2}{(s^2+1)(s-1)^2} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{Cs+D}{s^2+1},$$

where

$$A(s-1)(s^2+1) + B(s^2+1) + (Cs+D)(s-1)^2 = 3s+2. \tag{A}$$

Setting $s=1$ yields $2B=5$, so $B=\frac{5}{2}$. Substituting this into (A) shows that

$$\begin{aligned} A(s-1)(s^2+1) + (Cs+D)(s-1)^2 &= 3s+2-\frac{5}{2}(s^2+1) \\ &= -\frac{5s^2-6s+1}{2} = -\frac{(s-1)(5s-1)}{2}. \end{aligned}$$

Therefore,

$$A(s^2+1) + (Cs+D)(s-1) = \frac{1-5s}{2}.$$

$$\begin{aligned} 2A &= -2 \quad \text{(set } s=1\text{);} \\ A-D &= 1/2 \quad \text{(set } s=0\text{);} \\ A+C &= 0 \quad \text{(equate coefficients of } s^2\text{).} \end{aligned}$$

Solving this system yields $A=-1$, $C=1$, $D=-\frac{3}{2}$. Therefore,

$$\begin{aligned} \frac{3s+2}{(s^2+1)(s-1)^2} &= -\frac{1}{s-1} + \frac{5}{2}\frac{1}{(s-1)^2} + \frac{s-3/2}{s^2+1} \\ &\leftrightarrow -e^t + \frac{5}{2}te^t + \cos t - \frac{3}{2}\sin t. \end{aligned}$$

8.2.6. (a)

$$\frac{17s-15}{(s^2-2s+5)(s^2+2s+10)} = \frac{A(s-1)+B}{(s-1)^2+4} + \frac{C(s+1)+D}{(s+1)^2+9}$$

where

$$(A(s-1)+B)((s+1)^2+9) + (C(s+1)+D)((s-1)^2+4) = 17s-15.$$

$$
\begin{aligned}
13B + 8C + 4D &= 2 \quad \text{(set } s = 1\text{);}\\
-18A + 9B + 8D &= -32 \quad \text{(set } s = -1\text{);}\\
-10A + 10B + 5C + 5D &= -15 \quad \text{(set } s = 0\text{);}\\
A + C &= 0 \quad \text{(equate coefficients of } s^3\text{).}
\end{aligned}
$$

Solving this system yields $A = 1$, $B = 2$, $C = -1$, $D = -4$. Therefore,

$$
\begin{aligned}
\frac{17s - 15}{(s^2 - 2s + 5)(s^2 + 2s + 10)} &= \frac{(s-1)+2}{(s-1)^2+4} - \frac{(s+1)+4}{(s+1)^2+9}\\
&\leftrightarrow e^t(\cos 2t + \sin 2t) - e^{-t}\left(\cos 3t + \frac{4}{3}\sin 3t\right).
\end{aligned}
$$

(b)

$$
\frac{8s + 56}{(s^2 - 6s + 13)(s^2 + 2s + 5)} = \frac{A(s-3)+B}{(s-3)^2+4} + \frac{C(s+1)+D}{(s+1)^2+4}
$$

where

$$
(A(s-3)+B)((s+1)^2+4) + (C(s+1)+D)((s-3)^2+4) = 8s + 56.
$$

$$
\begin{aligned}
20B + 16C + 4D &= 80 \quad \text{(set } s = 3\text{);}\\
-16A + 4B + 20D &= 48 \quad \text{(set } s = -1\text{);}\\
-15A + 5B + 13C + 13D &= 56 \quad \text{(set } s = 0\text{);}\\
A + C &= 0 \quad \text{(equate coefficients of } s^3\text{).}
\end{aligned}
$$

Solving this system yields $A = -1$, $B = 3$, $C = 1$, $D = 1$. Therefore,

$$
\begin{aligned}
\frac{8s + 56}{(s^2 - 6s + 13)(s^2 + 2s + 5)} &= \frac{-(s-3)+3}{(s-3)^2+4} + \frac{(s+1)+1}{(s+1)^2+4}\\
&\leftrightarrow e^{3t}\left(-\cos 2t + \frac{3}{2}\sin 2t\right) + e^{-t}\left(\cos 2t + \frac{1}{2}\sin 2t\right).
\end{aligned}
$$

(c)

$$
\frac{s + 9}{(s^2 + 4s + 5)(s^2 - 4s + 13)} = \frac{A(s+2)+B}{(s+2)^2+1} + \frac{C(s-2)+D}{(s-2)^2+9}
$$

where

$$
(A(s+2)+B)((s-2)^2+9) + (C(s-2)+D)((s+2)^2+1) = s + 9.
$$

$$
\begin{aligned}
25B - 4C + D &= 7 \quad \text{(set } s = -2\text{);}\\
36A + 9B + 17D &= 11 \quad \text{(set } s = 2\text{);}\\
26A + 13B - 10C + 5D &= 9 \quad \text{(set } s = 0\text{);}\\
A + C &= 0 \quad \text{(equate coefficients of } s^3\text{).}
\end{aligned}
$$

Solving this system yields $A = 1/8$, $B = 1/4$, $C = -1/8$, $D = 1/4$. Therefore,

$$
\begin{aligned}
\frac{s + 9}{(s^2 + 4s + 5)(s^2 - 4s + 13)} &= = \left[\frac{1}{8}\frac{(s+2)+2}{(s+2)^2+1} - \frac{(s-2)-2}{(s-2)^2+3}\right]\\
&\leftrightarrow e^{-2t}\left(\frac{1}{8}\cos t + \frac{1}{4}\sin t\right) - e^{2t}\left(\frac{1}{8}\cos 3t - \frac{1}{12}\sin 3t\right).
\end{aligned}
$$

(d)

$$
\frac{3s - 2}{(s^2 - 4s + 5)(s^2 - 6s + 13)} = \frac{A(s-2)+B}{(s-2)^2+1} + \frac{C(s-3)+D}{(s-3)^2+4}
$$

where

$$
(A(s-2)+B)((s-3)^2+4) + (C(s-3)+D)((s-2)^2+1) = 3s - 2.
$$

$$\begin{aligned} 5B - C + D &= 4 \quad \text{(set } s = 2\text{);} \\ 4A + 4B + 2D &= 7 \quad \text{(set } s = 3\text{);} \\ -26A + 13B - 15C + 5D &= -2 \quad \text{(set } s = 0\text{);} \\ A + C &= 0 \quad \text{(equate coefficients of } s^3\text{).} \end{aligned}$$

Solving this system yields $A = 1$, $B = 1/2$, $C = -1$, $D = 1/2$. Therefore,

$$\begin{aligned} \frac{3s-2}{(s^2-4s+5)(s^2-6s+13)} &= \ = \frac{1}{2}\left[\frac{2(s-2)+1}{(s-2)^2+1} - \frac{2(s-3)-1}{(s-3)^2+4}\right] \\ &\leftrightarrow \ e^{2t}\left(\cos t + \frac{1}{2}\sin t\right) - e^{3t}\left(\cos 2t - \frac{1}{4}\sin 2t\right). \end{aligned}$$

(e)

$$\frac{3s-1}{(s^2-2s+2)(s^2+2s+5)} = \frac{A(s-1)+B}{(s-1)^2+1} + \frac{C(s+1)+D}{(s+1)^2+4}$$

where

$$(A(s-1)+B)((s+1)^2+4) + (C(s+1)+D)((s-1)^2+1) = 3s-1.$$

$$\begin{aligned} 8B + 2C + D &= 2 \quad \text{(set } s = 1\text{);} \\ -8A + 4B + 5D &= -4 \quad \text{(set } s = -1\text{);} \\ -5A + 5B + 2C + 2D &= -1 \quad \text{(set } s = 0\text{);} \\ A + 5B + C &= 0 \quad \text{(equate coefficients of } s^3\text{).} \end{aligned}$$

Solving this system yields $A = 1/5$, $B = 2/5$, $C = -1/5$, $D = -4/5$. Therefore,

$$\begin{aligned} \frac{3s-1}{(s^2-2s+2)(s^2+2s+5)} &= \frac{1}{5}\left[\frac{(s-1)+2}{(s-1)^2+1} - \frac{(s+1)+4}{(s+1)^2+4}\right]. \\ &\leftrightarrow \ e^{t}\left(\frac{1}{5}\cos t + \frac{2}{5}\sin t\right) - e^{-t}\left(\frac{1}{5}\cos 2t + \frac{2}{5}\sin 2t\right). \end{aligned}$$

(f)

$$\frac{20s+40}{(4s^2-4s+5)(4s^2+4s+5)} = [\frac{A(s-1/2)+B}{(s-1/2)^2+1} + \frac{C(s+1/2)+D}{(s+1/2)^2+1}$$

where

$$(A(s-1/2)+B)((s+1/2)^2+1) + (C(s+1/2)+D)((s-1/2)^2+1) = \frac{5s+10}{4}.$$

$$\begin{aligned} 2B + C + D &= 25/8 \quad \text{(set } s = 1/2\text{);} \\ -A + B + 2D &= 15/8 \quad \text{(set } s = -1/2\text{);} \\ -5A + 10B + 5C + 10D &= 20 \quad \text{(set } s = 0\text{);} \\ A + C &= 0 \quad \text{(equate coefficients of } s^3\text{).} \end{aligned}$$

Solving this system yields $A = -1$, $B = 9/8$, $C = 1$, $D = -1/8$. Therefore,

$$\begin{aligned} \frac{20s+40}{(4s^2-4s+5)(4s^2+4s+5)} &= \frac{1}{8}\left[\frac{-8(s-1/2)+9}{(s-1/2)^2+1} + \frac{8(s+1/2)-}{(s+1/2)^2+1}\right] \\ &\leftrightarrow \ e^{t/2}\left(-\cos t + \frac{9}{8}\sin t\right) + e^{-t/2}\left(\cos t - \frac{1}{8}\sin t\right). \end{aligned}$$

8.2.8. **(a)**

$$\frac{2s+1}{(s^2+1)(s-1)(s-3)} = \frac{A}{s-1} + \frac{B}{s-3} + \frac{Cs+D}{s^2+1}$$

where

$$(A(s-3)+B(s-1))(s^2+1)+(Cs+D)(s-1)(s-3)=2s+1.$$

$$\begin{aligned}
-4A &= 3 \quad (\text{set } s=1);\\
20B &= 7 \quad (\text{set } s=3);\\
-3A-B+3D &= 1 \quad (\text{set } s=0);\\
A+B+C &= 0 \quad (\text{equate coefficients of } s^3).
\end{aligned}$$

Solving this system yields $A=-3/4$, $B=7/20$, $C=2/5$, $D=-3/10$. Therefore,

$$\begin{aligned}
\frac{2s+1}{(s^2+1)(s-1)(s-3)} &= -\frac{3}{4}\frac{1}{s-1}+\frac{7}{20}\frac{1}{s-3}+\frac{2}{5}\frac{s}{s^2+1}-\frac{3}{10}\frac{1}{s^2+1}\\
&\leftrightarrow -\frac{3}{4}e^t+\frac{7}{20}e^{3t}+\frac{2}{5}\cos t-\frac{3}{10}\sin t.
\end{aligned}$$

(b)

$$\frac{s+2}{(s^2+2s+2)(s^2-1)}=\frac{A}{s-1}+\frac{B}{s+1}+\frac{C(s+1)+D}{(s+1)^2+1}$$

where

$$(A(s+1)+B(s-1))((s+1)^2+1)+(C(s+1)+D)(s^2-1)=s+2.$$

$$\begin{aligned}
10A &= 3 \quad (\text{set } s=1);\\
-2B &= 1 \quad (\text{set } s=-1);\\
2A-2B-C-D &= 2 \quad (\text{set } s=0);\\
A+B+C &= 0 \quad (\text{equate coefficients of } s^3).
\end{aligned}$$

Solving this system yields $A=3/10$, $B=-1/2$, $C=1/5$, $D=-3/5$. Therefore,

$$\begin{aligned}
\frac{s+2}{(s^2+2s+2)(s^2-1)} &= \frac{3}{10}\frac{1}{s-1}-\frac{1}{2}\frac{1}{s+1}+\frac{1}{5}\frac{s+1}{(s+1)^2+1}-\frac{3}{5}\frac{1}{(s+1)^2+1}\\
&\leftrightarrow \frac{3}{10}e^t-\frac{1}{2}e^{-t}+\frac{1}{5}e^{-t}\cos t e^{-t}\sin t.
\end{aligned}$$

(c)

$$\frac{2s-1}{(s^2-2s+2)(s+1)(s-2)}=\frac{A}{s-2}+\frac{B}{s+1}+\frac{C(s-1)+D}{(s-1)^2+1}$$

where

$$(A(s+1)+B(s-2))((s-1)^2+1)+(C(s-1)+D)(s-2)(s+1)=2s-1.$$

$$\begin{aligned}
6A &= 3 \quad (\text{set } s=2);\\
-15B &= -3 \quad (\text{set } s=-1);\\
2A-4B+2C-2D &= -1 \quad (\text{set } s=0);\\
A+B+C &= 0 \quad (\text{equate coefficients of } s^3).
\end{aligned}$$

Solving this system yields $A=1/2$, $B=1/5$, $C=-7/10$, $D=-1/10$. Therefore,

$$\begin{aligned}
\frac{2s-1}{(s^2-2s+2)(s+1)(s-2)} &= =\frac{1}{2}\frac{1}{s-2}+\frac{1}{5}\frac{1}{s+1}-\frac{7}{10}\frac{s-1}{(s-1)^2+1}-\frac{1}{10}\frac{1}{(s-1)^2+1}\\
&\leftrightarrow \frac{1}{2}e^{2t}+\frac{1}{5}e^{-t}-\frac{7}{10}e^t\cos t-\frac{1}{10}e^t\sin t.
\end{aligned}$$

(d)

$$\frac{s-6}{(s^2-1)(s^2+4)}=\frac{A}{s-1}+\frac{B}{s+1}+\frac{Cs+D}{s^2+4}$$

where

$$(A(s+1)+B(s-1))(s^2+4)+(Cs+D)(s^2-1)=s-6.$$

$$\begin{array}{rcll} 10A & = & -5 & \text{(set } s=1\text{);} \\ -10B & = & -7 & \text{(set } s=-1\text{);} \\ 4A-4B-D & = & -6 & \text{(set } s=0\text{);} \\ A+B+C & = & 0 & \text{(equate coefficients of } s^3\text{).} \end{array}$$

Solving this system yields $A=-1/2$, $B=7/10$, $C=-1/5$, $D=6/5$. Therefore,

$$\begin{aligned} \frac{s-6}{(s^2-1)(s^2+4)} &= = -\frac{1}{2}\frac{1}{s-1}+\frac{7}{10}\frac{1}{s+1}-\frac{1}{5}\frac{s}{s^2+4}+\frac{3}{5}+\frac{1}{s^2+4} \\ &\leftrightarrow -\frac{1}{2}e^t+\frac{7}{10}e^{-t}-\frac{1}{5}\cos 2t+\frac{3}{5}\sin 2t. \end{aligned}$$

(e)

$$\frac{2s-3}{s(s-2)(s^2-2s+5)}=\frac{A}{s}+\frac{B}{s-2}+\frac{C(s-1)+D}{(s-1)^2+4}$$

where

$$(A(s-2)+Bs)((s-1)^2+4)+(C(s-1)+D)s(s-2)=2s-3.$$

$$\begin{array}{rcll} -10A & = & -3 & \text{(set } s=0\text{);} \\ 10B & = & 1 & \text{(set } s=2\text{);} \\ -4A+4B-D & = & -1 & \text{(set } s=1\text{);} \\ A+B+C & = & 0 & \text{(equate coefficients of } s^3\text{).} \end{array}$$

Solving this system yields $A=3/10$, $B=1/10$, $C=-2/5$, $D=1/5$. Therefore,

$$\begin{aligned} \frac{2s-3}{s(s-2)(s^2-2s+5)} &= = \frac{3}{10s}+\frac{1}{10}\frac{1}{s-2}-\frac{2}{5}\frac{s-1}{(s-1)^2+4}+\frac{1}{5}\frac{1}{(s-1)^2+4} \\ &\leftrightarrow \frac{3}{10}+\frac{1}{10}e^{2t}-\frac{2}{5}e^t\cos 2t+\frac{1}{10}e^t\sin 2t. \end{aligned}$$

(f)

$$\frac{5s-15}{(s^2-4s+13)(s-2)(s-1)}=\frac{A}{s-1}+\frac{B}{s-2}+\frac{C(s-2)+D}{(s-2)^2+9}$$

where

$$(A(s-2)+B(s-1))((s-2)^2+9)+(C(s-2)+D)(s-1)(s-2)=5s-15.$$

$$\begin{array}{rcll} -10A & = & -10 & \text{(set } s=1\text{);} \\ 9B & = & -5 & \text{(set } s=2\text{);} \\ -26A-13B-4C+2D & = & -15 & \text{(set } s=0\text{);} \\ A+B+C & = & 0 & \text{(equate coefficients of } s^3\text{).} \end{array}$$

Solving this system yields $A=1$, $B=-5/9$, $C=-4/9$, $D=1$. Therefore,

$$\begin{aligned} \frac{5s-15}{(s^2-4s+13)(s-2)(s-1)} &= = \frac{1}{s-1}-\frac{5}{9}\frac{1}{s-2}-\frac{4}{9}\frac{s-2}{(s-2)^2+9}+\frac{1}{(s-2)^2+9} \\ &\leftrightarrow e^t-\frac{5}{9}e^{2t}-\frac{4}{9}e^{2t}\cos 3t+\frac{1}{3}e^{2t}\sin 3t. \end{aligned}$$

8.2.10. **(a)** Let $i = 1$. (The proof for $i = 2, \dots, n$) is similar. Multiplying the given equation through by $s - s_1$ yields

$$\frac{P(s)}{(s-s_2)\cdots(s-s_n)} = A_1 + (s-s_1)\left[\frac{A_2}{s-s_2} + \cdots + \frac{A_n}{s-s_n}\right],$$

and setting $s = s_1$ yields $A_1 = \dfrac{P(s_1)}{(s_1-s_2)\cdots(s_2-s_n)}$.

(b) From calculus we know that F has a partial fraction expansion of the form $\dfrac{P(s)}{(s-s_1)Q_1(s)} = \dfrac{A}{s-s_1} + G(s)$ where G is continuous at s_1. Multiplying through by $s - s_1$ shows that $\dfrac{P(s)}{Q_1(s)} = A + (s-s_1)G(s)$. Now set $s = s_1$ to obtain $A = \dfrac{P(s_1)}{Q(s_1)}$.

(c) The result in **(b)** is generalization of the result in **(a)**, since it shows that if s_1 is a simple zero of the denominator of the rational function, then Heaviside's method can be used to determine the coefficient of $1/(s-s_1)$ in the partial fraction expansion even if some of the other zeros of the denominator are repeated or complex.

8.3 SOLUTION OF INITIAL VALUE PROBLEMS

8.3.2.

$$(s^2 - s - 6)Y(s) = \frac{2}{s} + s - 1 = \frac{2 + s(s-1)}{s}.$$

Since $(s^2 - s - 6) = (s-3)(s+2)$,

$$Y(s) = \frac{2 + s(s-1)}{s(s-3)(s+2)} = -\frac{1}{3s} + \frac{8}{15}\frac{1}{s-3} + \frac{4}{5}\frac{1}{s+2}$$

and $y = -\dfrac{1}{3} + \dfrac{8}{15}e^{3t} + \dfrac{4}{5}e^{-2t}$.

8.3.4.

$$(s^2 - 4)Y(s) = \frac{2}{s-3} + (-1+s) = \frac{2 + (s-1)(s-3)}{s-3}.$$

Since $s^2 - 4 = (s-2)(s+2)$,

$$Y(s) = \frac{2 + (s-1)(s-3)}{(s-2)(s+2)(s-3)} = -\frac{1}{4}\frac{1}{s-2} + \frac{17}{20}\frac{1}{s+2} + \frac{2}{5}\frac{1}{s-3}$$

and $y = -\dfrac{1}{4}e^{2t} + \dfrac{17}{20}e^{-2t} + \dfrac{2}{5}e^{3t}$.

8.3.6.

$$(s^2 + 3s + 2)Y(s) = \frac{6}{s-1} + (-1+s) + 3 = \frac{6 + (s-1)(s+2)}{s-1}.$$

Since $s^2 + 3s + 2 = (s+2)(s+1)$,

$$Y(s) = \frac{6 + (s-1)(s+2)}{(s-1)(s+2)(s+1)} = \frac{1}{s-1} + \frac{2}{s+2} - \frac{2}{s+1}$$

and $y = e^t + 2e^{-2t} - 2e^{-t}$.

8.3.8.

$$(s^2 - 3s + 2)Y(s) = \frac{2}{s-3} + (-1+s) - 3 = \frac{2 + (s-3)(s-4)}{s-3}.$$

Since $s^2 - 3s + 2 = (s-1)(s-2)$,

$$Y(s) = \frac{2 + (s-3)(s-4)}{(s-1)(s-2)(s-3)} = \frac{4}{s-1} - \frac{4}{s-2} + \frac{1}{s-3}$$

and $y = 4e^t - 4e^{2t} + e^{3t}$.

8.3.10.

$$(s^2 - 3s + 2)Y(s) = \frac{1}{s-3} + (-4-s) + 3 = \frac{1 - (s-3)(s+1)}{s-3}.$$

Since $s^2 - 3s + 2 = (s-1)(s-2)$,

$$Y(s) = \frac{1 - (s-3)(s+1)}{(s-1)(s-2)(s-3)} = \frac{5}{2}\frac{1}{s-1} - \frac{4}{s-2} + \frac{1}{2}\frac{1}{s-3}$$

and $y = \frac{5}{2}e^t - 4e^{2t} + \frac{1}{2}e^{3t}$.

8.3.12.

$$(s^2 + s - 2)Y(s) = -\frac{4}{s} + (3+2s) + 2 = \frac{-4 + s(5+2s)}{s}.$$

Since $(s^2 + s - 2) = (s+2)(s-1)$,

$$Y(s) = \frac{-4 + s(5+2s)}{s(s+2)(s-1)} = \frac{2}{s} - \frac{1}{s+2} + \frac{1}{s-1},$$

and $y = 2 - e^{-2t} + e^t$.

8.3.14.

$$(s^2 - s - 6)Y(s) = \frac{2}{s} + s - 1 = \frac{2 + s(s-1)}{s}.$$

Since $s^2 - s - 6 = (s-3)(s+2)$,

$$Y(s) = \frac{2 + s(s-1)}{s(s-3)(s+2)} = -\frac{1}{3s} + \frac{8}{15}\frac{1}{s-3} + \frac{4}{5}\frac{1}{s+2}$$

and $y = -\frac{1}{3} + \frac{8}{15}e^{3t} + \frac{4}{5}e^{-2t}$.

8.3.16.

$$(s^2 - 1)Y(s) = \frac{1}{s} + s = \frac{1 + s^2}{s}.$$

Since $s^2 - 1 = (s-1)(s+1)$,

$$Y(s) = \frac{1 + s^2}{s(s-1)(s+1)} = -\frac{1}{s} + \frac{1}{s-1} + \frac{1}{s+1}$$

and $y = -1 + e^t + e^{-t}$.

8.3.18.

$$(s^2+s)Y(s) = \frac{2}{s-3} + (4-s) - 1 = \frac{2-(s-3)^2}{s-3}.$$

Since $s^2+s = s(s+1)$,

$$Y(s) = \frac{2-(s-3)^2}{s(s+1)(s-3)} = \frac{7}{3s} - \frac{7}{2}\frac{1}{s+1} + \frac{1}{6}\frac{1}{s-3}$$

and $y = \frac{7}{3} - \frac{7}{2}e^{-t} + \frac{1}{6}e^{3t}$.

8.3.20.

$$(s^2+1)Y(s) = \frac{1}{s^2} + 2, \quad \text{so} \quad Y(s) = \frac{1}{(s^2+1)s^2} + \frac{2}{s^2+1}.$$

Substituting $x = s^2$ into

$$\frac{1}{(x+1)x} = \frac{1}{x+1} - \frac{1}{x} \quad \text{yields} \quad \frac{1}{(s^2+1)s^2} = \frac{1}{s^2} - \frac{1}{s^2+1},$$

so $Y(s) = \frac{1}{s^2} + \frac{1}{s^2+1}$ and $y = t + \sin t$.

8.3.22.

$$(s^2+5s+6)Y(s) = \frac{2}{s+1} + (3+s) + 5 = \frac{2+(s+1)(s+8)}{s+1}.$$

Since $s^2+5s+6 = (s+2)(s+3)$,

$$Y(s) = \frac{2+(s+1)(s+8)}{(s+1)(s+2)(s+3)} = \frac{1}{s+1} + \frac{4}{s+2} - \frac{4}{s+3}$$

and $y = e^{-t} + 4e^{-2t} - 4e^{-3t}$.

8.3.24.

$$(s^2-2s-3)Y(s) = \frac{10s}{s^2+1} + (7+2s) - 4 = \frac{10s}{s^2+1} + (2s+3).$$

Since $s^2-2s-3 = (s-3)(s+1)$,

$$Y(s) = \frac{10s}{(s-3)(s+1)(s^2+1)} + \frac{2s+3}{(s-3)(s+1)}. \tag{A}$$

$$\frac{2s+3}{(s-3)(s+1)} = \frac{9}{4}\frac{1}{s-3} - \frac{1}{4}\frac{1}{s+1} \leftrightarrow \frac{9}{4}e^{3t} - \frac{1}{4}e^{-t}. \tag{B}$$

$$\frac{10s}{(s-3)(s+1)(s^2+1)} = \frac{A}{s-3} + \frac{B}{s+1} + \frac{Cs+D}{s^2+1}$$

where

$$(A(s+1)+B(s-3))(s^2+1) + (Cs+D)(s-3)(s+1) = 10s.$$

$$\begin{aligned} 40A &= 30 && \text{(set } s=3\text{);} \\ -8B &= -10 && \text{(set } s=-1\text{);} \\ A-3B-3D &= 0 && \text{(set } s=0\text{);} \\ A+B+C &= 0 && \text{(equate coefficients of } s^3\text{).} \end{aligned}$$

Solving this system yields $A = 3/4$, $B = 5/4$, $C = -2$, $D = -1$. Therefore,

$$\begin{aligned}\frac{10s}{(s-3)(s+1)(s^2+1)} &= \frac{3}{4}\frac{1}{s-3} + \frac{5}{4}\frac{1}{s+1} - \frac{2s+1}{s^2+1}\\ &\leftrightarrow \frac{3}{4}e^{3t} + \frac{5}{4}e^{-t} - 2\cos t - \sin t.\end{aligned}$$

From this, (A), and (B), $y = -\sin t - 2\cos t + 3e^{3t} + e^{-t}$.

8.3.26.

$$\begin{aligned}(s^2+4)Y(s) &= \frac{16}{s^2+4} + \frac{9s}{s^2+1} + s, \quad \text{so}\\ Y(s) &= \frac{16}{(s^2+4)^2} + \frac{9s}{(s^2+4)(s^2+1)} + \frac{s}{s^2+4}.\end{aligned}$$

From the table of Laplace transforms,

$$\begin{aligned}t\cos 2t \quad \leftrightarrow \quad \frac{s^2-4}{(s^2+4)^2} &= \frac{s^2+4}{(s^2+4)^2} - \frac{8}{(s^2+4)^2}\\ &= \frac{1}{s^2+4} - \frac{8}{(s^2+4)^2}.\end{aligned}$$

Therefore,

$$\frac{8}{(s^2+4)^2} = \frac{1}{s^2+4} - L(t\cos 2t), \text{ so } \frac{16}{(s^2+4)^2} \leftrightarrow \sin 2t - 2t\cos 2t. \tag{A}$$

Substituting $x = s^2$ into

$$\frac{9}{(x+4)(x+1)} = \frac{3}{x+1} - \frac{3}{x+4}$$

and multiplying by s yields

$$\frac{9s}{(s^2+4)(s^2+1)} = \frac{3s}{s^2+1} - \frac{3s}{s^2+4} \leftrightarrow 3\cos t - 3\cos 2t. \tag{B}$$

Finally,

$$\frac{s}{s^2+4} \leftrightarrow \cos 2t. \tag{C}$$

Adding (A), (B), and (C) yields $y = -(2t+2)\cos 2t + \sin 2t + 3\cos t$.

28.

$$(s^2+2s+2)Y(s) = \frac{2}{s^2} + (-7+2s) + 4.$$

Since $(s^2+2s+2) = (s+1)^2 + 1$,

$$Y(s) = \frac{2}{s^2((s+1)^2+1)} + \frac{2s-3}{(s+1)^2+1}. \tag{A}$$

$$\frac{2s-3}{(s+1)^2+1} = \frac{2(s+1)-5}{(s+1)^2+1} \leftrightarrow e^{-t}(2\cos t - 5\sin t). \tag{B}$$

$$\frac{2}{s^2((s+1)^2+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C(s+1)+D}{(s+1)^2+1},$$

where $(As+B)\left((s+1)^2+1\right)+s^2\left(C(s+1)+D\right)=2$.

$$\begin{array}{rcll} 2B &=& 2 & \text{(set } s=0\text{);}\\ -A+B+D &=& 2 & \text{(set } s=-1\text{);}\\ A+C &=& 0 & \text{(equate coefficients of } s^3\text{);}\\ 2A+B+C+D &=& 0 & \text{(equate coefficients of } s^2\text{).} \end{array}$$

Solving this system yields $A=-1$, $B=1$, $C=1$, $D=0$. Therefore,

$$\frac{2}{s^2((s+1)^2+1)} = -\frac{1}{s}+\frac{1}{s^2}+\frac{(s+1)}{(s+1)^2+1} \leftrightarrow -1+t+e^{-t}\cos t.$$

From this, (A), and (B), $y=-1+t+e^{-t}(\cos t-5\sin t)$.

8.3.30. $(s^2+4s+5)Y(s)=\dfrac{(s+1)+3}{(s+1)^2+1}+4$. Since $(s^2+4s+5)=(s+2)^2+1$,

$$Y(s)=\frac{s+4}{((s+1)^2+1)((s+2)^2+1)}+\frac{4}{(s+2)^2+1}. \qquad \text{(A)}$$

$$\frac{4}{(s+2)^2+1}\leftrightarrow 4e^{-2t}\sin t. \qquad \text{(B)}$$

$$\frac{s+4}{((s+1)^2+1)((s+2)^2+1)}=\frac{A(s+1)+B}{(s+1)^2+1}+\frac{C(s+2)+D}{(s+2)^2+1},$$

where $(A(s+1)+B)\left((s+2)^2+1\right)+(C(s+2)+D)\left((s+1)^2+1\right)=4+s$.

$$\begin{array}{rcll} 5A+5B+4C+2D &=& 4 & \text{(set } s=0\text{);}\\ 2B+C+D &=& 3 & \text{(set } s=-1\text{);}\\ -A+B+2D &=& 2 & \text{(set } s=-2\text{);}\\ A+C &=& 0 & \text{(equate coefficients of } s^3\text{).} \end{array}$$

Solving this system yields $A=-1$, $B=1$, $C=1$, $D=0$. Therefore,

$$\frac{s+4}{((s+1)^2+1)((s+2)^2+1)}=\frac{-(s+1)+1}{(s+1)^2+1}+\frac{s+2}{(s+2)^2+1}, \leftrightarrow e^{-t}(-\cos t+\sin t)+e^{-2t}\cos t.$$

From this, (A), and (B), $y=e^{t}(-\cos t+\sin t)+e^{-2t}(\cos t+4\sin t)$.

8.3.32.

$$(2s^2-3s-2)Y(s)=\frac{4}{s-1}+2(-2+s)-3=\frac{4+(2s-7)(s-1)}{s-1}$$

Since $2s^2-3s-2=(s-2)(2s+1)$,

$$Y(s)=\frac{4+(2s-7)(s-1)}{2(s-2)(s-1)(s+1/2)}=\frac{1}{5}\frac{1}{s-2}-\frac{4}{3}\frac{1}{s-1}+\frac{32}{15}\frac{1}{s+1/2}$$

and $y=\dfrac{1}{5}e^{2t}-\dfrac{4}{3}e^{t}+\dfrac{32}{15}e^{-t/2}$.

8.3.34.

$$(2s^2+2s+1)Y(s)=\frac{2}{s^2}+2(-1+s)+2=\frac{2}{s^2}+2s.$$

Since $2s^2+2s+1 = 2((s+1/2)^2+1/4)$,

$$Y(s) = \frac{1}{s^2((s+1/2)^2+1/4)} + \frac{s}{((s+1/2)^2+1/4)}. \tag{A}$$

$$\frac{s}{((s+1/2)^2+1/4)} \leftrightarrow e^{-t/2}(\cos(t/2)-\sin(t/2)). \tag{B}$$

$$\frac{1}{s^2((s+1/2)^2+1/4)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C(s+1/2)+D}{((s+1/2)^2+1/4)}$$

where

$$(As+B)((s+1/2)^2+1/4) + (C(s+1/2)+D)s^2 = 1.$$

$$\begin{array}{rcll} B &=& 2 & (\text{set } s=0); \\ -A+2B+2D &=& 8 & (\text{set } s=-1/2); \\ 5A+10B+2C+2D &=& 8 & (\text{set } s=1/2); \\ A+C &=& 0 & (\text{equate coefficients of } s^3). \end{array}$$

Solving this system yields $A=-4$, $B=2$, $C=4$, $D=0$. Therefore,

$$\begin{aligned} \frac{1}{s^2((s+1/2)^2+1/4)} &= -\frac{4}{s} + \frac{2}{s^2} + \frac{4(s+1/2)}{(s+1/2)^2+1/4} \\ &\leftrightarrow -4+2t+4e^{-t/2}\cos(t/2). \end{aligned}$$

This, (A), and (B) imply that $y = e^{-t/2}(5\cos(t/2)-\sin(t/2)) + 2t - 4$.

8.3.36.

$$(4s^2+4s+1)Y(s) = \frac{3+s}{s^2+1} + 4(-1+2s) + 8 = \frac{3+s}{s^2+1} + 4(-1+2s) + 8s + 4.$$

Since $4s^2+4s+1 = 4(s+1/2)^2$,

$$Y(s) = \frac{3+s}{4(s+1/2)^2(s^2+1)} + \frac{2}{s+1/2}. \tag{A}$$

$$\frac{3+s}{4(s+1/2)^2(s^2+1)} = \frac{A}{s+1/2} + \frac{B}{(s+1/2)^2} + \frac{Cs+D}{s^2+1}$$

where

$$(A(s+1/2)+B)(s^2+1) + (Cs+D)(s+1/2)^2 = \frac{3+s}{4}.$$

$$\begin{array}{rcll} 10B &=& 5 & (\text{set } s=-1/2); \\ 2A+4B+D &=& 3 & (\text{set } s=0); \\ 12A+8B+9C+9D &=& 4 & (\text{set } s=1); \\ A+C &=& 0 & (\text{equate coefficients of } s^3). \end{array}$$

Solving this system yields $A=3/5$, $B=1/2$, $C=-3/5$, $D=-1/5$. Therefore,

$$\begin{aligned} \frac{3+s}{4(s+1/2)^2(s^2+1)} &= \frac{3}{5}\frac{1}{s+1/2} + \frac{1}{2}\frac{1}{(s+1/2)^2} - \frac{1}{5}\frac{3s+1}{s^2+1}. \\ &\leftrightarrow \frac{3}{5}e^{-t/2} + \frac{1}{2}te^{-t/2} - \frac{1}{5}(3\cos t + \sin t). \end{aligned}$$

Since $\dfrac{2}{s+1/2} \leftrightarrow 2e^{-t/2}$, this and (A) imply that $y = \dfrac{e^{-t/2}}{10}(5t+26) - \dfrac{1}{5}(3\cos t + \sin t)$.

8.3.38. Transforming the initial value problem

$$ay'' + by' + cy = 0, \; y(0) = 1, \; y'(0) = 0$$

yields $(as^2 + bs + c)Y(s) = as + b$, so $Y(s) = \dfrac{as+b}{as^2+bs+c}$. Therefore, $y_1 = L^{-1}\left(\dfrac{as+b}{as^2+bs+c}\right)$ satisfies the initial conditions $y_1(0) = 1$, $y_1'(0) = 0$.

Transforming the initial value problem

$$ay'' + by' + cy = 0, \; y(0) = 0, \; y'(0) = 1$$

yields $(as^2 + bs + c)Y(s) = a$, so $Y(s) = \dfrac{a}{as^2+bs+c}$. Therefore, $y_2 = L^{-1}\left(\dfrac{a}{as^2+bs+c}\right)$ satisfies the initial conditions $y_1(0) = 0$, $y_1'(0) = 1$.

8.4 THE UNIT STEP FUNCTION

8.4.2.

$$L(f) = \int_0^\infty e^{-st} f(t)\,dt = \int_0^1 e^{-st} t\,dt + \int_1^\infty e^{-st}\,dt. \tag{A}$$

To relate the first term to a Laplace transform we add and subtract $\int_1^\infty e^{-st} t\,dt$ in (A) to obtain

$$L(f) = \int_0^\infty e^{-st} t\,dt + \int_1^\infty e^{-st}(1-t)\,dt = L(t) - \int_1^\infty e^{-st}(t-1)\,dt. \tag{B}$$

Letting $t = x + 1$ in the last integral yields

$$\int_1^\infty e^{-st}(t-1)\,dt = -\int_0^\infty e^{-s(x+1)} x\,dx = e^{-s} L(t).$$

This and (B) imply that $L(f) = (1 - e^{-s})L(t) = \dfrac{1-e^{-s}}{s^2}$.

Alternatively, $f(t) = t - u(t-1)(t-1) \leftrightarrow (1 - e^{-s})L(t) = \dfrac{1-e^{-s}}{s^2}$.

8.4.4.

$$L(f) = \int_0^\infty e^{-st} f(t)\,dt = \int_0^1 e^{-st}\,dt + \int_1^\infty e^{-st}(t+2)\,dt. \tag{A}$$

To relate the first term to a Laplace transform we add and subtract $\int_1^\infty e^{-st}\,dt$ in (A) to obtain

$$L(f) = \int_0^\infty e^{-st}\,dt + \int_1^\infty e^{-st}(t+1)\,dt = L(t) + \int_1^\infty e^{-st}(t+1)\,dt. \tag{B}$$

Letting $t = x + 1$ in the last integral yields

$$\int_1^\infty e^{-st}(t+1)\,dt = \int_0^\infty e^{-s(x+1)}(x+2)\,dx = e^{-s} L(t+2).$$

This and (B) imply that $L(f) = L(1) + e^{-s} L(t+2) = \dfrac{1}{s} + e^{-s}\left(\dfrac{1}{s^2} + \dfrac{2}{s}\right)$.

Alternatively,

$$f(t) = 1 + u(t-1)(t+1) \leftrightarrow L(1) + e^{-s}L(t+2) = \frac{1}{s} + e^{-s}\left(\frac{1}{s^2} + \frac{2}{s}\right).$$

8.4.6.

$$L(f) = \int_0^\infty e^{-st} f(t)\,dt = \int_0^1 e^{-st}t^2 = L(t^2) - \int_1^\infty t^2\,dt. \tag{A}$$

Letting $t = x+1$ in the last integral yields

$$\int_1^\infty e^{-st}t^2\,dt = \int_0^\infty e^{-s(x+1)}(t^2+2t+1)\,dx = e^{-s}L(t^2+2t+1).$$

This and (A) imply that

$$L(f) = L(t^2) + e^{-s}L(t^2+2t+1) = \frac{2}{s^3} - e^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right).$$

Alternatively,

$$f(t) = t^2\,(1-u(t-1)) \leftrightarrow L(t^2) + e^{-s}L(t^2+2t+1) = \frac{2}{s^3} - e^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right)$$

8.4.8. $f(t) = t^2 + 2 + u(t-1)(t - t^2 - 2)$. Since $t^2 + 2 \leftrightarrow \dfrac{2}{s^3} + \dfrac{2}{s}$ and

$$\begin{aligned} L\left(u(t-1)(t-t^2-2)\right) &= e^{-s}L\left((t+1) - (t+1)^2 - 2\right) \\ &= -e^{-s}L(t^2+t+2) = -e^{-s}\left(\frac{2}{s^3} + \frac{1}{s^2} + \frac{2}{s}\right), \end{aligned}$$

it follows that $F(s) = \dfrac{2}{s^3} + \dfrac{2}{s} - e^{-s}\left(\dfrac{2}{s^3} + \dfrac{1}{s^2} + \dfrac{2}{s}\right)$.

8.4.10. $f(t) = e^{-t} + u(t-1)(e^{-2t} - e^{-t}) \leftrightarrow L(e^{-t}) + e^{-s}L(e^{-2(t+1)}) - e^{-s}L(e^{-t-1}) = L(e^{-t}) + e^{-(s+2)}L(e^{-2t}) - e^{-(s+1)}L(e^{-t}) = \dfrac{1-e^{-(s+1)}}{s+1} + \dfrac{e^{-(s+2)}}{s+2}$.

8.4.12. $f(t) = [u(t-1) - u(t-2)]\,t \leftrightarrow e^{-s}L(t+1) - e^{-2s}L(t+2)$
$= e^{-s}\left(\dfrac{1}{s^2} + \dfrac{1}{s}\right) - e^{-2s}\left(\dfrac{1}{s^2} + \dfrac{2}{s}\right)$.

8.4.14.

$$\begin{aligned} f(t) &= t - 2u(t-1)(t-1) + u(t-2)(t+4) \leftrightarrow \frac{1}{s^2} - 2e^{-s}L(t) + e^{-2s}L(t+6) \\ &= \frac{1}{s^2} - \frac{2e^{-s}}{s^2} + e^{-2s}\left(\frac{1}{s^2} + \frac{6}{s}\right). \end{aligned}$$

8.4.16. $f(t) = 2 - 2u(t-1)t + u(t-3)(5t-2) \leftrightarrow L(2) - 2e^{-s}L(t+1) + e^{-3s}L(5t+13) = \dfrac{2}{s} - e^{-s}\left(\dfrac{2}{s^2} + \dfrac{2}{s}\right) + e^{-3s}\left(\dfrac{5}{s^2} + \dfrac{13}{s}\right)$.

8.4.18. $f(t) = (t+1)^2 + u(t-1)\left((t+2)^2 - (t+1)^2\right) = t^2 + 2t + 1 + u(t-1)(2t+3) \leftrightarrow$ $L(t^2+2t+1) + e^{-s}L(2t+5) = \dfrac{2}{s^3} + \dfrac{2}{s^2} + \dfrac{1}{s} + e^{-s}\left(\dfrac{2}{s^2} + \dfrac{5}{s}\right)$.

8.4.20. $\dfrac{1}{s(s+1)} = \dfrac{1}{s} - \dfrac{1}{s+1} \leftrightarrow 1 - e^{-t} \Rightarrow e^{-s}\dfrac{1}{s(s+1)} \leftrightarrow u(t-1)\left(1 - e^{-(t-1)}\right) = \begin{cases} 0, & 0 \le t < 1, \\ 1 - e^{-(t-1)}, & t \ge 1. \end{cases}$

8.4.22.

$$\frac{3}{s} - \frac{1}{s^2} \leftrightarrow 3 - t \Rightarrow e^{-s}\left(\frac{3}{s} - \frac{1}{s^2}\right) \leftrightarrow u(t-1)\left(3 - (t-1)\right) = u(t-1)(4-t);$$

$$\frac{1}{s} + \frac{1}{s^2} \leftrightarrow 1 + t \Rightarrow e^{-3s}\left(\frac{1}{s} + \frac{1}{s^2}\right) \leftrightarrow u(t-3)\left(1 + (t-3)\right) = u(t-3)(t-2);$$

therefore

$$h(t) = 2 + t + u(t-1)(4-t) + u(t-3)(t-2) = \begin{cases} 2+t, & 0 \le t < 1, \\ 6, & 1 \le t < 3, \\ t+4, & t \ge 3. \end{cases}$$

8.4.24.

$$\frac{1-2s}{s^2+4s+5} = \frac{5-2(s+2)}{(s+2)^2+1} \leftrightarrow e^{-2t}(5\sin t - 2\cos t);$$

therefore,

$$\begin{aligned} h(t) &= u(t-\pi)e^{-2(t-\pi)}\left(5\sin(t-\pi) - 2\cos(t-\pi)\right) \\ &= u(t-\pi)e^{-2(t-\pi)}(2\cos t - 5\sin t) \\ &= \begin{cases} 0, & 0 \le t < \pi, \\ e^{-2(t-\pi)}(2\cos t - 5\sin t), & t \ge \pi. \end{cases} \end{aligned}.$$

8.4.26. Denote $F(s) = \dfrac{3(s-3)}{(s+1)(s-2)} - \dfrac{s+1}{(s-1)(s-2)}$. Since $\dfrac{3(s-3)}{(s+1)(s-2)} = \dfrac{4}{s+1} - \dfrac{1}{s-2}$ and $\dfrac{s+1}{(s-1)(s-2)} = \dfrac{3}{s-2} - \dfrac{2}{s-1}$, $F(s) = \dfrac{4}{s+1} - \dfrac{4}{s-2} + \dfrac{2}{s-1} \leftrightarrow 4e^{-t} - 4e^{2t} + 2e^{t}$. Therefore,$e^{-2s}F(s) \leftrightarrow$ $u(t-2)\left(4e^{-(t-2)} - 4e^{2(t-2)} + 2e^{(t-2)}\right) = \begin{cases} 0, & 0 \le t < 2, \\ 4e^{-(t-2)} - 4e^{2(t-2)} + 2e^{(t-2)}, & t \ge 2. \end{cases}$

8.4.28.

$$\frac{3}{s} - \frac{1}{s^3} \leftrightarrow 3 - \frac{t^2}{2} \Rightarrow e^{-2s}\left(\frac{3}{s} - \frac{1}{s^3}\right) \leftrightarrow u(t-2)\left(3 - \frac{(t-2)^2}{2}\right) = u(t-2)\left(-\frac{t^2}{2} + 2t + 1\right);$$

$$\frac{1}{s^2} \leftrightarrow t \Rightarrow \frac{e^{-4s}}{s^2} \leftrightarrow u(t-4)(t-4);$$

therefore

$$\begin{aligned} h(t) &= 1-t^2+u(t-2)\left(-\frac{t^2}{2}+2t+1\right)+u(t-4)(t-4) \\ &= \begin{cases} 1-t^2, & 0 \le t < 2 \\ -\dfrac{3t^2}{2}+2t+2, & 2 \le t < 4, \\ -\dfrac{3t^2}{2}+3t-2, & t \ge 4. \end{cases} \end{aligned}$$

8.4.30. Let T be an arbitrary positive number. Since $\lim_{m\to\infty} t_m = \infty$, only finitely many members of $\{t_m\}$ are in $[0, T]$. Since f_m is continuous on $[t_m, \infty)$ for each m, f is piecewise continuous on $[0, T]$. If $t_M \le t < t_{M+1}$, then $u(t-t_m) = 1$ if $m \le M$, while $u(t-t_m) = 0$ if $m > M$. Therefore, $f(t) = f_0(t) + \sum_{m=1}^{M}(f_m(t) - f_{m-1}(t)) = f_M(t)$

8.4.32. Since $\sum_{m=0}^{\infty} e^{-\rho K m}$ converges if $\rho > 0$, $\sum_{m=0}^{\infty} e^{-\rho t_m}$ converges if $\rho > 0$, by the comparison test. Therefore,(C) of Exercise 8.3.31 holds if $s > s_0 + \rho$ if ρ is any positive number. This implies that it holds if $s > s_0$.

8.4.34. Let $t_m = m$ and $f_m(t) = (-1)^m$, $m = 0, 1, 2, \dots$. Then $f_m(t) - f_{m-1}(t) = (-1)^m 2$, so $f(t) = 1 + 2\sum_{m=1}^{\infty}(-1)^m u(t-m)$ and $F(s) = \frac{1}{s}\left(1 + 2\sum_{m=1}^{\infty}(-1)^m e^{-ms}\right)$ Substituting $x = e^{-s}$ in the identity $\sum_{m=1}^{\infty}(-1)^m x^m = -\frac{x}{1+x}$ $(|x| < 1)$ yields $F(s) = \frac{1}{s}\left(1 - \frac{2e^{-s}}{1+e^{-s}}\right) = \frac{1}{s}\frac{1-e^{-s}}{1+e^{-s}}$.

8.4.36. Let $t_m = m$ and $f_m(t) = (-1)^m m$, $m = 0, 1, 2, \dots$. Then $f_m(t) - f_{m-1}(t) = (-1)^m(2m-1)$, so $f(t) = \sum_{m=1}^{\infty}(-1)^m(2m-1)u(t-m)$and $F(s) = \frac{1}{s}\sum_{m=1}^{\infty}(-1)^m(2m-1)e^{-ms}$. Substituting $x = e^{-s}$ in the identities $\sum_{m=1}^{\infty}(-1)^m x^m = -\frac{x}{1+x}$ and $\sum_{m=1}^{\infty}(-1)^m m x^m = -\frac{x}{(1+x)^2}$ $(|x| < 1)$ yields $F(s) = \frac{1}{s}\left[\frac{e^{-s}}{1+e^{-s}} - \frac{2e^{-s}}{(1+e^{-s})^2}\right] = \frac{1}{s}\frac{(1-e^{s})}{(1+e^{s})^2}$.

8.5 CONSTANT COEEFFICIENT EQUATIONS WITH PIECEWISE CONTINUOUS FORCING FUNCTIONS

8.5.2. $y'' + y = 3 + u(t-4)(2t-8),\ y(0) = 1,\ y'(0) = 0$. Since

$$L\left(u(t-4)(2t-8)\right) = e^{-4s}L\left(2(t+4)-8\right) = e^{-4s}L(2t) = \frac{2e^{-4s}}{s^2},$$

$$(s^2+1)Y(s) = \frac{3}{s} + \frac{2e^{-4s}}{s^2} + s.$$

$$\begin{aligned}
Y(s) &= \frac{3}{s(s^2+1)} + \frac{2e^{-4s}}{s^2(s^2+1)} + \frac{s}{s^2+1} \\
&= 3\left(\frac{1}{s} - \frac{s}{s^2+1}\right) + 2e^{-4s}\left(\frac{1}{s^2} - \frac{1}{s^2+1}\right) + \frac{s}{s^2+1} \\
&= \frac{3}{s} - \frac{2s}{s^2+1} + 2e^{-4s}\left(\frac{1}{s^2} - \frac{1}{s^2+1}\right).
\end{aligned}$$

Since

$$\frac{1}{s^2} - \frac{1}{s^2+1} \leftrightarrow t - \sin t \Rightarrow e^{-4s}\left(\frac{1}{s^2} - \frac{1}{s^2+1}\right) \leftrightarrow u(t-4)\,(t-4-\sin(t-4)),$$

$y = 3 - 2\cos t + 2u(t-4)\,(t-4-\sin(t-4))$.

8.5.4. $y'' - y = e^{2t} + u(t-2)(1-e^{2t}),\ y(0) = 3,\ y'(0) = -1$. Since

$$L(u(t-2)(1-e^{2t})) = e^{-2s}L(1-e^{2(t+2)}) = e^{-2s}\left(\frac{1}{s} - \frac{e^4}{s-2}\right),$$

$$(s^2-1)Y(s) = \frac{1}{s-2} + e^{-2s}\left(\frac{1}{s} - \frac{e^4}{s-2}\right) + (-1+3s).$$

Therefore,

$$\begin{aligned}
Y(s) &= \frac{1}{(s-1)(s+1)(s-2)} + \frac{3s-1}{(s-1)(s+1)} \\
&\quad + e^{-2s}\left(\frac{1}{s(s-1)(s+1)} - \frac{e^4}{(s-1)(s+1)(s-2)}\right).
\end{aligned}$$

$$\begin{aligned}
\frac{1}{(s-1)(s+1)(s-2)} &= -\frac{1}{2}\frac{1}{s-1} + \frac{1}{6}\frac{1}{s+1} + \frac{1}{3}\frac{1}{s-2} \\
&\leftrightarrow -\frac{1}{2}e^t + \frac{1}{6}e^{-t} + \frac{1}{3}e^{2t};
\end{aligned}$$

$$\frac{e^{-2s}e^4}{(s-1)(s+1)(s-2)} \leftrightarrow u(t-2)\left(-\frac{1}{2}e^{t+2} + \frac{1}{6}e^{-(t-6)} + \frac{1}{3}e^{2t}\right);$$

$$\frac{1}{s(s-1)(s+1)} = -\frac{1}{s} + \frac{1}{2}\frac{1}{s-1} + \frac{1}{2}\frac{1}{s+1} \leftrightarrow -1 + \frac{1}{2}e^t + \frac{1}{2}e^{-t};$$

$$\frac{e^{-2s}}{s(s-1)(s+1)} \leftrightarrow u(t-2)\left(-1 + \frac{1}{2}e^{t-2} + \frac{1}{2}e^{-(t-2)}\right);$$

$$\frac{3s-1}{(s-1)(s+1)} = \frac{1}{s-1} + \frac{2}{s+1} \leftrightarrow e^t + 2e^{-t}.$$

Therefore,

$$y = \frac{1}{2}e^t + \frac{13}{6}e^{-t} + \frac{1}{3}e^{2t} + u(t-2)\left(-1 + \frac{1}{2}e^{t-2} + \frac{1}{2}e^{-(t-2)} + \frac{1}{2}e^{t+2} - \frac{1}{6}e^{-(t-6)} - \frac{1}{3}e^{2t}\right).$$

8.5.6. Note that $|\sin t| = \sin t$ if $0 \le t < \pi$, while $|\sin t| = -\sin t$ if $\pi \le t < 2\pi$. Rewrite the initial value problem as

$$y'' + 4y = \sin t - 2u(t-\pi)\sin t + u(t-2\pi)\sin t,\ y(0) = -3,\ y'(0) = 1.$$

Since

$$L\left(u(t-\pi)\sin t\right) = e^{-\pi s}L(\sin(t+\pi)) = -e^{-\pi s}L(\sin t)$$

and

$$L\left(u(t-2\pi)\sin t\right) = e^{-2\pi s}L(\sin(t+2\pi)) = e^{-2\pi s}L(\sin t),$$

$$(s^2+4)Y(s) = \frac{1+2e^{-\pi s}+e^{-2\pi s}}{(s^2+1)} + 1 - 3s, \text{ so } Y(s) = \frac{1+2e^{-\pi s}+e^{-2\pi s}}{(s^2+1)(s^2+4)} + \frac{1-3s}{s^2+4}.$$

$$\frac{1}{(s^2+1)(s^2+4)} = \left(\frac{1}{s^2+1} - \frac{1}{s^2+4}\right) \leftrightarrow \frac{1}{3}\sin t - \frac{1}{6}\sin 2t;$$

therefore

$$\begin{aligned}\frac{e^{-\pi s}}{(s^2+1)(s^2+4)} &\leftrightarrow u(t-\pi)\left(\frac{1}{3}\sin(t-\pi) - \frac{1}{6}\sin 2(t-\pi)\right)\\ &= -u(t-\pi)\left(\frac{1}{3}\sin t + \frac{1}{6}\sin 2t\right)\end{aligned}$$

and

$$\begin{aligned}\frac{e^{-2\pi s}}{(s^2+1)(s^2+4)} &\leftrightarrow u(t-2\pi)\left(\frac{1}{3}\sin(t-2\pi) - \frac{1}{6}\sin 2(t-2\pi)\right)\\ &= u(t-2\pi)\left(\frac{1}{3}\sin t - \frac{1}{6}\sin 2t\right);\end{aligned}$$

therefore

$$y = \frac{1}{3}\sin 2t - 3\cos 2t + \frac{1}{3}\sin t - 2u(t-\pi)\left(\frac{1}{3}\sin t + \frac{1}{6}\sin 2t\right) + u(t-2\pi)\left(\frac{1}{3}\sin t - \frac{1}{6}\sin 2t\right).$$

8.5.8. $y'' + 9y = \cos t + u(t-3\pi/2)(\sin t - \cos t),\ y(0) = 0,\ y'(0) = 0$. Since

$$\begin{aligned}L\left(u(t-3\pi/2)(\sin t - \cos t)\right) &= e^{-3\pi s/2}L\left(\sin(t+3\pi/2) - \cos(t+3\pi/2)\right)\\ &\quad -e^{-3\pi s/2}L(\cos t + \sin t),\end{aligned}$$

$$(s^2+9)Y(s) = \frac{1}{s^2+1} - e^{-3\pi s/2}\frac{s+1}{s^2+1}, \quad \text{so} \quad Y(s) = \frac{1}{(s^2+1)(s^2+9)} - e^{-3\pi s/2}\frac{s+1}{(s^2+1)(s^2+9)}.$$

$$\frac{1}{(s^2+1)(s^2+9)} = \frac{1}{8}\left(\frac{1}{s^2+1} - \frac{1}{s^2+9}\right) \leftrightarrow \frac{1}{8}\left(\sin t - \frac{1}{3}\sin 3t\right) \quad \text{and}$$

$$\frac{s}{(s^2+1)(s^2+9)} = \frac{1}{8}\left(\frac{s}{s^2+1} - \frac{s}{s^2+9}\right) \leftrightarrow \frac{1}{8}\left(\cos t - \cos 3t\right).$$

$$\begin{aligned}\frac{s+1}{(s^2+1)(s^2+9)} &= \frac{s+1}{(s^2+1)(s^2+9)} = \frac{s+1}{8}\left(\frac{s+1}{s^2+1} - \frac{s+1}{s^2+9}\right)\\ &\leftrightarrow \frac{1}{8}\left(\cos t + \sin t - \cos 3t - \frac{1}{3}\sin 3t\right), \text{ so}\end{aligned}$$

$$\begin{aligned}
e^{-3\pi s/2}\frac{s+1}{(s^2+1)(s^2+9)} \quad &\leftrightarrow \quad \frac{u(t-3\pi/2)}{8}\left(\cos(t-3\pi/2)+\sin(t-3\pi/2)\right. \\
&\qquad \left. -\cos 3(t-3\pi/2)-\frac{1}{3}\sin 3(t-\pi/2)\right) \\
&= \quad \frac{u(t-3\pi/2)}{8}\left(\sin t-\cos t+\sin 3t-\frac{1}{3}\cos 3t\right).
\end{aligned}$$

Therefore, $y=\frac{1}{8}(\cos t-\cos 3t)-\frac{1}{8}u(t-3\pi/2)\left(\sin t-\cos t+\sin 3t-\frac{1}{3}\cos 3t\right)$.

8.5.10. $y''+y=t-2u(t-\pi)t,\ y(0)=0,\ y'(0)=0$. Since

$$L\left(u(t-\pi)t\right)=e^{-\pi s}L\left(t+\pi\right)=e^{-\pi s}\left(\frac{1}{s^2}+\frac{\pi}{s}\right),$$

$$(s^2+1)Y(s)=\frac{1}{s^2}-2e^{-\pi s}\left(\frac{1}{s^2}+\frac{\pi}{s}\right);$$

$$\begin{aligned}
Y(s) &= \frac{1}{s^2(s^2+1)}-2e^{-\pi s}\left(\frac{1}{s^2(s^2+1)}+\frac{\pi}{s(s^2+1)}\right) \\
&= \left(\frac{1}{s^2}-\frac{1}{s^2+1}\right)-2e^{-\pi s}\left(\frac{1}{s^2}-\frac{1}{s^2+1}\right)-2\pi e^{-\pi s}\left(\frac{1}{s}-\frac{s}{s^2+1}\right).
\end{aligned}$$

Since

$$\begin{aligned}
\frac{1}{s^2}-\frac{1}{s^2+1} \quad &\leftrightarrow \quad t-\sin t \Rightarrow e^{-\pi s}\left(\frac{1}{s^2}-\frac{1}{s^2+1}\right) \\
&\leftrightarrow \quad u(t-\pi)\left(t-\pi-\sin(t-\pi)\right)=u(t-\pi)(t-\pi+\sin t)
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{s}-\frac{s}{s^2+1} \quad &\leftrightarrow \quad 1-\cos t \Rightarrow e^{-\pi s}\left(\frac{1}{s}-\frac{s}{s^2+1}\right) \\
&\leftrightarrow \quad u(t-\pi)\left(1-\cos(t-\pi)\right)=u(t-\pi)(1+\cos t),
\end{aligned}$$

$y=t-\sin t-2u(t-\pi)(t+\sin t+\pi\cos t)$.

8.5.12. $y''+y=t-3u(t-2\pi)t,\ y(0)=1,\ y'(0)=2$;

$$L(u(t-2\pi)t)=e^{-2\pi s}L(t+2\pi)=e^{-2\pi s}\left(\frac{1}{s^2}+\frac{2\pi}{s}\right);$$

$$(s^2+1)Y(s)=\frac{1-3e^{-2\pi s}}{s^2}-\frac{6\pi e^{-2\pi s}}{s}+2+s;$$

$$Y(s)=\frac{1-3e^{-2\pi s}}{s^2(s^2+1)}-\frac{6\pi e^{-2\pi s}}{s(s^2+1)}+\frac{2+s}{s^2+1};$$

$$\frac{1}{s^2(s^2+1)}=\frac{1}{s^2}-\frac{1}{s^2+1}\leftrightarrow t-\sin t;$$

$$\frac{e^{-2\pi s}}{s^2(s^2+1)}\leftrightarrow u(t-2\pi)((t-2\pi-\sin(t-2\pi))=u(t-2\pi)(t-2\pi-\sin t);$$

$$\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1} \leftrightarrow 1 - \cos t;$$

$$\frac{e^{-2\pi s}}{s(s^2+1)} \leftrightarrow u(t-2\pi)(1-\cos(t-2\pi)) = u(t-2\pi)(1-\cos t);$$

$$\frac{2+s}{s^2+1} \leftrightarrow 2\sin t + \cos t;$$

$y = t + \sin t + \cos t - u(t-2\pi)(3t - 3\sin t - 6\pi\cos t)$.

8.5.14. $y'' - 4y' + 3y = -1 + 2u(t-1),\ y(0) = 0,\ y'(0) = 0;$
$(s^2 - 4s + 3)Y(s) = \dfrac{-1+2e^{-s}}{s};\ Y(s) = \dfrac{-1+2e^{-s}}{s(s-1)(s-3)};$
$\dfrac{1}{s(s-1)(s-3)} = \dfrac{1}{3s} + \dfrac{1}{6}\dfrac{1}{s-3} - \dfrac{1}{2}\dfrac{1}{s-1} \leftrightarrow \dfrac{1}{3} + \dfrac{1}{6}e^{3t} - \dfrac{1}{2}e^{t};$
$\dfrac{e^{-s}}{s(s-1)(s-3)} \leftrightarrow u(t-1)\left(\dfrac{1}{3} + \dfrac{1}{6}e^{3(t-1)} - \dfrac{1}{2}e^{t-1}\right);$
$y = -\dfrac{1}{3} - \dfrac{1}{6}e^{3t} + \dfrac{1}{2}e^{t} + u(t-1)\left(\dfrac{2}{3} + \dfrac{1}{3}e^{3(t-1)} - e^{t-1}\right).$

8.5.16. $y'' + 2y' + y = 4e^t - 4u(t-1)e^t,\ y(0) = 0,\ y'(0) = 0$. Since

$$L\left(4u(t-1)e^t\right) = e^{-s}L\left(4e^{(t+1)}\right) = \frac{4e^{-s+1}}{s-1},$$

$$(s^2 + 2s + 1)Y(s) = \frac{4}{s-1} - \frac{4e^{-s+1}}{s-1}, \quad \text{so}$$

$$Y(s) = \frac{4}{(s-1)(s+1)^2} - \frac{4e^{-s+1}}{(s-1)(s+1)^2}.$$

$$\frac{1}{(s-1)(s+1)^2} = \frac{A}{s-1} + \frac{B}{s+1} + \frac{C}{(s+1)^2},$$

where

$$A(s+1)^2 + B(s-1)(s+1) + C(s-1) = 4.$$

$$\begin{array}{rcll} A & = & 1 & (\text{set } s = 1); \\ C & = & -2 & (\text{set } s = -1); \\ A + B & = & 0 & (\text{equate coefficients of } s^2). \end{array}$$

Solving this system yields $A = 1$, $B = -1$, $C = -2$. Therefore,

$$\frac{1}{(s-1)(s+1)^2} = \frac{1}{s-1} - \frac{1}{s+1} - \frac{2}{(s+1)^2} \quad \text{and}$$

$$\begin{aligned} y &= e^t - e^{-t} - 2te^{-t} - eu(t-1)\left(e^{t-1} - e^{-(t-1)} - 2(t-1)e^{-(t-1)}\right) \\ &= e^t - e^{-t} - 2te^{-t} - u(t-1)\left(e^t - e^{-(t-2)} - 2(t-1)e^{-(t-2)}\right). \end{aligned}$$

8.5.18. $y'' - 4y' + 4y = e^{2t} - 2u(t-2)e^{2t},\ y(0) = 0,\ y'(0) = -1$. Since

$$L\left(u(t-2)e^{2t}\right) = e^{-2s}L\left(e^{2t+4}\right) = \frac{e^{-2s+4}}{s-2},$$

$$(s^2-4s+4)Y(s)=\frac{1}{s-2}-\frac{2e^{-2s+4}}{s-2}-1, \quad \text{so}$$

$$Y(s)=\frac{1}{(s-2)^3}-\frac{2e^{-2s+4}}{(s-2)^3}-\frac{1}{(s-2)^2}.$$

$$\frac{1}{(s-2)^3}\leftrightarrow\frac{t^2e^{2t}}{2}\Rightarrow\frac{e^{-2s+4}}{(s-2)^3}\leftrightarrow\frac{e^4}{2}u(t-2)e^{2(t-2)}(t-2)^2=u(t-2)\frac{(t-2)^2e^{2t}}{2};$$

therefore $y=\dfrac{t^2e^{2t}}{2}-te^{2t}-u(t-2)(t-2)^2e^{2t}$.

8.5.20. $y''+2y'+2y=1+u(t-2\pi)(t-1)-u(t-3\pi)(t+1),\ y(0)=2,\ y'(0)=-1;$

$$L(u(t-2\pi)(t-1))=e^{-2\pi s}L((t+2\pi-1))=e^{-2\pi s}\left(\frac{1}{s^2}+\frac{2\pi-1}{s}\right);$$

$$L(u(t-3\pi)(t+1))=e^{-3\pi s}L((t+3\pi+1))=e^{-3\pi s}\left(\frac{1}{s^2}+\frac{3\pi+1}{s}\right);$$

$$(s^2+2s+2)Y(s)=\frac{1}{s}+e^{-2\pi s}\left(\frac{1}{s^2}+\frac{2\pi-1}{s}\right)-e^{-3\pi s}\left(\frac{1}{s^2}+\frac{3\pi+1}{s}\right)+(-1+2s)+4.$$

Let $G(s)=\dfrac{1}{s(s^2+2s+2)}$, $H(s)=\dfrac{1}{s(s^2+2s+2)}$; then

$$Y(s)=Y_1(s)+e^{-2\pi s}Y_2(s)-e^{-3\pi s}Y_3(s), \tag{A}$$

where

$$Y_1(s)=G(s)+\frac{2s+3}{s^2+2s+2}, \tag{B}$$

$$Y_2(s)=H(s)+(2\pi-1)G(s), \tag{C}$$

$$Y_3(s)=H(s)+(3\pi+1)G(s). \tag{D}$$

Let $y_i(t)=L^{-1}(Y_i(s))$, $(i=1,2,3)$. From (A),

$$y(t)=y_1(t)+u(t-2\pi)y_2(t-2\pi)-u(t-3\pi)y_3(t-3\pi). \tag{E}$$

Find $L^{-1}(G(s))$:

$$G(s)=\frac{A}{s}+\frac{B(s+1)+C}{(s+1)^2+1}$$

where $A((s+1)^2+1)+(B(s+1)+C)s=1$. Setting $s=0$ yields $A=1/2$; setting $s=-1$ yields $A-C=1$, so $C=-1/2$; since $A+B=0$ (coefficient of x^2), $B=-1/2$. Therefore,

$$G(s)=\frac{1}{2}\left(\frac{1}{s}-\frac{(s+1)+1}{(s+1)^2+1)}\right)\leftrightarrow\frac{1}{2}-\frac{1}{2}e^{-t}(\cos t+\sin t). \tag{F}$$

Find $L^{-1}(H(s))$:

$$H(s)=\frac{A}{s}+\frac{B}{s+2}+\frac{C(s+1)+D}{(s+1)^2+1}$$

where $(As+B)((s+1)^2+1)+(C(s+1)+D)s^2=1$.

$$\begin{array}{rcll} 2B &=& 1 & \text{(set } s=0);\\ -A+B+D &=& 1 & \text{(set } s=-1);\\ 5A+5B+2C+D &=& 1 & \text{(set } s=1);\\ A+B=0 &=& 0 & \text{(equate coefficients of } s^3). \end{array}$$

Solving this system yields $A=-1/2$, $B=1/2$, $C=1/2$, $D=0$; therefore

$$H(s)=-\frac{1}{2}\left(\frac{1}{s}-\frac{1}{s^2}-\frac{s+1}{(s+1)^2+1)}\right)\leftrightarrow -\frac{1}{2}(1-t-e^{-t}\cos t). \tag{G}$$

Since

$$\frac{2s+3}{s^2+2s+2}=\frac{2(s+1)+1}{(s+1)^2+1}\leftrightarrow e^{-t}(2\cos t+\sin t),$$

(B) and (F)) imply that

$$y_1(t)=\frac{1}{2}e^{-t}(3\cos t+\sin t)+\frac{1}{2}. \tag{H}$$

From (C), (F)), and (G),

$$y_2(t)=\pi-1+\frac{t}{2}+(\pi-1)e^{-t}\cos t-\frac{2\pi-1}{2}e^{-t}\sin t,$$

so

$$y_2(t-2\pi)=-\left(e^{-(t-2\pi)}\left((\pi-1)\cos t+\frac{2\pi-1}{2}\sin t\right)+1-\frac{t}{2}\right). \tag{I}$$

From (D), (F)), and (G),

$$y_3(t)=\frac{1}{2}\left(-e^{-t}(3\pi\cos t+(3\pi+1)\sin t+t+3\pi)\right),$$

so

$$y_3(t-3\pi)=\frac{1}{2}\left(e^{-(t-3\pi)}(3\pi\cos t+(3\pi+1)\sin t+t)\right). \tag{J}$$

Now (E), (20), (I), and (J)

$$\begin{aligned} y &= \frac{1}{2}e^{-t}(3\cos t+\sin t)+\frac{1}{2} \text{ imply that}\\ &-u(t-2\pi)\left(e^{-(t-2\pi)}\left((\pi-1)\cos t+\frac{2\pi-1}{2}\sin t\right)+1-\frac{t}{2}\right)\\ &-\frac{1}{2}u(t-3\pi)\left(e^{-(t-3\pi)}(3\pi\cos t+(3\pi+1)\sin t)+t\right). \end{aligned}$$

8.5.22. **(a)** $f(t)=\sum_{n=0}^{\infty}u(t-n\pi)$; $F(s)=\frac{1}{s}\sum_{n=0}^{\infty}e^{-n\pi s}$; $Y(s)=\frac{1}{s(s^2+1)}\sum_{n=0}^{\infty}e^{-n\pi s}$; $\frac{1}{s(s^2+1)}=\frac{1}{s}-\frac{s}{s^2+1}\leftrightarrow 1-\cos t$; $\frac{e^{-n\pi s}}{s(s^2+1)}\leftrightarrow u(t-n\pi)(1-\cos(t-n\pi))=u(t-n\pi)(1-(-1)^n\cos t)$; $y(t)=\sum_{n=0}^{\infty}u(t-n\pi)(1-(-1)^n\cos t)$. If $m\pi\le t<(m+1)\pi$, $y(t)=\sum_{n=0}^{m}(1-(-1)^n\cos t)$. Therefore,

$$y(t)=\begin{cases} 2m+1-\cos t, & 2m\pi\le t<(2m+1)\pi \quad (m=0,1,\dots)\\ 2m, & (2m-1)\pi\le t<2m\pi \quad (m=1,2,\dots)\end{cases}.$$

(b) $f(t) = \sum_{n=0}^{\infty} u(t-2n\pi)t$; $F(s) = \sum_{n=0}^{\infty} e^{-2n\pi s} L(t+2n\pi s) = \sum_{n=0}^{\infty} e^{-2n\pi s}\left(\frac{1}{s^2}+\frac{2n\pi}{s}\right)$; $Y(s) = \sum_{n=0}^{\infty} e^{-2n\pi s} Y_n(s)$, where $Y_n(s) = \frac{1}{s^2(s^2+1)} + \frac{2n\pi}{s(s^2+1)} = \frac{1}{s^2} - \frac{1}{s^2+1} + \frac{2n\pi}{s} - \frac{2n\pi}{s^2+1} \leftrightarrow y_n(t) = t - \sin t + 2n\pi - 2n\pi\cos t$. Since $\cos(t-2n\pi) = \cos t$ and $\sin(t-2n\pi) = \sin t$, $e^{-2n\pi s}Y_n(s) \leftrightarrow u(t-2n\pi)y_n(t) = u(t-2n\pi)(t-\sin t - 2n\pi\cos t)$; therefore $y(t) = \sum_{n=0}^{\infty} u(t-2n\pi)(t-\sin t-2n\pi\cos t)$. If $2m\pi \le t < 2(m+1)\pi$, then

$$y(t) = \sum_{n=0}^{m}(t-\sin t-2n\pi\cos t) = (m+1)(t-\sin t - m\pi\cos t).$$

(c) $f(t) = 1+2\sum_{n=1}^{\infty}(-1)^n u(t-n\pi)$; $F(s) = \frac{1}{s}\left(1+2\sum_{n=1}^{\infty}(-1)^n e^{-n\pi s}\right)$; $Y(s) = \frac{1}{s(s^2+1)}\left(1+2\sum_{n=1}^{\infty}(-1)^n e^{-n\pi s}\right)$; $\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1} \leftrightarrow 1-\cos t$; $\frac{e^{-n\pi s}}{s(s^2+1)} \leftrightarrow u(t-n\pi)(1-\cos(t-n\pi)) = u(t-n\pi)(1-(-1)^n\cos t)$; $y(t) = 1-\cos t + 2\sum_{n=1}^{\infty}(-1)^n u(t-n\pi)(1-(-1)^n\cos t)$. If $m\pi \le t < (m+1)\pi$,

$$y(t) = 1-\cos t + 2\sum_{n=1}^{m}(-1)^n(1-(-1)^n\cos t) = (-1)^m - (2m+1)\cos t.$$

(d) $f(t) = \sum_{n=0}^{\infty} u(t-n)$; $F(s) = \frac{1}{s}\sum_{n=0}^{\infty} e^{-ns}$; $Y(s) = \frac{1}{s(s^2-1)}\sum_{n=0}^{\infty} e^{-ns}$; $\frac{1}{s(s^2-1)} = \frac{1}{2}\frac{1}{s-1} + \frac{1}{2}\frac{1}{s+1} - \frac{1}{s} \leftrightarrow \frac{1}{2}(e^t+e^{-t}-2)$; $\frac{e^{-ns}}{s(s^2-1)} \leftrightarrow \frac{u(t-n)}{2}\left(e^t+e^{-t}-2\right)$; $y(t) = \frac{1}{2}\sum_{n=0}^{\infty} u(t-n)\left(e^{t-n}+e^{-(t-n)}-2\right)$. If $m \le t < (m+1)$,

$$\begin{aligned} y(t) &= \frac{1}{2}\sum_{n=0}^{m}\left(e^{t-n}+e^{-(t-n)}-2\right) = \frac{1}{2}(e^{t-m}+e^{-t})\sum_{n=0}^{m} e^n - m - 1 \\ &= \frac{1-e^{m+1}}{2(1-e)}(e^{t-m}+e^{-t}) - m - 1. \end{aligned}$$

(e) $f(t) = (\sin t + 2\cos t)\sum_{n=0}^{\infty} u(t-2n\pi)$; $F(s) = \frac{1+2s}{s^2+1}\sum_{n=0}^{\infty} e^{-2n\pi s}$; $Y(s) = \frac{1+2s}{(s^2+1)(s^2+2s+2)}\sum_{n=0}^{\infty} e^{-2n\pi s}$;

$$\frac{1+2s}{(s^2+1)(s^2+2s+2)} = \frac{As+B}{s^2+1} + \frac{C(s+1)+D}{(s+1)^2+1}$$

where

$$(As+B)((s+1)^2+1) + (C(s+1)+D)(s^2+1) = 1+2s.$$

$$\begin{aligned} 2B+C+D &= 1 \quad \text{(set } s=0\text{);} \\ -A+B+2D &= -1 \quad \text{(set } s=-1\text{);} \\ 5A+5B+4C+2D &= 3 \quad \text{(set } s=1\text{);} \\ A+C &= 0 \quad \text{(equate coefficients of } s^3\text{).} \end{aligned}$$

Solving this system yields $A = 0$, $B = 1$, $C = 0$, $D = -1$. Therefore,

$$\begin{aligned}\frac{1+2s}{(s^2+1)(s^2+2s+2)} &= \frac{1}{s^2+1} - \frac{1}{(s+1)^2+1}\\ &\leftrightarrow \left(1-e^{-t}\right)\sin t.\end{aligned}$$

Since $\sin(t-2n\pi) = \sin t$,

$$e^{-2n\pi s}\frac{1+2s}{(s^2+1)(s^2+2s+2)} \leftrightarrow u(t-2n\pi)\left(1-e^{-(t-2n\pi)}\right)\sin t,$$

so

$$y(t) = \sin t\sum_{n=0}^{\infty} u(t-2n\pi)\left(1-e^{-(t-2n\pi)}\right).$$

If $2m\pi \le t < 2(m+1)\pi$,

$$y(t) = \sin t\sum_{n=0}^{m}\left(1-e^{-(t-2n\pi)}\right) = \left(m+1-\left(\frac{1-e^{2(m+1)\pi}}{1-e^{2\pi}}\right)e^{-t}\right)\sin t.$$

(f) $f(t) = \sum_{n=0}^{\infty} u(t-n)$; $F(s) = \frac{1}{s}\sum_{n=0}^{\infty} e^{-ns}$; $Y(s) = \frac{1}{s(s-1)(s-2)}$;

$$\frac{1}{s(s-1)(s-2)} = \frac{1}{2s} - \frac{1}{s-1} + \frac{1}{2}\frac{1}{s-2} \leftrightarrow \frac{1}{2}\left(1-2e^t+e^{2t}\right);$$

$$\frac{e^{-ns}}{s(s-1)(s-2)} \leftrightarrow \frac{1}{2}u(t-n)\left(1-2e^{t-n}+e^{2(t-n)}\right);$$

$$y(t) = \frac{1}{2}\sum_{n=0}^{\infty} u(t-n)\left(1-2e^{t-n}+e^{2(t-n)}\right).$$

If $m \le t < m+1$,

$$\begin{aligned}y(t) &= \sum_{n=0}^{m}\left(1-2e^{t-n}+e^{2(t-n)}\right) = \frac{m+1}{2} - e^{t-m}\sum_{n=0}^{m} e^n + \frac{1}{2}e^{2(t-m)}\sum_{n=0}^{m} e^{2n}\\ &= \frac{m+1}{2} - e^{t-m}\frac{1-e^{m+1}}{1-e} + \frac{1}{2}e^{2(t-m)}\frac{1-e^{2m+2}}{1-e^2}.\end{aligned}$$

8.5.24. **(a)** The assumptions imply that $y''(t) = \dfrac{f(t)-by'(t)-cy(t)}{a}$ on (α, t_0) and (t_0, β), $y''(t_0+) = \dfrac{f(t_0+)-by'(t_0)-cy(t_0)}{a}$, and $y''(t_0-) = \dfrac{f(t_0-)-by'(t_0)-cy(t_0)}{a}$. This implies the conclusion.

(b) Since y'' has a junp discontinouity at t_0, applying Exercise 8.4.23**(c)** to y' shows that y' is not differentiable at t_0. Therefore,y cannot satisfy (A) on (α, β) if f has a jump discontinuity at some t_0 in (α, β).

8.5.26. If $0 \le t < t_0$, then $y(t) = z_0(t)$. Therefore, $y(0) = z_0(0) = k_0$ and $y'(0) = z'(0) = k_1$, and

$$ay'' + by' + cy = az_0'' + bz_0' + cz_0 = f_0(t) = f(t), \quad 0 < t < t_0.$$

Now suppose that $1 \le m \le n$. For convenience, define $t_{n+1} = \infty$. If $t_m \le t < t_{m+1}$, then $y(t) = \sum_{k=0}^{m} z_m(t)$, so

$$ay'' + by + cy = \sum_{k=0}^{m}(az_k'' + bz_k' + cz_k) = f_0 + \sum_{k=1}^{m}(f_k - f_{k-1}) = f_m = f, \quad t_m < t < t_{m+1}.$$

Thus, y satisfies $ay'' + by' + cy = f$ on any open interval that does not contain any of the points t_1, t_2,..., t_n.

Since $z(t_m) = z'(t_m)$ for $m = 1, 2, \dots$, y and y' are continuous on $[0, \infty)$. Since $y''(t) = -(by'(t) + cy(t))/a$ if $t \ne t_m$ $(m = 1, 2, \dots)$, y'' has limits from the left at $t_1, \dots, t_n$.

8.6 CONVOLUTION

8.6.2. **(a)** $\sin at \leftrightarrow \dfrac{a}{s^2 + a^2}$ and $\cos bt \leftrightarrow \dfrac{s}{s^2 + b^2}$, so $H(s) = \dfrac{as}{(s^2 + a^2)(s^2 + b^2)}$.

(b) $e^t \leftrightarrow \dfrac{1}{s-1}$ and $\sin at \leftrightarrow \dfrac{a}{s^2 + a^2}$, so $H(s) = \dfrac{a}{(s-1)(s^2 + a^2)}$.

(c) $\sinh at \leftrightarrow \dfrac{a}{s^2 - a^2}$ and $\cosh at \leftrightarrow \dfrac{1}{s^2 - a^2}$, so $H(s) = \dfrac{as}{(s^2 - a^2)^2}$.

(d) $t \sin \omega t \leftrightarrow \dfrac{2\omega s}{(s^2 + \omega^2)^2}$ and $t \cos \omega t \leftrightarrow \dfrac{s^2 - \omega^2}{(s^2 + \omega^2)^2}$, so $H(s) = \dfrac{2\omega s(s^2 - \omega^2)}{(s^2 + \omega^2)^4}$.

(e) $e^t \displaystyle\int_0^t \sin \omega\tau \cos \omega(t - \tau)\, d\tau = \int_0^t (e^\tau \sin \omega\tau)\left(e^{(t-\tau)} \cos \omega(t - \tau)\right) d\tau$; $e^t \sin \omega t \leftrightarrow \dfrac{\omega}{(s-1)^2 + \omega^2}$ and $e^t \cos \omega t \leftrightarrow \dfrac{s-1}{(s-1)^2 + \omega^2}$, so $H(s) = \dfrac{(s-1)\omega}{((s-1)^2 + \omega^2)^2}$.

(f) $e^t \displaystyle\int_0^t \tau^2(t - \tau)e^\tau\, d\tau = \int_0^t \tau^2 e^{2\tau}(t - \tau)e^{(t-\tau)}\, d\tau$; $t^2 e^{2t} \leftrightarrow \dfrac{2}{(s-2)^3}$ and $te^t \leftrightarrow \dfrac{1}{(s-1)^2}$, so $H(s) = \dfrac{2}{(s-2)^3(s-1)^2}$.

(g) $e^{-t} \displaystyle\int_0^t e^{-\tau}\tau \cos \omega(t - \tau)\, d\tau = \int_0^t \tau e^{-2\tau} e^{-(t-\tau)} \cos \omega(t-\tau)\, d\tau$; $te^{-2t} \leftrightarrow \dfrac{1}{(s+2)^2}$ and $e^{-t} \cos \omega t \leftrightarrow \dfrac{s+1}{(s+1)^2 + \omega^2}$, so $H(s) = \dfrac{s+1}{(s+2)^2\left[(s+1)^2 + \omega^2\right]}$.

(h) $e^t \displaystyle\int_0^t e^{2\tau} \sinh(t - \tau)\, d\tau = \int_0^t e^{3\tau}\left(e^{(t-\tau)} \sinh(t - \tau)\right) d\tau$; $e^{3t} \leftrightarrow \dfrac{1}{s-3}$ and $e^t \sinh t \leftrightarrow \dfrac{1}{(s-1)^2 - 1}$, so $H(s) = \dfrac{1}{(s-3)\left((s-1)^2 - 1\right)}$.

(i) $te^{2t} \leftrightarrow \dfrac{1}{(s-2)^2}$ and $\sin 2t \leftrightarrow \dfrac{2}{s^2 + 4}$, so $H(s) = \dfrac{2}{(s-2)^2(s^2 + 4)}$.

(j) $t^3 \leftrightarrow \dfrac{6}{s^4}$ and $e^t \leftrightarrow \dfrac{1}{s-1}$, so $H(s) = \dfrac{6}{s^4(s-1)}$.

(k) $t^6 \leftrightarrow \dfrac{6!}{s^7}$ and $e^{-t} \sin 3t \leftrightarrow \dfrac{3}{(s+1)^2 + 9}$, so $H(s) = \dfrac{3 \cdot 6!}{s^7\left[(s+1)^2 + 9\right]}$.

(l) $t^2 \leftrightarrow \dfrac{2}{s^3}$ and $t^3 \leftrightarrow \dfrac{6}{s^4}$, so $H(s) = \dfrac{12}{s^7}$.

(m) $t^7 \leftrightarrow \dfrac{7!}{s^8}$ and $e^{-t} \sin 2t \leftrightarrow \dfrac{2}{(s+1)^2 + 4}$, so $H(s) = \dfrac{2 \cdot 7!}{s^8\left[(s+1)^2 + 4\right]}$.

(n) $t^4 \leftrightarrow \dfrac{24}{s^5}$ and $\sin 2t \leftrightarrow \dfrac{2}{s^2 + 4}$, so $H(s) = \dfrac{48}{s^5(s^2 + 4)}$.

8.6.4. **(a)** $Y(s) = \frac{1}{s^2} - \frac{Y(s)}{s^2}$; $Y(s)\left(1 + \frac{1}{s^2}\right) = \frac{1}{s^2}$; $Y(s)\frac{s^2+1}{s^2} = \frac{1}{s^2}$; $Y(s) = \frac{1}{s^2+1}$, so $y = \sin t$.

(b) $Y(s) = \frac{1}{s^2+1} - \frac{2sY(s)}{s^2+1}$; $Y(s)\left(1 + \frac{2s}{s^2+1}\right) = \frac{1}{s^2+1}$; $Y(s)\frac{(s+1)^2}{s^2+1} = \frac{1}{s^2+1}$; $Y(s) = \frac{1}{(s+1)^2}$, so $y = te^{-t}$.

(c) $Y(s) = \frac{1}{s} + \frac{2sY(s)}{s^2+1}$; $Y(s)\left(1 - \frac{2s}{s^2+1}\right) = \frac{1}{s}$; $Y(s)\frac{(s-1)^2}{s^2+1} = \frac{1}{s}$; $Y(s) = \frac{(s^2+1)}{s(s-1)^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{(s-1)^2}$, where $A(s-1)^2 + Bs(s-1) + Cs = s^2+1$. Setting $s = 0$ and $s = 1$ shows that $A = 1$ and $C = 2$; equating coefficients of s^2 yields $A + B = 1$, so $B = 0$. Therefore, $Y(s) = \frac{1}{s} + \frac{1}{(s-1)^2}$, so $y = 1 + 2te^t$.

(d) $Y(s) = \frac{1}{s^2} + \frac{Y(s)}{s+1}$; $Y(s)\left(1 - \frac{1}{s+1}\right) = \frac{1}{s^2}$; $Y(s)\left(\frac{s}{s+1}\right) = \frac{1}{s^2}$; $Y(s) = \frac{s+1}{s^3} = \frac{1}{s^2} + \frac{1}{s^3}$, so $y = t + \frac{t^2}{2}$.

(e) $sY(s) - 4 = \frac{1}{s^2} + \frac{sY(s)}{s^2+1}$; $Y(s)\left(s - \frac{s}{s^2+1}\right) = 4 + \frac{1}{s^2}$; $Y(s)\frac{s^3}{s^2+1} = \frac{4s^2+1}{s^2}$; $Y(s) = \frac{(4s^2+1)(s^2+1)}{s^5} = \frac{4s^4+5s^2+1}{s^5} = \frac{4}{s} + \frac{5}{s^3} + \frac{1}{s^5}$, so $y = 4 + \frac{5}{2}t^2 + \frac{1}{24}t^4$.

(f) $Y(s) = \frac{s-1}{s^2+1} + \frac{Y(s)}{s^2+1}$; $Y(s)\left(1 - \frac{1}{s^2+1}\right) = \frac{s-1}{s^2+1}$; $Y(s)\frac{s^2}{s^2+1} = \frac{s-1}{s^2+1}$; $Y(s) = \frac{s-1}{s^2} = \frac{1}{s} - \frac{1}{s^2}$, so $y = 1 - t$.

8.6.6. Substituting $x = t - \tau$ yields $\int_0^t f(t-\tau)g(\tau)\,d\tau = -\int_t^0 f(x)g(t-x)(-dx) = \int_0^t f(x)g(t-x)\,dx = \int_0^t f(\tau)g(t-\tau)\,d\tau$.

8.6.8. $p(s)Y(s) = F(s) + a(k_1 + k_0 s) + bk_0$, so (A) $Y(s) = \frac{F(s)}{p(s)} + \frac{k_0(as+b)+k_1 a}{p(s)}$. Since $p(s) = a(s-r_1)(s-r_2)$ and therefore $b = -a(r_1+r_2)$, (A) can be rewritten as

$$Y(s) = \frac{F(s)}{a(s-r_1)(s-r_2)} + \frac{k_0(s-r_1-r_2)}{(s-r_1)(s-r_2)} + \frac{k_1}{(s-r_1)(s-r_2)}.$$

$$\frac{1}{(s-r_1)(s-r_2)} = \frac{1}{r_2-r_1}\left(\frac{1}{s-r_2} - \frac{1}{s-r_1}\right) \leftrightarrow \frac{e^{r_2 t} - e^{r_1 t}}{r_2 - r_1},$$

so the convolution theorem implies that

$$\frac{F(s)}{a(s-r_1)(s-r_2)} \leftrightarrow \frac{1}{a}\int_0^t \frac{e^{r_2\tau} - e^{r_1\tau}}{r_2-r_1} f(t-\tau)\,d\tau.$$

$$\frac{s-r_1-r_2}{(s-r_1)(s-r_2)} = \frac{r_2}{r_2-r_1}\frac{1}{s-r_1} - \frac{r_1}{r_2-r_1}\frac{1}{s-r_2} \leftrightarrow \frac{r_2 e^{r_1 t} - r_1 e^{r_2 t}}{r_2-r_1}.$$

Therefore,

$$y(t) = k_0\frac{r_2 e^{r_1 t} - r_1 e^{r_2 t}}{r_2 - r_1} + k_1\frac{e^{r_2 t} - e^{r_1 t}}{r_2 - r_1} + \frac{1}{a}a\int_0^t \frac{e^{r_2\tau} - e^{r_1\tau}}{r_2 - r_1} f(t-\tau)\,d\tau.$$

8.6.10. $p(s)Y(s) = F(s) + a(k_1 + k_0 s) + bk_0$, so (A) $Y(s) = \dfrac{F(s)}{p(s)} + \dfrac{k_0(as+b)+k_1a}{p(s)}$. Since $p(s) = a(s-\lambda)^2+\omega^2$ and therefore $b = -2a\lambda$, (A) can be rewritten as

$$Y(s) = \frac{F(s)}{a[(s-\lambda)^2+\omega^2}] + \frac{k_0(s-2\lambda)}{(s-\lambda)^2+\omega^2} + \frac{k_1}{(s-\lambda)^2+\omega^2}.$$

$\dfrac{1}{(s-\lambda)^2+\omega^2} \leftrightarrow \dfrac{1}{\omega}e^{\lambda t}\sin\omega t$, so the convolution theorem implies that

$$\frac{F(s)}{a[(s-\lambda)^2+\omega^2]} \leftrightarrow \frac{1}{a\omega}\int_0^t e^{\lambda t} f(t-\tau)\sin\omega\tau\,d\tau.$$

$$\frac{s-2\lambda}{(s-\lambda)^2+\omega^2} = \frac{(s-\lambda)-\lambda}{(s-\lambda)^2+\omega^2} \leftrightarrow e^{\lambda t}\left(\cos\omega t - \frac{\lambda}{\omega}\sin\omega t\right).$$

Therefore,

$$y(t) = e^{\lambda t}\left[k_0\left(\cos\omega t - \frac{\lambda}{\omega}\sin\omega t\right) + \frac{k_1}{\omega}\sin\omega t\right] + \frac{1}{a\omega}\int_0^t e^{\lambda t} f(t-\tau)\sin\omega\tau\,d\tau.$$

8.6.12. **(a)**

$$ay''+by'+cy = f_0(t)+u(t-t_1)(f_1(t)-f_0(t)),\ y(0)=0,\ y'(0)=0;$$

$$p(s)Y(s) = F_0(s) + L(u(t-t_1)(f_1(t)-f_0(t))) = F_0(s)+e^{-st_1}L(g);$$

$$Y(s) = \frac{F_0(s)+e^{-st_1}G(s)}{p(s)}. \qquad \text{(B)}$$

(b) Since $F_0(s) \leftrightarrow f_0(t)$, $G(s) \leftrightarrow g(t)$, and $\dfrac{1}{p(s)} \leftrightarrow w(t)$, the convolution theorem implies that

$$\frac{F_0(s)}{p(s)} \leftrightarrow \int_0^t w(t-\tau)f_0(\tau)\,d\tau \quad \text{and} \quad \frac{G(s)}{p(s)} \leftrightarrow \int_0^t w(t-\tau)g(\tau)\,d\tau.$$

Now Theorem 8.4.2 implies that $\dfrac{e^{-st_1}G(s)}{p(s)} \leftrightarrow u(t-t_1)\displaystyle\int_0^t w(t-t_1-\tau)g(\tau)\,d\tau$, and (B) implies that

$$y(t) = \int_0^t w(t-\tau)f_0(\tau)\,d\tau + u(t-t_1)\int_0^{t-t_1} w(t-t_1-\tau)g(\tau)\,d\tau.$$

(c) Let $z_0(t) = \int_0^t w(t-\tau)f_0(\tau)\,d\tau$ and $z_1(t) = \int_0^t w(t-\tau)g(\tau)\,d\tau$. Then $y(t) = z_0(t)+u(t-t_1)z_1(t-t_1)$. Using Leibniz's rule as in the solution of Exercise 8.6.11**(b)** shows that

$$z_0'(t) = \int_0^t w'(t-\tau)f_0(\tau)\,d\tau, \quad z_1'(t) = \int_0^t w'(t-\tau)g(\tau)\,d\tau, \quad t>0,$$

$$z_0''(t) = \frac{f_0(t)}{a} + \int_0^t w''(t-\tau)f_0(\tau)\,d\tau, z_1''(t) = \frac{g(t)}{a} + \int_0^t w''(t-\tau)g(\tau)\,d\tau, \quad t>0,$$

if $t>0$, and that

$$az_0''+bz_0'+cz_0 = f_0(t) \quad \text{and} \quad az_1''+bz_1'+cz_1 = f_1(t+t_1)-f_0(t+t_1), \quad t>0.$$

This implies the stated conclusion for y' and y'' on $(0,t)$ and (t,∞), and that $ay''+by'+cy=f(t)$ on these intervals.

(d) Since the functions $z_0(t)$ and $h(t)=u(t-t_1)z_1(t-t_1)$ are both continuous on $[0,\infty)$ and $h(t)=0$ if $0\le t\le t_1$, y is continuous on $[0,\infty)$. From **(c)**, y' is continuous on $[0,t_1)$ and (t_1,∞), so we need only show that y' is continuous at t_1. For this it suffices to show that $h'(t_1)=0$. Since $h(t_1)=0$ if $t\le t_1$, (B) $\lim\limits_{t\to t_1-}\dfrac{h(t)-h(t_1)}{t-t_1}=0$. If $t>t_1$, then $h(t)=\displaystyle\int_0^{t-t_1} w(t-t_1\tau)g(\tau)\,d\tau$. Since $h(t_1)=0$,

$$\left|\frac{h(t)-h(t_1)}{t-t_1}\right|\le\int_0^{t-t_1}|w(t-t_1-\tau)g(\tau)|\,d\tau. \tag{B}$$

Since g is continuous from the right at 0, we can choose constants $T>0$ and $M>0$ so that $|g(\tau)|<M$ if $0\le\tau\le T$. Then (B) implies that

$$\left|\frac{h(t)-h(t_1)}{t-t_1}\right|\le M\int_0^{t-t_1}|w(t-t_1-\tau)|\,d\tau,\quad t_1<t<t_1+T. \tag{C}$$

Now suppose $\epsilon>0$. Since $w(0)=0$, we can choose T_1 such that $0<T_1<T$ and $|w(x)|<\epsilon/M$ if $0\le x<T_1$. If $t_1<t<t_1+T_1$ and $0\le\tau\le t-t_1$, then $0\le t-t_1-\tau<T_1$, so (C) implies that

$$\left|\frac{h(t)-h(t_1)}{t-t_1}\right|<\epsilon,\quad t_1<t<t_1+T.$$

Therefore, $\lim\limits_{t\to t_1+}\dfrac{h(t)-h(t_1)}{t-t_1}=0$. This and (B) imply that $h'(t_1)=0$.

8.7 CONSTANT COEFFICIENT EQUATIONS WITH IMPULSES

8.7.2. $(s^2+s-2)\hat Y(s)=-\dfrac{10}{s+1}+(-9+7s)+7$; $\hat Y(s)=\dfrac{-10+(s+1)(7s-2)}{(s-1)(s+2)(s+1)}=\dfrac{2}{s+2}+\dfrac{5}{s+1}$; $\hat y=2e^{-2t}+5e^{-t}$; $\dfrac{1}{p(s)}=\dfrac{1}{(s+2)(s-1)}=\dfrac{1}{3}\left(\dfrac{1}{s-1}-\dfrac{1}{s+2}\right)$; $w=L^{-1}\left(\dfrac{1}{p(s)}\right)=\dfrac{e^t-e^{-2t}}{3}$; $y=2e^{-2t}+5e^{-t}+\dfrac{5}{3}u(t-1)\left(e^{(t-1)}-e^{-2(t-1)}\right)$.

8.7.4. $(s^2+1)\hat Y(s)=\dfrac{3}{s^2+9}-1+s$;
$\hat Y(s)=\dfrac{3}{(s^2+1)(s^2+9)}+\dfrac{s-1}{s^2+1}=\dfrac{3}{8}\left(\dfrac{1}{s^2+1}-\dfrac{1}{s^2+9}\right)+\dfrac{s-1}{s^2+1}=\dfrac{1}{8}\left(\dfrac{8s-5}{s^2+1}-\dfrac{3}{s^2+9}\right)$;
$\hat y=\dfrac{1}{8}(8\cos t-5\sin t-\sin 3t)$; $\dfrac{1}{p(s)}=\dfrac{1}{s^2+1}$; $w=L^{-1}\left(\dfrac{1}{p(s)}\right)=\sin t$; $y=\dfrac{1}{8}(8\cos t-5\sin t-\sin 3t)-2u(t-\pi/2)\cos t$.

8.7.6. $(s^2-1)\hat Y(s)=\dfrac{8}{s}+1-s$; $\hat Y(s)=\dfrac{8+s(1-s)}{s(s-1)(s+1)}=\dfrac{4}{s-1}+\dfrac{3}{s+1}-\dfrac{8}{s}$; $\hat y=4e^t+3e^{-t}-8$; $\dfrac{1}{p(s)}=\dfrac{1}{(s-1)(s+1)}=\dfrac{1}{2}\dfrac{1}{s-1}-\dfrac{1}{s+1}$; $w=L^{-1}\left(\dfrac{1}{p(s)}\right)=\dfrac{e^t+e^{-t}}{2}=\sinh t$; $y=4e^t+3e^{-t}-8+2u(t-2)\sinh(t-2)$;

8.7.8. $(s^2+4)\hat Y(s)=\dfrac{8}{s-2}+8s$; (A) $\hat Y(s)=\dfrac{8}{(s-2)(s^2+4)}+\dfrac{8s}{s^2+4}$; $\dfrac{8}{(s-2)(s^2+4)}=\dfrac{A}{s-2}+\dfrac{Bs+C}{s^2+4}$ where $A(s^2+4)+(Bs+C)(s-2)=8$. Setting $s=2$ yields $A=1$; setting $s=0$

yields $4A - 2C = 8$, so $C = -2$; $A + B = 0$ (coefficient of x^2), so $B = -A = -1$; therefore $\dfrac{8}{(s-2)(s^2+4)} = \dfrac{1}{s-2} - \dfrac{s+2}{s^2+4}$, so (A) implies that $\hat{y} = e^{2t} + 7\cos 2t - \sin 2t$; $\dfrac{1}{p(s)} = \dfrac{1}{s^2+4}$; $w = L^{-1}\left(\dfrac{1}{p(s)}\right) = \dfrac{1}{2}\sin 2t$. Since $\sin(2t-\pi) = -\sin 2t$, $y = e^{2t} + 7\cos 2t - \sin 2t - \dfrac{1}{2}u(t-\pi/2)\sin 2t$.

8.7.10. $(s^2+2s+1)\hat{Y}(s) = \dfrac{1}{s-1} + (2-s) - 2$; $\hat{Y}(s) = \dfrac{1-s(s-1)}{(s-1)(s+1)^2} = \dfrac{A}{s-1} + \dfrac{B}{s+1} + \dfrac{C}{(s+1)^2}$ where $A(s+1)^2 + (B(s+1)+C)(s-1) = 1 - s(s-1)$. Setting $s = 1$ yields $A = 1/4$; setting $s = -1$ yields $C = 1/2$; since $A + B = -1$ (coefficient of s^2), $B = -1 - A = -5/4$. Therefore, $\hat{Y}(s) = \dfrac{1}{4}\dfrac{1}{s-1} - \dfrac{5}{4}\dfrac{1}{s+1} + \dfrac{1}{2}\dfrac{1}{(s+1)^2}$; $\hat{y} = \dfrac{1}{4}e^t + \dfrac{1}{4}e^{-t}(2t-5)$; $\dfrac{1}{p(s)} = \dfrac{1}{(s+1)^2}$; $w = L^{-1}\left(\dfrac{1}{p(s)}\right) = te^{-t}$; $y = \dfrac{1}{4}e^t + \dfrac{1}{4}e^{-t}(2t-5) + 2u(t-2)(t-2)e^{-(t-2)}$.

8.7.12. $(s^2+2s+2)\hat{Y}(s) = (2-s) - 2$; $Y(s) = \dfrac{-(s+1)+1}{(s+1)^2+1}$; $\hat{y} = e^{-t}(\sin t - \cos t)$; $\dfrac{1}{p(s)} = \dfrac{1}{(s+1)^2+1}$; $w = L^{-1}\left(\dfrac{1}{p(s)}\right) = e^{-t}\sin t$. Since $\sin(t-\pi) = -\sin t$ and $\sin(t-2\pi) = \sin t$, $y = e^{-t}(\sin t - \cos t) - e^{-(t-\pi)}u(t-\pi)\sin t - 3u(t-2\pi)e^{-(t-2\pi)}\sin t$.

8.7.14. $(2s^2-3s-2)\hat{Y}(s) = \dfrac{1}{s} + 2(2-s) + 3$; $\hat{Y}(s) = \dfrac{1+s(7-2s)}{2s(s+1/2)(s-2)} = \dfrac{7}{10}\dfrac{1}{s-2} - \dfrac{6}{5}\dfrac{1}{s+1/2} - \dfrac{1}{2s}$; $\hat{y} = \dfrac{7}{10}e^{2t} - \dfrac{6}{5}e^{-t/2} - \dfrac{1}{2}$; $\dfrac{1}{p(s)} = \dfrac{1}{2(s+1/2)(s-2)} = \dfrac{1}{5}\left(\dfrac{1}{s-2} - \dfrac{1}{s+1/2}\right)$; $w = L^{-1}\left(\dfrac{1}{p(s)}\right) = \dfrac{1}{5}(e^{2t} - e^{-t/2})$; $y = \dfrac{7}{10}e^{2t} - \dfrac{6}{5}e^{-t/2} - \dfrac{1}{2} + \dfrac{1}{5}u(t-2)\left(e^{2(t-2)} - e^{-(t-2)/2}\right)$;

8.7.16. $(s^2+1)\hat{Y}(s) = \dfrac{s}{s^2+4} - 1$; $\hat{Y}(s) = \dfrac{s}{(s^2+1)(s^2+4)} - \dfrac{1}{s^2+1} = \dfrac{1}{3}\left(\dfrac{s}{s^2+1} - \dfrac{s}{s^2+4}\right) - \dfrac{1}{s^2+1}$; $\hat{y} = \dfrac{1}{3}(\cos t - \cos 2t - 3\sin t)$; $\dfrac{1}{p(s)} = \dfrac{1}{s^2+1}$; $w = L^{-1}\left(\dfrac{1}{p(s)}\right) = \sin t$. Since $\sin(t-\pi/2) = -\cos t$ and $\sin(t-\pi) = -\sin t$,

$$y = \frac{1}{3}(\cos t - \cos 2t - 3\sin t) - 2u(t-\pi/2)\cos t + 3u(t-\pi)\sin t.$$

8.7.18. $(s^2+2s+1)\hat{Y}(s) = \dfrac{1}{s-1} - 1$; (A) $\hat{Y}(s) = \dfrac{1}{(s-1)(s+1)^2} - \dfrac{1}{(s+1)^2}$;

$$\frac{1}{(s-1)(s+1)^2} = \frac{A}{s-1} + \frac{B}{s+1} + \frac{C}{(s+1)^2}$$

where $A(s+1)^2 + (B(s+1)+C)(s-1) = 1$. Setting $s = 1$ yields $A = 1/4$; setting $s = -1$ yields $C = -1/2$; since $A + B = 0$ (coefficient of s^2), $B = -A = -1/4$. Therefore,

$$\frac{1}{(s-1)(s+1)^2} = \frac{1}{4}\frac{1}{s-1} - \frac{1}{4}\frac{1}{s+1} - \frac{1}{2}\frac{1}{(s+1)^2}.$$

This and (A) imply that

$$\hat{Y}(s) = \frac{1}{4}\frac{1}{s-1} - \frac{1}{4}\frac{1}{s+1} - \frac{3}{2}\frac{1}{(s+1)^2};$$

$\hat{y} = \frac{1}{4}\left(e^t - e^{-t}(1+6t)\right)$; $\frac{1}{p(s)} = \frac{1}{(s+1)^2}$; $w = L^{-1}\left(\frac{1}{p(s)}\right) = te^{-t}$;

$$y = \frac{1}{4}\left(e^t - e^{-t}(1+6t)\right) - u(t-1)(t-1)e^{-(t-1)} + 2u(t-2)(t-2)e^{-(t-2)}.$$

8.7.20. $y'' + 4y = 1 - 2u(t-\pi/2) + \delta(t-\pi) - 3\delta(t-3\pi/2)$, $y(0) = 1$, $y'(0) = -1$. $(s^2+4)\hat{Y}(s) = \frac{1-2e^{-\pi s/2}}{s} + s - 1$; $\hat{Y}(s) = \frac{1-2e^{-\pi s/2}}{s(s^2+4)} + \frac{s-1}{s^2+4}$. Since $\frac{1}{s(s^2+4)} = \frac{1}{4}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)$, $\hat{Y}(s) = \frac{1}{4s} + \frac{3}{4}\frac{s}{s^2+4} - \frac{1}{s^2+4} - 2e^{-\pi s/2}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)$. $\hat{y} = \frac{3}{4}\cos 2t - \frac{1}{2}\sin 2t + \frac{1}{4} + \frac{1}{4}u(t-\pi/2)(1+\cos 2t)$. $w = L^{-1}\left(\frac{1}{p(s)}\right) = \frac{1}{2}\sin 2t$. Since $\sin 2(t-\pi) = \sin 2t$ and $\sin 2(t-3\pi/2) = -\sin 2t$,

$$y = \frac{3}{4}\cos 2t - \frac{1}{2}\sin 2t + \frac{1}{4} + \frac{1}{4}u(t-\pi/2)(1+\cos 2t) + \frac{1}{2}u(t-\pi)\sin 2t + \frac{3}{2}u(t-3\pi/2)\sin 2t.$$

8.7.26. $w(t) = e^{-t}\sin t$; $f_h(t) = \frac{u(t-t_0) - u(t-t_0-h)}{h}$; $(s^2+2s+2)Y_h(s) = \frac{1}{h}\frac{e^{-st_0} - e^{-s(t_0+h)}}{s}$; $Y_h(s) = \frac{1}{h}\frac{e^{-st_0} - e^{-s(t_0+h)}}{s(s^2+2s+2)}$; $\frac{1}{s(s^2+2s+2)} = -\frac{(s+1)+1}{2\left((s+1)^2+1\right)} + \frac{1}{2s} \leftrightarrow \frac{1}{2}\left(1 - e^{-t}(\cos t + \sin t)\right)$;

$$y_h(t) = \begin{cases} 0, & 0 \le t < t_0, \\ \dfrac{1}{2h}\left[1 - e^{-(t-t_0)}(\cos(t-t_0) + \sin(t-t_0))\right], & t_0 \le t < t_0 + h, \\ \dfrac{e^{-(t-t_0)}}{2h}\left[e^h\left(\cos(t-t_0-h) + \sin(t-t_0-h)\right) - \cos(t-t_0) - \sin(t-t_0)\right], & t \ge t_0 + h. \end{cases}$$

8.7.28. $w(t) = e^{-t} - e^{-2t}$; $f_h(t) = \frac{u(t-t_0) - u(t-t_0-h)}{h}$; $(s^2+3s+2)Y_h(s) = \frac{1}{h}\frac{e^{-st_0} - e^{-s(t_0+h)}}{s}$; $Y_h(s) = \frac{1}{h}\frac{e^{-st_0} - e^{-s(t_0+h)}}{s(s+1)(s+2)}$; $\frac{1}{s(s+1)(s+2)} = \frac{1}{2(s+2)} - \frac{1}{s+1} + \frac{1}{2s} \leftrightarrow \frac{e^{-2t}}{2} - e^{-t} + \frac{1}{2} = \frac{(e^{-t}-1)^2}{2}$;

$$y_h(t) = \begin{cases} 0, & 0 \le t < t_0, \\ \dfrac{(e^{-(t-t_0)} - 1)^2}{2h}, & t_0 \le t < t_0 + h, \\ \dfrac{(e^{-(t-t_0)} - 1)^2 - (e^{-(t-t_0-h)} - 1)^2}{2h}, & t \ge t_0 + h. \end{cases}$$

8.7.30. **(a)** $(s^2-1)\hat{Y}(s) = 1$, so $\hat{y} = w = L^{-1}\left(\frac{1}{s^2-1}\right) = \frac{1}{2}\left(e^t - e^{-t}\right)$; $y = \hat{y} + \sum_{k=0}^{\infty} u(t-k)w(t-k) = \frac{1}{2}\sum_{k=0}^{\infty} u(t-k)\left(e^{t-k} - e^{-t-k}\right)$. If $m \le t < m+1$, then $y = \frac{1}{2}\sum_{k=0}^{m}\left(e^{t-k} - e^{-t-k}\right) = \frac{1}{2}(e^{t-m} - e^{-t})\sum_{k=0}^{m} e^k \frac{e^{m+1}-1}{2(e-1)}(e^{t-m} - e^{-t})$.

(b) $(s^2+1)\hat{Y}(s) = 1$, so $\hat{y} = w = L^{-1}\left(\frac{1}{s^2+1}\right) = \sin t$; $y = \hat{y} + \sum_{k=0}^{\infty} u(t-2k\pi)w(t-2k\pi) =$

$\sin t \sum_{k=0}^{\infty} u(t-2k\pi)$. If $2m\pi \le t < 2(m+1)\pi$, then $y = (m+1)\sin t$.

(c) $(s^2 - 3s + 2)\hat{Y}(s) = 1$, so $\hat{y} = w = L^{-1}\left(\frac{1}{(s-1)(s-2)}\right) = (e^{2t} - e^t)$; $y = \hat{y} + \sum_{k=0}^{\infty} u(t - k)w(t-k) = \sum_{k=0}^{\infty} u(t-k)\left(e^{2(t-k)} - e^{t-k}\right)$. If $m \le t < m+1$, then $y = \sum_{k=0}^{m}\left(e^{2(t-k)} - e^{t-k}\right) = e^{2(t-m)}\sum_{k=0}^{m} e^{2k} - e^{t-m}\sum_{k=0}^{m} e^k = e^{2(t-m)}\frac{e^{2m+2}-1}{e^2-1} - e^{(t-m)}\frac{e^{m+1}-1}{e-1}$.

(d) $w = L^{-1}\left(\frac{1}{s^2+1}\right) = \sin t$; $y = \sum_{k=1}^{\infty} u(t-k\pi)w(t-k\pi) = \sin t \sum_{k=1}^{\infty}(-1)^k u(t-k\pi)$, so

$$y = \begin{cases} 0, & 2m\pi \le t < (2m+1)\pi, \\ -\sin t, & (2m+1)\pi \le t < (2m+2)\pi, \end{cases} \qquad (m = 0, 1, \ldots).$$

CHAPTER 9
Linear Higher Order Equations

9.1 INTRODUCTION TO LINEAR HIGHER ORDER EQUATIONS

9.1.2. From Example 9.1.1, $y = c_1x^2 + c_2x^3 + \dfrac{c_3}{x}$ $y' = 2c_1x + 3c_2x^2 - \dfrac{c_3}{x^2}$, and $y'' = 2c_1 + 6c_2x + \dfrac{2c_3}{x^3}$, where

$$\begin{array}{rcr} c_1 - c_2 - c_3 & = & 4 \\ -2c_1 + 3c_2 - c_3 & = & -14 \\ 2c_1 - 6c_2 - 2c_3 & = & 20, \end{array}$$

so $c_1 = 2$, $c_2 = -3$, $c_3 = 1$, and $y = 2x^2 - 3x^3 + \dfrac{1}{x}$.

9.1.4. The general solution of $y^{(n)} = 0$ can be written as $y(x) = \displaystyle\sum_{m=0}^{n-1} c_m(x - x_0)^m$. Since $y^{(j)}(x) = \displaystyle\sum_{m=j}^{n-1} m(m-1)\cdots(m-j+1)c_m(x-x_0)^{m-j}$, $y^{(j)}(x_0) = j!c_j$. Therefore, $y_i = \dfrac{(x - x_0)^{i-1}}{(i-1)!}$, $1 \le i \le n$.

9.1.6. We omit the verification that the given functions are solutions of the given equations.

(a) The equation is normal on $(-\infty, \infty)$. $W(x) = \begin{vmatrix} e^x & e^{-x} & xe^{-x} \\ e^x & -e^{-x} & e^{-x}(1-x) \\ e^x & e^{-x} & e^{-x}(x-2) \end{vmatrix}$; $W(0) = \begin{vmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & 1 & -2 \end{vmatrix} = 4$. Apply Theorem 9.1.4.

(b) The equation is normal on $(-\infty, \infty)$.

$$W(x) = \begin{vmatrix} e^x & e^x\cos 2x & e^x \sin 2x \\ e^x & e^x(\cos 2x - 2\sin 2x) & e^x(2\cos 2x + \sin 2x) \\ e^x & -e^x(3\cos 2x + 4\sin 2x) & e^x(4\cos 2x - 3\sin 2x) \end{vmatrix};$$

$W(0) = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & -3 & 4 \end{vmatrix}$ 8. Apply Theorem 9.1.4.

(c) The equation is normal on $(-\infty, 0)$ and $(0, \infty)$.

$W(x) = \begin{vmatrix} e^x & e^{-x} & x \\ e^x & -e^{-x} & 1 \\ e^x & e^{-x} & 0 \end{vmatrix} = 2x$. Apply Theorem 9.1.4.

(d) The equation is normal on $(-\infty, 0)$ and $(0, \infty)$.

$W(x) = \begin{vmatrix} e^x/x & e^{-x}/x & 1 \\ e^x(1/x - 1/x^2) & -e^{-x}(x+1)/x^2 & 0 \\ e^x(1/x - 2/x^2 + 2/x^3) & e^{-x}(x^2 + 2x + 2)/x^3 & 0 \end{vmatrix} = 2/x^2$. Apply Theorem 9.1.4.

(e) The equation is normal on $(-\infty,\infty)$. $W(x)=\begin{vmatrix} x & x^2 & e^x \\ 1 & 2x & e^x \\ 0 & 2 & e^x \end{vmatrix}$; $W(0)=\begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{vmatrix}=2$; Apply Theorem 9.1.4.

(f) The equation is normal on $(-\infty,1/2)$ and $(1/2,\infty)$.

$$W(x)=\begin{vmatrix} x & e^x & e^{-x} & e^{2x} \\ 1 & e^x & -e^{-x} & 2e^{2x} \\ 0 & e^x & e^{-x} & 4e^{(}2x) \\ 0 & e^x & -e^{-x} & 8e^{2x} \end{vmatrix}=e^{2x}(12x-6). \text{ Apply Theorem 9.1.4.}$$

(g) The equation is normal on $(-\infty,0)$ and $(0,\infty)$.

$$W(x)=\begin{vmatrix} 1 & x^2 & e^{2x} & e^{-2x} \\ 0 & 2x & 2e^{2x} & -2e^{-2x} \\ 0 & 2 & 4e^{2x} & 4e^{-2x} \\ 0 & 0 & 8e^{2x} & -8e^{-2x} \end{vmatrix}=-128x. \text{ Apply Theorem 9.1.4.}$$

9.1.8. From Abel's formula, (A) $W(x)=W(\pi/2)\exp\left(-\int_{\pi/4}^{x}\tan t\,dt\right)$; $\int_{\pi/4}^{x}\tan t\,dt=-\ln\cos t\Big|_{\pi/4}^{x}=-\ln(\sqrt{2}\cos x)$; therefore (A) implies that $W(x)=\sqrt{2}K\cos x$.

9.1.10. **(a)** $W(x)=\begin{vmatrix} 1 & e^x & e^{-x} \\ 0 & e^x & -e^{-x} \\ 0 & e^x & e^{-x} \end{vmatrix}=(e^x)(e^{-x})\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{vmatrix}=2.$

$$\begin{aligned}
\textbf{(b)}\quad W(x) &= \begin{vmatrix} e^x & e^x\sin x & e^x\cos x \\ e^x & e^x(\cos x+\sin x) & e^x(\cos x-\sin x) \\ e^x & 2e^x\cos x & -2e^x\sin x \end{vmatrix}= \\
&= e^{3x}\begin{vmatrix} 1 & \sin x & \cos x \\ 1 & \cos x+\sin x & \cos x-\sin x \\ 1 & 2\cos x & -2\sin x \end{vmatrix} \\
&= e^{3x}\begin{vmatrix} 1 & \sin x & \cos x \\ 0 & \cos x & -\sin x \\ 1 & 2\cos x-\sin x & -2\sin x-\cos x \end{vmatrix} \\
&= e^{3x}\begin{vmatrix} 1 & \sin x & \cos x \\ 0 & \cos x & -\sin x \\ 0 & -\sin x & -\cos x \end{vmatrix}=-e^{3x}
\end{aligned}$$

(c) $W(x)=\begin{vmatrix} 2 & x+1 & x^2+2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix}=4.$

$$\begin{aligned}
\textbf{(d)}\quad W(x) &= \begin{vmatrix} x & x\ln|x| & 1/x \\ 1 & \ln|x|+1 & -1/x^2 \\ 0 & 1/x & 2/x^3 \end{vmatrix}=\begin{vmatrix} 1 & \ln|x| & 1/x^2 \\ 1 & \ln|x|+1 & -1/x^2 \\ 0 & 1 & 2/x^2 \end{vmatrix} \\
&= \frac{1}{x^2}\begin{vmatrix} 1 & \ln|x| & 1 \\ 1 & \ln|x|+1 & -1 \\ 0 & 1 & 2 \end{vmatrix}=\frac{1}{x^2}\begin{vmatrix} 1 & \ln|x| & 1 \\ 0 & 1 & -2 \\ 0 & 1 & 2 \end{vmatrix} \\
&= \frac{1}{x^2}\begin{vmatrix} 1 & \ln|x| & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 4 \end{vmatrix}=4/x^2.
\end{aligned}$$

(e) $W(x)=\begin{bmatrix} 1 & x & x^2/2 & x^3/3 & \cdots & x^n/n! \\ 0 & 1 & x & x^2/2 & \cdots & x^{n-1}/(n-1)! \\ 0 & 0 & 1 & x & \cdots & x^{n-2}/(n-2)! \\ 0 & 0 & 0 & 1 & \cdots & x^{n-3}/(n-3)! \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}=1.$

(f) $W(x)=\begin{vmatrix} e^x & e^{-x} & x \\ e^x & -e^{-x} & 1 \\ e^x & e^{-x} & 0 \end{vmatrix}=\begin{vmatrix} 1 & 1 & x \\ 1 & -1 & 1 \\ 1 & 1 & 0 \end{vmatrix}=\begin{vmatrix} 1 & 1 & x \\ 0 & -2 & 1-x \\ 0 & 0 & -x \end{vmatrix}=2x.$

$$\begin{aligned}
\textbf{(g)}\quad W(x) &= \begin{vmatrix} e^x/x & e^{-x}/x & 1 \\ e^x/x-e^x/x^2 & -e^{-x}/x-e^{-x}/x^2 & 0 \\ e^x/x-2e^x/x^2+2e^x/x^3 & e^{-x}/x+2e^{-x}/x^2+2e^{-x}/x^3 & 0 \end{vmatrix} \\
&= \begin{vmatrix} 1/x & 1/x & 1 \\ 1/x-1/x^2 & -1/x-1/x^2 & 0 \\ 1/x-2/x^2+2/x^3 & 1/x+2/x^2+2/x^3 & 0 \end{vmatrix} \\
&= \begin{vmatrix} 1/x-1/x^2 & -1/x-1/x^2 \\ 1/x-2/x^2+2/x^3 & 1/x+2/x^2+2/x^3 \end{vmatrix} \\
&= \begin{vmatrix} 1/x-1/x^2 & -2/x \\ 1/x-2/x^2+2/x^3 & 4/x^2 \end{vmatrix} = 2/x^2.
\end{aligned}$$

(h) $W(x)=\begin{vmatrix} x & x^2 & e^x \\ 1 & 2x & e^x \\ 0 & 2 & e^x \end{vmatrix}=e^x\begin{vmatrix} x & x^2 & 1 \\ 1 & 2x & 1 \\ 0 & 2 & 1 \end{vmatrix}=e^x\left(x\begin{bmatrix} 2x & 1 \\ 2 & 1 \end{bmatrix}-\begin{bmatrix} x^2 & 1 \\ 2 & 1 \end{bmatrix}\right)$
$=e^x(x^2-2x+2).$

$$\begin{aligned}
\textbf{(i)}\quad W(x) &= \begin{vmatrix} x & x^3 & 1/x & 1/x^2 \\ 1 & 3x^2 & -1/x^2 & -2/x^3 \\ 0 & 6x & 2/x^3 & 6/x^4 \\ 0 & 6 & -6/x^4 & -24/x^5 \end{vmatrix} = \begin{vmatrix} 0 & -2x^3 & 2/x & 3/x^2 \\ 1 & 3x^2 & -1/x^2 & -2/x^3 \\ 0 & 6x & 2/x^3 & 6/x^4 \\ 0 & 6 & -6/x^4 & -24/x^5 \end{vmatrix} \\
&= -\begin{vmatrix} -2x^3 & 2/x & 3/x^2 \\ 6x & 2/x^3 & 6/x^4 \\ 6 & -6/x^4 & -24/x^5 \end{vmatrix} = -x^4\begin{vmatrix} -2 & 2/x^4 & 3/x^5 \\ 6 & 2/x^4 & 6/x^5 \\ 6 & -6/x^4 & -24/x^5 \end{vmatrix} \\
&= -x^4\begin{vmatrix} -2 & 2/x^4 & 3/x^5 \\ 0 & 8/x^4 & 15/x^5 \\ 0 & 0 & -15/x^5 \end{vmatrix} = -240/x^5.
\end{aligned}$$

$$\textbf{(j)}\quad W(x) = \begin{vmatrix} e^x & e^{-x} & x & e^{2x} \\ e^x & -e^{-x} & 1 & 2e^{2x} \\ e^x & e^{-x} & 0 & 4e^{2x} \\ e^x & -e^{-x} & 0 & 8e^{2x} \end{vmatrix} = e^{2x}\begin{vmatrix} 1 & 1 & x & 1 \\ 1 & -1 & 1 & 2 \\ 1 & 1 & 0 & 4 \\ 1 & -1 & 0 & 8 \end{vmatrix}$$

$$= e^{2x}\begin{vmatrix} 1-x & 1+x & 0 & 1-2x \\ 1 & -1 & 1 & 2 \\ 1 & 1 & 0 & 4 \\ 1 & -1 & 0 & 8 \end{vmatrix} = -e^{2x}\begin{vmatrix} 1-x & 1+x & 1-2x \\ 1 & 1 & 4 \\ 1 & -1 & 8 \end{vmatrix}$$

$$= -e^{2x}\begin{vmatrix} 2 & 1+x & 1-2x \\ 2 & 1 & 4 \\ 0 & -1 & 8 \end{vmatrix} = -e^{2x}\begin{vmatrix} 2 & 1+x & 1-2x \\ 0 & -x & 3+2x \\ 0 & -1 & 8 \end{vmatrix} = 6e^{2x}(2x-1).$$

$$\textbf{(k)}\quad W(x) = \begin{vmatrix} e^{2x} & e^{-2x} & 1 & x^2 \\ 2e^{2x} & -2e^{-2x} & 0 & 2x \\ 4e^{2x} & 4e^{-2x} & 0 & 2 \\ 8e^{2x} & -8e^{-2x} & 0 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & x^2 \\ 2 & -2 & 0 & 2x \\ 4 & 4 & 0 & 2 \\ 8 & -8 & 0 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & -2 & 2x \\ 4 & 4 & 2 \\ 8 & -8 & 0 \end{vmatrix} = \begin{vmatrix} 0 & -2 & 2x \\ 8 & 4 & 2 \\ 0 & -8 & 0 \end{vmatrix} = -128x.$$

9.1.12. Let y be an arbitrary solution of $Ly = 0$ on (a,b). Since $\{z_1,\dots,z_n\}$ is a fundamental set of solutions of $Ly = 0$ on (a,b), there are constants $c_1, c_2,\dots, c_n$ such that $y = \sum_{i=1}^n c_i y_i$. Therefore, $y = \sum_{i=1}^n c_i \sum_{j=1}^n a_{ij} y_j = \sum_{j=1}^n C_j y_j$, with $C_j = \sum_{i=1}^n a_{ij} c_i$. Hence $\{y_1,\dots,y_n\}$ is a fundamental set of solutions of $Ly = 0$ on (a,b).

9.1.14. Let y be a given solution of $Ly = 0$ and $z = \sum_{j=1}^n y^{(j-1)}(x_0) y_j$ Then $z^{(r)}(x_0) = y^{(r)}(x_0)$, $r = 0,\dots,n-1$. Since the solution of every initial value problem is unique (Theorem 9.1.1), $z = y$.

9.1.16. If $\{y_1, y_2,\dots, y_n\}$ is linearly dependent on (a,b) there are constants $c_1,\dots,c_n$, not all zeros, such that $c_1y_1 + c_2y_2 + \cdots + c_ny_n$. Let k be the smallest integer such that $c_k \ne 0$. If $k = 1$, then $y_1 = \frac{1}{c_1}(c_2y_2+\cdots+c_ny_n)$; if $1 < k < n$, then $y_k = 0\cdot y_1+\cdots+0\cdot y_{k-1}+\frac{1}{c_k}(c_{k+1}y_{k+1}+\cdots+c_ny_n)$; if $k = n$, then $y_n = 0$, so $y_n = 0\cdot y_1 + 0\cdot y_2 + \cdots + 0\cdot y_n$.

9.1.18. Since $F = \sum \pm f_{1i_1} f_{2i_2},\dots, f_{ni_n}$,

$$\begin{aligned} F' &= \sum \pm f'_{1i_1} f_{2i_2},\dots, f_{ni_n} + \sum \pm f_{1i_1} f'_{2i_2},\dots, f_{ni_n} + \cdots + \sum \pm f_{1i_1} f_{2i_2},\dots, f'_{ni_n} \\ &= F_1 + F_2 + \cdots + F_n. \end{aligned}$$

9.1.20. Since $y_j^{(n)} = -\sum_{k=1}^{n}(P_k/P_0)y_j^{(n-k)}$, Exercise 9.1.19 implies that

$$W' = -\begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_n^{(n-2)} \\ \sum_{k=1}^n (P_k/P_0)y_1^{(n-k)} & \sum_{k=1}^n (P_k/P_0)y_2^{(n-k)} & \cdots & \sum_{k=1}^n (P_k/P_0)y_n^{(n-k)} \end{vmatrix},$$

so Exercise 9.1.17 implies that

$$W' = -\sum_{k=1}^{n}\frac{P_k}{P_0}\begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_n^{(n-2)} \\ y_1^{(n-k)} & y_2^{(n-k)} & \cdots & y_n^{(n-k)} \end{vmatrix}.$$

However, the determinants on the right each have two identical rows if $k = 2, \dots, n$. Therefore, $W' = -\frac{P_1W}{P_0}$. Separating variables yields $\frac{W'}{W} = -\frac{P_1}{P_0}$; hence $\ln\frac{W(x)}{W(x_0)} = -\int_{x_0}^{x}\frac{P_1(t)}{P_0(t)}\,dt$, which implies Abel's formula.

9.1.22. See the proof of Theorem 5.3.3.

9.1.24. **(a)**

$$P_0(x) = -\begin{vmatrix} x & x^2-1 & x^2+1 \\ 1 & 2x & 2x \\ 0 & 2 & 2 \end{vmatrix} = -\begin{vmatrix} x & x^2-1 & x^2+1 \\ 1 & 0 & 0 \\ 0 & 2 & 2 \end{vmatrix} = \begin{vmatrix} x^2-1 & x^2+1 \\ 2 & 2 \end{vmatrix} = -4;$$

$P_1(x) = \begin{vmatrix} x & x^2-1 & x^2+1 \\ 1 & 2x & 2x \\ 0 & 0 & 0 \end{vmatrix} = 0$; $P_2(x) = -\begin{vmatrix} x & x^2-1 & x^2+1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{vmatrix} = 0$; $P_3(x) = \begin{vmatrix} 1 & 2x & 2x \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{vmatrix} = 0$. Therefore, $-4y''' = 0$, which is equivalent to $y''' = 0$.

(b)

$$P_0 = -\begin{vmatrix} e^x & e^{-x} & x \\ e^x & -e^{-x} & 1 \\ e^x & e^{-x} & 0 \end{vmatrix} = -\begin{vmatrix} 1 & 1 & x \\ 1 & -1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -\begin{vmatrix} 0 & 0 & x \\ 0 & -2 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -2x;$$

$$P_1 = \begin{vmatrix} e^x & e^{-x} & x \\ e^x & -e^{-x} & 1 \\ e^x & -e^{-x} & 0 \end{vmatrix} = \begin{vmatrix} 1 & 1 & x \\ 1 & -1 & 1 \\ 1 & -1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 2 & x \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{vmatrix} = 2;$$

$$P_2 = -\begin{vmatrix} e^x & e^{-x} & x \\ e^x & e^{-x} & 0 \\ e^x & -e^{-x} & 0 \end{vmatrix} = -\begin{vmatrix} 1 & 1 & x \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{vmatrix} = -\begin{vmatrix} 0 & 0 & x \\ 1 & 1 & 0 \\ 0 & -2 & 0 \end{vmatrix} = 2x;$$

$$P_3 = \begin{vmatrix} e^x & -e^{-x} & 1 \\ e^x & e^{-x} & 0 \\ e^x & -e^{-x} & 0 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ 1 & 2 & 0 \\ 1 & -1 & 0 \end{vmatrix} = -2.$$

Therefore, $-2xy''' + 2y'' + 2xy' - 2y = 0$, which is equivalent to $xy''' - y'' - xy' + y = 0$.

(c)

$$P_0(x) = -\begin{vmatrix} e^x & xe^{-x} & 1 \\ e^x & e^{-x}(1-x) & 0 \\ e^x & e^{-x}(x-2) & 0 \end{vmatrix} = -\begin{vmatrix} 1 & x & 1 \\ 1 & 1-x & 0 \\ 1 & x-2 & 0 \end{vmatrix} = -\begin{vmatrix} 1 & 1-x \\ 1 & x-2 \end{vmatrix} = 3-2x;$$

$$P_1(x) = \begin{vmatrix} e^x & xe^{-x} & 1 \\ e^x & e^{-x}(1-x) & 0 \\ e^x & e^{-x}(3-x) & 0 \end{vmatrix} = \begin{vmatrix} 1 & x & 1 \\ 1 & 1-x & 0 \\ 1 & 3-x & 0 \end{vmatrix} = \begin{vmatrix} 1 & 1-x \\ 1 & 3-x \end{vmatrix} = 2;$$

$$P_2(x) = -\begin{vmatrix} e^x & xe^{-x} & 1 \\ e^x & e^{-x}(x-2) & 0 \\ e^x & e^{-x}(3-x) & 0 \end{vmatrix} = -\begin{vmatrix} 1 & x & 1 \\ 1 & x-2 & 0 \\ 1 & 3-x & 0 \end{vmatrix} = -\begin{vmatrix} 1 & x-2 \\ 1 & 3-x \end{vmatrix} = 2x-5;$$

$$P_3(x) = \begin{vmatrix} e^x & e^{-x}(1-x) & 0 \\ e^x & e^{-x}(x-2) & 0 \\ e^x & e^{-x}(3-x) & 0 \end{vmatrix} = 0.$$

Therefore, $(3-2x)y''' + 2y'' + (2x-5)y' = 0$.

(d)

$$\begin{aligned} P_0(x) &= -\begin{vmatrix} x & x^2 & e^x \\ 1 & 2x & e^x \\ 0 & 2 & e^x \end{vmatrix} = -e^x \begin{vmatrix} x & x^2 & 1 \\ 1 & 2x & 1 \\ 0 & 2 & 1 \end{vmatrix} = -e^x \begin{vmatrix} x & x^2-2 & 0 \\ 1 & 2x-2 & 0 \\ 0 & 2 & 1 \end{vmatrix} \\ &= -e^x \begin{vmatrix} x & x^2-2 \\ 1 & 2x-2 \end{vmatrix} = -e^x(x^2-2x+2); \end{aligned}$$

$$P_1(x) = \begin{vmatrix} x & x^2 & e^x \\ 1 & 2x & e^x \\ 0 & 0 & e^x \end{vmatrix} = e^x \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = x^2 e^x;$$

$$P_2(x) = -\begin{vmatrix} x & x^2 & e^x \\ 0 & 2 & e^x \\ 0 & 0 & e^x \end{vmatrix} = -e^x \begin{vmatrix} x & x^2 \\ 0 & 2 \end{vmatrix} = -2xe^x;$$

$$P_3(x) = \begin{vmatrix} 1 & 2x & e^x \\ 0 & 2 & e^x \\ 0 & 0 & e^x \end{vmatrix} = e^x \begin{vmatrix} 1 & 2x \\ 0 & 2 \end{vmatrix} = 2e^x.$$

Therefore, $-e^x(x^2-2x+2)y''' + x^2e^xy'' - 2xe^xy' + 2e^xy = 0$; which is equivalent to $(x^2-2x+2)y''' - x^2y'' + 2xy' - 2y = 0$.

(e)

$$\begin{aligned} P_0(x) &= -\begin{vmatrix} x & x^2 & 1/x \\ 1 & 2x & -1/x^2 \\ 0 & 2 & 2/x^3 \end{vmatrix} = -x \begin{vmatrix} 1 & x & 1/x^2 \\ 1 & 2x & -1/x^2 \\ 0 & 2 & 2/x^3 \end{vmatrix} = -x \begin{vmatrix} 1 & x & 1/x^2 \\ 0 & x & -2/x^2 \\ 0 & 2 & 2/x^3 \end{vmatrix} \\ &= -x \begin{vmatrix} x & -2/x^2 \\ 2 & 2/x^3 \end{vmatrix} = -\frac{6}{x}; \end{aligned}$$

$$P_1(x) = \begin{vmatrix} x & x^2 & 1/x \\ 1 & 2x & -1/x^2 \\ 0 & 0 & -6/x^4 \end{vmatrix} = -\frac{6}{x^4}\begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = -\frac{6}{x^2};$$

$$P_2(x) = -\begin{vmatrix} x & x^2 & 1/x \\ 0 & 2 & 2/x^3 \\ 0 & 0 & -6/x^4 \end{vmatrix} = \frac{6}{x^4}\begin{vmatrix} x & x^2 \\ 0 & 2 \end{vmatrix} = \frac{12}{x^3};$$

$$P_3(x) = \begin{vmatrix} 1 & 2x & -1/x^2 \\ 0 & 2 & 2/x^3 \\ 0 & 0 & -6/x^4 \end{vmatrix} = -\frac{6}{x^4}\begin{vmatrix} 1 & 2x \\ 0 & 2 \end{vmatrix} = -\frac{12}{x^4}.$$

Therefore, $-\frac{6}{x}y''' - \frac{6}{x^2}y'' + \frac{12}{x^3}y' - \frac{12}{x^4}y = 0$, which is equivalent to $x^3y''' + x^2y'' - 2xy' + 2y = 0$.

(f)

$$\begin{aligned} P_0(x) &= -\begin{vmatrix} x+1 & e^x & e^{3x} \\ 1 & e^x & 3e^{3x} \\ 0 & e^x & 9e^{3x} \end{vmatrix} = -e^{4x}\begin{vmatrix} x+1 & 1 & 1 \\ 1 & 1 & 3 \\ 0 & 1 & 9 \end{vmatrix} = -e^{4x}\begin{vmatrix} x+1 & 0 & -8 \\ 1 & 0 & -6 \\ 0 & 1 & 9 \end{vmatrix} \\ &= e^{4x}\begin{vmatrix} x+1 & -8 \\ 1 & -6 \end{vmatrix} = 2e^{4x}(1-3x); \end{aligned}$$

$$\begin{aligned} P_1(x) &= \begin{vmatrix} x+1 & e^x & e^{3x} \\ 1 & e^x & 3e^{3x} \\ 0 & e^x & 27e^{3x} \end{vmatrix} = e^{4x}\begin{vmatrix} x+1 & 1 & 1 \\ 1 & 1 & 3 \\ 0 & 1 & 27 \end{vmatrix} = e^{4x}\begin{vmatrix} x+1 & 0 & -26 \\ 1 & 0 & -24 \\ 0 & 1 & 27 \end{vmatrix} \\ &= -e^{4x}\begin{vmatrix} x+1 & -26 \\ 1 & -24 \end{vmatrix} = 2e^{4x}(12x-1); \end{aligned}$$

$$\begin{aligned} P_2(x) &= -\begin{vmatrix} x+1 & e^x & e^{3x} \\ 0 & e^x & 9e^{3x} \\ 0 & e^x & 27e^{3x} \end{vmatrix} = -e^{4x}\begin{vmatrix} x+1 & 1 & 1 \\ 0 & 1 & 9 \\ 0 & 1 & 27 \end{vmatrix} \\ &= -e^{4x}\begin{vmatrix} x+1 & 1 & 1 \\ 0 & 1 & 9 \\ 0 & 0 & 18 \end{vmatrix} = -18e^{4x}(x+1); \end{aligned}$$

$$P_3(x) = \begin{vmatrix} 1 & e^x & 3e^{3x} \\ 0 & e^x & 9e^{3x} \\ 0 & e^x & 27e^{3x} \end{vmatrix} = e^{4x}\begin{vmatrix} 1 & 1 & 3 \\ 0 & 1 & 9 \\ 0 & 1 & 27 \end{vmatrix} = e^{4x}\begin{vmatrix} 1 & 1 & 3 \\ 0 & 1 & 9 \\ 0 & 0 & 18 \end{vmatrix} = 18e^{4x}.$$

Therefore,

$$2e^{4x}(1-3x)y''' + 2e^{4x}(12x-1)y'' - 18e^{4x}(x+1)y' + 18e^{4x}y = 0,$$

which is equivalent to

$$(3x-1)y''' - (12x-1)y'' + 9(x+1)y' - 9y = 0.$$

(g)

$$
\begin{aligned}
P_0(x) &= \begin{vmatrix} x & x^3 & 1/x & 1/x^2 \\ 1 & 3x^2 & -1/x^2 & -2/x^3 \\ 0 & 6x & 2/x^3 & 6/x^4 \\ 0 & 6 & -6/x^4 & -24/x^5 \end{vmatrix} = x \begin{vmatrix} 1 & x^2 & 1/x^2 & 1/x^3 \\ 1 & 3x^2 & -1/x^2 & -2/x^3 \\ 0 & 6x & 2/x^3 & 6/x^4 \\ 0 & 6 & -6/x^4 & -24/x^5 \end{vmatrix} \\
&= x \begin{vmatrix} 1 & x^2 & 1/x^2 & 1/x^3 \\ 0 & 2x^2 & -2/x^2 & -3/x^3 \\ 0 & 6x & 2/x^3 & 6/x^4 \\ 0 & 6 & -6/x^4 & -24/x^5 \end{vmatrix} = x \begin{vmatrix} 2x^2 & -2/x^2 & -3/x^3 \\ 6x & 2/x^3 & 6/x^4 \\ 6 & -6/x^4 & -24/x^5 \end{vmatrix} \\
&= x^2 \begin{vmatrix} 2x & -2/x^3 & -3/x^4 \\ 6x & 2/x^3 & 6/x^4 \\ 6 & -6/x^4 & -24/x^5 \end{vmatrix} = x^2 \begin{vmatrix} 2x & -2/x^3 & -3/x^4 \\ 0 & 8/x^3 & 15/x^4 \\ 0 & 0 & -15/x^5 \end{vmatrix} = -\frac{240}{x^5}
\end{aligned}
$$

$$
\begin{aligned}
P_1(x) &= -\begin{vmatrix} x & x^3 & 1/x & 1/x^2 \\ 1 & 3x^2 & -1/x^2 & -2/x^3 \\ 0 & 6x & 2/x^3 & 6/x^4 \\ 0 & 0 & 24/x^5 & 120/x^6 \end{vmatrix} = -x \begin{vmatrix} 1 & x^2 & 1/x^2 & 1/x^3 \\ 1 & 3x^2 & -1/x^2 & -2/x^3 \\ 0 & 6x & 2/x^3 & 6/x^4 \\ 0 & 0 & 24/x^5 & 120/x^6 \end{vmatrix} \\
&= -x \begin{vmatrix} 1 & x^2 & 1/x^2 & 1/x^3 \\ 0 & 2x^2 & -2/x^2 & -3/x^3 \\ 0 & 6x & 2/x^3 & 6/x^4 \\ 0 & 0 & 24/x^5 & 120/x^6 \end{vmatrix} = -x^2 \begin{vmatrix} 2x & -2/x^3 & -3/x^4 \\ 6x & 2/x^3 & 6/x^4 \\ 0 & 24/x^5 & 120/x^6 \end{vmatrix} \\
&= -x^2 \begin{vmatrix} 2x & -2/x^3 & -3/x^4 \\ 0 & 8/x^3 & 15/x^4 \\ 0 & 24/x^5 & 120/x^6 \end{vmatrix} = -2x^3 \begin{vmatrix} 8/x^3 & 15/x^4 \\ 24/x^5 & 120/x^6 \end{vmatrix} = -\frac{1200}{x^6};
\end{aligned}
$$

$$
\begin{aligned}
P_2(x) &= \begin{vmatrix} x & x^3 & 1/x & 1/x^2 \\ 1 & 3x^2 & -1/x^2 & -2/x^3 \\ 0 & 6 & -6/x^4 & -24/x^5 \\ 0 & 0 & 24/x^5 & 120/x^6 \end{vmatrix} = x \begin{vmatrix} 1 & x^2 & 1/x^2 & 1/x^3 \\ 1 & 3x^2 & -1/x^2 & -2/x^3 \\ 0 & 6 & -6/x^4 & -24/x^5 \\ 0 & 0 & 24/x^5 & 120/x^6 \end{vmatrix} \\
&= x \begin{vmatrix} 1 & x^2 & 1/x^2 & 1/x^3 \\ 0 & 2x^2 & -2/x^2 & -3/x^3 \\ 0 & 6 & -6/x^4 & -24/x^5 \\ 0 & 0 & 24/x^5 & 120/x^6 \end{vmatrix} = x^3 \begin{vmatrix} 2 & -2/x^4 & -3/x^5 \\ 6 & -6/x^4 & -24/x^5 \\ 0 & 24/x^5 & 120/x^6 \end{vmatrix} \\
&= x^3 \begin{vmatrix} 2 & -2/x^4 & -3/x^5 \\ 0 & 0 & -15/x^5 \\ 0 & 24/x^5 & 120/x^6 \end{vmatrix} = 2x^3 \begin{vmatrix} 0 & -15/x^5 \\ 24/x^5 & 120/x^6 \end{vmatrix} = \frac{720}{x^7};
\end{aligned}
$$

$$
\begin{aligned}
P_3(x) &= -\begin{vmatrix} x & x^3 & 1/x & 1/x^2 \\ 0 & 6x & 2/x^3 & 6/x^4 \\ 0 & 6 & -6/x^4 & -24/x^5 \\ 0 & 0 & 24/x^5 & 120/x^6 \end{vmatrix} = -x^2 \begin{vmatrix} 6 & 2/x^4 & 6/x^5 \\ 6 & -6/x^4 & -24/x^5 \\ 0 & 24/x^5 & 120/x^6 \end{vmatrix} \\
&= -x^2 \begin{vmatrix} 6 & 2/x^4 & 6/x^5 \\ 0 & -8/x^4 & -30/x^5 \\ 0 & 24/x^5 & 120/x^6 \end{vmatrix} = -6x^2 \begin{vmatrix} -8/x^4 & -30/x^5 \\ 24/x^5 & 120/x^6 \end{vmatrix} = \frac{1440}{x^6};
\end{aligned}
$$

$$
\begin{aligned}
P_4(x) &= \begin{vmatrix} 1 & 3x^2 & -1/x^2 & -2/x^3 \\ 0 & 6x & 2/x^3 & 6/x^4 \\ 0 & 6 & -6/x^4 & -24/x^5 \\ 0 & 0 & 24/x^5 & 120/x^6 \end{vmatrix} = x \begin{vmatrix} 6 & 2/x^4 & 6/x^5 & \\ 6 & -6/x^4 & -24/x^5 & \\ 0 & 0 & 24/x^5 & 120/x^6 \end{vmatrix} \\
&= x \begin{vmatrix} 6 & 2/x^4 & 6/x^5 & \\ 0 & -8/x^4 & -30/x^5 & \\ 0 & 0 & 24/x^5 & 120/x^6 \end{vmatrix} = 6x \begin{vmatrix} -8/x^4 & -30/x^5 \\ 24/x^5 & 120/x^6 \end{vmatrix} = -\frac{1440}{x^9}.
\end{aligned}
$$

Therefore,

$$
-\frac{240}{x^5}y^{(4)} - \frac{1200}{x^6}y''' + \frac{720}{x^7}y'' + \frac{1440}{x^8}y' - \frac{1440}{x^9}y = 0,
$$

which is equivalent to $x^4y^{(4)} + 5x^3y''' - 3x^2y'' - 6xy' + 6y = 0$.

(h)

$$
\begin{aligned}
P_0(x) &= \begin{vmatrix} x & x\ln|x| & 1/x & x^2 \\ 1 & \ln|x|+1 & -1/x^2 & 2x \\ 0 & 1/x & 2/x^3 & 2 \\ 0 & -1/x^2 & -6/x^4 & 0 \end{vmatrix} = x \begin{vmatrix} 1 & \ln|x| & 1/x^2 & x \\ 1 & \ln|x|+1 & -1/x^2 & 2x \\ 0 & 1/x & 2/x^3 & 2 \\ 0 & -1/x^2 & -6/x^4 & 0 \end{vmatrix} \\
&= x \begin{vmatrix} 1 & \ln|x| & 1/x^2 & x \\ 0 & 1 & -2/x^2 & x \\ 0 & 1/x & 2/x^3 & 2 \\ 0 & -1/x^2 & -6/x^4 & 0 \end{vmatrix} = x \begin{vmatrix} 1 & -2/x^2 & x \\ 1/x & 2/x^3 & 2 \\ -1/x^2 & -6/x^4 & 0 \end{vmatrix} \\
&= x \begin{vmatrix} 1 & -2/x^2 & x \\ 1/x & 2/x^3 & 2 \\ -1/x^2 & -6/x^4 & 0 \end{vmatrix} = x \begin{vmatrix} 1 & -2/x^2 & x \\ 0 & 4/x^3 & 1 \\ 0 & -8/x^4 & 1/x \end{vmatrix} = x \begin{vmatrix} 4/x^3 & 1 \\ -8/x^4 & 1/x \end{vmatrix} = \frac{12}{x^3};
\end{aligned}
$$

$$
\begin{aligned}
P_1(x) &= -\begin{vmatrix} x & x\ln|x| & 1/x & x^2 \\ 1 & \ln|x|+1 & -1/x^2 & 2x \\ 0 & 1/x & 2/x^3 & 2 \\ 0 & 2/x^3 & 24/x^5 & 0 \end{vmatrix} = -x \begin{vmatrix} 1 & \ln|x| & 1/x^2 & x \\ 1 & \ln|x|+1 & -1/x^2 & 2x \\ 0 & 1/x & 2/x^3 & 2 \\ 0 & 2/x^3 & 24/x^5 & 0 \end{vmatrix} \\
&= -x \begin{vmatrix} 1 & \ln|x| & 1/x^2 & x \\ 0 & 1 & -2/x^2 & x \\ 0 & 1/x & 2/x^3 & 2 \\ 0 & 2/x^3 & 24/x^5 & 0 \end{vmatrix} = -x \begin{vmatrix} 1 & -2/x^2 & x \\ 1/x & 2/x^3 & 2 \\ 2/x^3 & 24/x^5 & 0 \end{vmatrix} \\
&= -x \begin{vmatrix} 1 & -2/x^2 & x \\ 0 & 4/x^3 & 1 \\ 0 & 28/x^5 & -2/x^2 \end{vmatrix} = -x \begin{vmatrix} 4/x^3 & 1 \\ 28/x^5 & -2/x^2 \end{vmatrix} = \frac{36}{x^4};
\end{aligned}
$$

$$\begin{aligned}
P_2(x) &= \begin{vmatrix} x & x\ln|x| & 1/x & x^2 \\ 1 & \ln|x|+1 & -1/x^2 & 2x \\ 0 & -1/x^2 & -6/x^4 & 0 \\ 0 & 2/x^3 & 24/x^5 & 0 \end{vmatrix} = x\begin{vmatrix} 1 & \ln|x| & 1/x^2 & x \\ 1 & \ln|x|+1 & -1/x^2 & 2x \\ 0 & -1/x^2 & -6/x^4 & 0 \\ 0 & 2/x^3 & 24/x^5 & 0 \end{vmatrix} \\
&= x\begin{vmatrix} 1 & \ln|x| & 1/x^2 & x \\ 0 & 1 & -2/x^2 & x \\ 0 & -1/x^2 & -6/x^4 & 0 \\ 0 & 2/x^3 & 24/x^5 & 0 \end{vmatrix} = x\begin{vmatrix} 1 & -2/x^2 & x \\ -1/x^2 & -6/x^4 & 0 \\ 2/x^3 & 24/x^5 & 0 \end{vmatrix} \\
&= x\begin{vmatrix} 1 & -2/x^2 & x \\ 0 & -8/x^4 & 1/x \\ 0 & 28/x^5 & -2/x^2 \end{vmatrix} = x\begin{vmatrix} -8/x^4 & 1/x \\ 28/x^5 & -2/x^2 \end{vmatrix} = -\frac{12}{x^5};
\end{aligned}$$

$$\begin{aligned}
P_3(x) &= -\begin{vmatrix} x & x\ln|x| & 1/x & x^2 \\ 0 & 1/x & 2/x^3 & 2 \\ 0 & -1/x^2 & -6/x^4 & 0 \\ 0 & 2/x^3 & 24/x^5 & 0 \end{vmatrix} = -x\begin{vmatrix} 1/x & 2/x^3 & 2 \\ -1/x^2 & -6/x^4 & 0 \\ 2/x^3 & 24/x^5 & 0 \end{vmatrix} \\
&= -x\begin{vmatrix} 1/x & 2/x^3 & 2 \\ 0 & -4/x^4 & 2/x \\ 0 & 20/x^5 & -4/x^2 \end{vmatrix} = -\begin{vmatrix} -4/x^4 & 2/x \\ 20/x^5 & -4/x^2 \end{vmatrix} = \frac{24}{x^6};
\end{aligned}$$

$$\begin{aligned}
P_4(x) &= \begin{vmatrix} 1 & \ln|x|+1 & -1/x^2 & 2x \\ 0 & 1/x & 2/x^3 & 2 \\ 0 & -1/x^2 & -6/x^4 & 0 \\ 0 & 2/x^3 & 24/x^5 & 0 \end{vmatrix} = \begin{vmatrix} 1/x & 2/x^3 & 2 \\ -1/x^2 & -6/x^4 & 0 \\ 2/x^3 & 24/x^5 & 0 \end{vmatrix} \\
&= 2\begin{vmatrix} -1/x^2 & -6/x^4 \\ 2/x^3 & 24/x^5 \end{vmatrix} = -\frac{24}{x^7}.
\end{aligned}$$

Therefore,

$$\frac{12}{x^3}y^{(4)} + \frac{36}{x^4}y''' - \frac{12}{x^5}y'' + \frac{24}{x^6}y' - \frac{24}{x^7}y = 0,$$

which is equivalent to $x^4y^{(4)} + 3x^2y''' - x^2y'' + 2xy' - 2y = 0$.

(i)

$$\begin{aligned}
P_0(x) &= \begin{vmatrix} e^x & e^{-x} & x & e^{2x} \\ e^x & -e^{-x} & 1 & 2e^{2x} \\ e^x & e^{-x} & 0 & 4e^{2x} \\ e^x & -e^{-x} & 0 & 8e^{2x} \end{vmatrix} = e^{2x}\begin{vmatrix} 1 & 1 & x & 1 \\ 1 & -1 & 1 & 2 \\ 1 & 1 & 0 & 4 \\ 1 & -1 & 0 & 8 \end{vmatrix} \\
&= e^{2x}\begin{vmatrix} 2 & 1 & x & 1 \\ 0 & -1 & 1 & 2 \\ 2 & 1 & 0 & 4 \\ 0 & -1 & 0 & 8 \end{vmatrix} = e^{2x}\begin{vmatrix} 2 & 1 & x & 1 \\ 0 & -1 & 1 & 2 \\ 0 & 0 & -x & 3 \\ 0 & -1 & 0 & 8 \end{vmatrix} \\
&= 2e^{2x}\begin{vmatrix} -1 & 1 & 2 \\ 0 & -x & 3 \\ -1 & 0 & 8 \end{vmatrix} = 2e^{2x}\begin{vmatrix} -1 & 1 & 2 \\ 0 & -x & 3 \\ 0 & -1 & 6 \end{vmatrix} \\
&= -2e^{2x}\begin{vmatrix} -x & 3 \\ -1 & 6 \end{vmatrix} = 6e^{2x}(2x-1);
\end{aligned}$$

$$
\begin{aligned}
P_1(x) &= -\begin{vmatrix} e^x & e^{-x} & x & e^{2x} \\ e^x & -e^{-x} & 1 & 2e^{2x} \\ e^x & e^{-x} & 0 & 4e^{2x} \\ e^x & e^{-x} & 0 & 16e^{2x} \end{vmatrix} = -e^{2x}\begin{vmatrix} 1 & 1 & x & 1 \\ 1 & -1 & 1 & 2 \\ 1 & 1 & 0 & 4 \\ 1 & 1 & 0 & 16 \end{vmatrix} \\
&= -e^{2x}\begin{vmatrix} 0 & 1 & x & 1 \\ 2 & -1 & 1 & 2 \\ 0 & 1 & 0 & 4 \\ 0 & 1 & 0 & 16 \end{vmatrix} = 2e^{2x}\begin{vmatrix} 1 & x & 1 \\ 1 & 0 & 4 \\ 1 & 0 & 16 \end{vmatrix} \\
&= 2e^{2x}\begin{vmatrix} 1 & x & 1 \\ 0 & -x & 3 \\ 0 & -x & 15 \end{vmatrix} = 2e^{2x}\begin{vmatrix} -x & 3 \\ -x & 15 \end{vmatrix} = -24xe^{2x};
\end{aligned}
$$

$$
\begin{aligned}
P_2(x) &= \begin{vmatrix} e^x & e^{-x} & x & e^{2x} \\ e^x & -e^{-x} & 1 & 2e^{2x} \\ e^x & -e^{-x} & 0 & 8e^{2x} \\ e^x & e^{-x} & 0 & 16e^{2x} \end{vmatrix} = e^{2x}\begin{vmatrix} 1 & 1 & x & 1 \\ 1 & -1 & 1 & 2 \\ 1 & -1 & 0 & 8 \\ 1 & 1 & 0 & 16 \end{vmatrix} \\
&= e^{2x}\begin{vmatrix} 0 & 1 & x & 1 \\ 2 & -1 & 1 & 2 \\ 2 & -1 & 0 & 8 \\ 0 & 1 & 0 & 16 \end{vmatrix} = e^{2x}\begin{vmatrix} 0 & 1 & x & 1 \\ 2 & -1 & 1 & 2 \\ 0 & 0 & -1 & 6 \\ 0 & 1 & 0 & 16 \end{vmatrix} = -2e^{2x}\begin{vmatrix} 1 & x & 1 \\ 0 & -1 & 6 \\ 1 & 0 & 16 \end{vmatrix} \\
&= -2e^{2x}\begin{vmatrix} 1 & x & 1 \\ 0 & -1 & 6 \\ 0 & -x & 15 \end{vmatrix} = -2e^{2x}\begin{vmatrix} -1 & 6 \\ -x & 15 \end{vmatrix} = 6e^{2x}(5-2x);
\end{aligned}
$$

$$
\begin{aligned}
P_3(x) &= -\begin{vmatrix} e^x & e^{-x} & x & e^{2x} \\ e^x & e^{-x} & 0 & 4e^{2x} \\ e^x & -e^{-x} & 0 & 8e^{2x} \\ e^x & e^{-x} & 0 & 16e^{2x} \end{vmatrix} = -e^{2x}\begin{vmatrix} 1 & 1 & x & 1 \\ 1 & 1 & 0 & 4 \\ 1 & -1 & 0 & 8 \\ 1 & 1 & 0 & 16 \end{vmatrix} \\
&= -e^{2x}\begin{vmatrix} 0 & 1 & x & 1 \\ 0 & 1 & 0 & 4 \\ 2 & -1 & 0 & 8 \\ 0 & 1 & 0 & 16 \end{vmatrix} = -2e^{2x}\begin{vmatrix} 1 & x & 1 \\ 0 & -x & 3 \\ 0 & -x & 15 \end{vmatrix} = -2e^{2x}\begin{vmatrix} -x & 3 \\ -x & 15 \end{vmatrix} = 24xe^{2x};
\end{aligned}
$$

$$
\begin{aligned}
P_4(x) &= \begin{vmatrix} e^x & -e^{-x} & 1 & 2e^{2x} \\ e^x & e^{-x} & 0 & 4e^{2x} \\ e^x & -e^{-x} & 0 & 8e^{2x} \\ e^x & e^{-x} & 0 & 16e^{2x} \end{vmatrix} = e^{2x}\begin{vmatrix} 1 & -1 & 1 & 2 \\ 1 & 1 & 0 & 4 \\ 1 & -1 & 0 & 8 \\ 1 & 1 & 0 & 16 \end{vmatrix} \\
&= e^{2x}\begin{vmatrix} 0 & -1 & 1 & 2 \\ 2 & 1 & 0 & 4 \\ 0 & -1 & 0 & 8 \\ 2 & 1 & 0 & 16 \end{vmatrix} = e^{2x}\begin{vmatrix} 0 & -1 & 1 & 2 \\ 2 & 1 & 0 & 4 \\ 0 & -1 & 0 & 8 \\ 0 & 0 & 0 & 12 \end{vmatrix} \\
&= -2e^{2x}\begin{vmatrix} -1 & 1 & 2 \\ -1 & 0 & 8 \\ 0 & 0 & 12 \end{vmatrix} = -2e^{2x}\begin{vmatrix} -1 & 1 & 2 \\ 0 & -1 & 6 \\ 0 & 0 & 12 \end{vmatrix} = -24e^{2x}.
\end{aligned}
$$

Therefore,

$$6e^{2x}(2x-1)y^{(4)} - 24xe^{2x}y''' + 6e^{2x}(5-2x)y'' + 24xe^{2x}y' - 24e^{2x}y = 0,$$

which is equivalent to $(2x-1)y^{(4)} - 4xy''' + (5-2x)y'' + 4xy' - 4y = 0$.

9.1.24. **(j)**

$$\begin{aligned}
P_0(x) &= \begin{vmatrix} e^{2x} & e^{-2x} & 1 & x^2 \\ 2e^{2x} & -2e^{-2x} & 0 & 2x \\ 4e^{2x} & 4e^{-2x} & 0 & 2 \\ 8e^{2x} & -8e^{-2x} & 0 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & x^2 \\ 2 & -2 & 0 & 2x \\ 4 & 4 & 0 & 2 \\ 8 & -8 & 0 & 0 \end{vmatrix} \\
&= \begin{vmatrix} 2 & -2 & 2x \\ 4 & 4 & 2 \\ 8 & -8 & 0 \end{vmatrix} = \begin{vmatrix} 2 & -2 & 2x \\ 0 & 8 & 2-4x \\ 0 & 0 & -8x \end{vmatrix} = -128x
\end{aligned}$$

$$\begin{aligned}
P_1(x) &= -\begin{vmatrix} e^{2x} & e^{-2x} & 1 & x^2 \\ 2e^{2x} & -2e^{-2x} & 0 & 2x \\ 4e^{2x} & 4e^{-2x} & 0 & 2 \\ 16e^{2x} & 16e^{-2x} & 0 & 0 \end{vmatrix} = -\begin{vmatrix} 1 & 1 & 1 & x^2 \\ 2 & -2 & 0 & 2x \\ 4 & 4 & 0 & 2 \\ 16 & 16 & 0 & 0 \end{vmatrix} \\
&= -\begin{vmatrix} 2 & -2 & 2x \\ 4 & 4 & 2 \\ 16 & 16 & 0 \end{vmatrix} = -\begin{vmatrix} 2 & -2 & 2x \\ 0 & 8 & 2-4x \\ 0 & 0 & -8 \end{vmatrix} = 128;
\end{aligned}$$

$$\begin{aligned}
P_2(x) &= \begin{vmatrix} e^{2x} & e^{-2x} & 1 & x^2 \\ 2e^{2x} & -2e^{-2x} & 0 & 2x \\ 8e^{2x} & -8e^{-2x} & 0 & 0 \\ 16e^{2x} & 16e^{-2x} & 0 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & x^2 \\ 2 & -2 & 0 & 2x \\ 8 & -8 & 0 & 0 \\ 16 & 16 & 0 & 0 \end{vmatrix} \\
&= \begin{vmatrix} 2 & -2 & 2x \\ 8 & -8 & 0 \\ 16 & 16 & 0 \end{vmatrix} = 2x \begin{vmatrix} 8 & -8 \\ 16 & 16 \end{vmatrix} = 512x;
\end{aligned}$$

$$\begin{aligned}
P_3(x) &= -\begin{vmatrix} e^{2x} & e^{-2x} & 1 & x^2 \\ 4e^{2x} & 4e^{-2x} & 0 & 2 \\ 8e^{2x} & -8e^{-2x} & 0 & 0 \\ 16e^{2x} & 16e^{-2x} & 0 & 0 \end{vmatrix} = -\begin{vmatrix} 1 & 1 & 1 & x^2 \\ 4 & 4 & 0 & 2 \\ 8 & -8 & 0 & 0 \\ 16 & 16 & 0 & 0 \end{vmatrix} \\
&= -\begin{vmatrix} 4 & 4 & 2 \\ 8 & -8 & 0 \\ 16 & 16 & 0 \end{vmatrix} = -2 \begin{vmatrix} 8 & -8 \\ 16 & 16 \end{vmatrix} = -512;
\end{aligned}$$

$$P_4(x) = \begin{vmatrix} 2e^{2x} & -2e^{-2x} & 0 & 2x \\ 4e^{2x} & 4e^{-2x} & 0 & 2 \\ 8e^{2x} & -8e^{-2x} & 0 & 0 \\ 16e^{2x} & 16e^{-2x} & 0 & 0 \end{vmatrix} = 0.$$

Therefore, $-128xy^{(4)} + 128y''' + 512xy'' - 512y = 0$, which is equivalent to $xy^{(4)} - y''' - 4xy'' + 4y' = 0$.

9.2 HIGHER ORDER CONSTANT COEFFICIENT HOMOGENEOUS EQUATIONS

9.2.2. $p(r) = r^4 + 8r^2 - 9 = (r-1)(r+1)(r^2+9)$; $y = c_1e^x + c_2e^{-x} + c_3\cos 3x + c_4 \sin 3x$.

9.2.4. $p(r) = 2r^3 + 3r^2 - 2r - 3 = (r-1)(r+1)(2r+3)$; $y = c_1e^x + c_2e^{-x} + c_3e^{-3x/2}$.

9.2.6. $p(r) = 4r^3 - 8r^2 + 5r - 1 = (r-1)(2r-1)^2$; $y = c_1e^x + e^{x/2}(c_2 + c_3x)$.

9.2.8. $p(r) = r^4 + r^2 = r^2(r^2+1)$; $y = c_1 + c_2x + c_3\cos x + c_4\sin x$.

9.2.10. $p(r) = r^4 + 12r^2 + 36 = (r^2+6)^2$; $y = (c_1 + c_2x)\cos\sqrt{6}x + (c_3 + c_4x)\sin\sqrt{6}x$.

9.2.12. $p(r) = 6r^4 + 5r^3 + 7r^2 + 5r + 1 = (2r+1)(3r+1)(r^2+1)$; $y = c_1e^{-x/2} + c_2e^{-x/3} + c_3\cos x + c_4\sin x$.

9.2.14. $p(r) = r^4 - 4r^3 + 7r^2 - 6r + 2 = (r-1)^2(r^2-2r+2)$; $y = e^x(c_1 + c_2x + c_3\cos x + c_4\sin x)$.

9.2.16. $p(r) = r^3 + 3r^2 - r - 3 = (r-1)(r+1)(r+3)$;

$$\begin{array}{rcl} y &=& c_1e^x + c_2e^{-x} + c_3e^{-3x} \\ y' &=& c_1e^x - c_2e^{-x} - 3c_3e^{-3x} \\ y'' &=& c_1e^x + c_2e^{-x} + 9c_3e^{-3x} \end{array} ; \quad \begin{array}{rcr} c_1 + c_2 + c_3 &=& 0 \\ c_1 - c_2 - 3c_3 &=& 14 \\ c_1 + c_2 + 9c_3 &=& -40 \end{array} ;$$

$c_1 = 2, c_2 = 3, c_3 = -5$; $y = 2e^x + 3e^{-x} - 5e^{-3x}$.

9.2.18. $p(r) = r^3 - 2r - 4 = (r-2)(r^2+2r+2)$;

$$\begin{array}{rcl} y &=& e^{-x}(c_1\cos x + c_2\sin x) + c_3e^{2x} \\ y' &=& -e^{-x}((c_1 - c_2)\cos x + (c_1 + c_2)\sin x) + 2c_3e^{2x} \\ y'' &=& e^{-x}(2c_1\sin x - 2c_2\cos x) + 4c_3e^{2x} \end{array} ; \quad \begin{array}{rcr} c_1 + c_3 &=& 6 \\ -c_1 + c_2 + 2c_3 &=& 3 \\ -2c_2 + 4c_3 &=& 22 \end{array} ;$$

$c_1 = 2, c_2 = -3, c_3 = 4$; $y = 2e^{-x}\cos x - 3e^{-x}\sin x + 4e^{2x}$.

9.2.20. $p(r) = r^3 - 6r^2 + 12r - 8 = (r-2)^3$;

$$\begin{array}{rcl} y &=& e^{2x}(c_1 + c_2x + c_3x^2) \\ y' &=& e^{2x}(2c_1 + c_2 + (2c_2 + 2c_3)x + 2c_3x^2) \\ y'' &=& 2e^{2x}(2c_1 + 2c_2 + c_3 + 2(c_2 + 2c_3)x + 2c_3x^2) \end{array} ; \quad \begin{array}{rcr} c_1 &=& 1 \\ 2c_1 + c_2 &=& -1 \\ 4c_1 + 4c_2 + 2c_3 &=& -4 \end{array}$$

$c_1 = 1, c_2 = -3, c_3 = 2$; $y = e^{2x}(1 - 3x + 2x^2)$.

9.2.22. $p(r) = 8r^3 - 4r^2 - 2r + 1 = (2r+1)(2r-1)^2$;

$$\begin{array}{rcl} y &=& e^{x/2}(c_1 + c_2x) + c_3e^{-x/2} \\ y' &=& \frac{1}{2}e^{x/2}(c_1 + 2c_2 + c_2x) - \frac{1}{2}c_3e^{-x/2} \\ y'' &=& \frac{1}{4}e^{x/2}(c_1 + 4c_2 + c_2x) + \frac{1}{4}c_3e^{-x/2} \end{array} ; \quad \begin{array}{rcr} c_1 + c_3 &=& 4 \\ \frac{1}{2}c_1 + c_2 - \frac{1}{2}c_3 &=& -3 \\ \frac{1}{4}c_1 + c_2 + \frac{1}{4}c_3 &=& -1 \end{array} ;$$

$c_1 = 1, c_2 = -2, c_3 = 3$; $y = e^{x/2}(1 - 2x) + 3e^{-x/2}$.

9.2.24. $p(r) = r^4 - 6r^3 + 7r^2 + 6r - 8 = (r-1)(r-2)(r-4)(r+1)$;

$$\begin{array}{rcl} y &=& c_1e^x + c_2e^{2x} + c_3e^{4x} + c_4e^{-x} \\ y' &=& c_1e^x + 2c_2e^{2x} + 4c_3e^{4x} - c_4e^{-x} \\ y'' &=& c_1e^x + 4c_2e^{2x} + 16c_3e^{4x} + c_4e^{-x} \\ y''' &=& c_1e^x + 8c_2e^{2x} + 64c_3e^{4x} - c_4e^{-x} \end{array} ; \quad \begin{array}{rcr} c_1 + c_2 + c_3 + c_4 &=& -2 \\ c_1 + 2c_2 + 4c_3 - c_4 &=& -8 \\ c_1 + 4c_2 + 16c_3 + c_4 &=& -14 \\ c_1 + 8c_2 + 64c_3 - c_4 &=& -62 \end{array} ;$$

$c_1 = -4, c_2 = 1, c_3 = -1, c_4 = 2$; $y = -4e^x + e^{2x} - e^{4x} + 2e^{-x}$.

9.2.26. $p(r) = r^4 + 2r^3 - 2r^2 - 8r - 8 = (r-2)(r+2)(r^2+2r+2)$;

$$\begin{array}{rcl} y &=& c_1e^{2x} + c_2e^{-2x} + e^{-x}(c_3\cos x + c_4\sin x) \\ y' &=& 2c_1e^{2x} - 2c_2e^{-2x} - e^{-x}((c_3-c_4)\cos x + (c_3+c_4)\sin x) \\ y'' &=& 4c_1e^{2x} + 4c_2e^{-2x} + e^{-x}(2c_3\sin x - 2c_4\cos x) \\ y''' &=& 8c_1e^{2x} - 8c_2e^{-2x} + e^{-x}((2c_3+2c_4)\cos x + 2(c_4-c_3)\sin x) \end{array};$$

$$\begin{array}{rcr} c_1+c_2+c_3 &=& 5 \\ 2c_1-2c_2-c_3+c_4 &=& -2 \\ 4c_1+4c_2-2c_4 &=& 6 \\ 8c_1-8c_2+2c_3+2c_4 &=& 8 \end{array};$$

$c_1 = 1, c_2 = 1, c_3 = 3, c_4 = 1$; $y = e^{2x} + e^{-2x} + e^{-x}(3\cos x + \sin x)$.

9.2.28. **(a)** $W(x) = \begin{vmatrix} e^x & xe^x & e^{2x} \\ e^x & e^x(x+1) & 2e^{2x} \\ e^x & e^x(x+2) & 4e^{2x} \end{vmatrix}$; $W(0) = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} =$ 1.

(b) $W(x) = \begin{vmatrix} \cos 2x & \sin 2x & e^{3x} \\ -2\sin 2x & 2\cos 2x & 3e^{3x} \\ -4\cos 2x & -4\sin 2x & 9e^{3x} \end{vmatrix}$; $W(0) = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ -4 & 0 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 3 \\ -4 & 0 & 13 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 0 & 13 \end{vmatrix} =$ 26.

(c) $W(x) = \begin{vmatrix} e^{-x}\cos x & e^{-x}\sin x & e^x \\ -e^{-x}(\cos x + \sin x) & e^{-x}(\cos x - \sin x) & e^x \\ 2e^{-x}\sin x & -2e^{-x}\cos x & e^x \end{vmatrix}$; $W(0) = \begin{vmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & -2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & -2 & 1 \end{vmatrix} =$ $\begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix} = 5$.

(d) $W(x) = \begin{vmatrix} 1 & x & x^2 & e^x \\ 0 & 1 & 2x & e^x \\ 0 & 0 & 2 & e^x \\ 0 & 0 & 0 & e^x \end{vmatrix}$; $W(0) = \begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1$.

(e) $W(x) = \begin{vmatrix} e^x & e^{-x} & \cos x & \sin x \\ e^x & -e^{-x} & -\sin x & \cos x \\ e^x & e^{-x} & -\cos x & -\sin x \\ e^x & -e^{-x} & \sin x & -\cos x \end{vmatrix}$;

$$\begin{aligned} W(0) &= \begin{vmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & -2 \end{vmatrix} = -2\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & -1 \end{vmatrix} \\ &= -2\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{vmatrix} = 4\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -8. \end{aligned}$$

(f) $W(x) = \begin{vmatrix} \cos x & \sin x & e^x\cos x & e^x\sin x \\ -\sin x & \cos x & e^x(\cos x - \sin x) & e^x(\cos x + \sin x) \\ -\cos x & -\sin x & -2e^x\sin x & 2e^x\cos x \\ \sin x & -\cos x & -e^x(2\cos x + 2\sin x) & e^x(2\cos x - 2\sin x) \end{vmatrix}$;

$$W(0) = \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ -1 & 0 & 0 & 2 \\ 0 & -1 & -2 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & -1 & -2 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ -1 & -2 & 2 \end{vmatrix}$$
$$= \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & -1 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \end{vmatrix} = 5.$$

9.2.40. **(a)** Since $y = Q_1(D)P_1(D)y + Q_2(D)P_2(D)y$ and $P_1(D)y = P_2(D)y = 0$, it follows that $y = 0$.

(b) Suppose that (A) $a_1u_1 + \cdots + a_ru_r + b_1v_1 + \cdots + b_sv_s = 0$, where $a_1, \dots, a_r$ and $b_1, \dots, b_s$ are constants. Denote $u = a_1u_1 + \cdots + a_ru_r$ and $v = b_1v_1 + \cdots + b_sv_s$. Then (B) $P_1(D)u = 0$ and (C) $P_2(D)v = 0$. Since $u+v = 0$, $P_2(D)(u+v) = 0$. Therefore,$0 = P_2(D)(u+v) = P_2(D)u + P_2(D)v$. Now (C) implies that $P_2(D)u = 0$. This, (B), and **(a)** imply that $u = a_1u_1 + \cdots + a_ru_r = 0$, so $a_1 = \cdots = a_r = 0$, since $u_1, \dots, u_r$ are linearly independent. Now (A) reduces to $b_1v_1 + \cdots + b_sv_s = 0$, so $b_1 = c \cdots = b_s = 0$, since $v_1, \dots, v_s$ are linearly independent. Therefore,$u_1, \dots, u_r, v_1, \dots, v_r$ are linearly independent.

(c) It suffices to show that $\{y_1, y_2, \dots, y_n\}$ is linearly independent. Suppose that $c_1y_1 + \cdots + c_ny_n = 0$. We may assume that $y_1, \dots, y_r$ are linearly independent solutions of $p_1(D)y = 0$ and $y_{r+1}, \dots, y_n$ are solutions of $P_2(D) = p_2(D) \cdots p_k(D)y = 0$. Since $p_1(r)$ and $P_2(r)$ have no common factors, **(b)** implies that (A) $c_1y_1 + \cdots + c_ry_r = 0$ and (B) $c_{r+1}y_{r+1} + \cdots + c_ny_n = 0$. Now (A) implies that $c_1 = \cdots = c_r = 0$, since $y_1, \dots, y_r$ are linearly independent. If $k = 2$, then $y_{r+1}, \dots, y_n$ are linearly independent, so $c_{r+1} = \cdots = c_n = 0$, and the proof is complete. If $k > 2$ repeat this argument, starting from (B), with p_1 replaced by p_2, and P_2 replaced by $P_3 = p_3 \cdots p_n$.

9.2.42. **(a)**

$$\begin{aligned} (\cos A + i \sin A)(\cos B + i \sin B) &= (\cos A \cos B - \sin A \sin B) \\ &\quad + (\cos A \sin B + \sin A \cos B) \\ &= \cos(A+B) + i \sin(A+B). \end{aligned}$$

(b) Obvious for $n = 0$. If $n = -1$ write

$$\begin{aligned} \frac{1}{\cos\theta + i \sin\theta} &= \frac{1}{\cos\theta + i\sin\theta} \frac{\cos\theta - i\sin\theta}{\cos\theta - i\sin\theta} \\ &= \frac{\cos\theta - i\sin\theta}{\cos^2\theta + \sin^2\theta} = \cos\theta - i\sin\theta = \cos(-\theta) + i\sin(-\theta). \end{aligned}$$

(d) If n is a negative integer, then (B) $(\cos\theta + i\sin\theta)^n = \dfrac{1}{(\cos\theta + i\sin\theta)^{|n|}}$. From the hint, (C) $\dfrac{1}{(\cos\theta + i\sin\theta)^{|n|}} = (\cos\theta - i\sin\theta)^{|n|} = (\cos(-\theta) + i\sin(-\theta))^{|n|}$. Replacing θ by $-\theta$ and n by $|n|$ in (A) shows that (D) $(\cos(-\theta) + i\sin(-\theta))^{|n|} = \cos(-|n|\theta) + i\sin(-|n|\theta)$. Since $|n| = -n$, (E) $\cos(-|n|\theta) + i\sin(-|n|\theta) = \cos n\theta + i\sin n\theta$. Now (B), (C), (D), and (E) imply (A).

(e) From (A), $z_k^n = \cos 2k\pi + i\sin 2k\pi = 1$ and $\zeta_k^n = \cos(2k+1)\pi + i\sin(2k+1)\pi = \cos(2k+1)\pi = \cos\pi = -1$.

(f) From **(e)**, $\rho^{1/n}z_0, \dots, \rho^{1/n}z_{n-1}$ are all zeros of $z^n - \rho$. Since they are distinct numbers, $z^n - \rho$ has the stated factoriztion.

From **(e)**, $\rho^{1/n}\zeta_0, \dots, \rho^{1/n}\zeta_{n-1}$ are all zeros of $z^n + \rho$. Since they are distinct numbers, $z^n + \rho$ has the stated factoriztion.

9.2.43. **(a)** $p(r) = r^3 - 1 = (r - z_0)(r - z_1)(r - z_2)$ where $z_k = \cos\frac{2k\pi}{3} + i\sin\frac{2k\pi}{3}$, $k = 0, 1, 2$. Hence, $z_0 = 1$, $z_1 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$, and $z_2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$. Therefore, $p(r) = (r-1)\left(\left(r+\frac{1}{2}\right)^2 + \frac{3}{4}\right)$, so $\left\{e^x,\ e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right),\ e^{-x/2}\sin\left(\frac{\sqrt{3}}{2}x\right)\right\}$ is a fundamental set of solutions.

(b) $p(r) = r^3+1 = (r-\zeta_0)(r-\zeta_1)(r-\zeta_2)$ where $\zeta_k = \cos\frac{(2k+1)\pi}{3} + i\sin\frac{(2k+1)\pi}{3}$, $k = 0, 1, 2$. Hence, $\zeta_0 = \frac{1}{2} + i\frac{\sqrt{3}}{2}$, $\zeta_1 = -1$ $\zeta_2 = \frac{1}{2} - i\frac{\sqrt{3}}{2}$. Therefore, $p(r) = (r+1)\left(\left(r-\frac{1}{2}\right)^2 + \frac{3}{4}\right)$, so $\left\{e^{-x},\ e^{x/2}\cos\left(\frac{\sqrt{3}}{2}x\right),\ e^{x/2}\sin\left(\frac{\sqrt{3}}{2}x\right)\right\}$ is a fundamental set of solutions.

(c) $p(r) = r^4+64 = (r-2\sqrt{2}\zeta_0)(r-2\sqrt{2}\zeta_1)(r-2\sqrt{2}\zeta_2)(r-2\sqrt{2}\zeta_3)$, where $\zeta_k = \cos\frac{(2k+1)\pi}{4} + i\sin\frac{(2k+1)\pi}{4}$, $k = 0, 1, 2, 3$. Therefore, $\zeta_0 = \frac{1+i}{\sqrt{2}}$, $\zeta_1 = \frac{-1+i}{\sqrt{2}}$, $\zeta_2 = \frac{-1-i}{\sqrt{2}}$, and $\zeta_3 = \frac{1-i}{\sqrt{2}}$, so $p(r) = ((r-2)^2+4)((r+2)^2+4)$ and $\{e^{2x}\cos 2x,\ e^{2x}\sin 2x,\ e^{-2x}\cos 2x,\ e^{-2x}\sin 2x\}$ is a fundamental set of solutions.

(d) $p(r) = r^6-1 = (r-z_0)(r-z_1)(r-z_2)(r-z_3)(r-z_4)(r-z_5)$ where $z_k = \cos\frac{2k\pi}{6} + i\sin\frac{2k\pi}{6}$, $k = 0, 1, 2, 3, 4, 5$. Therefore, $z_0 = 1$, $z_1 = \frac{1}{2} + i\frac{\sqrt{3}}{2}$, $z_2 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$, $z_3 = -1$, $z_4 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$, and $z_5 = \frac{1}{2} - i\frac{\sqrt{3}}{2}$, so $p(r) = (r-1)(r+1)\left(\left(r-\frac{1}{2}\right)^2+\frac{3}{4}\right)\left(\left(r+\frac{1}{2}\right)^2+\frac{3}{4}\right)$ and

$\left\{e^x,\ e^{-x},\ e^{x/2}\cos\left(\frac{\sqrt{3}}{2}x\right),\ e^{x/2}\sin\left(\frac{\sqrt{3}}{2}x\right),\ e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right),\ e^{-x/2}\sin\left(\frac{\sqrt{3}}{2}x\right)\right\}$ is a fundamental set of solutions.

(e) $p(r) = r^6 + 64 = (r-2\zeta_0)(r-2\zeta_1)(r-2\zeta_2)(r-2\zeta_3)(r-2\zeta_4)(r-2\zeta_5)$ where $\zeta_k = \cos\frac{(2k+1)\pi}{6} + i\sin\frac{(2k+1)\pi}{6}$, $k = 0, 1, 2, 3, 4, 5$. Therefore, $\zeta_0 = \frac{\sqrt{3}}{2} + \frac{i}{2}$, $\zeta_1 = i$, $\zeta_2 = -\frac{\sqrt{3}}{2} + \frac{i}{2}$, $\zeta_3 = -\frac{\sqrt{3}}{2} - \frac{i}{2}$, $\zeta_4 = -i$, and $\zeta_5 = \frac{\sqrt{3}}{2} - \frac{i}{2}$, so $p(r) = (r^2+4)((r-\sqrt{3})^2+1)((r+\sqrt{3})^2+1)$ and $\{\cos 2x,\ \sin 2x,\ e^{-\sqrt{3}x}\cos x,\ e^{-\sqrt{3}x}\sin x,\ e^{\sqrt{3}x}\cos x,\ e^{\sqrt{3}x}\sin x\}$ is a fundamental set of solutions.

(f) $p(r) = (r-1)^6-1 = (r-1-z_0)(r-1-z_1)(r-1-z_2)(r-1-z_3)(r-1-z_4)(r-1-z_5)$ where $z_k = \cos\frac{2k\pi}{6} + i\sin\frac{2k\pi}{6}$, $k = 0, 1, 2, 3, 4, 5$. Therefore, $z_0 = 1$, $z_1 = \frac{1}{2} + i\frac{\sqrt{3}}{2}$, $z_2 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$, $z_3 = -1$, $z_4 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$, and $z_5 = \frac{1}{2} - i\frac{\sqrt{3}}{2}$, so $p(r) = r(r-2)\left(\left(r-\frac{3}{2}\right)^2+\frac{3}{4}\right)\left(\left(r-\frac{1}{2}\right)^2+\frac{3}{4}\right)$ and $\left\{1,\ e^{2x},\ e^{3x/2}\cos\left(\frac{\sqrt{3}}{2}x\right),\ e^{3x/2}\sin\left(\frac{\sqrt{3}}{2}x\right),\ e^{x/2}\cos\left(\frac{\sqrt{3}}{2}x\right),\ e^{x/2}\sin\left(\frac{\sqrt{3}}{2}x\right)\right\}$ is a fundamental set of solutions.

(g) $p(r) = r^5 + r^4 + r^3 + r^2 + r + 1 = \frac{r^6-1}{r-1}$. Therefore, from the solution of **(d)** $p(r) =$

$(r+1)\left(\left(r-\frac{1}{2}\right)^2+\frac{3}{4}\right)\left(\left(r+\frac{1}{2}\right)^2+\frac{3}{4}\right)$ and

$\left\{e^{-x}, e^{x/2}\cos\left(\frac{\sqrt{3}}{2}x\right), e^{x/2}\sin\left(\frac{\sqrt{3}}{2}x\right), e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right), e^{-x/2}\sin\left(\frac{\sqrt{3}}{2}x\right)\right\}$ is a fundamental set of solutions.

9.3 UNDETERMINED COEFFICIENTS FOR HIGHER ORDER EQUATIONS

9.3.2. If $y = u^{-3x}$, then $y''' - 2y'' - 5y' + 6y = e^{-3x}[(u''' - 11u'' + 34u' - 24u) - 2(u'' - 6u' + 9u) - 5(u' - 3u) + 6u] = e^{-3x}(u''' - 11u'' + 34u' - 24u)$. Let $u_p = A + Bx + Cx^2$, where $(-24A + 34B - 22C) + (-24B + 68C)x - 24Cx^2 = 32 - 23x + 6x^2$. Then $C = -1/4$, $B = 1/4$, $A = -3/4$ and $y_p = -\frac{e^{-3x}}{4}(3 - x + x^2)$.

9.3.4. If $y = ue^{-2x}$, then $y''' + 3y'' - y' - 3y = e^{-2x}[(u''' - 6u'' + 12u' - 8u) + 3(u'' - 4u' + 4u) - (u' - 2u) - 3u] = e^{-2x}(u''' - 3u'' - u' + 3u)$. Let $u_p = A + Bx + Cx^2$, where $(3A - B - 6C) + (3B - 2C)x + 3Cx^2 = 2 - 17x + 3x^2$. Then $C = 1$, $B = -5$, $A = 1$, and $y_p = e^{-2x}(1 - 5x + x^2)$.

9.3.6. If $y = ue^x$, then $y''' + y'' - 2y = e^x[(u''' + 3u'' + 3u' + u) + (u'' + 2u' + u) - 2u] = e^x(u''' + 4u'' + 5u')$. Let $u_p = x(A + Bx + Cx^2)$, where $(5A + 8B + 6C) + (10B + 24C)x + 15Cx^2 = 14 + 34x + 15x^2$. Then $C = 1$, $B = 1$, $A = 0$, and $y_p = x^2e^x(1 + x)$.

9.3.8. If $y = ue^x$, then $y''' - y'' - y' + y = e^x[(u''' + 3u'' + 3u' + u) - (u'' + 2u' + u) - (u' + u) + u] = e^x(u''' + 2u'')$. Let $u_p = x^2(A + Bx)$ where $(4A + 6B) + 12Bx = 7 + 6x$. Then $B = 1/2$, $A = 1$, and $y_p = \frac{x^2e^x}{2}(2 + x)$.

9.3.10. If $y = ue^{3x}$, then $y''' - 5y'' + 3y' + 9y = e^{3x}[(u''' + 9u'' + 27u' + 27u) - 5(u'' + 6u' + 9u) + 3(u' + 3u) + 9u] = e^{3x}(u''' + 4u'')$. Let $u_p = x^2(A + Bx + Cx^2)$, where $(8A + 6B) + (24B + 24C)x + 48Cx^2 = 22 - 48x^2$. Then $C = -1$, $B = 1$, $A = 2$, and $y_p = x^2e^{3x}(2 + x - x^2)$.

9.3.12. If $y = ue^{x/2}$, then $8y''' - 12y'' + 6y' - y = e^{x/2}[8(u''' + 3u''/2 + 3u'/4 + u/8) - 12(u'' + u' + u/4) + 6(u' + u/2) - u] = 8e^{x/2}u'''$, so $u''' = \frac{1 + 4x}{8}$. Integrating three times and taking the constants of integration to be zero yields $u_p = \frac{x^3}{48}(1 + x)$. Therefore, $y_p = \frac{x^3e^{x/2}}{48}(1 + x)$.

9.3.14. If $y = ue^{2x}$, then $y^{(4)} + 3y''' + y'' - 3y' - 2y = e^{2x}[(u^{(4)} + 8u''' + 24u'' + 32u' + 16u) + 3(u''' + 6u'' + 12u' + 8u) + (u'' + 4u' + 4u) - 3(u' + 2u) - 2u] = e^{2x}(u^{(4)} + 11u''' + 43u'' + 69u' + 36u)$. Let $u_p = A + Bx$ where $(36A + 69B) + 36Bx = -33 - 36x$. Then $B = -1$, $A = 1$, and $y_p = e^{2x}(1 - x)$.

9.3.16. If $y = ue^x$, then $4y^{(4)} - 11y'' - 9y' - 2y = e^x[4(u^{(4)} + 4u''' + 6u'' + 4u' + u) - 11(u'' + 2u' + u) - 9(u' + u) - 2u] = e^x(4u^{(4)} + 16u''' + 13u'' - 15u' - 18u)$. Let $u_p = A + Bx$ where $-(18A + 15B) - 18Bx = -1 + 6x$. Then $B = -1/3$, $A = 1/3$, and $y_p = \frac{e^x}{3}(1 - x)$.

9.3.18. If $y = ue^x$, then $y^{(4)} - 4y''' + 6y'' - 4y' + 2y = e^x[(u^{(4)} + 4u''' + 6u'' + 4u' + u) - 4(u''' + 3u'' + 3u' + u) + 6(u'' + 2u' + u) - 4(u' + u) + 2u] = e^x(u^{(4)} + u)$. Let $u_p = A + Bx + Cx^2 + Dx^3 + Ex^4$ where $(A + 24E) + Bx + Cx^2 + Dx^3 + Ex^4 = 24 + x + x^4$. Then $E = 1$ $D = 0$, $C = 0$ $B = 1$, $A = 0$, and $y_p = xe^x(1 + x^3)$.

9.3.20. If $y = ue^{2x}$, then $y^{(4)} + y''' - 2y'' - 6y' - 4y = e^{2x}[(u^{(4)} + 8u''' + 24u'' + 32u' + 16u) + (u''' + 6u'' + 12u' + 8u) - 2(u'' + 4u' + 4u) - 6(u' + 2u) - 4u] = e^{2x}(u^{(4)} + 9u''' + 28u'' + 30u')$. Let

$u_p = x(A+Bx+Cx^2)$ where $(30A+56B+54C)+(60B+168C)x+90Cx^2 = -(4+28x+15x^2)$. Then $C = -1/6$, $B = 0$, $A = 1/6$, and $y_p = \dfrac{xe^{2x}}{6}(1-x^2)$.

9.3.22. If $y = ue^x$, then $y^{(4)} - 5y'' + 4y = e^x[(u^{(4)}+4u'''+6u''+4u'+u)-5(u''+2u'+u)+4u] = e^x(u^{(4)} + 4u''' + u'' - 6u')$. Let $u_p = x(A + Bx + Cx^2)$ where $(-6A + 2B + 24C) + (-12B + 6C)x - 18Cx^2 = 3 + x - 3x^2$, so $C = 1/6$, $B = 0$, $A = 1/6$. Then $y_p = \dfrac{xe^x}{6}(1 + x^2)$.

9.3.24. If $y = ue^{2x}$, then $y^{(4)} - 3y''' + 4y' = e^{2x}[(u^{(4)} + 8u''' + 24u'' + 32u' + 16u) - 3(u''' + 6u'' + 12u' + 8u) + 4(u' + 2u)] = e^{2x}(u^{(4)} + 5u''' + 6u'')$. Let $u_p = x^2(A + Bx + Cx^2)$ where $(12A + 30B + 24C) + (36B + 120C)x + 72Cx^3 = 15 + 26x + 12x^2$. Then $C = 1/6$, $B = 1/6$, $A = 1/2$, and $y_p = \dfrac{x^2e^{2x}}{6}(3 + x + x^2)$.

9.3.26. If $y = ue^x$, then $2y^{(4)} - 5y''' + 3y'' + y' - y = e^x[2(u^{(4)} + 4u''' + 6u'' + 4u' + u) - 5(u''' + 3u'' + 3u' + u) + 3(u'' + 2u' + u) + (u' + u) - u] = e^x(2u^{(4)} + 3u''')$. Let $u_p = x^3(A + Bx)$, where $(18A + 48B) + 72Bx = 11 + 12x$. Then $B = 1/6$, $A = 1/6$, and $y_p = \dfrac{x^3e^x}{6}(1 + x)$.

9.3.28. If $y = ue^{2x}$, then $y^{(4)} - 7y''' + 18y'' - 20y' + 8y = e^{2x}[(u^{(4)} + 8u''' + 24u'' + 32u' + 16u) - 7(u''' + 6u'' + 12u' + 8u) + 18(u'' + 4u' + 4u) - 20(u' + 2u) + 8u] = e^{2x}(u^{(4)} + u''')$. Let $u_p = x^3(A + Bx + Cx^2)$ where $(6A + 24B) + (24B + 120C)x + 60Cx^2 = 3 - 8x - 5x^2$. Then so $C = -1/12$ $B = 1/12$, $A = 1/6$, and $y_p = \dfrac{x^3e^{2x}}{12}(2 + x - x^2)$.

9.3.30. If $y = ue^{-x}$, then $y''' + y'' - 4y' - 4y = e^{-x}[(u''' - 3u'' + 3u' - u) + (u'' - 2u' + u) - 4(u' - u) - 4u] = e^{-x}(u''' - 2u'' - 3u')$. Let $u_p = (A_0 + A_1x)\cos 2x + (B_0 + B_1x)\sin 2x$, where

$$\begin{aligned} 8A_1 - 14B_1 &= -22 \\ 14A_1 + 8B_1 &= -6 \\ 8A_0 - 14B_0 - 15A_1 - 8B_1 &= 1 \\ 14A_0 + 8B_0 + 8A_1 - 15B_1 &= -1. \end{aligned}$$

Then $A_1 = -1$, $B_1 = 1$, $A_0 = 1$, $B_0 = 1$, and $y_p = e^{-x}[(1 - x)\cos 2x + (1 + x)\sin 2x]$.

9.3.32. If $y = ue^x$, then $y'''-2y''+y'-2y = e^x[(u'''+3u''+3u'+u)-2(u''+2u'+u)+(u'+u)-2u] = e^x(u''' + u'' - 2u)$. Let $u_p = (A_0 + A_1x + A_2x^2)\cos 2x + (B_0 + B_1x + B_2x^2)\sin 2x$ where

$$\begin{aligned} -6A_2 - 8B_2 &= -4 \\ 8A_2 - 6B_2 &= -3 \\ -6A_1 - 8B_1 - 24A_2 + 8B_2 &= 5 \\ 8A_1 - 6B_1 - 8A_2 - 24B_2 &= -5 \\ -6A_0 - 8B_0 - 12A_1 + 4B_1 + 2A_2 + 12B_2 &= -9 \\ 8A_0 - 6B_0 - 4A_1 - 12B_1 - 12A_2 + 2B_2 &= 6. \end{aligned}$$

Then $A_2 = 0$, $B_2 = 1/2$; $A_1 = 1/2$, $B_1 = -1/2$; $A_0 = 1/1$, $B_0 = 1/2$; and $y_p = \dfrac{e^x}{2}[(1 + x)\cos 2x + (1 - x + x^2)\sin 2x]$.

9.3.34. If $y = ue^x$, then $y''' - y'' + 2y = e^x[(u''' + 3u'' + 3u' + u) - (u'' + 2u' + u) + 2u] = e^x(u''' + 2u'' + u' + 2u)$. Since $\cos x$ and $\sin x$ satisfy $u''' + 2u'' + u' + 2u = 0$, let $u_p = x[(A_0 +$

$A_1x)\cos x + (B_0 + B_1x)\sin x]$ where

$$\begin{aligned}
-4A_1 + 8B_1 &= 4\\
-8A_1 - 4B_1 &= -12\\
-2A_0 + 4B_0 + 4A_1 + 6B_1 &= 20\\
-4A_0 - 2B_0 - 6A_1 + 4B_1 &= -12.
\end{aligned}$$

Then $A_1 = 1$, $B_1 = 1$; $A_0 = 1$, $B_0 = 3$; and $y_p = xe^x[(1+x)\cos x + (3+x)\sin x]$.

9.3.36. If $y = ue^{3x}$, then $= e^{3x}[(u''' + 9u'' + 27u' + 27u) - 6(u'' + 6u' + 9u) + 18(u' + 3u)] = e^{3x}(u''' + 3u'' + 9u' + 27u)$. Since $\cos 3x$ and $\sin 3x$ satisfy $u''' + 3u'' + 9u' + 27u = 0$, let $u_p = x[(A_0 + A_1x)\cos 3x + (B_0 + B_1x)\sin 3x]$ where

$$\begin{aligned}
-36A_1 + 36B_1 &= 3\\
-36A_1 - 36B_1 &= 3\\
-18A_0 + 18B_0 + 6A_1 + 18B_1 &= -2\\
-18A_0 - 18B_0 - 18A_1 + 6B_1 &= 3.
\end{aligned}$$

Then $A_1 = -1/12$, $B_1 = 0$; $A_0 =$, $B_0 = -1/12$; and $y_p = -\dfrac{xe^{3x}}{12}(x\cos 3x + \sin 3x)$.

9.3.38. If $y = ue^x$, then $y^{(4)} - 3y''' + 2y'' + 2y' - 4y = e^x[(u^{(4)} + 4u''' + 6u'' + 4u' + u) - 3(u''' + 3u'' + 3u' + u) + 2(u'' + 2u' + u) + 4(u' + u) + u] = e^x(u^{(4)} + u''' - u'' + u' - 2u)$. Let $u_p = A\cos 2x + B\sin 2x$ where $18A - 6B = 2$ and $6A + 18B = -1$. Then $A = 1/12$, $B = -1/12$, and $y_p = \dfrac{e^x}{12}(\cos 2x - \sin 2x)$.

9.3.40. If $y = ue^{-x}$, then $y^{(4)} + 6y''' + 13y'' + 12y' + 4y = e^{-x}[(u^{(4)} - 4u''' + 6u'' - 4u' + u) + 6(u''' - 3u'' + 3u' - u) + 13(u'' - 2u' + u) + 12u' - u) + 4u] = e^{-x}(u^{(4)} + 2u''' + u'')$. Let $u_p = (A_0 + A_1x)\cos x + (B_0 + B_1x)\sin x$ where

$$\begin{aligned}
-2B_1 &= -1\\
2A_1 &= -1\\
-2B_0 - 6A_1 - 2B_1 &= 4\\
2A_0 + 2A_1 - 6B_1 &= -5.
\end{aligned}$$

Then $A_1 = -1/2$, $B_1 = 1/2$, $A_0 = -1/2$, $B_0 = -1$, and $y_p = -\dfrac{e^{-x}}{2}[(1+x)\cos x + (2-x)\sin x]$.

9.3.42. If $y = ue^x$, then $y^{(4)} - 5y''' + 13y'' - 19y' + 10y = e^x[(u^{(4)} + 4u''' + 6u'' + 4u' + u) - 5(u''' + 3u'' + 3u' + u) + 13(u'' + 2u' + u) - 19(u' + u) + 10u] = e^x(u^{(4)} - u''' + 4u'' - 4u')$. Since $\cos 2x$ and $\sin 2x$ satisfy $u^{(4)} - u''' + 4u'' - 4u' = 0$, let $u_p = x(A\cos 2x + B\sin 2x)$ where $8A - 16B = 1$ and $16A + 8B = 1$. Then $A = 3/40$, $B_0 = -1/40$, and $y_p = \dfrac{xe^x}{40}(3\cos 2x - \sin 2x)$.

9.3.44. If $y = ue^x$, then $y^{(4)} - 5y''' + 13y'' - 19y' + 10y = e^x[(u^{(4)} + 4u''' + 6u'' + 4u' + u) - 5(u''' + 3u'' + 3u' + u) + 13(u'' + 2u' + u) - 19(u' + u) + 10u] = e^x(u^{(4)} - u''' + 4u'' - 4u')$. Since $\cos 2x$ and $\sin 2x$ satisfy $u^{(4)} - u''' + 4u'' - 4u' = 0$, let $u_p = x[(A_0 + A_1x)\cos 2x + (B_0 + B_1x)\sin 2x)]$ where

$$\begin{aligned}
16A_1 - 32B_1 &= 8\\
32A_1 + 16B_1 &= -4\\
8A_0 - 16B_0 - 40A_1 - 12B_1 &= 7\\
16A_0 + 8B_0 + 12A_1 - 40B_1 &= 8.
\end{aligned}$$

Then $A_1 = 0$, $B_1 = -1/4$; $A_0 = 0$, $B_0 = -1/4$, and $y_p = -\dfrac{xe^x}{4}(1+x)\sin 2x$.

9.3.46. If $y = ue^{2x}$, then $y^{(4)} - 8y''' + 32y'' - 64y' + 64y + 4y = e^{2x}[(u^{(4)} + 8u''' + 24u'' + 32u' + 16u) - 8(u''' + 6u'' + 12u' + 8u) + 32(u'' + 4u' + 4u) - 64(u' + 2u) + 64u] = e^{2x}(u^{(4)} + 8u'' + 16u)$. Since $\cos 2x$, $\sin 2x$, $x\cos 2x$, and $x\sin 2x$ satisfy $u^{(4)} + 8u'' + 16u = 0$, let $u_p = x^2(A\cos 2x + B\sin 2x)$ where $-32A = 1$ and $-32B = -1$. Then $A = -1/32$, $B = 1/32$, and $y_p = -\dfrac{x^2e^{2x}}{32}(\cos 2x - \sin 2x)$.

9.3.48. Find particular solutions of **(a)** $y''' - 4y'' + 5y' - 2y = -4e^x$, **(b)** $y''' - 4y'' + 5y' - 2y = e^{2x}$, and **(c)** $y''' - 4y'' + 5y' - 2y = -2\cos x + 4\sin x$.

(a) If $y = ue^x$, then $y''' - 4y'' + 5y' - 2y = e^x[(u''' + 3u'' + 3u' + u) - 4(u'' + 2u' + u) + 5(u' + u) - 2u] = e^x(u''' - u'')$. Let $u_{1p} = Ax^2$ where $-2A = -4$. Then $A = 2$, and $y_{1p} = 2x^2e^x$.

(b) If $y = ue^{2x}$, then $y''' - 4y'' + 5y' - 2y = e^{2x}[(u''' + 6u'' + 12u' + 8u) - 4(u'' + 4u' + 4u) + 5(u' + 2u) - 2u] = e^{2x}(u''' + 2u'' + u')$. Let $u_{2p} = x$. Then $y_{2p} = xe^{2x}$.

(c) If $y_{3p} = A\cos x + B\sin x$, then $y_{3p}''' - 4y_{3p}'' + 5y_{3p}' - 2y_{3p} = (2A + 4B)\cos x + (-4A + 2B)\sin x = -2\cos x + 4\sin x$ if $A = -1$ and $B = 0$, so $y_{3p} = -\cos x$.

From the principle of superposition, $y_p = 2x^2e^x + xe^{2x} - \cos x$.

9.3.50. Find particular solutions of **(a)** $y''' - y' = -2(1+x)$, **(b)** $y''' - y' = 4e^x$, **(c)** $y''' - y' = -6e^{-x}$, and **(d)** $y''' - y' = 96e^{3x}$

(a) Let $y_{1p} = x(A + Bx)$. Then $y_{1p}''' - y_{1p}' = -A - 2Bx = -2(1+x)$ if $A = 2$ and $B = 1$; therefore $y_{1p} = 2x + 2x^2$.

(b) If $y = ue^x$, then $y''' - y' = e^x[(u''' + 3u'' + 3u' + u) - (u' + u)] = e^x(u''' + 3u'' + 2u')$. Let $u_{2p} = 4x$. Then $y_{2p} = 4xe^x$.

(c) If $y = ue^{-x}$, then $y''' - y' = e^{-x}[(u''' - 3u'' + 3u' - u) - (u' - u)] = e^{-x}(u''' - 3u'' + 2u')$. Let $u_{2p} = -3x$. Then $y_{2p} = -6xe^{-x}$.

(d) Since e^{3x} does not satisfy the complementary equation, let $y_{4p} = Ae^{3x}$. Then $y_{4p}''' - y_{4p}' = 24Ae^{3x}$. Let $A = 4$; then $y_{4p} = 4e^{4x}$.

From the principle of superposition, $y_p = 2x + x^2 + 2xe^x - 3xe^{-x} + 4e^{3x}$

9.3.52. Find particular solutions of **(a)** $y''' + 3y'' + 3y' + y = 12e^{-x}$ and **(b)** $y''' + 3y'' + 3y' + y = 9\cos 2x - 13\sin 2x$.

(a) If $y = ue^{-2x}$, then $y''' + 3y'' + 3y' + y = e^{-2x}[(u''' - 3u'' + 3u' - u) + 3(u'' - 2u' + u) + 3(u' - u) + u] = e^{-x}u'''$. Let $u_{1p}''' = 12$. Integrating three times and taking the constants of integration to be zero yields $u_{1p} = 2x^3$. Therefore,$y_{1p} = 2x^3$.

(b) Let $y_{2p} = A\cos 2x + B\sin 2x$ where $-11A - 2B = 9$ and $2A - 11B = -13$. Then $A = -1$, $B = 1$, and $y_{2p} = -\cos 2x + \sin 2x$.

From the principle of superposition, $y_p = 2x^3e^{-2x} - \cos 2x + \sin 2x$.

9.3.54. Find particular solutions of **(a)** $y^{(4)} - 5y'' + 4y = -12e^x$, **(b)** $y^{(4)} - 5y'' + 4y = 6e^{-x}$, and **(c)** $y^{(4)} - 5y'' + 4y = 10\cos x$.

(a) If $y = ue^x$, then $y^{(4)} - 5y'' + 4y = e^x[(u^{(4)} + 4u''' + 6u'' + 4u' + u) - 5(u'' + 2u' + u) + 4u] = e^x(u^{(4)} + 4u''' + u'' - 6u')$. Let $u_{1p} = 2x$. Then $y_{1p} = 2xe^x$.

(b) If $y = ue^{-x}$, then $y^{(4)} - 5y'' + 4y = e^{-x}[(u^{(4)} - 4u''' + 6u'' - 4u' + u) - 5(u'' - 2u' + u) + 4u] = e^{-x}(u^{(4)} - 4u''' + u'' + 6u')$. Let $u_{2p} = x$. Then $y_{2p} = xe^{-x}$.

(c) Let $y_{3p} = A\cos x + B\sin x$ where $10A = 10$ and $10B = 0$. Then $A = 1$, $B = 0$, and $y_{3p} = \cos x$.

From the principle of superposition, $y_p = 2xe^x + xe^{-x} + \cos x$.

9.3.56. Find particular solutions of **(a)** $y^{(4)} + 2y''' - 3y'' - 4y' + 4y = 2e^x(1+x)$ and **(b)** $y^{(4)} + 2y''' - 3y'' - 4y' + 4y = e^{-2x}$.

(a) If $y = ue^x$, then $y^{(4)} + 2y''' - 3y'' - 4y' + 4y = e^x[(u^{(4)} + 4u''' + 6u'' + 4u' + u) + 2(u''' + 3u'' + 3u' + u) - 3(u'' + 2u' + u) - 4(u' + u) + 4u] = e^x(u^{(4)} + 6u''' + 9u'')$. Let $u_{1p} = x^2(A + Bx)$ where $(18A + 36B) + 54Bx = 2 + 2x$. Then $B = 1/27$, $A = 1/27$, and $y_{1p} = \dfrac{x^2}{27}(1 + x)e^x$.

(b) If $y = ue^{-2x}$, then $y^{(4)} + 2y''' - 3y'' - 4y' + 4y = e^{-2x}[(u^{(4)} - 8u''' + 24u'' - 32u' + 16u) + 2(u''' - 6u'' + 12u' - 8u) - 3(u'' - 4u' + 4u) - 4(u' - 2u) + 4u] = e^{-2x}(u^{(4)} - 6u''' + 9u'')$. Let $u_{2p} = Ax^2$ where $18A = 1$. Then $A = 1/18$ and $y_p = \dfrac{x^2}{18}e^{-2x}$.

From the principle of superposition, $y_p = \dfrac{x^2}{54}[(2 + 2x)e^x + 3e^{-2x}]$.

9.3.58. Find particular solutions of **(a)** $y^{(4)} + 5y''' + 9y'' + 7y' + 2y = e^{-x}(30 + 24x)$ and **(b)** $y^{(4)} + 5y''' + 9y'' + 7y' + 2y = -e^{-2x}$.

(a) If $y = ue^{-x}$, then $y^{(4)} + 5y''' + 9y'' + 7y' + 2y = e^{-x}[(u^{(4)} - 4u''' + 6u'' - 4u' + u) + 5(u''' - 3u'' + 3u' - u) + 9(u'' - 2u' + u) + 7(u' - u) + 2u] = e^{-x}(u^{(4)} + u''')$. Let $u_{1p} = x^3(A + Bx)$ where $(6A + 24B) + 24Bx = 30 + 24x$. The $B = 1$, $A = 1$, and $y_{1p} = x^3(1 + x)e^{-x}$.

(b) If $y = ue^{-2x}$, then $y^{(4)} + 5y''' + 9y'' + 7y' + 2y = e^{-2x}[(u^{(4)} - 8u''' + 24u'' - 32u' + 16u) + 5(u''' - 6u'' + 12u' - 8u) + 9(u'' - 4u' + 4u) + 7(u' - 2u) + 2u] = e^{-2x}(u^{(4)} - 3u''' + 3u'' - u')$. Let $u_{2p} = x$. Then $y_{2p} = xe^{-2x}$.

From the principle of superposition, $y_p = x^3(1 + x)e^{-x} + xe^{-2x}$.

9.3.60. If $y = ue^{2x}$, then $y''' - y'' - y' + y = e^{2x}[(u''' + 6u'' + 12u' + 8u) - (u'' + 4u' + 4u) - (u' + 2u) + u] = e^{2x}(u''' + 5u'' + 7u' + 3u)$. Let $u_p = A + Bx$, where $(3A + 7B) + 3x = 10 + 3x$. Then $B = 1$, $A = 1$ and $y_p = e^{2x}(1 + x)$. Since $p(r) = (r + 1)(r - 1)^2$, $y = e^{2x}(1 + x) + c_1e^{-x} + e^x(c_2 + c_3x)$

9.3.62. If $y = ue^{2x}$, then $y''' - 6y'' + 11y' - 6y = e^{2x}[(u''' + 6u'' + 12u' + 8u) - 6(u'' + 4u' + 4u) + 11(u' + 2u) - 6u] = e^{2x}(u''' - u')$. Let $u_p = x(A + Bx + Cx^2)$ where $(-A + 6C) - 2Bx - 3Cx^2 = 5 - 4x - 3x^2$. Then $C = 1$, $B = 2$, $A = 1$, and $y_p = xe^{2x}(1 + x)^2$. Since $p(r) = (r - 1)(r - 2)(r - 3)$, $y = xe^{2x}(1 + x)^2 + c_1e^x + c_2e^{2x} + c_3e^{3x}$.

9.3.64. If $y = ue^x$, then $y''' - 3y'' + 3y' - y = e^x[(u''' + 3u'' + 3u' + u) - 3(u'' + 2u' + u) + 3(u' + u) - u] = e^xu'''$. Let $u''' = 1 + x$. Integrating three times and taking the constants of integration to be zero yields $u = \dfrac{x^3}{24}(4 + x)$. Therefore, $y_p = \dfrac{x^3e^x}{24}(4 + x)$. Since $p(r) = (r - 1)^3$, $y = \dfrac{x^3e^x}{24}(4 + x) + e^x(c_1 + c_2x + c_3x^2)$.

9.3.66. If $y = ue^{-2x}$, then $y''' + 2y'' - y' - 2y = e^{-2x}[(u''' - 6u'' + 12u' - 8u) + 2(u'' - 4u' + 4u) - (u' - 2u) - 2u] = e^{-2x}(u''' - 4u'' + 3u')$. Let $u_p = (A_0 + A_1x)\cos x + (B_0 + B_1x)\sin x$ where

$$\begin{aligned} 4A_1 + 2B_1 &= -2 \\ -2A_1 + 4B_1 &= -9 \\ 4A_0 + 2B_0 - 8B_1 &= 23 \\ -2A_0 + 4B_0 + 8A_1 &= 8. \end{aligned}$$

Then $A_1 = 1/2$, $B_1 = -2$; $A_0 = 1$, $B_0 = 3/2$, and $y_p = e^{-2x}\left[\left(1 + \dfrac{x}{2}\right)\cos x + \left(\dfrac{3}{2} - 2x\right)\sin x\right]$.

Since $p(r) = (r - 1)(r + 1)(r + 2)$, $y = e^{-2x}\left[\left(1 + \dfrac{x}{2}\right)\cos x + \left(\dfrac{3}{2} - 2x\right)\sin x\right] + c_1e^x + c_2e^{-x} + c_3e^{-2x}$.

9.3.68. If $y = ue^x$, then $y^{(4)} - 4y''' + 14y'' - 20y' + 25y = e^x[(u^{(4)} + 4u''' + 6u'' + 4u' + u) - 4(u''' + 3u'' + 3u' + u) + 14(u'' + 2u' + u) - 20(u' + u) + 25u] = e^x(u^{(4)} + 8u'' + 16u)$. Since $\cos 2x$, $\sin 2x$, $x\cos 2x$, and $x\sin 2x$ satisfy $u^{(4)} + 8u'' + 16u = 0$, let $u_p = x^2[(A_0 + A_1x)\cos 2x + (B_0 + B_1x)\sin 2x]$ where

$$\begin{aligned} -96A_1 &= 6 \\ -96B_1 &= 0 \\ -32A_0 + 48B_1 &= 2 \\ -48A_1 - 32B_0 &= 3. \end{aligned}$$

Then $A_1 = -1/16$, $B_1 = 0$; $A_0 = -1/16$, $B_0 = 0$, and $y_[= -\dfrac{x^2e^x}{16}(1 + x)\cos 2x$. Since $p(r) = [(r-1)^2 + 1]^2$, $y = -\dfrac{x^2e^x}{16}(1 + x)\cos 2x + e^x\,[(c_1 + c_2x)\cos 2x + (c_3 + c_4x)\sin 2x]$.

9.3.70. If $y = ue^{-x}$, then $y''' - y'' - y' + y = e^{-x}[(u''' - 3u'' + 3u' - u) - (u'' - 2u' + u) - (u' - u) + u] = e^{-x}(u''' - 4u'' + 4u')$. Let $u_p = x(A + Bx)$, where $(4A - 8B) + 8Bx = -4 + 8x$. Then $B = 1$, $A = 1$, and $y_p = x(1 + x)e^{-x}$. Since $p(r) = (r + 1)(r - 1)^2$ the general solution is $y = x(1 + x)e^{-x} + c_1e^{-x} + c_2e^x + c_3xe^x$. Therefore,

$$\begin{bmatrix} y \\ y' \\ y'' \end{bmatrix} = \begin{bmatrix} x(1+x)e^{-x} \\ -e^{-x}(x^2 - x - 1) \\ e^{-x}(x^2 - 3x) \end{bmatrix} + \begin{bmatrix} e^{-x} & e^x & xe^x \\ -e^{-x} & e^x & e^x(x+1) \\ e^{-x} & e^x & e^x(x+2) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}.$$

Setting $x = 0$ and imposing the initial conditions yields

$$\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix},$$

so $c_1 = 1$, $c_2 = 1$, $c_3 = -1$, and $y = e^{-x}(1 + x + x^2) + (1 - x)e^x$.

9.3.72. If $y = ue^{-x}$, then $y''' - 2y'' - 5y' + 6y = e^{-x}[(u^{(4)} - 4u''' + 6u'' - 4u' + u) + 2(u''' - 3u'' + 3u' - u) + 2(u'' - 2u' + u) + 2(u' - u) + u = e^{-x}(u^{(4)} - 2u''' + 2u'')$. Let $u_p = x^2(A + Bx)$, where $(4A - 12B) + 12Bx = 20 - 12x$. Then $B = -1$, $A = 2$, and $y_p = x^2(2 - x)e^{-x}$. Since $p(r) = (r+1)^2(r^2+1)$, the general solution is $y = x^2(2-x)e^{-x} + e^{-x}(c_1 + c_2x) + c_3\cos x + c_4\sin x$. Therefore,

$$\begin{bmatrix} y \\ y' \\ y'' \\ y''' \end{bmatrix} = \begin{bmatrix} x^2(2-x)e^{-x} \\ x(x^2 - 5x + 4)e^{-x} \\ -(x^3 - 8x^2 + 14x - 4)e^{-x} \\ (x^3 - 11x^2 + 30x - 18)e^{-x} \end{bmatrix} + \begin{bmatrix} e^{-x} & xe^{-x} & \cos x & \sin x \\ -e^{-x} & (1-x)e^{-x} & -\sin x & \cos x \\ e^{-x} & (x-2)e^{-x} & -\cos x & -\sin x \\ -e^{-x} & (3-x)e^{-x} & \sin x & -\cos x \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}$$

Setting $x = 0$ and imposing the initial conditions yields

$$\begin{bmatrix} 3 \\ -4 \\ 7 \\ -22 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 4 \\ -18 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 1 \\ 1 & -2 & -1 & 0 \\ -1 & 3 & 0 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix},$$

so $c_1 = 2$, $c_2 = -1$, $c_3 = 1$, $c_4 = -1$, and $y = (2 - x)(x^2 + 1)e^{-x} + \cos x - \sin x$.

9.3.74. If $y = ue^x$, then $y^{(4)} - 3y''' + 5y'' - 2y' = e^x[(u^{(4)} + 4u''' + 6u'' + 4u' + u) - 3(u''' + 3u'' + 3u' + u) + 4(u'' + 2u' + u) - 2(u' + u)] = e^x(u^{(4)} + u''' + u'' + u')$. Since $\cos x$ and $\sin x$ satisfy

$u^{(4)} + u''' + u'' + u' = 0$, let $u_p = x(A\cos x + B\sin x)$ where $-2A - 2B = -2$ and $2A - 2B = 2$. Then $A = 1$, $B = 0$, and $y_p = e^x\cos x$. Since $p(r) = r(r-1)[(r-1)^2+1]$ the general solution is $y = e^x\cos x + c_1 + e^x(c_2 + c_3\cos x + c_4\sin x)$. Therefore,

$$\begin{bmatrix} y \\ y' \\ y'' \\ y''' \end{bmatrix} = \begin{bmatrix} xe^x\cos x \\ e^x((x+1)\cos x - x\sin x) \\ e^x(2\cos x - 2(x+1)\sin x) \\ -e^x(2x\cos x + 2(x+3)\sin x) \end{bmatrix} + \begin{bmatrix} 1 & e^x & e^x\cos x & e^x\sin x \\ 0 & e^x & e^x(\cos x - \sin x) & e^x(\cos x + \sin x) \\ 0 & e^x & -2e^x\sin x & 2e^x\cos x \\ 0 & e^x & -e^x(2\cos x + 2\sin x) & e^x(2\cos x - 2\sin x) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}$$

Setting $x = 0$ and imposing the initial conditions yields

$$\begin{bmatrix} 2 \\ 0 \\ -1 \\ -5 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & -2 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix},$$

so $c_1 = 2$, $c_2 = -1$, $c_3 = 1$, $c_4 = -1$, and $2 + e^x[(1+x)\cos x - \sin x - 1]$.

9.4 VARIATION OF PARAMETERS FOR HIGHER ORDER EQUATIONS

9.4.2. $W = \begin{vmatrix} e^{-x^2} & xe^{-x^2} & x^2e^{-x^2} \\ -2xe^{-x^2} & e^{-x^2}(1-2x^2) & 2xe^{-x^2}(1-x^2) \\ e^{-x^2}(4x^2-2) & 2xe^{-x^2}(2x^2-3) & 2e^{-x^2}(2x^4-5x^2+1) \end{vmatrix} = 2e^{-3x^2}$; $W_1 = \begin{vmatrix} xe^{-x^2} & x^2e^{-x^2} \\ e^{-x^2}(1-2x^2) & 2xe^{-x^2}(1-x^2) \end{vmatrix} = x^2e^{-2x^2}$; $W_2 = \begin{vmatrix} e^{-x^2} & x^2e^{-x^2} \\ -2xe^{-x^2} & 2xe^{-x^2}(1-x^2) \end{vmatrix} = 2xe^{-2x^2}$; $W_3 = \begin{vmatrix} e^{-x^2} & xe^{-x^2} \\ -2xe^{-x^2} & e^{-x^2}(1-2x^2) \end{vmatrix} = e^{-2x^2}$; $u_1' = \dfrac{FW_1}{P_0W} = \dfrac{1}{2}x^{5/2}$; $u_2' = -\dfrac{FW_2}{P_0W} = -x^{3/2}$; $u_3' = \dfrac{FW_2}{P_0W} = \sqrt{x}/2$; $u_1 = x^{7/2}/7$; $u_2 = -\dfrac{2}{5}x^{5/2}$; $u_3 = x^{3/2}/3$; $y_p = u_1y_1 + u_2y_2 + u_3y_3 = \dfrac{8}{105}e^{-x^2}x^{7/2}$.

9.4.4. $W = \begin{vmatrix} 1 & \dfrac{e^x}{x} & \dfrac{e^{-x}}{x} \\ 0 & \dfrac{e^x(x-1)}{x^2} & -\dfrac{e^{-x}(x+1)}{x^2} \\ 0 & \dfrac{e^x(x^2-2x+2)}{x^3} & \dfrac{e^{-x}(x^2+2x+2)}{x^3} \end{vmatrix} = 2/x^2$; $W_1 = \begin{vmatrix} \dfrac{e^x}{x} & \dfrac{e^{-x}}{x} \\ \dfrac{e^x(x-1)}{x^2} & -\dfrac{e^{-x}(x+1)}{x^2} \end{vmatrix} = -\dfrac{2}{x^2}$; $W_2 = \begin{vmatrix} 1 & \dfrac{e^{-x}}{x} \\ 0 & -\dfrac{e^{-x}(x+1)}{x^2} \end{vmatrix} = -\dfrac{e^{-x}(x+1)}{x^2}$; $W_3 = \begin{vmatrix} 1 & \dfrac{e^x}{x} \\ 0 & \dfrac{e^x(x-1)}{x^2} \end{vmatrix} = \dfrac{e^x(x-1)}{x^2}$; $u_1' = \dfrac{FW_1}{P_0W} = -2$; $u_2' = -\dfrac{FW_2}{P_0W} = e^{-x}(x+1)$; $u_3' = \dfrac{FW_2}{P_0W} = e^x(x-1)$; $u_1 = -2x$; $u_2 = -e^{-x}(x+2)$; $u_3 = e^x(x-2)$; $y_p = u_1y_1 + u_2y_2 + u_3y_3 = -2(x^2+2)/x$.

9.4.6. $W = \begin{vmatrix} e^x & e^{-x} & \dfrac{1}{x} \\ e^x & -e^{-x} & -1/x^2 \\ e^x & e^{-x} & \dfrac{2}{x^3} \end{vmatrix} = \dfrac{2(x^2-2)}{x^3}$; $W_1 = \begin{vmatrix} e^{-x} & \dfrac{1}{x} \\ -e^{-x} & -\dfrac{1}{x^2} \end{vmatrix} = \dfrac{e^{-x}(x-1)}{x^2}$; $W_2 =$

$\begin{vmatrix} e^x & \dfrac{1}{x} \\ e^x & -\dfrac{1}{x^2} \end{vmatrix} = -\dfrac{e^x(x+1)}{x^2}$; $W_3 = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2$; $u_1' = \dfrac{FW_1}{P_0W} = e^{-x}(x-1)$; $u_2' = -\dfrac{FW_2}{P_0W} = e^x(x+1)$; $u_3' = \dfrac{FW_2}{P_0W} = -2x^2$; $y_p = u_1y_1 + u_2y_2 + u_3y_3 = -2\dfrac{x^2}{3}$.

9.4.8. $W = \begin{vmatrix} \sqrt{x} & \dfrac{1}{\sqrt{x}} & x^2 \\ \dfrac{1}{2\sqrt{x}} & -\dfrac{1}{2x^{3/2}} & 2x \\ -\dfrac{1}{4x^{3/2}} & \dfrac{3}{4x^{5/2}} & 2 \end{vmatrix} = -\dfrac{15}{4x}$; $W_1 = \begin{vmatrix} \dfrac{1}{\sqrt{x}} & x^2 \\ -\dfrac{1}{2x^{3/2}} & 2x \end{vmatrix} = \dfrac{5\sqrt{x}}{2}$; $W_2 = \begin{vmatrix} \sqrt{x} & x^2 \\ \dfrac{1}{2\sqrt{x}} & 2x \end{vmatrix} = \dfrac{3x^{3/2}}{2}$; $W_3 = \begin{vmatrix} \sqrt{x} & \dfrac{1}{\sqrt{x}} \\ \dfrac{1}{2\sqrt{x}} & -\dfrac{1}{2x^{3/2}} \end{vmatrix} = -\dfrac{1}{x}$; $u_1' = \dfrac{FW_1}{P_0W} = -5\sqrt{x}$; $u_2' = -\dfrac{FW_2}{P_0W} = 3x^{3/2}$; $u_3' = \dfrac{FW_2}{P_0W} = \dfrac{2}{x}$; $u_1 = -\dfrac{10}{3}x^{3/2}$; $u_2 = \dfrac{6}{5}x^{5/2}$; $u_3 = 2\ln|x|$; $y_p = u_1y_1 + u_2y_2 + u_3y_3 = 2x^2\ln|x| - \dfrac{32}{15}x^2$. Since $-\dfrac{32}{15}x^2$ satisfies the complementary equation we take $y_p = \ln|x|$.

9.4.10. $W = \begin{vmatrix} x & 1/x & \dfrac{e^x}{x} \\ 1 & -\dfrac{1}{x^2} & \dfrac{e^x(x-1)}{x^2} \\ 0 & \dfrac{2}{x^3} & \dfrac{e^x(x^2-2x+2)}{x^3} \end{vmatrix} = \dfrac{2e^x(1-x)}{x^3}$; $W_1 = \begin{vmatrix} \dfrac{1}{x} & \dfrac{e^x}{x} \\ -\dfrac{1}{x^2} & \dfrac{e^x(x-1)}{x^2} \end{vmatrix} = \dfrac{e^x}{x^2}$; $W_2 = \begin{vmatrix} x & \dfrac{e^x}{x} \\ 1 & \dfrac{e^x(x-1)}{x^2} \end{vmatrix} = \dfrac{e^x(x-2)}{x}$; $W_3 = \begin{vmatrix} x & \dfrac{1}{x} \\ 1 & -\dfrac{1}{x^2} \end{vmatrix} = -\dfrac{2}{x}$; $u_1' = \dfrac{FW_1}{P_0W} = 1$; $u_2' = -\dfrac{FW_2}{P_0W} = x(2-x)$; $u_3' = \dfrac{FW_2}{P_0W} = -2xe^{-x}$; $u_1 = x$; $u_2 = \dfrac{x^2(3-x)}{3}$; $u_3 = 2e^{-x}(x+1)$; $y_p = u_1y_1 + u_2y_2 + u_3y_3 = \dfrac{2x^3+3x^2+6x+6}{3x}$. Since $x + \dfrac{2}{x}$ satisfies the complementary equation we take $y_p = \dfrac{2x^2+6}{3}$.

9.4.12. $W = \begin{vmatrix} x & e^x & e^{-x} \\ 1 & e^x & -e^{-x} \\ 0 & e^x & e^{-x} \end{vmatrix} = 2x$; $W_1 = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2$; $W_2 = \begin{vmatrix} x & e^{-x} \\ 1 & -e^{-x} \end{vmatrix} = -e^{-x}(x+1)$; $W_3 = \begin{vmatrix} x & e^x \\ 1 & e^x \end{vmatrix} = e^x(x-1)$; $u_1' = \dfrac{FW_1}{P_0W} = -1$; $u_2' = -\dfrac{FW_2}{P_0W} = e^{-x}(x+1)/2$; $u_3' = \dfrac{FW_2}{P_0W} = e^x(x-1)/2$; $u_1 = -x$; $u_2 = -e^{-x}(x+2)/2$; $u_3 = e^x(x-2)/2$; $y_p = u_1y_1 + u_2y_2 + u_3y_3 = -x^2-2$.

9.4.14. $W = \begin{vmatrix} \sqrt{x} & 1/\sqrt{x} & x^{3/2} & \dfrac{1}{x^{3/2}} \\ \dfrac{1}{2\sqrt{x}} & -\dfrac{1}{2x^{3/2}} & \dfrac{3\sqrt{x}}{2} & -\dfrac{3}{2x^{5/2}} \\ -\dfrac{1}{4x^{3/2}} & \dfrac{3}{4x^{5/2}} & \dfrac{3}{4\sqrt{x}} & \dfrac{15}{4x^{7/2}} \\ \dfrac{3}{8x^{5/2}} & -\dfrac{15}{8x^{7/2}} & -\dfrac{3}{8x^{3/2}} & -\dfrac{105}{8x^{9/2}} \end{vmatrix} = \dfrac{12}{x^6}$; $W_1 = \begin{vmatrix} \dfrac{1}{\sqrt{x}} & x^{3/2} & \dfrac{1}{x^{3/2}} \\ -\dfrac{1}{2x^{3/2}} & \dfrac{3\sqrt{x}}{2} & -\dfrac{3}{2x^{5/2}} \\ \dfrac{3}{4x^{5/2}} & \dfrac{3}{4\sqrt{x}} & \dfrac{15}{4x^{7/2}} \end{vmatrix} =$ $\dfrac{6}{x^{7/2}}$; $W_2 = \begin{vmatrix} \sqrt{x} & x^{3/2} & \dfrac{1}{x^{3/2}} \\ \dfrac{1}{2\sqrt{x}} & \dfrac{3\sqrt{x}}{2} & -\dfrac{3}{2x^{5/2}} \\ -\dfrac{1}{4x^{3/2}} & \dfrac{3}{4\sqrt{x}} & \dfrac{15}{4x^{7/2}} \end{vmatrix} = \dfrac{6}{x^{5/2}}$; $W_3 = \begin{vmatrix} \sqrt{x} & \dfrac{1}{\sqrt{x}} & \dfrac{1}{x^{3/2}} \\ \dfrac{1}{2\sqrt{x}} & -\dfrac{1}{2x^{3/2}} & -\dfrac{3}{2x^{5/2}} \\ -\dfrac{1}{4x^{3/2}} & \dfrac{3}{4x^{5/2}} & \dfrac{15}{4x^{7/2}} \end{vmatrix} =$ $-\dfrac{2}{x^{9/2}}$; $W_4 = \begin{vmatrix} \sqrt{x} & \dfrac{1}{\sqrt{x}} & x^{3/2} \\ \dfrac{1}{2\sqrt{x}} & -\dfrac{1}{2x^{3/2}} & \dfrac{3\sqrt{x}}{2} \\ -\dfrac{1}{4x^{3/2}} & \dfrac{3}{4x^{5/2}} & \dfrac{3}{4\sqrt{x}} \end{vmatrix} = -\dfrac{2}{x^{3/2}}$; $u_1' = -\dfrac{FW_1}{P_0W} = -3x$; $u_2' = \dfrac{FW_2}{P_0W} =$ $3x^2$; $u_3' = -\dfrac{FW_2}{P_0W} = 1$; $u_4' = \dfrac{FW_4}{P_0W} = -x^3$; $u_1 = -\dfrac{3x^2}{2}$; $u_2 = x^3$; $u_3 = x$; $u_4 = -\dfrac{x^4}{4}$; $y_p = u_1y_1 + u_2y_2 + u_3y_3 = \dfrac{x^{5/2}}{4}$.

9.4.16. $W = \begin{vmatrix} x & x^2 & x^3 & x^4 \\ 1 & 2x & 3x^2 & 4x^3 \\ 0 & 2 & 6x & 12x^2 \\ 0 & 0 & 6 & 24x \end{vmatrix} = 12x^4$; $W_1 = \begin{vmatrix} x^2 & x^3 & x^4 \\ 2x & 3x^2 & 4x^3 \\ 2 & 6x & 12x^2 \end{vmatrix} = 2x^6$; $W_2 = \begin{vmatrix} x & x^3 & x^4 \\ 1 & 3x^2 & 4x^3 \\ 0 & 6x & 12x^2 \end{vmatrix} =$ $6x^5$; $W_3 = \begin{vmatrix} x & x^2 & x^4 \\ 1 & 2x & 4x^3 \\ 0 & 2 & 12x^2 \end{vmatrix} = 6x^4$; $W_4 = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} = 2x^3$; $u_1' = -\dfrac{FW_1}{P_0W} = -\dfrac{x^2}{6}$; $u_2' = \dfrac{FW_2}{P_0W} = \dfrac{x}{2}$; $u_3' = -\dfrac{FW_2}{P_0W} = -\dfrac{1}{2}$; $u_4' = \dfrac{FW_4}{P_0W} = \dfrac{1}{6x}$; $u_1 = -\dfrac{x^3}{18}$; $u_2 = \dfrac{x^2}{4}$; $u_3 = -\dfrac{x}{2}$; $u_4 = \dfrac{\ln|x|}{6}$; $y_p = u_1y_1 + u_2y_2 + u_3y_3 = \dfrac{x^4\ln|x|}{6} - \dfrac{11x^4}{36}$. Since $-\dfrac{11x^4}{36}$ satisfies the complementary equation we take $y_p = \dfrac{x^4\ln|x|}{6}$.

9.4.18. $W = \begin{vmatrix} x & x^2 & 1/x & 1/x^2 \\ 1 & 2x & -1/x^2 & -2/x^3 \\ 0 & 2 & 2/x^3 & 6/x^4 \\ 0 & 0 & -6/x^4 & -24/x^5 \end{vmatrix} = -72/x^6$; $W_1 = \begin{vmatrix} x^2 & 1/x & 1/x^2 \\ 2x & -1/x^2 & -2/x^3 \\ 2 & 2/x^3 & 6/x^4 \end{vmatrix} = -12/x^4$; $W_2 = \begin{vmatrix} x & 1/x & 1/x^2 \\ 1 & -1/x^2 & -2/x^3 \\ 0 & 2/x^3 & 6/x^4 \end{vmatrix} = -6/x^5$; $W_3 = \begin{vmatrix} x & x^2 & 1/x^2 \\ 1 & 2x & -2/x^3 \\ 0 & 2 & 6/x^4 \end{vmatrix} = 12/x^2$; $W_4 = \begin{vmatrix} x & x^2 & 1/x \\ 1 & 2x & -1/x^2 \\ 0 & 2 & 2/x^3 \end{vmatrix} =$ $6/x$; $u_1' = -\dfrac{FW_1}{P_0W} = -2$; $u_2' = \dfrac{FW_2}{P_0W} = 1/x$; $u_3' = -\dfrac{FW_2}{P_0W} = 2x^2$; $u_4' = \dfrac{FW_4}{P_0W} = -x^3$; $u_1 = -2x$; $u_2 = \ln|x|$; $u_3 = 2x^3/3$; $u_4 = -x^4/4$; $y_p = u_1y_1 + u_2y_2 + u_3y_3 = x^2\ln|x| - 19x^2/12$. Since $-19x^2/12$ satisfies the complementary equation we take $y_p = x^2\ln|x|$.

9.4.20. $W = \begin{vmatrix} e^x & e^{2x} & e^x/x & \dfrac{e^{2x}}{x} \\ e^x & 2e^{2x} & \dfrac{e^x(x-1)}{x^2} & \dfrac{e^{2x}(2x-1)}{x^2} \\ e^x & 4e^{2x} & \dfrac{e^x(x^2-2x+2)}{x^3} & \dfrac{2e^{2x}(2x^2-2x+1)}{x^3} \\ e^x & 8e^{2x} & \dfrac{e^x(x^3-3x^2+6x-6)}{x^4} & \dfrac{2e^{2x}(4x^3-6x^2+6x-3)}{x^4} \end{vmatrix} = -\dfrac{e^{6x}}{x^4}$;

$W_1 = \begin{vmatrix} e^{2x} & \dfrac{e^x}{x} & \dfrac{e^{2x}}{x} \\ 2e^{2x} & \dfrac{e^x(x-1)}{x^2} & \dfrac{e^{2x}(2x-1)}{x^2} \\ 4e^{2x} & \dfrac{e^x(x^2-2x+2)}{x^3} & \dfrac{2e^{2x}(2x^2-2x+1)}{x^3} \end{vmatrix} = \dfrac{e^{5x}}{x^3}$;

$W_2 = \begin{vmatrix} e^x & \dfrac{e^x}{x} & \dfrac{e^{2x}}{x} \\ e^x & \dfrac{e^x(x-1)}{x^2} & \dfrac{e^{2x}(2x-1)}{x^2} \\ e^x & \dfrac{e^x(x^2-2x+2)}{x^3} & \dfrac{2e^{2x}(2x^2-2x+1)}{x^3} \end{vmatrix} = -\dfrac{e^{4x}}{x^3}$;

$W_3 = \begin{vmatrix} e^x & e^{2x} & \dfrac{e^{2x}}{x} \\ e^x & 2e^{2x} & \dfrac{e^{2x}(2x-1)}{x^2} \\ e^x & 4e^{2x} & \dfrac{2e^{2x}(2x^2-2x+1)}{x^3} \end{vmatrix} = \dfrac{e^{5x}(2-x)}{x^3}$;

$W_4 = \begin{vmatrix} e^x & e^{2x} & \dfrac{e^x}{x} \\ e^x & 2e^{2x} & \dfrac{e^x(x-1)}{x^2} \\ e^x & 4e^{2x} & \dfrac{e^x(x^2-2x+2)}{x^3} \end{vmatrix} = \dfrac{e^{4x}(x+2)}{x^3}$;

$u_1' = -\dfrac{FW_1}{P_0W} = 3$; $u_2' = \dfrac{FW_2}{P_0W} = 3e^{-x}$; $u_3' = -\dfrac{FW_2}{P_0W} = 3(2-x)$; $u_4' = \dfrac{FW_4}{P_0W} = -3e^{-x}(x+2)$; $u_1 = 3x$; $u_2 = -3e^{-x}$; $u_3 = 3x(4-x)/2$; $u_4 = 3e^{-x}(x+3)$; $y_p = u_1y_1 + u_2y_2 + u_3y_3 = \dfrac{3e^x(x^2+4x+6)}{2x}$. Since $\dfrac{3e^x(2x+3)}{x}$ is a solution of the complementary equation we take $y_p = \dfrac{3xe^x}{2}$.

9.4.22. $W = \begin{vmatrix} x & x^3 & x\ln x \\ 1 & 3x^2 & 1+\ln x \\ 0 & 6x & 1/x \end{vmatrix} = -4x^2$; $W_1 = \begin{vmatrix} x^3 & x\ln x \\ 3x^2 & 1+\ln x \end{vmatrix} = x^3 - 2x^3\ln x$; $W_2 = \begin{vmatrix} x & x\ln x \\ 1 & 1+\ln x \end{vmatrix} = x$; $W_3 = \begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix} = 2x^3$; $u_1' = \dfrac{FW_1}{P_0W} = 2\ln x/x - \dfrac{1}{x}$; $u_2' = -\dfrac{FW_2}{P_0W} = \dfrac{1}{x^3}$; $u_3' = \dfrac{FW_2}{P_0W} = -\dfrac{2}{x}$; $u_1 = (\ln x)^2 - \ln x$; $u_2 = -\dfrac{1}{2x^2}$; $u_3 = -2\ln x$; $y_p = u_1y_1 + u_2y_2 + u_3y_3 = -x(\ln x)^2 - x\ln x - \dfrac{x}{2}$. Since $-x\ln x - \dfrac{x}{2}$ satisfies the complementary equation we take $y_p = -x(\ln x)^2$

The general solution is $y = -x(\ln x)^2 + c_1x + c_2x^3 + c_3x\ln x$, so

$$\begin{bmatrix} y \\ y' \\ y'' \end{bmatrix} = \begin{bmatrix} -x(\ln x)^2 \\ -(\ln x)^2 - 2\ln x \\ -\frac{2\ln x}{x} - \frac{2}{x} \end{bmatrix} + \begin{bmatrix} x & x^3 & x\ln x \\ 1 & 3x^2 & 1+\ln x \\ 0 & 6x & \frac{1}{x} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

Setting $x = 1$ and imposing the initial conditions yields

$$\begin{bmatrix} 4 \\ 4 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 6 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}.$$

Solving this system yields $c_1 = 3$, $c_2 = 1$, $c_3 = -2$. Therefore, $y = -x(\ln x)^2 + 3x + x^3 - 2x\ln x$.

9.4.24. $W = \begin{vmatrix} e^x & e^{2x} & xe^{-x} \\ e^x & 2e^{2x} & e^{-x}(1-x) \\ e^x & 4e^{2x} & e^{-x}(x-2) \end{vmatrix} = e^{2x}(6x-5)$; $W_1 = \begin{vmatrix} e^{2x} & xe^{-x} \\ 2e^{2x} & e^{-x}(1-x) \end{vmatrix} = e^x(1-3x)$; $W_2 = \begin{vmatrix} e^x & xe^{-x} \\ e^x & e^{-x}(1-x) \end{vmatrix} = 1-2x$; $W_3 = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = e^{3x}$; $u_1' = \frac{FW_1}{P_0W} = 1-3x$; $u_2' = -\frac{FW_2}{P_0W} = e^{-x}(2x-1)$; $u_3' = \frac{FW_2}{P_0W} = e^{2x}$; $u_1 = \frac{x(2-3x)}{2}$; $u_2 = -e^{-x}(2x+1)$; $u_3 = \frac{e^{2x}}{2}$; $y_p = u_1y_1 + u_2y_2 + u_3y_3 = -e^x(3x^2+x+2)/2$. Since $-\frac{e^x}{2}$ is a solution of the complementary equation we take $y_p = -\frac{e^x(3x+1)x}{2}$.

The general solution is $y = -\frac{e^x(3x+1)x}{2} + c_1e^x + c_2e^{2x} + c_3xe^{-x}$, so

$$\begin{bmatrix} y \\ y' \\ y'' \end{bmatrix} = \begin{bmatrix} -\frac{e^x(3x+1)x}{2} \\ -\frac{e^x(3x^2+7x+1)}{2} \\ -\frac{e^x(3x^2+13x+8)}{2} \end{bmatrix} + \begin{vmatrix} e^x & e^{2x} & xe^{-x} \\ e^x & 2e^{2x} & e^{-x}(1-x) \\ e^x & 4e^{2x} & e^{-x}(x-2) \end{vmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}.$$

Setting $x = 0$ and imposing the initial conditions yields

$$\begin{bmatrix} -4 \\ -\frac{3}{2} \\ -19 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{2} \\ -4 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 4 & -2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}.$$

Solving this system yields $c_1 = -3$, $c_2 = -1$, $c_3 = 4$. Therefore, $y = -\frac{e^x(3x+1)x}{2} - 3e^x - e^{2x} + 4xe^{-x}$.

9.4.26. $W = \begin{vmatrix} x & x^2 & e^x \\ 1 & 2x & e^x \\ 0 & 2 & e^x \end{vmatrix} = e^x(x^2-2x+2)$; $W_1 = \begin{vmatrix} x^2 & e^x \\ 2x & e^x \end{vmatrix} = e^x(x^2-2x)$; $W_2 = \begin{vmatrix} x & e^x \\ 1 & e^x \end{vmatrix} = e^x(x-1)$; $W_3 = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = x^2$; $u_1' = \frac{FW_1}{P_0W} = x(x-2)$; $u_2' = -\frac{FW_2}{P_0W} = 1-x$; $u_3' = \frac{FW_2}{P_0W} = x^2e^{-x}$; $y_p = u_1y_1 + u_2y_2 + u_3y_3 = -\frac{x^4+6x^2+12x+12}{6}$. Since $-\frac{6x^2+12x}{6}$ is a solution of the complementary equations we take $y_p = -\frac{x^4+12}{6}$.

The general solution is $y = -\dfrac{x^4+12}{6} + c_1x + c_2x^2 + c_3e^x$, so

$$\begin{bmatrix} y \\ y' \\ y'' \end{bmatrix} = -\begin{bmatrix} (x^4+12)/6 \\ 2x^3/3 \\ 2x^2 \end{bmatrix} + \begin{vmatrix} x & x^2 & e^x \\ 1 & 2x & e^x \\ 0 & 2 & e^x \end{vmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}.$$

Setting $x = 0$ and imposing the initial conditions yields

$$\begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix} = -\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}.$$

Solving this system yields $c_1 = 3$, $c_2 = -1$, $c_3 = 2$. Therefore, $y = -\dfrac{x^4+12}{6} + 3x - x^2 + 2e^x$.

9.4.28. $W = \begin{vmatrix} x+1 & e^x & e^{3x} \\ 1 & e^x & 3e^{3x} \\ 0 & e^x & 9e^{3x} \end{vmatrix} = 2e^{4x}(3x-1)$; $W_1 = \begin{vmatrix} e^x & e^{3x} \\ e^x & 3e^{3x} \end{vmatrix} = 2e^{4x}$; $W_2 = \begin{vmatrix} x+1 & e^{3x} \\ 1 & 3e^{3x} \end{vmatrix} = e^{3x}(3x+2)$; $W_3 = \begin{vmatrix} x+1 & e^x \\ 1 & e^x \end{vmatrix} = xe^x$; $u_1' = \dfrac{FW_1}{P_0W} = 2e^x$; $u_2' = -\dfrac{FW_2}{P_0W} = -3x-2$; $u_3' = \dfrac{FW_2}{P_0W} = xe^{-2x}$; $u_1 = 2e^x$; $u_2 = -\dfrac{x(3x+4)}{2}$; $u_3 = -\dfrac{e^{-2x}(2x+1)}{4}$; $y_p = u_1y_1 + u_2y_2 + u_3y_3 = -\dfrac{e^x(6x^2+2x-7)}{4}$. Since $\dfrac{7e^x}{4}$ is a solution of the complementary equation we take $y_p = -\dfrac{xe^x(3x+1)}{2}$

The general solution is $y = -\dfrac{xe^x(3x+1)}{2} + c_1(x+1) + c_2e^x + c_3e^{2x}$, so

$$\begin{bmatrix} y \\ y' \\ y'' \end{bmatrix} = \begin{bmatrix} -xe^x(3x+1)/2 \\ -e^x(3x^2+7x+1)/2 \\ -e^x(3x^2+13x+8)/2 \end{bmatrix} + \begin{vmatrix} x+1 & e^x & e^{3x} \\ 1 & e^x & 3e^{3x} \\ 0 & e^x & 9e^{3x} \end{vmatrix}.$$

Setting $x = 0$ and imposing the initial conditions yields

$$\begin{bmatrix} \frac{3}{4} \\ \frac{5}{4} \\ \frac{1}{4} \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{2} \\ -4 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 0 & 1 & 9 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}.$$

Solving this system yields $c_1 = \dfrac{1}{2}$, $c_2 = -\dfrac{1}{4}$, $c_3 = \dfrac{1}{2}$. Therefore, $y = -\dfrac{xe^x(3x+1)}{2} + \dfrac{x+1}{2} - \dfrac{e^x}{4} + \dfrac{e^{2x}}{2}$.

9.4.30. $W = \begin{vmatrix} x & x^2 & \frac{1}{x} & x\ln x \\ 1 & 2x & -\frac{1}{x^2} & \ln x + 1 \\ 0 & 2 & \frac{2}{x^3} & \frac{1}{x} \\ 0 & 0 & -\frac{6}{x^2} & -\frac{1}{x^2} \end{vmatrix} = -\dfrac{12}{x^3}$; $W_1 = \begin{vmatrix} x^2 & \frac{1}{x} & x\ln x \\ 2x & -\frac{1}{x^2} & \ln x + 1 \\ 2 & \frac{2}{x^3} & \frac{1}{x} \end{vmatrix} = 6\dfrac{\ln x}{x} - \dfrac{3}{x}$;

$W_2 = \begin{vmatrix} x & \frac{1}{x} & x\ln x \\ 1 & -\frac{1}{x^2} & \ln x + 1 \\ 0 & \frac{2}{x^3} & \frac{1}{x} \end{vmatrix} = -\frac{4}{x^2}$; $W_3 = \begin{vmatrix} x & x^2 & x\ln x \\ 1 & 2x & \ln x + 1 \\ 0 & 2 & \frac{1}{x} \end{vmatrix} = -x$; $W_4 = \begin{vmatrix} x & x^2 & \frac{1}{x} \\ 1 & 2x & -\frac{1}{x^2} \\ 0 & 2 & \frac{2}{x^3} \end{vmatrix} = \frac{6}{x}$; $u_1' = -\frac{FW_1}{P_0W} = 9\ln x/2 - \frac{9}{4}$; $u_2' = \frac{FW_2}{P_0W} = \frac{3}{x}$; $u_3' = -\frac{FW_2}{P_0W} = -\frac{3x^2}{4}$; $u_4' = \frac{FW_4}{P_0W} = -\frac{9}{2}$; $u_1 = \frac{9x\ln x}{2} - \frac{27x}{4}$; $u_2 = 3\ln x$; $u_3 = -\frac{x^3}{4}$; $u_4 = -\frac{9x}{2}$; $y_p = u_1y_1 + u_2y_2 + u_3y_3 = 3x^2\ln x - 7x^2$. Since $-7x^2$ satisfies the complementary equation we take $y_p = 3x^2\ln x$.

The general solution is $y = 3x^2\ln x + c_1x + c_2x^2 + \frac{c_3}{x} + c_4x\ln x$, so

$$\begin{bmatrix} y \\ y' \\ y'' \\ y''' \end{bmatrix} = \begin{bmatrix} 3x^2\ln x \\ 6x\ln x + 3x \\ 6\ln x + 9 \\ \frac{6}{x} \end{bmatrix} + \begin{vmatrix} x & x^2 & \frac{1}{x} & x\ln x \\ 1 & 2x & -\frac{1}{x^2} & \ln x + 1 \\ 0 & 2 & \frac{2}{x^3} & \frac{1}{x} \\ 0 & 0 & -\frac{6}{x^2} & -\frac{1}{x^2} \end{vmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}.$$

Setting $x = 1$ and imposing the initial conditions yields

$$\begin{bmatrix} -7 \\ -11 \\ -5 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 9 \\ 6 \end{bmatrix} + \begin{vmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & -1 & 1 \\ 0 & 2 & 2 & 1 \\ 0 & 0 & -6 & -1 \end{vmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}.$$

Solving this system yields $c_1 = 0$, $c_2 = -7$, $c_3 = 0$, $c_4 = 0$. Therefore, $y = 3x^2\ln x - 7x^2$.

9.4.32. $W = \begin{vmatrix} x & \sqrt{x} & 1/x & 1/\sqrt{x} \\ 1 & 1/2\sqrt{x} & -1/x^2 & -1/2x^{3/2} \\ 0 & -1/4x^{3/2} & 2/x^3 & 3/4x^{5/2} \\ 0 & 3/8x^{5/2} & -6/x^4 & -15/8x^{7/2} \end{vmatrix} = -9/8x^6$; $W_1 = \begin{vmatrix} \sqrt{x} & 1/x & 1/\sqrt{x} \\ 1/2\sqrt{x} & -1/x^2 & -1/2x^{3/2} \\ -1/4x^{3/2} & 2/x^3 & 3/4x^{5/2} \end{vmatrix} = 3/4x^4$; $W_2 = \begin{vmatrix} x & 1/x & 1/\sqrt{x} \\ 1 & -1/x^2 & -1/2x^{3/2} \\ 0 & 2/x^3 & 3/4x^{5/2} \end{vmatrix} = 3/2x^{7/2}$; $W_3 = \begin{vmatrix} x & \sqrt{x} & 1/\sqrt{x} \\ 1 & 1/2\sqrt{x} & -1/2x^{3/2} \\ 0 & -1/4x^{3/2} & 3/4x^{5/2} \end{vmatrix} = -3/4x^2$; $W_4 = \begin{vmatrix} x & \sqrt{x} & 1/x \\ 1 & 1/2\sqrt{x} & -1/x^2 \\ 0 & -1/4x^{3/2} & 2/x^3 \end{vmatrix} = -3/2x^{5/2}$; $u_1' = -\frac{FW_1}{P_0W} = 1/x$; $u_2' = \frac{FW_2}{P_0W} = -2/\sqrt{x}$; $u_3' = -\frac{FW_2}{P_0W} = -x$; $u_4' = \frac{FW_4}{P_0W} = 2\sqrt{x}$; $u_1 = \ln x$; $u_2 = -4\sqrt{x}$; $u_3 = -x^2/2$; $u_4 = 4x^{3/2}/3$; $y_p = u_1y_1 + u_2y_2 + u_3y_3 = x\ln x - 19x/6$. since $-19x/6$ satisfies the complementary equation we take $y_p = x\ln x$.

The general solution is $y = x\ln x + c_1x + c_2\sqrt{x} + c_3/x + c_4/\sqrt{x}$, so

$$\begin{bmatrix} y \\ y' \\ y'' \\ y''' \end{bmatrix} = \begin{bmatrix} x\ln x \\ \ln x + 1 \\ 1/x \\ -1/x^2 \end{bmatrix} + \begin{vmatrix} x & \sqrt{x} & 1/x & 1/\sqrt{x} \\ 1 & 1/2\sqrt{x} & -1/x^2 & -1/2x^{3/2} \\ 0 & -1/4x^{3/2} & 2/x^3 & 3/4x^{5/2} \\ 0 & 3/8x^{5/2} & -6/x^4 & -15/8x^{7/2} \end{vmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}.$$

Setting $x = 1$ and imposing the initial conditions yields

$$\begin{bmatrix} 2 \\ 0 \\ 4 \\ \frac{37}{4} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix} + \left| \begin{array}{cccc} 1 & 1 & 1 & 1 \\ 1 & 1/2 & -1 & -1/2 \\ 0 & -1/4 & 2 & 3/4 \\ 0 & 3/8 & -6 & -15/8 \end{array} \right| \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}.$$

Solving this system yields $c_1 = 1, c_2 = -1, c_3 = 1, c_4 = 1$. Therefore, $y = x\ln x + x - \sqrt{x} + \dfrac{1}{x} + \dfrac{1}{\sqrt{x}}$.

9.4.34. **(a)** Since $u_j' = (-1)^{n-j}\dfrac{FW_j}{P_0W}$ $(1 \le j \le n)$, the argument used in the derivation of the method of variation of parameters implies that y_p is a solution of (A).

(b) Follows immediately from **(a)**, since $u_j(x_0) = 0,\ j = 1, 2, \dots, n$.

(c) Expand the determinant in cofactors of its nth row.

(d) Just differentiate the determinant $n - 1$ times.

(e) If $0 \le j \le n-2$, then $\left.\dfrac{\partial^j G(x,t)}{\partial x^j}\right|_{x=t}$ has two identical rows, and is therefore zero, while $\left.\dfrac{\partial^{n-1} G(x,t)}{\partial x^j}\right|_{x=t} = W(t)$

(f) Since $y_p(x) = \displaystyle\int_{x_0}^x G(x,t)F(t)\,dt$, $y_p'(x) = G(x,x)F(x) + \displaystyle\int_{x_0}^x \frac{\partial G(x,t)}{\partial x}F(t)\,dt$. But $G(x,x) = 0$ from **(e)**, so $y_p'(x) = \displaystyle\int_{x_0}^x \frac{\partial G(x,t)}{\partial x}F(t)\,dt$. Repeating this argument for $j = 1, \dots, n$ and invoking **(e)** each time yields the conclusion.

9.4.36.

$$\begin{aligned}
\left| \begin{array}{ccc} y_1(t) & y_1(t) & y_2(t) \\ y_1'(t) & y_1'(t) & y_2'(t) \\ y_1(x) & y_1(x) & y_2(x) \end{array} \right| &= \left| \begin{array}{ccc} t & t^2 & 1/t \\ 1 & 2t & -1/t^2 \\ x & x^2 & 1/x \end{array} \right| \\
&= x \left| \begin{array}{cc} t^2 & 1/t \\ 2t & -1/t^2 \end{array} \right| - x^2 \left| \begin{array}{cc} t & 1/t \\ 1 & -1/t^2 \end{array} \right| + \frac{1}{x} \left| \begin{array}{cc} t & t^2 \\ 1 & 2t \end{array} \right| \\
&= -3x + 2\frac{x^2}{t} + \frac{t^2}{x} = \frac{(x-t)^2(2x+t)}{xt}.
\end{aligned}$$

Since $P_0(t) = t^3$ and $W(t) = \left| \begin{array}{ccc} t & t^2 & 1/t \\ 1 & 2t & -1/t^2 \\ x & x^2 & 1/x \end{array} \right| = \dfrac{6}{t}$, $G(x,t) = \dfrac{(x-t)^2(2x+t)}{6xt^3}$, so $y_p = \displaystyle\int_{x_0}^x \frac{(x-t)^2(2x+t)}{6xt^3}F(t)\,dt$.

9.4.38.

$$\begin{aligned}
\left| \begin{array}{ccc} y_1(t) & y_1(t) & y_2(t) \\ y_1'(t) & y_1'(t) & y_2'(t) \\ y_1(x) & y_1(x) & y_2(x) \end{array} \right| &= \left| \begin{array}{ccc} t & 1/t & e^t/t \\ 1 & -1/t^2 & e^t(1/t - 1/t^2) \\ x & 1/x & e^x/x \end{array} \right| \\
&= x \left| \begin{array}{cc} 1/t & e^t/t \\ -1/t^2 & e^t(1/t - 1/t^2) \end{array} \right| - \frac{1}{x} \left| \begin{array}{cc} t & e^t/t \\ 1 & e^t(1/t - 1/t^2) \end{array} \right| + \frac{e^x}{x} \left| \begin{array}{cc} t & 1/t \\ 1 & -1/t^2 \end{array} \right| \\
&= \frac{xe^t}{t^2} - \frac{e^t(t-2)}{xt} - \frac{2e^x}{xt} = \frac{x^2e^t - e^t t(t-2) - 2te^x}{xt^2}
\end{aligned}$$

Since $P_0(t) = t(1-t)$ and $W(t) = \begin{vmatrix} t & 1/t & e^t/t \\ 1 & -1/t^2 & e^t(1/t - 1/t^2) \\ 0 & 2/t^3 & e^t(1/t - 2/t^2 + 2/t^3) \end{vmatrix} = \dfrac{2e^t(1-t)}{t^3}$, $G(x,t) = \dfrac{x^2 - t(t-2) - 2te^{(x-t)}}{2x(t-1)^2}$, so $y_p = \displaystyle\int_{x_0}^x \frac{x^2 - t(t-2) - 2te^{(x-t)}}{2x(t-1)^2} F(t)\,dt$.

9.4.40.

$$\begin{aligned}
\begin{vmatrix} y_1(t) & y_1(t) & y_2(t) & y_3(t) \\ y_1'(t) & y_1'(t) & y_2'(t) & y_3'(t) \\ y_1''(t) & y_1''(t) & y_2''(t) & y_3''(t) \\ y_1(x) & y_1(x) & y_2(x) & y_3(x) \end{vmatrix} &= \begin{vmatrix} 1 & t & t^2 & 1/t \\ 0 & 1 & 2t & -1/t^2 \\ 0 & 0 & 2 & 2/t^3 \\ 1 & x & x^2 & 1/x \end{vmatrix} \\
&= -\begin{vmatrix} t & t^2 & 1/t \\ 1 & 2t & -1/t^2 \\ 0 & 2 & 2/t^3 \end{vmatrix} + x\begin{vmatrix} 1 & t^2 & 1/t \\ 0 & 2t & -1/t^2 \\ 0 & 2 & 2/t^3 \end{vmatrix} \\
&\quad -x^2\begin{vmatrix} 1 & t & 1/t \\ 0 & 1 & -1/t^2 \\ 0 & 0 & 2/t^3 \end{vmatrix} + \frac{1}{x}\begin{vmatrix} 1 & t & t^2 \\ 0 & 1 & 2t \\ 0 & 0 & 2 \end{vmatrix} \\
&= -\frac{6}{t} + \frac{6x}{t^2} - \frac{2x^2}{t^3} + \frac{2}{x} = \frac{2(t-x)^3}{xt^3}.
\end{aligned}$$

Since $P_0(t) = t$ and $W(t) = \begin{vmatrix} 1 & t & t^2 & 1/t \\ 0 & 1 & 2t & -1/t^2 \\ 0 & 0 & 2 & 2/t^3 \\ 0 & 0 & 0 & -6/t^4 \end{vmatrix} = -\dfrac{12}{t^4}$, $G(x,t) = \dfrac{(x-t)^3}{6x}$, so $y_p = \displaystyle\int_{x_0}^x \frac{(x-t)^3}{6x} F(t)\,dt$.

9.4.42.

$$\begin{aligned}
\begin{vmatrix} y_1(t) & y_1(t) & y_2(t) & y_3(t) \\ y_1'(t) & y_1'(t) & y_2'(t) & y_3'(t) \\ y_1''(t) & y_1''(t) & y_2''(t) & y_3''(t) \\ y_1(x) & y_1(x) & y_2(x) & y_3(x) \end{vmatrix} &= \begin{vmatrix} 1 & t^2 & e^{2t} & e^{-2t} \\ 0 & 2t & 2e^{2t} & -2e^{-2t} \\ 0 & 2 & 4e^{2t} & 4e^{-2t} \\ 1 & x^2 & e^{2x} & e^{-2x} \end{vmatrix} \\
&= -\begin{vmatrix} t^2 & e^{2t} & e^{-2t} \\ 2t & 2e^{2t} & -2e^{-2t} \\ 2 & 4e^{2t} & 4e^{-2t} \end{vmatrix} + x^2\begin{vmatrix} 1 & e^{2t} & e^{-2t} \\ 0 & 2e^{2t} & -2e^{-2t} \\ 0 & 4e^{2t} & 4e^{-2t} \end{vmatrix} \\
&\quad -e^{2x}\begin{vmatrix} 1 & t^2 & e^{-2t} \\ 0 & 2t & -2e^{-2t} \\ 0 & 2 & 4e^{-2t} \end{vmatrix} + e^{-2x}\begin{vmatrix} 1 & t^2 & e^{2t} \\ 0 & 2t & 2e^{2t} \\ 0 & 2 & 4e^{2t} \end{vmatrix} \\
&= -(16t^2 - 8) + 16x^2 - e^{2(x-t)t}(8t+4) + e^{-2(x-t)}(8t-4).
\end{aligned}$$

Since $P_0(t) = t$ and $W(t) = \begin{vmatrix} 1 & t^2 & e^{2t} & e^{-2t} \\ 0 & 2t & 2e^{2t} & -2e^{-2t} \\ 0 & 2 & 4e^{2t} & 4e^{-2t} \\ 0 & 0 & 8e^{2t} & -8e^{-2t} \end{vmatrix} = -128t$,

$G(x,t) = \dfrac{e^{2(x-t)}(1+2t) + e^{-2(x-t)}(1-2t) - 4x^2 + 4t^2 - 2}{32t^2}$, so

$y_p = \displaystyle\int_{x_0}^x \frac{e^{2(x-t)}(1+2t) + e^{-2(x-t)}(1-2t) - 4x^2 + 4t^2 - 2}{32t^2} F(t)\,dt$.

CHAPTER 10
Linear Systems of Differential Equations

10.1 INTRODUCTION TO SYSTEMS OF DIFFERENTIAL EQUATIONS

10.1.2. $Q_1' = (\text{rate in})_1 - (\text{rate out})_1$ and $Q_2' = (\text{rate in})_2 - (\text{rate out})_2$.

The volumes of the solutions in T_1 and T_2 are $V_1(t) = 100 + 2t$ and $V_2(t) = 100 + 3t$, respectively. T_1 receives salt from the external source at the rate of (2 lb/gal) $\times$ (6 gal/min) $=$ 12 lb/min, and from T_2 at the rate of (lb/gal in T_2) $\times$ (1 gal/min) $= \dfrac{1}{100+3t}Q_2$ lb/min. Therefore, (A) $(\text{rate in})_1 = 12 + \dfrac{1}{100+3t}Q_2$. Solution leaves T_1 at 5 gal/min, since 3 gal/min are drained and 2 gal/min are pumped to T_2; hence (B) $(\text{rate out})_1 =$ (lb/gal in T_1) $\times$ (5 gal/min) $= \dfrac{1}{100+2t}Q_1 \times 5 = \dfrac{5}{100+2t}Q_1$. Now (A) and (B) imply that (C) $Q_1' = 12 - \dfrac{5}{100+2t}Q_1 + \dfrac{1}{100+3t}Q_2$.

T_2 receives salt from the external source at the rate of (1 lb/gal) $\times$ (5 gal/min) $=$ 5 lb/min, and from T_1 at the rate of (lb/gal in T_1) $\times$ (2 gal/min) $= \dfrac{1}{100+2t}Q_1 \times 2 = \dfrac{1}{50+t}Q_1$ lb/min. Therefore, (D) $(\text{rate in})_2 = 5 + \dfrac{1}{50+t}Q_1$. Solution leaves T_2 at 4 gal/min, since 3 gal/min are drained and 1 gal/min is pumped to T_1; hence (E) $(\text{rate out})_2 =$ (lb/gal in T_2) $\times$ (4 gal/min) $= \dfrac{1}{100+3t}Q_2 \times 4 = \dfrac{4}{100+3t}Q_2$. Now (D) and (E) imply that (F) $Q_2' = 5 + \dfrac{1}{50+t}Q_1 - \dfrac{4}{100+3t}Q_2$. Now (C) and (F) form the desired system.

10.1.4. $m\mathbf{X}'' = -\alpha\mathbf{X}' - mgR^2\dfrac{\mathbf{X}}{\|\mathbf{X}\|^3}$; see Example 10.1.3.

$$\begin{aligned}
I_{1i} &= g_1(t_i, y_{1i}, y_{2i}),\\
J_{1i} &= g_2(t_i, y_{1i}, y_{2i}),\\
I_{2i} &= g_1\left(t_i + h, y_{1i} + hI_{1i}, y_{2i} + hJ_{1i}\right),\\
J_{2i} &= g_2\left(t_i + h, y_{1i} + hI_{1i}, y_{2i} + hJ_{1i}\right),\\
y_{1,i+1} &= y_{1i} + \frac{h}{2}(I_{1i} + I_{2i}),\\
y_{2,i+1} &= y_{2i} + \frac{h}{2}(J_{1i} + J_{2i}).
\end{aligned}$$

10.2 LINEAR SYSTEMS OF DIFFERENTIAL EQUATIONS

10.2.6. Let $y_i = y^{(i-1)}$, $i = 1, 2, \dots, n$; then $y_i' = y_{i+1}$, $i = 1, 2, \dots, n-1$ and $P_0(t)y_n' + P_1(t)y_n + \cdots + P_n(t)y_1 = F(t)$, so

$$A = -\frac{1}{P_0}\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ P_n & P_{n-1} & P_{n-2} & \cdots & P_1 \end{bmatrix} \quad \text{and} \quad \mathbf{f} = \frac{1}{P_0}\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ F \end{bmatrix}.$$

If $P_0, P_1, \dots, P_n$ and F are continuous and P_0 has no zeros on (a, b), then $P_1/P_0, \dots, P_n/P_0$ and F/P_0 are contnuous on (a, b).

10.2.7. **(a)** $(c_1P + c_2Q)_{ij}' = (c_1p_{ij} + c_2q_{ij})' = c_1p_{ij}' + c_2q_{ij}' = (c_1P' + c_2Q')_{ij}$; hence $(c_1P + c_2Q)' = c_1P' + c_2Q'$.

(b) Let P be $k \times r$ and Q be $r \times s$; then PQ is $k \times s$ and $(PQ)_{ij} = \sum_{\ell=1}^{r} p_{i\ell}q_{\ell j}$. Therefore, $(PQ)_{ij}' = \sum_{\ell=1}^{r} p_{i\ell}'q_{\ell j} + \sum_{\ell=1}^{r} p_{i\ell}q_{\ell j}' = (P'Q)_{ij} + (PQ')_{ij}$. Therefore, $(PQ)' = P'Q + PQ'$.

10.2.10. **(a)** From Exercise 10.2.7**(b)** with $P = Q = X$, $(X^2)' = (XX)' = X'X + XX'$.

(b) By starting from Exercise 10.2.7**(b)** and using induction it can be shown if $P_1, P_2, \dots, P_n$ are square matrices of the same order, then $(P_1P_2\cdots P_n)' = P_1'P_2\cdots P_n + P_1P_2'\cdots P_n + \cdots + P_1P_2\cdots P_n'$. Taking $P_1 = P_2 = \cdots = P_n = X$ yields (A) $(Y^n)' = Y'Y^{n-1} + YY'Y^{n-2} + Y^2Y'Y^{n-3} + \cdots + Y^{n-1}Y' = \sum_{r=0}^{n-1} Y^rY'Y^{n-r-1}$.

(c) If Y is a scalar function, then (A) reduces to the familiar result $(Y^n)' = nY^{n-1}Y'$.

10.2.12. From Exercise 10.2.6, the initial value problem (A) $P_0(x)y^{(n)} + P_1(x)y^{(n-1)} + \cdots + P_n(x)y = F(x)$, $y(x_0) = k_0$, $y'(x_0) = k_1, \dots, y^{(n-1)}(x_0) = k_{n-1}$ is equivalent to the initial value problem (B) $\mathbf{y}' = A(t)\mathbf{y} + \mathbf{f}(t)$, with

$$A = -\frac{1}{P_0}\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ P_n & P_{n-1} & P_{n-2} & \cdots & P_1 \end{bmatrix}, \mathbf{f} = \frac{1}{P_0}\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ F \end{bmatrix}, \text{ and } \mathbf{k} = \begin{bmatrix} k_0 \\ k_1 \\ \vdots \\ k_{n-1} \end{bmatrix}.$$

Since Theorem 10.2.1 implies that (B) has a unique solution on (a, b), it follows that (A) does also.

10.3 BASIC THEORY OF HOMOGENEOUS LINEAR SYSTEM

10.3.2. **(a)** The system equivalent of (A) is (B) $\mathbf{y}' = -\frac{1}{P_0(x)}\begin{bmatrix} 0 & 1 \\ P_2(x) & P_1(x) \end{bmatrix}\mathbf{y}$, where $\mathbf{y} = \begin{bmatrix} y \\ y' \end{bmatrix}$. Let $\mathbf{y}_1 = \begin{bmatrix} y_1 \\ y_1' \end{bmatrix}$ and $\mathbf{y}_1 = \begin{bmatrix} y_2 \\ y_2' \end{bmatrix}$. Then the Wronskian of $\{\mathbf{y}_1, \mathbf{y}_2\}$ as defined in this section is $\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = W$.

(b) The trace of the matrix in (B) is $-P_1(x)/P_0(x)$, so Eqn. 10.3.6 implies that $W(x) = W(x_0)\exp\left\{-\int_{x_0}^{x}\frac{P_1(s)}{P_0(s)}\,ds\right\}$.

10.3.4. **(a)** See the solution of Exercise 9.1.18.

(c) $\begin{vmatrix} y_{11}' & y_{12}' \\ y_{21} & y_{22}\end{vmatrix} = \begin{vmatrix} a_{11}y_{11}+a_{12}y_{21} & a_{11}y_{12}+a_{12}y_{22} \\ y_{21} & y_{22}\end{vmatrix} = a_{11}\begin{vmatrix} y_{11} & y_{12} \\ y_{21} & y_{22}\end{vmatrix} + a_{12}\begin{vmatrix} y_{21} & y_{22} \\ y_{21} & y_{22}\end{vmatrix} =$ $a_{11}W + a_{12}0 = a_{11}W$. Similarly, $\begin{bmatrix} y_{11} & y_{12} \\ y_{21}' & y_{22}'\end{bmatrix} = a_{22}W$.

10.3.6. **(a)** From the equivalence of Theorem 10.3.3**(b)**and **(e)**, $Y(t_0)$ is invertible.

(b) From the equivalence of Theorem 10.3.3**(a)**and **(b)**, the solution of the initial value problem is $\mathbf{y} = Y(t)\mathbf{c}$, where $\mathbf{c}$ is a constant vector. To satisfy $\mathbf{y}(t_0) = \mathbf{k}$, we must have $Y(t_0)\mathbf{c} = \mathbf{k}$, so $\mathbf{c} = Y^{-1}(t_0)\mathbf{k}$ and $\mathbf{y} = Y^{-1}(t_0)Y(t)\mathbf{k}$.

10.3.8. **(b)** $\mathbf{y} = \begin{bmatrix} e^{-4t} \\ e^{-4t}\end{bmatrix} + c_2\begin{bmatrix} -2e^{3t} \\ 5e^{3t}\end{bmatrix}$ where $\begin{array}{rcl} c_1 - 2c_2 & = & 10 \\ c_1 + 5c_2 & = & -4\end{array}$, so $c_1 = 6$, $c_2 = -2$, and $\mathbf{y} = \begin{bmatrix} 6e^{-4t} + 4e^{3t} \\ 6e^{-4t} - 10e^{3t}\end{bmatrix}$.

(c) $Y(t) = \begin{bmatrix} e^{-4t} & -2e^{3t} \\ e^{-4t} & 5e^{3t}\end{bmatrix}$; $Y(0) = \begin{bmatrix} 1 & -2 \\ 1 & 5\end{bmatrix}$; $Y^{-1}(0) = \frac{1}{7}\begin{bmatrix} 5 & 2 \\ -1 & 1\end{bmatrix}$; $\mathbf{y} = Y(t)Y^{-1}(0)\mathbf{k} =$ $\frac{1}{7}\begin{bmatrix} 5e^{-4t} + 2e^{3t} & 2e^{-4t} - 2e^{3t} \\ 5e^{-4t} - 5e^{3t} & 2e^{-4t} + 5e^{3t}\end{bmatrix}\mathbf{k}$.

10.3.10. **(b)** $\mathbf{y}_1 = c_1\begin{bmatrix} e^{3t} \\ e^{3t}\end{bmatrix} + c_2\begin{bmatrix} e^{t} \\ -e^{t}\end{bmatrix}$, where $\begin{array}{rcl} c_1 + c_2 & = & 2 \\ c_1 - c_2 & = & 8\end{array}$, so $c_1 = 5$, $c_2 = -3$, and $\mathbf{y} = \begin{bmatrix} 5e^{3t} - 3e^{t} \\ 5e^{3t} + 3e^{t}\end{bmatrix}$.

(c) $Y(t) = \begin{bmatrix} e^{3t} & e^{t} \\ 3e^{3t} & -e^{t}\end{bmatrix}$; $Y(0) = \begin{bmatrix} 1 & 1 \\ 1 & -1\end{bmatrix}$;

$Y^{-1}(0) = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1\end{bmatrix}$; $\mathbf{y} = Y(t)Y^{-1}(0)\mathbf{k} = \frac{1}{2}\begin{bmatrix} e^{3t} + e^{t} & e^{3t} - e^{t} \\ e^{3t} - e^{t} & e^{3t} + e^{t}\end{bmatrix}\mathbf{k}$.

10.3.12. **(b)** $\mathbf{y} = c_1\begin{bmatrix} -e^{-2t} \\ 0 \\ e^{-2t}\end{bmatrix} + c_2\begin{bmatrix} -e^{-2t} \\ e^{-2t} \\ 0\end{bmatrix} + c_3\begin{bmatrix} e^{4t} \\ e^{4t} \\ e^{4t}\end{bmatrix}$, where $\begin{array}{rcl} -c_1 - c_2 + c_3 & = & 0 \\ c_2 + c_3 & = & -9 \\ c_1 + c_3 & = & 12\end{array}$, so $c_1 = 11$, $c_2 = -10$, $c_3 = 1$, and $\mathbf{y} = \frac{1}{3}\begin{bmatrix} -e^{-2t} + e^{4t} \\ -10e^{-2t} + e^{4t} \\ 11e^{-2t} + e^{4t}\end{bmatrix}$.

(c) $Y(t) = \begin{bmatrix} -e^{-2t} & -e^{-2t} & e^{4t} \\ 0 & e^{-2t} & e^{4t} \\ e^{-2t} & 0 & e^{4t}\end{bmatrix}$; $Y(0) = \begin{bmatrix} -1 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1\end{bmatrix}$; $Y^{-1}(0) = \frac{1}{3}\begin{bmatrix} -1 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & 1 & 1\end{bmatrix}$;

$\mathbf{y} = Y(t)Y^{-1}(0)\mathbf{k} = \frac{1}{3}\begin{bmatrix} 2e^{-2t} + e^{4t} & -e^{-2t} + e^{4t} & -e^{-2t} + e^{4t} \\ -e^{-2t} + e^{4t} & 2e^{-2t} + e^{4t} & -e^{-2t} + e^{4t} \\ -e^{-2t} + e^{4t} & -e^{-2t} + e^{4t} & 2e^{-2t} + e^{4t}\end{bmatrix}\mathbf{k}$.

10.3.14. If Y and Z are both fundamental matrices for $\mathbf{y}' = A(t)\mathbf{y}$, then $Z = CY$, where C is a constant invertible matrix. Therefore, $ZY^{-1} = C$ and $YZ^{-1} = C^{-1}$.

10.3.16. **(a)** The Wronskian of $\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n\}$ equals one when $t = t_0$. Apply Theorem 10.3.3.

(b) Let Y be the matrix with columns $\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n\}$. From **(a)**, Y is a fundamental matrix for $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b). From Exercise 10.3.15**(b)**, so is $Z = YC$ if C is any invertible constant matrix.

10.3.18. **(a)** $\Gamma_1'(t) = Z'(t)Z(s) = AZ(t)Z(s) = A\Gamma(t)$ and $\Gamma_1(0) = Z(s)$, since $Z(0) = I$. $\Gamma_2'(t) = Z'(t+s) = AZ(t+s) = A\Gamma_2(t)$ (since A is constant) and $\Gamma_2(0) = Z(s)$. Applying Theorem 10.2.1 to the columns of Γ_1 and Γ_2 shows that $\Gamma_1 = \Gamma_2$.

(b) With $s = -t$, **(a)** implies that $Z(t)Z(-t) = Z(0) = I$; therefore $(Z(t))^{-1} = Z(-t)$.

(c) $e^{0\cdot A} = I$ is analogous to $e^{o\cdot a} = e^0 = 1$ when a is a scalar, while $e^{(t+s)A} = e^{tA}e^{sA}$ is analagous to $e^{(t+s)a} = e^{ta}e^{sa}$ when a is a scalar.

10.4 CONSTANT COEFFICIENT HOMOGENEOUS SYSTEMS I

10.4.2. $\dfrac{1}{4}\begin{vmatrix} -5-4\lambda & 3 \\ 3 & -5-4\lambda \end{vmatrix} = (\lambda+1/2)(\lambda+2)$. Eigenvectors associated with $\lambda_1 = -1/2$ satisfy $\begin{bmatrix} -3 & 3 \\ -4 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, so $x_1 = x_2$. Taking $x_2 = 1$ yields $\mathbf{y}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{-t/2}$. Eigenvectors associated with $\lambda_2 = -2$ satisfy $\begin{bmatrix} \frac{3}{4} & \frac{3}{4} \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, so $x_1 = -x_2$. Taking $x_2 = 1$ yields $\mathbf{y}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}e^{-2t}$. Hence $\mathbf{y} = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{-t/2} + c_2\begin{bmatrix} -1 \\ 1 \end{bmatrix}e^{-2t}$.

10.4.4. $\begin{vmatrix} -1-\lambda & -4 \\ -1 & -1-\lambda \end{vmatrix} = (\lambda-1)(\lambda+3)$. Eigenvectors associated with $\lambda_1 = -3$ satisfy $\begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, so $x_1 = 2x_2$. Taking $x_2 = 1$ yields $\mathbf{y}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}e^{-3t}$. Eigenvectors associated with $\lambda_2 = 1$ satisfy $\begin{bmatrix} -2 & -4 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, so $x_1 = -2x_2$. Taking $x_2 = 1$ yields $\mathbf{y}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}e^{t}$. Hence $\mathbf{y} = c_1\begin{bmatrix} 2 \\ 1 \end{bmatrix}e^{-3t} + c_2\begin{bmatrix} -2 \\ 1 \end{bmatrix}e^{t}$.

10.4.6. $\begin{vmatrix} 4-\lambda & -3 \\ 2 & -1-\lambda \end{vmatrix} = (\lambda-2)(\lambda-1)$. Eigenvectors associated with $\lambda_1 = 2$ satisfy $\begin{bmatrix} 2 & -3 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, so $x_1 = \dfrac{3}{2}x_2$. Taking $x_2 = 2$ yields $\mathbf{y}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}e^{2t}$. Eigenvectors associated with $\lambda_2 = 1$ satisfy $\begin{bmatrix} 3 & -3 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, so $x_1 = x_2$. Taking $x_2 = 1$ yields $\mathbf{y}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{t}$. Hence $\mathbf{y} = c_1\begin{bmatrix} 3 \\ 2 \end{bmatrix}e^{2t} + c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{t}$.

10.4.8. $\begin{vmatrix} 1-\lambda & -1 & -2 \\ 1 & -2-\lambda & -3 \\ -4 & 1 & -1-\lambda \end{vmatrix} = -(\lambda+3)(\lambda+1)(\lambda-2)$. The eigenvectors associated with with $\lambda_1 = -3$ satisfy the system with augmented matrix $\left[\begin{array}{rrrcr} 4 & -1 & -2 & \vdots & 0 \\ 1 & 1 & -3 & \vdots & 0 \\ -4 & 1 & 2 & \vdots & 0 \end{array}\right]$, which is row equivalent to $\left[\begin{array}{rrrcr} 1 & 0 & -1 & \vdots & 0 \\ 0 & 1 & -2 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{array}\right]$. Hence $x_1 = x_3$ and $x_2 = 2x_3$. Taking $x_3 = 1$ yields $\mathbf{y}_1 =$

$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} e^{-3t}$. The eigenvectors associated with with $\lambda_2 = -1$ satisfy the system with augmented matrix $\begin{bmatrix} 2 & -1 & -2 & \vdots & 0 \\ 1 & -1 & -3 & \vdots & 0 \\ -4 & 1 & 0 & \vdots & 0 \end{bmatrix}$, which is row equivalent to $\begin{bmatrix} 1 & 0 & 1 & \vdots & 0 \\ 0 & 1 & 4 & 0\vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$. Hence $x_1 = -x_3$ and $x_2 = -4x_3$. Taking $x_3 = 1$ yields $\mathbf{y}_2 = \begin{bmatrix} -1 \\ -4 \\ 1 \end{bmatrix} e^{-t}$. The eigenvectors associated with with $\lambda_3 = 2$ satisfy the system with augmented matrix $\begin{bmatrix} -1 & -1 & -2 & \vdots & 0 \\ 1 & -4 & -3 & \vdots & 0 \\ -4 & 1 & -3 & \vdots & 0 \end{bmatrix}$, which is row equivalent to $\begin{bmatrix} 1 & 0 & 1 & \vdots & 0 \\ 0 & 1 & 1 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$. Hence $x_1 = -x_3$ and $x_2 = -x_3$. Taking $x_3 = 1$ yields $\mathbf{y}_3 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} e^{2t}$.

Hence $\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} -1 \\ -4 \\ 1 \end{bmatrix} e^{-t} + c_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} e^{2t}$.

10.4.10. $\begin{vmatrix} 3-\lambda & 5 & 8 \\ 1 & -1-\lambda & -2 \\ -1 & -1 & -1-\lambda \end{vmatrix} = -(\lambda - 1)(\lambda + 2)(\lambda - 2)$. The eigenvectors associated with with $\lambda_1 = 1$ satisfy the system with augmented matrix $\begin{bmatrix} 2 & 5 & 8 & \vdots & 0 \\ 1 & -2 & -2 & \vdots & 0 \\ -1 & -1 & -2 & \vdots & 0 \end{bmatrix}$, which is row equivalent to $\begin{bmatrix} 1 & 0 & \frac{2}{3} & \vdots & 0 \\ 0 & 1 & \frac{4}{3} & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$. Hence $x_1 = -\frac{2}{3}x_3$ and $x_2 = -\frac{4}{3}x_3$. Taking $x_3 = 3$ yields $\mathbf{y}_1 = \begin{bmatrix} -2 \\ -4 \\ 3 \end{bmatrix} e^{t}$. The eigenvectors associated with with $\lambda_2 = -2$ satisfy the system with augmented matrix $\begin{bmatrix} 5 & 5 & 8 & \vdots & 0 \\ 1 & 1 & -2 & \vdots & 0 \\ -1 & -1 & 1 & \vdots & 0 \end{bmatrix}$, which is row equivalent to $\begin{bmatrix} 1 & 1 & 0 & \vdots & 0 \\ 0 & 0 & 1 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$. Hence $x_1 = -x_2$ and $x_3 = 0$. Taking $x_2 = 1$ yields $\mathbf{y}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{-2t}$. The eigenvectors associated with with

$\lambda_3 = 2$ satisfy the system with augmented matrix $\left[\begin{array}{rrrcr} 1 & 5 & 8 & \vdots & 0 \\ 1 & -3 & -2 & \vdots & 0 \\ -1 & -1 & -3 & \vdots & 0 \end{array}\right]$, which is row equivalent to $\left[\begin{array}{rrrcr} 1 & 0 & \frac{7}{4} & \vdots & 0 \\ 0 & 1 & \frac{5}{4} & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{array}\right]$. Hence $x_1 = -\frac{7}{4}x_3$ and $x_2 = -\frac{5}{4}x_3$. Taking $x_3 = 4$ yields $\mathbf{y}_3 = \left[\begin{array}{r} -7 \\ -5 \\ 4 \end{array}\right] e^{2t}$.

Hence $\mathbf{y} = c_1 \left[\begin{array}{r} -2 \\ -4 \\ 3 \end{array}\right] e^{t} + c_2 \left[\begin{array}{r} -1 \\ 1 \\ 0 \end{array}\right] e^{-2t} + c_3 \left[\begin{array}{r} -7 \\ -5 \\ 4 \end{array}\right] e^{2t}$.

10.4.12. $\left|\begin{array}{ccc} 4-\lambda & -1 & -4 \\ 4 & -3-\lambda & -2 \\ 1 & -1 & -1-\lambda \end{array}\right| = -(\lambda-3)(\lambda+2)(\lambda+1)$. The eigenvectors associated with $\lambda_1 = 3$ satisfy the system with augmented matrix $\left[\begin{array}{rrrcr} 1 & -1 & -4 & \vdots & 0 \\ 4 & -6 & -2 & \vdots & 0 \\ 1 & -1 & -4 & \vdots & 0 \end{array}\right]$, which is row equivalent to $\left[\begin{array}{rrrcr} 1 & 0 & -11 & \vdots & 0 \\ 0 & 1 & -7 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{array}\right]$. Hence $x_1 = 11x_3$ and $x_2 = 7x_3$. Taking $x_3 = 1$ yields $\mathbf{y}_1 = \left[\begin{array}{r} 11 \\ 7 \\ 1 \end{array}\right] e^{3t}$. The eigenvectors associated with $\lambda_2 = -2$ satisfy the system with augmented matrix $\left[\begin{array}{rrrcr} 6 & -1 & -4 & \vdots & 0 \\ 4 & -1 & -2 & \vdots & 0 \\ 1 & -1 & 1 & \vdots & 0 \end{array}\right]$, which is row equivalent to $\left[\begin{array}{rrrcr} 1 & 0 & -1 & \vdots & 0 \\ 0 & 1 & -2 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{array}\right]$. Hence $x_1 = x_3$ and $x_2 = 2x_3$. Taking $x_3 = 1$ yields $\mathbf{y}_2 = \left[\begin{array}{r} 1 \\ 2 \\ 1 \end{array}\right] e^{-2t}$. The eigenvectors associated with $\lambda_3 = -1$ satisfy the system with augmented matrix $\left[\begin{array}{rrrcr} 5 & -1 & -4 & \vdots & 0 \\ 4 & -2 & -2 & \vdots & 0 \\ 1 & -1 & 0 & \vdots & 0 \end{array}\right]$, which is row equivalent to $\left[\begin{array}{rrrcr} 1 & 0 & -1 & \vdots & 0 \\ 0 & 1 & -1 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{array}\right]$.

Hence $x_1 = x_3$ and $x_2 = x_3$. Taking $x_3 = 1$ yields $\mathbf{y}_3 = \left[\begin{array}{r} 1 \\ 1 \\ 1 \end{array}\right] e^{-t}$. Hence $\mathbf{y} = c_1 \left[\begin{array}{r} 11 \\ 7 \\ 1 \end{array}\right] e^{3t} + c_2 \left[\begin{array}{r} 1 \\ 2 \\ 1 \end{array}\right] e^{-2t} + c_3 \left[\begin{array}{r} 1 \\ 1 \\ 1 \end{array}\right] e^{-t}$

10.4.14. $\left|\begin{array}{ccc} 3-\lambda & 2 & -2 \\ -2 & 7-\lambda & -2 \\ -10 & 10 & -5-\lambda \end{array}\right| = -(\lambda+5)(\lambda-5)^2$. The eigenvectors associated with $\lambda_1 =$

-5 satisfy the system with augmented matrix $\left[\begin{array}{rrrcr} 8 & 2 & -2 & \vdots & 0 \\ -2 & 12 & -2 & \vdots & 0 \\ -10 & 10 & 0 & \vdots & 0 \end{array}\right]$, which is row equivalent to $\left[\begin{array}{rrrcr} 1 & 0 & -\frac{1}{5} & \vdots & 0 \\ 0 & 1 & -\frac{1}{5} & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{array}\right]$. Hence $x_1 = \dfrac{1}{5}x_3$ and $x_2 = \dfrac{1}{5}x_3$. Taking $x_3 = 5$ yields $\mathbf{y}_1 = \left[\begin{array}{r} 1 \\ 1 \\ 5 \end{array}\right] e^{-5t}$. The eigenvectors associated with $\lambda_2 = 5$ satisfy the system with augmented matrix $\left[\begin{array}{rrrcr} -2 & 2 & -2 & \vdots & 0 \\ -2 & 2 & -2 & \vdots & 0 \\ -10 & 10 & -10 & \vdots & 0 \end{array}\right]$, which is row equivalent to $\left[\begin{array}{rrrcr} 1 & -1 & 1 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{array}\right]$. Hence $x_1 = x_2 - x_3$. Taking $x_2 = 0$ and $x_3 = 1$ yields $\mathbf{y}_2 = \left[\begin{array}{r} -1 \\ 0 \\ 1 \end{array}\right] e^{5t}$. Taking $x_2 = 1$ and $x_3 = 0$ yields $\mathbf{y}_3 = \left[\begin{array}{r} 1 \\ 1 \\ 0 \end{array}\right] e^{5t}$. Hence $\mathbf{y} = c_1 \left[\begin{array}{r} 1 \\ 1 \\ 5 \end{array}\right] e^{-5t} + c_2 \left[\begin{array}{r} -1 \\ 0 \\ 1 \end{array}\right] e^{5t} + c_3 \left[\begin{array}{r} 1 \\ 1 \\ 0 \end{array}\right] e^{5t}$.

10.4.16. $\left|\begin{array}{cc} -7-\lambda & 4 \\ -6 & 7-\lambda \end{array}\right| = (\lambda-5)(\lambda+5)$. Eigenvectors associated with $\lambda_1 = 5$ satisfy $\left[\begin{array}{rr} -12 & 4 \\ -6 & 2 \end{array}\right] \left[\begin{array}{c} x_1 \\ x_2 \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \end{array}\right]$, so $x_1 = \dfrac{x_2}{3}$. Taking $x_2 = 3$ yields $\mathbf{y}_1 = \left[\begin{array}{r} 1 \\ 3 \end{array}\right] e^{5t}$. Eigenvectors associated with $\lambda_2 = 5$ satisfy $\left[\begin{array}{rr} -2 & 4 \\ -6 & 12 \end{array}\right] \left[\begin{array}{c} x_1 \\ x_2 \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \end{array}\right]$, so $x_1 = 2x_2$. Taking $x_2 = 1$ yields $\mathbf{y}_2 = \left[\begin{array}{r} 2 \\ 1 \end{array}\right] e^{-5t}$. The general solution is $\mathbf{y} = c_1 \left[\begin{array}{r} 1 \\ 3 \end{array}\right] e^{5t} + c_2 \left[\begin{array}{r} 2 \\ 1 \end{array}\right] e^{-5t}$. Now $\mathbf{y}(0) = \left[\begin{array}{r} 2 \\ -4 \end{array}\right] \Rightarrow c_1 \left[\begin{array}{r} 1 \\ 3 \end{array}\right] + c_2 \left[\begin{array}{r} 2 \\ 1 \end{array}\right] = \left[\begin{array}{r} 2 \\ -4 \end{array}\right]$, so $c_1 = -2$ and $c_2 = 2$. Therefore, $\mathbf{y} = -\left[\begin{array}{r} 2 \\ 6 \end{array}\right] e^{5t} + \left[\begin{array}{r} 4 \\ 2 \end{array}\right] e^{-5t}$.

10.4.18. $\left|\begin{array}{cc} 21-\lambda & -12 \\ 24 & -15-\lambda \end{array}\right| = (\lambda - 9)(\lambda + 3)$. Eigenvectors associated with $\lambda_1 = 9$ satisfy $\left[\begin{array}{rr} 12 & -12 \\ 24 & -24 \end{array}\right] \left[\begin{array}{c} x_1 \\ x_2 \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \end{array}\right]$, so $x_1 = x_2$. Taking $x_2 = 1$ yields $\mathbf{y}_1 = \left[\begin{array}{r} 1 \\ 1 \end{array}\right] e^{9t}$. Eigenvectors associated with $\lambda_2 = -3$ $\left[\begin{array}{rr} 24 & -12 \\ 24 & -12 \end{array}\right] \left[\begin{array}{c} x_1 \\ x_2 \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \end{array}\right]$, so $x_1 = \dfrac{1}{2}x_2$. Taking $x_2 = 2$ yields $\mathbf{y}_2 = \left[\begin{array}{r} 1 \\ 2 \end{array}\right] e^{-3t}$. The general solution is $\mathbf{y} = c_1 \left[\begin{array}{r} 1 \\ 1 \end{array}\right] e^{9t} + c_2 \left[\begin{array}{r} 1 \\ 2 \end{array}\right] e^{-3t}$. Now $\mathbf{y}(0) = \left[\begin{array}{r} 5 \\ 3 \end{array}\right] \Rightarrow c_1 \left[\begin{array}{r} 1 \\ 1 \end{array}\right] + c_2 \left[\begin{array}{r} 1 \\ 2 \end{array}\right] = \left[\begin{array}{r} 5 \\ 3 \end{array}\right]$, so $c_1 = 7$ and $c_2 = -2$. Therefore, $\mathbf{y} = \left[\begin{array}{r} 7 \\ 7 \end{array}\right] e^{9t} - \left[\begin{array}{r} 2 \\ 4 \end{array}\right] e^{-3t}$.

10.4.20. $\begin{vmatrix} \frac{1}{6}-\lambda & \frac{1}{3} & 0 \\ \frac{2}{3} & -\frac{1}{6}-\lambda & 0 \\ 0 & 0 & \frac{1}{2}-\lambda \end{vmatrix} = -(\lambda+1/2)(\lambda-1/2)^2$. The eigenvectors associated with with $\lambda_1 = -1/2$ satisfy the system with augmented matrix $\left[\begin{array}{ccccc} \frac{2}{3} & \frac{1}{3} & 0 & \vdots & 0 \\ \frac{2}{3} & \frac{1}{3} & 0 & \vdots & 0 \\ 0 & 0 & 1 & \vdots & 0 \end{array}\right]$, which is row equivalent to $\left[\begin{array}{ccccc} 1 & \frac{1}{2} & 0 & \vdots & 0 \\ 0 & 0 & 1 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{array}\right]$. Hence $x_1 = -\dfrac{x_2}{2}$ and $x_3 = 0$. Taking $x_2 = 2$ yields $\mathbf{y}_1 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} e^{-t/2}$. The eigenvectors associated with with $\lambda_2 = \lambda_3 = 1/2$ satisfy the system with augmented matrix $\left[\begin{array}{ccccc} -\frac{1}{3} & \frac{1}{3} & 0 & \vdots & 0 \\ \frac{2}{3} & -\frac{2}{3} & 0 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{array}\right]$, which is row equivalent to $\left[\begin{array}{ccccc} 1 & -1 & 0 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{array}\right]$. Hence $x_1 = x_2$ and x_3 is arbitrary. Taking $x_2 = 1$ and $x_3 = 0$ yields $\mathbf{y}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{t/2}$. Taking $x_2 = 0$ and $x_3 = 1$ yields $\mathbf{y}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{t/2}$. The general solution is $\mathbf{y} = c_1 \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} e^{-t/2} + c_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{t/2} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{t/2}$. Now $\mathbf{y}(0) = \begin{bmatrix} 4 \\ 7 \\ 1 \end{bmatrix} \Rightarrow c_1 \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{t/2} = \begin{bmatrix} 4 \\ 7 \\ 1 \end{bmatrix}$, so $c_1 = 1$, $c_2 = 5$, and $c_3 = 1$. Hence $\mathbf{y} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} e^{-t/2} + \begin{bmatrix} 5 \\ 5 \\ 0 \end{bmatrix} e^{t/2} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{t/2}$.

10.4.22. $\begin{vmatrix} 6-\lambda & -3 & -8 \\ 2 & 1-\lambda & -2 \\ 3 & -3 & -5-\lambda \end{vmatrix} = -(\lambda-1)(\lambda+2)(\lambda-3)$. The eigenvectors associated with $\lambda_1 = 1$ satisfy the system with augmented matrix $\left[\begin{array}{ccccc} 5 & -3 & -8 & \vdots & 0 \\ 2 & 0 & -2 & \vdots & 0 \\ 3 & -3 & -6 & \vdots & 0 \end{array}\right]$, which is row equivalent to $\left[\begin{array}{ccccc} 1 & 0 & -1 & \vdots & 0 \\ 0 & 1 & 1 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{array}\right]$. Hence $x_1 = x_3$ and $x_2 = -x_3$. Taking $x_3 =$ yields $\mathbf{y}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^t$. The eigenvectors associated with $\lambda_2 = -2$ satisfy the system with augmented matrix $\left[\begin{array}{ccccc} 8 & -3 & -8 & \vdots & 0 \\ 2 & 3 & -2 & \vdots & 0 \\ 3 & -3 & -3 & \vdots & 0 \end{array}\right]$, which is row equivalent to $\begin{bmatrix} 1 & 0 & -1 & \vdots & 0 \\ 0 & 1 & 0 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$. Hence $x_1 = x_3$ and $x_2 = 0$. Taking $x_3 = 1$ yields $\mathbf{y}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{-2t}$. The eigenvectors associated with $\lambda_3 = 3$ satisfy the system with augmented matrix $\begin{bmatrix} 3 & -3 & -8 & \vdots & 0 \\ 2 & -2 & -2 & \vdots & 0 \\ 3 & -3 & -8 & \vdots & 0 \end{bmatrix}$, which is row equivalent to $\begin{bmatrix} 1 & -1 & 0 & \vdots & 0 \\ 0 & 0 & 1 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$. Hence $x_1 = x_2$ and $x_3 = 0$. Taking $x_2 = 1$ yields $\mathbf{y}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{3t}$. The general solution is $\mathbf{y} = c_1 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{-2t} + c_3 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{3t}$. Now $\mathbf{y}(0) = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} \Rightarrow c_1 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}$, so $c_1 = 2$, $c_2 = -3$, and $c_3 = 1$. Therefore, $\mathbf{y} = \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix} e^{t} - \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix} e^{-2t} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{3t}$.

10.4.24. $\begin{vmatrix} 3-\lambda & 0 & 1 \\ 11 & -2-\lambda & 7 \\ 1 & 0 & 3-\lambda \end{vmatrix} = -(\lambda - 2)(\lambda + 2)(\lambda - 4)$. The eigenvectors associated with with $\lambda_1 = 2$ satisfy the system with augmented matrix $\begin{bmatrix} 1 & 0 & 1 & \vdots & 0 \\ 11 & -4 & 7 & \vdots & 0 \\ 1 & 0 & 1 & \vdots & 0 \end{bmatrix}$, which is row equivalent to $\begin{bmatrix} 1 & 0 & 1 & \vdots & 0 \\ 0 & 1 & 1 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$. Hence $x_1 = -x_3$ and $x_2 = -x_3$. Taking $x_3 = 1$ yields $\mathbf{y}_1 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} e^{2t}$. The eigenvectors associated with with $\lambda_2 = -2$ satisfy the system with augmented matrix $\begin{bmatrix} 5 & 0 & 1 & \vdots & 0 \\ 11 & 0 & 7 & \vdots & 0 \\ 1 & 0 & 5 & \vdots & 0 \end{bmatrix}$, which is row equivalent to $\begin{bmatrix} 1 & 0 & 0 & \vdots & 0 \\ 0 & 0 & 1 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$. Hence $x_1 = x_3 = 0$ and x_2 is arbitrary. Taking $x_3 = 1$ yields $\mathbf{y}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{-2t}$. The eigenvectors associated with with

$\lambda_3 = 4$ satisfy the system with augmented matrix $\left[\begin{array}{rrrcr} -1 & 0 & 1 & \vdots & 0 \\ 11 & -6 & 7 & \vdots & 0 \\ 1 & 0 & -1 & \vdots & 0 \end{array}\right]$, which is row equivalent to $\left[\begin{array}{rrrcr} 1 & 0 & -1 & \vdots & 0 \\ 0 & 1 & -3 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{array}\right]$. Hence $x_1 = x_3$ and $x_2 = 3x_3$. Taking $x_3 = 1$ yields $\mathbf{y}_3 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} e^{4t}$.
The general solution is $\mathbf{y} = c_1 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{-2t} + c_3 \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} e^{4t}$. Now $\mathbf{y}(0) = \begin{bmatrix} 2 \\ 7 \\ 6 \end{bmatrix} \Rightarrow$ $c_1 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \\ 6 \end{bmatrix}$, so $c_1 = 2$, $c_2 = -3$, and $c_3 = 4$. Hence $\mathbf{y} =$ $\begin{bmatrix} -2 \\ -2 \\ 2 \end{bmatrix} e^{2t} - \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} e^{-2t} + \begin{bmatrix} 4 \\ 12 \\ 4 \end{bmatrix} e^{4t}$

10.4.26. $\begin{vmatrix} 3-\lambda & -1 & 0 \\ 4 & -2-\lambda & 0 \\ 4 & -4 & 2-\lambda \end{vmatrix} = -(\lambda+1)(\lambda-2)^2$. The eigenvectors associated with $\lambda_1 = -1$ satisfy the system with augmented matrix $\left[\begin{array}{rrrcr} 4 & -1 & 0 & \vdots & 0 \\ 4 & -1 & 0 & \vdots & 0 \\ 4 & -4 & 3 & \vdots & 0 \end{array}\right]$, which is row equivalent to $\left[\begin{array}{rrrcr} 1 & 0 & -\frac{1}{4} & \vdots & 0 \\ 0 & 1 & -1 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{array}\right]$ Hence $x_1 = x_2/4$ and $x_2 = x_3$. Taking $x_3 = 4$ yields $\mathbf{y}_1 = \begin{bmatrix} 1 \\ 4 \\ 4 \end{bmatrix} e^{-t}$. The eigenvectors associated with with $\lambda_2 = \lambda_3 = 2$ satisfy the system with augmented matrix $\left[\begin{array}{rrrcr} 1 & -1 & 0 & \vdots & 0 \\ 4 & -4 & 0 & \vdots & 0 \\ 4 & -4 & 0 & \vdots & 0 \end{array}\right]$, which is row equivalent to $\left[\begin{array}{rrrcr} 1 & -1 & 0 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{array}\right]$. Hence $x_1 = x_2$ and x_3 is arbitrary. Taking $x_2 = 1$ and $x_3 = 0$ yields $\mathbf{y}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{2t}$. Taking $x_2 = 0$ and $x_3 = 1$ yields $\mathbf{y}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{2t}$. The general solution is $\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 4 \\ 4 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{2t} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{2t}$. Now $\mathbf{y}(0) = \begin{bmatrix} 7 \\ 10 \\ 2 \end{bmatrix} \Rightarrow c_1 \begin{bmatrix} 1 \\ 4 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} +$ $c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 10 \\ 2 \end{bmatrix}$, so $c_1 = 1$, $c_2 = 6$, and $c_3 = -2$. Hence $\mathbf{y} = \begin{bmatrix} 1 \\ 4 \\ 4 \end{bmatrix} e^{-t} + \begin{bmatrix} 6 \\ 6 \\ -2 \end{bmatrix} e^{2t}$.

10.4.28. **(a)** If $\mathbf{y}(t_0) = 0$, then $\mathbf{y}$ is the solution of the initial value problem $\mathbf{y}' = A\mathbf{y},\ \mathbf{y}(t_0) = \mathbf{0}$. Since $\mathbf{y} \equiv 0$ is a solution of this problem, Theorem 10.2.1 implies the conclusion.

(b) It is given that $\mathbf{y}_1'(t) = A\mathbf{y}_1(t)$ for all t. Replacing t by $t-\tau$ shows that $\mathbf{y}_1'(t-\tau) = A\mathbf{y}_1(t-\tau) = A\mathbf{y}_2(t)$ for all t. Since $\mathbf{y}_2'(t) = \mathbf{y}_1'(t-\tau)$ by the chain rule, this implies that $\mathbf{y}_2'(t) = A\mathbf{y}_2(t)$ for all t.

(c) If $\mathbf{z}(t) = \mathbf{y}_1(t-\tau)$, then $\mathbf{z}(t_2) = \mathbf{y}_1(t_1) = \mathbf{y}_2(t_2)$; therefore $\mathbf{z}$ and $\mathbf{y}_2$ are both solutions of the initial value problem $\mathbf{y}' = A\mathbf{y},\ \mathbf{y}(t_2) = \mathbf{k}$, where $\mathbf{k} = \mathbf{y}_2(t_2)$.

10.4.42. The characteristic polynomial of A is $p(\lambda) = \lambda^2 - (a+b) + ab - \alpha\beta$, so the eigenvalues of A are $\lambda_1 = \dfrac{a+b-\gamma}{2}$ and $\lambda_1 = \dfrac{a+b+\gamma}{2}$, where $\gamma = \sqrt{(a-b)^2 + 4\alpha\beta}$; $\mathbf{x}_1 = \begin{bmatrix} b-a+\gamma \\ 2\beta \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} b-a-\gamma \\ 2\beta \end{bmatrix}$ are associated eigenvectors. Since $\gamma > |b-a|$, if L_1 and L_2 are lines through the origin parallel to $\mathbf{x}_1$ and $\mathbf{x}_2$, then L_1 is in the first and third quadrants and L_2 is in the second and fourth quadrants. The slope of L_1 is $\rho = \dfrac{2\beta}{b-a+\gamma} > 0$. If $Q_0 = \rho P_0$ there are three possibilities: (i) if $\alpha\beta = ab$, then $\lambda_1 = 0$ and $P(t) = P_0$, $Q(t) = Q_0$ for all $t > 0$; (ii) if $\alpha\beta < ab$, then $\lambda_1 > 0$ and $\lim_{t\to\infty} P(t) = \lim_{t\to\infty} Q(t) = \infty$ (monotonically); (iii) if $\alpha\beta > ab$, then $\lambda_1 < 0$ and $\lim_{t\to\infty} P(t) = \lim_{t\to\infty} Q(t) = 0$ (monotonically). Now suppose $Q_0 \ne \rho P_0$, so that the trajectory cannot intersect L_1, and assume for the moment that (A) makes sense for all $t > 0$; that is, even if one or the other of P and Q is negative. Since $\lambda_2 > 0$ it follows that either $\lim_{t\to\infty} P(t) = \infty$ or $\lim_{t\to\infty} Q(t) = \infty$ (or both), and the trajectory is asymptotically parallel to L_2. Therefore,the trajectory must cross into the third quadrant (so $P(T) = 0$ and $Q(T) > 0$ for some finite T) if $Q_0 > \rho P_0$, or into the fourth quadrant (so $Q(T) = 0$ and $P(T) > 0$ for some finite T) if $Q_0 < \rho P_0$.

10.5 CONSTANT COEFFICIENT HOMOGENEOUS SYSTEMS II

10.5.2. $\begin{vmatrix} -\lambda & -1 \\ 1 & -2-\lambda \end{vmatrix} = (\lambda+1)^2$. Hence $\lambda_1 = -1$. Eigenvectors satisfy $\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, so $x_1 = x_2$. Taking $x_2 = 1$ yields $\mathbf{y}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}$. For a second solution we need a vector $\mathbf{u}$ such that $\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Let $u_1 = 1$ and $u_2 = 0$. Then $\mathbf{y}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-t} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-t}$. The general solution is $\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + c_2 \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-t} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-t} \right)$.

10.5.4. $\mathbf{y}' = \begin{vmatrix} 3-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = (\lambda-2)^2$. Hence $\lambda_1 = 2$. Eigenvectors satisfy $\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, so $x_1 = -x_2$. Taking $x_2 = 1$ yields $\mathbf{y}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{2t}$. For a second solution we need a vector $\mathbf{u}$ such that $\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Let $u_1 = -1$ and $u_2 = 0$. Then $\mathbf{y}_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix} e^{2t} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} te^{2t}$. The general solution is $\mathbf{y} = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{2t} + c_2 \left(\begin{bmatrix} -1 \\ 0 \end{bmatrix} e^{2t} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} te^{2t} \right)$.

10.5.6. $\begin{vmatrix} -10-\lambda & 9 \\ -4 & 2-\lambda \end{vmatrix} = (\lambda+4)^2$. Hence $\lambda_1 = -4$. Eigenvectors satisfy $\begin{bmatrix} -6 & 9 \\ -4 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, so $x_1 = \frac{3}{2}x_2$. Taking $x_2 = 2$ yields $\mathbf{y}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{-4t}$. For a second solution we need a vector $\mathbf{u}$ such that $\begin{bmatrix} -6 & 9 \\ -4 & 6 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$. Let $u_1 = -\frac{1}{2}$ and $u_2 = 0$. Then $\mathbf{y}_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \dfrac{e^{-4t}}{2} + \begin{bmatrix} 3 \\ 2 \end{bmatrix} te^{-4t}$. The general solution is $\mathbf{y} = c_1 \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{-4t} + c_2 \left(\begin{bmatrix} -1 \\ 0 \end{bmatrix} \frac{e^{-4t}}{2} + \begin{bmatrix} 3 \\ 2 \end{bmatrix} te^{-4t} \right)$.

10.5.8. $\begin{bmatrix} -\lambda & 2 & 1 \\ -4 & 6-\lambda & 1 \\ 0 & 4 & 2-\lambda \end{bmatrix} = -\lambda(\lambda-4)^2$. Hence $\lambda_1 = 0$ and $\lambda_2 = \lambda_3 = 4$. The eigenvectors associated with $\lambda_1 = 0$ satisfy the system with augmented matrix $\left[\begin{array}{rrrcr} 0 & 2 & 1 & \vdots & 0 \\ -4 & 6 & 1 & \vdots & 0 \\ 0 & 4 & 2 & \vdots & 0 \end{array}\right]$, which is row equivalent to $\left[\begin{array}{rrrcr} 1 & 0 & \frac{1}{2} & \vdots & 0 \\ 0 & 1 & \frac{1}{2} & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{array}\right]$. Hence $x_1 = -\frac{1}{2}x_3$ and $x_2 = -\frac{1}{2}x_3$. Taking $x_3 = 2$ yields $\mathbf{y}_1 = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$. The eigenvectors associated with $\lambda_2 = 4$ satisfy the system with augmented matrix $\left[\begin{array}{rrrcr} -4 & 2 & 1 & \vdots & 0 \\ -4 & 2 & 1 & \vdots & 0 \\ 0 & 4 & -2 & \vdots & 0 \end{array}\right]$, which is row equivalent to $\left[\begin{array}{rrrcr} 1 & 0 & -\frac{1}{2} & \vdots & 0 \\ 0 & 1 & -\frac{1}{2} & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{array}\right]$. Hence $x_1 = \frac{1}{2}x_3$ and $x_2 = \frac{1}{2}x_3$. Taking $x_3 = 2$ yields $\mathbf{y}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} e^{4t}$. For a third solution we need a vector $\mathbf{u}$ such that $\begin{bmatrix} -4 & 2 & 1 \\ -4 & 2 & 1 \\ 0 & 4 & -2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$. The augmented matrix of this system is row equivalent to $\left[\begin{array}{rrrcr} 1 & 0 & -\frac{1}{2} & \vdots & 0 \\ 0 & 1 & -\frac{1}{2} & \vdots & \frac{1}{2} \\ 0 & 0 & 0 & \vdots & 0 \end{array}\right]$. Let $u_3 = 0$, $u_1 = 0$, and $u_2 = \frac{1}{2}$. Then $\mathbf{y}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \frac{e^{4t}}{2} + \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} te^{4t}$. The general solution is

$$\mathbf{y} = c_1 \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} e^{4t} + c_3 \left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \frac{e^{4t}}{2} + \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} te^{4t} \right).$$

10.5.10. $\begin{bmatrix} -1-\lambda & 1 & -1 \\ -2 & -\lambda & 2 \\ -1 & 3 & -1-\lambda \end{bmatrix} = -(\lambda-2)(\lambda+2)^2$. Hence $\lambda_1 = 2$ and $\lambda_2 = \lambda_3 = -2$. The eigenvectors associated with $\lambda_1 = 2$ satisfy the system with augmented matrix $\left[\begin{array}{rrrcr} -3 & 1 & -1 & \vdots & 0 \\ -2 & -2 & 2 & \vdots & 0 \\ -1 & 3 & -3 & \vdots & 0 \end{array}\right]$,

which is row equivalent to $\begin{bmatrix} 1 & 0 & 0 & \vdots & 0 \\ 0 & 1 & -1 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$. Hence $x_1 = 0$ and $x_2 = x_3$. Taking $x_3 = 1$ yields $\mathbf{y}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} e^{2t}$. The eigenvectors associated with $\lambda_2 = -2$ satisfy the system with augmented matrix $\begin{bmatrix} 1 & 1 & -1 & \vdots & 0 \\ -2 & 2 & 2 & \vdots & 0 \\ -1 & 3 & 1 & \vdots & 0 \end{bmatrix}$, which is row equivalent to $\begin{bmatrix} 1 & 0 & -1 & \vdots & 0 \\ 0 & 1 & 0 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$. Hence $x_1 = x_3$ and $x_2 = 0$. Taking $x_3 = 1$ yields $\mathbf{y}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{-2t}$. For a third solution we need a vector $\mathbf{u}$ such that $\begin{bmatrix} 1 & 1 & -1 \\ -2 & 2 & 2 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. The augmented matrix of this system is row equivalent to $\begin{bmatrix} 1 & 0 & -1 & \vdots & \frac{1}{2} \\ 0 & 1 & 0 & \vdots & \frac{1}{2} \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$. Let $u_3 = 0$, $u_1 = \dfrac{1}{2}$, and $u_2 = \dfrac{1}{2}$. Then $\mathbf{y}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \dfrac{e^{-2t}}{2} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} te^{-2t}$.
The general solution is

$$\mathbf{y} = c_1 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{-2t} + c_3 \left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \frac{e^{-2t}}{2} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} te^{-2t} \right).$$

10.5.12. $\begin{vmatrix} 6-\lambda & -5 & 3 \\ 2 & -1-\lambda & 3 \\ 2 & 1 & 1-\lambda \end{vmatrix} = -(\lambda+2)(\lambda-4)^2$. Hence $\lambda_1 = -2$ and $\lambda_2 = \lambda_3 = 4$. The eigenvectors associated with $\lambda_1 = -2$ satisfy the system with augmented matrix $\begin{bmatrix} 8 & -5 & 3 & \vdots & 0 \\ 2 & 1 & 3 & \vdots & 0 \\ 2 & 1 & 3 & \vdots & 0 \end{bmatrix}$, which is row equivalent to $\begin{bmatrix} 1 & 0 & 1 & \vdots & 0 \\ 0 & 1 & 1 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$. Hence $x_1 = -x_3$ and $x_2 = -x_3$. Taking $x_3 = 1$ yields $\mathbf{y}_1 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} e^{-2t}$. The eigenvectors associated with $\lambda_2 = 4$ satisfy the system with augmented matrix $\begin{bmatrix} 2 & -5 & 3 & \vdots & 0 \\ 2 & -5 & 3 & \vdots & 0 \\ 2 & 1 & -3 & \vdots & 0 \end{bmatrix}$, which is row equivalent to $\begin{bmatrix} 1 & 0 & -1 & \vdots & 0 \\ 0 & 1 & -1 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$. Hence $x_1 = x_3$

and $x_2 = x_3$. Taking $x_3 = 1$ yields $\mathbf{y}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{4t}$. For a third solution we need a vector $\mathbf{u}$ such that $\begin{bmatrix} 2 & -5 & 3 \\ 2 & -5 & 3 \\ 2 & 1 & -3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. The augmented matrix of this system is row equivalent to $\left[\begin{array}{ccc|c} 1 & 0 & -1 & \frac{1}{2} \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$. Let $u_3 = 0$, $u_1 = \dfrac{1}{2}$, and $u_2 = 0$. Then $\mathbf{y}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \dfrac{e^{4t}}{2} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} te^{4t}$. The general solution is $\mathbf{y} = c_1 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{4t} + c_3 \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \dfrac{e^{4t}}{2} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} te^{4t} \right)$.

10.5.14. $\begin{vmatrix} 15-\lambda & -9 \\ 16 & -9-\lambda \end{vmatrix} = (\lambda - 3)^2$. Hence $\lambda_1 = 3$. Eigenvectors satisfy $\begin{bmatrix} 12 & -9 \\ 16 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, so $x_1 = \dfrac{3}{4} x_2$. Taking $x_2 = 4$ yields $\mathbf{y}_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix} e^{3t}$. For a second solution we need a vector $\mathbf{u}$ such that $\begin{bmatrix} 12 & -9 \\ 16 & -12 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$. Let $u_1 = \dfrac{1}{4}$ and $u_2 = 0$. Then $\mathbf{y}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \dfrac{e^{3t}}{4} + \begin{bmatrix} 3 \\ 4 \end{bmatrix} te^{3t}$. The general solution is $\mathbf{y} = c_1 \begin{bmatrix} 3 \\ 4 \end{bmatrix} e^{3t} + c_2 \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \dfrac{e^{3t}}{4} + \begin{bmatrix} 3 \\ 4 \end{bmatrix} te^{3t} \right)$. Now $\mathbf{y}(0) = \begin{bmatrix} 5 \\ 8 \end{bmatrix} \Rightarrow c_1 \begin{bmatrix} 3 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} \frac{1}{4} \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$, so $c_1 = 2$ and $c_2 = -4$. Therefore, $\mathbf{y} = \begin{bmatrix} 5 \\ 8 \end{bmatrix} e^{3t} - \begin{bmatrix} 12 \\ 16 \end{bmatrix} te^{3t}$.

10.5.16. $\begin{vmatrix} -7-\lambda & 24 \\ -6 & 17-\lambda \end{vmatrix} = (\lambda - 5)^2$. Hence $\lambda_1 = 5$. Eigenvectors satisfy $\begin{bmatrix} -12 & 24 \\ -6 & 12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, so $x_1 = 2x_2$. Taking $x_2 = 1$ yields $\mathbf{y}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{5t}$. For a second solution we need a vector $\mathbf{u}$ such that $\begin{bmatrix} -12 & 24 \\ -6 & 12 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Let $u_1 = \dfrac{1}{6}$ and $u_2 = 0$. Then $\mathbf{y}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \dfrac{e^{5t}}{6} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} te^{5t}$. The general solution is $\mathbf{y} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{5t} + c_2 \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \dfrac{e^{5t}}{6} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} te^{5t} \right)$. Now $\mathbf{y}(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \Rightarrow c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} \frac{1}{6} \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, so $c_1 = 1$ and $c_2 = 6$. Therefore, $\mathbf{y} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{5t} - \begin{bmatrix} 12 \\ 6 \end{bmatrix} te^{5t}$.

10.5.18. $\begin{bmatrix} -1-\lambda & 1 & 0 \\ 1 & -1-\lambda & -2 \\ -1 & -1 & -1-\lambda \end{bmatrix} = -(\lambda - 1)(\lambda + 2)^2$. Hence $\lambda_1 = 1$ and $\lambda_2 = \lambda_3 = -2$. The eigenvectors associated with $\lambda_1 = 1$ satisfy the system with augmented matrix $\left[\begin{array}{ccc|c} -2 & 1 & 0 & 0 \\ 1 & -2 & -2 & 0 \\ -1 & -1 & -2 & 0 \end{array}\right]$, which is row equivalent to $\left[\begin{array}{ccc|c} 1 & 0 & \frac{2}{3} & 0 \\ 0 & 1 & \frac{4}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$. Hence $x_1 = -\dfrac{2}{3} x_3$ and $x_2 = -\dfrac{4}{3} x_3$. Taking

$x_3 = 3$ yields The eigenvectors associated with $\lambda_2 = -2$ satisfy the system with augmented matrix $\left[\begin{array}{rrrcr} 1 & 1 & 0 & \vdots & 0 \\ 1 & 1 & -2 & \vdots & 0 \\ -1 & -1 & 1 & \vdots & 0 \end{array}\right]$, which is row equivalent to $\left[\begin{array}{rrrcr} 1 & 1 & 0 & \vdots & 0 \\ 0 & 0 & 1 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{array}\right]$. Hence $x_1 = -x_2$ and $x_3 = 0$. Taking $x_2 = 1$ yields $\mathbf{y}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{-2t}$. For a third solution we need a vector $\mathbf{u}$ such that $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & -2 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$. The augmented matrix of this system is row equivalent to $\left[\begin{array}{rrrcr} 1 & 1 & 0 & \vdots & -1 \\ 0 & 0 & 1 & \vdots & -1 \\ 0 & 0 & 0 & \vdots & 0 \end{array}\right]$. Let $u_2 = 0$, $u_1 = -1$, and $u_3 = -1$. Then $\mathbf{y}_3 = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix} e^{-2t} + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} te^{-2t}$. The general solution is $\mathbf{y} = c_1 \begin{bmatrix} -2 \\ -4 \\ 3 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{-2t} + c_3 \left(\begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix} e^{-2t} + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} te^{-2t} \right)$.

Now $\mathbf{y}(0) = \begin{bmatrix} 6 \\ 5 \\ -7 \end{bmatrix} \Rightarrow c_1 \begin{bmatrix} -2 \\ -4 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ -7 \end{bmatrix}$, so $c_1 = -2$, $c_2 = -3$, and $c_3 = 1$. Therefore, $\mathbf{y} = \begin{bmatrix} 4 \\ 8 \\ -6 \end{bmatrix} e^{t} + \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix} e^{-2t} + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} te^{-2t}$.

10.5.20. $\begin{bmatrix} -7-\lambda & -4 & 4 \\ -1 & 0-\lambda & 1 \\ -9 & -5 & 6-\lambda \end{bmatrix} = -(\lambda+3)(\lambda-1)^2$. Hence $\lambda_1 = -3$ and $\lambda_2 = \lambda_3 = 1$. The eigenvectors associated with $\lambda_1 = -3$ satisfy the system with augmented matrix $\left[\begin{array}{rrrcr} -4 & -4 & 4 & \vdots & 0 \\ -1 & 3 & 1 & \vdots & 0 \\ -9 & -5 & 9 & \vdots & 0 \end{array}\right]$, which is row equivalent to $\left[\begin{array}{rrrcr} 1 & 0 & -1 & \vdots & 0 \\ 0 & 1 & 0 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{array}\right]$. Hence $x_1 = x_3$ and $x_2 = 0$. Taking $x_3 = 1$ yields $\mathbf{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{-3t}$. The eigenvectors associated with $\lambda_2 = 1$ satisfy the system with augmented matrix $\left[\begin{array}{rrrcr} -8 & -4 & 4 & \vdots & 0 \\ -1 & -1 & 1 & \vdots & 0 \\ -9 & -5 & -5 & \vdots & 0 \end{array}\right]$, which is row equivalent to $\left[\begin{array}{rrrcr} 1 & 0 & 0 & \vdots & 0 \\ 0 & 1 & -1 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{array}\right]$. Hence $x_1 = 0$

and $x_2 = x_3$. Taking $x_3 = 1$ yields $\mathbf{y}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} e^t$. For a third solution we need a vector $\mathbf{u}$ such that $\begin{bmatrix} -8 & -4 & 4 \\ -1 & -1 & 1 \\ -9 & -5 & 5 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. The augmented matrix of this system is row equivalent to $\begin{bmatrix} 1 & 0 & 0 & \vdots & 1 \\ 0 & 1 & -1 & \vdots & -2 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$. Let $u_3 = 0$, $u_1 = 1$, and $u_2 = -2$. Then $\mathbf{y}_3 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} e^t + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} te^t$.

The general solution is $\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} e^t + c_3 \left(\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} e^t + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} te^t \right)$. Now $\mathbf{y}(0) = \begin{bmatrix} -6 \\ 9 \\ -1 \end{bmatrix} \Rightarrow c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} -6 \\ 9 \\ -1 \end{bmatrix}$, so $c_1 = -2$, $c_2 = 1$, and $c_3 = -4$. Therefore, $\mathbf{y} = -\begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} e^{-3t} + \begin{bmatrix} -4 \\ 9 \\ 1 \end{bmatrix} e^t - \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix} te^t$.

10.5.22. $\begin{vmatrix} 4-\lambda & -8 & -4 \\ -3 & -1-\lambda & -3 \\ 1 & -1 & 9-\lambda \end{vmatrix} = -(\lambda+4)(\lambda-8)^2$. Hence $\lambda_1 = -4$ and $\lambda_2 = \lambda_3 = 8$. The eigenvectors associated with $\lambda_1 = -4$ satisfy the system with augmented matrix $\begin{bmatrix} 8 & -8 & -4 & \vdots & 0 \\ -3 & 3 & -3 & \vdots & 0 \\ 1 & -1 & 13 & \vdots & 0 \end{bmatrix}$, which is row equivalent to $\begin{bmatrix} 1 & -1 & 0 & \vdots & 0 \\ 0 & 0 & 1 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$. Hence $x_1 = x_2$ and $x_3 = 0$. Taking $x_2 = 1$ yields $\mathbf{y}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^t$. The eigenvectors associated with $\lambda_2 = 8$ satisfy the system with augmented matrix $\begin{bmatrix} -4 & -8 & -4 & \vdots & 0 \\ -3 & -9 & -3 & \vdots & 0 \\ 1 & -1 & 1 & \vdots & 0 \end{bmatrix}$, which is row equivalent to $\begin{bmatrix} 1 & 0 & 1 & \vdots & 0 \\ 0 & 1 & 0 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$. Hence $x_1 = -x_3$ and $x_2 = 0$. Taking $x_3 = 1$ yields $\mathbf{y}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{8t}$. For a third solution we need a vector $\mathbf{u}$ such that $\begin{bmatrix} -4 & -8 & -4 \\ -3 & -9 & -3 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. The augmented matrix of this system is row equivalent to

$\left[\begin{array}{rrrcr} 1 & 0 & 1 & \vdots & \frac{3}{4} \\ 0 & 1 & 0 & \vdots & -\frac{1}{4} \\ 0 & 0 & 0 & \vdots & 0 \end{array}\right]$. Let $u_3 = 0$, $u_1 = \dfrac{3}{4}$, and $u_2 = -\dfrac{1}{4}$. Then $\mathbf{y}_3 = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} \dfrac{e^{8t}}{4} + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} te^{8t}$.

The general solution is $c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{8t} + c_3 \left(\begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} \dfrac{e^{8t}}{4} + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} te^{8t} \right)$. Now $\mathbf{y}(0) = \begin{bmatrix} -4 \\ 1 \\ -3 \end{bmatrix} \Rightarrow c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} \frac{3}{4} \\ -\frac{1}{4} \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \\ -3 \end{bmatrix}$, so $c_1 = -1$, $c_2 = -3$, and $c_3 = -8$.

Therefore, $\mathbf{y} = -\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{-4t} + \begin{bmatrix} -3 \\ 2 \\ -3 \end{bmatrix} e^{8t} + \begin{bmatrix} 8 \\ 0 \\ -8 \end{bmatrix} te^{8t}$.

10.5.24. $\begin{bmatrix} 5-\lambda & -1 & 1 \\ -1 & 9-\lambda & -3 \\ -2 & 2 & 4-\lambda \end{bmatrix} = -(\lambda-6)^3$. Hence $\lambda_1 = 6$. The eigenvectors satisfy the system with augmented matrix $\left[\begin{array}{rrrcr} 1 & -1 & 1 & \vdots & 0 \\ -1 & 3 & -3 & \vdots & 0 \\ -2 & 2 & -2 & \vdots & 0 \end{array}\right]$, which is row equivalent to $\left[\begin{array}{rrrcr} 1 & 0 & 0 & \vdots & 0 \\ 0 & 1 & -1 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{array}\right]$.

Hence $x_1 = 0$ and $x_2 = x_3$. Taking $x_3 = 1$ yields $\mathbf{y}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} e^{6t}$. For a second solution we need a vector $\mathbf{u}$ such that $\begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -3 \\ -2 & 2 & -2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. The augmented matrix of this system is row equivalent to $\left[\begin{array}{rrrcr} 1 & 0 & 0 & \vdots & -\frac{1}{4} \\ 0 & 1 & -1 & \vdots & \frac{1}{4} \\ 0 & 0 & 0 & \vdots & 0 \end{array}\right]$. Let $u_3 = 0$, $u_1 = -\dfrac{1}{4}$, and $u_2 = \dfrac{1}{4}$. Then $\mathbf{y}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \dfrac{e^{6t}}{4} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} te^{6t}$. For a third solution we need a vector $\mathbf{v}$ such that $\begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -3 \\ -2 & 2 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} \\ \frac{1}{4} \\ 0 \end{bmatrix}$.

The augmented matrix of this system is row equivalent to $\left[\begin{array}{rrrcr} 1 & 0 & 0 & \vdots & \frac{1}{8} \\ 0 & 1 & -1 & \vdots & \frac{1}{8} \\ 0 & 0 & 0 & \vdots & 0 \end{array}\right]$. Let $v_3 = 0$, $v_1 = \dfrac{1}{8}$, and $v_2 = \dfrac{1}{8}$. Then $\mathbf{y}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \dfrac{e^{6t}}{8} + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \dfrac{te^{6t}}{4} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \dfrac{t^2e^{6t}}{2}$. The general solution is $\mathbf{y} =$

$$c_1\begin{bmatrix}0\\1\\1\end{bmatrix}e^{6t}+c_2\left(\begin{bmatrix}-1\\1\\0\end{bmatrix}\frac{e^{6t}}{4}+\begin{bmatrix}0\\1\\1\end{bmatrix}te^{6t}\right)$$
$$+c_3\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\frac{e^{6t}}{8}+\begin{bmatrix}-1\\1\\0\end{bmatrix}\frac{te^{6t}}{4}+\begin{bmatrix}0\\1\\1\end{bmatrix}\frac{t^2e^{6t}}{2}\right).$$

10.5.26. $\begin{bmatrix}-6-\lambda & -4 & -4\\ 2 & -1-\lambda & 1\\ 2 & 3 & 1-\lambda\end{bmatrix}=-(\lambda+2)^3$. Hence $\lambda_1=-2$. The eigenvectors satisfy the system with augmented matrix $\begin{bmatrix}-4 & -4 & -4 & \vdots & 0\\ 2 & 1 & 1 & \vdots & 0\\ 2 & 3 & 3 & \vdots & 0\end{bmatrix}$, which is row equivalent to $\begin{bmatrix}1 & 0 & 0 & \vdots & 0\\ 0 & 1 & 1 & \vdots & 0\\ 0 & 0 & 0 & \vdots & 0\end{bmatrix}$. Hence $x_1=0$ and $x_2=-x_3$. Taking $x_3=1$ yields $\mathbf{y}_1=\begin{bmatrix}0\\-1\\1\end{bmatrix}e^{-2t}$. For a second solution we need a vector $\mathbf{u}$ such that $\begin{bmatrix}-4 & -4 & -4\\ 2 & 1 & 1\\ 2 & 3 & 3\end{bmatrix}\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}=\begin{bmatrix}0\\-1\\1\end{bmatrix}$. The augmented matrix of this system is row equivalent to $\begin{bmatrix}1 & 0 & 0 & \vdots & -1\\ 0 & 1 & 1 & \vdots & 1\\ 0 & 0 & 0 & \vdots & 0\end{bmatrix}$. Let $u_3=0$, $u_1=-1$, and $u_2=1$. Then $\mathbf{y}_2=\begin{bmatrix}-1\\1\\0\end{bmatrix}e^{-2t}+\begin{bmatrix}0\\-1\\1\end{bmatrix}te^{-2t}$. For a third solution we need a vector $\mathbf{v}$ such that $\begin{bmatrix}-4 & -4 & -4\\ 2 & 1 & 1\\ 2 & 3 & 3\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}=\begin{bmatrix}-1\\1\\0\end{bmatrix}$. The augmented matrix of this system is row equivalent to $\begin{bmatrix}1 & 0 & 0 & \vdots & \frac{3}{4}\\ 0 & 1 & 1 & \vdots & -\frac{1}{2}\\ 0 & 0 & 0 & \vdots & 0\end{bmatrix}$. Let $v_3=0$, $v_1=\dfrac{3}{4}$, and $v_2=-\dfrac{1}{2}$. Then $\mathbf{y}_3=\begin{bmatrix}3\\-2\\0\end{bmatrix}\dfrac{e^{-2t}}{4}+\begin{bmatrix}-1\\1\\0\end{bmatrix}te^{-2t}+\begin{bmatrix}0\\-1\\1\end{bmatrix}\dfrac{t^2e^{-2t}}{2}$. The general solution is $\mathbf{y}=c_1\begin{bmatrix}0\\-1\\1\end{bmatrix}e^{-2t}+c_2\left(\begin{bmatrix}-1\\1\\0\end{bmatrix}e^{-2t}+\begin{bmatrix}0\\-1\\1\end{bmatrix}te^{-2t}\right)$

$$+c_3\left(\begin{bmatrix}3\\-2\\0\end{bmatrix}\frac{e^{-2t}}{4}+\begin{bmatrix}-1\\1\\0\end{bmatrix}te^{-2t}+\begin{bmatrix}0\\-1\\1\end{bmatrix}\frac{t^2e^{-2t}}{2}\right)$$

10.5.28. $\begin{bmatrix}-2-\lambda & -12 & 10\\ 2 & -24-\lambda & 11\\ 2 & -24 & 8-\lambda\end{bmatrix}=-(\lambda+6)^3$. Hence $\lambda_1=-6$. The eigenvectors satisfy the system with augmented matrix $\left[\begin{array}{rrrcr} 4 & -12 & 10 & \vdots & 0 \\ 2 & -18 & 11 & \vdots & 0 \\ 2 & -24 & 14 & \vdots & 0 \end{array}\right]$, which is row equivalent to $\left[\begin{array}{rrrcr} 1 & 0 & 1 & \vdots & 0 \\ 0 & 1 & -\frac{1}{2} & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{array}\right]$.

Hence $x_1 = -x_3$ and $x_2 = \dfrac{x_3}{2}$. Taking $x_3 = 2$ yields $\mathbf{y}_1 = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} e^{-6t}$. For a second solution we need a vector $\mathbf{u}$ such that $\begin{bmatrix} 4 & -12 & 10 \\ 2 & -18 & 11 \\ 2 & -24 & 14 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$. The augmented matrix of this system is row equivalent to $\left[\begin{array}{rrrcr} 1 & 0 & 1 & \vdots & -1 \\ 0 & 1 & -\frac{1}{2} & \vdots & -\frac{1}{6} \\ 0 & 0 & 0 & \vdots & 0 \end{array}\right]$. Let $u_3 = 0$, $u_1 = -1$, and $u_2 = -\dfrac{1}{6}$. Then $\mathbf{y}_2 = -\begin{bmatrix} 6 \\ 1 \\ 0 \end{bmatrix} \dfrac{e^{-6t}}{6} + \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} te^{-6t}$. For a third solution we need a vector $\mathbf{v}$ such that $\begin{bmatrix} 4 & -12 & 10 \\ 2 & -18 & 11 \\ 2 & -24 & 14 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -1 \\ -\frac{1}{6} \\ 0 \end{bmatrix}$. The augmented matrix of this system is row equivalent to $\left[\begin{array}{rrrcr} 1 & 0 & 1 & \vdots & -\frac{1}{3} \\ 0 & 1 & -\frac{1}{2} & \vdots & -\frac{1}{36} \\ 0 & 0 & 0 & \vdots & 0 \end{array}\right]$. Let $v_3 = 0$, $v_1 = -\dfrac{1}{3}$, and $v_2 = -\dfrac{1}{36}$. Then $\mathbf{y}_3 = -\begin{bmatrix} 12 \\ 1 \\ 0 \end{bmatrix} \dfrac{e^{-6t}}{36} - \begin{bmatrix} 6 \\ 1 \\ 0 \end{bmatrix} \dfrac{te^{-6t}}{6} + \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} \dfrac{t^2e^{-6t}}{2}$. The general solution is $\mathbf{y} = c_1 \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} e^{-6t} + c_2 \left(-\begin{bmatrix} 6 \\ 1 \\ 0 \end{bmatrix} \dfrac{e^{-6t}}{6} + \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} te^{-6t} \right)$

$$+c_3 \left(-\begin{bmatrix} 12 \\ 1 \\ 0 \end{bmatrix} \frac{e^{-6t}}{36} - \begin{bmatrix} 6 \\ 1 \\ 0 \end{bmatrix} \frac{te^{-6t}}{6} + \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} \frac{t^2e^{-6t}}{2} \right).$$

10.5.30. $\begin{bmatrix} -4-\lambda & 0 & -1 \\ -1 & -3-\lambda & -1 \\ 1 & 0 & -2-\lambda \end{bmatrix} = -(\lambda+3)^3$. Hence $\lambda_1 = 3$. The eigenvectors satisfy the system with augmented matrix $\left[\begin{array}{rrrcr} -1 & 0 & -1 & \vdots & 0 \\ -1 & 0 & -1 & \vdots & 0 \\ 1 & 0 & 1 & \vdots & 0 \end{array}\right]$, which is row equivalent to $\left[\begin{array}{rrrcr} 1 & 0 & 1 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{array}\right]$.

Hence $x_1 = -x_3$ and x_2 is arbitrary. Taking $x_2 = 0$ and $x_3 = 1$ yields $\mathbf{y}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{-3t}$. Taking $x_2 = 1$ and $x_3 = 0$ yields $\mathbf{y}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{-3t}$. For a third solution we need constants α and β and a vector $\mathbf{u}$ such that $\begin{bmatrix} -1 & 0 & -1 \\ -1 & 0 & -1 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \alpha\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \beta\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. The augmented matrix of this system is row equivalent to $\left[\begin{array}{ccccc} 1 & 0 & 1 & \vdots & \alpha \\ 0 & 0 & 0 & \vdots & \alpha+\beta \\ 0 & 0 & 0 & \vdots & 0 \end{array}\right]$; hence the system has a solution if $\alpha = -\beta = 1$, which yields the eigenvector $\mathbf{x}_3 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$. Taking $u_1 = 1$ and $u_2 = u_3 = 0$ yields the solution $\mathbf{y}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{-3t} + \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} te^{-3t}$. The general solution is $\mathbf{y} = c_1\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{-3t} + c_2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{-3t} + c_3\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{-3t} + \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} te^{-3t}\right)$

10.5.32. $\begin{bmatrix} -3-\lambda & -1 & 0 \\ 1 & -1-\lambda & 0 \\ -1 & -1 & -2-\lambda \end{bmatrix} = -(\lambda+2)^3$. Hence $\lambda_1 = -2$. The eigenvectors satisfy the system with augmented matrix $\left[\begin{array}{ccccc} -1 & -1 & 0 & \vdots & 0 \\ 1 & 1 & 0 & \vdots & 0 \\ -1 & -1 & 0 & \vdots & 0 \end{array}\right]$, which is row equivalent to $\left[\begin{array}{ccccc} 1 & 1 & 0 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{array}\right]$. Hence $x_1 = -x_2$ and x_3 is arbitrary. Taking $x_2 = 1$ and $x_3 = 0$ yields $\mathbf{y}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{-2t}$. Taking $x_2 = 0$ and $x_3 = 1$ yields $\mathbf{y}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{-2t}$. For a third solution we need constants α and β and a vector $\mathbf{u}$ such that $\begin{bmatrix} -1 & -1 & 0 \\ 1 & 1 & 0 \\ -1 & -1 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \alpha\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \beta\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. The augmented matrix of this system is row equivalent to $\left[\begin{array}{ccccc} 1 & 1 & 0 & \vdots & \alpha \\ 0 & 0 & 0 & \vdots & \alpha+\beta \\ 0 & 0 & 0 & \vdots & 0 \end{array}\right]$; hence the system has a solution if $\alpha = -\beta = 1$, which yields the eigenvector $\mathbf{x}_3 = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$. Taking $u_1 = 1$ and $u_2 = u_3 = 0$ yields the solution $\mathbf{y}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{-2t} + \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} te^{-2t}$. The general solution is $\mathbf{y} = c_1\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{-2t} + c_2\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{-2t} + c_3\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{-2t} + \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} te^{-2t}\right)$.

10.5.34.

$$\begin{aligned} \mathbf{y}_3' - A\mathbf{y}_3 &= (\lambda_1 I - A)\mathbf{v}e^{\lambda_1 t} + (\lambda_1 I - A)\mathbf{u}te^{\lambda_1 t} + \mathbf{u}e^{\lambda_1 t} \\ &\quad +(\lambda_1 I - A)\mathbf{x}\frac{t^2 e^{\lambda_1 t}}{2} + \mathbf{x}te^{\lambda_1 t} \\ &= -\mathbf{u}e^{\lambda_1 t} - \mathbf{x}te^{\lambda_1 t} + \mathbf{u}e^{\lambda_1 t} + 0 + \mathbf{x}te^{\lambda_1 t} = 0. \end{aligned}$$

Now suppose that $c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + c_3\mathbf{y}_3 = \mathbf{0}$. Then

$$c_1\mathbf{x} + c_2(\mathbf{u} + t\mathbf{x}) + c_3\left(\mathbf{v} + t\mathbf{u} + \frac{t^2}{2}\mathbf{x}\right) = \mathbf{0}. \tag{A}$$

Differentiating this twice yields $c_3\mathbf{x} = 0$, so $c_3 = 0$ since $\mathbf{x} \neq \mathbf{0}$. Therefore,(A) reduces to (B) $c_1\mathbf{x} + c_2(\mathbf{u} + t\mathbf{x}) = \mathbf{0}$. Differentiating this yields $c_2\mathbf{x} = 0$, so $c_2 = 0$ since $\mathbf{x} \neq \mathbf{0}$. Therefore,(B) reduces to $c_3\mathbf{x} = \mathbf{0}$, so $c_1 = 0$ since $\mathbf{x} \neq \mathbf{0}$. Therefore,$\mathbf{y}_1$, $\mathbf{y}_2$, and $\mathbf{y}_3$ are linearly independendent.

10.6 CONSTANT COEFFICIENT HOMOGENEOUS SYSTEMS III

10.6.2. $\begin{vmatrix} -11-\lambda & 4 \\ -26 & 9-\lambda \end{vmatrix} = (\lambda + 1)^2 + 4$. The augmented matrix of $(A - (-1+2i)\,I)\,\mathbf{x} = \mathbf{0}$ is $\left[\begin{array}{cc|c} -10-2i & 4 & 0 \\ -26 & 10-2i & 0 \end{array}\right]$, which is row equivalent to $\left[\begin{array}{cc|c} 1 & \frac{-5+i}{13} & 0 \\ 0 & 0 & 0 \end{array}\right]$. Therefore,$x_1 = (5-i)x_2/13$. Taking $x_2 = 13$ yields the eigenvector $\mathbf{x} = \begin{bmatrix} 5-i \\ 13 \end{bmatrix}$. Taking real and imaginary parts of $e^{-t}(\cos 2t + i\sin 2t))\begin{bmatrix} 5-i \\ 13 \end{bmatrix}$ yields

$$\mathbf{y} = c_1e^{-t}\begin{bmatrix} 5\cos 2t + \sin 2t \\ 13\cos 2t \end{bmatrix} + c_2e^{-t}\begin{bmatrix} 5\sin 2t - \cos 2t \\ 13\sin 2t \end{bmatrix}.$$

10.6.4. $\begin{vmatrix} 5-\lambda & -6 \\ 3 & -1-\lambda \end{vmatrix} = (\lambda - 2)^2 + 9$. Hence, $\lambda = 2+3i$ is an eigenvalue of A. The associated eigenvectors satisfy $(A - (2+3i)\,I)\,\mathbf{x} = \mathbf{0}$. The augmented matrix of this system is $\left[\begin{array}{cc|c} 3-3i & -6 & 0 \\ 3 & -3-3i & 0 \end{array}\right]$, which is row equivalent to $\left[\begin{array}{cc|c} 1 & -1-i & 0 \\ 0 & 0 & 0 \end{array}\right]$. Therefore,$x_1 = (1+i)x_2$. Taking $x_2 = 1$ yields $x_1 = 1+i$, so $\mathbf{x} = \begin{bmatrix} 1+i \\ 1 \end{bmatrix}$ is an eigenvector. Taking real and imaginary parts of $e^{2t}(\cos 3t + i\sin 3t)\begin{bmatrix} 1+i \\ 1 \end{bmatrix}$ yields $\mathbf{y} = c_1e^{2t}\begin{bmatrix} \cos 3t - \sin 3t \\ \cos 3t \end{bmatrix} + c_2e^{2t}\begin{bmatrix} \sin 3t + \cos 3t \\ \sin 3t \end{bmatrix}$.

10.6.6. $\begin{vmatrix} -3-\lambda & 3 & 1 \\ 1 & -5-\lambda & -3 \\ -3 & 7 & 3-\lambda \end{vmatrix} = -(\lambda + 1)\left((\lambda+2)^2 + 4\right)$. The augmented matrix of $(A + I)\mathbf{x} = \mathbf{0}$ is $\left[\begin{array}{ccc|c} -2 & 3 & 1 & 0 \\ 1 & -4 & -3 & 0 \\ -3 & 7 & 4 & 0 \end{array}\right]$, which is row equivalent to $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$. Therefore,$x_1 = x_2 = -x_3$. Taking $x_3 = 1$ yields $\mathbf{y}_1 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} e^{-t}$. The augmented matrix of $(A - (-2+2i)I)\,\mathbf{x} = \mathbf{0}$

is $\left[\begin{array}{ccccc} -1-2i & 3 & 1 & \vdots & 0 \\ 1 & -3-2i & -3 & \vdots & 0 \\ -3 & 7 & 5-2i & \vdots & 0 \end{array}\right]$, which is row equivalent to $\left[\begin{array}{ccccc} 1 & 0 & -\frac{1+i}{2} & \vdots & 0 \\ 0 & 1 & \frac{1-i}{2} & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{array}\right]$. Therefore,$x_1 = \dfrac{(1+i)}{2}x_3$ and $x_2 = -\dfrac{(1-i)}{2}x_3$. Taking $x_3 = 2$ yields the eigenvector $\mathbf{x}_2 = \begin{bmatrix} 1+i \\ -1+i \\ 2 \end{bmatrix}$. The real and imaginary parts of $e^{-2t}(\cos 2t + i\sin 2t)\begin{bmatrix} 1+i \\ -1+i \\ 2 \end{bmatrix}$ are $\mathbf{y}_2 = e^{-2t}\begin{bmatrix} \cos 2t - \sin 2t \\ -\cos 2t - \sin 2t \\ 2\cos 2t \end{bmatrix}$ and $\mathbf{y}_3 = e^{-2t}\begin{bmatrix} \sin 2t + \cos 2t \\ -\sin 2t + \cos 2t \\ 2\sin 2t \end{bmatrix}$. Therefore,

$$\mathbf{y} = c_1\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}e^{-t} + c_2e^{-2t}\begin{bmatrix} \cos 2t - \sin 2t \\ -\cos 2t - \sin 2t \\ 2\cos 2t \end{bmatrix} + c_3e^{-2t}\begin{bmatrix} \sin 2t + \cos 2t \\ -\sin 2t + \cos 2t \\ 2\sin 2t \end{bmatrix}.$$

10.6.8. $\begin{vmatrix} -3-\lambda & 1 & -3 \\ 4 & -1-\lambda & 2 \\ 4 & -2 & 3-\lambda \end{vmatrix} = -(\lambda-1)\left((\lambda+1)^2+4\right)$. The augmented matrix of $(A-I)\mathbf{x} = \mathbf{0}$ is $\left[\begin{array}{ccccc} -4 & 1 & -3 & \vdots & 0 \\ 4 & -2 & 2 & \vdots & 0 \\ 4 & -2 & 2 & \vdots & 0 \end{array}\right]$, which is row equivalent to $\left[\begin{array}{ccccc} 1 & 0 & 1 & \vdots & 0 \\ 0 & 1 & 1 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{array}\right]$. Therefore,$x_1 = x_2 = -x_3$. Taking $x_3 = 1$ yields $\mathbf{y}_1 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}e^t$. The augmented matrix of $(A-(-1+2i)I)\,\mathbf{x} = \mathbf{0}$ is $\left[\begin{array}{ccccc} -2-2i & 1 & -3 & \vdots & 0 \\ 4 & -2i & 2 & \vdots & 0 \\ 4 & -2 & 4-2i & \vdots & 0 \end{array}\right]$, which is row equivalent to $\left[\begin{array}{ccccc} 1 & 0 & \frac{1-i}{2} & \vdots & 0 \\ 0 & 1 & -1 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{array}\right]$. Therefore,$x_1 = -\dfrac{1-i}{2}x_3$ and $x_2 = x_3$. Taking $x_3 = 2$ yields the eigenvector $\mathbf{x}_2 = \begin{bmatrix} -1+i \\ 2 \\ 2 \end{bmatrix}$. The real and imaginary parts of $e^{-t}(\cos 2t + i\sin 2t)\begin{bmatrix} -1+i \\ 2 \\ 2 \end{bmatrix}$ are $\mathbf{y}_2 = e^{-t}\begin{bmatrix} -\sin 2t - \cos 2t \\ 2\cos 2t \\ 2\cos 2t \end{bmatrix}$ and $\mathbf{y}_3 = e^{-t}\begin{bmatrix} \cos 2t - \sin 2t \\ 2\sin 2t \\ 2\sin 2t \end{bmatrix}$. Therefore,

$$\mathbf{y} = c_1\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}e^{-t} + c_2e^{-t}\begin{bmatrix} -\sin 2t - \cos 2t \\ 2\cos 2t \\ 2\cos 2t \end{bmatrix} + c_3e^{-t}\begin{bmatrix} \cos 2t - \sin 2t \\ 2\sin 2t \\ 2\sin 2t \end{bmatrix}.$$

10.6.10. $\dfrac{1}{3}\begin{vmatrix} 7-3\lambda & -5 \\ 2 & 5-3\lambda \end{vmatrix} = (\lambda-2)^2+1$. The augmented matrix of $(A-(2+i)I)\mathbf{x} = \mathbf{0}$ is $\dfrac{1}{3}\left[\begin{array}{cccc} 1-3i & -5 & \vdots & 0 \\ 2 & -1-3i & \vdots & 0 \end{array}\right]$, which is row equivalent to $\left[\begin{array}{cccc} 1 & -\frac{1+3i}{2} & \vdots & 0 \\ 0 & 0 & \vdots & 0 \end{array}\right]$. Therefore,$x_1 = \dfrac{1+3i}{2}x_2$. Taking $x_2 = 2$ yields the eigenvector $\mathbf{x} = \begin{bmatrix} 1+3i \\ 2 \end{bmatrix}$. Taking real and imaginary parts of $e^{2t}(\cos t + i\sin t)\begin{bmatrix} 1+3i \\ 2 \end{bmatrix}$ yields

$$\mathbf{y} = c_1e^{2t}\begin{bmatrix} \cos t - 3\sin t \\ 2\cos t \end{bmatrix} + c_2e^{2t}\begin{bmatrix} \sin t + 3\cos t \\ 2\sin t \end{bmatrix}.$$

10.6.12. $\begin{vmatrix} 34-\lambda & 52 \\ -20 & -30-\lambda \end{vmatrix} = (\lambda-2)^2+16$. The augmented matrix of $(A-(2+4i)\,I)\,\mathbf{x} = \mathbf{0}$ is $\left[\begin{array}{cccc} 32-4i & 52 & \vdots & 0 \\ -20 & -32-4i & \vdots & 0 \end{array}\right]$, which is row equivalent to $\left[\begin{array}{cccc} 1 & \frac{8+i}{5} & \vdots & 0 \\ 0 & 0 & \vdots & 0 \end{array}\right]$. Therefore,$x_1 = -\dfrac{(8+i)}{5}x_2$. Taking $x_2 = 5$ yields the eigenvector $\mathbf{x} = \begin{bmatrix} -8-i \\ 5 \end{bmatrix}$. Taking real and imaginary parts of $e^{2t}(\cos 4t + i\sin 4t)\begin{bmatrix} -8-i \\ 5 \end{bmatrix}$ yields $\mathbf{y} = c_1e^{2t}\begin{bmatrix} \sin 4t - 8\cos 4t \\ 5\cos 4t \end{bmatrix} + c_2e^{2t}\begin{bmatrix} -\cos 4t - 8\sin 4t \\ 5\sin 4t \end{bmatrix}$.

10.6.14. $\begin{vmatrix} 3-\lambda & -4 & -2 \\ -5 & 7-\lambda & -8 \\ -10 & 13 & -8-\lambda \end{vmatrix} = -(\lambda+2)\left((\lambda-2)^2+9\right)$. The augmented matrix of $(A + 2I)\mathbf{x} = \mathbf{0}$ is $\left[\begin{array}{ccccc} 5 & -4 & -2 & \vdots & 0 \\ -5 & 9 & -8 & \vdots & 0 \\ -10 & 13 & -6 & \vdots & 0 \end{array}\right]$, which is row equivalent to $\left[\begin{array}{ccccc} 1 & 0 & -2 & \vdots & 0 \\ 0 & 1 & -2 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{array}\right]$. Therefore,$x_1 = x_2 = 2x_3$. Taking $x_3 = 1$ yields $\mathbf{y}_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}e^{-2t}$. The augmented matrix of $(A-(2+3i)I)\,\mathbf{x} = \mathbf{0}$ is $\left[\begin{array}{ccccc} 1-3i & -4 & -2 & \vdots & 0 \\ -5 & 5-3i & -8 & \vdots & 0 \\ -10 & 13 & -10-3i & \vdots & 0 \end{array}\right]$, which is row equivalent to $\left[\begin{array}{ccccc} 1 & 0 & 1-i & \vdots & 0 \\ 0 & 1 & -i & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{array}\right]$. Therefore,$x_1 = -(1-i)x_3$ and $x_2 = ix_3$. Taking $x_3 = 1$ yields the eigenvector $\mathbf{x}_2 = \begin{bmatrix} -1+i \\ i \\ 1 \end{bmatrix}$. The real and imaginary parts of $e^{2t}(\cos 3t + i\sin 3t)\begin{bmatrix} -1+i \\ i \\ 1 \end{bmatrix}$ are $\mathbf{y}_2 = e^{2t}\begin{bmatrix} -\cos 3t - \sin 3t \\ -\sin 3t \\ \cos 3t \end{bmatrix}$ and

$\mathbf{y}_3 = c_3 e^{2t} \begin{bmatrix} -\sin 3t + \cos 3t \\ \cos 3t \\ \sin 3t \end{bmatrix}$. Therefore,

$$\mathbf{y} = c_1 \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} e^{-2t} + c_2 e^{2t} \begin{bmatrix} -\cos 3t - \sin 3t \\ -\sin 3t \\ \cos 3t \end{bmatrix} + c_3 e^{2t} \begin{bmatrix} -\sin 3t + \cos 3t \\ \cos 3t \\ \sin 3t \end{bmatrix}.$$

10.6.16. $\begin{vmatrix} 1-\lambda & 2 & -2 \\ 0 & 2-\lambda & -1 \\ 1 & 0 & -\lambda \end{vmatrix} = -(\lambda-2)\left((\lambda-1)^2+1\right)$. The augmented matrix of $(A-I)\mathbf{x} = \mathbf{0}$ is $\begin{bmatrix} 0 & 2 & -2 & \vdots & 0 \\ 0 & 1 & -1 & \vdots & 0 \\ 1 & 0 & -1 & \vdots & 0 \end{bmatrix}$, which is row equivalent to $\begin{bmatrix} 1 & 0 & -1 & \vdots & 0 \\ 0 & 1 & -1 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$. Therefore,$x_1 = x_2 = 1$.

Taking $x_3 = 1$ yields $\mathbf{y}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^t$. The augmented matrix of $(A-(1+i)I)\,\mathbf{x} = \mathbf{0}$ is $\begin{bmatrix} -i & 2 & -2 & \vdots & 0 \\ 0 & 1-i & -1 & \vdots & 0 \\ 1 & 0 & -1-i & \vdots & 0 \end{bmatrix}$, which is row equivalent to $\begin{bmatrix} 1 & 0 & -1-i & \vdots & 0 \\ 0 & 1 & -\frac{1+i}{2} & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$. Therefore,$x_1 = (1+i)x_3$ and $x_2 = \dfrac{(1+i)}{2}x_3$.

Taking $x_3 = 2$ yields the eigenvector $\mathbf{x}_2 = \begin{bmatrix} 2+2i \\ 1+i \\ 2 \end{bmatrix}$. The real and imaginary parts of $e^{4t}(\cos t + i \sin t) \begin{bmatrix} 2+2i \\ 1+i \\ 2 \end{bmatrix}$ are $\mathbf{y}_2 = e^t \begin{bmatrix} 2\cos t - 2\sin t \\ \cos t - \sin t \\ 2\cos t \end{bmatrix}$ and $\mathbf{y}_3 = c_3 e^t \begin{bmatrix} 2\sin t + 2\cos t \\ \cos t + \sin t \\ 2\sin t \end{bmatrix}$. Therefore,

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^t + c_2 e^t \begin{bmatrix} 2\cos t - 2\sin t \\ \cos t - \sin t \\ 2\cos t \end{bmatrix} + c_3 e^t \begin{bmatrix} 2\sin t + 2\cos t \\ \cos t + \sin t \\ 2\sin t \end{bmatrix}.$$

10.6.18. $\begin{vmatrix} 7-\lambda & 15 \\ -3 & 1-\lambda \end{vmatrix} = (\lambda-4)^2 + 36$. The augmented matrix of $(A-(4+6i)\,I)\,\mathbf{x} = \mathbf{0}$ is $\begin{bmatrix} 3-6i & 15 & \vdots & 0 \\ -3 & -3-6i & \vdots & 0 \end{bmatrix}$, which is row equivalent to $\begin{bmatrix} 1 & 1+2i & \vdots & 0 \\ 0 & 0 & \vdots & 0 \end{bmatrix}$. Therefore,$x_1 = -(1+2i)x_2$. Taking $x_2 = 1$ yields the eigenvector $\mathbf{x} = \begin{bmatrix} -1-2i \\ 1 \end{bmatrix}$. Taking real and imaginary parts of $e^{4t}(\cos 6t + i \sin 6t) \begin{bmatrix} -1-2i \\ 1 \end{bmatrix}$ yields $\mathbf{y} = c_1 e^{4t} \begin{bmatrix} 2\sin 6t - \cos 6t \\ \cos 6t \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} -2\cos 6t - \sin 6t \\ \sin 6t \end{bmatrix}$.

Now $\mathbf{y}(0) = \begin{bmatrix} 5 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$, so $c_1 = 1, c_2 = -3$, and $\mathbf{y} = e^{4t} \begin{bmatrix} 5\cos 6t + 5\sin 6t \\ \cos 6t - 3\sin 6t \end{bmatrix}$.

10.6.20. $\frac{1}{6}\begin{vmatrix} 4-6\lambda & -2 \\ 5 & 2-6\lambda \end{vmatrix} = \left(\lambda-\frac{1}{2}\right)^2+\frac{1}{4}$. The augmented matrix of $\left(A-\frac{1+i}{2}I\right)\mathbf{x}=\mathbf{0}$ is $\frac{1}{6}\left[\begin{array}{cccc} 1-3i & -2 & \vdots & 0 \\ 5 & -1-3i & \vdots & 0 \end{array}\right]$, which is row equivalent to $\left[\begin{array}{cccc} 1 & -\frac{1+3i}{5} & \vdots & 0 \\ 0 & 0 & \vdots & 0 \end{array}\right]$. Therefore, $x_1 = \frac{1+3i}{5}x_2$. Taking $x_2=5$ yields the eigenvector $\mathbf{x}=\begin{bmatrix} 1+3i \\ 5 \end{bmatrix}$. Taking real and imaginary parts of $e^{t/2}(\cos t/2+i\sin t/2)\begin{bmatrix} 1+3i \\ 5 \end{bmatrix}$ yields $\mathbf{y}=c_1e^{t/2}\begin{bmatrix} \cos t/2-3\sin t/2 \\ 5\cos t/2 \end{bmatrix}+c_2e^{t/2}\begin{bmatrix} \sin t/2+3\cos t/2 \\ 5\sin t/2 \end{bmatrix}$. Now $\mathbf{y}(0)=\begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 3 \\ 5 & 0 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}=\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, so $c_1=-\frac{1}{5}$, $c_2=\frac{2}{5}$, and $\mathbf{y}=e^{t/2}\begin{bmatrix} \cos(t/2)+\sin(t/2) \\ -\cos(t/2)+2\sin(t/2) \end{bmatrix}$.

10.6.22. $\begin{vmatrix} 4-\lambda & 4 & 0 \\ 8 & 10-\lambda & -20 \\ 2 & 3 & -2-\lambda \end{vmatrix} = -(\lambda-8)\left((\lambda-2)^2+4\right)$. The augmented matrix of $(A-8I)\mathbf{x}=\mathbf{0}$ is $\left[\begin{array}{ccccc} 0 & 4 & 0 & \vdots & 0 \\ 8 & 6 & -20 & \vdots & 0 \\ 2 & 3 & -6 & \vdots & 0 \end{array}\right]$, which is row equivalent to $\left[\begin{array}{ccccc} 1 & 0 & -2 & \vdots & 0 \\ 0 & 1 & -2 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{array}\right]$. Therefore, $x_1 = x_2 = 2x_3$. Taking $x_3=2$ yields $\mathbf{y}_1=\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}e^{8t}$. The augmented matrix of $(A-(2+2i)I)\mathbf{x}=\mathbf{0}$ is $\left[\begin{array}{ccccc} 2-2i & 4 & 0 & \vdots & 0 \\ 8 & 8-2i & -20 & \vdots & 0 \\ 2 & 3 & -4-2i & \vdots & 0 \end{array}\right]$, which is row equivalent to $\left[\begin{array}{ccccc} 1 & 0 & -2+2i & \vdots & 0 \\ 0 & 1 & -2i & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{array}\right]$. Therefore, $x_1 = (2-2i)x_3$ and $x_2=2ix_3$. Taking $x_3=1$ yields the eigenvector $\mathbf{x}_2=\begin{bmatrix} 2-2i \\ 2i \\ 1 \end{bmatrix}$. The real and imaginary parts of $e^{2t}(\cos 2t+i\sin 2t)\begin{bmatrix} 2-2i \\ 2i \\ 1 \end{bmatrix}$ are $\mathbf{y}_2=e^{2t}\begin{bmatrix} 2\cos 2t+2\sin 2t \\ -2\sin 2t \\ 2\cos 2t \end{bmatrix}$ and $\mathbf{y}_3=e^{2t}\begin{bmatrix} 2\sin 2t+2\cos 2t \\ 2\cos 2t \\ \sin 2t \end{bmatrix}$, so the general solution is $\mathbf{y}=c_1\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}e^{8t}+c_2e^{2t}\begin{bmatrix} 2\cos 2t+2\sin 2t \\ -2\sin 2t \\ 2\cos 2t \end{bmatrix}+c_3e^{2t}\begin{bmatrix} 2\sin 2t+2\cos 2t \\ 2\cos 2t \\ \sin 2t \end{bmatrix}$. Now $\mathbf{y}(0)=\begin{bmatrix} 8 \\ 6 \\ 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 2 & -2 \\ 2 & 0 & 2 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}=\begin{bmatrix} 8 \\ 6 \\ 5 \end{bmatrix}$, so $c_1=2$, $c_2=3$, $c_3=1$, and $\mathbf{y}=\begin{bmatrix} 4 \\ 4 \\ 2 \end{bmatrix}e^{8t}+e^{2t}\begin{bmatrix} 4\cos 2t+8\sin 2t \\ -6\sin 2t+2\cos 2t \\ 3\cos 2t+\sin 2t \end{bmatrix}$.

10.6.24. $\begin{vmatrix} 4-\lambda & -4 & 4 \\ -10 & 3-\lambda & 15 \\ 2 & -3 & 1-\lambda \end{vmatrix} = -(\lambda-8)(\lambda^2+16)$. The augmented matrix of $(A-8I)\mathbf{x}=\mathbf{0}$ is

$\left[\begin{array}{rrr|r} -4 & -4 & 4 & 0 \\ -10 & -5 & 15 & 0 \\ 2 & -3 & -7 & 0 \end{array}\right]$, which is row equivalent to $\left[\begin{array}{rrr|r} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$. Therefore,$x_1 = 2x_3$ and $x_2 = -x_3$. Taking $x_3 = 1$ yields $\mathbf{y}_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} e^{8t}$. The augmented matrix of $(A - 4iI)\mathbf{x} = \mathbf{0}$ is $\left[\begin{array}{ccc|c} 4-4i & -4 & 4 & 0 \\ -10 & 3-4i & 15 & 0 \\ 2 & -3 & 1-4i & 0 \end{array}\right]$, which is row equivalent to $\left[\begin{array}{ccc|c} 1 & 0 & -1+i & 0 \\ 0 & 1 & -1+2i & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$. Therefore,$x_1 = (1-i)x_3$ and $x_2 = (1-2i)x_3$. Taking $x_3 = 1$ yields the eigenvector $\mathbf{x}_2 = \begin{bmatrix} 1-i \\ 1-2i \\ 1 \end{bmatrix}$. The real and imaginary parts of $(\cos 4t + i \sin 4t) \begin{bmatrix} 1-i \\ 1-2i \\ 1 \end{bmatrix}$ are $\mathbf{y}_2 = \begin{bmatrix} \cos 4t + \sin 4t \\ \cos 4t + 2\sin 4t \\ \cos 4t \end{bmatrix}$ and $\mathbf{y}_3 = \begin{bmatrix} \sin 4t - \cos 4t \\ \sin 4t - 2\cos 4t \\ \sin 4t \end{bmatrix}$, so the general solution is $\mathbf{y} = c_1 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} e^{8t} + c_2 \begin{bmatrix} \cos 4t + \sin 4t \\ \cos 4t + 2\sin 4t \\ \cos 4t \end{bmatrix} + c_3 \begin{bmatrix} \sin 4t - \cos 4t \\ \sin 4t - 2\cos 4t \\ \sin 4t \end{bmatrix}$. Now $\mathbf{y}(0) = \begin{bmatrix} 16 \\ 14 \\ 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 1 & -1 \\ -1 & 1 & -2 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 16 \\ 14 \\ 6 \end{bmatrix}$, so $c_1 = 3$, $c_2 = 3$, $c_3 = -7$, and $\mathbf{y} = \begin{bmatrix} 6 \\ -3 \\ 3 \end{bmatrix} e^{8t} + \begin{bmatrix} 10\cos 4t - 4\sin 4t \\ 17\cos 4t - \sin 4t \\ 3\cos 4t - 7\sin 4t \end{bmatrix}$.

10.6.28. **(a)** From the quadratic formula the roots are

$$\begin{aligned} k_1 &= \frac{\|\mathbf{u}\|^2 - \|\mathbf{v}^2\| + \sqrt{(\|\mathbf{u}\|^2 - \|\mathbf{v}^2\|)^2 + 4(\mathbf{u}, \mathbf{v})^2}}{2(\mathbf{u}, \mathbf{v})} \\ k_2 &= \frac{\|\mathbf{u}\|^2 - \|\mathbf{v}^2\| - \sqrt{(\|\mathbf{u}\|^2 - \|\mathbf{v}^2\|)^2 + 4(\mathbf{u}, \mathbf{v})^2}}{2(\mathbf{u}, \mathbf{v})}. \end{aligned}$$

Clearly $k_1 > 0$ and $k_2 < 0$. Moreover,

$$k_1 k_2 = \frac{(\|\mathbf{u}\|^2 - \|\mathbf{v}^2\|)^2 - \left[(\|\mathbf{u}\|^2 - \|\mathbf{v}^2\|)^2 + 4(\mathbf{u}, \mathbf{v})^2\right]}{4(\mathbf{u}, \mathbf{v})^2} = -1.$$

(b) Since $k_2 = -1/k_1$,

$$\begin{aligned} \mathbf{u}_1^{(2)} &= \mathbf{u} - k_2\mathbf{v} = \mathbf{u} + \frac{1}{k_1}\mathbf{v} = \frac{1}{k_1}(\mathbf{v} + k_1\mathbf{u}) = \frac{1}{k_1}\mathbf{v}_1^{(1)} \\ \mathbf{v}_1^{(2)} &= \mathbf{v} + k_2\mathbf{u} = \mathbf{v} - \frac{1}{k_1}\mathbf{u} = -\frac{1}{k_1}(\mathbf{u} - k_1\mathbf{v}) = -\frac{1}{k_1}\mathbf{u}_1^{(1)}. \end{aligned}$$

10.6.30. $\begin{vmatrix} -15-\lambda & 10 \\ -25 & 15-\lambda \end{vmatrix} = \lambda^2+25$. The augmented matrix of $(A-5iI)\mathbf{x} = \mathbf{0}$ is $\left[\begin{array}{cc|c} -15-5i & 10 & 0 \\ -25 & 15-5i & 0 \end{array}\right]$, which is row equivalent to $\left[\begin{array}{cc|c} 1 & \frac{-3+i}{5} & 0 \\ 0 & 0 & 0 \end{array}\right]$. Therefore, $x_1 = \dfrac{(3-i)}{5}x_2$. Taking $x_2 = 5$ yields the eigenvector $\mathbf{x} = \begin{bmatrix} 3-i \\ 5 \end{bmatrix}$, so $\mathbf{u} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$. The quadratic equation is $-3k^2-33k+3 = 0$, with positive root $k \approx .0902$. Routine calculations yield $\mathbf{U} \approx \begin{bmatrix} .5257 \\ .8507 \end{bmatrix}$, $\mathbf{V} \approx \begin{bmatrix} -.8507 \\ .5257 \end{bmatrix}$.

10.6.32. $\begin{vmatrix} -3-\lambda & -15 \\ 3 & 3-\lambda \end{vmatrix} = \lambda^2+36$. The augmented matrix of $(A-6iI)\mathbf{x} = \mathbf{0}$ is $\left[\begin{array}{cc|c} -3-6i & -15 & 0 \\ 3 & 3-6i & 0 \end{array}\right]$, which is row equivalent to $\left[\begin{array}{cc|c} 1 & 1-2i & 0 \\ 0 & 0 & 0 \end{array}\right]$. Therefore, $x_1 = -(1-2i)x_2$. Taking $x_2 = 1$ yields the eigenvector $\mathbf{x} = \begin{bmatrix} -1+2i \\ 1 \end{bmatrix}$, so $\mathbf{u} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$. The quadratic equation is $-2k^2 + 2k + 2 = 0$, with positive root $k \approx 1.6180$. Routine calculations yield $\mathbf{U} \approx \begin{bmatrix} -.9732 \\ .2298 \end{bmatrix}$, $\mathbf{V} \approx \begin{bmatrix} .2298 \\ .9732 \end{bmatrix}$.

10.6.34. $\begin{vmatrix} 5-\lambda & -12 \\ 6 & -7-\lambda \end{vmatrix} = (\lambda+1)^2 + 36$. The augmented matrix of $(A-(-1+6i)I)\mathbf{x} = \mathbf{0}$ is $\left[\begin{array}{cc|c} 6-6i & -12 & 0 \\ 6 & -6-6i & 0 \end{array}\right]$, which is row equivalent to $\left[\begin{array}{cc|c} 1 & -(1+i) & 0 \\ 0 & 0 & 0 \end{array}\right]$. Therefore, $x_1 = (1+i)x_2$. Taking $x_2 = 1$ yields the eigenvector $\mathbf{x} = \begin{bmatrix} 1+i \\ 1 \end{bmatrix}$, so $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. The quadratic equation is $k^2 - k - 1 = 0$, with positive root $k \approx 1.6180$. Routine calculations yield $\mathbf{U} \approx \begin{bmatrix} -.5257 \\ .8507 \end{bmatrix}$, $\mathbf{V} \approx \begin{bmatrix} .8507 \\ .5257 \end{bmatrix}$.

10.6.36. $\begin{vmatrix} -4-\lambda & 9 \\ -5 & 2-\lambda \end{vmatrix} = (\lambda+1)^2 + 36$. The augmented matrix of $(A-(-1+6i)I)\mathbf{x} = \mathbf{0}$ is $\left[\begin{array}{cc|c} -3-6i & 9 & 0 \\ -5 & 3-6i & 0 \end{array}\right]$, which is row equivalent to $\left[\begin{array}{cc|c} 1 & -\frac{3-6i}{5} & 0 \\ 0 & 0 & 0 \end{array}\right]$. Therefore, $x_1 = \dfrac{3-6i}{5}x_2$. Taking $x_2 = 5$ yields the eigenvector $\mathbf{x} = \begin{bmatrix} 3-6i \\ 5 \end{bmatrix}$, so $\mathbf{u} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -6 \\ 0 \end{bmatrix}$. The quadratic equation is $-18k^2 + 2k + 18 = 0$, with positive root $k \approx 1.0571$. Routine calculations yield $\mathbf{U} \approx \begin{bmatrix} .8817 \\ .4719 \end{bmatrix}$, $\mathbf{V} \approx \begin{bmatrix} -.4719 \\ .8817 \end{bmatrix}$.

10.6.38. $\begin{vmatrix} -1-\lambda & -5 \\ 20 & -1-\lambda \end{vmatrix} = (\lambda+1)^2 + 100$. The augmented matrix of $(A-(-1+10i)I)\mathbf{x} = \mathbf{0}$

is $\left[\begin{array}{cc:c} -10i & -5 & 0 \\ 20 & -10i & 0 \end{array}\right]$, which is row equivalent to $\left[\begin{array}{cc:c} 1 & -\frac{i}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$. Therefore,$x_1 = \dfrac{i}{2}x_2$. Taking $x_2 = 2$ yields the eigenvector $\mathbf{x} = \begin{bmatrix} i \\ 2 \end{bmatrix}$, so $\mathbf{u} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Since $(\mathbf{u}, \mathbf{v}) = 0$ we just normalize $\mathbf{u}$ and $\mathbf{v}$ to obtain $\mathbf{U} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\mathbf{V} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

10.6.40. $\begin{vmatrix} -7-\lambda & 6 \\ -12 & 5-\lambda \end{vmatrix} = (\lambda + 1)^2 + 36$. The augmented matrix of $(A - (-1 + 6i)I)\mathbf{x} = \mathbf{0}$ is $\left[\begin{array}{cc:c} -6-6i & 6 & 0 \\ -12 & 6-6i & 0 \end{array}\right]$, which is row equivalent to $\left[\begin{array}{cc:c} 1 & -\frac{1-i}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$. Therefore,$x_1 = \dfrac{1-i}{2}x_2$. Taking $x_2 = 2$ yields the eigenvector $\mathbf{x} = \begin{bmatrix} 1-i \\ 2 \end{bmatrix}$, so $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$. The quadratic equation is $-k^2 - 4k + 1 = 0$, with positive root $k \approx .2361$. Routine calculations yield $\mathbf{U} \approx \begin{bmatrix} .5257 \\ .8507 \end{bmatrix}$, $\mathbf{V} \approx \begin{bmatrix} -.8507 \\ .5257 \end{bmatrix}$.

10.7 VARIATION OF PARAMETERS FOR NONHOMOGENEOUS LINEAR SYSTEMS

10.7.2. $Y = \begin{bmatrix} -3e^{-t} & -e^{-2t} \\ e^{-t} & 2e^{-2t} \end{bmatrix}$; $\mathbf{u}' = Y^{-1}\mathbf{f} = \dfrac{1}{5}\begin{bmatrix} -2e^{t} & -e^{t} \\ e^{2t} & 3e^{2t} \end{bmatrix}\begin{bmatrix} 50e^{37} \\ 10e^{-3t} \end{bmatrix} = \begin{bmatrix} -20e^{4t} - 2e^{-2t} \\ 10e^{5t} + 6e^{-t} \end{bmatrix}$; $\mathbf{u} = \begin{bmatrix} e^{-2t} - 5e^{4t} \\ 2e^{5t} - 6e^{-t} \end{bmatrix}$; $\mathbf{y}_p = Y\mathbf{u} = \begin{bmatrix} 13e^{3t} + 3e^{-3t} \\ -e^{3t} - 11e^{-3t} \end{bmatrix}$.

10.7.4. $Y = \begin{bmatrix} -e^{2t} & -e^{-t} \\ 2e^{2t} & e^{-t} \end{bmatrix}$; $\mathbf{u}' = Y^{-1}\mathbf{f} = \begin{bmatrix} e^{-2t} & e^{-2t} \\ -2e^{t} & -e^{t} \end{bmatrix}\begin{bmatrix} 2 \\ -2e^{t} \end{bmatrix} = \begin{bmatrix} 2e^{-2t} - 2e^{-t} \\ 2e^{2t} - 4e^{t} \end{bmatrix}$; $\mathbf{u} = \begin{bmatrix} 2e^{-t} - e^{-2t} \\ e^{2t} - 4e^{t} \end{bmatrix}$; $\mathbf{y}_p = Y\mathbf{u} = \begin{bmatrix} 5 - 3e^{t} \\ 5e^{t} - 6 \end{bmatrix}$.

10.7.6. $Y = \begin{bmatrix} \sin t & -\cos t \\ \cos t & \sin t \end{bmatrix}$; $\mathbf{u}' = Y^{-1}\mathbf{f} = \begin{bmatrix} \sin t & \cos t \\ -\cos t & \sin t \end{bmatrix}\begin{bmatrix} 1 \\ t \end{bmatrix} = \begin{bmatrix} t\cos t + \sin t \\ t\sin t - \cos t \end{bmatrix}$; $\mathbf{u} = \begin{bmatrix} t\sin t \\ -t\cos t \end{bmatrix}$; $\mathbf{y}_p = Y\mathbf{u} = \begin{bmatrix} t \\ 0 \end{bmatrix}$.

10.7.8. $Y = \begin{bmatrix} e^{3t} & e^{2t} & e^{-t} \\ -e^{3t} & 0 & -3e^{-t} \\ e^{3t} & e^{2t} & 7e^{-t} \end{bmatrix}$; $\mathbf{u}' = Y^{-1}\mathbf{f} = \dfrac{1}{6}\begin{bmatrix} 3e^{-3t} & -6e^{-3t} & -3e^{-3t} \\ 4e^{-2t} & 6e^{-2t} & 2e^{-2t} \\ -e^{t} & 0 & e^{t} \end{bmatrix}\begin{bmatrix} 1 \\ e^{t} \\ e^{t} \end{bmatrix} = \dfrac{1}{6}\begin{bmatrix} 3e^{-3t} - 9e^{-2t} \\ 8e^{-t} + 4e^{-2t} \\ e^{2t} - e^{t} \end{bmatrix}$; $\mathbf{u} = \dfrac{1}{12}\begin{bmatrix} 9e^{-2t} - 2e^{-3t} \\ -16e^{-t} - 4e^{-2t} \\ e^{2t} - 2e^{t} \end{bmatrix}$; $\mathbf{y}_p = Y\mathbf{u} = -\dfrac{1}{6}\begin{bmatrix} 3e^{t} + 4 \\ 6e^{t} - 4 \\ 10 \end{bmatrix}$.

10.7.10. $Y = \begin{bmatrix} -e^{t} & e^{-t} & te^{-t} \\ e^{t} & -e^{-t} & 3e^{-t} - te^{-t} \\ e^{t} & e^{-t} & te^{-t} \end{bmatrix}$; $\mathbf{u}' = Y^{-1}\mathbf{f} = \dfrac{1}{18}\begin{bmatrix} -9e^{-t} & 0 & 9e^{-t} \\ 3e^{t}(3-2t) & -6te^{t} & 9e^{t} \\ 6e^{t} & 6e^{t} & 0 \end{bmatrix}\begin{bmatrix} e^{t} \\ e^{t} \\ e^{t} \end{bmatrix} = \dfrac{1}{3}\begin{bmatrix} 0 \\ e^{2t}(3-2t) \\ 2e^{2t} \end{bmatrix}$; $\mathbf{u} = \dfrac{1}{3}\begin{bmatrix} 0 \\ e^{2t}(2-t) \\ e^{2t} \end{bmatrix}$; $\mathbf{y}_p = Y\mathbf{u} = \dfrac{1}{3}\begin{bmatrix} 2e^{t} \\ e^{t} \\ 2e^{t} \end{bmatrix}$.

10.7.12. $\mathbf{u}' = Y^{-1}\mathbf{f} = \frac{1}{2t}\begin{bmatrix} e^{-t} & e^{-t} \\ e^{t} & -e^{t} \end{bmatrix}\begin{bmatrix} t \\ t^2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} e^{-t}(t+1) \\ e^{t}(1-t) \end{bmatrix}$; $\mathbf{u} = \frac{1}{2}\begin{bmatrix} -e^{-t}(t+2) \\ e^{t}(2-t) \end{bmatrix}$;
$\mathbf{y}_p = Y\mathbf{u} = \frac{t}{2}\begin{bmatrix} e^{t} & e^{-t} \\ e^{t} & -e^{-t} \end{bmatrix}\begin{bmatrix} -e^{-t}(t+2) \\ e^{t}(2-t) \end{bmatrix} = -\begin{bmatrix} t^2 \\ 2t \end{bmatrix}$.

10.7.14. $\mathbf{u}' = Y^{-1}\mathbf{f} = \frac{1}{3}\begin{bmatrix} 2 & -e^{-t} \\ -e^{t} & 2 \end{bmatrix}\begin{bmatrix} e^{2t} \\ e^{-2t} \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2e^{2t} - e^{-3t} \\ 2e^{-2t} - e^{3t} \end{bmatrix}$; $\mathbf{u} = \frac{1}{9}\begin{bmatrix} 3e^{2t} + e^{-3t} \\ -e^{3t} - 3e^{-2t} \end{bmatrix}$;
$\mathbf{y}_p = Y\mathbf{u} = \frac{1}{9}\begin{bmatrix} 2 & e^{-t} \\ e^{t} & 2 \end{bmatrix}\begin{bmatrix} 3e^{2t} + e^{-3t} \\ -e^{3t} - 3e^{-2t} \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 5e^{2t} - e^{-3t} \\ e^{3t} - 5e^{-2t} \end{bmatrix}$.

10.7.16. $\mathbf{u}' = Y^{-1}\mathbf{f} = \frac{1}{t^2-1}\begin{bmatrix} t & -e^{-t} \\ -e^{t} & t \end{bmatrix}\begin{bmatrix} t^2-1 \\ t^2-1 \end{bmatrix} = \begin{bmatrix} t - e^{-t} \\ t - e^{t} \end{bmatrix}$; $\mathbf{u} = \frac{1}{2}\begin{bmatrix} 2e^{-t} + t^2 \\ t^2 - 2e^{t} \end{bmatrix}$;
$\mathbf{y}_p = Y\mathbf{u} = \frac{1}{2}\begin{bmatrix} t & e^{-t} \\ e^{t} & t \end{bmatrix}\begin{bmatrix} 2e^{-t} + t^2 \\ t^2 - 2e^{t} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} te^{-t}(t+2) + t^3 - 2 \\ te^{t}(t-2) + t^3 + 2 \end{bmatrix}$.

10.7.18. $\mathbf{u}' = Y^{-1}\mathbf{f} = \frac{1}{3}\begin{bmatrix} e^{-5t} & e^{-4t} & e^{-3t} \\ 2e^{-2t} & -e^{-t} & -1 \\ -e^{-2t} & 2e^{-t} & -1 \end{bmatrix}\begin{bmatrix} e^{t} \\ 0 \\ 0 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} e^{-4t} \\ 2e^{-t} \\ -e^{-t} \end{bmatrix}$; $\mathbf{u} = \frac{1}{12}\begin{bmatrix} -e^{-4t} \\ -8e^{-t} \\ 4e^{-t} \end{bmatrix}$;
$\mathbf{y}_p = Y\mathbf{u} = \frac{1}{12}\begin{bmatrix} e^{5t} & e^{2t} & 0 \\ e^{4t} & 0 & e^{t} \\ e^{3t} & -1 & -1 \end{bmatrix}\begin{bmatrix} -e^{-4t} \\ -8e^{-t} \\ 4e^{-t} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -3e^{t} \\ 1 \\ e^{-t} \end{bmatrix}$.

10.7.20. $\mathbf{u}' = Y^{-1}\mathbf{f} = \frac{1}{2t}\begin{bmatrix} -e^{-2t} & te^{-t} & te^{-t} + e^{-2t} \\ 1 & -te^{t} & te^{t} - 1 \\ 2 & 0 & -2 \end{bmatrix}\begin{bmatrix} e^{t} \\ 0 \\ e^{t} \end{bmatrix} = \begin{bmatrix} 1 \\ e^{2t} \\ 0 \end{bmatrix}$; $\mathbf{u} = \frac{1}{4}\begin{bmatrix} 2t \\ e^{2t} \\ 0 \end{bmatrix}$;
$\mathbf{y}_p = Y\mathbf{u} = Y = \frac{1}{4t}\begin{bmatrix} e^{t} & e^{-t} & t \\ e^{t} & -e^{-t} & e^{-t} \\ e^{t} & e^{-t} & 0 \end{bmatrix}\begin{bmatrix} 2t \\ e^{2t} \\ 0 \end{bmatrix} = \frac{e^{t}}{4t}\begin{bmatrix} 2t+1 \\ 2t-1 \\ 2t+1 \end{bmatrix}$.

10.7.22. **(c)** If $\mathbf{y}_p = Y\mathbf{u}$, then $\mathbf{y}_p' = Y'\mathbf{u} + Y\mathbf{u}' = AY\mathbf{u} + Y\mathbf{u}'$, so (E) $\mathbf{y}_p' = A\mathbf{y_p} + Y\mathbf{u}'$. However, from the derivation of the method of variation of parameters in Section 9.4, $Y\mathbf{u}' = \mathbf{f}$ as defined in the solution of **(a)**. This and (E) imply the conclusion.

(d) Since $Y\mathbf{u}' = \mathbf{f}$ with $\mathbf{f}$ as defined in the solution of **(a)**, $u_1, u_2, \dots, u_n$ satisfy the conditions required in the derivation of the method of variation of parameters in Section 9.4; hence, $y_p = c_1y_1 + c_2y_2 + \cdots + c_ny_n$ is a particular solution of (A).

CHAPTER 11
Boundary Value Problems and Fourier Expansions

11.1 EIGENVALUE PROBLEMS FOR $y'' + \lambda y = 0$

11.1.2. From Theorem 11.1.2 with $L = \pi$, $\lambda_n = n^2$, $y_n = \sin nx$, $n = 1, 2, 3, \dots$

11.1.4. From Theorem 11.1.4 with $L = \pi$, $\lambda_n = \dfrac{(2n-1)^2}{4}$, $y_n = \sin \dfrac{(2n-1)x}{2}$, $n = 1, 2, 3, \dots,$

11.1.6. From Theorem 11.1.6 with $L = \pi$, $\lambda_0 = 0$, $y_0 = 1$, $\lambda_n = n^2$, $y_{1n} = \cos nx$, $y_{2n} = \sin nx$, $n = 1, 2, 3, \dots$

11.1.8. From Theorem 11.1.5 with $L = 1$, $\lambda_n = \dfrac{(2n-1)^2\pi^2}{4}$, $y_n = \cos \dfrac{(2n-1)\pi x}{2}$, $n = 1, 2, 3, \dots$

11.1.10. From Theorem 11.1.6 with $L = 1$, $\lambda_0 = 0$, $y_0 = 1$, $\lambda_n = n^2\pi^2$, $y_{1n} = \cos n\pi x$, $y_{2n} = \sin n\pi x$, $n = 1, 2, 3, \dots$

11.1.12. From Theorem 11.1.6 with $L = 2$, $\lambda_0 = 0$, $y_0 = 1$, $\lambda_n = \dfrac{n^2\pi^2}{4}$, $y_{1n} = \cos \dfrac{n\pi x}{2}$, $y_{2n} = \sin \dfrac{n\pi x}{2}$, $n = 1, 2, 3, \dots$

11.1.14. From Theorem 11.1.5 with $L = 3$, $\lambda_n = \dfrac{(2n-1)^2\pi^2}{36}$, $y_n = \cos \dfrac{(2n-1)\pi x}{6}$, $n = 1, 2, 3, \dots$

11.1.16. From Theorem 11.1.3 with $L = 5$, $\lambda_n = \dfrac{n^2\pi^2}{25}$, $y_n = \cos \dfrac{n\pi x}{5}$, $n = 1, 2, 3, \dots$

11.1.18. From Theorem 11.1.1, any eigenvalues of Problem 11.1.4 must be positive. If $\lambda > 0$, then every solution of $y'' + \lambda y = 0$ is of the form $y = c_1 \cos \sqrt{\lambda}\, x + c_2 \sin \sqrt{\lambda}\, x$ where c_1 and c_2 are constants. Therefore, $y' = \sqrt{\lambda}\,(-c_1 \sin \sqrt{\lambda}\, x + c_2 \cos \sqrt{\lambda}\, x)$. Since $y'(0) = 0$, $c_2 = 0$. Therefore, $y = c_1 \cos \sqrt{\lambda}\, x$. Since $y(L) = 0$, $c_1 \cos \sqrt{\lambda}\, L = 0$. To make $c_1 \cos \sqrt{\lambda}\, L = 0$ with $c_1 \neq 0$ we must choose $\sqrt{\lambda} = \dfrac{(2n-1)\pi}{2L}$, where n is a positive integer. Therefore,$\lambda_n = \dfrac{(2n-1)^2\pi^2}{4L^2}$ is an eigenvalue and $y_n = \cos \dfrac{(2n-1)\pi x}{2L}$ is an associated eigenfunction.

11.1.20. If r is a positive integer, then $\int_{-L}^{L} \cos\frac{r\pi x}{L}\,dx = \frac{L}{r\pi}\sin\frac{r\pi x}{L}\Big|_{-L}^{L} = 0$, so $y_0 = 1$ is orthogonal to all the other eigenfunctions. If m and n are distinct positive integers, then $\int_0^L \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx = \frac{1}{2}\int_{-L}^{L}\cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx = 0$, from Example 11.1.4.

11.1.22. Let m and n be distinct positive integers. From the identity $\cos A\cos B = \frac{1}{2}[\cos(A-B)+\cos(A+B)]$ with $A=(2m-1)\pi x/2L$ and $B=(2n-1)\pi x/2L$,

$$\int_0^L \cos\frac{(2m-1)\pi x}{2L}\cos\frac{(2n-1)\pi x}{2L}\,dx = \frac{1}{2}\int_0^L\left[\cos\frac{(m-n)\pi x}{L}+\cos\frac{(m+n-1)\pi x}{L}\right]dx = 0.$$

11.1.24. If $y = c_1 + c_2x$, then $y'(0) = 0$ implies that $c_2 = 0$, so $y = c_1$. Now $\int_0^L y(x)\,dx = c_1\int_0^L dx = c_1L = 0$ only if $c_1 = 0$. Therefore,zero is not an eigenvalue.

If $y = c_1\cosh kx + c_2\sinh kx$, then $y'(0) = 0$ implies that $c_2 = 0$, so $y = c_1\cosh kx$. Now $\int_0^L y(x)\,dx = c_1\int_0^L \cosh kx\,dx = c_1\frac{\sinh kL}{k} = 0$ with $k>0$ only if $c_1 = 0$. Therefore, there are no negative eigenvalues.

If $y = c_1\cos kx + c_2\sin kx$, then $y'(0) = 0$ implies that $c_2 = 0$, so $y = c_1\cos kx$. Now $\int_0^L y(x)\,dx = c_1\int_0^L \cos kx\,dx = c_2\frac{\sin kL}{k} = 0$ if $k = \frac{n\pi}{L}$, where n is a positive integer. Therefore, $\lambda_n = \frac{n^2\pi^2}{L^2}$ and $y_n = \cos\frac{n\pi x}{L}$, $n = 1,2,3,\dots$.

11.1.26. If $y = c_1 + c_2(x-L)$, then $y'(L) = 0$ implies that $c_2 = 0$, so $y = c_1$. Now $\int_0^L y(x)\,dx = c_1\int_0^L dx = c_1L = 0$ only if $c_1 = 0$. Therefore,zero is not an eigenvalue.

If $y = c_1\cosh k(x-L) + c_2\sinh k(x-L)$, then $y'(L) = 0$ implies that $c_2 = 0$, so $y = c_1\cosh k(x-L)$. Now $\int_0^L y(x)\,dx = c_1\int_0^L \cosh k(x-L)\,dx = c_1\frac{\sinh kL}{k} = 0$ with $k>0$ only if $c_1 = 0$. Therefore,there are no negative eigenvalues.

If $y = c_1\cos k(x-L) + c_2\sin k(x-L)$, then $y'(L) = 0$ implies that $c_2 = 0$, so $y = c_1\cos k(x-L)$. Now $\int_0^L y(x)\,dx = c_1\int_0^L \cos k(x-L)\,dx = c_2\frac{\sin kL}{k} = 0$ if $k = \frac{n\pi}{L}$, where n is a positive integer. Therefore, $\lambda_n = \frac{n^2\pi^2}{L^2}$ and $y_n = \cos\frac{n\pi(x-L)}{L}$, or, equivalently, $y_n = \cos\frac{n\pi x}{L}$, $n = 1,2,3,\dots$.

11.2 FOURIER EXPANSIONS I

11.2.2.

$$
\begin{aligned}
a_0 &= \frac{1}{2}\int_{-1}^{1}(2-x)\,dx = \int_0^1 2\,dx = 2;\\
a_n &= \int_{-1}^{1}(2-x)\cos n\pi x\,dx = 4\int_0^1 \cos n\pi x\,dx = \frac{4}{n\pi}\sin n\pi x\Big|_0^1 = 0;\\
b_n &= \int_{-1}^{1}(2-x)\sin n\pi x\,dx = -2\int_0^1 x\sin n\pi x\,dx\\
&= \frac{2}{n\pi}\left[x\cos n\pi x\Big|_0^1 - \int_0^1 \cos n\pi x\,dx\right]\\
&= \frac{2}{n\pi}\left[\cos n\pi - \frac{1}{n\pi}\sin n\pi x\Big|_0^1\right] = (-1)^n\frac{2}{n\pi};
\end{aligned}
$$

$F(x) = 2 + \dfrac{2}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin n\pi x$. From Theorem 11.2.4,

$$
F(x) = \begin{cases} 2, & x = -1,\\ 2-x, & -1 < x < 1,\\ 2, & x = 1.\end{cases}
$$

11.2.4. Since f is even, $b_n = 0$ for $n \ge 1$; $a_0 = \displaystyle\int_0^1 (1-3x^2)\,dx = (x-x^3)\Big|_0^1 = 0$; if $n \ge 1$, then

$$
\begin{aligned}
a_n &= 2\int_0^1 (1-3x^2)\cos n\pi x\,dx = \frac{2}{n\pi}\left[(1-3x^2)\sin n\pi x\Big|_0^1 + 6\int_0^1 x\sin n\pi x\,dx\right]\\
&= -\frac{12}{n^2\pi^2}\left[x\cos n\pi x\Big|_0^1 - \int_0^1 \cos n\pi x\,dx\right]\\
&= -\frac{12}{n^2\pi^2}\left[\cos n\pi - \frac{1}{n\pi}\sin n\pi x\Big|_0^1\right] = (-1)^{n+1}\frac{12}{n^2\pi^2};
\end{aligned}
$$

$F(x) = -\dfrac{12}{\pi^2}\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{\cos n\pi x}{n^2}$. From Theorem 11.2.4, $F(x) = 1-3x^2$, $-1 \le x \le 1$.

11.2.6. Since f is odd, $a_n = 0$ if $n \ge 0$;

$$
\begin{aligned}
b_1 &= \frac{2}{\pi}\int_0^{\pi} x\cos x\sin x\,dx = \frac{1}{\pi}\int_0^{\pi} x\sin 2x\,dx\\
&= -\frac{1}{2\pi}\left[x\cos 2x\Big|_0^{\pi} - \int_0^{\pi}\cos 2x\,dx\right] = -\frac{1}{2\pi}\left[\pi - \frac{\sin 2x}{2}\Big|_0^{\pi}\right] = -\frac{1}{2}.
\end{aligned}
$$

if $n \geq 2$, then

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^\pi x\cos x\sin nx\,dx = \frac{1}{\pi}\int_0^\pi x[\sin(n+1)x + \sin(n-1)x]\,dx \\ &= -\frac{1}{\pi}\left[x\left[\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]\Bigg|_0^\pi - \int_0^\pi\left[\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]dx\right] \\ &= (-1)^n\left[\frac{1}{n+1} + \frac{1}{n-1}\right] + \frac{1}{\pi}\left[\frac{\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2}\right]\Bigg|_0^\pi = (-1)^n\frac{2n}{n^2-1}; \end{aligned}$$

$F(x) = -\dfrac{1}{2}\sin x + 2\displaystyle\sum_{n=2}^{\infty}(-1)^n\frac{n}{n^2-1}\sin nx$. From Theorem 11.2.4, $F(x) = x\cos x$, $-\pi \leq x \leq \pi$.

11.2.8. Since f is even, $b_n = 0$ if $n \geq 1$; $a_0 = \dfrac{1}{\pi}\displaystyle\int_0^\pi x\sin x\,dx = -\frac{1}{\pi}\left[x\cos x\Big|_0^\pi - \int_0^\pi \cos x\,dx\right] =$ $1 + \dfrac{\sin x}{\pi}\Big|_0^\pi = 1$; $a_1 = \dfrac{2}{\pi}\displaystyle\int_0^\pi x\sin x\cos x\,dx = \frac{1}{\pi}\int_0^\pi x\sin 2x\,dx = -\frac{1}{2\pi}\left[x\cos 2x\Big|_0^\pi - \int_0^\pi\cos 2x\,dx\right] =$ $-\dfrac{1}{2} + \dfrac{\sin 2x}{4\pi}\Big|_0^\pi = -\dfrac{1}{2}$; if $n \geq 2$, then

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^\pi x\sin x\cos nx\,dx = \frac{1}{\pi}\int_0^\pi x[\sin(n+1)x - \sin(n-1)x]\,dx \\ &= \frac{1}{\pi}\left[x\left[\frac{\cos(n-1)x}{n-1} - \frac{\cos(n+1)x}{n+1}\right]\Bigg|_0^\pi - \int_0^\pi\left[\frac{\cos(n-1)x}{n-1} - \frac{\cos(n+1)x}{n+1}\right]dx\right] \\ &= (-1)^{n+1}\left[\frac{1}{n-1} - \frac{1}{n+1}\right] - \frac{1}{\pi}\left[\frac{\sin(n-1)x}{(n-1)^2} - \frac{\sin(n+1)x}{(n+1)^2}\right]\Bigg|_0^\pi = (-1)^{n+1}\frac{2}{n^2-1}; \end{aligned}$$

$F(x) = 1 - \dfrac{1}{2}\cos x - 2\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2-1}\cos nx$. From Theorem 11.2.4, $F(x) = x\sin x$, $-\pi \leq x \leq \pi$.

11.2.10. Since f is even, $b_n = 0$ if $n \geq 0$; $a_0 = \displaystyle\int_0^{1/2}\cos\pi x\,dx = \frac{\sin\pi x}{\pi}\Big|_0^{1/2} = \frac{1}{\pi}$; $a_1 =$ $2\displaystyle\int_0^{1/2}\cos^2\pi x\,dx = \int_0^{1/2}(1-\cos 2\pi x)\,dx = \frac{1}{2} - \frac{\sin 2\pi x}{2\pi}\Big|_0^{1/2} = \frac{1}{2}$; if $n \geq 2$, then

$$\begin{aligned} a_n &= 2\int_0^{1/2}\cos\pi x\cos n\pi x\,dx = \int_0^{1/2}[\cos(n+1)x + \cos(n-1)x]\,dx \\ &= \frac{1}{\pi}\left[\frac{\sin(n+1)\pi x}{n+1} + \frac{\sin(n-1)\pi x}{n-1}\right]\Bigg|_0^{1/2} = \frac{1}{\pi}\left[\frac{1}{n+1} - \frac{1}{n-1}\right]\cos\frac{n\pi}{2} \\ &= -\frac{2}{(n^2-1)\pi}\cos\frac{n\pi}{2} = \begin{cases} (-1)^{m+1}\dfrac{2}{(4m^2-1)\pi} & \text{if } n = 2m, \\ 0 & \text{if } n = 2m+1; \end{cases} \end{aligned}$$

$F(x) = \dfrac{1}{\pi} + \dfrac{1}{2}\cos\pi x - \dfrac{2}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1}\cos 2n\pi x$. From Theorem 11.2.4, $F(x) = f(x)$, $-1 \leq x \leq 1$.

11.2.12. Since f is odd, $a_n = 0$ if $n \ge 0$; $b_1 = 2\int_0^{1/2} \sin^2 2\pi x\, dx = \int_0^{1/2} (1 - \cos 4\pi x)\, dx = \frac{1}{2} - \frac{\sin 4\pi x}{4\pi}\bigg|_0^{1/2} = \frac{1}{2}$; if $n \ge 2$, then

$$\begin{aligned}
b_n &= 2\int_0^{1/2} \sin \pi x \sin n\pi x\, dx = \int_0^{1/2} [\cos(n-1)\pi x - \cos(n+1)\pi x]\, dx \\
&= \frac{1}{\pi}\left[\frac{\sin(n-1)\pi x}{n-1} - \frac{\sin(n+1)\pi x}{n+1}\right]\bigg|_0^{1/2} \\
&= -\frac{2n}{(n^2-1)\pi}\cos\frac{n\pi}{2} = \begin{cases} (-1)^{m+1}\dfrac{4m}{4m^2-1} & \text{if } n = 2m, \\ 0 & \text{if } n = 2m+1; \end{cases}
\end{aligned}$$

$F(x) = \frac{1}{2}\sin \pi x - \frac{4}{\pi}\sum_{n=1}^{\infty}(-1)^n \frac{n}{4n^2-1}\sin 2n\pi x$. From Theorem 11.2.4,

$$F(x) = \begin{cases} 0, & -1 \le x < \frac{1}{2}, \\ -\frac{1}{2}, & x = -\frac{1}{2}, \\ \sin \pi x, & -\frac{1}{2} < x < \frac{1}{2}, \\ \frac{1}{2}, & x = \frac{1}{2}, \\ 0, & \frac{1}{2} < x \le 1. \end{cases}$$

11.2.14. Since f is even, $b_n = 0$ if $n \ge 1$;

$$a_0 = \int_0^{1/2} x \sin \pi x\, dx = -\frac{1}{\pi}\left[x \cos \pi x \bigg|_0^{1/2} - \int_0^{1/2} \cos \pi x\, dx\right] = \frac{\sin \pi x}{\pi^2}\bigg|_0^{1/2} = \frac{1}{\pi^2};$$

$$\begin{aligned}
a_1 &= 2\int_0^{1/2} x \sin \pi x \cos \pi x\, dx = \int_0^{1/2} x \sin 2x\, dx \\
&= -\frac{1}{2\pi}\left[x \cos 2\pi x \bigg|_0^{1/2} - \int_0^{1/2} \cos 2\pi x\, dx\right] = \frac{1}{4\pi} + \frac{\sin 2\pi x}{4\pi^2}\bigg|_0^{1/2} = \frac{1}{4\pi};
\end{aligned}$$

if $n \ge 2$, then

$$\begin{aligned}
a_n &= 2\int_0^{1/2} x \sin \pi x \cos n\pi x\, dx = \int_0^{1/2} x[\sin(n+1)\pi x - \sin(n-1)\pi x]\, dx \\
&= \frac{1}{\pi}\left[x\left[\frac{\cos(n-1)\pi x}{n-1} - \frac{\cos(n+1)\pi x}{n+1}\right]\bigg|_0^{1/2} - \int_0^{1/2}\left[\frac{\cos(n-1)\pi x}{n-1} - \frac{\cos(n+1)\pi x}{n+1}\right] dx\right] \\
&= \frac{1}{2\pi}\left[\frac{\cos(n-1)\pi/2}{n-1} - \frac{\cos(n+1)\pi/2}{n+1}\right] - \frac{1}{\pi^2}\left[\frac{\sin(n-1)\pi/2}{(n-1)^2} - \frac{\sin(n+1)\pi/2}{(n+1)^2}\right] \\
&= \frac{1}{\pi}\frac{n}{n^2-1}\sin\frac{n\pi}{2} + \frac{2}{\pi^2}\frac{n^2+1}{(n^2-1)^2}\cos\frac{n\pi}{2} = \begin{cases} (-1)^m \dfrac{2}{\pi^2}\dfrac{4m^2+1}{(4m^2-1)^2} & \text{if } n = 2m, \\ (-1)^m \dfrac{1}{4\pi}\dfrac{2m+1}{m(m+1)} & \text{if } n = 2m+1; \end{cases}
\end{aligned}$$

$$F(x) = \frac{1}{\pi^2} + \frac{1}{4\pi}\cos\pi x + \frac{2}{\pi^2}\sum_{n=1}^{\infty}(-1)^n\frac{4n^2+1}{(4n^2-1)^2}\cos 2n\pi x + \frac{1}{4\pi}\sum_{n=1}^{\infty}(-1)^n\frac{2n+1}{n(n+1)}\cos(2n+1)\pi x.$$

From Theorem 11.2.4,

$$F(x) = \begin{cases} 0, & -1 \le x < \frac{1}{2}, \\ \frac{1}{4}, & x = -\frac{1}{2}, \\ x\sin\pi x, & -\frac{1}{2} < x < \frac{1}{2}, \\ \frac{1}{4}, & x = \frac{1}{2}, \\ 0, & \frac{1}{2} < x \le 1. \end{cases}$$

11.2.16. Note that $\int_{-1}^{0} x^2 g(x)\,dx = \int_0^1 x^2 g(-x)\,dx$; therefore, $a_0 = \frac{1}{2}\left[\int_{-1}^{0} x^2\,dx + \int_0^1 (1-x^2)\,dx\right] = \frac{1}{2}\int_0^1 dx = \frac{1}{2}$, and if $n \ge 1$, then

$$a_n = \int_{-1}^{0} x^2\cos n\pi x\,dx + \int_0^1 (1-x^2)\cos n\pi x\,dx = \int_0^1 \cos n\pi x\,dx = \left.\frac{\sin n\pi x}{n\pi}\right|_0^1 = 0$$

and

$$\begin{aligned} b_n &= \int_{-1}^{0} x^2\sin n\pi x\,dx + \int_0^1 (1-x^2)\sin n\pi x\,dx = \int_0^1 (1-2x^2)\sin n\pi x\,dx \\ &= -\frac{1}{n\pi}\left[\left.(1-2x^2)\cos n\pi x\right|_0^1 + 4\int_0^1 x\cos n\pi x\,dx\right] \\ &= \frac{1+\cos n\pi}{n\pi} - \frac{4}{n^2\pi^2}\left[\left.x\sin n\pi x\right|_0^1 - \int_0^1 \sin n\pi x\,dx\right] \\ &= \frac{1+\cos n\pi}{n\pi} - \left.\frac{4\cos n\pi x}{n^3\pi^3}\right|_0^1 = \frac{1+\cos n\pi}{n\pi} + \frac{4(1-\cos n\pi)}{n^3\pi^3} \\ &= \begin{cases} \dfrac{1}{m\pi} & \text{if } n = 2m, \\ \dfrac{8}{(2m+1)^3\pi^3} & \text{if } n = 2m+1; \end{cases} \end{aligned}$$

$$F(x) = \frac{1}{2} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin 2n\pi x + \frac{8}{\pi^3}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^3}\sin(2n+1)\pi x.$$

From Theorem 11.2.4,

$$F(x) = \begin{cases} \frac{1}{2}, & x = -1, \\ x^2, & -1 < x < 0, \\ \frac{1}{2}, & x = 0, \\ 1-x^2, & 0 < x < 1, \\ \frac{1}{2}, & x = 1. \end{cases}$$

11.2.18. $a_0 = \dfrac{1}{6}\left[\int_{-3}^{-2} 2\,dx + \int_{-2}^{2} 3\,dx + \int_{2}^{3} 1\,dx\right] = \dfrac{5}{2}$. If $n \ge 1$, then

$$\begin{aligned}
a_n &= \frac{1}{3}\int_{-3}^{3} f(x)\cos\frac{n\pi x}{3}\,dx \\
&= \frac{1}{3}\left[\int_{-3}^{-2} 2\cos\frac{n\pi x}{3}\,dx + \int_{-2}^{2} 3\cos\frac{n\pi x}{3}\,dx + \int_{2}^{3}\cos\frac{n\pi x}{3}\,dx\right] = \frac{3}{n\pi}\sin\frac{2n\pi}{3}, \\
b_n &= \frac{1}{3}\left[\int_{-3}^{-2} 2\sin\frac{n\pi x}{3}\,dx + \int_{-2}^{2} 3\sin\frac{n\pi x}{3}\,dx + \int_{2}^{3}\sin\frac{n\pi x}{3}\,dx\right] = \frac{1}{n\pi}\left(\cos n\pi - \cos\frac{2n\pi}{3}\right);
\end{aligned}$$

$$F(x) = \frac{5}{2} + \frac{3}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{2n\pi}{3}\cos\frac{n\pi x}{3} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(\cos n\pi - \cos\frac{2n\pi}{3}\right)\sin\frac{n\pi x}{3}.$$

11.2.20. (a) $a_0 = \dfrac{1}{2\pi}\int_{-\pi}^{\pi} e^x\,dx = \dfrac{e^x}{2\pi}\Big|_{-\pi}^{\pi} = \dfrac{\sinh\pi}{\pi}$. If $n \ge 1$ then (A) $a_n = \dfrac{1}{\pi}\int_{-\pi}^{\pi} e^x\cos nx\,dx$ and (B) $b_n = \dfrac{1}{\pi}\int_{-\pi}^{\pi} e^x\sin nx\,dx$. Integrating (B) by parts yields

$$b_n = \frac{1}{\pi}\left[e^x\sin nx\Big|_{-\pi}^{\pi} - n\int_{-\pi}^{\pi} e^x\cos nx\,dx\right] = -na_n. \tag{C}$$

Integrating (A) by parts yields

$$a_n = \frac{1}{\pi}\left[e^x\cos nx\Big|_{-\pi}^{\pi} + n\int_{-\pi}^{\pi} e^x\sin nx\,dx\right] = (-1)^n\frac{2\sinh\pi}{\pi} + nb_n = (-1)^n\frac{2\sinh\pi}{\pi} - n^2a_n,$$

from (C). Therefore, $a_n = \dfrac{2\sinh\pi}{\pi}\dfrac{(-1)^n}{n^2+1}$. Now (C) implies that $b_n = \dfrac{2\sinh\pi}{\pi}\dfrac{(-1)^{n+1}n}{n^2+1}$. Therefore,

$$F(x) = \frac{\sinh\pi}{\pi}\left(1 + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}\cos nx - 2\sum_{n=1}^{\infty}\frac{(-1)^n n}{n^2+1}\sin nx\right).$$

(b) From Theorem 11.2.4, $F(\pi) = \cosh\pi$, so $\dfrac{\sinh\pi}{\pi}\left(1 + 2\sum_{n=1}^{\infty}\dfrac{1}{n^2+1}\right) = \cosh\pi$, which implies the stated result.

11.2.24. Since f is even, $b_n = 0$, $n \ge 1$, $a_0 = \dfrac{1}{\pi}\int_0^{\pi}\cos kx\,dx = \dfrac{\sin kx}{k\pi}\Big|_0^{\pi} = \dfrac{\sin k\pi}{k\pi}$; if $n \ge 1$ then

$$\begin{aligned}
a_n &= \frac{2}{\pi}\int_0^{\pi}\cos kx\cos nx\,dx = \frac{1}{\pi}\int_0^{\pi}[\cos(n-k)x + \cos(n+k)x]\,dx \\
&= \frac{1}{\pi}\left[\frac{\sin(n-k)x}{n-k} + \frac{\sin(n+k)x}{n+k}\right]\Bigg|_0^{\pi} \\
&= \frac{\cos n\pi\sin kx}{\pi}\left[\frac{1}{n+k} - \frac{1}{n-k}\right] = (-1)^{n+1}\frac{2k\sin k\pi}{(n^2-k^2)\pi};
\end{aligned}$$

$$F(x) = \frac{\sin k\pi}{\pi}\left[\frac{1}{k} - 2k\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2-k^2}\cos nx\right].$$

11.2.26. Since f is continuous on $[-L,L]$ and $f(-L)=f(L)$, Theorem 11.2.4 implies that

$$f(x)=a_0+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L}+b_n\sin\frac{n\pi x}{L}\right),\quad -L\le x\le L,$$

if $a_0=\dfrac{1}{2L}\displaystyle\int_{-L}^{L} f(x)\,dx$, and, for $n\ge 1$,

$$\begin{aligned}
a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx=\frac{1}{n\pi}\left[f(x)\sin\frac{n\pi x}{L}\Big|_{-L}^{L}-\int_{-L}^{L} f'(x)\sin\frac{n\pi x}{L}\,dx\right]\\
&= \frac{L}{n^2\pi^2}\left[f'(x)\cos\frac{n\pi x}{L}\Big|_{-L}^{L}-\int_{-L}^{L} f''(x)\cos\frac{n\pi x}{L}\,dx\right]=-\frac{L}{n^2\pi^2}\int_{-L}^{L} f''(x)\cos\frac{n\pi x}{L}\,dx
\end{aligned}$$

(since $f'(-L)=f'(L)$), and

$$\begin{aligned}
b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx=-\frac{1}{n\pi}\left[f(x)\cos\frac{n\pi x}{L}\Big|_{-L}^{L}-\int_{-L}^{L} f'(x)\cos\frac{n\pi x}{L}\,dx\right]\\
&= \frac{1}{n\pi}\int_{-L}^{L} f'(x)\cos\frac{n\pi x}{L}\,dx \text{ (since } f(-L)=f(L))\\
&= \frac{L}{n^2\pi^2}\left[f'(x)\sin\frac{n\pi x}{L}\Big|_{-L}^{L}-\int_{-L}^{L} f''(x)\sin\frac{n\pi x}{L}\,dx\right]=-\frac{L}{n^2\pi^2}\int_{-L}^{L} f''(x)\sin\frac{n\pi x}{L}\,dx.
\end{aligned}$$

If f''' is integrable on $[-L,L]$, then

$$\begin{aligned}
a_n &= -\frac{L}{n^2\pi^2}\int_{-L}^{L} f''(x)\cos\frac{n\pi x}{L}\,dx=-\frac{L^2}{n^3\pi^3}\left[f'''(x)\sin\frac{n\pi x}{L}\Big|_{-L}^{L}-\int_{-L}^{L} f'''(x)\sin\frac{n\pi x}{L}\,dx\right]\\
&= \frac{L^2}{n^3\pi^3}\int_{-L}^{L} f'''(x)\sin\frac{n\pi x}{L}\,dx.
\end{aligned}$$

11.2.28. The Fourier series is $a_0+\displaystyle\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L}+b_n\sin\frac{n\pi x}{L}\right)$ where

$$a_0=\frac{1}{2L}\int_{-L}^{L} f(x)\,dx=\frac{1}{2L}\left[\int_{-L}^{0} f(x)\,dx+\int_{0}^{L} f(x)\,dx\right]. \tag{A}$$

Since $\displaystyle\int_{-L}^{0} f(x)\,dx=-\int_{-L}^{0} f(x+L)\,dx=-\int_{0}^{L} f(x)\,dx$, (A) implies that $a_0=0$. If $n\ge 1$, then

$$a_n=\frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx=\frac{1}{L}\left[\int_{-L}^{0} f(x)\cos\frac{n\pi x}{L}\,dx+\int_{0}^{L} f(x)\cos\frac{n\pi x}{L}\,dx\right]. \tag{B}$$

Since

$$\begin{aligned}
\int_{-L}^{0} f(x)\cos\frac{n\pi x}{L}\,dx &= -\int_{-L}^{0} f(x+L)\cos\frac{n\pi x}{L}\,dx=-\int_{0}^{L} f(x)\cos\frac{n\pi(x+L)}{L}\,dx\\
&= (-1)^{n+1}\int_{0}^{L} f(x)\cos\frac{n\pi x}{L}\,dx,
\end{aligned}$$

(B) implies that $a_{2n-1} = A_n$ and $a_{2n} = 0$, $n \ge 1$. A similar argument showsthat $b_{2n-1} = B_n$ and $b_{2n} = 0$, $n \ge 1$.

11.2.30.(b) Let $\phi_0 = 1$ and $\phi_{2m} = \cos\dfrac{m\pi x}{L}$, $\phi_{2m-1} = \sin\dfrac{m\pi x}{L}$, $m \ge 1$. Then $c_0 = a_0$ and $c_{2m} = a_m$, $c_{2m-1} = b_m$, $m \ge 1$. Since $\int_{-L}^{L} \phi_0^2(x)\,dx = 2L$ and $\int_{-L}^{L} \phi_{2m-1}^2(x)\,dx = \frac{1}{2}\int_{-L}^{L}\left(1-\cos\frac{2mx}{L}\right)dx = L$, $\int_{-L}^{L}\phi_{2m}^2(x)\,dx = \frac{1}{2}\int_{-L}^{L}\left(1+\cos\frac{2mx}{L}\right)dx = L$, $m \ge 1$, Exercise 12.2.29**(d)** implies the conclusion.

11.3 FOURIER EXPANSIONS II

11.3.2. $a_0 = \int_0^1 (1-x)\,dx = -\left.\dfrac{(1-x)^2}{2}\right|_0^1 = \dfrac{1}{2}$; if $n \ge 1$,

$$\begin{aligned}
a_n &= 2\int_0^1 (1-x)\cos n\pi x\,dx = \frac{2}{n\pi}\left[(1-x)\sin n\pi x\Big|_0^1 + \int_0^1 \sin n\pi x\,dx\right] \\
&= -\frac{2}{n^2\pi^2}\cos n\pi x\Big|_0^1 = \frac{2}{n^2\pi^2}[1-(-1)^n] = \begin{cases} \dfrac{4}{(2m-1)^2\pi^2} & \text{if } n = 2m-1, \\ 0 & \text{if } n = 2m; \end{cases}
\end{aligned}$$

$$C(x) = \frac{1}{2} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos(2n-1)\pi x.$$

11.3.4. $a_0 = \dfrac{1}{\pi}\int_0^{\pi}\sin kx\,dx = -\left.\dfrac{\cos kx}{k}\right|_0^{\pi} = \dfrac{1-\cos k\pi}{k\pi}$; if $n \ge 1$, then

$$\begin{aligned}
a_n &= \frac{2}{\pi}\int_0^{\pi}\sin kx\cos nx\,dx = \frac{1}{\pi}\int_0^{\pi}[\sin(n+k)x - \sin(n-k)x]\,dx \\
&= \frac{1}{\pi}\left[\frac{\cos(n-k)x}{n-k} - \frac{\cos(n+k)x}{n+k}\right]\Bigg|_0^{\pi} = \frac{1}{\pi}\left[\frac{\cos(n-k)\pi - 1}{n-k} - \frac{\cos(n+k)\pi - 1}{n+k}\right] \\
&= -\frac{2k[1-(-1)^n\cos k\pi]}{(n^2-k^2)\pi};
\end{aligned}$$

$$C(x) = \frac{1-\cos k\pi}{k\pi} - \frac{2k}{\pi}\sum_{n=1}^{\infty}\frac{[1-(-1)^n\cos k\pi]}{n^2-k^2}\cos nx.$$

11.3.6. $a_0 = \dfrac{1}{L}\int_0^L (x^2-L^2)\,dx = \dfrac{1}{L}\left(\dfrac{x^3}{3} - L^2x\right)\Bigg|_0^L = -\dfrac{2L^2}{3}$; if $n \ge 1$,

$$\begin{aligned}
a_n &= \frac{2}{L}\int_0^L (x^2-L^2)\cos\frac{n\pi x}{L}\,dx = \frac{2}{n\pi}\left[(x^2-L^2)\sin\frac{n\pi x}{L}\Big|_0^L - 2\int_0^L x\sin\frac{n\pi x}{L}\,dx\right] \\
&= \frac{4L}{n^2\pi^2}\left[x\cos\frac{n\pi x}{L}\Big|_0^L - \int_0^L\cos\frac{n\pi x}{L}\,dx\right] = (-1)^n\frac{4L^2}{n^2\pi^2} - \frac{4L^2}{n^3\pi^3}\sin\frac{n\pi x}{L}\Big|_0^L = (-1)^n\frac{4L^2}{n^2\pi^2};
\end{aligned}$$

$$C(x) = -\frac{2L^2}{3} + \frac{4L^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\frac{n\pi x}{L}.$$

11.3.8. $a_0 = \int_0^\pi e^x\,dx = e^x\Big|_0^\pi = \frac{e^\pi - 1}{\pi}$; if $n \ge 1$, then

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^\pi e^x \cos nx\,dx = \frac{2}{\pi}\left[e^x \cos nx\Big|_0^\pi - n\int_0^\pi e^x \sin nx\,dx\right] \\ &= \frac{2}{\pi}\left[(-1)^n e^\pi - 1 - ne^x \sin nx\Big|_0^\pi - n^2\int_0^\pi e^x \cos nx\,dx\right] = \frac{2}{\pi}[(-1)^n e^\pi - 1] - n^2 a_n; \end{aligned}$$

$(1+n^2)a_n = \frac{2}{\pi}[(-1)^n e^\pi - 1]$; $a_n = \frac{2}{(n^2+1)\pi}[(-1)^n e^\pi - 1]$;

$$C(x) = \frac{e^\pi - 1}{\pi} + \frac{2}{\pi}\sum_{n=1}^\infty \frac{[(-1)^n e^\pi - 1]}{(n^2+1)}\cos nx.$$

11.3.10. $a_0 = \frac{1}{L}\int_0^L (x^2 - 2Lx)\,dx = \frac{1}{L}\left(\frac{x^3}{3} - Lx^2\right)\Big|_0^L = -\frac{2L^2}{3}$; if $n \ge 1$,

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L (x^2 - 2Lx)\cos\frac{n\pi x}{L}\,dx = \frac{2}{n\pi}\left[(x^2 - 2Lx)\sin\frac{n\pi x}{L}\Big|_0^L - 2\int_0^L (x-L)\sin\frac{n\pi x}{L}\,dx\right] \\ &= \frac{4L}{n^2\pi^2}\left[(x-L)\cos\frac{n\pi x}{L}\Big|_0^L - \int_0^L \cos\frac{n\pi x}{L}\,dx\right] = \frac{4L^2}{n^2\pi^2} - \frac{4L^3}{n^3\pi^3}\sin\frac{n\pi x}{L}\Big|_0^L = \frac{4L^2}{n^2\pi^2}; \end{aligned}$$

$$C(x) = -\frac{2L^2}{3} + \frac{4L^2}{\pi^2}\sum_{n=1}^\infty \frac{1}{n^2}\cos\frac{n\pi x}{L}.$$

11.3.12. $b_n = 2\int_0^1 (1-x)\sin n\pi x\,dx = -\frac{2}{n\pi}\left[(1-x)\cos n\pi x\Big|_0^1 + \int_0^1 \cos n\pi x\,dx\right] = \frac{2}{n\pi} + \frac{2}{n^2\pi^2}\sin n\pi x\Big|_0^1 = \frac{2}{n\pi}$; $S(x) = \frac{2}{\pi}\sum_{n=1}^\infty \frac{1}{n}\sin n\pi x$.

11.3.14. $b_n = \frac{2}{L}\int_0^{L/2} \sin\frac{n\pi x}{L}\,dx = -\frac{2}{n\pi}\cos\frac{n\pi x}{L}\Big|_0^{L/2} = \frac{2}{n\pi}\left[1 - \cos\frac{n\pi}{2}\right]$;

$$S(x) = \frac{2}{\pi}\sum_{n=1}^\infty \frac{1}{n}\left[1 - \cos\frac{n\pi}{2}\right]\sin\frac{n\pi x}{L}.$$

11.3.16.

$$\begin{aligned} b_1 &= \frac{2}{\pi}\int_0^\pi x\sin^2 x\,dx = \frac{1}{\pi}\int_0^\pi x(1 - \cos 2x)\,dx = \frac{x^2}{2\pi}\Big|_0^\pi - \frac{1}{\pi}\int_0^\pi x\cos 2x\,dx \\ &= \frac{\pi}{2} - \frac{1}{2\pi}\left[x\sin 2x\Big|_0^\pi - \int_0^\pi \cos 2x\,dx\right] = \frac{\pi}{2} + \frac{\sin 2x}{4\pi}\Big|_0^\pi = \frac{\pi}{2}; \end{aligned}$$

if $n \ge 2$, then

$$
\begin{aligned}
b_n &= \frac{2}{\pi}\int_0^\pi x\sin x\sin nx\,dx = \frac{1}{\pi}\int_0^\pi x[\cos(n-1)x - \cos(n+1)x]\,dx \\
&= \frac{1}{\pi}\left[x\left[\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right]\Bigg|_0^\pi - \int_0^\pi \left[\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right]dx\right] \\
&= \frac{1}{\pi}\left[\frac{\cos(n-1)x}{(n-1)^2} - \frac{\cos(n+1)x}{(n+1)^2}\right]\Bigg|_0^\pi = \frac{1}{\pi}\left[\frac{1}{(n-1)^2} - \frac{1}{(n+1)^2}\right]\left[(-1)^{n+1}-1\right] \\
&= \frac{4n}{(n^2-1)^2\pi}\left[(-1)^{n+1}-1\right] = \begin{cases} 0 & \text{if } n = 2m-1, \\ -\dfrac{16m}{(4m^2-1)\pi} & \text{if } n = 2m; \end{cases}
\end{aligned}
$$

$$
S(x) = \frac{\pi}{2}\sin x - \frac{16}{\pi}\sum_{n=1}^\infty \frac{n}{(4n^2-1)^2}\sin 2nx.
$$

11.3.18. $c_n = \dfrac{2}{L}\displaystyle\int_0^L \cos\frac{(2n-1)\pi x}{2L}\,dx = \frac{4}{(2n-1)\pi}\sin\frac{(2n-1)\pi x}{2L}\Bigg|_0^L = (-1)^{n+1}\frac{4}{(2n-1)\pi};$

$$
C_M(x) = -\frac{4}{\pi}\sum_{n=1}^\infty \frac{(-1)^n}{2n-1}\cos\frac{(2n-1)\pi x}{2L}.
$$

11.3.20.

$$
\begin{aligned}
d_n &= 2\int_0^1 x\cos\frac{(2n-1)\pi x}{2}\,dx \\
&= \frac{4}{(2n-1)\pi}\left[x\sin\frac{(2n-1)\pi x}{2}\Bigg|_0^1 - \int_0^1 \sin\frac{(2n-1)\pi x}{2}\,dx\right] \\
&= \frac{4}{(2n-1)\pi}\left[(-1)^{n+1} + \frac{2}{(2n-1)\pi}\cos\frac{(2n-1)\pi x}{2}\Bigg|_0^1\right] \\
&= -\frac{4}{(2n-1)\pi}\left[(-1)^n + \frac{2}{(2n-1)\pi}\right];
\end{aligned}
$$

$$
C_M(x) = -\frac{4}{\pi}\sum_{n=1}^\infty \left[(-1)^n + \frac{2}{(2n-1)\pi}\right]\cos\frac{(2n-1)\pi x}{2}.
$$

11.3.22.

$$
\begin{aligned}
c_n &= \frac{2}{\pi}\int_0^\pi \cos x\cos\frac{(2n-1)x}{2}\,dx = \frac{1}{\pi}\int_0^\pi \left[\frac{\cos(2n+1)x}{2} + \frac{\cos(2n-3)x}{2}\right]dx \\
&= \frac{2}{\pi}\left[\frac{\sin(2n+1)x/2}{2n+1} + \frac{\sin(2n-3)x/2}{2n-3}\right]\Bigg|_0^\pi \\
&= (-1)^n\frac{2}{\pi}\left[\frac{1}{2n+1} + \frac{1}{2n-3}\right] = (-1)^n\frac{4(2n-1)}{(2n-3)(2n+1)\pi};
\end{aligned}
$$

$$C_M(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}(-1)^n \frac{2n-1}{(2n-3)(2n+1)}\cos\frac{(2n-1)x}{2}.$$

11.3.24.

$$\begin{aligned}
c_n &= \frac{2}{L}\int_0^L (Lx-x^2)\cos\frac{(2n-1)\pi x}{2L}\,dx \\
&= \frac{4}{(2n-1)\pi}\left[(Lx-x^2)\sin\frac{(2n-1)\pi x}{2L}\bigg|_0^L - \int_0^L (L-2x)\sin\frac{(2n-1)\pi x}{2L}\,dx\right] \\
&= \frac{8L}{(2n-1)^2\pi^2}\left[(L-2x)\cos\frac{(2n-1)\pi x}{2L}\bigg|_0^L + 2\int_0^L \cos\frac{(2n-1)\pi x}{2L}\,dx\right] \\
&= -\frac{8L^2}{(2n-1)^2\pi^2} + \frac{32L^2}{(2n-1)^3\pi^3}\sin\frac{(2n-1)\pi x}{2L}\bigg|_0^L = \frac{32L^2}{(2n-1)^3\pi^3}\sin\frac{(2n-1)\pi x}{2} \\
&= -\frac{8L^2}{(2n-1)^2\pi^2} + (-1)^{n-1}\frac{32L^2}{(2n-1)^3\pi^3};
\end{aligned}$$

$$C_M(x) = -\frac{8L^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\left[1+\frac{4(-1)^n}{(2n-1)\pi}\right]\cos\frac{(2n-1)\pi x}{2L}.$$

11.3.26.

$$\begin{aligned}
d_n &= \frac{2}{L}\int_0^L x^2\sin\frac{(2n-1)\pi x}{2L}\,dx \\
&= -\frac{4}{(2n-1)\pi}\left[x^2\cos\frac{(2n-1)\pi x}{2L}\bigg|_0^L - 2\int_0^L x\cos\frac{(2n-1)\pi x}{2L}\,dx\right] \\
&= \frac{16L}{(2n-1)^2\pi^2}\left[x\sin\frac{(2n-1)\pi x}{2L}\bigg|_0^L - \int_0^L \sin\frac{(2n-1)\pi x}{2L}\,dx\right] \\
&= (-1)^{n+1}\frac{16L^2}{(2n-1)^2\pi^2} + \frac{32L^2}{(2n-1)^3\pi^3}\cos\frac{(2n-1)\pi x}{2L}\bigg|_0^L \\
&= (-1)^{n+1}\frac{16L^2}{(2n-1)^2\pi^2} - \frac{32L^2}{(2n-1)^3\pi^3};
\end{aligned}$$

$$S_M(x) = -\frac{16L^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\left[(-1)^n+\frac{2}{(2n-1)\pi}\right]\sin\frac{(2n-1)\pi x}{2L}.$$

11.3.28.

$$\begin{aligned}
d_n &= \frac{2}{\pi}\int_0^\pi \cos x\sin\frac{(2n-1)x}{2}\,dx = \frac{1}{\pi}\int_0^\pi\left[\frac{\sin(2n+1)x}{2}+\frac{\sin(2n-3)x}{2}\right]dx \\
&= -\frac{2}{\pi}\left[\frac{\cos(2n+1)x/2}{2n-1}+\frac{\cos(2n-3)x/2}{2n-3}\right]\bigg|_0^\pi \\
&= \frac{2}{\pi}\left[\frac{1}{2n+1}+\frac{1}{2n-3}\right] = \frac{4(2n-1)}{(2n-3)(2n+1)\pi};
\end{aligned}$$

$$S_M(x) = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{2n-1}{(2n-3)(2n+1)} \sin\frac{(2n-1)x}{2}.$$

11.3.30.

$$\begin{aligned}
d_n &= \frac{2}{L}\int_0^L (Lx-x^2)\sin\frac{(2n-1)\pi x}{2L}\,dx \\
&= -\frac{4}{(2n-1)\pi}\left[(Lx-x^2)\cos\frac{(2n-1)\pi x}{2L}\bigg|_0^L - \int_0^L (L-2x)\cos\frac{(2n-1)\pi x}{2L}\,dx\right] \\
&= \frac{8L}{(2n-1)^2\pi^2}\left[(L-2x)\sin\frac{(2n-1)\pi x}{2L}\bigg|_0^L + 2\int_0^L \sin\frac{(2n-1)\pi x}{2L}\,dx\right] \\
&= (-1)^n\frac{8L^2}{(2n-1)^2\pi^2} - \frac{32L^2}{(2n-1)^3\pi^3}\cos\frac{(2n-1)\pi x}{2L}\bigg|_0^L \\
&= (-1)^n\frac{8L^2}{(2n-1)^2\pi^2} + \frac{32L^2}{(2n-1)^3\pi^3};
\end{aligned}$$

$$S_M(x) = \frac{8L^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\left[(-1)^n + \frac{4}{(2n-1)\pi}\right]\sin\frac{(2n-1)\pi x}{2L}.$$

11.3.32. $a_0 = \frac{1}{L}\int_0^L (3x^4-4Lx^3)\,dx = \frac{1}{L}\left(\frac{3x^5}{5} - Lx^4\right)\bigg|_0^L = -\frac{2L^4}{5}$. Since $f'(0) = f'(L) = 0$ and $f'''(x) = 24(3x-L)$,

$$\begin{aligned}
a_n &= \frac{48L^2}{n^3\pi^3}\int_0^L (3x-L)\sin\frac{n\pi x}{L}\,dx = -\frac{48L^3}{n^4\pi^4}\left[(3x-L)\cos\frac{n\pi x}{L}\bigg|_0^L - 3\int_0^L \cos\frac{n\pi x}{L}\,dx\right] \\
&= -\frac{48L^3}{n^4\pi^4}\left[(-1)^n 2L + L\right] + \frac{144L^4}{n^5\pi^5}\sin\frac{n\pi x}{L}\bigg|_0^L = -\frac{48L^4}{n^4\pi^4}\left[1+(-1)^n 2\right], \quad n \ge 1;
\end{aligned}$$

$$C(x) = -\frac{2L^4}{5} - \frac{48L^4}{\pi^4}\sum_{n=1}^{\infty}\frac{1+(-1)^n 2}{n^4}\cos\frac{n\pi x}{L}.$$

11.3.34. $a_0 = \frac{1}{L}\int_0^L (x^4 - 2Lx^3 + L^2x^2)\,dx = \frac{1}{L}\left(\frac{x^5}{5} - \frac{Lx^4}{2} + \frac{L^2x^3}{3}\right)\bigg|_0^L = \frac{L^4}{30}$. Since $f'(0) = f'(L) = 0$ and $f'''(x) = 12(2x-L)$,

$$\begin{aligned}
a_n &= \frac{24L^2}{n^3\pi^3}\int_0^L (2x-L)\sin\frac{n\pi x}{L}\,dx = -\frac{24L^3}{n^4\pi^4}\left[(2x-L)\cos\frac{n\pi x}{L}\bigg|_0^L - 2\int_0^L \cos\frac{n\pi x}{L}\,dx\right] \\
&= -\frac{24L^3}{n^4\pi^4}\left[(-1)^n L + L\right] + \frac{48L^4}{n^5\pi^5}\sin\frac{n\pi x}{L}\bigg|_0^L = -\frac{24L^4}{n^4\pi^4}\left[1+(-1)^n\right] \\
&= \left\{\begin{array}{cl} 0 & \text{if } n = 2m-1, \\ -\dfrac{3L^4}{m^4\pi^4} & \text{if } n = 2m, \end{array}\right. \quad n \ge 1.
\end{aligned}$$

$$C(x) = \frac{L^4}{30} - \frac{3L^4}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4}\cos\frac{2n\pi x}{L}.$$

11.3.36. Since $f(0) = f(L) = 0$ and $f''(x) = -2$,

$$\begin{aligned} b_n &= \frac{4L}{n^2\pi^2}\int_0^L \sin\frac{n\pi x}{L}\,dx = -\frac{4L^2}{n^3\pi^3}\cos\frac{n\pi x}{L}\Big|_0^L = -\frac{4L^2}{n^3\pi^2}(\cos n\pi - 1) \\ &= \begin{cases} \dfrac{8L^2}{(2m-1)^3\pi^3}, & \text{if } n = 2m-1, \\ 0, & \text{if } n = 2m; \end{cases} \end{aligned}$$

$$S(x) = \frac{8L^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\sin\frac{(2n-1)\pi x}{L}.$$

11.3.38. Since $f(0) = f(L) = 0$ and $f''(x) = -6x$,

$$b_n = \frac{12L}{n^2\pi^2}\int_0^L x\sin\frac{n\pi x}{L}\,dx = -\frac{12L^2}{n^3\pi^3}\left[x\cos\frac{n\pi x}{L}\Big|_0^L - \int_0^L \cos\frac{n\pi x}{L}\,dx\right] = (-1)^{n+1}\frac{12L^3}{n^3\pi^3};$$

$$S(x) = -\frac{12L^3}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin\frac{n\pi x}{L}.$$

11.3.40. Since $f(0) = f(L) = f''(0) = f''(L) = 0$ and $f^{(4)} = 360x$,

$$\begin{aligned} b_n &= \frac{720L^3}{n^4\pi^4}\int_0^L x\sin\frac{n\pi x}{L}\,dx = -\frac{720L^4}{n^5\pi^5}\left[x\cos\frac{n\pi x}{L}\Big|_0^L - \int_0^L \cos\frac{n\pi x}{L}\,dx\right] \\ &= (-1)^{n+1}\frac{720L^5}{n^5\pi^5} + \frac{720L^5}{n^6\pi^6}\sin\frac{n\pi x}{L}\Big|_0^L = (-1)^{n+1}\frac{720L^5}{n^5\pi^5}; \end{aligned}$$

$$S(x) = -\frac{720L^5}{\pi^5}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^5}\sin\frac{n\pi x}{L}.$$

11.3.42. (a) Since f is continuous on $[0, L]$ and $f(L) = 0$, Theorem 11.3.3 implies that $f(x) = \sum_{n=1}^{\infty} c_n \cos\frac{(2n-1)\pi x}{2L}$, $-L \le x \le L$, with

$$\begin{aligned} c_n &= \frac{2}{L}\int_0^L f(x)\cos\frac{(2n-1)\pi x}{2L}\,dx \\ &= \frac{4}{(2n-1)\pi}\left[f(x)\sin\frac{(2n-1)\pi x}{2L}\Big|_0^L - \int_0^L f'(x)\sin\frac{(2n-1)\pi x}{2L}\,dx\right] \\ &= -\frac{4}{(2n-1)\pi}\int_0^L f'(x)\sin\frac{(2n-1)\pi x}{2L}\,dx \text{ (since } f(L) = 0) \\ &= \frac{8L^2}{(2n-1)^2\pi^2}\left[f'(x)\cos\frac{(2n-1)\pi x}{2L}\Big|_0^L - \int_0^L f''(x)\cos\frac{(2n-1)\pi x}{2L}\,dx\right] \\ &= -\frac{8L}{(2n-1)^2\pi^2}\int_0^L f''(x)\cos\frac{(2n-1)\pi x}{2L}\,dx \text{ (since } f'(0) = 0). \end{aligned}$$

(b) Continuing the integration by parts yields

$$\begin{aligned} c_n &= -\frac{16L^2}{(2n-1)^3\pi^3}\left[f''(x)\sin\frac{(2n-1)\pi x}{2L}\bigg|_0^L - \int_0^L f'''(x)\sin\frac{(2n-1)\pi x}{2L}\,dx\right] \\ &= \frac{16L^2}{(2n-1)^3\pi^3}\int_0^L f'''(x)\sin\frac{(2n-1)\pi x}{2L}\,dx. \end{aligned}$$

11.3.44. Since $f'(0) = f(L) = 0$ and $f''(x) = -2$,

$$\begin{aligned} c_n &= \frac{16L}{(2n-1)^2\pi^2}\int_0^L \cos\frac{(2n-1)\pi x}{2L}\,dx \\ &= \frac{32L^2}{(2n-1)^3\pi^3}\sin\frac{(2n-1)\pi x}{2L}\bigg|_0^L = (-1)^{n+1}\frac{32L^2}{(2n-1)^3\pi^3}; \end{aligned}$$

$$C_M(x) = -\frac{32L^2}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^3}\cos\frac{(2n-1)\pi x}{2L}.$$

11.3.46. Since $f'(0) = f(L) = 0$ and $f''(x) = 6(2x+L)$,

$$\begin{aligned} c_n &= -\frac{48L}{(2n-1)^2\pi^2}\int_0^L (2x+L)\cos\frac{(2n-1)\pi x}{2L}\,dx \\ &= -\frac{96L^2}{(2n-1)^3\pi^3}\left[(2x+L)\sin\frac{(2n-1)\pi x}{2L}\bigg|_0^L - 2\int_0^L \sin\frac{(2n-1)\pi x}{2L}\right]dx \\ &= -\frac{96L^2}{(2n-1)^3\pi^3}\left[(-1)^{n+1}3L - \frac{4L}{(2n-1)\pi}\cos\frac{(2n-1)\pi x}{2L}\bigg|_0^L\right] \\ &= \frac{96L^3}{(2n-1)^3\pi^3}\left[(-1)^n 3 + \frac{4}{(2n-1)\pi}\right]; \end{aligned}$$

$$C_M(x) = \frac{96L^3}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\left[(-1)^n 3 + \frac{4}{(2n-1)\pi}\right]\cos\frac{(2n-1)\pi x}{2L}.$$

11.3.48. Since $f'(0) = f(L) = f''(L) = 0$ and $f'''(x) = 12(2x-L)$,

$$\begin{aligned} c_n &= \frac{192L^2}{(2n-1)^3\pi^3}\int_0^L (2x-L)\sin\frac{(2n-1)\pi x}{2L}\,dx \\ &= -\frac{384L^3}{(2n-1)^4\pi^4}\left[(2x-L)\cos\frac{(2n-1)\pi x}{2L}\bigg|_0^L - 2\int_0^L \cos\frac{(2n-1)\pi x}{2L}\right]dx \\ &= -\frac{384L^3}{(2n-1)^4\pi^4}\left[L - \frac{4L}{(2n-1)\pi}\sin\frac{(2n-1)\pi x}{2L}\bigg|_0^L\right] \\ &= -\frac{384L^4}{(2n-1)^4\pi^4}\left[1 + \frac{(-1)^n 4}{(2n-1)\pi}\right]; \end{aligned}$$

$$C_M(x) = -\frac{384L^4}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}\left[1 + \frac{(-1)^n 4}{(2n-1)\pi}\right]\cos\frac{(2n-1)\pi x}{2L}.$$

11.3.50. **(a)** Since f is continuous on $[0, L]$ and $f(0) = 0$, Theorem 11.3.4 implies that $f(x) = \sum_{n=1}^{\infty} d_n \sin \frac{(2n-1)\pi x}{2L}$, $-L \le x \le L$, with

$$\begin{aligned}
d_n &= \frac{2}{L}\int_0^L f(x)\sin\frac{(2n-1)\pi x}{2L}\,dx \\
&= -\frac{4}{(2n-1)\pi}\left[f(x)\cos\frac{(2n-1)\pi x}{2L}\Big|_0^L - \int_0^L f'(x)\cos\frac{(2n-1)\pi x}{2L}\,dx\right] \\
&= \frac{4}{(2n-1)\pi}\int_0^L f'(x)\cos\frac{(2n-1)\pi x}{2L}\,dx \text{ (since } f(0)=0) \\
&= \frac{8L}{(2n-1)^2\pi^2}\left[f'(x)\sin\frac{(2n-1)\pi x}{2L}\Big|_0^L - \int_0^L f''(x)\sin\frac{(2n-1)\pi x}{2L}\,dx\right] \\
&= -\frac{8L}{(2n-1)^2\pi^2}\int_0^L f''(x)\sin\frac{(2n-1)\pi x}{2L}\,dx \text{ since } f'(L)=0.
\end{aligned}$$

(b) Continuing the integration by parts yields

$$\begin{aligned}
d_n &= \frac{16L^2}{(2n-1)^3\pi^3}\left[f''(x)\cos\frac{(2n-1)\pi x}{2L}\Big|_0^L - \int_0^L f'''(x)\cos\frac{(2n-1)\pi x}{2L}\,dx\right] \\
&= -\frac{16L^2}{(2n-1)^3\pi^3}\int_0^L f'''(x)\cos\frac{(2n-1)\pi x}{2L}\,dx.
\end{aligned}$$

11.3.52. Since $f(0) = f'(L) = 0$, and $f''(x) = 6(L - 2x)$

$$\begin{aligned}
d_n &= -\frac{48L}{(2n-1)^2\pi^2}\int_0^L (L-2x)\sin\frac{(2n-1)\pi x}{2L}\,dx \\
&= \frac{96L^2}{(2n-1)^3\pi^3}\left[(L-2x)\cos\frac{(2n-1)\pi x}{2L}\Big|_0^L + 2\int_0^L \cos\frac{(2n-1)\pi x}{2L}\,dx\right] \\
&= \frac{96L^2}{(2n-1)^3\pi^3}\left[-L + \frac{4L}{(2n-1)\pi}\sin\frac{(2n-1)\pi x}{2L}\Big|_0^L\right] \\
&= -\frac{96L^3}{(2n-1)^3\pi^3}\left[1 + (-1)^n\frac{4}{(2n-1)\pi}\right];
\end{aligned}$$

$$S_M(x) = -\frac{96L^3}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\left[1 + (-1)^n\frac{4}{(2n-1)\pi}\right]\sin\frac{(2n-1)\pi x}{2L}.$$

11.3.54. Since $f(0) = f'(L) = f''(0) = 0$ and $f'''(x) = 6$,

$$\begin{aligned}
d_n &= -\frac{96L^2}{(2n-1)^3\pi^3}\int_0^L \cos\frac{(2n-1)\pi x}{2L}\,dx \\
&= -\frac{192L^3}{(2n-1)^4\pi^4}\sin\frac{(2n-1)\pi x}{2L}\Big|_0^L = (-1)^n\frac{192L^3}{(2n-1)^4\pi^4};
\end{aligned}$$

$$S_M(x) = \frac{192L^3}{\pi^4}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^4}\sin\frac{(2n-1)\pi x}{2L}.$$

11.3.56. Since $f(0) = f'(L) = f''(0) = 0$ and $f'''(x) = 12(2x-L)$,

$$\begin{aligned}
d_n &= -\frac{192L^2}{(2n-1)^3\pi^3}\int_0^L (2x-L)\cos\frac{(2n-1)\pi x}{2L}\,dx \\
&= -\frac{384L^3}{(2n-1)^4\pi^4}\left[(2x-L)\sin\frac{(2n-1)\pi x}{2L}\Big|_0^L - 2\int_0^L \sin\frac{(2n-1)\pi x}{2L}\,dx\right] \\
&= -\frac{384L^3}{(2n-1)^4\pi^4}\left[(-1)^{n+1}L + \frac{4L}{(2n-1)\pi}\cos\frac{(2n-1)\pi x}{2L}\Big|_0^L\right] \\
&= -\frac{384L^3}{(2n-1)^4\pi^4}\left[(-1)^{n+1}L - \frac{4L}{(2n-1)\pi}\right] \\
&= \frac{384L^4}{(2n-1)^4\pi^4}\left[(-1)^n + \frac{4}{(2n-1)\pi}\right];
\end{aligned}$$

$$S_M(x) = \frac{384L^4}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}\left[(-1)^n + \frac{4}{(2n-1)\pi}\right]\sin\frac{(2n-1)\pi x}{2L}.$$

11.3.58. The Fourier sine series of f_4 on $[0,2L]$ is $\displaystyle\sum_{n=1}^{\infty} B_n\sin\frac{n\pi x}{2L}$, where

$$B_n = \frac{1}{L}\int_0^{2L} f_4(x)\sin\frac{n\pi x}{2L}\,dx = \frac{1}{L}\left[\int_0^L f(x)\sin\frac{n\pi x}{2L}\,dx + \int_L^{2L} f(2L-x)\sin\frac{n\pi x}{2L}\,dx\right].$$

Replacing x by $2L-x$ yields $\displaystyle\int_L^{2L} f(2L-x)\sin\frac{n\pi x}{2L}\,dx = \int_0^L f(x)\sin\frac{n\pi(2L-x)}{2L}\,dx$. Since $\displaystyle\sin\frac{n\pi(2L-x)}{2L} = (-1)^{n+1}\sin\frac{n\pi x}{2L}$,

$$\int_L^{2L} f(2L-x)\sin\frac{n\pi x}{2L}\,dx = (-1)^{n+1}\int_0^L f(x)\sin\frac{n\pi x}{2L}\,dx,$$

so

$$B_n = \frac{1+(-1)^{n+1}}{L}\int_0^L f(x)\sin\frac{n\pi x}{2L}\,dx = \begin{cases} \dfrac{2}{L}\displaystyle\int_0^L f(x)\sin\frac{(2m-1)\pi x}{2L}\,dx & \text{if } n = 2m-1, \\ 0 & \text{if } n = 2m. \end{cases}$$

Therefore,the Fourier sine series of f_4 on $[0,2L]$ is $\displaystyle\sum_{n=1}^{\infty} d_n\sin\frac{(2n-1)\pi x}{2L}$ with

$$d_n = \frac{2}{L}\int_0^L f(x)\sin\frac{(2n-1)\pi x}{2L}\,dx.$$

11.3.60. The Fourier cosine series of f_4 on $[0, 2L]$ is $A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{2L}$, where

$$A_0 = \frac{1}{2L}\int_0^{2L} f_4(x)\,dx = \frac{1}{2L}\left[\int_0^L f(x)\,dx + \int_L^{2L} f(2L-x)\,dx\right] = \frac{1}{L}\int_0^L f(x)\,dx$$

and

$$A_n = \frac{1}{L}\int_0^{2L} f_4(x)\cos\frac{n\pi x}{2L}\,dx = \frac{1}{L}\left[\int_0^L f(x)\cos\frac{n\pi x}{2L}\,dx + \int_L^{2L} f(2L-x)\cos\frac{n\pi x}{2L}\,dx\right].$$

Replacing x by $2L - x$ yields

$$\int_L^{2L} f(2L-x)\cos\frac{n\pi x}{2L}\,dx = -\int_L^0 f(x)\cos\frac{n\pi(2L-x)}{2L}\,dx = \int_0^L f(x)\cos\frac{n\pi(2L-x)}{2L}\,dx.$$

Since $\cos\frac{n\pi(2L-x)}{2L} = \cos n\pi \cos\frac{n\pi x}{2L} = (-1)^n \cos\frac{n\pi x}{2L}$,

$$A_n = \frac{1+(-1)^n}{L}\int_0^L f(x)\cos\frac{n\pi x}{2L}\,dx = \begin{cases} 0 & \text{if } n = 2m-1 \\ \dfrac{2}{L}\displaystyle\int_0^L f(x)\cos\frac{m\pi x}{L}\,dx & \text{if } n = 2m. \end{cases}$$

Therefore,the Fourier cosine series of f_4 on $[0, 2L]$ is $A_0 + \sum_{n=0}^{\infty} A_{2n} cos\frac{n\pi x}{L} = a_0 + \sum_{n=0}^{\infty} a_n \cos\frac{n\pi x}{L}$.

CHAPTER 12
Fourier Solutions of Partial Differential

12.1 THE HEAT EQUATION

12.1.2. $X(x)T(t)$ satisfies $u_t = a^2u_{xx}$ if $X'' + \lambda X = 0$ and (A) $T' = -a^2\lambda T$ for the same value of λ. The product also satisfies the boundary conditions $u(0,t) = u_x(L,t) = 0$, $t > 0$, if and only if $X(0) = X'(L) = 0$. Since we are interested in nontrivial solutions, X must be a nontrivial solution of (B) $X'' + \lambda X = 0$, $X(0) = 0$, $X'(L) = 0$. From Theorem 11.1.4, $\lambda_n = (2n-1)^2\pi^2/4L^2$ is an eigenvalue of (B) with associated eigenfunction $X_n = \sin\dfrac{(2n-1)\pi x}{2L}$, $n = 1, 2, 3, \dots$. Substituting $\lambda = (2n-1)^2\pi^2/4L^2$ into (A) yields $T' = -((2n-1)^2\pi^2a^2/4L^2)T$, which has the solution $T_n = e^{-(2n-1)^2\pi^2a^2t/4L^2}$.

We have now shown that the functions $u_n(x,t) = e^{-(2n-1)^2\pi^2a^2t/4L^2}\sin\dfrac{(2n-1)\pi x}{2L}, n = 1, 2, 3, \dots$ satisfy $u_t = a^2u_{xx}$ and the boundary conditions $u(0,t) = u_x(L,t) = 0$, $t > 0$. Any finite sum $\displaystyle\sum_{n=1}^{m} d_n e^{-(2n-1)^2\pi^2a^2t/4L^2}\sin\frac{(2n-1)\pi x}{2L}$ also has these properties. Therefore,it is plausible to expect that that this is also true of the infinite series (C) $u(x,t) = \displaystyle\sum_{n=1}^{\infty} d_n e^{-(2n-1)^2\pi^2a^2t/4L^2}\sin\frac{(2n-1)\pi x}{2L}$ under suitable conditions on the coefficients $\{d_n\}$. Since $u(x,0) = \displaystyle\sum_{n=1}^{\infty} d_n \sin\frac{(2n-1)\pi x}{2L}$, if $\{d_n\}$ are the mixed Fourier sine coefficients of f on $[0, L]$, then $u(x,0) = f(x)$ at all points x in $[0, L]$ where the mixed Fourier sine series converges to $f(x)$. In this case (C) is a formal solution of the initial-boundary value problem of Definition 12.1.3.

12.1.8. Since $f(0) = f(1) = 0$ and $f''(x) = -2$, Theorem 11.3.5(**b**) implies that

$$\begin{aligned}
\alpha_n &= \frac{4}{n^2\pi^2}\int_0^1 \sin n\pi x\,dx = -\frac{4}{n^3\pi^3}\cos n\pi x\Big|_0^1 = -\frac{4}{n^3\pi^2}(\cos n\pi - 1) \\
&= \begin{cases} \dfrac{8}{(2m-1)^3\pi^3}, & \text{if } n = 2m-1, \\ 0, & \text{if } n = 2m; \end{cases}
\end{aligned}$$

$S(x) = \dfrac{8}{\pi^3}\displaystyle\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\sin\frac{(2n-1)\pi x}{L}$. From Definition 12.1.1,

$$u(x,t) = \frac{8}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}e^{-(2n-1)^2\pi^2t}\sin(2n-1)\pi x.$$

12.1.10.

$$\begin{aligned}\alpha_1 &= \frac{2}{\pi}\int_0^\pi x\sin^2 x\,dx = \frac{1}{\pi}\int_0^\pi x(1-\cos 2x)\,dx = \frac{x^2}{2\pi}\Big|_0^\pi - \frac{1}{\pi}\int_0^\pi x\cos 2x\,dx \\ &= \frac{\pi}{2} - \frac{1}{2\pi}\left[x\sin 2x\Big|_0^\pi - \int_0^\pi \cos 2x\,dx\right] = \frac{\pi}{2} + \frac{\sin 2x}{4\pi}\Big|_0^\pi = \frac{\pi}{2};\end{aligned}$$

if $n \geq 2$, then

$$\begin{aligned}\alpha_n &= \frac{2}{\pi}\int_0^\pi x\sin x\sin nx\,dx = \frac{1}{\pi}\int_0^\pi x[\cos(n-1)x - \cos(n+1)x]\,dx \\ &= \frac{1}{\pi}\left[x\left[\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right]\Big|_0^\pi - \int_0^\pi \left[\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right]dx\right] \\ &= \frac{1}{\pi}\left[\frac{\cos(n-1)x}{(n-1)^2} - \frac{\cos(n+1)x}{(n+1)^2}\right]\Big|_0^\pi = \frac{1}{\pi}\left[\frac{1}{(n-1)^2} - \frac{1}{(n+1)^2}\right]\left[(-1)^{n+1}-1\right] \\ &= \frac{4n}{(n^2-1)^2\pi}\left[(-1)^{n+1}-1\right] = \begin{cases} 0 & \text{if } n = 2m-1, \\ -\dfrac{16m}{(4m^2-1)\pi} & \text{if } n = 2m;\end{cases}\end{aligned}$$

$S(x) = \dfrac{\pi}{2}\sin x - \dfrac{16}{\pi}\displaystyle\sum_{n=1}^\infty \frac{n}{(4n^2-1)^2}\sin 2nx$. From Definition 12.1.1,

$$u(x,t) = \frac{\pi}{2}e^{-3t}\sin x - \frac{16}{\pi}\sum_{n=1}^\infty \frac{n}{(4n^2-1)^2}e^{-12n^2t}\sin 2nx.$$

12.1.12. Since $f(0) = f(L) = 0$ and $f''(x) = -6x$, Theorem 11.3.5(**b**) implies that

$$\begin{aligned}\alpha_n &= \frac{36}{n^2\pi^2}\int_0^3 x\sin\frac{n\pi x}{3}\,dx = -\frac{108}{n^3\pi^3}\left[x\cos\frac{n\pi x}{3}\Big|_0^3 - \int_0^3 \cos\frac{n\pi x}{3}\,dx\right] \\ &= (-1)^{n+1}\frac{108}{n^3\pi^3} + \frac{108}{n^4\pi^4}\sin\frac{n\pi x}{3}\Big|_0^3 = (-1)^{n+1}\frac{324}{n^3\pi^3};\end{aligned}$$

$S(x) = -\dfrac{324}{\pi^3}\displaystyle\sum_{n=1}^\infty \frac{(-1)^n}{n^3}\sin\frac{n\pi x}{3}$. From Definition 12.1.1, $u(x,t) = -\dfrac{324}{\pi^3}\displaystyle\sum_{n=1}^\infty \frac{(-1)^n}{n^3}e^{-4n^2\pi^2 t/9}\sin\frac{n\pi x}{3}$.

12.1.14. Since $f(0) = f(1) = f''(0) = f''(L) = 0$ and $f^{(4)} = 360x$, Theorem 11.3.5(**b**) and Exercise 35(**b**) of Section 11.3 imply that

$$\begin{aligned}\alpha_n &= \frac{720}{n^4\pi^4}\int_0^1 x\sin n\pi x\,dx = -\frac{720}{n^5\pi^5}\left[x\cos n\pi x\Big|_0^1 - \int_0^1 \cos n\pi x\,dx\right] \\ &= (-1)^{n+1}\frac{720}{n^5\pi^5} + \frac{720}{n^6\pi^6}\sin\frac{n\pi x}{L}\Big|_0^1 = (-1)^{n+1}\frac{720}{n^5\pi^5};\end{aligned}$$

$S(x) = -\dfrac{720}{\pi^5}\displaystyle\sum_{n=1}^\infty \frac{(-1)^n}{n^5}\sin n\pi x$. From Definition 12.1.1, $u(x,t) = -\dfrac{720}{\pi^5}\displaystyle\sum_{n=1}^\infty \frac{(-1)^n}{n^5}e^{-7n^2\pi^2 t}\sin n\pi x$.

12.1.16. Since $f(0) = f(1) = f''(0) = f''(L) = 0$ and $f^{(4)} = 120(3x-1)$, Theorem 11.3.5(**b**) and Exercise 35(**b**) of Section 11.3 imply that

$$\begin{aligned}\alpha_n &= \frac{240}{n^4\pi^4}\int_0^1 (3x-1)\sin n\pi x\,dx = -\frac{240}{n^5\pi^5}\left[(3x-1)\cos n\pi x\Big|_0^1 - 3\int_0^1 \cos n\pi x\,dx\right]\\ &= -\frac{240}{n^5\pi^5}\left[(-1)^n 2+1\right] + \frac{720}{n^6\pi^6}\sin n\pi x\Big|_0^1 = -\frac{240}{n^5\pi^5}\left[1+(-1)^n 2\right];\end{aligned}$$

$S(x) = -\dfrac{240}{\pi^5}\displaystyle\sum_{n=1}^{\infty}\frac{1+(-1)^n 2}{n^5}\sin n\pi x$. From Definition 12.1.1,

$$u(x,t) = -\frac{240}{\pi^5}\sum_{n=1}^{\infty}\frac{1+(-1)^n 2}{n^5}e^{-2n^2\pi^2 t}\sin n\pi x.$$

12.1.18. $\alpha_0 = \dfrac{1}{2}\displaystyle\int_0^2 (x^2-4x)\,dx = \frac{1}{L}\left(\frac{x^3}{3}-2x^2\right)\Big|_0^2 = -\frac{8}{3}$; if $n \ge 1$,

$$\begin{aligned}\alpha_n &= \int_0^2 (x^2-4x)\cos\frac{n\pi x}{2}\,dx = \frac{2}{n\pi}\left[(x^2-4x)\sin\frac{n\pi x}{2}\Big|_0^2 - 2\int_0^2 (x-2)\sin\frac{n\pi x}{2}\,dx\right]\\ &= \frac{8}{n^2\pi^2}\left[(x-2)\cos\frac{n\pi x}{2}\Big|_0^2 - \int_0^2 \cos\frac{n\pi x}{2}\,dx\right] = \frac{16}{n^2\pi^2} - \frac{32}{n^3\pi^3}\sin\frac{n\pi x}{2}\Big|_0^2 = \frac{16}{n^2\pi^2};\end{aligned}$$

$C(x) = -\dfrac{8}{3}+\dfrac{16}{\pi^2}\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\frac{n\pi x}{2}$. From Definition 12.1.3, $u(x,t) = -\dfrac{8}{3}+\dfrac{16}{\pi^2}\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}e^{-n^2\pi^2 t}\cos\frac{n\pi x}{2}$.

12.1.20. From Example 11.3.5, $C(x) = 4-\dfrac{384}{\pi^4}\displaystyle\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}\cos\frac{(2n-1)\pi x}{2}$. From Definition 12.1.3, $u(x,t) = 4-\dfrac{384}{\pi^4}\displaystyle\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}e^{-3(2n-1)^2\pi^2 t/4}\cos\frac{(2n-1)\pi x}{2}$.

12.1.22. $\alpha_0 = \dfrac{1}{L}\displaystyle\int_0^1 (3x^4-4Lx^3)\,dx = \frac{1}{L}\left(\frac{3x^5}{5}-x^4\right)\Big|_0^1 = -\frac{2}{5}$. Since $f'(0) = f'(1) = 0$ and $f'''(x) = 24(3x-1)$, Theorem 11.3.5(**a**) implies that

$$\begin{aligned}\alpha_n &= \frac{48}{n^3\pi^3}\int_0^1 (3x-1)\sin n\pi x\,dx = -\frac{48}{n^4\pi^4}\left[(3x-1)\cos n\pi x\Big|_0^1 - 3\int_0^1 \cos n\pi x\,dx\right]\\ &= -\frac{48}{n^4\pi^4}\left[(-1)^n 2+1\right] + \frac{144}{n^5\pi^5}\sin n\pi x\Big|_0^1 = -\frac{48}{n^4\pi^4}\left[1+(-1)^n 2\right], \quad n \ge 1;\end{aligned}$$

$C(x) = -\dfrac{2}{5}-\dfrac{48}{\pi^4}\displaystyle\sum_{n=1}^{\infty}\frac{1+(-1)^n 2}{n^4}\cos n\pi x$. From Definition 12.1.3,

$$u(x,t) = -\frac{2}{5}-\frac{48}{\pi^4}\sum_{n=1}^{\infty}\frac{1+(-1)^n 2}{n^4}e^{-3n^2\pi^2 t}\cos n\pi x.$$

12.1.24. $\alpha_0 = \dfrac{1}{\pi}\displaystyle\int_0^\pi (x^4 - 2\pi x^3 + \pi^2 x^2)\,dx = \dfrac{1}{\pi}\left(\dfrac{x^5}{5} - \dfrac{\pi x^4}{2} + \dfrac{\pi^2 x^3}{3}\right)\Big|_0^\pi = \dfrac{\pi^4}{30}$. Since $f'(0) = f'(\pi) = 0$ and $f'''(x) = 12(2x - \pi)$, Theorem 11.3.5**(a)** implies that

$$\begin{aligned}
\alpha_n &= \frac{24}{n^3\pi}\int_0^\pi (2x-\pi)\sin nx\,dx = -\frac{24}{n^4\pi}\left[(2x-\pi)\cos nx\Big|_0^\pi - 2\int_0^\pi \cos nx\,dx\right] \\
&= -\frac{24}{n^4\pi}\left[(-1)^n\pi + \pi\right] + \frac{48}{n^5\pi}\sin nx\Big|_0^\pi = -\frac{24}{n^4}\left[1 + (-1)^n\right] \\
&= \begin{cases} 0 & \text{if } n = 2m-1, \\ -\dfrac{3}{m^4} & \text{if } n = 2m, \end{cases} \qquad n \ge 1;
\end{aligned}$$

$C(x) = \dfrac{\pi^4}{30} - 3\displaystyle\sum_{n=1}^\infty \dfrac{1}{n^4}\cos 2nx$. From Definition 12.1.3, $u(x,t) = \dfrac{\pi^4}{30} - 3\displaystyle\sum_{n=1}^\infty \dfrac{1}{n^4}e^{-4n^2 t}\cos 2nx$.

12.1.26.

$$\begin{aligned}
\alpha_n &= \frac{2}{\pi}\int_0^\pi (\pi x - x^2)\sin\frac{(2n-1)x}{2}\,dx \\
&= -\frac{4}{(2n-1)\pi}\left[(\pi x - x^2)\cos\frac{(2n-1)x}{2}\Big|_0^\pi - \int_0^\pi (\pi - 2x)\cos\frac{(2n-1)x}{2}\,dx\right] \\
&= \frac{8}{(2n-1)^2\pi}\left[(\pi - 2x)\sin\frac{(2n-1)x}{2}\Big|_0^\pi + 2\int_0^\pi \sin\frac{(2n-1)x}{2}\,dx\right] \\
&= (-1)^n\frac{8}{(2n-1)^2} - \frac{32}{(2n-1)^3\pi}\cos\frac{(2n-1)x}{2}\Big|_0^\pi \\
&= (-1)^n\frac{8}{(2n-1)^2} + \frac{32}{(2n-1)^3\pi};
\end{aligned}$$

$S_M(x) = 8\displaystyle\sum_{n=1}^\infty \dfrac{1}{(2n-1)^2}\left[(-1)^n + \dfrac{4}{(2n-1)\pi}\right]\sin\dfrac{(2n-1)x}{2}$. From Definition 12.1.4,

$$u(x,t) = 8\sum_{n=1}^\infty \frac{1}{(2n-1)^2}\left[(-1)^n + \frac{4}{(2n-1)\pi}\right]e^{-3(2n-1)^2 t/4}\sin\frac{(2n-1)x}{2}.$$

12.1.28. Since $f(0) = f'(1) = 0$, and $f''(x) = 6(1-2x)$, Theorem 11.3.5**(d)** implies that

$$\begin{aligned}
\alpha_n &= -\frac{48}{(2n-1)^2\pi^2}\int_0^1 (1-2x)\sin\frac{(2n-1)\pi x}{2}\,dx \\
&= \frac{96}{(2n-1)^3\pi^3}\left[(1-2x)\cos\frac{(2n-1)\pi x}{2}\Big|_0^1 + 2\int_0^1 \cos\frac{(2n-1)\pi x}{2}\,dx\right] \\
&= \frac{96}{(2n-1)^3\pi^3}\left[-1 + \frac{4}{(2n-1)\pi}\sin\frac{(2n-1)\pi x}{2}\Big|_0^1\right] \\
&= -\frac{96}{(2n-1)^3\pi^3}\left[1 + (-1)^n\frac{4}{(2n-1)\pi}\right];
\end{aligned}$$

$S_M(x) = -\frac{96}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\left[1+(-1)^n\frac{4}{(2n-1)\pi}\right]\sin\frac{(2n-1)\pi x}{2}$. From Definition 12.1.4,

$$u(x,t) = -\frac{96}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\left[1+(-1)^n\frac{4}{(2n-1)\pi}\right]e^{-(2n-1)^2\pi^2 t}\sin\frac{(2n-1)\pi x}{2}.$$

12.1.30. Since $f(0) = f'(L) = f''(0) = 0$ and $f'''(x) = 6$, Theorem 11.3.5**(d)** and Exercise 11.3.50**(b)** imply that

$$\begin{aligned} \alpha_n &= -\frac{96}{(2n-1)^3\pi^3}\int_0^1 \cos\frac{(2n-1)\pi x}{2}\,dx \\ &= -\frac{192}{(2n-1)^4\pi^4}\sin\frac{(2n-1)\pi x}{2}\bigg|_0^1 = (-1)^n\frac{192}{(2n-1)^4\pi^4}; \end{aligned}$$

$S_M(x) = \frac{192}{\pi^4}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^4}\sin\frac{(2n-1)\pi x}{2}$. From Definition 12.1.4,

$$u(x,t) = \frac{192}{\pi^4}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^4}e^{-(2n-1)^2\pi^2 t}\sin\frac{(2n-1)\pi x}{2}.$$

12.1.32. Since $f(0) = f'(1) = f''(0) = 0$ and $f'''(x) = 12(2x-1)$, Theorem 11.3.5**(d)** and Exercise 11.3.50**(b)** imply that

$$\begin{aligned} \alpha_n &= -\frac{192}{(2n-1)^3\pi^3}\int_0^1 (2x-1)\cos\frac{(2n-1)\pi x}{2}\,dx \\ &= -\frac{384}{(2n-1)^4\pi^4}\left[(2x-1)\sin\frac{(2n-1)\pi x}{2}\bigg|_0^1 - 2\int_0^1 \sin\frac{(2n-1)\pi x}{2}\,dx\right] \\ &= -\frac{384}{(2n-1)^4\pi^4}\left[(-1)^{n+1}1 + \frac{4}{(2n-1)\pi}\cos\frac{(2n-1)\pi x}{2}\bigg|_0^1\right] \\ &= -\frac{384}{(2n-1)^4\pi^4}\left[(-1)^{n+1}1 - \frac{4}{(2n-1)\pi}\right] = \frac{384}{(2n-1)^4\pi^4}\left[(-1)^n + \frac{4}{(2n-1)\pi}\right]; \end{aligned}$$

$S_M(x) = \frac{384}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}\left[(-1)^n + \frac{4}{(2n-1)\pi}\right]\sin\frac{(2n-1)\pi x}{2}$. From Definition 12.1.4,

$$u(x,t) = \frac{384}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}\left[(-1)^n + \frac{4}{(2n-1)\pi}\right]e^{-(2n-1)^2\pi^2 t}\sin\frac{(2n-1)\pi x}{2}.$$

12.1.36. From Example 11.3.3, $C_M(x) = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi x}{2}$. From Definition 12.1.5,

$$u(x,t) = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}e^{-3(2n-1)^2\pi^2 t/4}\cos\frac{(2n-1)\pi x}{2}.$$

12.1.38. Since $f'(0) = f(\pi) = 0$ and $f''(x) = -2$, Theorem 11.3.5**(c)** implies that

$$\alpha_n = \frac{16}{(2n-1)^2\pi}\int_0^\pi \cos\frac{(2n-1)x}{2}\,dx = \frac{32}{(2n-1)^3\pi}\sin\frac{(2n-1)x}{2}\bigg|_0^\pi = (-1)^{n+1}\frac{32}{(2n-1)^3\pi};$$

$C_M(x) = -\dfrac{32}{\pi}\displaystyle\sum_{n=1}^\infty \frac{(-1)^n}{(2n-1)^3}\cos\frac{(2n-1)x}{2}$. From Definition 12.1.5,

$$u(x,t) = -\frac{32}{\pi}\sum_{n=1}^\infty \frac{(-1)^n}{(2n-1)^3} e^{-7(2n-1)^2t/4}\cos\frac{(2n-1)x}{2}.$$

12.1.40. Since $f'(0) = f(1) = 0$ and $f''(x) = 6(2x+1)$, Theorem 11.3.5**(c)** implies that

$$\begin{aligned}
\alpha_n &= -\frac{48L}{(2n-1)^2\pi^2}\int_0^1 (2x+1)\cos\frac{(2n-1)\pi x}{2}\,dx \\
&= -\frac{96}{(2n-1)^3\pi^3}\left[(2x+1)\sin\frac{(2n-1)\pi x}{2}\bigg|_0^1 - 2\int_0^1 \sin\frac{(2n-1)\pi x}{2}\right]dx \\
&= -\frac{96}{(2n-1)^3\pi^3}\left[(-1)^{n+1}3 - \frac{4}{(2n-1)\pi}\cos\frac{(2n-1)\pi x}{2}\bigg|_0^1\right] \\
&= \frac{96}{(2n-1)^3\pi^3}\left[(-1)^n 3 + \frac{4}{(2n-1)\pi}\right];
\end{aligned}$$

$C_M(x) = \dfrac{96}{\pi^3}\displaystyle\sum_{n=1}^\infty \frac{1}{(2n-1)^3}\left[(-1)^n 3 + \frac{4}{(2n-1)\pi}\right]\cos\frac{(2n-1)\pi x}{2}$. From Definition 12.1.5,

$$u(x,t) = \frac{96}{\pi^3}\sum_{n=1}^\infty \frac{1}{(2n-1)^3}\left[(-1)^n 3 + \frac{4}{(2n-1)\pi}\right] e^{-(2n-1)^2\pi^2t/4}\cos\frac{(2n-1)\pi x}{2}.$$

12.1.42. Theorem 11.3.5**(c)** and Exercise 11.3.42**(b)** imply that Since $f'(0) = f(1) = f''(1) = 0$ and $f'''(x) = 12(2x-1)$,

$$\begin{aligned}
\alpha_n &= \frac{192}{(2n-1)^3\pi^3}\int_0^1 (2x-1)\sin\frac{(2n-1)\pi x}{2}\,dx \\
&= -\frac{384}{(2n-1)^4\pi^4}\left[(2x-1)\cos\frac{(2n-1)\pi x}{2}\bigg|_0^1 - 2\int_0^1 \cos\frac{(2n-1)\pi x}{2}\right]dx \\
&= -\frac{384}{(2n-1)^4\pi^4}\left[1 - \frac{4}{(2n-1)\pi}\sin\frac{(2n-1)\pi x}{2}\bigg|_0^1\right] = -\frac{384}{(2n-1)^4\pi^4}\left[1 + \frac{(-1)^n 4}{(2n-1)\pi}\right];
\end{aligned}$$

$C_M(x) = -\dfrac{384}{\pi^4}\displaystyle\sum_{n=1}^\infty \frac{1}{(2n-1)^4}\left[1 + \frac{(-1)^n 4}{(2n-1)\pi}\right]\cos\frac{(2n-1)\pi x}{2}$. From Definition 12.1.5,

$$u(x,t) = -\frac{384}{\pi^4}\sum_{n=1}^\infty \frac{1}{(2n-1)^4}\left[1 + \frac{(-1)^n 4}{(2n-1)\pi}\right] e^{-(2n-1)^2\pi^2t/4}\cos\frac{(2n-1)\pi x}{2}.$$

12.1.44. $\alpha_n = \dfrac{2}{L}\displaystyle\int_0^{L/2} \sin\dfrac{n\pi x}{L}\,dx = -\dfrac{2}{n\pi}\cos\dfrac{n\pi x}{L}\Big|_0^{L/2} = \dfrac{2}{n\pi}\left[1-\cos\dfrac{n\pi}{2}\right]$;
$S(x) = \dfrac{2}{\pi}\displaystyle\sum_{n=1}^{\infty}\dfrac{1}{n}\left[1-\cos\dfrac{n\pi}{2}\right]\sin\dfrac{n\pi x}{L}$. From Definition 12.1.1,

$$u(x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[1-\cos\frac{n\pi}{2}\right]e^{-n^2\pi^2t^2/L^2}\sin\frac{n\pi x}{L}.$$

12.1.46.

$$\begin{aligned}\alpha_n &= \frac{2}{L}\int_0^{L/2}\sin\frac{(2n-1)\pi x}{2L}\,dx = -\frac{4}{(2n-1)\pi}\cos\frac{(2n-1)\pi x}{2L}\Big|_0^{L/2}\\ &= \frac{4}{(2n-1)\pi}\left[1-\cos\frac{(2n-1)\pi)}{4}\right];\end{aligned}$$

$S_M(x) = \dfrac{4}{\pi}\displaystyle\sum_{n=1}^{\infty}\dfrac{1}{2n-1}\left[1-\cos\dfrac{(2n-1)\pi)}{4}\right]\sin\dfrac{(2n-1)\pi x}{2L}$. From Definition 12.1.4,

$$u(x,t) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\left[1-\cos\frac{(2n-1)\pi)}{4}\right]e^{-(2n-1)^2\pi^2a^2t/4L^2}\sin\frac{(2n-1)\pi x}{2L}.$$

12.1.48. Let $u(x,t) = v(x,t)+q(x)$; then $u_t = v_t$ and $u_{xx} = v_{xx}+q''$, so

$$\begin{array}{c} v_t = 9v_{xx}+9q''-54x, \quad 0<x<4, \quad t>0,\\ v(0,t) = 1-q(0), \quad v(4,t) = 61-q(4), \quad t>0,\\ v(x,0) = 2-x+x^3-q(x), \quad 0\le x\le 4.\end{array} \tag{A}$$

We want $q''(x) = 6x$, $q(0) = 1$, $q(4) = 61$; $q(x) = x^3+a_1+a_2x$; $q(0) = 1 \Rightarrow a_1 = 1$; $q(x) = x^3+1+a_2x$; $q(4) = 61 \Rightarrow a_2 = -1$; $q(x) = x^3+1-x$. Now (A) reduces to

$$\begin{array}{c} v_t = 9v_{xx}, \quad 0<x<4, \quad t>0,\\ v(0,t) = 0, \quad v(4,t) = 0, \quad t>0,\\ v(x,0) = 1, \quad 0\le x\le 4,\end{array}$$

which we solve by separation of variables.

$$\begin{aligned}\alpha_n &= \frac{1}{2}\int_0^L\sin\frac{n\pi x}{4}\,dx = -\frac{2}{n\pi}\cos\frac{n\pi x}{4}\Big|_0^4\\ &= \frac{2}{n\pi}[1-(-1)^n] = \begin{cases}\dfrac{4}{(2m-1)\pi} & \text{if } n = 2m-1,\\ 0 & \text{if } n = 2m;\end{cases}\end{aligned}$$

$S(x) = \dfrac{4}{\pi}\displaystyle\sum_{n=1}^{\infty}\dfrac{1}{(2n-1)}\sin\dfrac{(2n-1)\pi x}{4}$. From Definition 12.1.1,

$$v(x,t) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}e^{-9\pi^2n^2t/16}\sin\frac{(2n-1)\pi x}{4}.$$

Therefore,

$$u(x,t) = 1 - x + x^3 + \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{e^{-9\pi^2 n^2 t/16}}{(2n-1)} \sin\frac{(2n-1)\pi x}{4}.$$

12.1.50. Let $u(x,t) = v(x,t) + q(x)$; then $u_t = v_t$ and $u_{xx} = v_{xx} + q''$, so

$$\begin{array}{c} v_t = 3u_{xx} + 3q'' - 18x, \quad 0 < x < 1, \quad t > 0, \\ v_x(0,t) = -1 - q'(0), \quad v(1,t) = -1 - q(1), \quad t > 0, \\ v(x,0) = x^3 - 2x - q(x), \quad 0 \le x \le 1. \end{array} \qquad \text{(A)}$$

We want $q''(x) = 6x$, $q'(0) = -1$, $q(1) = -1$; $q'(x) = 3x^2 + a_2$; $q'(0) = -1 \Rightarrow a_2 = -1$; $q'(x) = 3x^2 - 1$; $q(x) = x^3 - x + a_1 x$; $q(1) = -1 \Rightarrow a_1 = -1$; $q(x) = x^3 - x - 1$. Now (A) reduces to

$$\begin{array}{c} v_t = 3u_{xx}, \quad 0 < x < 1, \quad t > 0, \\ v_x(0,t) = 0, \quad v(1,t) = 0, \quad t > 0, \\ v(x,0) = 1 - x, \quad 0 \le x \le 1. \end{array}$$

From Example 11.3.3, $C_M(x) = \dfrac{8}{\pi^2}\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi x}{2}$. From Definition 12.1.5, $v(x,t) = \dfrac{8}{\pi^2}\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} e^{-3(2n-1)^2\pi^2 t/4}\cos\frac{(2n-1)\pi x}{2}$. Therefore,

$$u(x,t) = -1 - x + x^3 + \frac{8}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} e^{-3(2n-1)^2\pi^2 t/4}\cos\frac{(2n-1)\pi x}{2}.$$

12.1.52. Let $u(x,t) = v(x,t) + q(x)$; then $u_t = v_t$ and $u_{xx} = v_{xx} + q''$, so

$$\begin{array}{c} v_t = v_{xx} + q'' + \pi^2 \sin\pi x, \quad 0 < x < 1, \quad t > 0, \\ v(0,t) = -q(0), \quad v_x(1,t) = -\pi - q'(1), \quad t > 0, \\ v(x,0) = 2\sin\pi x - q(x), \quad 0 \le x \le 1. \end{array} \qquad \text{(A)}$$

We want $q''(x) = -\pi^2 \sin\pi x$, $q(0) = 0$, $q'(1) = -\pi$; $q'(x) = \pi\cos\pi x + a_2$; $q'(1) = -\pi \Rightarrow a_2 = 0$; $q'(x) = \pi\cos\pi x$; $q(x) = \sin\pi x + a_1$; $q(0) = 0 \Rightarrow a_1 = 0$; $q(x) = \sin\pi x$. Now (A) reduces to

$$\begin{array}{c} v_t = v_{xx}, \quad 0 < x < 1, \quad t > 0, \\ v(0,t) = 0, \quad v_x(1,t) = 0, \quad t > 0, \\ v(x,0) = \sin\pi x, \quad 0 \le x \le 1. \end{array}$$

$$\begin{aligned} \alpha_n &= 2\int_0^1 \sin\pi x \sin\frac{(2n-1)\pi x}{2}\,dx = \int_0^1 \left[\frac{\cos(2n-3)\pi x}{2} - \frac{\cos(2n+1)\pi x}{2}\right] dx \\ &= \frac{2}{\pi}\left[\frac{\sin(2n-3)\pi x/2}{(2n-3)} - \frac{\sin(2n+1)\pi x/2}{(2n+1)}\right]\Bigg|_0^1 \\ &= (-1)^n \frac{2}{\pi}\left[\frac{1}{2n-3} - \frac{1}{2n+1}\right] = (-1)^n \frac{8}{\pi}\frac{1}{(2n+1)(2n-3)}; \end{aligned}$$

$S_M(x) = \dfrac{8}{\pi}\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)(2n-3)} \sin\frac{(2n-1)\pi x}{2}$. From Definition 12.1.4,

$$v(x,t) = \frac{8}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)(2n-3)} e^{-(2n-1)^2\pi^2 t/4} \sin\frac{(2n-1)\pi x}{2}.$$

Therefore, $u(x,t) = \sin \pi x + \dfrac{8}{\pi}\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)(2n-3)} e^{-(2n-1)^2\pi^2 t/4} \sin \frac{(2n-1)\pi x}{2}$.

12.1.54. **(a)** Since f is piecewise smooth of $[0, L]$, there is a constant K such that $|f(x)| \le K$, $0 \le x \le L$. Therefore, $|\alpha_n| = \dfrac{2}{L}\left|\displaystyle\int_0^L f(x) \sin \frac{n\pi x}{L}\, dx\right| \le \dfrac{2}{L}\displaystyle\int_0^L |f(x)|\, dx = 2K$. Hence, $|\alpha_n e^{-n^2\pi^2 a^2 t/L}| \le 2Ke^{-n^2\pi^2 a^2 t/L}$, so $u(x,t)$ converges for all x if $t > 0$, by the comparison test.

(b) Let t be a fixed positive number. Apply Theorem 12.1.2 with $z = x$ and $w_n(x) = \alpha_n e^{-n^2\pi^2 t/L^2} \sin \dfrac{n\pi x}{L}$. Then $w_n'(x) = \dfrac{\pi}{L} n\alpha_n e^{-n^2\pi^2 t/L^2} \cos \dfrac{n\pi x}{L}$, so $|w_n'(x)| \le \dfrac{2K\pi}{L} n e^{-n^2\pi^2 t/L^2}$, $-\infty < x < \infty$. Since $\displaystyle\sum_{n=1}^{\infty} n e^{-n^2\pi^2 a^2 t/L^2}$ converges if $t > 0$, Theorem 12.1.1 (with $z_1 = x_1$ and $z_2 = x_2$ arbitrary) implies the concclusion.

(c) Since $\displaystyle\sum_{n=1}^{\infty} n^2 e^{-n^2\pi^2 a^2 t/L^2}$ also converges if $t > 0$, an argument like that in **(b)** with $w_n(x) = n\alpha_n e^{-n^2\pi^2 t/L^2} \cos \dfrac{n\pi x}{L}$ yields the conclusion.

(d) Let x be arbitrary, but fixed. Apply Theorem 12.1.2 with $z = t$ and $w_n(t) = \alpha_n e^{-n^2\pi^2 a^2 t/L^2} \sin \dfrac{n\pi x}{L}$. Then $w_n'(t) = -\dfrac{\pi^2 a^2}{L^2} n^2 \alpha_n e^{-n^2\pi^2 a^2 t/L^2} \sin \dfrac{n\pi x}{L}$, so $|w_n'(t)| \le \dfrac{2K\pi^2 a^2}{L} n^2 e^{-n^2\pi^2 a^2 t_0/L^2}$ if $t > t_0$. Since $\displaystyle\sum_{n=1}^{\infty} n^2 e^{-n^2\pi^2 a^2 t_0/L^2}$ converges, Theorem 12.1.1 (with $z_1 = t_0 > 0$ and $z_2 = t_1$ arbitrary implies the conclusion for $t \ge t_0$. However, since t_0 is an arbitrary positive number, this holds for $t > 0$.

12.2 THE WAVE EQUATION

12.2.1. $\beta_n = 2\left[\int_0^{1/2} x \sin n\pi x + \int_{1/2}^1 (1-x) \sin n\pi x\, dx\right]$;

$$\begin{aligned}
\int_0^{1/2} x \sin n\pi x\, dx &= -\frac{1}{n\pi}\left[x \cos n\pi x\Big|_0^{1/2} - \int_0^{1/2} \cos n\pi x\, dx\right] \\
&= -\frac{1}{2n\pi}\cos\frac{n\pi}{2} + \frac{1}{n^2\pi^2} \sin n\pi x\Big|_0^{1/2} = -\frac{1}{2n\pi}\cos\frac{n\pi}{2} + \frac{1}{n^2\pi^2}\sin\frac{n\pi}{2};
\end{aligned}$$

$$\begin{aligned}
\int_0^{1/2} (1-x) \sin n\pi x\, dx &= -\frac{1}{n\pi}\left[(1-x)\cos n\pi x\Big|_{1/2}^{1} + \int_0^{1/2} \cos n\pi x\, dx\right] \\
&= \frac{1}{2n\pi}\cos\frac{n\pi}{2} - \frac{1}{n^2\pi^2} \sin n\pi x\Big|_{1/2}^{1} = \frac{1}{2n\pi}\cos\frac{n\pi}{2} + \frac{1}{n^2\pi^2}\sin\frac{n\pi}{2};
\end{aligned}$$

$$b_n = \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2} = \begin{cases} (-1)^{m+1}\dfrac{4}{(2m-1)^2\pi^2} & \text{if } n = 2m-1 \\ 0 & \text{if } n = 2m; \end{cases}$$

$S_g(x) = \dfrac{4}{\pi^2} \displaystyle\sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{(2n-1)^2} \sin(2n-1)\pi x$. From Definition 12.1.1,

$$u(x,t) = \frac{4}{3\pi^3} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^3} \sin 3(2n-1)\pi t \sin(2n-1)\pi x.$$

12.2.2. Since $f(0) = f(1) = 0$ and $f''(x) = -2$, Theorem 11.3.5**(b)** implies that

$$\begin{aligned} \alpha_n &= \frac{4}{n^2\pi^2} \int_0^1 \sin n\pi x \, dx = -\frac{4}{n^3\pi^3} \cos n\pi x \Big|_0^1 = -\frac{4}{n^3\pi^2} (\cos n\pi - 1) \\ &= \begin{cases} \dfrac{8}{(2m-1)^3\pi^3}, & \text{if } n = 2m-1, \\ 0, & \text{if } n = 2m; \end{cases} \end{aligned}$$

$S_f(x) = \dfrac{8}{\pi^3} \displaystyle\sum_{n=1}^{\infty} \dfrac{1}{(2n-1)^3} \sin(2n-1)\pi x$. From Definition 12.1.1,

$$u(x,t) = \frac{8}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} \cos 3(2n-1)\pi t \sin(2n-1)\pi x.$$

12.2.4. Since $g(0) = g(1) = 0$ and $g''(x) = -2$, Theorem 11.3.5**(b)** implies that

$$\begin{aligned} \beta_n &= \frac{4}{n^2\pi^2} \int_0^1 \sin n\pi x \, dx = -\frac{4}{n^3\pi^3} \cos n\pi x \Big|_0^1 = -\frac{4}{n^3\pi^2} (\cos n\pi - 1) \\ &= \begin{cases} \dfrac{8}{(2m-1)^3\pi^3}, & \text{if } n = 2m-1, \\ 0, & \text{if } n = 2m; \end{cases} \end{aligned}$$

$S_g(x) = \dfrac{8}{\pi^3} \displaystyle\sum_{n=1}^{\infty} \dfrac{1}{(2n-1)^3} \sin(2n-1)\pi x$. From Definition 12.1.1,

$$u(x,t) = \frac{8}{3\pi^4} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} \sin 3(2n-1)\pi t \sin(2n-1)\pi x.$$

12.2.6. From Example 11.2.6, $S_f(x) = \dfrac{324}{\pi^3} \displaystyle\sum_{n=1}^{\infty} \dfrac{(-1)^n}{n^3} \sin \dfrac{n\pi x}{3}$. From Definition 12.1.1,

$$u(x,t) = \frac{324}{\pi^3} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \cos \frac{8\pi n t}{3} \sin \frac{n\pi x}{3}.$$

12.2.8. From Example 11.2.6 $S_g(x) = \dfrac{324}{\pi^3} \displaystyle\sum_{n=1}^{\infty} \dfrac{(-1)^n}{n^3} \sin \dfrac{n\pi x}{3}$. From Definition 12.1.1,

$$u(x,t) = \frac{81}{2\pi^4} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4} \sin \frac{8\pi n t}{3} \sin \frac{n\pi x}{3}.$$

12.2.10.

$$\begin{aligned}\alpha_1 &= \frac{2}{\pi}\int_0^\pi x\sin^2 x\,dx = \frac{1}{\pi}\int_0^\pi x(1-\cos 2x)\,dx = \frac{x^2}{2\pi}\bigg|_0^\pi - \frac{1}{\pi}\int_0^\pi x\cos 2x\,dx \\ &= \frac{\pi}{2} - \frac{1}{2\pi}\left[x\sin 2x\bigg|_0^\pi - \int_0^\pi \cos 2x\,dx\right] = \frac{\pi}{2} + \frac{\sin 2x}{4\pi}\bigg|_0^\pi = \frac{\pi}{2};\end{aligned}$$

if $n \ge 2$, then

$$\begin{aligned}\alpha_n &= \frac{2}{\pi}\int_0^\pi x\sin x\sin nx\,dx = \frac{1}{\pi}\int_0^\pi x[\cos(n-1)x - \cos(n+1)x]\,dx \\ &= \frac{1}{\pi}\left[x\left[\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right]\bigg|_0^\pi - \int_0^\pi\left[\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right]dx\right] \\ &= \frac{1}{\pi}\left[\frac{\cos(n-1)x}{(n-1)^2} - \frac{\cos(n+1)x}{(n+1)^2}\right]\bigg|_0^\pi = \frac{1}{\pi}\left[\frac{1}{(n-1)^2} - \frac{1}{(n+1)^2}\right]\left[(-1)^{n+1}-1\right] \\ &= \frac{4n}{(n^2-1)^2\pi}\left[(-1)^{n+1}-1\right] = \begin{cases} 0 & \text{if } n = 2m-1, \\ -\dfrac{16m}{(4m^2-1)\pi} & \text{if } n = 2m;\end{cases}\end{aligned}$$

$S_f(x) = \dfrac{\pi}{2}\sin x - \dfrac{16}{\pi}\displaystyle\sum_{n=1}^\infty \frac{n}{(4n^2-1)^2}\sin 2nx$. From Definition 12.1.1,

$$u(x,t) = \frac{\pi}{2}\cos\sqrt{5}\,t\sin x - \frac{16}{\pi}\sum_{n=1}^\infty \frac{n}{(4n^2-1)^2}\cos 2n\sqrt{5}\,t\sin 2nx.$$

12.2.12.

$$\begin{aligned}\beta_1 &= \frac{2}{\pi}\int_0^\pi x\sin^2 x\,dx = \frac{1}{\pi}\int_0^\pi x(1-\cos 2x)\,dx = \frac{x^2}{2\pi}\bigg|_0^\pi - \frac{1}{\pi}\int_0^\pi x\cos 2x\,dx \\ &= \frac{\pi}{2} - \frac{1}{2\pi}\left[x\sin 2x\bigg|_0^\pi - \int_0^\pi \cos 2x\,dx\right] = \frac{\pi}{2} + \frac{\sin 2x}{4\pi}\bigg|_0^\pi = \frac{\pi}{2};\end{aligned}$$

if $n \ge 2$ then

$$\begin{aligned}\beta_n &= \frac{2}{\pi}\int_0^\pi x\sin x\sin nx\,dx = \frac{1}{\pi}\int_0^\pi x[\cos(n-1)x - \cos(n+1)x]\,dx \\ &= \frac{1}{\pi}\left[x\left[\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right]\bigg|_0^\pi - \int_0^\pi\left[\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right]dx\right] \\ &= \frac{1}{\pi}\left[\frac{\cos(n-1)x}{(n-1)^2} - \frac{\cos(n+1)x}{(n+1)^2}\right]\bigg|_0^\pi = \frac{1}{\pi}\left[\frac{1}{(n-1)^2} - \frac{1}{(n+1)^2}\right]\left[(-1)^{n+1}-1\right] \\ &= \frac{4n}{(n^2-1)^2\pi}\left[(-1)^{n+1}-1\right] = \begin{cases} 0 & \text{if } n = 2m-1, \\ -\dfrac{16m}{(4m^2-1)\pi} & \text{if } n = 2m;\end{cases}\end{aligned}$$

$S_g(x) = \dfrac{\pi}{2}\sin x - \dfrac{16}{\pi}\displaystyle\sum_{n=1}^\infty \frac{n}{(4n^2-1)^2}\sin 2nx$. From Definition 12.1.1,

$$u(x,t) = \frac{\pi}{2\sqrt{5}}\sin\sqrt{5}\,t\sin x - \frac{8}{\pi\sqrt{5}}\sum_{n=1}^\infty \frac{1}{(4n^2-1)^2}\sin 2n\sqrt{5}\,t\sin 2nx.$$

12.2.14. Since $f(0) = f(1) = f''(0) = f''(L) = 0$ and $f^{(4)} = 360x$, Theorem 11.3.5(**b**) and Exercise 35(**b**) of Section 11.3 imply that

$$\begin{aligned}\alpha_n &= \frac{720}{n^4\pi^4}\int_0^1 x\sin n\pi x\,dx = -\frac{720}{n^5\pi^5}\left[x\cos n\pi x\Big|_0^1 - \int_0^1 \cos n\pi x\,dx\right] \\ &= (-1)^{n+1}\frac{720}{n^5\pi^5} + \frac{720}{n^6\pi^6}\sin\frac{n\pi x}{L}\Big|_0^1 = (-1)^{n+1}\frac{720}{n^5\pi^5};\end{aligned}$$

$S_f(x) = -\dfrac{720}{\pi^5}\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n^5}\sin n\pi x$. From Definition 12.1.1, $u(x,t) = -\dfrac{720}{\pi^5}\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n^5}\cos 3n\pi t\sin n\pi x$.

12.2.16. **(a)** t must be in some interval of the form $[mL/a, (m+1)L/a]$. If $\dfrac{mL}{a} \le t \le \left(m+\dfrac{1}{2}\right)\dfrac{L}{a}$, then **(i)** holds with $0 \le \tau \le L/2a$. If $\left(m+\dfrac{1}{2}\right)\dfrac{L}{a} \le t \le \dfrac{(m+1)L}{a}$, then **(ii)** holds with $0 \le \tau \le L/2a$.

(b) Suppose that **(i)** holds. Since

$$\cos\frac{(2n-1)\pi a}{L}\left(\tau + \frac{mL}{a}\right) = \cos\frac{(2n-1)\pi a\tau}{L}\cos(2n-1)m\pi = (-1)^m\cos\frac{(2n-1)\pi a\tau}{L},$$

(A) implies that $u(x,t) = (-1)^m u(x,\tau)$.

Suppose that **(ii)** holds. Since

$$\begin{aligned}\cos\frac{(2n-1)\pi a}{L}\left(-\tau + \frac{(m+1)L}{a}\right) &= \cos\frac{(2n-1)\pi a\tau}{L}\cos(2n-1)(m+1)\pi \\ &= (-1)^{m+1}\cos\frac{(2n-1)\pi a\tau}{L},\end{aligned}$$

(B) implies that that $u(x,t) = (-1)^{m+1}u(x,\tau)$.

12.2.18. Since $f'(0) = f(2) = 0$ and $f''(x) = -2$, Theorem 11.3.5(c) implies that

$$\begin{aligned}\alpha_n &= \frac{32}{(2n-1)^2\pi^2}\int_0^2 \cos\frac{(2n-1)\pi x}{4}\,dx \\ &= \frac{128}{(2n-1)^3\pi^3}\sin\frac{(2n-1)\pi x}{4}\Big|_0^4 = (-1)^{n+1}\frac{128}{(2n-1)^3\pi^3};\end{aligned}$$

$C_{Mf}(x) = -\dfrac{128}{\pi^3}\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^3}\cos\frac{(2n-1)\pi x}{4}$. From Exercise 12.2.17,

$$u(x,t) = -\frac{128}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^3}\cos\frac{3(2n-1)\pi t}{4}\cos\frac{(2n-1)\pi x}{4}.$$

12.2.20. Since $g'(0) = g(2) = 0$ and $g''(x) = -2$, Theorem 11.3.5(c) implies that

$$\begin{aligned}\beta_n &= \frac{32}{(2n-1)^2\pi^2}\int_0^2 \cos\frac{(2n-1)\pi x}{4}\,dx \\ &= \frac{128}{(2n-1)^3\pi^3}\sin\frac{(2n-1)\pi x}{4}\Big|_0^4 = (-1)^{n+1}\frac{128}{(2n-1)^3\pi^3};\end{aligned}$$

$C_{Mf}(x) = -\dfrac{128}{\pi^3}\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^3}\cos\frac{(2n-1)\pi x}{4}$. From Exercise 12.2.17,

$$u(x,t) = -\frac{512}{3\pi^4}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^4}\sin\frac{3(2n-1)\pi t}{4}\cos\frac{(2n-1)\pi x}{4}.$$

12.2.22. Since $f'(0) = f(1) = 0$ and $f''(x) = 6(2x+1)$, Theorem 11.3.5**(c)** implies that

$$\begin{aligned}
\alpha_n &= -\frac{48L}{(2n-1)^2\pi^2}\int_0^1 (2x+1)\cos\frac{(2n-1)\pi x}{2}\,dx \\
&= -\frac{96}{(2n-1)^3\pi^3}\left[(2x+1)\sin\frac{(2n-1)\pi x}{2}\Big|_0^1 - 2\int_0^1 \sin\frac{(2n-1)\pi x}{2}\right]dx \\
&= -\frac{96}{(2n-1)^3\pi^3}\left[(-1)^{n+1}3 - \frac{4}{(2n-1)\pi}\cos\frac{(2n-1)\pi x}{2}\Big|_0^1\right] \\
&= \frac{96}{(2n-1)^3\pi^3}\left[(-1)^n 3 + \frac{4}{(2n-1)\pi}\right];
\end{aligned}$$

$$C_{Mf}(x) = \frac{96}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\left[(-1)^n 3 + \frac{4}{(2n-1)\pi}\right]\cos\frac{(2n-1)\pi x}{2}.$$

From Exercise 12.2.17,

$$u(x,t) = \frac{96}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\left[(-1)^n 3 + \frac{4}{(2n-1)\pi}\right]\cos\frac{(2n-1)\sqrt{5}\pi t}{2}\cos\frac{(2n-1)\pi x}{2}.$$

12.2.24. Since $g'(0) = g(1) = 0$ and $g''(x) = 6(2x+1)$, Theorem 11.3.5**(c)** implies that

$$\begin{aligned}
\beta_n &= -\frac{48L}{(2n-1)^2\pi^2}\int_0^1 (2x+1)\cos\frac{(2n-1)\pi x}{2}\,dx \\
&= -\frac{96}{(2n-1)^3\pi^3}\left[(2x+1)\sin\frac{(2n-1)\pi x}{2}\Big|_0^1 - 2\int_0^1 \sin\frac{(2n-1)\pi x}{2}\right]dx \\
&= -\frac{96}{(2n-1)^3\pi^3}\left[(-1)^{n+1}3 - \frac{4}{(2n-1)\pi}\cos\frac{(2n-1)\pi x}{2}\Big|_0^1\right] \\
&= \frac{96}{(2n-1)^3\pi^3}\left[(-1)^n 3 + \frac{4}{(2n-1)\pi}\right];
\end{aligned}$$

$C_{Mg}(x) = \dfrac{96}{\pi^3}\displaystyle\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\left[(-1)^n 3 + \frac{4}{(2n-1)\pi}\right]\cos\frac{(2n-1)\pi x}{2}$. From Exercise 12.2.17,

$$u(x,t) = \frac{192}{\pi^4\sqrt{5}}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}\left[(-1)^n 3 + \frac{4}{(2n-1)\pi}\right]\sin\frac{(2n-1)\sqrt{5}\pi t}{2}\cos\frac{(2n-1)\pi x}{2}.$$

12.2.26. Since $f'(0) = f(1) = f''(1) = 0$ and $f'''(x) = 24(x-1)$, Theorem 11.3.5**(c)** and Exercise 42**(b)** of Section 11.3 imply that

$$\begin{aligned}
\alpha_n &= \frac{384}{(2n-1)^3\pi^3}\int_0^1 (x-1)\sin\frac{(2n-1)\pi x}{2}\,dx \\
&= -\frac{768}{(2n-1)^4\pi^4}\left[(x-1)\cos\frac{(2n-1)\pi x}{2}\Big|_0^1 - \int_0^1 \cos\frac{(2n-1)\pi x}{2}\right]dx \\
&= -\frac{768}{(2n-1)^4\pi^4}\left[1 - \frac{2}{(2n-1)\pi}\sin\frac{(2n-1)\pi x}{2}\Big|_0^1\right] \\
&= -\frac{768}{(2n-1)^4\pi^4}\left[1 + \frac{(-1)^n 2}{(2n-1)\pi}\right];
\end{aligned}$$

$C_{Mf}(x) = -\dfrac{384}{\pi^4}\displaystyle\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}\left[1+\frac{(-1)^n 4}{(2n-1)\pi}\right]\cos\frac{(2n-1)\pi x}{2}$. From Exercise 12.2.17,

$$u(x,t) = -\frac{384}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}\left[1+\frac{(-1)^n 4}{(2n-1)\pi}\right]\cos\frac{3(2n-1)\pi t}{2}\cos\frac{(2n-1)\pi x}{2}.$$

12.2.28. Since $g'(0) = g(1) = g''(1) = 0$ and $g'''(x) = 24(x-1)$, Theorem 11.3.5**(c)** and Exercise 11.2.42**(b)** imply that

$$\begin{aligned}
\beta_n &= \frac{384}{(2n-1)^3\pi^3}\int_0^1 (x-1)\sin\frac{(2n-1)\pi x}{2}\,dx \\
&= -\frac{768}{(2n-1)^4\pi^4}\left[(x-1)\cos\frac{(2n-1)\pi x}{2}\Big|_0^1 - \int_0^1 \cos\frac{(2n-1)\pi x}{2}\right]dx \\
&= -\frac{768}{(2n-1)^4\pi^4}\left[1 - \frac{2}{(2n-1)\pi}\sin\frac{(2n-1)\pi x}{2}\Big|_0^1\right] \\
&= -\frac{768}{(2n-1)^4\pi^4}\left[1 + \frac{(-1)^n 2}{(2n-1)\pi}\right];
\end{aligned}$$

$$C_{Mg}(x) = -\frac{384}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}\left[1+\frac{(-1)^n 4}{(2n-1)\pi}\right]\cos\frac{(2n-1)\pi x}{2}.$$

From Exercise 12.2.17,

$$u(x,t) = -\frac{768}{3\pi^5}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^5}\left[1+\frac{(-1)^n 4}{(2n-1)\pi}\right]\sin\frac{3(2n-1)\pi t}{2}\cos\frac{(2n-1)\pi x}{2}.$$

12.2.30. Since $f'(0) = f(1) = f''(1) = 0$ and $f'''(x) = 24(x-1)$, Theorem 11.3.5**(c)** and Exercise 42(**b**) of Section 11.3 imply that

$$\begin{aligned}
\alpha_n &= \frac{384}{(2n-1)^3\pi^3}\int_0^1 (x-1)\sin\frac{(2n-1)\pi x}{2}\,dx \\
&= -\frac{768}{(2n-1)^4\pi^4}\left[(x-1)\cos\frac{(2n-1)\pi x}{2}\Big|_0^1 - \int_0^1 \cos\frac{(2n-1)\pi x}{2}\right]dx \\
&= -\frac{768}{(2n-1)^4\pi^4}\left[1-\frac{2}{(2n-1)\pi}\sin\frac{(2n-1)\pi x}{2}\Big|_0^1\right] \\
&= -\frac{768}{(2n-1)^4\pi^4}\left[1+\frac{(-1)^n 2}{(2n-1)\pi}\right];
\end{aligned}$$

$C_{Mf}(x) = -\dfrac{768}{\pi^4}\displaystyle\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}\left[1+\frac{(-1)^n 2}{(2n-1)\pi}\right]\cos\frac{(2n-1)\pi x}{2}$. From Exercise 12.2.17,

$$u(x,t) = -\frac{768}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}\left[1+\frac{(-1)^n 2}{(2n-1)\pi}\right]\cos\frac{(2n-1)\pi t}{2}\cos\frac{(2n-1)\pi x}{2}.$$

12.2.32. Setting $A = (2n-1)\pi x/2L$ and $B = (2n-1)\pi at/2L$ in the identities
$\cos A\cos B = \frac{1}{2}[\cos(A+B)+\cos(A-B)]$ and $\cos A\sin B = \frac{1}{2}[\sin(A+B)-\sin(A-B)]$ yields

$$\cos\frac{(2n-1)\pi at}{2L}\cos\frac{(2n-1)\pi x}{2L} = \frac{1}{2}\left[\cos\frac{(2n-1)\pi(x+at)}{2L}+\cos\frac{(2n-1)\pi(x-at)}{2L}\right] \tag{A}$$

and

$$\begin{aligned}
\sin\frac{(2n-1)\pi at}{2L}\cos\frac{(2n-1)\pi x}{2L} &= \frac{1}{2}\left[\sin\frac{(2n-1)\pi(x+at)}{2L}-\sin\frac{(2n-1)\pi(x-at)}{2L}\right] \\
&= \frac{(2n-1)\pi}{4L}\int_{x-at}^{x+at}\cos\frac{(2n-1)\pi\tau}{2L}\,d\tau.
\end{aligned} \tag{B}$$

Since $C_{Mf}(x) = \displaystyle\sum_{n=1}^{\infty}\alpha_n\cos\frac{(2n-1)\pi x}{2L}$, (A) implies that

$$\sum_{n=1}^{\infty}\alpha_n\cos\frac{(2n-1)\pi at}{2L}\cos\frac{(2n-1)\pi x}{2L} = \frac{1}{2}[C_{Mf}(x+at)+C_{Mf}(x-at)]. \tag{C}$$

Since it can be shown that a mixed Fourier cosine series can be integrated term by term between any two limits, (B) implies that

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{2L\beta_n}{(2n-1)\pi a}\sin\frac{(2n-1)\pi at}{2L}\cos\frac{(2n-1)\pi x}{2L} &= \frac{1}{2a}\sum_{n=1}^{\infty}\beta_n\int_{x-at}^{x+at}\cos\frac{(2n-1)\pi\tau}{2L}\,d\tau \\
&= \frac{1}{2a}\int_{x-at}^{x+at}\left(\sum_{n=1}^{\infty}\beta_n\cos\frac{(2n-1)\pi\tau}{2L}\right)d\tau \\
&= \frac{1}{2a}\int_{x-at}^{x+at}C_{Mg}(\tau)\,d\tau.
\end{aligned}$$

This and (C) imply that

$$u(x,t) = \frac{1}{2}[C_{Mf}(x+at) + C_{Mf}(x-at)] + \frac{1}{2a}\int_{x-at}^{x+at} C_{Mg}(\tau)\,d\tau.$$

12.2.34. We begin by looking for functions of the form $v(x,t) = X(x)T(t)$ that are not identically zero and satisfy $v_{tt} = a^2 v_{xx}$, $v(0,t) = 0$, $v_x(L,t) = 0$ for all (x,t). As shown in the text, X and T must satisfy $X'' + \lambda X = 0$ and (B) $T'' + a^2\lambda T = 0$ for the same value of λ. Since $v(0,t) = X(0)T(t)$ and $v_x(L,t) = X'(L)T(t)$ and we don't want T to be identically zero, $X(0) = 0$ and $X'(L) = 0$. Therefore,λ must be an eigenvalue of (C) $X'' + \lambda X = 0$, $X(0) = 0$, $X'(L) = 0$, and X must be a λ-eigenfunction. From Theorem 11.1.4, the eigenvalues of (C) are $\lambda_n = \dfrac{(2n-1)^2}{\pi^2/4L^2}$, integer), with associated eigenfunctions $X_n = \sin\dfrac{(2n-1)\pi x}{2L}$, $n = 1, 2, 3, \dots$. Substituting $\lambda = \dfrac{(2n-1)^2\pi^2}{4L^2}$ into (B) yields $T'' + ((2n-1)^2\pi^2 a^2/4L^2)T = 0$, which has the general solution

$$T_n = \alpha_n \cos\frac{(2n-1)\pi at}{2L} + \frac{2\beta_n L}{(2n-1)\pi a}\sin\frac{(2n-1)\pi at}{2L},$$

where α_n and β_n are constants. Now let

$$v_n(x,t) = X_n(x)T_n(t) = \left(\alpha_n \cos\frac{(2n-1)\pi at}{2L} + \frac{2\beta_n L}{(2n-1)\pi a}\sin\frac{(2n-1)\pi at}{2L}\right)\sin\frac{(2n-1)\pi x}{2L}.$$

Then

$$\frac{\partial v_n}{\partial t}(x,t) = \left(-\frac{(2n-1)\pi a}{2L}\alpha_n \sin\frac{(2n-1)\pi at}{2L} + \beta_n \cos\frac{(2n-1)\pi at}{2L}\right)\sin\frac{(2n-1)\pi x}{2L},$$

so

$$v_n(x,0) = \alpha_n \sin\frac{(2n-1)\pi x}{2L} \quad \text{and} \quad \frac{\partial v_n}{\partial t}(x,0) = \beta_n \sin\frac{(2n-1)\pi x}{2L}.$$

Therefore,v_n satisfies (A) with $f(x) = \alpha_n \sin\dfrac{(2n-1)\pi x}{2L}$ and $g(x) = \beta_n \sin\dfrac{(2n-1)\pi x}{2L}$. More generally, if $\alpha_1, \alpha_2, \dots, \alpha_m$ and $\beta_1, \beta_2, \dots, \beta_m$ are constants and

$$u_m(x,t) = \sum_{n=1}^{m}\left(\alpha_n \cos\frac{(2n-1)\pi at}{2L} + \frac{2\beta_n L}{(2n-1)\pi a}\sin\frac{(2n-1)\pi at}{2L}\right)\sin\frac{(2n-1)\pi x}{2L},$$

then u_m satisfies (A) with

$$f(x) = \sum_{n=1}^{m}\alpha_n \sin\frac{(2n-1)\pi x}{2L} \quad \text{and} \quad g(x) = \sum_{n=1}^{m}\beta_n \sin\frac{(2n-1)\pi x}{2L}.$$

This motivates the definition.

12.2.36. Since $f(0) = f'(1) = 0$, and $f''(x) = 6(1-2x)$, Theorem 11.3.5**(d)** implies that

$$\begin{aligned}
\alpha_n &= -\frac{48}{(2n-1)^2\pi^2}\int_0^1 (1-2x)\sin\frac{(2n-1)\pi x}{2}\,dx \\
&= \frac{96}{(2n-1)^3\pi^3}\left[(1-2x)\cos\frac{(2n-1)\pi x}{2}\Big|_0^1 + 2\int_0^1 \cos\frac{(2n-1)\pi x}{2}\,dx\right] \\
&= \frac{96}{(2n-1)^3\pi^3}\left[-1 + \frac{4}{(2n-1)\pi}\sin\frac{(2n-1)\pi x}{2}\Big|_0^1\right] \\
&= -\frac{96}{(2n-1)^3\pi^3}\left[1 + (-1)^n\frac{4}{(2n-1)\pi}\right];
\end{aligned}$$

$S_{Mf}(x) = -\dfrac{96}{\pi^3}\displaystyle\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\left[1 + (-1)^n\frac{4}{(2n-1)\pi}\right]\sin\frac{(2n-1)\pi x}{2}$. From Exercise 12.2.34,

$$u(x,t) = -\frac{96}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\left[1 + (-1)^n\frac{4}{(2n-1)\pi}\right]\cos\frac{3(2n-1)\pi t}{2}\sin\frac{(2n-1)\pi x}{2}.$$

12.2.38. Since $g(0) = g'(1) = 0$, and $g''(x) = 6(1-2x)$, Theorem 11.3.5**(d)** implies that

$$\begin{aligned}
\beta_n &= -\frac{48}{(2n-1)^2\pi^2}\int_0^1 (1-2x)\sin\frac{(2n-1)\pi x}{2}\,dx \\
&= \frac{96}{(2n-1)^3\pi^3}\left[(1-2x)\cos\frac{(2n-1)\pi x}{2}\Big|_0^1 + 2\int_0^1 \cos\frac{(2n-1)\pi x}{2}\,dx\right] \\
&= \frac{96}{(2n-1)^3\pi^3}\left[-1 + \frac{4}{(2n-1)\pi}\sin\frac{(2n-1)\pi x}{2}\Big|_0^1\right] \\
&= -\frac{96}{(2n-1)^3\pi^3}\left[1 + (-1)^n\frac{4}{(2n-1)\pi}\right];
\end{aligned}$$

$S_{Mg}(x) = -\dfrac{96}{\pi^3}\displaystyle\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\left[1 + (-1)^n\frac{4}{(2n-1)\pi}\right]\sin\frac{(2n-1)\pi x}{2}$. From Exercise 12.2.34,

$$u(x,t) = -\frac{64}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}\left[1 + (-1)^n\frac{4}{(2n-1)\pi}\right]\sin\frac{3(2n-1)\pi t}{2}\sin\frac{(2n-1)\pi x}{2}.$$

12.2.40. Since $f(0) = f'(\pi) = f''(0) = 0$ and $f'''(x) = 6$, Theorem 11.3.5**(d)** and Exercise 11.3.50**(b)** imply that

$$\alpha_n = -\frac{96}{(2n-1)^3\pi}\int_0^{\pi}\cos\frac{(2n-1)x}{2}\,dx = -\frac{192}{(2n-1)^4\pi}\sin\frac{(2n-1)x}{2}\Big|_0^{\pi} = (-1)^n\frac{192}{(2n-1)^4\pi};$$

$S_{Mf}(x) = \dfrac{192}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^4}\sin\frac{(2n-1)x}{2}$. From Exercise 12.2.34,

$$u(x,t) = \frac{192}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^4}\cos\frac{(2n-1)\sqrt{3}\,t}{2}\sin\frac{(2n-1)x}{2}.$$

12.2.42. Since $g(0) = g'(\pi) = g''(0) = 0$ and $g'''(x) = 6$, Theorem 11.3.5(**d**) and Exercise 50(**b**) imply that

$$\beta_n = -\frac{96}{(2n-1)^3\pi}\int_0^\pi \cos\frac{(2n-1)x}{2}\,dx = -\frac{192}{(2n-1)^4\pi}\sin\frac{(2n-1)x}{2}\Big|_0^\pi = (-1)^n\frac{192}{(2n-1)^4\pi};$$

$S_{Mg}(x) = \dfrac{192}{\pi}\displaystyle\sum_{n=1}^\infty \frac{(-1)^n}{(2n-1)^4}\sin\frac{(2n-1)x}{2}$. From Exercise 12.2.34,

$$u(x,t) = \frac{384}{\sqrt{3}\,\pi}\sum_{n=1}^\infty \frac{(-1)^n}{(2n-1)^5}\sin\frac{(2n-1)\sqrt{3}\,t}{2}\sin\frac{(2n-1)x}{2}.$$

12.2.44. Since $f(0) = f'(1) = f''(0) = 0$ and $f'''(x) = 12(2x-1)$, Theorem 11.3.5(**d**) and Exercise 11.3.50(**b**) imply that

$$\begin{aligned}
\alpha_n &= -\frac{192}{(2n-1)^3\pi^3}\int_0^1 (2x-1)\cos\frac{(2n-1)\pi x}{2}\,dx \\
&= -\frac{384}{(2n-1)^4\pi^4}\left[(2x-1)\sin\frac{(2n-1)\pi x}{2}\Big|_0^1 - 2\int_0^1 \sin\frac{(2n-1)\pi x}{2}\,dx\right] \\
&= -\frac{384}{(2n-1)^4\pi^4}\left[(-1)^{n+1} + \frac{4}{(2n-1)\pi}\cos\frac{(2n-1)\pi x}{2}\Big|_0^1\right] \\
&= -\frac{384}{(2n-1)^4\pi^4}\left[(-1)^{n+1} - \frac{4}{(2n-1)\pi}\right] = \frac{384}{(2n-1)^4\pi^4}\left[(-1)^n + \frac{4}{(2n-1)\pi}\right];
\end{aligned}$$

$S_{Mf}(x) = \dfrac{384}{\pi^4}\displaystyle\sum_{n=1}^\infty \frac{1}{(2n-1)^4}\left[(-1)^n + \frac{4}{(2n-1)\pi}\right]\sin\frac{(2n-1)\pi x}{2}$. From Exercise 12.2.34,

$$u(x,t) = \frac{384}{\pi^4}\sum_{n=1}^\infty \frac{1}{(2n-1)^4}\left[(-1)^n + \frac{4}{(2n-1)\pi}\right]\cos(2n-1)\pi t\,\sin\frac{(2n-1)\pi x}{2}.$$

12.2.46. Since $g(0) = g'(1) = g''(0) = 0$ and $g'''(x) = 12(2x-1)$, Theorem 11.3.5(**d**) and Exercise 11.3.50(**b**) imply that

$$\begin{aligned}
\beta_n &= -\frac{192}{(2n-1)^3\pi^3}\int_0^1 (2x-1)\cos\frac{(2n-1)\pi x}{2}\,dx \\
&= -\frac{384}{(2n-1)^4\pi^4}\left[(2x-1)\sin\frac{(2n-1)\pi x}{2}\Big|_0^1 - 2\int_0^1 \sin\frac{(2n-1)\pi x}{2}\,dx\right] \\
&= -\frac{384}{(2n-1)^4\pi^4}\left[(-1)^{n+1} + \frac{4}{(2n-1)\pi}\cos\frac{(2n-1)\pi x}{2}\Big|_0^1\right] \\
&= -\frac{384}{(2n-1)^4\pi^4}\left[(-1)^{n+1} - \frac{4}{(2n-1)\pi}\right] = \frac{384}{(2n-1)^4\pi^4}\left[(-1)^n + \frac{4}{(2n-1)\pi}\right];
\end{aligned}$$

$S_{Mg}(x) = \dfrac{384}{\pi^4}\displaystyle\sum_{n=1}^\infty \frac{1}{(2n-1)^4}\left[(-1)^n + \frac{4}{(2n-1)\pi}\right]\sin\frac{(2n-1)\pi x}{2}$. From Exercise 12.2.34,

$$u(x,t) = \frac{384}{\pi^5}\sum_{n=1}^\infty \frac{1}{(2n-1)^5}\left[(-1)^n + \frac{4}{(2n-1)\pi}\right]\sin(2n-1)\pi t\,\sin\frac{(2n-1)\pi x}{2}.$$

12.2.48. Since f is continuous on $[0, L]$ and $f'(L) = 0$, Theorem 11.3.4 implies that $S_{Mf}(x) = f(x)$, $0 \le x \le L$. From Exercise 11.3.58, S_{Mf} is the odd periodic extension (with period $2L$) of the function $r(x) = \begin{cases} f(x), & 0 \le x \le L, \\ f(2L - x), & L < x \le 2L, \end{cases}$ which is continuous on $[0, 2L]$. Since $r(0) = r(2L) = f(0) = 0$, S_{Mf} is continuous on $(-\infty, \infty)$. Moreover, $r'(x) = \begin{cases} f'(x), & 0 < x < L, \\ -f'(2L - x), & L < x < 2L, \end{cases}$ $r'_+(0) = f'_+(0)$, $r'_-(2L) = -f'_+(0)$, and, since $f'_-(L) = 0$, $r'(L) = 0$. Hence, r is differentiable on $[0, 2L]$. Since $r(0) = r(2L) = f(0) = 0$, Theorem 12.2.3**(a)** with $h = r$, $p = S_{Mf}$, and L replaced by $2L$ implies that S_{Mf} is differentiable on $(-\infty, \infty)$. Similarly, S_{Mg} is differentiable on $(-\infty, \infty)$.

Now we note that $r''(x) = \begin{cases} f''(x), & 0 < x < L, \\ f''(2L - x), & L < x < 2L, \end{cases}$ $r''(L) = f''_-(L)$, and $r''_+(0) = r''_-(2L) = f''_+(0) = 0$. Since S'_{Mf} is the even periodic extension of r', Theorem 12.2.3**(b)** with $h = r'$, $q = S'_{Mf}$, and L replaced by $2L$ implies that S'_{Mf} is differentiable on $(-\infty, \infty)$. Now follow the argument used to complete the proof of Theorem 12.2.4.

12.2.50. From Example 11.3.5, $C_f(x) = 4 - \dfrac{768}{\pi^4} \displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} \cos \frac{(2n-1)\pi x}{2}$. From Exercise 12.2.49,

$$u(x, t) = 4 - \frac{768}{\pi^4} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} \cos \frac{\sqrt{5}(2n-1)\pi t}{2} \cos \frac{(2n-1)\pi x}{2}.$$

12.2.52. $\alpha_0 = \displaystyle\int_0^\pi (3x^4 - 4Lx^3)\, dx = \left(\frac{3x^5}{5} - \pi x^4 \right) \Bigg|_0^\pi = -\frac{2\pi^4}{5}$. Since $f'(0) = f'(\pi) = 0$ and $f'''(x) = 24(3x - \pi)$, Theorem 11.3.5**(a)** implies that

$$\begin{aligned} \alpha_n &= \frac{48}{n^3\pi} \int_0^\pi (3x - \pi) \sin nx\, dx = -\frac{48}{n^4\pi} \left[(3x - \pi) \cos nx \Big|_0^\pi - 3 \int_0^\pi \cos nx\, dx \right] \\ &= -\frac{48}{n^4\pi} \left[(-1)^n 2\pi + \pi \right] + \frac{144}{n^5\pi} \sin nx \Big|_0^\pi = -\frac{48}{n^4} \left[1 + (-1)^n 2 \right], \quad n \ge 1; \end{aligned}$$

$C_f(x) = -\dfrac{2\pi^4}{5} - 48 \displaystyle\sum_{n=1}^{\infty} \frac{1 + (-1)^n 2}{n^4} \cos nx$. From Exercise 12.2.49,

$$u(x, t) = -\frac{2\pi^4}{5} - 48 \sum_{n=1}^{\infty} \frac{1 + (-1)^n 2}{n^4} \cos 2nt \cos nx.$$

12.2.54. $\beta_0 = \displaystyle\int_0^\pi (3x^4 - 4Lx^3)\, dx = \frac{1}{\pi} \left(\frac{3x^5}{5} - \pi x^4 \right) \Bigg|_0^\pi = -\frac{2\pi^4}{5}$. Since $g'(0) = g'(\pi) = 0$ and $g'''(x) = 24(3x - \pi)$, Theorem 11.3.5**(a)** implies that

$$\begin{aligned} \beta_n &= \frac{48}{n^3\pi} \int_0^\pi (3x - \pi) \sin nx\, dx = -\frac{48}{n^4\pi} \left[(3x - \pi) \cos nx \Big|_0^\pi - 3 \int_0^\pi \cos nx\, dx \right] \\ &= -\frac{48}{n^4\pi} \left[(-1)^n 2\pi + \pi \right] + \frac{144}{n^5\pi} \sin nx \Big|_0^\pi = -\frac{48}{n^4} \left[1 + (-1)^n 2 \right], \quad n \ge 1; \end{aligned}$$

$C_g(x) = -\dfrac{2\pi^4}{5} - 48\displaystyle\sum_{n=1}^{\infty} \dfrac{1+(-1)^n 2}{n^4}\cos nx$. From Exercise 12.2.49,

$$u(x,t) = -\frac{2\pi^4 t}{5} - 24\sum_{n=1}^{\infty} \frac{1+(-1)^n 2}{n^5}\sin 2nt\cos nx.$$

12.2.56. $\alpha_0 = \dfrac{1}{\pi}\displaystyle\int_0^\pi (x^4 - 2\pi x^3 + \pi^2 x^2)\,dx = \dfrac{1}{\pi}\left(\dfrac{x^5}{5} - \dfrac{\pi x^4}{2} + \dfrac{\pi^2 x^3}{3}\right)\Bigg|_0^\pi = \dfrac{\pi^4}{30}$. Since $f'(0) = f'(\pi) = 0$ and $f'''(x) = 12(2x-\pi)$, Theorem 11.3.5**(a)** implies that

$$\begin{aligned}
\alpha_n &= \frac{24}{n^3\pi}\int_0^\pi (2x-\pi)\sin nx\,dx = -\frac{24}{n^4\pi}\left[(2x-\pi)\cos nx\Big|_0^\pi - 2\int_0^\pi \cos nx\,dx\right] \\
&= -\frac{24}{n^4\pi}[(-1)^n\pi + \pi] + \frac{48}{n^5\pi}\sin nx\Big|_0^\pi = -\frac{24}{n^4}[1+(-1)^n] \\
&= \begin{cases} 0 & \text{if } n = 2m-1, \\ -\dfrac{3}{m^4} & \text{if } n = 2m, \end{cases} \quad n \ge 1;
\end{aligned}$$

$C_f(x) = \dfrac{\pi^4}{30} - 3\displaystyle\sum_{n=1}^{\infty}\dfrac{1}{n^4}\cos 2nx$. From Exercise 12.2.49, $u(x,t) = \dfrac{\pi^4}{30} - 3\displaystyle\sum_{n=1}^{\infty}\dfrac{1}{n^4}\cos 8nt\cos 2nx$.

12.2.58. $\beta_0 = \dfrac{1}{\pi}\displaystyle\int_0^\pi (x^4 - 2\pi x^3 + \pi^2 x^2)\,dx = \dfrac{1}{\pi}\left(\dfrac{x^5}{5} - \dfrac{\pi x^4}{2} + \dfrac{\pi^2 x^3}{3}\right)\Bigg|_0^\pi = \dfrac{\pi^4}{30}$. Since $g'(0) = g'(\pi) = 0$ and $g'''(x) = 12(2x-\pi)$, Theorem 11.3.5**(a)** implies that

$$\begin{aligned}
\beta_n &= \frac{24}{n^3\pi}\int_0^\pi (2x-\pi)\sin nx\,dx = -\frac{24}{n^4\pi}\left[(2x-\pi)\cos nx\Big|_0^\pi - 2\int_0^\pi \cos nx\,dx\right] \\
&= -\frac{24}{n^4\pi}[(-1)^n\pi + \pi] + \frac{48}{n^5\pi}\sin nx\Big|_0^\pi = -\frac{24}{n^4}[1+(-1)^n] \\
&= \begin{cases} 0 & \text{if } n = 2m-1, \\ -\dfrac{3}{m^4} & \text{if } n = 2m, \end{cases} \quad \ge 1;
\end{aligned}$$

$C_g(x) = \dfrac{\pi^4}{30} - 3\displaystyle\sum_{n=1}^{\infty}\dfrac{1}{n^4}\cos 2nx$. From Exercise 12.2.49, $u(x,t) = \dfrac{\pi^4 t}{30} - \dfrac{3}{8}\displaystyle\sum_{n=1}^{\infty}\dfrac{1}{n^5}\sin 8nt\cos 2nx$.

12.2.60. Setting $A = n\pi x/L$ and $B = n\pi at/L$ in the identities $\cos A\cos B = \dfrac{1}{2}[\cos(A+B)+\cos(A-B)]$ and $\cos A\sin B = \dfrac{1}{2}[\sin(A+B)-\sin(A-B)]$ yields

$$\cos\frac{n\pi at}{L}\cos\frac{n\pi x}{L} = \frac{1}{2}\left[\cos\frac{n\pi(x+at)}{L} + \cos\frac{n\pi(x-at)}{L}\right] \tag{A}$$

and

$$\begin{aligned}
\sin\frac{n\pi at}{L}\cos\frac{n\pi x}{L} &= \frac{1}{2}\left[\sin\frac{n\pi(x+at)}{L} - \sin\frac{n\pi(x-at)}{L}\right] \\
&= \frac{n\pi}{2L}\int_{x-at}^{x+at}\cos\frac{n\pi\tau}{L}\,d\tau.
\end{aligned} \tag{B}$$

Since $C_f(x) = \alpha_0 + \sum_{n=1}^{\infty} \alpha_n \sin \frac{n\pi x}{L}$, (A) implies that

$$\alpha_0 + \sum_{n=1}^{\infty} \alpha_n \cos \frac{n\pi at}{L} \cos \frac{n\pi x}{L} = \frac{1}{2}[C_f(x+at) + C_f(x-at)]. \tag{C}$$

Since it can be shown that a Fourier sine series can be integrated term by term between any two limits, (B) implies that

$$\begin{aligned} \beta_0 t + \sum_{n=1}^{\infty} \frac{\beta_n L}{n\pi a} \sin \frac{n\pi at}{L} \cos \frac{n\pi x}{L} &= \beta_0 t + \frac{1}{2a} \sum_{n=1}^{\infty} \beta_n \int_{x-at}^{x+at} \cos \frac{n\pi \tau}{L}\, d\tau \\ &= \frac{1}{2a} \int_{x-at}^{x+at} \left(\beta_0 + \sum_{n=1}^{\infty} \beta_n \cos \frac{n\pi \tau}{L} \right) d\tau \\ &= \frac{1}{2a} \int_{x-at}^{x+at} C_g(\tau)\, d\tau. \end{aligned}$$

This and (C) imply that

$$u(x,t) = \frac{1}{2}[C_f(x+at) + C_f(x-at)] + \frac{1}{2a} \int_{x-at}^{x+at} C_g(\tau)\, d\tau.$$

12.2.62.(a). Since $|p_n(x)| \le 1$ and $|q_n(t)| \le 1$ for all t, $|k_n p_n(x) q_n(t)| \le |k_n|$ for all (x,t), and the comparison test implies the conclusion.

(b) If t is fixed but arbitrary, then $|k_n p_n'(x) q_n(t)| \le |\lambda| n |k_n|$, so Theorem 12.1.2 with $z = x$ and $w_n(x) = k_n p_n(x) q_n(t)$ justifies term by term differentiation with respect to x on $(-\infty, \infty)$. If x is fixed but arbitrary, then $|k_n p_n(x) q_n'(t)| \le |\mu| n |k_n|$, so Theorem 12.1.2 with $z = t$ and $w_n(t) = k_n p_n(x) q_n(t)$ justifies term by term differentiation with respect to t on $(-\infty, \infty)$.

(c) The argument is similar to argument use in **(b)**.

(d) Apply **(b)** and **(c)** to the series $\sum_{n=1}^{\infty} \alpha_n \cos \frac{n\pi at}{L} \sin \frac{n\pi x}{L}$ and $\sum_{n=1}^{\infty} \frac{\beta_n L}{n\pi a} \sin \frac{n\pi at}{L} \sin \frac{n\pi x}{L}$, recalling that the individual terms in the series satisfy $u_{tt} = a^2 u_{xx}$ for all (x,t).

(d) $u(x,t) = \dfrac{f(x+ct) + f(x-ct)}{2} + \dfrac{1}{2a} \displaystyle\int_{x-at}^{x+at} g(u)\, du$.

12.2.64.

$$\begin{aligned} u(x,t) &= \frac{(x+at) + (x-at)}{2} + \frac{1}{2a} \int_{x-at}^{x+at} 4au\, du = x + 2 \int_{x-at}^{x+at} u\, du = x + u^2 \Big|_{x-at}^{x+at} \\ &= x + (x+at)^2 - (x-at)^2 = x(1+4at). \end{aligned}$$

12.2.66.

$$\begin{aligned} u(x,t) &= \frac{\sin(x+at) + \sin(x-at)}{2} + \frac{1}{2a} \int_{x-at}^{x+at} a \cos u\, du \\ &= \frac{\sin(x+at) + \sin(x-at)}{2} + \frac{\sin(x+at) - \sin(x-at)}{2} = \sin(x+at). \end{aligned}$$

12.2.68.

$$\begin{aligned} u(x,t) &= \frac{(x+at)\sin(x+at)+(x-at)\sin(x-at)}{2}+\frac{1}{2a}\int_{x-at}^{x+at}\sin u\,du \\ &= \frac{x[\sin(x+at)+\sin(x-at)]}{2}+\frac{at[\sin(x+at)-\sin(x-at)]}{2} \\ &\quad +\frac{\cos(x-at)-\cos(x+at)}{2a} \end{aligned}$$

$= x\sin x\cos at + at\cos x\sin at + \dfrac{\sin x\sin at}{a}.$

12.3 LAPLACE'S EQUATION IN RECTANGULAR COORDINATES

12.3.2. Since $f(0)=f(1)=0$ and $f''(x)=2-6x$, Theorem 11.3.5**(b)** implies that

$$\begin{aligned} \alpha_n &= -\frac{4}{n^2\pi^2}\int_0^1(4-6x)\sin\frac{n\pi x}{2}\,dx=\frac{8}{n^3\pi^3}\left[(4-6x)\cos\frac{n\pi x}{2}\Big|_0^2+6\int_0^2\cos\frac{n\pi x}{2}\,dx\right] \\ &= -\frac{32}{n^3\pi^3}(1+(-1)^n2)+\frac{96}{n^4\pi^4}\sin\frac{n\pi x}{2}\Big|_0^2=-\frac{32}{n^3\pi^3}[1+(-1)^n2]; \end{aligned}$$

$S(x)=-\dfrac{32}{\pi^3}\displaystyle\sum_{n=1}^{\infty}\frac{[1+(-1)^n2]}{n^3}\sin\frac{n\pi x}{2}$. From Example 12.3.1,

$$u(x,y)=-\frac{32}{\pi^3}\sum_{n=1}^{\infty}\frac{[1+(-1)^n2]\sinh n\pi(3-y)/2}{n^3\sinh 3n\pi/2}\sin\frac{n\pi x}{2}.$$

12.3.4.

$$\begin{aligned} \alpha_1 &= \frac{2}{\pi}\int_0^\pi x\sin^2 x\,dx=\frac{1}{\pi}\int_0^\pi x(1-\cos 2x)\,dx=\frac{x^2}{2\pi}\Big|_0^\pi-\frac{1}{\pi}\int_0^\pi x\cos 2x\,dx \\ &= \frac{\pi}{2}-\frac{1}{2\pi}\left[x\sin 2x\Big|_0^\pi-\int_0^\pi\cos 2x\,dx\right]=\frac{\pi}{2}+\frac{\sin 2x}{4\pi}\Big|_0^\pi=\frac{\pi}{2}; \end{aligned}$$

if $n\ge 2$, then

$$\begin{aligned} \alpha_n &= \frac{2}{\pi}\int_0^\pi x\sin x\sin nx\,dx=\frac{1}{\pi}\int_0^\pi x[\cos(n-1)x-\cos(n+1)x]\,dx \\ &= \frac{1}{\pi}\left[x\left[\frac{\sin(n-1)x}{n-1}-\frac{\sin(n+1)x}{n+1}\right]\Big|_0^\pi-\int_0^\pi\left[\frac{\sin(n-1)x}{n-1}-\frac{\sin(n+1)x}{n+1}\right]dx\right] \\ &= \frac{1}{\pi}\left[\frac{\cos(n-1)x}{(n-1)^2}-\frac{\cos(n+1)x}{(n+1)^2}\right]\Big|_0^\pi=\frac{1}{\pi}\left[\frac{1}{(n-1)^2}-\frac{1}{(n+1)^2}\right]\left[(-1)^{n+1}-1\right] \\ &= \frac{4n}{(n^2-1)^2\pi}\left[(-1)^{n+1}-1\right]=\begin{cases}0 & \text{if } n=2m-1,\\ -\dfrac{16m}{(4m^2-1)\pi} & \text{if } n=2m;\end{cases} \end{aligned}$$

$S(x)=\dfrac{\pi}{2}\sin x-\dfrac{16}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{n}{(4n^2-1)^2}\sin 2nx$. From Example 12.3.1,

$$u(x,y)=\frac{\pi}{2}\frac{\sinh(1-y)}{\sinh 1}\sin x-\frac{16}{\pi}\sum_{n=1}^{\infty}\frac{n\sinh 2n(1-y)}{(4n^2-1)^2\sinh 2n}\sin 2nx.$$

12.3.6. $\alpha_0 = \int_0^1 (1-x)\,dx = -\frac{(1-x)^2}{2}\Big|_0^1 = \frac{1}{2}$; if $n \ge 1$,

$$\begin{aligned}\alpha_n &= 2\int_0^1 (1-x)\cos n\pi x\,dx = \frac{2}{n\pi}\left[(1-x)\sin n\pi x\Big|_0^1 + \int_0^1 \sin n\pi x\,dx\right] \\ &= -\frac{2}{n^2\pi^2}\cos n\pi x\Big|_0^1 = \frac{2}{n^2\pi^2}[1-(-1)^n] = \begin{cases} \dfrac{4}{(2m-1)^2\pi^2} & \text{if } n = 2m-1, \\ 0 & \text{if } n = 2m;\end{cases}\end{aligned}$$

$C(x) = \frac{1}{2} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos(2n-1)\pi x$. From Example 12.3.3,

$$u(x,y) = \frac{y}{2} + \frac{4}{\pi^3}\sum_{n=1}^{\infty}\frac{\sinh(2n-1)\pi y}{(2n-1)^3\cosh 2(2n-1)\pi}\cos(2n-1)\pi x.$$

12.3.8. $\alpha_0 = \int_0^1 (x-1)^2\,dx = \frac{(x-1)^3}{3}\Big|_0^1 = \frac{1}{3}$; if $n \ge 1$, then

$$\begin{aligned}\alpha_n &= 2\int_0^1 (x-1)^2\cos n\pi x\,dx = \frac{2}{n\pi}\left[(x-1)^2\sin n\pi x\Big|_0^1 - 2\int_0^1 (x-1)\sin n\pi x\,dx\right] \\ &= \frac{4}{n^2\pi^2}\left[(x-1)\cos n\pi x\Big|_0^1 - \int_0^1 \cos n\pi x\,dx\right] = \frac{4}{n^2\pi^2} - \frac{4}{n^3\pi^3}\sin n\pi x\Big|_0^1 = \frac{4}{n^2\pi^2};\end{aligned}$$

$C(x) = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos n\pi x$. From Example 12.3.3, $u(x,y) = \frac{y}{3} + \frac{4}{\pi^3}\sum_{n=1}^{\infty}\frac{\sinh n\pi y}{n^3\cosh n\pi}\cos n\pi x$.

12.3.10. Since $g(0) = g'(1) = 0$, and $g''(y) = 6(1-2y)$, Theorem 11.3.5**(d)** implies that

$$\begin{aligned}\alpha_n &= -\frac{48}{(2n-1)^2\pi^2}\int_0^1 (1-2y)\sin\frac{(2n-1)\pi y}{2}\,dy \\ &= \frac{96}{(2n-1)^3\pi^3}\left[(1-2y)\cos\frac{(2n-1)\pi y}{2}\Big|_0^1 + 2\int_0^1 \cos\frac{(2n-1)\pi y}{2}\,dy\right] \\ &= \frac{96}{(2n-1)^3\pi^3}\left[-1 + \frac{4}{(2n-1)\pi}\sin\frac{(2n-1)\pi y}{2}\Big|_0^1\right] \\ &= -\frac{96}{(2n-1)^3\pi^3}\left[1 + (-1)^n\frac{4}{(2n-1)\pi}\right];\end{aligned}$$

$S_M(y) = -\frac{96}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\left[1+(-1)^n\frac{4}{(2n-1)\pi}\right]\sin\frac{(2n-1)\pi y}{2}$. From Example 12.3.5,

$$u(x,y) = -\frac{96}{\pi^3}\sum_{n=1}^{\infty}\left[1+(-1)^n\frac{4}{(2n-1)\pi}\right]\frac{\cosh(2n-1)\pi(x-2)/2}{(2n-1)^3\cosh 2(2n-1)\pi/2}\sin\frac{(2n-1)\pi y}{2}.$$

12.3.12. From Example 11.3.8.3,

$$S_M(y) = \frac{96}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} \left[3 + (-1)^n \frac{4}{(2n-1)\pi}\right] \sin \frac{(2n-1)\pi y}{2}.$$

From Example 12.3.5,

$$u(x,y) = \frac{96}{\pi^3} \sum_{n=1}^{\infty} \left[3 + (-1)^n \frac{4}{(2n-1)\pi}\right] \frac{\cosh(2n-1)\pi(x-3)/2}{(2n-1)^3 \cosh 3(2n-1)\pi/2} \sin \frac{(2n-1)\pi y}{2}.$$

12.3.14.

$$\begin{aligned}
c_n &= \frac{2}{3} \int_0^3 (3y - y^2) \cos \frac{(2n-1)\pi y}{6}\, dy \\
&= \frac{4}{(2n-1)\pi} \left[(3y - y^2) \sin \frac{(2n-1)\pi y}{6} \Big|_0^3 - \int_0^3 (3-2y) \sin \frac{(2n-1)\pi y}{6}\, dy \right] \\
&= \frac{24}{(2n-1)^2\pi^2} \left[(3-2y) \cos \frac{(2n-1)\pi y}{6} \Big|_0^3 + 2 \int_0^3 \cos \frac{(2n-1)\pi y}{6}\, dy \right] \\
&= -\frac{72}{(2n-1)^2\pi^2} + \frac{288}{(2n-1)^3\pi^3} \sin \frac{(2n-1)\pi y}{6} \Big|_0^3 = \frac{288}{(2n-1)^3\pi^3} \sin \frac{(2n-1)\pi y}{2} \\
&= -\frac{72}{(2n-1)^2\pi^2} + (-1)^{n-1} \frac{288}{(2n-1)^3\pi^3};
\end{aligned}$$

$C_M(y) = -\dfrac{72}{\pi^2} \displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \left[1 + \frac{4(-1)^n}{(2n-1)\pi}\right] \cos \frac{(2n-1)\pi y}{6}$. From Example 12.3.7,

$$u(x,y) = -\frac{432}{\pi^3} \sum_{n=1}^{\infty} \left[1 + \frac{4(-1)^n}{(2n-1)\pi}\right] \frac{\cosh(2n-1)\pi x/6}{(2n-1)^3 \sinh(2n-1)\pi/3} \cos \frac{(2n-1)\pi y}{6}.$$

12.3.16. Since $g'(0) = g(1) = 0$ and $g''(y) = -6y$, Theorem 11.3.5**(c)** implies that

$$\begin{aligned}
\alpha_n &= \frac{48}{(2n-1)^2\pi^2} \int_0^1 y \cos \frac{(2n-1)\pi y}{2}\, dy \\
&= \frac{96}{(2n-1)^3\pi^3} \left[y \sin \frac{(2n-1)\pi y}{2} \Big|_0^1 - \int_0^1 \sin \frac{(2n-1)\pi y}{2}\, dy \right] \\
&= \frac{96}{(2n-1)^3\pi^3} \left[(-1)^{n+1} + \frac{2}{(2n-1)\pi} \cos \frac{(2n-1)\pi y}{2} \Big|_0^1 \right] \\
&= -\frac{96}{(2n-1)^3\pi^3} \left[(-1)^n + \frac{2}{(2n-1)\pi} \right] dy;
\end{aligned}$$

$C_M(y) = -\dfrac{96}{\pi^3} \displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} \left[(-1)^n + \frac{2}{(2n-1)\pi}\right] \cos \frac{(2n-1)\pi y}{2}$. From Example 12.3.7,

$$u(x,y) = -\frac{192}{\pi^4} \sum_{n=1}^{\infty} \frac{\cosh(2n-1)\pi x/2}{(2n-1)^4 \sinh(2n-1)\pi/2} \left[(-1)^n + \frac{2}{(2n-1)\pi}\right] \cos \frac{(2n-1)\pi y}{2}.$$

12.3.18. The boundary conditions require products $v(x,y) = X(x)Y(y)$ such that (A) $X'' + \lambda X = 0$, $X'(0) = 0$, $X'(a) = 0$, and (B) $Y'' - \lambda Y = 0$, $Y(0) = 1$, $Y(b) = 0$. From Theorem 11.1.3, the eigenvalues of (A) are $\lambda = 0$, with associated eigenfunction $X_0 = 1$, and $\lambda_n = \dfrac{n^2\pi^2}{a^2}$, with associated eigenfunctions $Y_n = \cos\dfrac{n\pi x}{a}$, $n = 1, 2, 3, \dots$. Substituting $\lambda = 0$ into (B) yields $Y_0'' = 0$, $Y_0(0) = 1$, $Y_0(b) = 0$, so $Y_0(y) = 1 - \dfrac{y}{b}$. Substituting $\lambda = \dfrac{n^2\pi^2}{a^2}$ into (B) yields $Y_n'' - (n^2\pi^2/a^2)Y_n = 0$, $Y_n(0) = 1$, $Y_n(b) = 0$, so $Y_n = \dfrac{\sinh n\pi(b-y)/a}{\sinh n\pi b/a}$. Then $v_n(x,y) = X_n(x)Y_n(y) = \dfrac{\sinh n\pi(b-y)/a}{\sinh n\pi b/a}\cos\dfrac{n\pi x}{a}$, so $v_n(x,0) = \cos\dfrac{n\pi x}{a}$. Therefore,$v_n$ is solution of the given problem with $f(x) = \cos\dfrac{n\pi x}{a}$. More generally, if $\alpha_0, \dots, \alpha_m$ are arbitrary constants, then $u_m(x,y) = \alpha_0\left(1 - \dfrac{y}{b}\right) + \displaystyle\sum_{n=1}^{m}\alpha_n\dfrac{\sinh n\pi(b-y)/a}{\sinh n\pi b/a}\cos\dfrac{n\pi x}{a}$ is a solution of the given problem with $f(x) = \alpha_0 + \displaystyle\sum_{n=1}^{m}\alpha_n\cos\dfrac{n\pi x}{a}$. Therefore, if f is an arbitrary piecewise smooth function on $[0,a]$ we define the formal solution of the given problem to be $u(x,y) = \alpha_0\left(1 - \dfrac{y}{b}\right) + \displaystyle\sum_{n=1}^{\infty}\alpha_n\dfrac{\sinh n\pi(b-y)/a}{\sinh n\pi b/a}\cos\dfrac{n\pi x}{a}$, where $C(x) = \alpha_0 + \displaystyle\sum_{n=1}^{\infty}\alpha_n\cos\dfrac{n\pi x}{a}$ is the Fourier cosine series of f on $[0,a]$; that is, $\alpha_0 = \dfrac{1}{a}\displaystyle\int_0^a f(x)\,dx$ and $\alpha_n = \dfrac{2}{a}\displaystyle\int_0^a f(x)\cos\dfrac{n\pi x}{a}\,dx$, $n \geq 1$.

Now consider the special case. $\alpha_0 = \dfrac{1}{2}\displaystyle\int_0^2 (x^4 - 4x^3 + 4x^2)\,dx = \dfrac{1}{2}\left(\dfrac{x^5}{5} - x^4 + \dfrac{4x^3}{3}\right)\Bigg|_0^2 = \dfrac{6}{5}$. Since $f'(0) = f'(2) = 0$ and $f'''(x) = 12(2x-2)$, Theorem 11.3.5**(a)** implies that

$$
\begin{aligned}
\alpha_n &= \frac{96}{n^3\pi^3}\int_0^2 (2x-2)\sin\frac{n\pi x}{2}\,dx = -\frac{192}{n^4\pi^4}\left[(2x-2)\cos\frac{n\pi x}{2}\Bigg|_0^2 - 2\int_0^2 \cos\frac{n\pi x}{2}\,dx\right] \\
&= -\frac{192}{n^4\pi^4}\left[(-1)^n 2 + 2\right] + \frac{768}{n^5\pi^5}\sin\frac{n\pi x}{2}\Bigg|_0^2 = -\frac{384}{n^4\pi^4}\left[1 + (-1)^n\right] \\
&= \begin{cases} 0 & \text{if } n = 2m-1, \\ -\dfrac{48}{m^4\pi^4} & \text{if } n = 2m, \end{cases} \qquad n \geq 1.
\end{aligned}
$$

$C(x) = \dfrac{8}{15} - \dfrac{48}{\pi^4}\displaystyle\sum_{n=1}^{\infty}\dfrac{1}{n^4}\cos n\pi x$; $u(x,y) = \dfrac{8(1-y)}{15} - \dfrac{48}{\pi^4}\displaystyle\sum_{n=1}^{\infty}\dfrac{1}{n^4}\dfrac{\sinh n\pi(1-y)}{\sinh n\pi}\cos n\pi x$.

12.3.20. The boundary conditions require products $v(x,y) = X(x)Y(y)$ such that (A) $X'' + \lambda X = 0$, $X(0) = 0$, $X'(a) = 0$, and (B) $Y'' - \lambda Y = 0$, $Y(0) = 1$, $Y(b) = 0$. From Theorem 11.1.4, the eigenvalues of (A) are $\lambda_n = \dfrac{(2n-1)^2\pi^2}{4a^2}$, with associated eigenfunctions $Y_n = \sin\dfrac{(2n-1)\pi x}{2a}$, $n = 1, 2, 3, \dots$. Substituting $\lambda = \dfrac{(2n-1)^2\pi^2}{4a^2}$ into (B) yields $Y_n'' - ((2n-1)^2\pi^2/4a^2)Y_n = 0$, $Y_n(0) = 1$, $Y_n(b) = 0$, so $Y_n = \dfrac{\sinh(2n-1)\pi(b-y)/2a}{\sinh(2n-1)\pi b/2a}$. Then $v_n(x,y) = X_n(x)Y_n(y) = \dfrac{\sinh(2n-1)\pi(b-y)/2a}{\sinh(2n-1)\pi b/2a}\sin\dfrac{(2n-1)\pi x}{2a}$, so $v_n(x,0) = \sin\dfrac{(2n-1)\pi x}{2a}$. Therefore,$v_n$ is solution of the given problem with $f(x) = \sin\dfrac{(2n-1)\pi x}{2a}$. More generally, if $\alpha_1,\dots,\alpha_m$ are arbitrary constants, then $u_m(x,y) = \displaystyle\sum_{n=1}^{m}\alpha_n\frac{\sinh(2n-1)\pi(b-y)/2a}{\sinh(2n-1)\pi b/2a}\cos\frac{(2n-1)\pi x}{2a}$ is a solution of the given problem with $f(x) = \displaystyle\sum_{n=1}^{m}\alpha_n\sin\frac{(2n-1)\pi x}{2a}$. Therefore, if f is an arbitrary piecewise smooth function on $[0,a]$ we define the formal solution of the given problem to be $u(x,y) = \displaystyle\sum_{n=1}^{\infty}\alpha_n\frac{\sinh(2n-1)\pi(b-y)/2a}{\sinh(2n-1)\pi b/2a}\sin\frac{(2n-1)\pi x}{2a}$, where $S_m(x) = \displaystyle\sum_{n=1}^{\infty}\alpha_n\sin\frac{(2n-1)\pi x}{2a}$ is the mixed Fourier sine series of f on $[0,a]$; that is, $\alpha_n = \dfrac{2}{a}\displaystyle\int_0^a f(x)\sin\frac{(2n-1)\pi x}{2a}$.

Now consider the special case. Since $f(0) = f'(L) = 0$ and $f''(x) = -2$, Theorem 11.3.5(**d**) implies that

$$\alpha_n = \frac{48}{(2n-1)^2\pi^2}\int_0^3\sin\frac{(2n-1)\pi x}{6}\,dx = -\frac{288}{(2n-1)^3\pi^3}\cos\frac{(2n-1)\pi x}{6}\bigg|_0^3 = \frac{288}{(2n-1)^3\pi^3};$$

$$S_M(x) = \frac{288}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\sin\frac{(2n-1)\pi x}{6};$$

$$u(x,y) = \frac{288}{\pi^3}\sum_{n=1}^{\infty}\frac{\sinh(2n-1)\pi(2-y)/6}{(2n-1)^3\sinh(2n-1)\pi/3}\sin\frac{(2n-1)\pi x}{6}.$$

12.3.22. The boundary conditions require products $v(x,y) = X(x)Y(y)$ such that (A) $X'' + \lambda X = 0$, $X'(0) = 0$, $X'(a) = 0$, and (B) $Y'' - \lambda Y = 0$, $Y'(0) = 0$, $Y(b) = 1$. From Theorem 11.1.3, the eigenvalues of (A) are $\lambda = 0$, with associated eigenfunction $X_0 = 1$, and $\lambda_n = \dfrac{n^2\pi^2}{a^2}$, with associated eigenfunctions $Y_n = \cos\dfrac{n\pi x}{a}$, $n = 1, 2, 3,\dots$. Substituting $\lambda = 0$ into (B) yields $Y_0'' = 0$, $Y_0'(0) = 0$, $Y_0(b) = 1$, so $Y_0 = 1$. Substituting $\lambda = \dfrac{n^2\pi^2}{a^2}$ into (B) yields $Y_n'' - (n^2\pi^2/a^2)Y_n = 0$, $Y_n'(0) = 0$, $Y_n(b) = 1$, so $Y_n = \dfrac{\cosh n\pi y/a}{\cosh n\pi b/a}$. Then $v_n(x,y) = X_n(x)Y_n(y) = \dfrac{\sinh n\pi y/a}{\cosh n\pi b/a}\cos\dfrac{n\pi x}{a}$, so $v_n(x,b) = \cos\dfrac{n\pi x}{a}$. Therefore,$v_n$ is solution of the given problem with $f(x) = \cos\dfrac{n\pi x}{a}$. More generally, if $\alpha_0,\dots,\alpha_m$ are arbitrary constants, then $u_m(x,y) = \alpha_0 + \displaystyle\sum_{n=1}^{m}\alpha_n\frac{\cosh n\pi y/a}{\cosh n\pi b/a}\cos\frac{n\pi x}{a}$ is a solution of the given problem with $f(x) = \alpha_0 + \displaystyle\sum_{n=1}^{m}\alpha_n\cos\frac{n\pi x}{a}$. Therefore, if f is an arbitrary piecewise smooth function on $[0,a]$ we define the formal solution of the given problem to be $u(x,y) = \alpha_0 + \displaystyle\sum_{n=1}^{\infty}\alpha_n\frac{\cosh n\pi y/a}{\cosh n\pi b/a}\cos\frac{n\pi x}{a}$, where $C(x) = \alpha_0 + \displaystyle\sum_{n=1}^{\infty}\alpha_n\cos\frac{n\pi x}{a}$ is the Fourier cosine series of f on $[0,a]$; that is, $\alpha_0 = \dfrac{1}{a}\displaystyle\int_0^a f(x)\,dx$ and $\alpha_n = \dfrac{2}{a}\displaystyle\int_0^a f(x)\cos\frac{n\pi x}{a}\,dx$, $n \ge 1$.

Now consider the special case.

$$\alpha_0 = \frac{1}{\pi}\int_0^\pi (x^4 - 2\pi x^3 + \pi^2 x^2)\,dx = \frac{1}{\pi}\left(\frac{x^5}{5} - \frac{\pi x^4}{2} + \frac{\pi^2 x^3}{3}\right)\Big|_0^\pi = \frac{\pi^4}{30}.$$

Since $f'(0) = f'(\pi) = 0$ and $f'''(x) = 12(2x - \pi)$, Theorem 11.3.5**(a)** implies that

$$\begin{aligned}\alpha_n &= \frac{24}{n^3\pi}\int_0^\pi (2x-\pi)\sin nx\,dx = -\frac{24}{n^4\pi}\left[(2x-\pi)\cos nx\Big|_0^\pi - 2\int_0^\pi \cos nx\,dx\right]\\ &= -\frac{24}{n^4\pi}[(-1)^n\pi + \pi] + \frac{48}{n^5\pi}\sin nx\Big|_0^\pi = -\frac{24}{n^4}[1 + (-1)^n]\\ &= \begin{cases} 0 & \text{if } n = 2m-1,\\ -\dfrac{3}{m^4} & \text{if } n = 2m,\end{cases} \quad n \ge 1;\end{aligned}$$

$$C(x) = \frac{\pi^4}{30} - 3\sum_{n=1}^\infty \frac{1}{n^4}\cos 2nx;\ u(x,y) = \frac{\pi^4}{30} - 3\sum_{n=1}^\infty \frac{1}{n^4}\frac{\cosh 2ny}{\cos 2n}\cos 2nx$$

12.3.24. The boundary conditions require products $v(x,y) = X(x)Y(y)$ such that (A) $X'' - \lambda X = 0$, $X'(0) = 0$, $X(a) = 1$, and (B) $Y'' + \lambda Y = 0$, $Y(0) = 0$, $Y(b) = 0$. From Theorem 11.1.2, the eigenvalues of (B) are $\lambda_n = \dfrac{n^2\pi^2}{b^2}$, with associated eigenfunctions $Y_n = \sin\dfrac{n\pi y}{b}$, $n = 1, 2, 3, \dots$. Substituting $\lambda = \dfrac{n^2\pi^2}{b^2}$ into (A) yields $X_n'' - (n^2\pi^2/b^2)X_n = 0$, $X_n'(0) = 0$, $X_n(a) = 1$, so $X_n = \dfrac{\cosh n\pi x/b}{\cosh n\pi a/b}$. Then $v_n(x,y) = X_n(x)Y_n(y) = \dfrac{\cosh n\pi x/b}{\cosh n\pi a/b}\sin\dfrac{n\pi y}{b}$, so $v_n(a,y) = \sin\dfrac{n\pi y}{b}$. Therefore,$v_n$ is solution of the given problem with $g(y) = \sin\dfrac{n\pi y}{b}$. More generally, if $\alpha_1, \dots, \alpha_m$ are arbitrary constants, then $u_m(x,y) = \displaystyle\sum_{n=1}^m \alpha_n \frac{\cosh n\pi x/b}{\cosh n\pi a/b}\sin\frac{n\pi y}{b}$ is a solution of the given problem with $g(y) = \displaystyle\sum_{n=1}^m \alpha_n \sin\frac{n\pi y}{b}$. Therefore, if g is an arbitrary piecewise smooth function on $[0,b]$ we define the formal solution of the given problem to be $u(x,y) = \displaystyle\sum_{n=1}^\infty \alpha_n \frac{\cosh n\pi x/b}{\cosh n\pi a/b}\sin\frac{n\pi y}{b}$, where $S(y) = \displaystyle\sum_{n=1}^\infty \alpha_n \sin\frac{n\pi y}{b}$ is the Fourier sine series of g on $[0,b]$; that is, $\alpha_n = \dfrac{2}{b}\displaystyle\int_0^b g(y)\sin\frac{n\pi y}{b}\,dy$.

Now consider the special case. Since $g(0) = g(1) = g''(0) = g''(L) = 0$ and $f^{(4)}(y) = 24$, Theorem 11.3.5**(b)** and Exercise 35**(b)** of Section 11.3 imply that

$$\begin{aligned}\alpha_n &= \frac{48}{n^4\pi^4}\int_0^1 \sin n\pi y\,dy = -\frac{48}{n^5\pi^5}\cos n\pi y\Big|_0^1\\ &= -\frac{48}{n^5\pi^5}[(-1)^n - 1] = \begin{cases} \dfrac{96}{(2m-1)^5\pi^5} & \text{if } n = 2m-1\\ 0 & \text{if } n = 2m;\end{cases}\end{aligned}$$

$$S(y) = \frac{96}{\pi^5}\sum_{n=1}^\infty \frac{1}{(2n-1)^5}\sin(2n-1)\pi y;\ u(x,y) = \frac{96}{\pi^5}\sum_{n=1}^\infty \frac{\cosh(2n-1)\pi x}{(2n-1)^5\cosh(2n-1)\pi}\sin(2n-1)\pi y.$$

12.3.26. The boundary conditions require products $v(x,y) = X(x)Y(y)$ such that (A) $X'' - \lambda X = 0$, $X'(0) = 0$, $X'(a) = 1$, and (B) $Y'' + \lambda Y = 0$, $Y(0) = 0$, $Y(b) = 0$. From Theorem 11.1.2, the eigenvalues of (B) are $\lambda_n = \dfrac{n^2\pi^2}{b^2}$, with associated eigenfunctions $Y_n = \sin\dfrac{n\pi y}{b}$, $n = 1, 2, 3, \dots$. Substituting $\lambda = \dfrac{n^2\pi^2}{b^2}$ into (A) yields $X_n'' - (n^2\pi^2/b^2)X_n = 0$, $X_n'(0) = 0$, $X_n'(a) = 1$, so $X_n = \dfrac{b}{n\pi}\dfrac{\cosh n\pi x/b}{\sinh n\pi a/b}$. Then $v_n(x,y) = X_n(x)Y_n(y) = \dfrac{b}{n\pi}\dfrac{\cosh n\pi x/b}{\sinh n\pi a/b}\sin\dfrac{n\pi y}{b}$, so $\dfrac{\partial v_n}{\partial x}(a,y) = \sin\dfrac{n\pi y}{b}$. Therefore,$v_n$ is solution of the given problem with $g(y) = \sin\dfrac{n\pi y}{b}$. More generally, if $\alpha_1, \dots, \alpha_m$ are arbitrary constants, then $u_m(x,y) = \dfrac{b}{\pi}\displaystyle\sum_{n=1}^{m}\alpha_n\frac{\cosh n\pi x/b}{n\sinh n\pi a/b}\sin\frac{n\pi y}{b}$ is a solution of the given problem with $g(y) = \displaystyle\sum_{n=1}^{m}\alpha_n\sin\frac{n\pi y}{b}$. Therefore, if g is an arbitrary piecewise smooth function on $[0,b]$ we define the formal solution of the given problem to be $u(x,y) = \dfrac{b}{\pi}\displaystyle\sum_{n=1}^{\infty}\alpha_n\frac{\cosh n\pi x/b}{n\sinh n\pi a/b}\sin\frac{n\pi y}{b}$, where $S(y) = \displaystyle\sum_{n=1}^{\infty}\alpha_n\sin\frac{n\pi y}{b}$ is the Fourier sine series of g on $[0,b]$; that is, $\alpha_n = \dfrac{2}{b}\displaystyle\int_0^b g(y)\sin\frac{n\pi y}{b}\,dy$.

Now consider the special case. $\alpha_n = \dfrac{1}{2}\left[\displaystyle\int_0^2 y\sin\frac{n\pi y}{4} + \int_2^4(4-y)\sin\frac{n\pi y}{4}\,dy\right]$;

$$\begin{aligned}
\int_0^2 y\sin\frac{n\pi y}{4}\,dy &= -\frac{4}{n\pi}\left[y\cos\frac{n\pi y}{4}\Big|_0^2 - \int_0^2\cos\frac{n\pi y}{4}\,dy\right] \\
&= -\frac{2}{n\pi}\cos\frac{n\pi}{2} + \frac{4}{n^2\pi^2}\sin\frac{n\pi y}{4}\Big|_0^2 = -\frac{2}{n\pi}\cos\frac{n\pi}{2} + \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2};
\end{aligned}$$

$$\begin{aligned}
\int_0^2(4-y)\sin\frac{n\pi y}{4}\,dy &= -\frac{2}{n\pi}\left[(4-y)\cos\frac{n\pi y}{4}\Big|_2^4 + \int_0^2\cos\frac{n\pi y}{4}\,dy\right] \\
&= \frac{2}{n\pi}\cos\frac{n\pi}{2} - \frac{4}{n^2\pi^2}\sin\frac{n\pi y}{4}\Big|_2^4 = \frac{2}{n\pi}\cos\frac{n\pi}{2} + \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2};
\end{aligned}$$

$$\alpha_n = \frac{16}{n^2\pi^2}\sin\frac{n\pi}{2} = \begin{cases} (-1)^{m+1}\dfrac{16}{(2m-1)^2\pi^2} & \text{if } n = 2m-1 \\ 0 & \text{if } n = 2m; \end{cases}$$

$$S(y) = \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\sin\frac{(2n-1)\pi y}{4};$$

$$u(x,y) = \frac{64}{\pi^3}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cosh(2n-1)\pi x/4}{(2n-1)^3\sinh(2n-1)\pi/4}\sin\frac{(2n-1)\pi y}{4}.$$

12.3.28. The boundary conditions require products $v(x,y) = X(x)Y(y)$ such that (A) $X'' - \lambda X = 0$, $X'(0) = 1$, $X(a) = 0$, and (B) $Y'' + \lambda Y = 0$, $Y'(0) = 0$, $Y'(b) = 0$. From Theorem 11.1.3, the eigenvalues of (B) are $\lambda_0 = 0$, with associated eigenfunction $Y_0 = 1$, and $\lambda_n = \dfrac{n^2\pi^2}{b^2}$, with associated eigenfunctions $Y_n = \cos\dfrac{n\pi y}{b}$, $n = 1, 2, 3, \dots$ Substituting $\lambda_0 = 0$ into (A) yields $X_0'' = 0$, $X_0'(0) = 1$ $X(a) = 0$, so $X_0 = x - a$. Substituting $\lambda = \dfrac{n^2\pi^2}{b^2}$ into (A) yields $X_n'' - (n^2\pi^2/b^2)X_n = 0$, $X_n'(0) = 1$, $X_n(a) = 0$, so $X_n = \dfrac{b}{n\pi}\dfrac{\sinh n\pi(x-a)/b}{\cosh n\pi a/b}$. Then $v_n(x,y) = X_n(x)Y_n(y) = \dfrac{b}{n\pi}\dfrac{\sinh n\pi(x-a)/b}{\cosh n\pi a/b}\cos\dfrac{n\pi y}{b}$, so $\dfrac{\partial v_n}{\partial x}(0,y) = \cos\dfrac{n\pi y}{b}$. Therefore,$v_n$ is solution of the given problem with $g(y) = \cos\dfrac{n\pi y}{b}$. More generally, if $\alpha_0, \dots, \alpha_m$ are arbitrary constants, then $u_m(x,y) = \alpha_0(x-a) + \dfrac{b}{\pi}\displaystyle\sum_{n=1}^{m}\alpha_n\frac{\sinh n\pi(x-a)/b}{n\cosh n\pi a/b}\cos\frac{n\pi y}{b}$ is a solution of the given problem with $g(y) = \displaystyle\sum_{n=1}^{m}\alpha_n\cos\frac{n\pi y}{b}$. Therefore, if g is an arbitrary piecewise smooth function on $[0,b]$ we define the formal solution of the given problem to be $u(x,y) = \alpha_0(x-a) + \dfrac{b}{\pi}\displaystyle\sum_{n=1}^{\infty}\alpha_n\frac{\sinh n\pi(x-a)/b}{n\cosh n\pi a/b}\cos\frac{n\pi y}{b}$ where $C(y) = \alpha_0 + \displaystyle\sum_{n=1}^{\infty}\alpha_n\cos\frac{n\pi y}{b}$ is the Fourier cosine series of g on $[0,b]$; that is, $\alpha_0 = \dfrac{1}{b}\displaystyle\int_0^b g(y)\cos\frac{n\pi y}{b}\,dy$, $\alpha_n = \dfrac{2}{b}\displaystyle\int_0^b g(y)\cos\frac{n\pi y}{b}\,dy$, $n \ge 1$.

Now consider the special case. From Example 11.3.1,

$$C(y) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos(2n-1)y;$$

$$u(x,y) = \frac{\pi(x-2)}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sinh(2n-1)(x-2)}{(2n-1)^3\cosh 2(2n-1)}\cos(2n-1)y.$$

12.3.30. The boundary conditions require products $v(x,y) = X(x)Y(y)$ such that (A) $X'' + \lambda X = 0$, $X'(0) = 0$, $X(a) = 0$, and (B) $Y'' - \lambda Y = 0$, $Y(0) = 1$, and Y is bounded. From Theorem 11.1.5, the eigenvalues of (A) are $\lambda_n = \dfrac{(2n-1)^2\pi^2}{4a^2}$, with associated eigenfunctions $Y_n = \cos\dfrac{(2n-1)\pi x}{2a}$, $n = 1, 2, 3, \dots$ Substituting $\lambda = \dfrac{(2n-1)^2\pi^2}{4a^2}$ into (B) yields $Y_n'' - ((2n-1)^2\pi^2/4a^2)Y_n = 0$, $Y_n(0) = 1$, so $Y_n = e^{-(2n-1)\pi y/2a}$. Then $v_n(x,y) = X_n(x)Y_n(y) = e^{-(2n-1)\pi y/2a}\cos\dfrac{(2n-1)\pi x}{2a}$, so $v_n(x,0) = \cos\dfrac{(2n-1)\pi x}{2a}$. Therefore,$v_n$ is solution of the given problem with $f(x) = \cos\dfrac{(2n-1)\pi x}{2a}$. More generally, if $\alpha_1, \dots, \alpha_m$ are arbitrary constants, then $u_m(x,y) = \displaystyle\sum_{n=1}^{m}\alpha_n e^{-(2n-1)\pi y/2a}\cos\frac{(2n-1)\pi x}{2a}$ is a solution of the given problem with $f(x) = \displaystyle\sum_{n=1}^{m}\alpha_n\cos\frac{(2n-1)\pi x}{2a}$. Therefore, if f is an arbitrary piecewise smooth function on $[0,a]$ we define the formal solution of the given problem to be

$u(x,y) = \sum_{n=1}^{\infty} \alpha_n e^{-(2n-1)\pi y/2a} \cos \frac{(2n-1)\pi x}{2a}$, where $C_m(x) = \sum_{n=1}^{\infty} \alpha_n \cos \frac{(2n-1)\pi x}{2a}$ is the mixed Fourier cosine series of f on $[0,a]$; that is, $\alpha_n = \frac{2}{a}\int_0^a f(x)\cos\frac{(2n-1)\pi x}{2a}$.

Now consider the special case. Since $f'(0) = f(L) = 0$ and $f''(x) = -2$, Theorem 11.3.5**(d)** implies that

$$\begin{aligned} \alpha_n &= \frac{16L}{(2n-1)^2\pi^2}\int_0^3 \cos\frac{(2n-1)\pi x}{6}\,dx \\ &= \frac{288}{(2n-1)^3\pi^3}\sin\frac{(2n-1)\pi x}{6}\Big|_0^3 = (-1)^{n+1}\frac{288}{(2n-1)^3\pi^3}; \end{aligned}$$

$$C_M(x) = -\frac{288}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^3}\cos\frac{(2n-1)\pi x}{6};$$

$$u(x,y) = -\frac{288}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^3}e^{-(2n-1)\pi y/6}\cos\frac{(2n-1)\pi x}{6}.$$

12.3.32. The boundary conditions require products $v(x,y) = X(x)Y(y)$ such that (A) $X'' + \lambda X = 0$, $X(0) = 0$, $X(a) = 0$, and (B) $Y'' - \lambda Y = 0$, $Y'(0) = 1$, and Y is bounded. From Theorem 11.1.2, the eigenvalues of (A) are $\lambda_n = \frac{n^2\pi^2}{a^2}$, with associated eigenfunctions $Y_n = \sin\frac{n\pi x}{a}$, $n = 1, 2, 3, \dots$. Substituting $\lambda = \frac{n^2\pi^2}{a^2}$ into (B) yields $Y_n'' - (n^2\pi^2/a^2)Y_n = 0$, $Y_n'(0) = 1$, so $Y_n = -\frac{a}{n\pi}e^{-n\pi y/a}$. Then $v_n(x,y) = X_n(x)Y_n(y) = -\frac{a}{n\pi}e^{-n\pi y/a}\sin\frac{n\pi x}{a}$, so $\frac{\partial v_n}{\partial y}(x,0) = \sin\frac{n\pi x}{a}$. Therefore,$v_n$ is solution of the given problem with $f(x) = \sin\frac{n\pi x}{a}$. More generally, if $\alpha_1,\dots,\alpha_m$ are arbitrary constants, then $u_m(x,y) = -\frac{a}{\pi}\sum_{n=1}^{m}\frac{\alpha_n}{n}e^{-n\pi y/a}\sin\frac{n\pi x}{a}$ is a solution of the given problem with $f(x) = \sum_{n=1}^{m}\alpha_n\sin\frac{n\pi x}{a}$. Therefore, if f is an arbitrary piecewise smooth function on $[0,a]$ we define the formal solution of the given problem to be $u(x,y) = -\frac{a}{\pi}\sum_{n=1}^{\infty}\frac{\alpha_n}{n}e^{-n\pi y/a}\sin\frac{n\pi x}{a}$, where $C(x) = \sum_{n=1}^{\infty}\alpha_n\sin\frac{n\pi x}{a}$ is the Fourier sine series of f on $[0,a]$; that is, $\alpha_n = \frac{2}{a}\int_0^a f(x)\sin\frac{n\pi x}{a}\,dx$.

Now consider the special case. Since $f(0) = f(\pi) = 0$ and $f''(x) = 2\pi - 6x$, Theorem 11.3.5**(b)** implies that

$$\begin{aligned} \alpha_n &= -\frac{2}{n^2\pi}\int_0^\pi (2\pi - 6x)\sin nx\,dx = \frac{2}{n^3\pi}\left[(2\pi-6x)\cos nx\Big|_0^\pi + 6\int_0^\pi \cos nx\,dx\right] \\ &= -\frac{4}{n^3}\left[1 + (-1)^n 2\right] + \frac{12}{n^4\pi}\sin nx\Big|_0^\pi = -\frac{4}{n^3}\left[1+(-1)^n 2\right]; \end{aligned}$$

$S(x) = -4\sum_{n=1}^{\infty}\frac{[1+(-1)^n 2]}{n^3}\sin nx$; $u(x) = 4\sum_{n=1}^{\infty}\frac{[1+(-1)^n 2]}{n^4}e^{-ny}\sin nx$;

12.3.34. The boundary conditions require products $v(x,y) = X(x)Y(y)$ such that (A) $X'' + \lambda X = 0$, $X(0) = 0$, $X'(a) = 0$, and (B) $Y'' - \lambda Y = 0$, $Y'(0) = 1$, and Y is bounded. From Theorem 11.1.4, the eigenvalues of (A) are $\lambda_n = \dfrac{(2n-1)^2\pi^2}{4a^2}$, with associated eigenfunctions $Y_n = \sin\dfrac{(2n-1)\pi x}{2a}$, $n = 1$, 2, 3,…. Substituting $\lambda = \dfrac{(2n-1)^2\pi^2}{4a^2}$ into (B) yields $Y_n'' - ((2n-1)^2\pi^2/4a^2)Y_n = 0$, $Y_n'(0) = 1$, so $Y_n = -\dfrac{2a}{(2n-1)\pi}e^{-(2n-1)\pi y/2a}$. Then $v_n(x,y) = X_n(x)Y_n(y) = -\dfrac{2a}{(2n-1)\pi}e^{-(2n-1)\pi y/2a}\sin\dfrac{(2n-1)\pi x}{2a}$, so $\dfrac{\partial v_n}{\partial y}(x,0) = \sin\dfrac{(2n-1)\pi x}{2a}$. Therefore,$v_n$ is solution of the given problem with $f(x) = \sin\dfrac{(2n-1)\pi x}{2a}$. More generally, if $\alpha_1,\dots,\alpha_m$ are arbitrary constants, then $u_m(x,y) = -\dfrac{2a}{\pi}\displaystyle\sum_{n=1}^{m}\dfrac{\alpha_n}{2n-1}e^{-(2n-1)\pi y/2a}\sin\dfrac{(2n-1)\pi x}{2a}$ is a solution of the given problem with $f(x) = \displaystyle\sum_{n=1}^{m}\alpha_n\sin\dfrac{(2n-1)\pi x}{2a}$. Therefore, if f is an arbitrary piecewise smooth function on $[0,a]$ we define the formal solution of the given problem to be $u(x,y) = -\dfrac{2a}{\pi}\displaystyle\sum_{n=1}^{\infty}\dfrac{\alpha_n}{2n-1}e^{-(2n-1)\pi y/2a}\sin\dfrac{(2n-1)\pi x}{2a}$,where $S_m(x) = \displaystyle\sum_{n=1}^{\infty}\alpha_n\sin\dfrac{(2n-1)\pi x}{2a}$ is the mixed Fourier sine series of f on $[0,a]$; that is, $\alpha_n = \dfrac{2}{a}\displaystyle\int_0^a f(x)\sin\dfrac{(2n-1)\pi x}{2a}$.

Now consider the special case.

$$
\begin{aligned}
\alpha_n &= \frac{2}{5}\int_0^5 (5x - x^2)\sin\frac{(2n-1)\pi x}{10}\,dx \\
&= -\frac{4}{(2n-1)\pi}\left[(5x-x^2)\cos\frac{(2n-1)\pi x}{10}\Big|_0^5 - \int_0^5 (5-2x)\cos\frac{(2n-1)\pi x}{10}\,dx\right] \\
&= \frac{40}{(2n-1)^2\pi^2}\left[(5-2x)\sin\frac{(2n-1)\pi x}{10}\Big|_0^5 + 2\int_0^5 \sin\frac{(2n-1)\pi x}{10}\,dx\right] \\
&= (-1)^n\frac{200}{(2n-1)^2\pi^2} - \frac{800}{(2n-1)^3\pi^3}\cos\frac{(2n-1)\pi x}{10}\Big|_0^5 \\
&= (-1)^n\frac{200}{(2n-1)^2\pi^2} + \frac{800}{(2n-1)^3\pi^3};
\end{aligned}
$$

$$
S_M(x) = \frac{200}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\left[(-1)^n + \frac{4}{(2n-1)\pi}\right]\sin\frac{(2n-1)\pi x}{10};
$$

$$
u(x,y) = -\frac{2000}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\left[(-1)^n + \frac{4}{(2n-1)\pi}\right]e^{-(2n-1)\pi y/10}\sin\frac{(2n-1)\pi x}{10}.
$$

12.3.36. Solving BVP$(1,1,1,1)(f_0,0,0,0)$ requires products $X(x)Y(y)$ such that

$$
X'' + \lambda X = 0, \quad X'(0) = 0, \quad X'(a) = 0; \quad Y'' - \lambda Y = 0, \quad Y'(b) = 0, \quad Y'(0) = 1.
$$

Hence, $X_n = \cos\dfrac{n\pi x}{a}$, $Y_n = -\dfrac{a\cosh n\pi(y-b)/a}{n\pi\sinh n\pi b/a}$, and $c_1 - \dfrac{a}{\pi}\displaystyle\sum_{n=1}^{\infty}A_n\dfrac{\cosh n\pi(y-b)/a}{n\pi\sinh n\pi b/a}\cos\dfrac{n\pi x}{a}$ is a formal solution of BVP$(1,1,1,1)(f_0,0,0,0)$ if c_1 is any constant and $\sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{a}$ is the Fourier cosine expansion of f_0 on $[0,a]$, which is possible if and only if $\int_0^a f_0(x)\,dx = 0$.

Similarly, $c_2 + \frac{a}{\pi}\sum_{n=1}^{\infty} B_n \frac{\cosh n\pi y/a}{n\pi \sinh n\pi b/a}\cos\frac{n\pi x}{a}$ is a formal solution of BVP$(1,1,1,1)(0,f_1,0,0)$ if c_2 is any constant and $\sum_{n=1}^{\infty} B_n \cos\frac{n\pi x}{a}$ is the Fourier cosine expansion of f_1 on $[0,a]$, which is possible if and only if $\int_0^a f_1(x)\,dx = 0$.

Interchanging x and y and a and b shows that $c_3 - \frac{b}{\pi}\sum_{n=1}^{\infty} C_n \frac{\cosh n\pi(x-a)/b}{n\pi \sinh n\pi a/b}\cos\frac{n\pi y}{b}$ is a formal solution of BVP$(1,1,1,1)(0,0,g_0,0)$ if c_3 is any constant and $\sum_{n=1}^{\infty} C_n \cos\frac{n\pi y}{b}$ is the Fourier cosine expansion of g_0 on $[0,b]$, which is possible if and only if $\int_0^b g_0(x)\,dx = 0$, and $c_4 + \frac{b}{\pi}\sum_{n=1}^{\infty} D_n \frac{\cosh n\pi x/b}{n\pi \sinh n\pi a/b}\cos\frac{n\pi y}{b}$ is a formal solution of BVP$(1,1,1,1)(0,0,0,g_1)$ if c_4 is any constant and $\sum_{n=1}^{\infty} D_n \cos\frac{n\pi y}{b}$ is the Fourier cosine expansion of g_1 on $[0,b]$, which is possible if and only if $\int_0^b g_1(x)\,dx = 0$.

Adding the four solutions yields

$$\begin{aligned} u(x,y) &= C + \frac{a}{\pi}\sum_{n=1}^{\infty} \frac{B_n \cosh n\pi y/a - A_n \cosh n\pi(y-b)/a}{n \sinh n\pi b/a}\cos\frac{n\pi x}{a} \\ &\quad + \frac{b}{\pi}\sum_{n=1}^{\infty} \frac{D_n \cosh n\pi x/b - C_n \cosh n\pi(x-a)/b}{n \sinh n\pi a/b}\cos\frac{n\pi y}{b}, \end{aligned}$$

where C is an arbitrary constant.

12.4 LAPLACE'S EQUATION IN POLAR COORDINATES

12.4.2. $v(r,\theta) = R(r)\Theta(\theta)$ where (A) $r^2R'' + rR' - \lambda R = 0$ and $\Theta'' + \lambda\Theta = 0$, $\Theta(0) = 0$, $\Theta(\gamma) = 0$. From Theorem 11.1.2, $\lambda_n = \frac{n^2\pi^2}{\gamma^2}$, $\Theta_n = \sin\frac{n\pi\theta}{\gamma}$, $n = 1, 2, 3, \dots$. Substituting $\lambda = \frac{n^2\pi^2}{\gamma^2}$ into (A) yields the Euler equation $r^2R_n'' + rR_n' - \frac{n^2\pi^2}{\gamma^2}R_n = 0$ for R_n. The indicial polynomial is $\left(s - \frac{n\pi}{\gamma}\right)\left(s + \frac{n\pi}{\gamma}\right)$, so $R_n = c_1 r^{n\pi/\gamma} + c_2 r^{-n\pi/\gamma}$, by Theorem 7.4.3. We want $R_n(\rho) = 1$ and $R_n(\rho_0) = 0$, so $R_n(r) = \frac{\rho_0^{-n\pi/\gamma} r^{n\pi/\gamma} - \rho_0^{n\pi/\gamma} r^{-n\pi/\gamma}}{\rho_0^{-n\pi/\gamma}\rho^{n\pi/\gamma} - \rho_0^{n\pi/\gamma}\rho^{-n\pi/\gamma}}$;

$$v_n(r,\theta) = \frac{\rho_0^{-n\pi/\gamma} r^{n\pi/\gamma} - \rho_0^{n\pi/\gamma} r^{-n\pi/\gamma}}{\rho_0^{-n\pi/\gamma}\rho^{n\pi/\gamma} - \rho_0^{n\pi/\gamma}\rho^{-n\pi/\gamma}}\sin\frac{n\pi\theta}{\gamma};$$

$u(r,\theta) = \sum_{n=1}^{\infty} \alpha_n \frac{\rho_0^{-n\pi/\gamma} r^{n\pi/\gamma} - \rho_0^{n\pi/\gamma} r^{-n\pi/\gamma}}{\rho_0^{-n\pi/\gamma} \rho^{n\pi/\gamma} - \rho_0^{n\pi/\gamma} \rho^{-n\pi/\gamma}} \sin \frac{n\pi\theta}{\gamma}$, where $S(\theta) = \sum_{n=1}^{\infty} \gamma_n \sin \frac{n\pi\theta}{\gamma}$ is the Fourier sine series if f on $[0,\gamma]$; that is, $\alpha_n = \frac{1}{\gamma}\int_0^{\gamma} f(\theta)\sin\frac{n\pi\theta}{\gamma}\,d\theta$, $n = 1, 2, 3, \dots$.

12.4.4. $v(r,\theta) = R(r)\Theta(\theta)$ where (A) $r^2R'' + rR' - \lambda R = 0$ and $\Theta'' + \lambda\Theta = 0$, $\Theta'(0) = 0$, $\Theta(\gamma) = 0$. From Theorem 11.1.5, $\lambda_n = \frac{(2n-1)^2\pi^2}{4\gamma^2}$, $\Theta_n = \cos\frac{(2n-1)\pi\theta}{2\gamma}$, $n = 1, 2, 3, \dots$. Substituting $\lambda = \frac{(2n-1)^2\pi^2}{4\gamma^2}$ into (A) yields the Euler equation $r^2R_n'' + rR_n' - \frac{(2n-1)^2\pi^2}{4\gamma^2}R_n = 0$ for R_n. The indicial polynomial is $\left(s - \frac{(2n-1)\pi}{2\gamma}\right)\left(s + \frac{(2n-1)\pi}{2\gamma}\right)$, so $R_n = c_1 r^{(2n-1)\pi/2\gamma} + c_2 r^{-(2n-1)\pi/2\gamma}$, by Theorem 7.4.3. We want R_n to be bounded as $r \to 0+$ and $R_n(\rho) = 1$, so we take $R_n(r) = \frac{r^{(2n-1)\pi/2\gamma}}{\rho^{(2n-1)\pi/2\gamma}}$; $v_n(r,\theta) = \frac{r^{(2n-1)\pi/2\gamma}}{\rho^{(2n-1)\pi/2\gamma}} \cos\frac{(2n-1)\pi\theta}{2\gamma}$; $u(r,\theta) = \sum_{n=1}^{\infty} \alpha_n \frac{r^{(2n-1)\pi/2\gamma}}{\rho^{(2n-1)\pi/2\gamma}} \cos\frac{(2n-1)\pi\theta}{2\gamma}$, where $C_M(\theta) = \sum_{n=1}^{\infty} \alpha_n \cos\frac{(2n-1)\pi\theta}{2\gamma}$ is the mixed Fourier cosine series of f on $[0,\gamma]$; that is, $\alpha_n = \frac{2}{\gamma}\int_0^{\gamma} f(\theta)\cos\frac{(2n-1)\pi\theta}{2\gamma}\,d\theta$, $\quad n = 1, 2, 3, \dots$.

12.4.6. $v(r,\theta) = R(r)\Theta(\theta)$ where (A) $r^2R'' + rR' - \lambda R = 0$ and $\Theta'' + \lambda\Theta = 0$, $\Theta'(0) = 0$, $\Theta'(\gamma) = 0$. From Theorem 11.1.3, $\lambda_0 = 0$, $\Theta_0 = 1$; $\lambda_n = \frac{n^2\pi^2}{\gamma^2}$, $\Theta_n = \cos\frac{n\pi\theta}{\gamma}$, $n = 1, 2, 3, \dots$. Substituting $\lambda = 0$ into (A) yields the equation $r^2R_0'' + rR_0' = 0$ for R_0; $R_0 = c_1 + c_2\ln r$. Since we want R_0 to be bounded as $r \to 0+$ and $R_0(\rho) = 1$, $R_0(r) = 1$; therefore $v_0(r,\theta) = 1$.

Substituting $\lambda = \frac{n^2\pi^2}{\gamma^2}$ into (A) yields the Euler equation $r^2R_n'' + rR_n' - \frac{n^2\pi^2}{\gamma^2}R_n = 0$ for R_n. The indicial polynomial is $\left(s - \frac{n\pi}{\gamma}\right)\left(s + \frac{n\pi}{\gamma}\right)$, so $R_n = c_1 r^{n\pi/\gamma} + c_2 r^{-n\pi/\gamma}$, by Theorem 7.4.3. Since we want R_n to be bounded as $r \to 0+$ and $R_n(\rho) = 1$, $R_n(r) = \frac{r^{n\pi/\gamma}}{\rho^{n\pi/\gamma}}$; $v_n(r,\theta) = \frac{r^{n\pi/\gamma}}{\rho^{n\pi/\gamma}} \cos\frac{n\pi\theta}{\gamma}$, $n = 1, 2, 3, \dots$; $u(r,\theta) = \alpha_0 + \sum_{n=1}^{\infty} \alpha_n \frac{r^{n\pi/\gamma}}{\rho^{n\pi/\gamma}} \cos\frac{n\pi\theta}{\gamma}$, where $F(\theta) = \alpha_0 + \sum_{n=1}^{\infty} \alpha_n \cos\frac{n\pi\theta}{\gamma}$ is the Fourier cosine series of f on $[0,\gamma]$; that is, $\alpha_0 = \frac{1}{\gamma}\int_0^{\gamma} f(\theta)\,d\theta$ and $\alpha_n = \frac{2}{\gamma}\int_0^{\gamma} f(\theta)\cos\frac{n\pi\theta}{\gamma}\,d\theta$, $n = 1, 2, 3, \dots$.

CHAPTER 13
Boundary Value Problems for Second Order Ordinary Differential Equations

13.1 BOUNDARY VALUE PROBLEMS

13.1.2. By inspection, $y_p = -x$; $y = -x + c_1e^x + c_2e^{-x}$; $y(0) = -2 \implies c_1 + c_2 = -2$; $y(1) = 1 \implies -1 + c_1e + c_2/e = 1$; $\begin{bmatrix} 1 & 1 \\ e & 1/e \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$; $c_1 = \dfrac{2}{e-1}$; $c_2 = \dfrac{2e}{1-e}$; $y = -x + \dfrac{2\left(e^x - e^{-(x-1)}\right)}{e-1}$

13.1.4. By inspection, $y_p = -x$; $y = -x + c_1e^x + c_2e^{-x}$; $y' = -1 + c_1e^x - c_2e^{-x}$; $y(0) + y'(0) = 3 \implies -1 + 2c_1 = 3$; $c_1 = 2$; $y(1) - y'(1) = 2 \implies \dfrac{2c_2}{e} = 2$; $c_2 = e$; $y = -x + 2e^x + e^{-(x-1)}$

13.1.6. $y_p = Ax^2e^x$; $y_p' = A(x^2 + 2xe^x)$; $y_p'' = A(x^2e^x + 4xe^x + 2e^x)$; $y_p'' - 2y_p' + y_p = 2Ae^x = e^x$ if $A = 1$; $y_p = x^2e^x$; $y = (x^2 + c_1 + c_2x)e^x$; $y' = (x^2 + 2x + c_1 + c_2 + c_2x)e^x$; $B_1(y) = 3$ and $B_2(y) = 6e \implies -c_1 - 2c_2 = 3, 2c_1 + 3c_2 = 24$; $c_1 = 13$; $c_2 = -8$; $y = (x^2 - 8x + 13)e^x$.

13.1.8. $B_1(y) = y(0)$; $B_2(y) = y(1) - y'(1)$. Let $y_1 = x$, $y_2 = 1$; $B_1(y_1) = B_2(y_1) = 0$. By variation of parameters, if $y_p = u_1x + u_2$ where $u_1'x + u_2' = 0$ and $u_1' = F$, then $y_p'' = F(x)$. Let $u_1' = F$, $u_2' = -xF$; $u_1 = -\displaystyle\int_x^1 F(t)\,dt$, $u_2 = -\displaystyle\int_0^x tF(t)\,dt$; $y_p = -x\displaystyle\int_x^1 F(t)\,dt - \int_0^x tF(t)\,dt$; $y_p' = -\displaystyle\int_x^1 F(t)\,dt$; $y = y_p + c_1x + c_2$. Since $B_1(y_p) = 0$, $B_1(x) = 0$ and $B_1(1) = 1$, $B_1(y) = 0 \implies c_2 = 0$; hence, $y = y_p + c_1x$. Since $B_2(y_p) = -\displaystyle\int_0^1 tF(t)\,dt$ and $B_2(x) = 0$, $B_2(y) = 0 \implies \displaystyle\int_0^1 tF(t)\,dt = 0$. There is no solution if this conditions does not hold. If it does hold, then the solutions are $y = y_p + c_1x$, with c_1 arbitrary.

13.1.10. **(a)** The condition is $b-a \neq (k+1/2)\pi$ (k = integer). Let $y_1 = \sin(x-a)$ and $y_2 = \cos(x-b)$. Then $y_1(a) = y_2'(b) = 0$ and $\{y_1, y_2\}$ is linearly independent if $b - a \neq (k + 1/2)\pi$ (k = integer), since

$$\begin{vmatrix} \sin(x-a) & \cos(x-b) \\ \cos(x-a) & -\sin(x-b) \end{vmatrix} = -\cos(b-a) \neq 0.$$

Now Theorem 13.1.2 implies that (A) has a unique solution for any continuous F and constants k_1 and k_2. If $y = u_1 \sin(x-a) + u_2 \cos(x-b)$ where

$$\begin{aligned} u_1' \sin(x-a) + u_2' \cos(x-b) &= 0 \\ u_1' \cos(x-a) - u_2' \sin(x-b) &= F, \end{aligned}$$

then $y'' + y = F$.

$$u_1' = F(x)\frac{\cos(x-b)}{\cos(b-a)}, \quad u_2' = -F(x)\frac{\sin(x-a)}{\cos(b-a)},$$

$$u_1 = -\frac{1}{\cos(b-a)}\int_x^b F(t)\cos(t-b)\,dt, \quad u_2 = -\frac{1}{\cos(b-a)}\int_a^x F(t)\sin(t-b)\,dt;$$

$$y = -\frac{\sin(x-a)}{\cos(b-a)}\int_x^b F(t)\cos(t-b)\,dt - \frac{\cos(x-b)}{\cos(b-a)}\int_a^x F(t)\sin(t-a)\,dt.$$

(b) If $b-a = (k+1/2)\pi$ (k = integer), then $y_1 = \sin(x-a)$ satisfies both boundary conditions $y(a) = 0$ and $y'(b) = 0$. Let $y_2 = \cos(x-a)$. If $y_p = u_1 \sin(x-a) + u_2 \cos(x)$ where

$$\begin{aligned} u_1' \sin(x-a) + u_2' \cos(x-a) &= 0, \\ u_1' \cos(x-a) - u_2' \sin(x-a) &= F, \end{aligned}$$

then $y_p'' + y_p = F$; $u_1' = F\cos(x-a)$; $u_2' = -F\sin(x-a)$;

$$u_1 = -\int_x^b F(t)\cos(t-a)\,dt, \quad u_2 = -\int_a^x F(t)\sin(t-a)\,dt;$$

$$y_p = -\sin(x-a)\int_x^b F(t)\cos(t-a)\,dt - \cos(x-a)\int_a^x F(t)\sin(t-a)\,dt;$$

$$y_p' = -\cos(x-a)\int_x^b F(t)\cos(t-a)\,dt + \sin(x-a)\int_a^x F(t)\sin(t-a)\,dt.$$

The general solution of $y''+y = F$ is $y = y_p + c_1\sin(x-a) + c_2\cos(x-a)$. Since $b-a = (k+1/2)\pi$, $y'(b) = 0 \implies y_p'(b) = (-1)^k\int_a^b F(t)\sin(t-a)\,dt = 0$; therefore, F must satisfy $\int_a^b F(t)\sin(t-a)\,dt = 0$. In this case, the solutions of the boundary value problem are $y = y_p + c_1\sin(x-a)$, with c_1 arbitrary.

13.1.12. Let $y_1 = \sinh(x-a)$ and $y_2 = \sinh(x-b)$. Then $y_1(a) = 0$, $y_2(b) = 0$, and

$$W(x) = \begin{vmatrix} \sinh(x-a) & \sinh(x-b) \\ \cosh(x-a) & \cosh(x-b) \end{vmatrix} = \sinh(b-a) \neq 0.$$

(Since W is constant (Theorem 5.1.4), evaluate it by setting $x = b$.) From Theorem 13.1.2, (A) has a unique solution for any continuous F and constants k_1 and k_2. If $y = u_1\sinh(x-a) + u_2\sinh(x-b)$ where

$$\begin{aligned} u_1' \sinh(x-a) + u_2' \sinh(x-b) &= 0 \\ u_1' \cosh(x-a) + u_2' \cosh(x-b) &= F, \end{aligned}$$

then $y'' - y = F$.

$$u_1' = -F(x)\frac{\sinh(x-b)}{\sinh(b-a)}, \quad u_2' = F(x)\frac{\sinh(x-a)}{\sinh(b-a)},$$

$$u_1 = \frac{1}{\sinh(b-a)}\int_x^b F(t)\sinh(t-b)\,dt, \quad u_2 = \frac{1}{\sinh(b-a)}\int_a^x F(t)\sinh(t-a)\,dt;$$
$$y = \frac{\sinh(x-a)}{\sinh(b-a)}\int_x^b F(t)\sinh(t-b)\,dt + \frac{\sinh(x-b)}{\sinh(b-a)}\int_a^x F(t)\sinh(t-a)\,dt.$$

13.1.14. Let $y_1 = \cosh(x-a)$ and $y_2 = \cosh(x-b)$. Then $y_1'(a) = y_2'(b) = 0$ and

$$W(x) = \begin{vmatrix} \cosh(x-a) & \cosh(x-b) \\ \sinh(x-a) & \sinh(x-b) \end{vmatrix} = -\sinh(b-a) \neq 0.$$

(Since W is constant (Theorem 5.1.40, evaluate it by setting $x = b$.) If $y = u_1\cosh(x-a) + u_2\cosh(x-b)$ where

$$\begin{aligned} u_1'\cosh(x-a) + u_2'\cosh(x-b) &= 0 \\ u_1'\sinh(x-a) + u_2'\sinh(x-b) &= F, \end{aligned}$$

then $y'' - y = F$.

$$u_1' = F(x)\frac{\cosh(x-b)}{\sinh(b-a)}, \quad u_2' = -F(x)\frac{\cosh(x-a)}{\sinh(b-a)},$$
$$u_1 = -\frac{1}{\sinh(b-a)}\int_x^b F(t)\cosh(t-b)\,dt, \quad u_2 = -\frac{1}{\sinh(b-a)}\int_a^x F(t)\cosh(t-a)\,dt;$$
$$y = -\frac{\cosh(x-a)}{\sinh(b-a)}\int_x^b F(t)\cosh(t-b)\,dt - \frac{\cosh(x-b)}{\sinh(b-a)}\int_a^x F(t)\cosh(t-a)\,dt.$$

13.1.16. Let $y_1 = \sin\omega x$, $y_2 \sin\omega(x-\pi)$; then $y_1(0) = 0$, $y_2(\pi) = 0$,

$$W(x) = \begin{vmatrix} \sin\omega x & \sin\omega(x-\pi) \\ \omega\cos\omega x & \omega\cos\omega(x-\pi) \end{vmatrix} = \omega\sin\omega\pi \neq 0$$

if and only if ω is not an integer. If this is so, then $y = u_1\sin\omega x + u_2\sin\omega(x-\pi)$ if

$$\begin{aligned} u_1'\sin\omega x + u_2'\sin\omega(x-\pi) &= 0 \\ \omega(u_1'\cos\omega x + u_2'\cos\omega(x-\pi)) &= F; \end{aligned}$$

$$u_1' = -F\frac{\sin\omega(x-\pi)}{\omega\sin\omega\pi}, \quad u_2' = F\frac{\sin\omega x}{\omega\sin\omega\pi};$$
$$u_1 = \frac{1}{\omega\sin\omega\pi}\int_x^\pi F(t)\sin\omega(t-\pi)\,dt; \quad u_2 = \frac{1}{\omega\sin\omega\pi}\int_0^x F(t)\sin\omega t\,dt;$$
$$y = \frac{1}{\omega\sin\omega\pi}\left(\sin\omega x\int_x^\pi F(t)\sin\omega(t-\pi)\,dt + \sin\omega(x-\pi)\int_0^\pi F(t)\sin\omega t\,dt\right).$$

If $\omega = n$ (positive integer), then $y_1 = \sin nx$ is a nontrivial solution of $y'' + y = 0$, $y(0) = 0$, $y(\pi) = 0$. Let $y_2 = \cos nx$; then $W(x) = \begin{vmatrix} \sin nx & \cos nx \\ n\cos nx & -n\sin nx \end{vmatrix} = -n$, and $y_p = u_1\sin nx + u_2\cos nx$ satisfies $y_p'' + n^2 y_p = 0$ if

$$\begin{aligned} u_1'\sin nx + u_2'\cos nx &= 0 \\ nu_1'\cos nx - nu_2'\sin nx &= F; \end{aligned}$$

$u_1' = \frac{1}{n} F \cos nx$, $u_2' = -\frac{1}{n} F \sin nx$; $u_1 = -\frac{1}{n} \int_x^\pi F(t) \cos nt\, dt$; $u_2 = -\frac{1}{n} \int_0^x F(t) \sin nt\, dt$;

$$y_p = -\frac{1}{n} \left(\sin nx \int_x^\pi F(t) \cos nt\, dt + \cos nx \int_0^x F(t) \sin nt\, dt \right);$$

$y = y_p + c_1 \sin nx + c_2 \cos nx$. Since $y_p = 0$, $y(0) = 0$, so $c_2 = 0$; $y = y_p + c_1 \sin nx$. Since $y(\pi) = 0$, $\int_0^\pi F(t) \sin nt\, dt = 0$ is necessary for existence of a solution. If this hold, then the solutions are $y = y_p + c_1 \sin nx$, with c_1 arbitrary.

13.1.18. Let $y_1 = \cos \omega x$; $y_2 = \sin \omega(x - \pi)$; then $y_1'(0) = y_2(\pi) = 0$, and

$$W(x) = \begin{vmatrix} \cos \omega x & \sin \omega(x-\pi) \\ -\omega \sin \omega x & \omega \cos \omega(x-\pi) \end{vmatrix} = \omega \cos \omega\pi \neq 0$$

if and only if $\omega \neq n + 1/2$ (n = integer). If this is so, then $y = u_1 \cos \omega x + u_2 \sin \omega(x - \pi)$ satisfies $y'' + \omega^2 y = F(x)$ if

$$\begin{aligned} u_1' \cos \omega x + u_2' \sin \omega(x-\pi) &= 0 \\ \omega(-u_1' \sin \omega x + u_2' \cos \omega(x-\pi))\omega &= F; \end{aligned}$$

then

$$u_1' = -\frac{F \sin \omega(x-\pi)}{\omega \cos \omega\pi}, \quad u_2' = \frac{F \cos \omega x}{\omega \cos \omega\pi},$$

$$u_1 = \frac{1}{\omega \cos \omega\pi} \int_x^\pi F(t) \sin \omega(t-\pi), dt, \quad u_2 = \frac{1}{\omega \cos \omega\pi} \int_0^x F(t) \cos \omega t\, dt,$$

$$y = \frac{1}{\omega \cos \omega\pi} \left(\sin \omega x \int_x^\pi F(t) \sin \omega(t-\pi), dt + \sin \omega(x-\pi) \int_0^x F(t) \cos \omega t\, dt \right).$$

If $\omega = n + 1/2$ (n = integer), then $y_1 = \cos(n + 1/2)x$ is a nontrivial solution $y'' + y = 0$, $y'(0) = y(\pi) = 0$. Let $y_2 = \sin(n + 1/2)x$; then

$$W(x) = \begin{vmatrix} \cos(n+1/2)x & \sin(n+1/2)x \\ -(n+1/2)\sin(n+1/2)x & (n+1/2)\cos(n+1/2)x \end{vmatrix} = n + 1/2,$$

so $y_p = u_1 \cos(n + 1/2)x + u_2 \sin(n + 1/2)x$ satisfies $y_p'' + (n + 1/2)^2 y_p = F$ if

$$\begin{aligned} u_1' \cos(n+1/2)x + u_2' \sin(n+1/2)x &= 0 \\ -(n+1/2)u_1' \sin(n+1/2)x + (n+1/2)u_2' \cos(n+1/2)x &= F. \end{aligned}$$

$$u_1' = -\frac{F \sin(n+1/2)x}{n+1/2}; \quad u_2' = \frac{F \cos(n+1/2)x}{n+1/2};$$

$$u_1 = \int_x^\pi \frac{F(t) \sin(n+1/2)t}{n+1/2}\, dt; \quad u_2 = \int_0^x \frac{F(t) \cos(n+1/2)t}{n+1/2}\, dt;$$

$$y_p = \frac{1}{n+1/2} \left(\cos(n+1/2)x \int_x^\pi F(t) \sin(n+1/2)t\, dt + \sin(n+1/2)x \int_0^x F(t) \cos(n+1/2)t\, dt \right);$$

$$y_p' = -\sin(n+1/2)x \int_x^\pi F(t) \sin(n+1/2)t\, dt + \cos(n+1/2)x \int_0^x F(t) \cos(n+1/2)t\, dt;$$

$$y = y_p + c_1\cos(n+1/2)x + c_2\sin(n+1/2)x;$$
$$y' = y_p' + (n+1/2)(-c_1\sin(n+1/2)x + c_2\cos(n+1/2))x.$$

Since $y_p'(0) = 0$, $y'(0) = 0 \implies c_2 = 0$; $y = y_p + c_1\cos(n+1/2)x$;
$y' = y_p' + (n+1/2)c_1(n+1/2)x$; $y(\pi) = 0 \implies y_p(\pi) = 0$. Hence, $\int_0^\pi F(y)\cos(n+1/2)t\,dt = 0$ is a necessary condition for existence of a solution. If this holds, then the solutions are $y = y_p + c_1\cos(n+1/2)x$ with c_1 arbitrary.

13.1.20. Suppose $y = c_1z_1 + c_2z_2$ is a nontrivial solution of the homogeneous boundary value problem. Then $B_1(y) = c_1B_1(z_1) + c_2B_1(z_2) = 0$. From Theorem 13.1.1 we may assume without loss of generality that $B_1(z_2) \neq 0$. Then $c_2 = -\dfrac{B_1(z_1)}{B_1(z_2)}c_1$. Therefore, y is constant multiple of $y_0 = B_1(z_2)z_1 - B_1(z_1)z_2 \neq 0$. To that check y satisfies the boundary conditions, note that $B_1(y_0) = B_1(z_2)B_1(z_1) - B_1(z_1)B_1(z_2) = 0$ $B_2(y_0) = B_1(z_2)B_2(z_1) - B_1(z_1)B_2(z_2) = 0$, by Theorem 13.1.2.

13.1.22. $y_1 = a_1 + a_2x$; $y_1(0) - 2y_1'(0) = a_1 - 2a_2 = 0$ if $a_1 = 2, a_2 = 1$; $y_1 = 2 + x$. $y_2 = b_1 + b_2x$; $y_2(1) = 2y_1'(1) = b_1 + 3b_2 = 0$ if $b_1 = 3, b_2 = -1$; $y_2 = 3 - x$.

$$W(x) = \begin{vmatrix} 2+x & 3-x \\ 1 & -1 \end{vmatrix} = -5; \quad G(x,t) = \begin{cases} -\dfrac{(2+t)(3-x)}{5}, & 0 \le t \le x, \\ -\dfrac{(2+x)(3-t)}{5}, & x \le t \le 1. \end{cases}$$

$$y = -\frac{1}{5}\left[(2+x)\int_x^1 (3-t)F(t)\,dt + (3-x)\int_0^x (2+t)F(t)\,dt\right]. \tag{B}$$

(a) With $F(x) = 1$, (B) becomes

$$\begin{aligned} y &= -\frac{1}{5}\left[(2+x)\int_x^1 (3-t)\,dt + (3-x)\int_0^x (2+t)\,dt\right] \\ &= -\frac{1}{5}\left[(2+x)\left(\frac{x^2-6x+5}{2}\right) + (3-x)\left(\frac{x^2+4x}{2}\right)\right] \\ &= \frac{x^2-x-2}{2}. \end{aligned}$$

(b) With $F(x) = x$, (B) becomes

$$\begin{aligned} y &= -\frac{1}{5}\left[(2+x)\int_x^1 (3t-t^2)\,dt + (3-x)\int_0^x (2t+t^2)\,dt\right] \\ &= -\frac{1}{5}\left[(2+x)\left(\frac{2x^3-9x^2+7}{6}\right) + (3-x)\left(\frac{x^3+3x^2}{3}\right)\right] \\ &= \frac{5x^3-7x-14}{30}. \end{aligned}$$

(b) With $F(x) = x^2$, (B) becomes

$$\begin{aligned} y &= -\frac{1}{5}\left[(2+x)\int_x^1 (3t^2-t^3)\,dt + (3-x)\int_0^x (2t^2+t^3)\,dt\right] \\ &= -\frac{1}{5}\left[(2+x)\left(\frac{x^4-4x^2+3}{4}\right) + (3-x)\left(\frac{3x^4+8x^3}{12}\right)\right] \\ &= \frac{5x^4-9x-18}{60}. \end{aligned}$$

13.1.24. $y_1 = x^2 - x$, $y_2 = x^2 - 2x$; then $y_1(1) = 0$, $y_2(2) = 0$; $W(x) = \begin{vmatrix} x^2 - x & x^2 - 2x \\ 2x - 1 & 2x - 2 \end{vmatrix} = x^2$.

Since $P_0(x) = x^2$, $G(x,t) = \begin{cases} \dfrac{(t-1)x(x-2)}{t^3}, & 1 \le t \le x, \\ \dfrac{x(x-1)(t-2)}{t^3}, & x \le t \le 2. \end{cases}$.

$$y = x(x-1)\int_x^2 \frac{t-2}{t^3}F(t)\,dt + x(x-2)\int_1^x F(t)\,dt. \tag{B}$$

(a) With $F(x) = 2x^3$, (B) becomes

$$\begin{aligned} y &= 2x(x-1)\int_x^2 (t-2)\,dt + 2x(x-2)\int_1^x (t-1)\,dt \\ &= -x(x-1)(x-2)^2 + x(x-2)(x-1)^2 = x(x-1)(x-2). \end{aligned}$$

(b) With $F(x) = 6x^4$, (B) becomes

$$\begin{aligned} y &= 6x(x-1)\int_x^2 (t-2)t\,dt + 6x(x-2)\int_1^x (t-1)t\,dt \\ &= -2x(x-1)(x+1)(x-2)^2 + x(x-2)(x-1)^2(2x+1) = x(x-1)(x-2)(x+3). \end{aligned}$$

13.1.26. $y_1 = a_1 + a_2x$; $y_1' = a_2$; $B_1(y_1) = \alpha a_1 + \beta a_2 = 0$ if $a_1 = \beta$, $a_2 = -\alpha$; $y_1 = \beta - \alpha x$. $y_2 = b_1 + b_2x$; $y_2' = b_2$; $B_2(y_2) = \rho b_1 + (\rho + \delta)b_2 = 0$ if $b_1 = \rho + \delta$, $b_2 = -\rho$; $y_2 = \rho + \delta - \rho x$; $W(x) = \begin{bmatrix} \beta - \alpha x & \rho + \delta - \rho x \\ -\alpha & -\rho \end{bmatrix} = \alpha(\rho + \delta) - \beta\rho$. From Theorem 13.1.2, (A) has a unique solution if and only if $\alpha(\rho + \delta) - \beta\rho \ne 0$. Then

$$G(x,t) = \begin{cases} \dfrac{(\beta - \alpha t)(\rho + \delta - \rho x)}{\alpha(\rho + \delta) - \beta\rho}, & 0 \le t \le x, \\ \dfrac{(\beta - \alpha x)(\rho + \delta - \rho t)}{\alpha(\rho + \delta) - \beta\rho}, & x \le t \le 1. \end{cases}$$

13.1.28. $y_1 = a_1\cos x + a_2\sin x$; $y_1' = -a_1\sin x + a_2\cos x$; $B_1(y_1) = \alpha a_1 + \beta a_2 = 0$ if $a_1 = \beta$, $a_2 = -\alpha$. $y_1 = \beta\cos x - \alpha\sin x$. $y_2 = b_1\cos x + b_2\sin x$; $y_2' = -b_1\sin x + b_2\cos x$; $B_2(y_2) = \rho b_2 - \delta b_1 = 0$ if $b_1 = \rho$, $b_2 = \delta$; $y_2 = \rho\cos x + \delta\sin x$; $W(x) = \begin{bmatrix} \beta\cos x - \alpha\sin x & \rho\cos x + \delta\sin x \\ -\beta\sin x - \alpha\cos x & -\rho\sin x + \delta\cos x \end{bmatrix}$ Since W is constant, we can evaluate it with $x = 0$: $W = \begin{bmatrix} \beta & \rho \\ -\alpha & \delta \end{bmatrix} = \alpha\rho + \beta\delta$. From Theorem 13.1.2, (A) has a unique solution if and only if $\alpha\rho + \beta\delta \ne 0$. Then

$$G(x,t) = \begin{cases} \dfrac{(\beta\cos t - \alpha\sin t)(\rho\cos x + \delta\sin x)}{\alpha\rho + \beta\delta}, & 0 \le t \le x, \\ \dfrac{(\beta\cos x - \alpha\sin x)(\rho\cos t + \delta\sin t))}{\alpha\rho + \beta\delta}, & x \le t \le \pi. \end{cases}$$

13.1.30. $y_1 = e^x(a_1\cos x + a_2\sin x)$; $y_1' = e^x[a_1(\cos x - \sin x) + a_2(\sin x + \cos x)]$;
$B_1(y_1) = (\alpha + \beta)a_1 + \beta a_2 = 0$ if $a_1 = \beta$, $a_2 = -(\alpha + \beta)$.

$y_1 = e^x(\beta\cos x - (\alpha+\beta)\sin x$. $y_2 = e^x(b_1\cos x + b_2\sin x)$;
$y_2' = e^x[(b_1(\cos x - \sin x)) + b_2(\sin x + \cos x)]$;
$B_2(y_2) = -e^{\pi/2}[(\rho+\delta)b_2 - \delta b_1] = 0$ if $b_1 = \delta$, $b_2 = (\rho+\delta)$;
$y_2 = e^x[(\rho+\delta)\cos x + \delta\sin x)$;
To evaluate $W(x)$, we write $y_1 = e^x v_1$ and $y_2 = e^x v_2$, where
$v_1 = \beta\cos x - (\alpha+\beta)\sin x$ and $v_2 = (\rho+\delta)\cos x + \delta\sin x$.
Then $y_1' = y_1 + e^x v_1'$ and $y_2 = y_2 + e^x y_2'$,

$$\begin{aligned} W(x) &= \begin{vmatrix} y_1 & y_2 \\ y_1 + e^x v_1' & y_2 + e^x v_2' \end{vmatrix} = \begin{vmatrix} y_1 & y_2 \\ e^x v_1' & x v_2' \end{vmatrix} = e^{2x}\begin{vmatrix} v_1 & v_2 \\ v_1' & x v_2' \end{vmatrix} \\ &= \begin{vmatrix} (\beta\cos x - (\alpha+\beta)\sin x & (\rho+\delta)\cos x + \delta\sin x \\ (-\beta\sin x - (\alpha+\beta)\cos x & (-(\rho+\delta)\sin x + \delta\cos x \end{vmatrix}. \end{aligned}$$

Since $v_i'' + v_i = 0$, $i = 1, 2$, Theorem 5.1.4 implies that $W(x) = Ke^{2x}$, where is a constant that can be determined by setting $x = 0$ in the determinant:

$$W(x)e^{2x}\begin{vmatrix} \beta & \rho+\delta \\ -\alpha-\beta & \delta \end{vmatrix} = [\beta\delta + (\alpha+\beta)(\rho+\delta)].$$

From Theorem 13.1.2, the boundary value problem has a unique solution if and only if $\beta\delta + (\alpha+\beta)(\rho+\delta) \neq 0$. In this case the Green's function is

$$G(x,t) = \begin{cases} e^{x-t}\dfrac{[\beta\cos t - (\alpha+\beta)\sin t][\rho+\delta)\cos x + \delta\sin x]}{\beta\delta + (\alpha+\beta)(\rho+\delta)}, & a \le t \le x \\ e^{x-t}\dfrac{[\beta\cos x - (\alpha+\beta)\sin x][\rho+\delta)\cos t + \delta\sin t]}{\beta\delta + (\alpha+\beta)(\rho+\delta)}, & x \le t \le \pi/2. \end{cases}$$

13.1.32. Let $y_p = \int_a^b G(x,t)F(t)\,dt$. From Theorem 13.1.3, $Ly_p = F$, $B_1(y_p) = 0$, and $B_2(y_p) = 0$. The solution of $Ly = F$, $B_1(y) = k_1$, and $B_2(y) = k_2$ is of the form $y = y_p + c_1y_1 + c_2y_2$. Since $B_1(y_p) = 0$ and $B_1(y_1) = 0$, $B_1(y) = k_1 \implies k_1 = c_2B_1(y_2) \implies c_2 = \dfrac{k_1}{B_1(y_2)}$. Since $B_2(y_p) = 0$ and $B_2(y_2) = 0$, $B_2(y) = k_2 \implies k_2 = c_1B_2(y_1) \implies c_1 = \dfrac{k_2}{B_2(y_1)}$.

13.2 STURM-LIOUVILLE PROBLEMS

13.2.2. $y'' + \dfrac{1}{x}y' + \left(1 - \dfrac{\nu^2}{x^2}\right)y = 0$; $\dfrac{p'}{p} = \dfrac{1}{x}$; $\ln|p| = \ln|x|$; $p = x$; $xy'' + y' = \left(x - \dfrac{\nu^2}{x}\right)y = 0$; $(xy')' + \left(x - \dfrac{\nu^2}{x}\right)y = 0$.

13.2.4. $y'' + \dfrac{b}{x}y' + \dfrac{c}{x^2}y = 0$; $\dfrac{p'}{p} = \dfrac{b}{x}$; $\ln|p| = b\ln|x|$; $p = x^b$;
$x^by'' + bx^{b-1}y' + cx^{b-2}y = 0$; $(x^by')' + cx^{b-2}y = 0$.

13.2.6. $xy'' + (1-x)y' + \alpha y = 0$; $y'' + \left(\dfrac{1}{x} - 1\right)y' + \dfrac{\alpha}{x}y = 0$; $\dfrac{p'}{p} = \dfrac{1}{x} - 1$; $\ln|p| = \ln|x| - x$; $p = xe^{-x}$; $xe^{-x}y'' + (1-x)y' + \alpha e^{-x}y = 0$; $(xe^{-x}y')' + \alpha e^{-x}y = 0$.

13.2.8. If λ is an eigenvalues of (A) and y is a λ-eigenfunction, multiplying the differential equation in (B) by y yields $(xy')y + \dfrac{\lambda}{x}y^2 = 0$;

$$\lambda \int_1^2 \frac{y^2(x)}{x^2}\,dx = -\int_1^2 (xy'(x))'y(x)\,dx = -xy'(x)y(x)\Big|_1^2 + \int_1^2 x(y'(x))^2\,dx;$$

$$y(1) = y(2) = 0 \implies xy'(x)y(x)\Big|_1^2 = 0; \quad \lambda \int_1^2 \frac{y^2(x)}{x}\,dx = \int_1^2 x(y'(x))^2\,dx.$$

Therefore $\lambda \ge 0$. We must still show that $\lambda = 0$ is not an eigenvalue. To this end, suppose that $(xy')' = 0$; then $xy' = c_1$; $y' = \dfrac{c_1}{x}$; $y = c_1 \ln|x| + c_2$; $y(1) = 0 \implies c_2 = 0$; $y = c_1 \ln|x|$; $y(2) = 0 \implies c_1 = 0$; $y \equiv 0$; therefore $\lambda = 0$ is not an eigenvalue.

13.2.10. Characteristic equation: $r^2 + 2r + 1 + \lambda = 0$; $r = -1 \pm \sqrt{-\lambda}$.

$\lambda = 0$: $y = e^{-x}(c_1 + c_2x)$; $y' = -e^{-x}(c_1 - c_2 + c_2x)$; $y'(0) = 0 \implies c_1 = c_2$; $y' = -c_2xe^{-x}$; $y'(1) = 0 \implies -c_2/e = 0 \implies c_2 = 0$; $\lambda = 0$ is not an eigenvalue.

$\lambda = -k^2$, $k > 0$: $r = -1 \pm k$; $y = e^{-x}(c_1 \cosh kx + c_2 \sinh kx)$;

$$y' = -c_1e^{-x}(-\cosh kx + k \sinh kx) + c_2e^{-x}(-\sinh kx + k \cosh kx).$$

The boundary conditions require that

$$c_1 + c_2k = 0 \quad \text{and} \quad (-\cosh k + k \sinh k)c_1 + (-\sinh k + k \cosh k)c_2 = 0.$$

This system has a nontrivial solution if and only if $(1 - k^2)\sinh k = 0$. Let $k = 1$ and $c_1 = c_2 = 1$; then $\lambda = -1$ is the only negative eigenvalue, with associated eigenfunction $y = 1$.

$\lambda = k^2$, $k > 0$: $r = -1 \pm ik$; $y = e^{-x}(c_1 \cos kx + c_2 \sin kx)$;

$y' = c_1e^{-x}(-\cos kx - k \sin kx) + c_2e^{-x}(-\sin x + k \cos kx)$. The boundary conditions require that

$$-c_1 + c_2k = 0 \quad \text{and} \quad (-\cos k - k \sin k)c_1 + (-\sin k + k \cos k)c_2 = 0.$$

This system has a nontrivial solution if and only if $(1 + k^2)\sin k = 0$. Let $k = n\pi$ (k a positive integer)and $c_1 = n\pi$, $c_2 = 1$; then $\lambda_n = n^2\pi^2$ is an eigenvalue, with associated eigenfunction $y_n = e^{-x}(n\pi \cos n\pi x + \sin n\pi x)$.

13.2.12. Characteristic equation: $r^2 + \lambda = 0$.

$\lambda = 0 : y = c_1 + c_2x$. $y(0) = 0 \implies c_1 = 0$, so $y = c_2x$. Now $y(1) - 2y'(1) = 0 \implies c_2 = 0$. Therefore $\lambda = 0$ is not an eigenvalue.

$\lambda = -k^2$, $k > 0$: $y = c_1 \cosh kx + c_2 \sinh kx$; $y' = k(c_1 \sinh kx + c_2 \cosh kx)$. $y'(0) \implies c_2 = 0$, so $y = c_1 \cosh kx$. Now $y(1) - 2y'(1) = 0 \implies c_1(\cosh k - 2k \sinh k) = 0$, which is possible with $c_1 \ne 0$ if and only if $\tanh k = \dfrac{1}{2k}$. Graphing both sides of this equation on the same axes show that it has one positive solution k_0; $y_0 = \cosh k_0x$ is a $-k_0^2$-eigefunction.

$\lambda = k^2$, $k > 0$: $y = c_1 \cos kx + c_2 \sin kx$; $y' = k(-c_1 \sin kx + c_2 \cos kx)$. $y'(0) \implies c_2 = 0$, so $y = c_1 \cos kx$. Now $y(1) - 2y'(1) = 0 \implies c_1(\cos k + 2k \sin k) = 0$, which is possible with $c_1 \ne 0$ if and only if $\tan k = -\dfrac{1}{2k}$. Graphing both sides of this equation on the same axes shows that it has a solution k_n in $((2n-1)\pi/2, n\pi)$, $n = 1, 2, 3, \dots$; $y_n = \cos k_nx$ is a k_n^2-eigenfunction.

13.2.14. Characteristic equation: $r^2 + \lambda = 0$.

$\lambda = 0 : y = c_1 + c_2x$. The boundary conditions require that $c_1 + 2c_2 = 0$ and $c_1 + \pi c_2 = 0$, which imply that $c_1 = c_2 = 0$, so $\lambda = 0$ is not an eigenvalue.

$\lambda = -k^2$, $k > 0$: $y = c_1 \cosh kx + c_2 \sinh kx$; $y' = k(c_1 \sinh kx + c_2 \cosh kx)$. The boundary conditions require that

$$c_1 + 2kc_2 = 0 \quad \text{and} \quad c_1 \cosh k\pi + c_2 \sinh 2k\pi = 0.$$

This system has a nontrivial solution if and only if $\tanh k\pi = 2k$. Graphing both sides of this equation on the same axes shows that it has a solution k_0 in $(0,\pi)$; $y_0 = 2k_0\cosh k_0x - \sinh k_0x$ is a $-k_0^2$-eigenfunction.

$\lambda = k^2$, $k > 0$: $y = c_1\cos kx + c_2\sin kx$; $y' = k(-c_1\sin kx + c_2\cos kx)$. The boundary conditions require that

$$c_1 + 2kc_2 = 0 \text{ and } c_1\cos k\pi + c_2\sin k\pi = 0.$$

This system has a nontrivial solution if and only if $\tan k\pi = 2k$. Graphing both sides of this equation on the same axes shows that it has a solution k_n in $(n, n+1/2)$, $n = 1,2,3,\dots$; $y_n = 2k_n\cos k_nx - \sin k_nx$ is a k_n^2-eigenfunction.

13.2.16. Characteristic equation: $r^2 + \lambda = 0$.

$\lambda = 0 : y = c_1 + c_2x$. The boundary conditions require that $c_1 + c_2 = 0$ and $c_1 + 4c_2 = 0$, so $c_1 = c_2 = 0$. Therefore $\lambda = 0$ is not an eigenvalue.

$\lambda = -k^2$, $k > 0$: $y = c_1\cosh kx + c_2\sinh kx$; $y' = k(c_1\sinh kx + c_2\cosh kx)$. The boundary conditions require that

$$c_1 + kc_2 = 0 \quad\text{and}\quad (\cosh 2k + 2k\sinh 2k)c_1 + (\sinh 2k + 2k\cosh 2k)c_2 = 0.$$

This system has a nontrivial solution if and only if $\tanh 2k = -\dfrac{k}{1-2k^2}$. Graphing both sides of this equation on the same axes shows that it has a solution k_0 in $(1/\sqrt{2})$; $y_0 = k_0\cosh k_0x - \sinh k_0x$ is a $-k_0^2$- eigenfunction.

$\lambda = k^2$, $k > 0$: $y = c_1\cos kx + c_2\sin kx$; $y' = k(-c_1\sin kx + c_2\cos kx)$. The boundary conditions require that

$$c_1 + kc_2 = 0 \quad\text{and}\quad (\cos 2k - 2k\sin 2k)c_1 + (\sin 2k + 2k\cos 2k)c_2 = 0.$$

This system has a nontrivial solution if and only if $\tan 2k = -\dfrac{k}{1+2k^2}$. Graphing both sides of this equation on the same axes shows that it has a solution k_n in $((2n-1)\pi/4, n\pi/2)$, $n = 1, 2,3, \dots$; $y_n = k_n\cos k_nx - \sin k_nx$ is a k_n^2-eigenfunction.

13.2.18. Characteristic equation: $r^2 + \lambda = 0$.

$\lambda = 0 : y = c_1 + c_2x$. The boundary conditions require that $3c_1 + 2c_2 = 0$ and $3c_1 + 4c_2 = 0$, so $c_1 = c_2 = 0$. Therefore $\lambda = 0$ is not an eigenvalue.

$\lambda = -k^2$, $k > 0$: $y = c_1\cosh kx + c_2\sinh kx$; $y' = k(c_1\sinh kx + c_2\cosh kx)$. The boundary conditions require that

$$3c_1 + kc_2 = 0 \quad\text{and}\quad (3\cosh 2k - 2k\sinh 2k)c_1 + (3\sinh 3k - 2k\cosh 2k)c_2 = 0.$$

This system has a nontrivial solution if and only if $\tanh 2k = \dfrac{9k}{9+2k^2}$. Graphing both sides of this equation on the same axes shows that it has solutions y_1 in $(1,2)$ and y_2 in $(5/2, 7/2)$; $y_n = k_n\cosh k_nx - 3\sinh k_nx$ is a $-k_n^2$-eigenfunction, $k = 1, 2$.

$\lambda = k^2$, $k > 0$: $y = c_1\cos kx + c_2\sin kx$; $y' = k(-c_1\sin kx + c_2\cos kx)$. The boundary conditions require that

$$3c_1 + kc_2 = 0 \quad\text{and}\quad (3\cos 2k + 2k\sin 2k)c_1 + (3\sin 2k - 2k\cos 2k)c_2 = 0.$$

This system has a nontrivial solution if and only if $\tan 2k = -\dfrac{9k}{9-2k^2}$. Graphing both sides of this equation on the same axes shows that it has solutions k_0 in $(3/\sqrt{2}, \pi)$ and k_n in $((2n+3)\pi/4, (n+2)\pi/3)$, $n = 1, 2,3, \dots$; $y_n = k_n\cos k_nx - 3\sin k_nx$ is a k_n^2-eigenfunction.

13.2.20. Characteristic equation: $r^2 + \lambda = 0$.

$\lambda = 0 : y = c_1 + c_2x$. The boundary conditions require that $5c_1 + 2c_2 = 0$ and $5c_1 + 3c_2$, so $c_1 = c_2 = 0$. Therefore $\lambda = 0$ is not an eigenvalue.

$\lambda = -k^2$, $k > 0$: $y = c_1\cosh kx + c_2\sinh kx$; $y' = k(c_1\sinh kx + c_2\cosh kx)$. The boundary conditions require that

$$5c_1 + 2kc_2 = 0 \quad\text{and}\quad (5\cosh k - 2k\sinh k)c_1 + (5\sinh k - 2k\cosh k)c_2 = 0.$$

This system has a nontrivial solution if and only if $\tanh k = \dfrac{20k}{25+4k^2}$. Graphing both sides of this equation on the same axes shows that it has solutions k_1 in $(1,2)$ and k_2 in $(5/2, 7/2)$; $y_n = 2k_n \cosh k_n x - \sinh k_n x$ is $-k_n$-eigenfunction, $n = 1, 2$.

$\lambda = k^2$, $k > 0$: $y = c_1 \cos kx + c_2 \sin kx$; $y' = k(-c_1 \sin kx + c_2 \cos kx)$. The boundary conditions require that

$$5c_1 + 2kc_2 = 0 \quad \text{and} \quad (5\cos k + 2k \sin k)c_1 + (5 \sin k - 2k \cos k)c_2 = 0.$$

This system has a nontrivial solution if and only if $\tan k = -\dfrac{20k}{25-4k^2}$. Graphing both sides of this equation on the same axes shows that it has a solution k_n in $((2n+1)\pi/2, (n+1)\pi)$, $n = 1, 2, 3, \dots$; $y_n = 2k_n \cos k_n x - 3 \sin k_n x$ is a k_n^2-eigenfunction.

13.2.22. $\lambda = 0$: $x^2y'' - 2xy' + 2y = 0$ is an Euler equation with indicial equation $r(r-1) - 2r + 2 = (r-1)(r-2) = 0$. $y = x(c_1 + c_2x)$; $y(1) = y(2) = 0 \implies c_1 + c_2 = c_1 + 2c_2 = 0 \implies c_1 = c_2 = 0$, so $\lambda = 0$ is not an eigenvalue.

$\lambda = -k^2$; $k > 0$: $y = x(c_1 \cosh k(x-1) + c_2 \sinh k(x-1))$; $y(1) = 0 \implies c_1 = 0$; $y = c_2 x \sinh k(x-1)$; $y(2) = 0 \implies 2c_2 \sinh k = 0 \implies c_2 = 0$; λ is not an eigenvalue.

$\lambda = k^2$; $k > 0$: $y = x(c_1 \cos k(x-1) + c_2 \sin k(x-1))$; $y(1) = 0 \implies c_1 = 0$; $y = c_2 x \sin k(x-1)$; $y(2) = 0$ with $c_2 \neq 0$ if $k = n\pi$ (n a positive integer); $\lambda_n = n^2\pi^2$; $y_n = x \sin n\pi(x-1)$ is a k_n^2-eigenfunction.

13.2.24. $\lambda = 0$: $x^2y'' - 2xy' + 2y = 0$ is an Euler equation with indicial equation $r(r-1) - 2r + 2 = (r-1)(r-2) = 0$. $y = x(c_1 + c_2x)$; $y' = c_1 + 2c_2x$; $y(1) = y'(2) = 0 \implies c_1 + c_2 = c_1 + 4c_2 = 0 \implies c_1 = c_2 = 0$, so $\lambda = 0$ is not an eigenvalue.

$\lambda = -k^2$; $k > 0$: $y = x(c_1 \cosh k(x-1) + c_2 \sinh k(x-1))$; $y(1) = 0 \implies c_1 = 0$; $y = c_2 x \sinh k(x-1)$; $y' = c_2(\sinh k(x-1) + kx \cosh k(x-1))$; $y'(2) = 0 \implies c_2(\sinh k + k \cosh k) \implies c_2 = 0$; λ is not an eigenvalue.

$\lambda = k^2$; $k > 0$: $y = x(c_1 \cos k(x-1) + c_2 \sin k(x-1))$; $y(1) = 0 \implies c_1 = 0$; $y = c_2 x \sin k(x-1)$; $y' = c_2(\sin k(x-1) + kx \cos k(x-1))$; $y'(2) = 0$ with $c_2 \neq 0$ if and only if $\sin k + 2k \cos k = 0$ or, equivalently, $\tan k = -2k$. Graphing both sides of this equation on the same axes shows that it has a solution k_n in $((2n-1)\pi/2, n\pi)$, $n = 1, 2, 3, \dots$; $y_n = x \sin k_n(x-1)$ is a k_n^2-eigenfunction.

13.2.26. $\lambda = 0$: $y = c_1 + c_2x$. The boundary conditions require that $c_1 + \alpha c_2 = 0$ and $c_1 + (\pi + \alpha)c_2 = 0$, so $c_1 = c_2 = 0$. Therefore $\lambda = 0$ is not an eigenvalue of (A).

$\lambda = -k^2$, $k > 0$: $y = c_1 \cosh kx + c_2 \sinh kx$; $y' = k(c_1 \sinh kx + c_2 \cosh kx)$. The boundary conditions require that

$$\begin{aligned} c_1 + \alpha k c_2 &= 0 \\ (\cosh k\pi + \alpha k \sinh k\pi)c_1 + (\sinh k\pi + \alpha k \cosh k\pi)c_2 &= 0. \end{aligned} \tag{D}$$

This system has a nontrivial solution if and only if $(1 - k^2\alpha^2)\sinh k\pi = 0$, which holds with $k > 0$ if and only if $k^2 = \pm 1/\alpha$. Therefore $\lambda = -1/\alpha^2$ is the only negative eigenvalue. We can choose $k = \pm 1/\alpha$. Either way, the first equation in (D) implies that $e^{-x/\alpha}$ is an associated eigenfunction.

$\lambda = k^2$, $k > 0$: $y = c_1 \cos kx + c_2 \sin kx$; $y' = k(-c_1 \sin kx + c_2 \cos kx)$. The boundary conditions require that

$$\begin{aligned} c_1 + \alpha k c_2 &= 0 \\ (\cos k\pi - \alpha k \sin k\pi)c_1 + (\sin k\pi + \alpha k \cos k\pi)c_2 &= 0. \end{aligned} \tag{E}$$

This system has a nontrivial solution if and only if $(1 + k^2\alpha^2)\sin k\pi = 0$. Choosing $k = n$ produces eigenvalues $\lambda_n = n^2\pi^2$. Setting $k = n$ in the first equation in (E) yields $c_1 + \alpha n c_2 = 0$, so $y_n = n\alpha \cos nx - \sin nx$.

13.2.28. $y = c_1 + c_2 x$. The boundary conditions require that

$$\alpha c_1 + \beta c_2 = 0 \quad \text{and} \quad \rho c_1 + (\rho L + \delta) c_2 = 0.$$

This system has a nontrivial solution if and only if $\alpha(\rho L + \delta) - \beta\rho = 0$.

13.2.30. **(a)** $y = c_1 \cos kx + c_2 \sin kx$; $y' = k(-c_1 \sin kx + c_2 \cos kx)$. The boundary conditions require that

$$\alpha c_1 + \beta k c_2 = 0 \quad \text{and} \quad (\rho \cos kL - \delta k \sin kL) c_1 + (\rho \sin kL + \delta k \cos kL) c_2 = 0.$$

This system has a nontrivial solution if and only if its determinant is zero. This implies the conclusion.

(b) If $\alpha\delta - \beta\rho = 0$, (A) reduces to

$$(\alpha\rho + k^2 \beta\delta) \sin kL = 0. \tag{B}$$

From the solution of Exercise 13.2.29**(b)**, $\alpha\rho + k^2\beta\delta > 0$ for all $k > 0$. Therefore the positive zeros of (B) are $k_n = n\pi/L$, $n = 1, 2, 3, \ldots$, so the positive eigenvalues (SL) are $\lambda_n = n^2\pi^2/L^2$, $n = 1, 2, 3, \ldots$.

13.2.32. Suppose λ is an eigenvalue and y is an associated eigenfunction. From the solution of Exercise 13.2.31,

$$\lambda \int_a^b r(x) y^2(x)\, dx = p(a)y(a)y'(a) - p(b)y(b)y'(b) + \int_a^b p(x)(y'(x))^2\, dx. \tag{A}$$

If $\alpha\beta = 0$ then either $y(a) = 0$ or $y'(a) = 0$, so $y(a)y'(a) = 0$. If $\alpha\beta < 0$ then $y(a) = -\dfrac{\beta}{\alpha} y'(a)$, so

$$y(a)y'(a) = -\frac{\beta}{\alpha}(y'(a))^2. \tag{B}$$

Moreover, $y'(a) \neq 0$ because if $y'(a) = 0$ then $y(a) = 0$, from (B), and $y \equiv 0$, a contradiction. Since $-\dfrac{\beta}{\alpha} > 0$ if $\alpha\beta < 0$, we conclude that if $\alpha\beta \le 0$, then

$$p(a)y(a)y'(a) \ge 0, \tag{C}$$

with equality if and only if $\rho\delta = 0$. A similar argument shows that if $\rho\delta \ge 0$, then

$$p(b)y(b)y'(b) \ge 0, \tag{D}$$

with equality if and only if $\alpha\beta = 0$. Since $(\alpha\beta)^2 + (\rho\delta)^2 > 0$, the inequality must hold in at least one of (C) and (D). Now (A) implies that $\lambda > 0$.