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Long-time Asymptotics for mKdV Type Reduced Equations of the AKNS Hierarchy in Weighted L^2 Sobolev Spaces

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by

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Abstract

The long-time asymptotics of nonlinear integrable partial differential equations is one of important research areas in the field of integrable systems. The main tool to analysis the long-time behaviors is the so-called nonlinear steepest descent method, or Deift-Zhou's method, which was born in 1993. To apply Deift-Zhou's method, one first uses the inverse scattering transform to formulate the nonlinear PDEs in terms of an oscillatory 2 by 2 matrix Riemann-Hilbert problem (RHP). After about 15 years of development, a generalized version of Deift-Zhou's method, the $\bar{\partial}-$steepest method, came out. The $\bar{\partial}-$steepest descent method is a useful method for analyzing RHP with rough scattering data, or in the view of inverse scattering, with rough initial data. Recently, the $\bar{\partial}-$steepest descent method has been applied to study long-time behaviors of NLS, DNLS, mKdV and sine-Gordon equations as well as their corresponding soliton resolutions.

In this dissertation, we use the $\bar{\partial}-$ steepest descent method to fully study the asymptotic behavior of a class of oscillatory Riemann-Hilbert problems. We restrict ourselves to the case of defocusing mKdV type of reductions of the AKNS hierarchy and consider the initial data in $H^{n-1,1}$ for the nth member of the hierarchy, n is an odd integer. The formulas for the long-time asymptotics in three regions are presented. The three regions are defined in the main results. In the oscillatory region (i.e. region I), we find

$$q(x,t) = q_{as}(x,t) + \mathcal{O}(t^{-3/4}), \quad t \to \infty,$$

where

$$q_{as}(x,t) = -2i \sum_{j=1}^{l} \frac{|\eta(z_j)|^{1/2}}{\sqrt{2t\theta''(z_j)}} e^{i\varphi(t)},$$

where $\theta, \eta, \varphi(t)$ and z_j's are defined in the introduction.

In the case of mKdV hierarchy, we derive some interesting result for the other two regions. In this case, $\theta = xz + ctz^n, n$ is odd.

In the Painlevé region (i.e. region II), we find

$$q(x,t) = (nt)^{-\frac{1}{n}} u_n(x(nt)^{-\frac{1}{n}}) + \mathcal{O}(t^{-\frac{3}{2n}}), \quad t \to \infty,$$

where $u_n(x)$ belongs to Painlevé II hierarchy.

In the fast decay region (i.e. region III), we find

$$q(x,t) = \mathcal{O}(t^{-1}), \quad t \to \infty.$$

In the second part, we study exact solutions to the focusing 5th-order mKdV and formulate multi-poles soliton solutions, i.e., solitons associated with the reflection coefficients having arbitrary order poles. We use the generalized Vandermonde-like determinant to present the resulting solitons, which reduces the complexity of the involved computation..

In conclusion, in the first part, we develop a generic scheme of applying the $\bar{\partial}$-steepest descent method to an oscillatory RHP with arbitrary stationary phase points. Our results can be directly applied to any nonlinear PDEs generated from defocusing reductions of the AKNS hierarchy. In the second part, we show that the generalized Vandermonde-like determinant is a more efficient way to present higher order pole soliton solutions, based on generalized Darboux transformations.

Chapter 1

Introduction and main results

1.1 Introduction

The AKNS system [2], named after Mark J. Ablowitz, David J. Kaup, Alan C. Newell and Segur, Harvey, is one of the most famous models in the study of integrable systems. In this dissertation, for a reduced integrable AKNS hierarchy, two aspects have been carefully studied : the long-time asymptotics and exact solutions. Both can be handled within the framework of inverse scattering transforms (IST) associated with Riemann-Hilbert problems.

The IST of first order linear systems with initial data in L^1 has been well studied by Beals and Coifman [5]. Beals-Coifman's main theorem states that there always exist a dense subset of L^1 such that the IST, considered as a map between initial data and reflection coefficients, is bijective. Later on, Xin Zhou generalized the result to some L^2-Sobolev space [26]. This provides a framework to study long-time asymptotics and exact solutions for the AKNS hierarchy with low-regularity initial data.

In the first half of the dissertation, we will carefully study the defocusing type long-time asymptotics of the following 2 by 2 AKNS hierarchy (Lax pair form):

$$\psi_x(x,t;z)\psi(x,t;z)^{-1} = iz\sigma_3 + Q(x,t), \tag{1.1}$$

$$\psi_t(x,t;z)\psi(x,t;z)^{-1} = a_n z^n + V_0(x,t;z), \tag{1.2}$$

where Q is off-diagonal, V_0 is a polynomial of z with matrix-valued coefficients and the degree of V_0 is less than n. We are interested in mKdV-type reductions, which corresponding to odd $n \geq 3$. For $n = 3$, the compatibility condition gives the mKdV equation, and its long-time asymptotics with Schwartz initial data has been well-studied in Deift-Zhou's paper [11], where the nonlinear steepest descent method was raised. The method has been successfully applied to study the long-time asymptotics of most of the classical integrable equation as well as their multi-components version [20, 19]. Recently, the long-time asymptotics for the mKdV with low-regularity data has been studied by Chen and Liu [8] using the $\bar{\partial}-$steepest descent method. $\bar{\partial}-$steepest descent was originally applied to solving the asymptotics problem in the field of orthogonal polynomial [22] and later applied to long-time asymptotics for nonlinear Schrödinger (NLS) equation[13]. One advantage of the $\bar{\partial}-$steepest descent method is that the analysis for the oscillating Cauchy integrals becomes some relatively simple double integrals.

The rest of the dissertation will mainly study the exact solution (including solitons) to the AKNS hierarchy. Since the V_0 will becomes extremely complicated as n increases, we will only focus on the case $n = 5$. We consider the Cauchy problem of the corresponding nonlinear integrable PDE:

$$\begin{cases} q_t = 30q^4 q_x - 10q^2 q_{xxx} - 40q q_{xx} q_x - 10q_x^3 + q_{xxxxx} \\ q(x, t = 0) = q_0(x) \to 0, |x| \to \infty. \end{cases} \tag{1.3}$$

Constructing solitons within the framework of inverse scattering or RHP is straightforward but that is not an efficient way. In this dissertation, we will mainly apply Darboux transformation(DT) to construct the N-soliton for the 5th-order mKdV. And, by a slight generalization of the classical DT, we construct the so-called double pole soliton [25]. The whole procedure of DT can be compactly represented by a Darboux matrix and the potentials can be recovered from the quotients of some Vandermonde-like determinants. We will introduce a method to reduce the complexity of computing the Vandermonde-like determinants from order $(2N)!$ to order $\binom{2N}{N}$.

1.2 Main results

Let's denote the weighted Sobolev space by

$$H^{k,j}(\mathbb{R}) = \{f \in L^2(\mathbb{R}) | \partial_x^k f, x^j f \in L^2(\mathbb{R})\}, \tag{1.4}$$

with norm

$$\|f\|_{H^{k,j}} = \left(\|f\|_{L^2}^2 + \sum_{l=1}^{k} \|\partial_x^l f\|_{L^2}^2 + \sum_{m=1}^{j} \|x^m f\|_{L^2}^2 \right)^{1/2}. \tag{1.5}$$

First, we consider the defocusing reduction (see, (1.1)) : $Q^* = Q$, where $*$ means complex conjugate transpose. We mainly focus on the following three regions [1]: denote $z_0 = \max_j\{|z_j| : \theta'(z_j) = 0\}$ and $\tau = tz_0^n$, where θ is the phase function and is a polynomial of degree n, then[2]

[1] A more detailed definition was given in [11].

[2] In the region I and III, $|x/t|$ is bounded, while in region II $|x/t^{1/n}|$ is bounded.

- Region I (the oscillatory region) Fix $M > 1$, $z_0 \le M,\ x > 0, \quad \tau \to \infty$.
- Region II (the Painlevé region) Fix $M' > 1$, $z_0 < M',\ (M')^{-1} \le \tau \le M'$.
- Region III (the fast decaying region) $z_0 > M, x \to -\infty$.

In each region, as $t \to \infty$, solution $q(x,t)$, with initial data in the some weighted Sobolev spaces, has uniform leading asymptotics described (fix a large t) in each region. The leading asymptotic presented in theorems (1.1), (1.5) and (1.7) have to be understood in such way: fix t large, the asymptotics valid uniformly for any compact intervals of x which satisfy the definition of the regions. Due to the fact that x, t are connected by z_0, the compact interval of x in each region will become larger as t becomes larger. The main results are as follows:

Theorem 1.1. *In the region I, the leading asymptotics for the potentials $q(x,t)$, for fixed large t, associated with a generic oscillatory RHP whose phase function is $\theta(z;x,t)$* [3] *and initial data $q(x,0)$* [4] *in some weighted Sobolev space, reads*

$$q(x,t) = q_{as}(x,t) + \mathcal{O}(t^{-3/4}), \tag{1.6}$$

where

$$q_{as}(x,t) = -2i \sum_{j=1}^{l} \frac{|\eta(z_j)|^{1/2}}{\sqrt{2t\theta''(z_j)}} e^{i\varphi_j(t)}, \tag{1.7}$$

and

$$\varphi_j(t) = \frac{\pi}{4} - \arg\Gamma(-i\eta(z_j)) - 2t\theta(z_j) - \frac{\eta(z_j)}{2}\ln|2t\theta''(z_j)| + 2\arg(\delta_j) + \arg(R_j), \tag{1.8}$$

in which $\{z_j\}_{j=1}^{l}$ are the stationary phase points, and

$$\begin{aligned} \eta(z) &= -\frac{1}{2\pi}\ln(1-|R(z)|^2), z \in \mathbb{R}, \\ R_j &= R(z_j), \\ \delta_j &= \lim_{\substack{z = z_j + \rho e^{i\phi}, \\ \rho \to 0, \\ \phi \in (0,\pi/2)}} \delta(z)(z - z_j)^{i\eta(z_j)}. \end{aligned}$$

[3] The $\theta(z,x,t)$ will be determined in a specific problem. In the current theorem, θ is a polynomial of z with only l simple stationary phase points.

[4] From the view of inverse scattering, $q(x,0)$ determines the reflection coefficient $R(z;t)$.

Corollary 1.2. *For the AKNS system, the phase function $\theta(z) = xz + ctz^n$. For the mKdV hierarchy, n is odd and θ only has two real stationary phase points: $z_\pm = \pm\left|-\frac{x}{nct}\right|^{\frac{1}{n-1}}$, then suppose the initial data is in $H^{n-1,1}$, the long-time asymptotics for the potentials in the AKNS systems is a special case of the Theorem 1.1, say, $l = 2$.*

Remark 1.3. In the case of the mKdV hierarchy, the stationary points are coming in pairs, then $q_{as}(x,t)$ is easily seen to be a real function. The graph of q_{as}, for fix t, is oscillating over a compact interval of x.

Remark 1.4. It is well-known that linear combination of the members from the same hierarchy gives a new integrable PDEs, in such case, the phase function is a generic polynomial. So it is meaningful to study the asymptotics for generic phase function θ in the region I. However, in other two region, the situation seems to be very complicated. The author will study those problems in the future. In current dissertation, we only deal with the mKdV type reduction.

The following two theorems are under the mKdV type reductions of the AKNS hierarchy. So the phase function $\theta(z; x, t) = xz + ctz^n$, where n is odd and c is some constant depends on n.

Theorem 1.5. *In the region II, the leading asymptotics for the potentials reads*

$$q(x,t) = (nt)^{-\frac{1}{n}} u_n(x(nt)^{-\frac{1}{n}}) + \mathcal{O}(t^{-\frac{3}{2n}}), \tag{1.9}$$

where u_n solves the n^{th} member of the Painlevé II hierarchy.

Remark 1.6. The asymptotics for the Painlevé II equation was presented in the book [16]. There are also some recent works on the asymptotics or special solutions to the Painlevé II equations, to name few such as [23],[7],[9].

Theorem 1.7. *In the region III, the leading asymptotics for the potentials reads*

$$q(x,t) = \mathcal{O}(t^{-1}). \tag{1.10}$$

Remark 1.8. To justify our results, one can compare our result with two recently published work on the long-time asymptotics of the defocusing mKdV equation [8] and the fifth order defocusing mKdV equation [17].

Remark 1.9. Historically, the first general result regarding the AKNS system was due to Varzugin [24]. Varzugin considered the case where the initial data was in the Schwartz class and the phase functions were smooth with first order stationary phase points. His analysis was mainly based on the approximation and decomposition of some Cauchy type integrals. Then later, the asymptotic problem of the general oscillatory matrix RHP was considered by Yen Do [15], where the method was heavily based on the harmonic analysis and the author considered arbitrary order stationary points of the phase. While, now, the $\bar{\partial}-$steepest descent method has been used to significantly simplify the analysis of the asymptotics for the oscillatory RHP.

The second part of the dissertation is devoted to the study of exact solutions of the focusing reductions, which is in general much harder than the defocusing case due to the occurrence of solitons. For that reason, we will focus only on a specific example, the fifth-order focusing mKdV equation.

Theorem 1.10 ($N-$fold generalized DT for the AKNS hierarchy)**.** *Suppose $q_0(x,t) = Q_{1,2}$ solves the nonlinear PDE generated by the lax pair (M_0, N_0), the the new potential generated by the N-fold DT is*

$$q(x,t) = q_0 - 2iB_{N-1}, \tag{1.11}$$

where B_{N-1} is defined in the equation (6.61)*, respectively.*

Remark 1.11. In Chapter V, we present several kinds of interesting exact solutions of the focusing 5th order mKdV by applying the Theorem 1.10.

1.3 Organization of the dissertation

First, we formulate the direct scattering for the AKNS hierarchy in some L^2 weighted Sobolev space. The main purpose of the second chapter is to show the bijectivity of the scattering map from the initial data to scattering data. Most of the theorems in the second chapter is a reformulation of Zhou's work [26]. A minimal knowledge on the matrix RHP in L^2 space will be introduced, which serves as fundamental tool to formulate the inverse scattering transform. After that, we will attack the main problem of this dissertation: looking for the long-time asymptotics for the mKdV type defocusing reduced equations. More specifically, we divide the $(x,t) \in \mathbb{R} \times \mathbb{R}_+$ into three regions: the oscillatory region, the fast decay region and the Painlevé region. After that, we will discuss how to use the Darboux transformation to construct several kinds of soliton solutions. Finally, those solutions will be represented by quotients of some Vandermonde-like matrices. In the last chapter, we simply discuss some further questions and possible generalizations.

Chapter 2

Direct scattering

2.1 Constructing scattering data and Riemann-Hilbert problems

Consider the spectral problem

$$\psi_x(x,t;z) = \left(iz\sigma_3 + \begin{pmatrix} 0 & q(x,t) \\ \epsilon q(x,t) & 0 \end{pmatrix} \right) \psi(x,t;z), \tag{2.1}$$

and one can see that the $z = \infty$ is an essential singularity to the solution of the equation. To remove it, make the following transformation,

$$\psi(x,t;z) = \mu(x,t;z)e^{ixz\sigma_3},$$

and then the original equation is transformed into the following one:

$$\mu_x = iz[\sigma_3, \mu] + Q\mu, \tag{2.2}$$

for which we can impose normalization condition as $x \to \pm\infty$:

$$\mu^{(\pm)}(x,z) \to I$$

The task is equivalent to establish the following two integral equations:

$$\begin{aligned} \mu^{(\pm)}(x,z) &= I + \int_{\pm\infty}^{x} e^{iz(x-y)\text{ ad }\sigma_3} Q(y)\mu^{(\pm)}(y;z)dy \\ &= I + K_{q,z,\pm}\mu^{(\pm)}(x,z). \end{aligned} \tag{2.3}$$

Note that for now, the time variable t is a dummy variable, and it is fixed. So in the remainder of this chapter we will not write t explicitly.

From now on, the superscripts indicate the normalization of $x \to \pm\infty$ while the subscripts relate to the analytic continuation to the upper/lower half z-planes. As usual, define the scattering matrix as follows,

$$\psi^{(+)}(x,t) = \psi^{(-)}(x,t)S(z), \tag{2.4}$$

where

$$\psi^{(\pm)} = \mu^{(\pm)} e^{ixz\sigma_3},$$

and set

$$S(z) = \begin{bmatrix} a(z) & \breve{b}(z) \\ b(z) & \breve{a}(z) \end{bmatrix}, z \in \mathbb{R}.$$

Next we will discuss defocussing/focusing cases separately. First, for the defocussing case, we have

$$Q = \begin{bmatrix} 0 & q \\ q & 0 \end{bmatrix},$$

which satisfies the following two obvious symmetries:

$$\bar{Q} = Q, \tag{2.5}$$

$$\sigma Q \sigma = Q, \sigma = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{2.6}$$

Substituting back to the spectral problem (2.1), one can see that the solution ψ also satisfies the following two properties:

$$\bar{\psi}_x(-\bar{z}) = (iz\sigma_3 + Q)\bar{\psi}(-\bar{z}), \tag{2.7}$$

$$(\sigma\psi(-z)\sigma)_x = (iz\sigma_3 + Q)\sigma\psi(-z)\sigma, \tag{2.8}$$

which in turn implies that ψ enjoys the following symmetry property:

$$\bar{\psi}(-\bar{z}) = \sigma\psi(-z)\sigma = \psi(z). \tag{2.9}$$

Then by the definition of $S(z)$, we have

$$\bar{S}(-z) = S(z) = \sigma S(-z)\sigma, \tag{2.10}$$

and thus,

$$\breve{a}(z) = a(-z) = \bar{a}(z), \tag{2.11}$$

$$\breve{b}(z) = b(-z) = \bar{b}(z). \tag{2.12}$$

Since the coefficient matrix of the spectral problem is trace-free, the Wronskian of the fundamental system of solutions is independent of x. Combining with the normalization conditions at infinite, we have for $z \in \mathbb{R}$,

$$|a(z)|^2 - |b(z)|^2 = 1. \tag{2.13}$$

For the focusing case, we have

$$Q = \begin{bmatrix} 0 & q \\ -q & 0 \end{bmatrix}, \tag{2.14}$$

which has the symmetries

$$Q^T = -Q, \tag{2.15}$$

$$\bar{Q} = Q. \tag{2.16}$$

From those we found that

$$[\psi^{-T}(-z)]_x = [iz\sigma_3 + Q]\psi^{-T}(-z). \tag{2.17}$$

Thus, again due to the uniqueness of the first-order ODE, we have

$$\psi(-z)\psi^T(z) = I. \tag{2.18}$$

Similarly, the scattering matrix satisfies

$$S(-z)S^T(z) = I, \tag{2.19}$$

which implies

$$\breve{a}(z) = a(-z), \tag{2.20}$$

$$\breve{b}(z) = -b(-z), \tag{2.21}$$

and the determinant det $S(z) = 1$ leads to

$$|a(z)|^2 + |b(z)|^2 = 1. \tag{2.22}$$

Here we have also applied that $S(-z) = \bar{S}(z)$ which comes from the assumption that the potential matrix Q is real. Historically, one can then define the so-called Jost solutions and rewrite the spectral problem as Volterra-type integral equations, from where one can then use Fourier techniques to construct the Gelfand-Levitain-Marchenko (GLM) equation. Later, A.B.Shabat formulated the direct problem as a Riemann-Hilbert problem. These two methods have their own advantages. In the one hand, it is better to apply the RHP when studying the long-time asymptotics to the solutions since the Deift-Zhou's steepest descent method relies on the RHP formulation. On the other hand, for constructing exact solutions such as N-soliton/breathers, especially, for the multiple-pole solutions (see Wadati [25]) or for studying nonlocal integrable equations (see Ablowitz [3], [4] and also our recent work [21]), the GLM equation seems more convenient than the RHP formulation. A good survey of the inverse scattering and linear spectral problem is [6].

In this section, we will stay with a 2×2 matrix RHP and assume that the potential $q\in L^1(\mathbb{R})$. In order to formulate an RHP, one needs to study the analyticity of the singular integral equation (2.3). The following proposition shows where $\mu(z)$ can be analytically extended in the z plane.

Proposition 2.1. *$\mu_1^{(\pm)}(z)$ and $\mu_2^{(\mp)}(z)$ are respectively analytic in the half plane $\mathbb{C}^{(\pm)}$, where $\mu^{(\pm)}=[\mu_1^{(\pm)},\mu_2^{(\pm)}]$ and $\mathbb{C}^{(\pm)}=\{z|\pm\operatorname{Im}(z)>0\}$.*

Proof. First, expanding the matrix integral equation (2.3) component-wise, we have a pair of independent scalar singular integral equations, which are related to $(\mu_{11}^{(\pm)},\mu_{21}^{(\pm)})$ and $(\mu_{22}^{(\pm)},\mu_{12}^{(\pm)})$, where $\mu_{jk}^{(\pm)}$ are the (j,k)-entry of $\mu^{(\pm)}$. For each pair, the proofs are almost the same. Thus we will only proof the first pair, i.e.,

$$\mu_{11}^{(\pm)}=1+\int_{\pm\infty}^{x}q(y)\mu_{21}^{(+)}(y,z)dy, \tag{2.23}$$

$$\mu_{21}^{(\pm)}=\int_{\pm\infty}^{x}e^{-2i(x-y)z}q(y)\mu_{11}^{(+)}(y,z)dy. \tag{2.24}$$

Substituting the second equation into the first one, and changing the order of the integrals, we arrived at the following integral equation:

$$\mu_{11}^{(+)}(x,z)=1+\int_{x}^{\infty}ds\int_{x}^{s}dy\ e^{-2i(y-s)z}q(y)q(s)\mu_{11}^{(+)}(s,z). \tag{2.25}$$

In the following analysis under this proof, without causing any confusions, we will drop the subscript and superscript of $\mu_{11}^{(\pm)}$.

Denoting $U(s,x,z):=q(s)\int_x^s e^{-2i(y-s)z}q(y)dy$, we obtain

$$\mu(x,z)=1+\int_{x}^{\infty}U(x,s,z)\mu(s,z)ds. \tag{2.26}$$

Following the standard Neumann series technique, we construct the Neumann series as follows:

$$\mu^{[1]}(x,z)=1, \tag{2.27}$$

$$\mu^{[n]}(x,z)=1+\int_{x}^{\infty}U(s,x,z)\mu^{[n-1]}(s,z)ds, n=2,3,\cdots. \tag{2.28}$$

Since $\mathrm{Im}(z) > 0$ and $y < s$,

$$|U(s, x, z)| \leq |q(s)| \int_x^s |q(y)| dy \leq \|q\|_{L^1(\mathbb{R})} |q(s)|. \tag{2.29}$$

Noticing that

$$\left| \int_x^\infty U(s_1, x, z) \int_{s_1}^\infty U(s_2, s_1, z) \cdots \int_{s_{n-1}}^\infty U(s_n, s_{n-1}, z) ds_n ds_{n-1} \cdots ds_1 \right| \leq \frac{(\int_x^\infty |q(s)| ds)^n}{n!} \|q\|_{L^1(\mathbb{R})}^n, \tag{2.30}$$

we have

$$\left| \mu^{[N]}(x, z) \right| \leq \sum_{k=0}^{N-1} \frac{(\|q\|_{L^1(\mathbb{R})} \int_x^\infty |q(s)| ds)^k}{k!} \leq e^{\|q\|_{L^1(\mathbb{R})}^2}.$$

So letting n goes to infinite, the series converges uniformly as long as q is L^1 and $\mathrm{Im}(z) > 0$. Also, since each $\mu^{[n]}$ is analytic in z on $\mathbb{C}^+$, the limiting function is analytic and satisfies the integral equation (2.26). Thus the proof is done. $\square$

In the next proposition, we will present the entries of the scattering matrix in terms the limiting value of the solutions $\mu^{(\pm)}$ of the ODE (2.2).

Proposition 2.2.

$$a(z) = \mu_{11}^{(+)}(x \to -\infty) = 1 - \int_{\mathbb{R}} q(y) \mu_{21}^{(+)}(y, z) dy, \tag{2.31}$$

$$b(z) = \mu_{12}^{(+)}(x \to -\infty) = - \int_{\mathbb{R}} e^{2iyz} q(y) \mu_{22}^{(+)}(y, z) dy, \tag{2.32}$$

$$\breve{a}(z) = \mu_{22}^{(+)}(x \to -\infty) = 1 - \int_{\mathbb{R}} q(y) \mu_{12}^{(+)}(y, z) dy, \tag{2.33}$$

$$\breve{b}(z) = \mu_{21}^{(+)}(x \to -\infty) = - \int_{\mathbb{R}} e^{-2iyz} q(y) \mu_{11}^{(+)}(y, z) dy. \tag{2.34}$$

Or in matrix form as:

$$S(z) = I - \int_{\mathbb{R}} e^{-izy\, ad\sigma_3} Q(y) \mu^{(+)}(y, z) dy. \tag{2.35}$$

Proof. Noting from the definition of the scattering matrix (2.4), we have

$$e^{-ixz\, \mathrm{ad}\sigma_3} [\mu^{(-)-1}(x, z), \mu^{(+)}(x, z)] = S(z).$$

Then expanding the left hand side and letting $x \to -\infty$, one has

$$a(z) = 1 - \int_{\mathbb{R}} q(y) \mu_{21}^{(+)}(y, z) dy.$$

The integral is well-defined provided that $q(x,t) \in L^2(\mathbb{R}, dx) \cap L^1(\mathbb{R}, dx)$ and $\mu^{(+)}(x,z) \in I + L^2(\mathbb{R}, dx)$. $\square$

From those representations, the following analytic properties of $a(z)$ naturally follows

Proposition 2.3. $a(z)$ *extends analytically to the upper half-plane, and* $a(z) = 1 + O(1/z), z \to \infty$. $b(z) = O(1/z), |z| \to \infty, \operatorname{Im} z = 0$.

Proof. To show the asymptotic for $a(s)$, it is sufficient to show that $\mu_{11}^{(+)}(x,z) = 1 + O(1/z), z \to \infty, \operatorname{Im} z \geq 0$. From the Neumann series, we know for $\mu^{[1]}$ it is true. Now suppose for $k = 1, 2, \cdots, n-1$, we have $\mu^{[k]} = 1 + O(1/z)$, we will prove it is also true for $k = n$. In fact,

$$\begin{aligned} \mu^{[n]} &= 1 + \int_x^\infty q(s) \int_x^s e^{-2i(y-s)z} q(y) dy (1 + O(1/z)) ds \\ &= 1 + \int_x^\infty q(s) \int_x^s e^{-2i(y-s)z} q(y) dy ds + (\int_x^\infty q(s) \int_x^s e^{-2i(y-s)z} q(y) dy ds) O(1/z). \end{aligned}$$

For real z, if $q \in L^1(\mathbb{R})$. Then by Riemann-Lebesgue lemma, we have

$$|\int_x^\infty q(s) \int_x^s e^{-2i(y-s)z} q(y) dy| = \|q\|_{L^1(\mathbb{R})} O(1/z), z \to \infty.$$

For $\operatorname{Im} z > 0$, we have $e^{-2i(y-s)z} = O(e^{-\operatorname{Im} z}) = O(1/z), z \to \infty$. Thus, we have shown that

$$\mu^{[n]}(z,x) = 1 + \|q\|_{L^1} O(1/z) = 1 + O(1/z), z \to \infty, \operatorname{Im} z \geq 0.$$

Since $\mu^{[n]}$ converges uniformly, we conclude that $\mu_{11}^{(+)}(x,z) = 1 + O(1/z), z \to \infty, \operatorname{Im} z \geq 0$.

Similarly, one can also show $\mu_{22}^{(+)}(x,z) = 1 + O(1/z)$ for real z. Moreover, due to the representation (2.32) of $b(z)$, via the Riemann-Lebesgue lemma, $b(z) = O(1/z)$. Thus, the proof is done. $\square$

Remark 2.4. For sufficiently smooth initial data, one can always perform integration by parts to derive similar results.

2.2 Bijectivity of the direct scattering map

Now, we formulate the RHP as follows. Motivated by the proposition 2.1 and for the purpose of normalization, we define

$$m_+(x,z) = (\mu_1^{(+)}(x,z)/a(z), \mu_2^{(-)}(x,z)), \quad \operatorname{Im} z \geq 0, \tag{2.36}$$

$$m_-(x,z) = (\mu_1^{(-)}(x,z), \mu_2^{(+)}(x,z)/\breve{a}(z)), \quad \operatorname{Im} z \leq 0. \tag{2.37}$$

Such m is normalized at $x = \pm\infty$.

From the previous analysis on the defocusing type reduction, we have, for all $z \in \overline{\mathbb{C}^+}$

$$|a(z)|^2 = 1 + |b(z)|^2 > 0.$$

This implies that $a(z)$ has no zeros and hence $m_\pm(z) := m_\pm(x, z)$ are continuous in $\overline{\mathbb{C}^\pm}$ and analytic in $C^\pm$. Since both $m_\pm$ are solutions to the same ODE(2.2), it is straightforward to check that

$$m_-^{-1} m_+ = e^{ixz \, \mathrm{ad}\sigma_3} v(z)$$

for some matrix $v(z)$ which dose not depend on x. Thus, we define the jump matrix as

$$v(z) = e^{-izx \, \mathrm{ad}\sigma_3}[m_-^{-1} m_+]. \tag{2.38}$$

Direct computation gives us

$$v(z) = \begin{pmatrix} 1 & \frac{\breve{b}}{\breve{a}} \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ \frac{b}{a} & 1 \end{pmatrix} = \begin{pmatrix} 1 - |r|^2 & r \\ -\bar{r} & 1 \end{pmatrix},$$

where $r(z) = \frac{-\bar{b}(z)}{\bar{a}(z)}$.

We use the following notation to denote the RHP: $(m, e^{ixz \, \mathrm{ad}\sigma_3} v(z), \mathbb{R})$, which means we are seeking a piecewise analytic/meromorphic matrix-valued function in $\mathbb{C} - \mathbb{R}$ with the jump condition(matrix) $e^{ixz \, \mathrm{ad}\sigma_3} v(z)$. In the scattering theory, v is often called the scattering data. The procedure from the potential q to the scattering data v is called the direct scattering and the inverse scattering is to recovery the potentials from the given scattering data.

We define the direct scattering map and the inverse scattering map as follows,

$$\mathcal{D} : q \in H^{j,k}(\mathbb{R}) \mapsto r \in H^{k,j}(\mathbb{R}), \tag{2.39}$$

$$\mathcal{I} : r \in H^{j,k}(\mathbb{R}) \mapsto q \in H^{k,j}(\mathbb{R}). \tag{2.40}$$

The following theorem is well-known. See for example [12]. Here we give a more detailed proof than the original proof in the paper [12].

Theorem 2.5. *$\mathcal{D}$ maps $H^{j,k}(\mathbb{R}, dx)$ into $H^{k,j}(\mathbb{R}, dz)$, $j \geq 1, k \geq 0$.*

Proof. Since $a(z) = 1 + O(1/z), z \to \infty$, it is sufficient to show the L^2-decay of $\mu^{(+)} - I$ knowing that

$$\begin{pmatrix} a(z) - 1 & \breve{b}(z) \\ b(z) & \breve{a}(z) - 1 \end{pmatrix} = \mu^{(+)}(-\infty, z) - I \tag{2.41}$$

Since the $\mu^{(+)}$ satisfies the integral equation (2.3), it can be represented as

$$\mu^{(+)}(x,z) - I = ((I - K_{q,z,+})^{-1} K_{q,z,+} I)(x,z).$$

Provided that $q \in H^{j,k}(\mathbb{R}) \subset H^{0,0} = L^2$, we have

$$\begin{aligned}\|(K_{q,z,+}I)(x,z)\|_{L^2(dz)L^\infty(dx)} &= \sup_{x\in\mathbb{R}} \sup_{\substack{\phi\in C_0^\infty \\ \|\phi\|_{L^2}=1}} \operatorname{Tr}\left(\left|\int_{\mathbb{R}}\int_x^\infty e^{i(x-y)z \text{ ad } \sigma_3} Q(y)\phi(z)dydz\right|\right) \\ &\text{(noting } Q \text{ is off-diagonal)} \\ &\le \sup_{x\in\mathbb{R}} \sup_{\substack{\phi\in C_0^\infty \\ \|\phi\|_{L^2}=1}} \int_x^\infty (|\hat{\phi}_{12}(2(y-x))| + |\hat{\phi}_{21}(2(x-y))|)|q(y)|dy \qquad (2.42) \\ &\text{(by the Cauchy-Schwartz inequality and the Plancherel's theorem)} \\ &\le \|q(y)\|_{L^2}.\end{aligned}$$

And the method of Neumann series gives us the estimate for

$$\|(1-K_{q,z,+})^{-1}\|_{L^2(dz)L^\infty(dx)\circlearrowleft} \le e^{\|Q\|_{L^1}}. \tag{2.43}$$

Combining all two inequalities together, we arrive at

$$\|\mu^{(+)}(x,z) - I\|_{L^2(dz)L^\infty(dx)\circlearrowleft} \le e^{\|Q\|_{L^1}}\|q\|_{L^2}, \tag{2.44}$$

which implies that if $q \in L^2(dx)$. Then $r(z) \in L^2$. Next, add one L^2-decay to the potential q, i.e. $q \in H^{1,0}$, we will study the regularity of the reflection coefficient $r(z)$.

Now let us consider the z derivative of $\mu^{(+)}(x,z)$, which enjoys the following integral equation:

$$\begin{aligned}\partial_z \mu^{(+)}(x,z) = &\int_\infty^x i(x-y) \text{ ad } \sigma_3[e^{i(x-y)z \text{ ad } \sigma_3} Q(y)\mu^{(+)}(y,z)]dy \\ &+ \int_\infty^x e^{i(x-y)z \text{ ad } \sigma_3} Q(y)\partial_z\mu^{(+)}(y,z)dy.\end{aligned}$$

If we set $D_z(x)\mu(x,z) := (\partial_z - ix \text{ ad } \sigma_3)\mu$. Then we have

$$\begin{aligned}D_z(x)\mu^{(+)}(x,z) = &-i\int_\infty^x e^{i(x-y)z \text{ ad } \sigma_3}[\text{ ad } \sigma_3 yQ(y)]\mu^{(+)}(y,z)dy \\ &+ \int_\infty^x e^{i(x-y)z \text{ ad } \sigma_3} Q(y)D_z(y)\mu^{(+)}(y,z)dy.\end{aligned}$$

It is easily seen that the first integral belongs to $L^2(dz)L^\infty(dx)$ by writing $\mu^{(+)} = \mu^{(+)} - I + I$ and the condition that $xq(x) \in L^2(dx)$, and denoting it by S_1. Then the new integral equation reads

$$D_z(x)\mu^{(+)}(x,z) = S_1 + K_{q,z,+}D_z\mu^{(+)}, \tag{2.45}$$

which implies

$$\|D_z\mu^{(+)}\|_{L^2(dz)} = \|(1-K_{q,z,+})^{-1}S_1\|_{L^2(dz)} \le e^{\|Q\|_{L^1}}\|S_1(x,z)\|_{L^2L^\infty}. \tag{2.46}$$

The $L^\infty(dx)$ enables us to set $x = 0$. Thus $\partial_z\mu^{(+)}(0,z) = D_z(0)\mu^{(+)}(0,z) \in L^2(dz)$, and so is $\partial_z(\mu^{(+)}(0,z) - I)$. Similarly, one can show that $\partial_z(\mu^{(-)}(0,z) - I)$ is in $L^2(dz)$. Then by the triangle inequality, $r(z) \in H^{1,0}(dz)$.

Next, we add one regularity to the potential, (so $q \in H^{1,1}$ now), and study the L^2 decay of the reflection coefficient $r(z)$. We know

$$b(z) = -\int_{\mathbb{R}} e^{2iyz}q(y)(\mu_{22}^{(+)}(y,z) - 1)dy + \int_{\mathbb{R}} e^{2iyz}q(y)dy. \tag{2.47}$$

Clearly, by Plancherel's theorem, the second integral is in $H^{1,1}$ provided $q \in H^{1,1}$. To show the first integral is also in $H^{1,1}$, it is sufficient to show

$$\int_{\mathbb{R}} e^{2iyz}q(y)(\mu^{(+)}(y,z) - I)dy \in H^{1,1}. \tag{2.48}$$

The regularity has now proved, next we need to show the L^2-decay. First, let $K = K_{q,z,+}$, and integration by parts gives us

$$(KI)(x) = \frac{-1}{2iz}\sigma_3 Q(x) - \int_\infty^x e^{i(x-y)z\,\mathrm{ad}\,\sigma_3}Q'(y)dy := I_1(x,z) + I_2(x,z).$$

Then

$$\begin{aligned}
\mu^{(+)} - I &= (1-K)^{-1}(I_1 + I_2) \\
&= (1 + K + (1-K)^{-1}K^2)I_1 + (1-K)^{-1}I_2 \\
&= I_1 + g_1 + g_2 + g_3.
\end{aligned}$$

We need to show that $\int_{\mathbb{R}} e^{2iyz}q(y)[I_1(y,z) + g_1 + g_2 + g_3]dy \in H^{1,0}(dz)$.

Since $q \in H^{0,1}$, so is q^2 since $H^{0,1}$ can be regarded as a Banach algebra. Thus Plancherel's theorem tells us that

$$\int_{\mathbb{R}} e^{2iyz}q(y)I_1(y,z)dy \in H^{2,0}(dz) \subset H^{1,0}(dz).$$

Observing that

$$g_1 = KI_1 = \frac{-1}{2iz}\int_\infty^x e^{i(x-y)z \text{ ad } \sigma_3} q^2(y)\sigma_3 dy = \frac{-1}{2iz}\int_\infty^x q^2(y)\sigma_3 dy,$$

thus

$$q(x)\int_\infty^x q^2(y)\sigma_3 dy \in H^{0,1},$$

and we have

$$\int_{\mathbb{R}} e^{2iyz} q(y) g_1(y,z) dy \in H^{2,0}(dz) \subset H^{1,0}(dz).$$

By replacing the I by $Q\int_\infty^y Q^2\sigma_3$ in equation (2.42) and by the same argument, we have $K^2 I_1 \in H^{0,1}(dz)L^\infty(dx)$ and since $(1-K)^{-1}$ is bounded in L^2L^∞, $g_2 \in H^{0,1}(dz)L^\infty(dx)$ and thus

$$\begin{aligned}(\int_{\mathbb{R}} |\int_{\mathbb{R}} ze^{2iyz} q(y) g_2(y,z) dy|^2 dz)^{1/2} &\leq \int_{\mathbb{R}} |q(y)| \|zg_2\|_{L^2(dz)} dy \\ &\leq \pi \|q\|_{H^{0,1}}.\end{aligned}$$

Similarly, $I_2 \in H^{0,1}L^\infty$, so is g_3 and thus

$$\int_{\mathbb{R}} e^{2iyz} q(y) g_3 dy \in H^{1,0}.$$

Thus, so far we have proved that $\mathcal{D}$ maps $H^{1,1}(dx)$ to $H^{1,1}(dz)$.

Finally, by performing integration by parts several times, the general statement follows. □

Remark 2.6. Here $j \geq 1$ is for keeping the function in L^1.

Remark 2.7. For the general case, such as the focusing type reduction with arbitrary singularities (zeros/poles or spectral singularities), one can refer to Zhou's fundamental work [26].

Chapter 3

Inhomogeneous Riemann-Hilbert Problems in weighted Sobolev spaces

3.1 Introduction to the Cauchy operator in L^2

In this chapter, we will review some basic properties of the Cauchy operator and how to apply it to solve the so-called inhomogeneous Riemann-Hilbert problem (RHP). Note that most notations follow from [12]. As usual, given a smooth oriented contour $\Gamma \subset \mathbb{C}$, define the Cauchy operator as

$$Cf(z) := \int_\Gamma \frac{f(s)}{s-z}\frac{ds}{2\pi i}, z \in \mathbb{C}\backslash\Gamma. \tag{3.1}$$

Also define the Cauchy boundary value on the contour Γ:

$$C^\pm f(z) := \lim_{\epsilon\downarrow 0} Cf(z \pm i\epsilon z), z \in \Gamma. \tag{3.2}$$

It is well known (see for example, Duren's book) that such limit (as a special case of non-tangential limits) exit for a.a. $z \in \Gamma$ provided $f \in L^2(\Gamma, |dz|)$.

Corollary 3.1 (M. Riesz)**.** *The Hilbert transform* $Hf(z) := \lim_{\epsilon\downarrow 0}\int_{\Gamma\backslash\{s:|s-z|\le\epsilon\}} \frac{f(s)}{z-s}\frac{ds}{i\pi}$ *is a bounded linear operator in* $L^2(\Gamma)$*. Moreover,* $\|H\|_{L^2} = 1$.

Theorem 3.2 (Sokhotski-Plemelj)**.** $C^+ - C^- = 1$ *and* $C^+ + C^- = -H$.

Corollary 3.3. $\|C^\pm\|_{L^2} = 1$.

All above mentioned theorems are classical and well-know, so their proofs are omitted here. However, it is worth mentioning the representation of H or C by the Fourier transform in L^2 distributional sense. That is to say, for any $f \in L^2$, we have

$$\mathcal{F}[Hf](z) = (-\operatorname{sgn}(z))\mathcal{F}[f](z), \quad z \in \mathbb{R}. \tag{3.3}$$

Also for the Cauchy operator $C^\pm$, we have

$$\mathcal{F}[C^\pm f](z) = \frac{\pm 1 + \operatorname{sgn}(z)}{2}\mathcal{F}[f](z). \tag{3.4}$$

The operator $C^{\pm}_{\Gamma\to\Gamma'}$ is defined by $f \mapsto$ boundary value $(C_\Gamma f)_\pm$ on Γ', and it is bounded from $L^2(\Gamma)$ to $L^2(\Gamma')$, i.e., $\|C^\pm f\|_{L^2(\Gamma')} \le c\|f\|_{L^2(\Gamma)}$. In this section, we will revisit some basic knowledge about the RHP in L^2 sense with jumps in $GL(\mathbb{C},2)$. In this section we will consider an oriented smooth contour Γ with a 2×2 jump matrix $v(z)$ defined on Γ whose determinant is 1. One can think of this v as the scattering data.

Definition 3.4. Given a contour Γ, we say a pair of $L^2(\Gamma)$ function $f_\pm \in \partial C(L^2)$ if there is a function $h \in L^2(\Gamma)$ such that

$$f_\pm(z) = (C^\pm h)(z), z \in \Gamma.$$

And call $f(z) = Ch(z)$ the extension of $f_\pm$ off the contour Γ.

Definition 3.5. Given Γ and v, and a function $f \in (L^\infty + L^2)(\Gamma)$, we say $m_\pm \in f + \partial C(L^2)$ solves $IRHP1(v, f, \Gamma)$ if

$$m_+(z) = m_-(z)v(z), z \in \Gamma.$$

Definition 3.6. Given Γ, v and a function $F(z) \in L^2(\Gamma)$, we say $M_\pm \in \partial C(L^2)$ solves $IRHP2(v, F, \Gamma)$ if

$$M_+(z) = M_-(z)v(z) + F(z), z \in \Gamma.$$

Definition 3.7. We say $m_\pm$ solves the normalized RHP if $m_\pm$ solves $IRHP1(v, I, \Gamma)$.

Proposition 3.8. *Given Γ, v, f such that $f(z)(v(z) - I) \in L^2$. Then*

$$m_\pm = M_\pm + f \tag{3.5}$$

solves $IRHP1(v, f, \Gamma)$ if $M_\pm$ solves $IRHP2(v, F, \Gamma)$ with $F = f(z)(v(z)-I) \in L^2(\Gamma, |dz|)$. Conversely, for a given $F \in L^2$,

$$M_+ = m_+ + F, \quad M_- = m_-, \tag{3.6}$$

solve $IRHP2(v, F, \Gamma)$ if $m_\pm$ solve $IRHP1(v, f, \Gamma)$ with $f = C^- F$.

Proof. On the one hand, by assumption, $f(v-I) \in L^2$, and $M_\pm$ solves $IRHP2(v,F,\Gamma)$ with $F = f(v-I)$, and then

$$\begin{aligned} M_+ &= M_- v + F \\ &= M_- v + fv - f \\ &= (M_- + f)v - f. \end{aligned}$$

Thus

$$(M_+ + f) = (M_- + f)v,$$

and $m_\pm = M_\pm + f \in f + \partial C(L^2)$ solve $IRHP1(v,f,\Gamma)$ by definition.

On the other hand, if $m_\pm$ solve the $IRHP1(v,f,\Gamma)$ and let $f = C^- F$, by the definition of solution to $IRPH1(v,f,\Gamma)$, we have

$$m_\pm - f \in \partial C(L^2).$$

That is to say there is a $h \in L^2$ such that

$$m_\pm = f + C^\pm h = C^- F + C^\pm h, \tag{3.7}$$

which in turn implies that

$$m_+ + F = F + C^- F + C^+ h = C^+(F+h) \tag{3.8}$$

$$m_- = C^-(F+h). \tag{3.9}$$

Then $M_+ = m_+ + F = m_- v + F = M_- v + F$ implies $M_\pm$ solve the $IRHP2(v,F,\Gamma)$, and the proof is done. $\square$

3.3 The Beals-Coifman operator

In this section, we will revisit the Beals-Coifman operator $C_w f$, which was applied in [5] to the inverse scattering for the first order system.

Definition 3.9. Given a contour Γ, and a pair of weight 2×2 matrix valued functions $w_\pm$, and then for each $f \in (L^\infty + L^2)(\Gamma)$, define

$$C_w f(z) = C^+(f w_-) + C^-(f w_+). \tag{3.10}$$

From the definition we easily see that if $w_\pm \in L^\infty(\Gamma) \cup L^2(\Gamma)$. Then C_w maps $L^\infty + L^2$ to L^2. The weight functions are usually from the factorization of the jump matrix v. Suppose v admits a factorization $v = v^{-{-1}} v^+$, and suppose $m_\pm$ solve $IRHP1(v, I, \Gamma)$. Then $m_+ v^{+{-1}} = m_- v^{-{-1}} := \mu$. Moreover, one can show that μ solves the singular integral equation

$$\mu = I + C_w \mu, \quad w_\pm = \pm v^\pm \mp I, \tag{3.11}$$

if and only if $m_\pm$ solves $IRHP1(v, I, \Gamma)$. The following proposition connects the bijectivity of the operator $1 - C_w$ with the solvability of the $IRHP1$ and $IRHP2$. In the future, we will estimate the L^2 norm of the operator $(1 - C_w)^{-1}$ which plays a fundamental role in performing the nonlinear steepest descent method to the oscillatory RHPs.

Proposition 3.10. *The following statements are equivalent:*

(1). $1 - C_w$ *is bijective from* L^2 *to* L^2.

(2). $IRHP1(v, f, \Gamma)$ *has a unique solution for all* $f \in L^2$.

(3). $IRHP2(v, F, \Gamma)$ *has a unique solution for all* $F \in L^2$.

Proof. The equivalence of the last two statements has been done by the previous proposition. We only need to show that the first two are equivalent. In fact, if $m_\pm$ solve the $IRHP1(v, f, \Gamma)$. Then $m_+ = m_- v$. Noting that w comes from a decomposition of the jump matrix v, say $v = v^{-{-1}} v^+$, we have $w^\pm = \pm v^\pm \mp I$. And letting $\mu := m_+ v^{+{-1}} = m_- v^{-{-1}}$. Then μ is entire since the jump matrix is I and belongs to L^2 since $v^{\pm -1} \in L^\infty$. Further denoting $H = C(\mu(w^+ + w^-))$ off Γ, it is easy to check that the boundary values on Γ are

$$\begin{aligned}
H_+ &= C^+(\mu(w^+ + w^-)) \\
&= \mu w^+ + C_w \mu \\
&= \mu(w^+ + I) + (C_w - I)\mu, \\
&= m_+ + (C_w - I)\mu, \\
H_- &= C^-(\mu(w^+ + w^-)) \\
&= -\mu w^- + C_w \mu \\
&= \mu(I - w^-) + (C_w - I)\mu \\
&= m_- + (C_w - I)\mu,
\end{aligned}$$

which gives us

$$m_{\pm} - H_{\pm} - f \in \partial C(L^2), \tag{3.12}$$

and then by Louisville's argument, we have $m_{\pm} - H_{\pm} - f = 0$ and hence $(1 - C_w)\mu = f$.

Conversely, if μ solves the singular integral equation $(1 - C_w)\mu = f$, and for any appropriate decomposition $v^{\pm}$ of the jump matrix v, $m_{\pm} = \mu v^{\pm}$ indeed solve that $IRHP1(v, f, \Gamma)$. In fact, we have

$$m_{\pm} - f = C^{\pm}(\mu(w^{+} + w^{-})) \in \partial C(L^2). \tag{3.13}$$

Next we will show that the injectivity of $1 - C_w$ is equivalent to the unique solvability of the $IRHP1(v, f, \Gamma)$. In fact, taking $\mu \in L^2$ in the kernel of $1 - C_w$. Then $m_{\pm} = H^{\pm} \in L^2$ solve the $IRHP1(v, 0, \Gamma)$, and vice verse. Here the uniqueness is in the sense of L^2 space. □

Next, let us consider the uniqueness for the solution of the normalized RHP in L^2, which will be sufficient for studying the 5th-order mKdV equation. The following proposition is rather general but useful in practice. One can easily generalize even to the L^p space for $1 < p < \infty$; see [12] Theorem 2.9. We only consider the case in the Hilbert space.

Proposition 3.11. *Suppose the piecewise analytic matrix-valued function m solves the $IRHP(v, I, \Gamma)$ by definition ($v^{\pm 1} \in L^{\infty}$) given before, and m^{-1} exists for all z off Γ such that*

$$(m^{-1})_{\pm} - I \in \partial C(L^2(\Gamma)).$$

Then m is unique in the L^2 sense.

Proof. Let m_1, m_2 both solve the $IRHP(v, I, \Gamma)$. Then by assumption, we know m_2^{-1} satisfies

$$m_{2+}^{-1} = v^{-1} m_{2-}^{-1},$$

and noting that for $f, g \in L^2$,

$$CfCg = -\frac{1}{2} C\left((Hf)g + fHg\right).$$

Both sides are in L^1 by the Cauchy-Schwartz inequality, and hence the Plemelj formula works, which gives us

$$C^{+}fC^{+}g - C^{-}fC^{-}g = -\frac{1}{2}\left((Hf)g + fHg\right).$$

Since

$$m_1 m_2^{-1} = (m_1 - I)(m_2^{-1} - I) + (m_1 - I) + (m_2^{-1} - I) \in L^1 + L^2,$$

by the Plemelj's formula, we have

$$(m_1 m_2^{-1} - I)_+ - (m_1 m_2^{-1} - I)_- = 0, \quad \text{a.e. on } \Gamma.$$

Combining with the normalization condition, we have shown that

$$m_1 m_2^{-1} - I = 0,$$

i.e.,

$$m_1 = m_2 \tag{3.14}$$

in the L^2 sense. □

Corollary 3.12. *For the* 2×2 *matrix case and* $\det v = 1$*, the solution for the normalized RHP is unique.*

Proof. Since the $\det v = 1$, $\det m$ is entire and equals 1 by the normalization condition. By Louisville's argument, $\det m = 1$ for all z. Hence the inverse of m is

$$\begin{pmatrix} m_{22} & -m_{12} \\ -m_{21} & m_{11} \end{pmatrix},$$

and so the boundary values belong to $I + \partial C(L^2)$. Then by previous proposition, the solution is unique. □

The following proposition is crucial when obtaining the decomposition data, say $w^{\pm}$ from the scattering data v.

Proposition 3.13. *Suppose* $r \in L^2 \cap L^\infty$*, and* $\|r\|_{L^\infty} \leq \rho < 1$*. Then the* $IRHP1(1 - |r|^2, 1, \Gamma = [-z_0, z_0])$*, where* z_0 *is a positive real number, has a unique solution, denoted by* $\delta_\pm$*, is given by :*

$$\delta_\pm(z) = e^{C_\Gamma^\pm \log{(1-|r|^2)}}, \quad z \in \mathbb{R}. \tag{3.15}$$

Its extension of Γ *has the following properties:*

$$\delta(z)\bar{\delta}(\bar{z}) = 1, \tag{3.16}$$

$$|\delta^{\pm 1}(z)| \in [(1-\rho)^{1/2}, (1-\rho)^{-1/2}]. \tag{3.17}$$

For $z \in \mathbb{R}$*,*

$$|\delta_+(z)\delta_-(z)| = 1, \quad z \in R, \tag{3.18}$$

$$|\delta_\pm(z)| = 1, \quad z \in \mathbb{R}\backslash\Gamma, \tag{3.19}$$

$$|\delta_\pm(z)| = (1-|r|^2)^{1/2}, \quad z \in \Gamma. \tag{3.20}$$

Moreover,

$$\|\delta_\pm - 1\|_{L^2} \leq \frac{c\|r\|_{L^2}}{1-\rho}. \tag{3.21}$$

Proof. Since $|r| < 1$, the winding number of the $IRHP1$ is zero, which implies there exists a unique solution. Then by the Plemelj's formula, it is easy to get

$$\delta_\pm = e^{C_\Gamma^\pm \log(1-|r|^2)}, \quad z \in \mathbb{R}. \tag{3.22}$$

Here we have used the fact the Cauchy transform of an analytic function is 0.

Next to check the extension $\delta(z) = e^{C_\Gamma(\log(1-|r|^2))}$, since $\log(1-|r|^2)$ is real for any z,

$$\overline{\frac{1}{2\pi i}\int_\Gamma \frac{\log(1-|r|^2)}{s-\bar{z}}ds} = -\frac{1}{2\pi i}\int_\Gamma \frac{\log(1-|r|^2)}{s-z}ds,$$

thus,

$$\delta(z)\bar{\delta}(\bar{z}) = 1.$$

Setting $z = a + ib$. Then

$$\begin{aligned}
|\delta(z)| &= e^{\frac{1}{2\pi}\int_{-z_0}^{z_0} \frac{b\log(1-|r(s)|^2)}{(s-a)^2+b^2}ds} \\
&= e^{\frac{1}{2\pi}\int_{-z_0}^{z_0} \frac{\log(1-|r(sb+a)|^2)}{s^2+1}ds} \\
&\geq e^{\frac{1}{2\pi}\int_{-z_0}^{z_0} \frac{\log(1-\rho^2)}{s^2+1}ds} \\
&= (1-\rho^2)^{\frac{\arctan(z_0)}{\pi}} \\
&\geq (1-\rho^2)^{1/2} \\
&\geq (1-\rho)^{1/2}.
\end{aligned}$$

Here we have used $\rho < 1$, $1-\rho^2 \geq 1-\rho > 0$. Since $\delta(z)\bar{\delta}(\bar{z}) = 1$, we have $|\delta^{-1}| \leq (1-\rho)^{-1/2}$.

Now back to the boundary value on the real line, again by the Plemelj formula, $C^{\pm} = \pm\frac{1}{2} - \frac{1}{2}H$, and $C^{+} + C^{-} = -H$, and knowing that H acts on a real valued function one gets a pure imaginary number, and thus

$$|\delta_{+}\delta_{-}| = e^{\mathrm{Im}(-H(\log(1-|r|^2)))} = 1, \tag{3.23}$$

from which it immediately follows that

$$|\delta_{\pm}(z)| = 1, \quad z \in \mathbb{R}\backslash\Gamma, \tag{3.24}$$

$$|\delta_{\pm}(z)| = (1-|r|^2)^{1/2}, \quad z \in \Gamma. \tag{3.25}$$

To show the last inequality, note the fact that for $z \in \mathbb{C}$,

$$|e^{z} - 1| \leq |z|e^{\mathrm{Re}(z)}. \tag{3.26}$$

Thus,

$$\begin{aligned}
\|\delta_{\pm} - 1\|_{L^2} &= \|e^{C^{\pm}\log(1-|r|^2)} - 1\|_{L^2} \\
&\leq \|C^{\pm}\log(1-|r|^2)\|_{L^2} \\
&= \|\log(1-|r|^2)\|_{L^2} \\
&= (\int_{\mathbb{R}} (\log(1-|r|^2))^2 dz)^{\frac{1}{2}} \\
&\leq c(\int_{\mathbb{R}} (\log(1-|r|))^2 dz)^{\frac{1}{2}} \\
&\leq c(\int_{\mathbb{R}} (\frac{|r|}{1-|r|})^2 dz)^{\frac{1}{2}} \\
&\leq c(\int_{\mathbb{R}} (\frac{|r|}{1-\rho})^2 dz)^{\frac{1}{2}} \\
&= \frac{c\|r\|_{L^2}}{1-\rho}.
\end{aligned}$$

□

3.4 Inverse scattering problem

In this section, we will begin with the $IRHP1(v_x, I, \mathbb{R})$, using the operator C_w, construct the solution of the RHP, and then the potential q. Here $v_x(z) = v(z)$ from the equation (2.38). Note that $v_x(z)$ has the following upper/lower triangular factorization:

$$v = \begin{pmatrix} 1 & -re^{2izx} \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ -\bar{r}e^{-2ixz} & 1 \end{pmatrix},$$

from which we construct

$$w_x^- = \begin{pmatrix} 0 & re^{2izx} \\ 0 & 0 \end{pmatrix},$$

$$w_x^+ = \begin{pmatrix} 0 & 0 \\ -\bar{r}e^{-2ixz} & 0 \end{pmatrix}.$$

Then it is easy to check that $C_{w_x(z)}$ is a bounded operator in $L^2(dz)$ due to the boundedness of $r, \bar{r}$. In the focusing case, we always have $\|r\|_{L^\infty} \leq \rho < 1$, which implies that $\|C_{w_x}\|_{L^2} \leq \rho$ which further implies that $(1 - C_{w_x})^{-1}$ exists and its L^2 bound is $\frac{1}{1-\rho}$, together with Proposition 3.10, we have the following proposition:

Proposition 3.14. *$IRHP(v_x, I, \mathbb{R})$ has a unique solution for each $x \in \mathbb{R}$ and the solution can be represented by*

$$m_\pm(x, z) = I + C^\pm(\mu(w_x^+ + w_x^-)) = I + C^\pm(\mu(v_x^+ - v_x^-)),$$

where $\mu \in I + L^2$ solves $(1 - C_{w_x})\mu = I$ uniquely.

Proof. By the proposition (3.10) and the fact $\|r\|_{L^\infty} \leq \rho < 1$, the proposition follows. □

Also, since the the solution to the RHP solves the ODE(2.2), and since $r \in L^2$, by the Riemann-Lebesgue lemma, $m \to I, \quad x \to \pm\infty$. And taking the limit as x goes to infinity in the ODE(2.2), we have

$$Q(x) = -i[\sigma_3, m_1], \tag{3.27}$$

where we have assumed the expansion of m as $z \to \infty$, $m(x, z) = I + m_1(x)/z + O(z^{-2})$.

So the potential q can be recovered by

$$q(x) = 2i(m_1(x))_{12}. \tag{3.28}$$

Now we are in the position to study the inverse map:

$$\mathcal{I} : r \in H^{j,k}(dz) \mapsto q \in H^{k,j}(dx). \tag{3.29}$$

Theorem 3.15. *If* $r \in H^{j,k}$. *Then* $q = \mathcal{I}(r) \in H^{k,j}$.

Before we prove the theorem, we first prove some lemmas.

Lemma 3.16. *For* $x < 0$,

$$\|C_{w_x} I\|_{L^2} \leq 2(1 + x^2/\pi^2)^{-j/2} \|r\|_{H^{j,0}}.$$

Proof. First we consider the $C^+_{w_x^-}$, where only the $(1,2)$ entry is nonzero, which is $r(z)e^{2izx}$. Then by Plancherel's theorem,

$$\begin{aligned}
\|C^+(r(s)e^{2isx})\|_{L^2(ds)} &= \|\mathcal{F}[C^+(r(s)e^{2isx})]\|_{L^2(ds)} \\
&= \| \int_0^\infty \hat{r}(s - x/\pi) e^{2\pi s z} ds \|_{L^2(dz)} \\
&\leq \| \int_{-x/\pi}^\infty \hat{r}(s) e^{2\pi i (s + x/\pi)} ds \|_{L^2(dz)} \\
&\leq (\int_{-x/\pi}^\infty |\hat{r}(s)|^2 ds)^{\frac{1}{2}} \\
&\leq (\int_{-x/\pi}^\infty (1+s^2)^{-j} (1+s^2)^j |\hat{r}(s)|^2 ds)^{\frac{1}{2}} \\
&\leq (1 + x^2/\pi^2)^{-j/2} \|\hat{r}\|_{H^{0,j}} \\
&\leq (1 + x^2/\pi^2)^{-j/2} \|r\|_{H^{j,0}}.
\end{aligned}$$

Also, similarly, we have

$$\|C^-(-\bar{r}(s)e^{-2isx})\|_{L^2(ds)} \leq (1 + x^2/\pi^2)^{-j/2} \|r\|_{H^{j,0}}.$$

Thus noting that $C_{w_x} I = C^+ w_- + C^- w_+$, the lemma follows. ☐

Lemma 3.17. *If* $r \in H^{1,0}_1$. *Then* $q \in H^{0,1}$.

Proof. By analytic continuation, $m = I + C(\mu(w_+ + w_-))$ and $Q = \lim_{z\to\infty} iz[m, \sigma_3] = \frac{\mathrm{ad}(\sigma)}{2\pi} \int_{\mathbb{R}} \mu(w_x^+ + w_x^-)$, and so it is sufficient to show that $\int_{\mathbb{R}} \mu w_x^+ dz \in L^2((1+x^2)dx)$. In fact,

$$\begin{aligned}
x \int_{\mathbb{R}} \mu w_x^+ &= x \int (\mu - I) w_x^+ + x \int w^+ \\
&= I_1 + I_2 \\
&= x \int (1 - C_{w_x})^{-1} (C_{w_x} I) w_x^+ + x \int w^+.
\end{aligned}$$

Since $r \in H^{1,0}$, so $x\hat{r}(x) \in L^2$, which implies that $I_2 = x\int w^+ \in L^2(dx)$. For I_1, note that via a limiting process, we know that the weak derivative $\frac{d}{dz}\mu = (1 - C_{w_x})^{-1}(C_{\frac{dw_x}{dz}}I)$ belongs to L^2. And then we have

$$\begin{aligned}
\|x\int(\mu - I))w_x^+\|_{L^2(dx)} &= c\|x\mathcal{F}[(\mu - I))r(z)](x)\|_{L^2(dx)} \\
&= c\|\partial_z((\mu - I))r(z))\|_{L^2(dz)} \\
&\leq c\|(\partial_z\mu)r(z)\|_{L^2} + c\|(\mu - I)\partial_z r(z)\|_{L^2}.
\end{aligned}$$

To estimate the first integral, note that by a limit process in the Sobolev space, we can determine the L^2 weak derivative $\partial_z\mu = (1 - C_{w_x})^{-1}(C_{\partial_x w_x}I)$, and again due to that $r \in H^{1,0}$, the weight function $\partial_z w_x$ also belongs to L^2, so by the boundedness of the Cauchy operator, we have $\partial_z\mu \in L^2$. The second integral is finite since $r \in H^{1,0}$ and due to the Sobolev embedding, and Lemma 3.16, we have

$$\|\mu - I\|_{L^\infty} \leq \|\mu - I\|_{L^2} + \|\partial_z\mu\|_{L^2}.$$

Thus we have shown that the second integral is also bounded in L^2. Similarly, we can show $x\int \mu w_x^+ \in L^2(dx)$ then by triangularity, we see that $q \in H^{0,1}$. $\square$

Lemma 3.18. *If $r \in H_1^{1,1}$. Then $q \in H^{1,1}$.*

Proof. By the Dominated Convergence theorem, we have

$$\partial_x\mu = (iz\text{ad}\sigma_3 + Q)\mu.$$

Thus,

$$\begin{aligned}
\partial_x(\mu(w_x^+ + w_x^-)) &= \partial_x\mu(w_x^+ + w_x^-) + \mu\partial_x(w_x^+ + w_x^-) \\
&= (iz\text{ad}\sigma_3 + Q)\mu(w_x^+ + w_x^-) + \mu\text{ad}\sigma_3(w_x^+ + w_x^-) \\
&= (iz\text{ad}\sigma_3 + Q)(\mu(w_x^+ + w_x^-)),
\end{aligned}$$

which in turn implies

$$Q' = \frac{\text{ad}\sigma_3}{2\pi}Q\int\mu(w_x^+ + w_x^-)dz + \frac{\text{ad}\sigma_3}{2\pi}\int iz\text{ad}\sigma_3\mu(w_x^+ + w_x^-).$$

The first term is in L^2 since $Q \in L^2$ and since $r \in H^{0,1}$, we have

$$\begin{aligned}
|\int \mu w_x^+| &\leq |\int (\mu - I) w_x^+| + |\int w_x^+| \\
&\leq \|\mu - I\|_{L^2} \|r(z)\|_{L^2} + \|\frac{1}{(1+z^2)^{1/2}}\|_{L^2} \|r(z)\|_{L^2((1+z^2)dz)} \\
&\leq \|(1 - C_{w_x})^{-1} C_{w_x} I\|_{L^2} \|r(z)\|_{L^2} + \sqrt{\pi} \|r(z)\|_{L^2((1+z^2)dz)}.
\end{aligned}$$

Then by triangularity, the first term is in L^2.

For the second term, knowing that the function of adσ_3 only generates a constant, it is sufficient to show $\int \mu z w_x^+ dz$ in L^2. Actually, applying the Cauchy-Schwartz inequality, we can compute

$$\begin{aligned}
\|\int \mu z w_x^+\|_{L^2(dx)} &= \|\int (\mu - I) z r(z) e^{2izx} dz\| + |\int z r(z) e^{2izx} dz| \\
&\leq \big\| \|\mu - I\|_{L^2} \|zr\|_{L^2(dz)} \big\|_{L^2(dx)} + \|\mathcal{F}[zr]\|_{L^2}(dx) \\
&\leq \|\frac{1}{(1+x^2)^{1/2}}\|_{L^2(dx)} + \|zr\|_{L^2(dz)} \\
&< \infty.
\end{aligned}$$

So we have shown that $q \in H^{1,0}$. Together with the previous lemma, the whole proof is done. ☐

Proof. [Theorem 3.15] The proof follows mainly from the previous two lemmas and performing integration by parts. The rest is trivial. ☐

Remark 3.19. Here the RHP is established based on Proposition 2.2. However, there is a counterpart problem when $x \to \infty$, and the new RHP whose scattering data has a opposite factorization. Combining with the estimate of δ, one can move all the proofs for $x \to -\infty$ to the $x \to \infty$ case. We will skip the proof.

Finally, we will show that the maps $\mathcal{D}$ and $\mathcal{I}$ are bijections and inverse of each other. Let's first consider the case for $x \to \infty$, we have a new $IRHP1(\tilde{v}_x, I, \mathbb{R})$, where $\tilde{v}_x = \delta_-^{\sigma_3} v_x \delta_+^{\sigma_3}$. Let $\tilde{m}$ be the solution to the $IRHP1(\tilde{v}_x, I, \mathbb{R})$, and then since both $m, \tilde{m}$ solve the fundamental ODE 2.2, it is easy to check that they are connected by the following relation:

$$m = \tilde{m} \delta^{\sigma_3}. \tag{3.30}$$

This relation gives us that m is bounded as $x \to \infty$, by definition, such m is a unique solution to the fundamental ODE 2.2. That means for any given r, which is contained in the scattering matrix v_x, one can always find a solution m and hence the potential such that both solve the fundamental ODE. This means the map $\mathcal{D}$ is onto. The uniqueness comes from the fact that $\|r\|_{L^\infty} \leq \rho < 1$ so Ker$(1 - C_{w_x}) = \{0\}$. Then apply Proposition 3.10, the solution to the $IRHP1(v_x, I, \mathbb{R})$ is unique for given r, and so is q which is uniquely determined by the solution to the $IRHP1(v_x, I, \mathbb{R})$.

Chapter 4

Asymptotic analysis of the Oscillatory Riemann-Hilbert Problem

4.1 Introduction

In this chapter, we will study the phase function θ(or weight function) with the following setup:

1). θ is a real polynomial of degree N.

2). $\theta'(z_j) = 0, \theta''(z_j) \neq 0$ for $j = 1, \cdots, l$.

Now let us begin with a RHP, in terms of the previous notation, $IRHP1(v_\theta, I, \mathbb{R})$. And denote the solution by $m^{[0]} \in I + \partial C(L^2(\mathbb{R}))$. We will first consider the defocusing case and then the focusing case by an algebraically modification. So we begin with the jump matrix

$$v_\theta(z) = \begin{pmatrix} 1 - |R(z)|^2 & -\bar{R}(z)e^{-2it\theta(z)} \\ R(z)e^{2it\theta(z)} & 1 \end{pmatrix} \tag{4.1}$$

In the following sections, we will execute several steps which have been commonly shown in many literatures. The first step is called conjugation, which coincides with the previous chapters. After conjugation, we are able to to factorize the jump matrix into lower/upper or upper/lower matrices whose diagonals are all one and the exponential terms will decay (as $t \to \infty$) due to the signature of $\mathrm{Re}(it\theta)$. The next step is the so-called "lenses opening". In each interval where θ is monotonic, we can deform those intervals into new contours off the real line such that those exponential terms will decay as t goes to infinity. Also in this step, we will use a $\bar{\partial}$-RHP. In the spirit of the method of steepest descent, the asymptotics are dominated near those saddle points. Hence the next step is to study how one can separate the contributions from each saddle point. Then in each saddle point, one can study the RHP part of the $\bar{\partial}-$RHP through the so-called small norm technique. The last step is to estimate the errors from the pure $\bar{\partial}$ problems which dominate the errors generated from the small norm approximation. Then we conclude this chapter by representing a general formula for our setup of the phase function.

4.2 Conjugation

As we have already knew that that our jump matrix can be factorized into upper/lower triangular matrices in the interval where θ is increasing and lower/diagonal/upper matrices in where θ is decreasing. So let us denote $D_{\pm} = \{z \in \mathbb{R} : \pm\theta'(z) > 0\}$. To eliminate the diagonal matrix in the second factorization, we introduction a new scalar RHP, $IRHP1((1-|R|^2)\chi_{D_-} + \chi_{D_+}, I, \mathbb{R})$, whose solution is denoted by δ as usual. Then we conjugate v_θ by δ and arrive at a new RHP, $IRHP1(\delta_-^{\sigma_3} v_\theta \delta_+^{\sigma_3}, I, \mathbb{R})$, and denoting the solution by $m^{[1]}(z) \in I + \partial C(L^2)$.

Here δ shares all the properties shown previously in the last chapter. In this section, we will study $IRHP1((1-|R|^2)\chi_{D_-} + \chi_{D_+}, I, \mathbb{R})$ in more detail.

This scalar RHP can be easily solved by the Plemelj formula, one can obtain

$$\ln(\delta(z)) = (C_{D_-}(\ln(1-|R|^2)))(z), z \in \mathbb{C}\backslash D_-, \tag{4.2}$$

where the Cauchy operator $C_{D_-} f = \frac{1}{2\pi i}\int_{D_-} \frac{f(s)}{s-z} ds$. Now if we assume $R \in H_1^{1,1}(\mathbb{R})$. Then it is obvious that $\ln(1-|R|^2)$ is in $H^{1,0}$. Then by the Sobolev embedding, we know it is also Hölder continuous with index $1/2$. Then the Plemelj-Privalov theorem, which says that the Cauchy operator perseveres H Older continuity with index less than 1, tells us $\ln(\delta(z))$ is Hölder continuous with index $1/2$ except for those end points. So we need to study the behavior near those points. Let us denote

$$\eta(z) = -\frac{1}{2\pi}\ln(1-|R(z)|^2), z \in \mathbb{R}. \tag{4.3}$$

We will prove the following proposition.

First we define a function supported on the interval $[-1, 1]$,

$$s_\epsilon(z) = \begin{cases} 0, & |z| \geq \epsilon, \\ \pm\frac{1}{\epsilon}z + 1, & 0 < \pm z < \epsilon. \end{cases} \tag{4.4}$$

Proposition 4.1. *For each $\epsilon > 0$, and $\epsilon \leq \frac{1}{3}\min_{j\neq k}|z_j - z_k|$,there exists an interval $I = I(\epsilon)$, such that* *the identity*

$$\ln(\delta(z)) = i\int_{D_-\setminus I}\frac{\eta(s)}{s-z}ds + i\sum_{j=1}^{l}\left[\eta(z_j)(1+\ln(z-z_j))\right]\varepsilon_j \tag{4.5}$$

$$+\, i\sum_{j=1}^{l}\int_{I\cap D_-}\frac{\eta(s)-\eta_j(s)}{s-z}ds \tag{4.6}$$

$$+\, i\sum_{j=1}^{l}\frac{1}{\epsilon}\eta(z_j)[(z-z_j)\ln(z-z_j)-(z-z_j+\varepsilon_j\epsilon)\ln(z-z_j+\varepsilon_j\epsilon)] \tag{4.7}$$

is true, where $\varepsilon_j = sgn(\theta''(z_j))$ *and* $\eta_j(z) = \eta(z_j)s_\epsilon(z-z_j)$ *and for the logarithm function, the branch is chosen such that* $arg \in (-\pi,\pi)$.

Proof. Let $I = \cup_{j=1}^{l}(I_{j+}\cup I_{j-})$, where $I_{j\pm} = \{z : 0 < \pm(z-z_j) < \epsilon\}$. Now we have

$$\ln(\delta(z)) = i\int_{D_-\setminus I}\frac{\eta(s)}{s-z}ds$$
$$+\, i\sum_{j=1}^{l}\Big(\int_{I_{j-}\cap D_-}+\int_{I_{j+}\cap D_-}\frac{\eta(s)}{s-z}ds\Big).$$

For each j, we have

$$\int_{I_{j-}}\frac{\eta(s)}{s-z}ds = \int_{I_{j-}}\frac{\eta(s)-\eta_j(s)}{s-z}ds + \int_{I_{j-}}\frac{\eta_j(s)}{s-z}ds.$$

The first integral on the right hand side has non-tangential limit as $z\to z_j$ and the second one generates a logarithm singularity near z_j.

In fact, direct computation shows

$$\int_{I_{j-}}\frac{\eta_j(s)}{s-z}ds = \eta(z_j) + \frac{1}{\epsilon}[(z-z_j)\ln(z-z_j)-(z-z_j+\epsilon)\ln(z-z_j+\epsilon)]\eta(z_j)$$
$$+\,\eta(z_j)\ln(z-z_j).$$

Similarly, for I_{j+},

$$\int_{I_{j+}}\frac{\eta_j(s)}{s-z}ds = -\eta(z_j) + \frac{1}{\epsilon}[(z-z_j)\ln(z-z_j)-(z-z_j-\epsilon)\ln(z-z_j-\epsilon)]$$
$$-\,\eta(z_j)\ln(z-z_j).$$

And noting that only one of the $I_{j\pm}\cap D_-$ is nonempty, which depends on the sign of the second derivative of the phase function θ. Assembling all together, the proof is done. $\square$

Remark 4.2. The above proposition tells us how the function $\delta(z)$ behaviors near those saddle points. In fact, near those saddle points, there are mild singularities $(z-z_j)^{i\eta(z_j)}$. Fortunately, those singularities are bounded along any rays off $\mathbb{R}$ and hence in some sense they do not affect asymptotics much. Also it is worth mentioning that this general treatment has been taken in [24] and [15] before.

Let us denote $v^{[1]}(z) = \delta_-^{\sigma_3} v_\theta \delta_+^{\sigma_3}$. Now the conjugated jump matrix $v^{[1]}$ enjoys the following factorization:

$$v^{[1]} = \begin{cases} \begin{pmatrix} 1 & -\bar{R}(z)\delta^2(z)e^{-2it\theta(z)} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ R(z)\delta^{-2}(z)e^{2it\theta(z)} & 1 \end{pmatrix}, & z \in D_+ \\ \begin{pmatrix} 1 & 0 \\ \frac{R(z)\delta_-^{-2}(z)e^{2it\theta(z)}}{1-|R(z)|^2} & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{\bar{R}(z)\delta_+^2(z)e^{-2it\theta(z)}}{1-|R(z)|^2} \\ 0 & 1 \end{pmatrix}, & z \in D_-. \end{cases}$$

4.3 Lenses opening

The purpose of opening lenses is to deform the real line to some new contours where the exponential terms will decay as $t \to \infty$. We begin with a study of the signature of $\operatorname{Im}\theta$ near the saddle points z_j. Since there

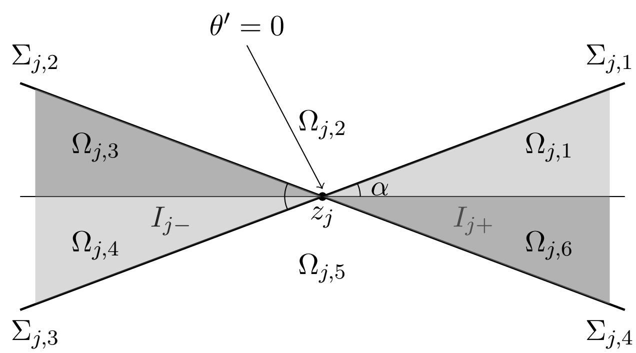

Figure 1. Notations for studying signatures of $\operatorname{Im}(\theta(z))$ near z_j

are only a finite number of saddle points, we choose α sufficiently small such that $\operatorname{Im}(\theta)$ shares the same sign in the shaded region. Let $I_{j+} = [z_j, \frac{z_j+z_{j+1}}{2}]$ and $I_{j-} = [\frac{z_j+z_{j-1}}{2}, z_j]$. Two cases need to be discussed. The first case, if $\theta''(z_j) > 0$. Then we have $I_{j\pm} \subset D_\pm$. Recall the factorization of the conjugated jump matrix, and to deform I_{j+} to $\Sigma_{j,1}$ and keep the exponential term ($e^{2iy\theta(z)}$) decay, we need to discuss $\operatorname{Im}\theta$ on $\Sigma_{j,1}$. Consider the Taylor expansion of $\theta(z)$ at z_j, and we denote $\theta(z) = \theta(z_j) + \varepsilon_j A_j (z-z_j)^2 + O((z-z_j)^3)$, where $A_j = \left|\frac{\theta''(z_j)}{2}\right|$.

Let $z - z_j = u + iv = \rho e^{i\phi}$. Then $\mathrm{Im}(\theta(z)) = \varepsilon_j A_j \rho^2 \sin(2\phi) + O(\rho^3)$, where $\phi \in (0, \alpha]$ is fixed. Since α is sufficiently small, say less then $\pi/2$. Then 2ϕ will be less than $\pi/2$. Note that we have

$$\rho^2 \sin(2\phi) = 2uv.$$

And for those higher order terms, we have

$$\rho^k \sin(k\phi) \geq c\rho^k \sin(\phi)\cos^{k-1}(\phi) = cu^{k-1}v.$$

Putting all together, we conclude that $|e^{2it\theta(z)}|$ is dominated by $e^{-4t|A_j|uv}$. Similarly, one can show in all other contours $\Sigma_{j,k}, k = 2, 3, 4$, the exponential terms will decay as $t \to \infty$. And now we are in the position to open lenses.

First we introduce a bounded smooth function $\mathcal{K}$ defined on $[0, \alpha]$ such that $\mathcal{K}(0) = 1$ and $\mathcal{K}(\alpha) = 0$. Consider $\varepsilon_j = 1$ first. And $\bar{\partial}$ extension functions are defined as follows. Let $z - z_j = u + iv, |z - z_j| = \rho$ and the $\arg z - z_j$ at the right upper corner is denoted by ϕ, then we can define

$$E_{j,1}(z) = \mathcal{K}(\phi)R(u + z_j)\delta^{-1}(z) + [1 - \mathcal{K}(\phi)]R(z_j)\delta_j^{-2}(z - z_j)^{-2i\varepsilon_j\eta(z_j)}, \quad z \in \Omega_{j,1}, \tag{4.8}$$

$$E_{j,3}(z) = \mathcal{K}(\pi - \phi)(-\frac{\bar{R}(u + z_j)}{1 - |R(u + z_j)|^2}\delta_+^2(z)) + [1 - \mathcal{K}(\pi - \phi)](-\frac{\bar{R}(z_j)}{1 - |R(z_j)|^2}\delta_j^2(z - z_j)^{2i\varepsilon_j\eta(z_j)}), \quad z \in \Omega_{j,3}, \tag{4.9}$$

$$E_{j,4}(z) = \mathcal{K}(\pi + \phi)(\frac{R(z_j + u)}{1 - |R(z_j + u)|^2}\delta_-^{-2}(z)) + [1 - \mathcal{K}(\pi + \phi)](\frac{R(z_j)}{1 - |R(z_j)|^2}\delta_j^{-1}(z - z_j)^{-2i\varepsilon_j\eta(z_j)}), \quad z \in \Omega_{j,4}, \tag{4.10}$$

$$E_{j,6}(z) = \mathcal{K}(-\phi)(-\bar{R}(z_j + u)\delta^2(z)) + [1 - \mathcal{K}(-\phi)](-\bar{R}(z_j)\delta_j(z - z_j)^{2i\varepsilon_j\eta(z_j)}), \quad z \in \Omega_{j,6}. \tag{4.11}$$

For the case $\varepsilon_j = -1$, one only needs to switch the index 1 with 3 and 4 with 6 due to the difference of factorization, based on the local monotonicity of the phase function. The boundary values of those $E_{j,k}, k = 1, 3, 4, 6$ at $\mathbb{R}$ are just the original conjugated jump matrices. And the boundary values on the new contours are just some function with a mild singularity compared to the exponential decay. Now using those

$E_{j,k}$, we construct the lens-opening matrix $O(z)$ as follows:

$$O(z) = \begin{cases} O_n(z) = \begin{pmatrix} 1 & 0 \\ (-1)^n E_{j,n} e^{2it\theta(z)} & 1 \end{pmatrix}, \quad z \in \Omega_{j,n}, \quad n = 1, 4, \\ O_m(z) = \begin{pmatrix} 1 & (-1)^m E_{j,m} e^{-2it\theta(z)} \\ 0 & 1 \end{pmatrix}, \quad z \in \Omega_{j,m}, \quad m = 3, 6, \\ O_k(z) = I, \quad z \in \Omega_{j,k}, \quad k = 2, 5. \end{cases} \tag{4.12}$$

Now let $m^{[2]}(z) = m^{[1]}(z)O(z), z \in \mathbb{C}\backslash\mathbb{R}$, due to a lack of analyticity of $O(z)$, we arrive at a mixed $\bar{\partial}$-RHP:

1. The RHP

   (1.a). $m^{[2]}(z) = m^{[2]}(u, v) \in C^1(\mathbb{R}^2\backslash\Sigma)$ and $m^{[2]}(z) = I + \mathcal{O}(z^{-1}), z \to \infty$.

   (1.b). On the new contour $\Sigma_{j,k}, j = 1, \cdots, l, k = 1, 2, 3, 4$, $v^{[2]}(z) = O_j(z), z \in \Sigma_{j,k}$.

2. The $\bar{\partial}$ problem

   For $z \in \mathbb{C}$, we have

$$\bar{\partial} m^{[2]}(z) = m^{[2]}(z)\bar{\partial}O(z). \tag{4.13}$$

Remark 4.3. By multiplying $m^{[1]}(z)$ by $O(z)$, we can actually remove the jumps on the real line, where the exponential factor $e^{\pm 2it\theta(z)}$ is oscillating. The regularity of $m^{[2]}$ is determined by the regularity of $O(z)$ which is inherited from the construction of $E_{j,k}$, and the fact $R(z_j + u)$ is no longer analytic but is C^1 in the weak sense. Moreover, due to the boundedness of $E_{j,k}(z)$ along any non-real ray, and the fact the exponential factors are all exponential decaying as $z \to \infty$, we will have $O(z) = I + o(1), z \to \infty$, i.e., uniformly in t. In the next sections, we will see the error eventually dominated by the $\bar{\partial}$ problem.

To close the section, we formulate a bound for $\bar{\partial}E_{j,k}$ which will be used in later sections.

Lemma 4.4. *For $j = 1, \cdots, l,\ k = 1, 2, 3, 4$, and $z \in \Omega_{j,k}, u = \mathrm{Re}(z - z_j)$,*

$$|\bar{\partial}E_{j,k}(z)| \le c(|z - z_j|^{-1/2} + |R'(u + z_j)|). \tag{4.14}$$

Proof. In polar coordinates, $\bar{\partial} = \frac{e^{i\phi}}{2}(\partial_\rho + i\rho^{-1}\partial_\phi)$. And thus for z in any ray (not parallel to $\mathbb{R}$) and away from z_j, we have

$$\begin{aligned} \bar{\partial}E_{j,1}(z) = &\frac{ie^{i\phi}\mathcal{K}'(\phi)}{2\rho}[R(u + z_j)\delta^{-2}(z) - R(z_j)\delta_j^{-2}(z - z_j)^{-2i\eta(z_j)}] \\ &+ \mathcal{K}(\phi)R'(u + z_j)\delta^{-2}(z), \end{aligned}$$

where

$$\delta_j = \lim_{\substack{z=z_j+\rho e^{i\phi},\\ \rho\to 0,\\ \phi\in(0,\pi/2)}} \delta(z)(z-z_j)^{i\eta(z_j)}.$$

From the Proposition 4.1, one can easily see that $|\delta(z)-\delta_j(z-z_j)^{i\eta(z_j)}| \le c|z-z_j|^{1/2}$. In fact,

$$\begin{aligned}
|\delta(z)-\delta_j(z-z_j)^{i\eta(z_j)}| &\le |\ln(\delta(z))-\ln(\delta_j)-i\eta(z_j)\ln(z-z_j)| \\
&\le |\int_{D_-\setminus I}\frac{\eta(s)}{s-z}ds+\sum_{k\neq j}^{l}\int_{I\cap D_-}\frac{\eta(s)-\eta_k(s)}{s-z}ds \\
&+\sum_{k\neq j}^{l}\frac{1}{\epsilon}\eta(z_k)[(z-z_k)\ln(z-z_k)-(z-z_k+\varepsilon_k\epsilon)\ln(z-z_k+\varepsilon_j\epsilon)]-\ln(\delta_j)|.
\end{aligned}$$

Since $R\in H^{1,1}$, from the standard Sobolev embedding, we know η is Hölder continuous with index $1/2$ and then an application of the Privalov-Plemelj theorem leads to the Hölder continuity with index $1/2$ of $\int_{D_-\setminus I}\frac{\eta(s)}{s-z}ds$. Similarly one can show $\int_{I\cap D_-}\frac{\eta(s)-\eta_k(s)}{s-z}ds$ is also Hölder continuous with index $1/2$. Now let us denote

$$g(z)=(z-z_k)\ln(z-z_k)-(z-z_k+\varepsilon_k\epsilon)\ln(z-z_k+\varepsilon_j\epsilon).$$

Direct computation (using the fact that $\ln(z)$ has a mild singularity at 0 which is integrable.) shows that $g'\in L^2$ along any rays that are not parallel to $\mathbb{R}$. And again by the Sobolev embedding, this term is also Hölder continuous with index $1/2$. All those three together eventually approach $\ln(\delta_j)$ and the rate of convergence is controlled by $|z-z_j|^{1/2}$.

Thus

$$\begin{aligned}
|\bar{\partial}E_{j,1}(z)| &\le c\rho^{-1}|z-z_j|^{1/2}+c|R'(u+z_j)| \\
&\le c(|z-z_j|^{-1/2}+|R'(u+z_j)|).
\end{aligned}$$

Note that $\rho=|z-z_j|$ and $\delta(z)$ and $\mathcal{K}(\phi)$ are bounded along any rays that is not parallel to $\mathbb{R}$.

Note also that $\sup|R|<1$, we have $\frac{R}{1-|R|^2}\le\frac{R}{1-\sup|R|}$, and thus by the dominated converge theorem, all estimations for $E_{j,1}$ can be smoothly moved to $E_{j,k}, k=3,4,6$. □

4.4 Separate Contributions and reduce the degree of the phase

Since there are multiple saddle points on the real line, we have to separate contributions from each saddle point. And at each saddle point, we may approximate the RHP by a model RHP locally which will be discussed in the next section. Also as we assumed all saddle points are of order 1, so the phase function can be approximated by $\theta(z_j) + \frac{\theta''(z_j)}{2}(z - z_j)^2$. Thus we need to estimate the error generated by reduce the order of the phase function. Since our phase function is a polynomial, we can always choose sufficient small α, such that the small triangular region along two saddle points shares the same signature of $\mathrm{Im}(\theta(z))$. And a difference between two situations of multiple saddle points and a single saddle point is that there are jumps in that small triangular region. So in this section, we will show that those jumps can be ignored with a sufficiently fast decaying error, say faster than the error generated from the pure $\bar{\partial}-$problem.

Let us consider two saddle points z_j, z_{j+1}, and discuss $\varepsilon_j = 1 = -\varepsilon_{j+1}$ for example.

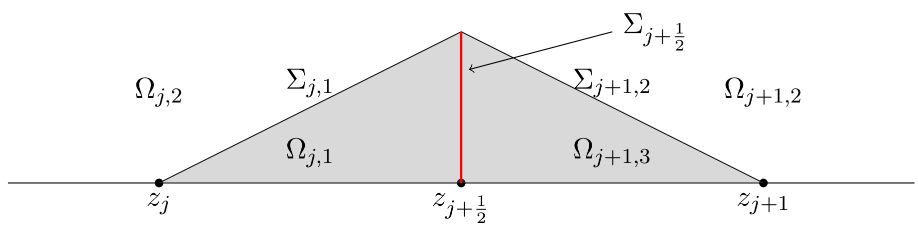

Figure 2. Jumps in a small triangular region.

Recall the constructions of $E_{j,1}$ and $E_{j+1,3}$, and the boundary value of $m^{[2]}(z)$ on $\Sigma_{j+\frac{1}{2}}$ from $\Omega_{j,1}$ is

$$m^{[1]}(z_{j+1/2} + iv)O_{j,1}(z_{j+1/2} + iv),$$

while from $\Omega_{j+1,3}$ is

$$m^{[1]}(z_{j+1/2} + iv)O_{j+1,3}(z_{j+1/2} + iv).$$

Both correspond to locally increasing part of the phase function, and thus correspond to a upper/lower factorization. So the jump on the new contour $\Sigma_{j+1/2}$ is $O_{j+1,3}O_{j,1}^{-1}$, where the nontrivial entry is

$$(1-\mathcal{K}(\phi))[R(z_j)\delta_j^{-2}(z_{j+1/2}-z_j+iv)^{-2i\eta(z_j)} - R(z_{j+1})\delta_{j+1}^{-2}(z_{j+1/2}-z_{j+1}+iv)^{-2i\eta(z_{j+1})}]e^{2it\theta(z_{j+1/2}+iv)},$$

where $v \in (0, (z_{j+1/2} - z_j)\tan(\alpha))$.

Note that

$$|(z_{j+1/2} - z_j + iv)^{-2i\eta(z_j)}| = e^{2\eta(z_j)\phi} \leq e^{2\eta(z_j)\alpha},$$

and

$$|e^{2it\theta(z_{j+1/2}+iv)}| \leq ce^{-2tdv}, \quad d = (z_{j+1} + z_j)/2.$$

We actually have shown that

$$O_{j+1,3}O_{j,1}^{-1} = I + \mathcal{O}(e^{-ct}), c > 0, \quad t \to \infty. \tag{4.15}$$

Since $m^{[1]}$ is analytic in a neighborhood of $\Sigma_{j+1/2}$, $O(z_{j+1/2} + iv)$ is at least C^1 with respect to v. Then we have

$$\begin{aligned}\lim_{z\to\infty} |z(m^{[2]}(z) - I)| &\leq \frac{1}{2\pi}\int_0^{d\tan(\alpha)} |m_-^{[1]}(z_{j+1/2} + s)|e^{-2tds}ds \\ &\text{by integration by parts since } O \text{ is } C^1 \text{ near } \Sigma_{j+1/2} \\ &= \mathcal{O}(t^{-1}).\end{aligned}$$

This limit is in fact from the construction of potential, and this estimation show that if we drop the contour $\Sigma_{j+1/2}$, the potential induced by the new RHP will generate an error term $\mathcal{O}(t^{-1})$, which is dominated by the error generated by the $\bar{\partial}-$problem. Actually, we will later show that the $\bar{\partial}-$problem will generate an error $\mathcal{O}(t^{-3/4})$.

For the triangular region below and for the cases when $\varepsilon_j = -1$, one can do slight modification to guarantee that the errors are still $\mathcal{O}(t^{-1})$. Thus we will remove the jumps on $\Sigma_{j+1/2}$ for the RHP of $m^{[2]}$.

Now for the $\bar{\partial}O_{j,1}$ and $\bar{\partial}O_{j+1,3}$,we have, for $z - z_j = \rho e^{i\phi}$, $\rho = d/\tan(\phi)$,

$$\begin{aligned}|\bar{\partial}Q_{j,1}| &\leq c_1|\frac{ie^{i\phi}\cos(\phi)}{d}\mathcal{K}'(\phi)| \\ &\leq c.\end{aligned}$$

The implicit constant comes from the fact that both $R(z)$ and $\delta(z)$ are bounded on the contour $\Sigma_{j+1/2}$. Comparing to the estimation on Lemma 4.4, we see that the $\bar{\partial}-$estimations on $\Sigma_{j+1/2}$ are also dominated by $|z - z_j|^{-1/2} + |R'(\mathrm{Re}(z))|$.

Moreover, one can actually drop segments away from the stationary phase points. It is well-known [11, 15] that the $|E_{j,1}e^{2it\theta}| \leq ce^{-2t\tan(\alpha)u^2}$, where letting $u \geq u_0 > 0$, then the jump matrix will go to I with a decaying rate at $\mathcal{O}(e^{-c_0t}), c_0 = c_0(u_0) > 0$. Together with the analysis of RHP on $\Sigma_{j+1/2}$, and a priori estimate that the pure $\bar{\partial}-$problem will generate an error $\mathcal{O}(t^{-3/4})$, we can in fact truncate our contours to a new one by dropping $\Sigma_{j+1/2}, j = 1, \cdots, l$. And we arrive at a new $\bar{\partial}-$RHP by simply dropping those contours which contribute less than the $\bar{\partial}-$problem. See Figure 3 about the new contours, without causing more confusion, we will still denote the new contours by $\Sigma_{j,k}, j = 1, c \ldots, l, k = 1, 2, 3, 4$.

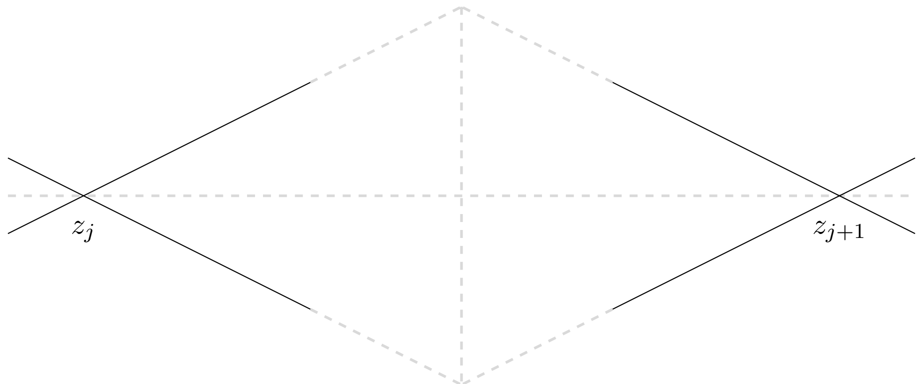

Figure 3. New contours, dashed line segments are those deleted parts.

Now we will separate contributions from the RHPs on each crosses. A countless number of literatures (especially, [11, 24, 15]) about the nonlinear steepest descent for cases of multiple stationary phase points, which are based on the analysis of the Beals-Coifman operators, have already shown the following lemma. For convenience, we will reprove it based on our settings.

Lemma 4.5. *As* $t \to \infty$*,*

$$\int_{\Sigma}((1-C_w)^{-1}\, I)w = \sum_{j=1}^{l}\int_{\Sigma_j}((1-C_{w_j})^{-1}I)w_j + \mathcal{O}(t^{-1}), \tag{4.16}$$

where w_j *is the factorization data supported on* $\Sigma_j = \cup_{k=1}^{4}\Sigma_{j,k}$*,* $w = \sum_{j=1}^{l} w_j$ *and* $\Sigma = \cup_j^l \Sigma_j$*.*

Proof. First, recall the following observation by Varzugin [24],

$$(1-C_w)(1+\sum_j C_{w_j}(1-C_{w_j})^{-1}) = 1 - \sum_{j\neq k} C_{w_j}C_{w_k}(1-C_{w_k})^{-1}.$$

With the hints from this observation, we need to estimate the norms of $C_{w_j}C_{w_k}$ from L^∞ to L^2 and from L^2 to L^2. Also from the next section (by a small norm argument), we know $(1-C_{w_j})^{-1}$ are uniformly bounded in L^2 sense. Now let us focus on the contour $\Sigma_{j,1}$, and $\varepsilon = 1$. Then the nontrivial entry of the factorization data is $E_{j,1}(z)e^{-2it\theta(z)}$, $z \in \Sigma_{j,1}$, and thus we have

$$|w_j \restriction_{\Sigma_{j,1}}| \le ce^{-2t\tan(\alpha)u^2},$$

which implies that $\|w_j \restriction_{\Sigma_{j,1}}\|_{L^1} = \mathcal{O}(t^{-1/2})$ and $\|w_j \restriction_{\Sigma_{j,1}}\|_{L^2} = \mathcal{O}(t^{-1/4})$. Then follow exactly the same steps in the proof of [11], Lemma 3.5, we have for $j \neq k$

$$\|C_{w_j} C_{w_k}\|_{L^2(\Sigma)} = \mathcal{O}(t^{-1/2}),$$
$$\|C_{w_j} C_{w_k}\|_{L^\infty \to L^2(\Sigma)} = \mathcal{O}(t^{-3/4}).$$

Then use the resolvent identities and Cauchy-Schwartz inequality,

$$\begin{aligned}((1-C_w)^{-1}I) = I &+ \sum_{j=1}^{l} C_{w_j}(1-C_{w_j})^{-1}I \\ &+ [1+\sum_{j=1}^{l} C_{w_j}(1-C_{w_j})^{-1}](1-\sum_{j\neq k} C_{w_j}C_{w_k}(1-C_{w_k})^{-1})^{-1} \\ &(\sum_{j\neq k} C_{w_j}C_{w_k}(1-C_{w_k})^{-1})I \\ &= I + \sum_{j=1}^{l} C_{w_j}(1-C_{w_j})^{-1}I + ABCI.\end{aligned}$$

thus

$$\begin{aligned}|\int_\Sigma ABCIw| &\leq \|A\|_{L^2}\|B\|_{L^2}\|C\|_{L^\infty\to L^2}\|w\|_{L^2} \\ &\leq ct^{-3/4}t^{-1/4} = \mathcal{O}(t^{-1}).\end{aligned}$$

In the proof over the L^2 boundedness, we actually use the triangularity of w_j's, which gives us a mild orthogonality [15]. Then applying the restriction lemma([11],Lemma 2.56), and also by the mild orthogonality, one can obtain

$$\begin{aligned}\int_\Sigma (I + C_{w_j}(1-C_{w_j})^{-1}I)w \restriction_{\Sigma_j} &= \int_{\Sigma_j} (I + C_{w_j}(1-C_{w_j})^{-1}I)w \\ &= \int_{\Sigma_j} ((1-C_{w_j})^{-1}I)w_j\end{aligned}$$

All together, the proof is done. □

Remark 4.6. However there is a hole in the proof, which based on a priori estimate on $\|1 - C_{w_j}\|_{L^2}$. We will show that those bounds are uniform and they can be approximated by solving an explicitly solvable model RHP in the following sections.

4.5 Model Riemann-Hilbert problem

In this section, we discuss a model RHP which can be solved explicitly by solving a parabolic-cylinder equation. Consider the following RHP:

1. $P_+(\xi;R) = P_-(\xi;R)J(R), \xi \in \mathbb{R}$, where

$$J(\xi) = \begin{pmatrix} 1-|R|^2 & -\bar{R} \\ R & 1 \end{pmatrix}$$

   is a constant matrix with respect to ξ.

2. $P(\xi;R) = \xi^{i\eta\sigma_3} e^{-i\frac{\xi^2}{4}\sigma_3}(I + P_1\xi^{-1} + \mathcal{O}(\xi^{-2})), \quad \xi \to \infty$, where $P_1 = \begin{pmatrix} 0 & \beta \\ \bar{\beta} & 0 \end{pmatrix}$

Then by Liouville's argument, $P'P^{-1}$ is analytic and thus

$$P'(\xi) = (-\frac{i\xi}{2}\sigma_3 - \frac{i}{2}[\sigma_3, P_1])P(\xi), \tag{4.17}$$

which can be solved in terms parabolic-cylinder equation, and apply the asymptotics formulas we can eventually determine that

$$\beta = \frac{\sqrt{2\pi}e^{i\pi/4}e^{-\pi\eta/2}}{R\Gamma(-a)}, \tag{4.18}$$

where

$$a = i\eta. \tag{4.19}$$

The above result has been presented in a considerable literature in many ways. Here we followed the representations in [11]. Next, we connect this model RHP to our RHP. Recall, near the stationary phase point z_j, we need to estimate the integral $\int_{\Sigma_j}((1-C_{w_j})^{-1}I)(w_{j+}+w_{j-})$, which is equivalent to solve the following RHP:

1. $m_+^{[3,j]}(z) = m_-^{[3,j]}(z)v^{[3,j]}(z), z \in \Sigma_j$. The jump matrix ($\varepsilon_j > 0$) is

$$v^{[3,j]}(z) = \begin{cases} \begin{pmatrix} 1 & 0 \\ R_j^\sharp (z-z_j)^{-2i\eta(z_j)} e^{it\theta''(z_j)(z-z_j)^2} & 1 \end{pmatrix}, z \in \Sigma_{j,1}, \\ \begin{pmatrix} 1 & -\frac{\bar{R}_j^\sharp}{1-|R_j^\sharp|^2}(z-z_j)^{2i\eta(z_j)} e^{-it\theta''(z_j)(z-z_j)^2} \\ 0 & 1 \end{pmatrix}, z \in \Sigma_{j,2}, \\ \begin{pmatrix} 1 & 0 \\ \frac{R_j^\sharp}{1-|R_j^\sharp|^2}(z-z_j)^{2i\eta(z_j)} e^{it\theta''(z_j)(z-z_j)^2} & 1 \end{pmatrix}, z \in \Sigma_{j,3}, \\ \begin{pmatrix} 1 & -\bar{R}_j^\sharp (z-z_j)^{-2i\eta(z_j)} e^{-it\theta''(z_j)(z-z_j)^2} \\ 0 & 1 \end{pmatrix}, z \in \Sigma_{j,4}, \end{cases} \tag{4.20}$$

where $R_j^\sharp = R_j \delta_j^{-2} e^{2it\theta(z_j)}$.

2. $m^{[3,j]} = I + \mathcal{O}(z^{-1}), \quad z \to \infty$.

Setting $\xi = (2t\theta''(z_j))^{1/2}(z - z_j)$ and by closing lenses, we arrive at an equivalent RHP on the real line:

1. $m^{[4,j]}(\xi)_+ = m_-^{[4]}v^{[4]}(\xi), \xi \in \mathbb{R}$. The new jump is

$$v^{[4,j]}(\xi) = (2\theta''(z_j)t)^{-\frac{i\eta(z_j)}{2}\,\mathrm{ad}\,\sigma_3} \xi^{i\eta(z_j)\,\mathrm{ad}\,\sigma_3} e^{-\frac{i\xi^2}{4}\,\mathrm{ad}\,\sigma^3} \begin{pmatrix} 1-|R_j^\sharp|^2 & -\bar{R}_j^\sharp \\ R_j^\sharp & 1 \end{pmatrix}. \tag{4.21}$$

2. $m^{[4,j]} = I + \mathcal{O}(\xi^{-1}), \xi \to \infty$.

Compared with the model RHP, we observe that $m^{[4,j]}(\xi)(2\theta''(z_j)t)^{-\frac{i\eta(z_j)}{2}\sigma_3}\xi^{i\eta(z_j)\sigma_3}e^{-\frac{i\xi^2}{4}\sigma_3}$ solves the model RHP, which leads to

$$m_{1,12}^{[4]} = \frac{\sqrt{2\pi}e^{i\pi/4}e^{-\pi\eta(z_j)/2}}{R_j^\sharp\Gamma(-i\eta(z_j))} \tag{4.22}$$

$$m_{1,21}^{[4]} = \frac{-\sqrt{2\pi}e^{-i\pi/4}e^{-\pi\eta(z_j)/2}}{\bar{R}_j^\sharp\Gamma(i\eta(z_j))} \tag{4.23}$$

Change the variable ξ back to z, we have

$$m_{1,12}^{[3]}(t) = \sum_{j=1}^{l}(2t\theta''(z_j))^{-\frac{1}{2}-\frac{i\eta(z_j)}{2}}\frac{\sqrt{2\pi}e^{i\pi/4}e^{-\pi\eta(z_j)/2}}{R_j^\sharp\Gamma(-i\eta(z_j))}, \tag{4.24}$$

$$m_{1,12}^{[3]}(t) = -\sum_{j=1}^{l}(2t\theta''(z_j))^{-\frac{1}{2}+\frac{i\eta(z_j)}{2}}\frac{\sqrt{2\pi}e^{-i\pi/4}e^{-\pi\eta(z_j)/2}}{\bar{R}_j^\sharp\Gamma(i\eta(z_j))}. \tag{4.25}$$

Note that $R_j^\sharp = R_j\delta_j^{-2}e^{2it\theta(z_j)}$, one can rewrite in a neat way:

$$m_{1,12}^{[3]}(t) = \sum_{j=1}^{l} \frac{|\eta(z_j)|^{1/2}}{\sqrt{2t\theta''(z_j)}} e^{i\varphi(t)}, \tag{4.26}$$

$$m_{1,21}^{[3]}(t) = \sum_{j=1}^{l} \frac{|\eta(z_j)|^{1/2}}{\sqrt{2t\theta''(z_j)}} e^{-i\varphi(t)}, \tag{4.27}$$

where the phase is

$$\varphi(t) = \frac{\pi}{4} - \arg\Gamma(-i\eta(z_j)) - 2t\theta(z_j) - \frac{\eta(z_j)}{2}\ln|2t\theta''(z_j)| + 2\arg(\delta_j) + \arg(R_j). \tag{4.28}$$

Here we have used the fact that $|\beta|^2 = \eta$. From the relation connecting the RHP and the potential, we have

$$q_{as}(x,t) = -2i\sum_{j=1}^{l} \frac{|\eta(z_j)|^{1/2}}{\sqrt{2t\theta''(z_j)}} e^{i\varphi(t)}, \tag{4.29}$$

The variable x is implicitly contained in z_j's.

4.6 Errors from pure $\bar{\partial}-$problem

In this section, we will discuss the error generated from the pure $\bar{\partial}-$problem of $m^{[2]}$. Let us denote

$$E(z) = m^{[2]}(m^{[3]})^{-1}. \tag{4.30}$$

Since $m^{[k]} = I + m_1^{[k]}z^{-1} + \mathcal{O}(z^{-2}), k = 2, 3$, we have

$$E(z) = 1 + (m_1^{[2]} - m_1^{[3]})z^{-1} + \mathcal{O}(z^{-2}), \tag{4.31}$$

which can be regarded as the error of replacing $m^{[2]}$ by $m^{[3]}$. Moreover, from this construction, there is no jump on the contours $\Sigma_{j,k}, k = 1, 2, 3, 4$ and only a pure $\bar{\partial}-$problem is left due to the non-analyticity, which reads

$$\bar{\partial}E = EW, \tag{4.32}$$

where

$$W(z) = m^{[3]}\bar{\partial}O(z)(m^{[3]})^{-1}. \tag{4.33}$$

From the normalization condition of $m^{[3]}$, we see it is uniformly bounded by $\frac{c}{1-\sup R}$. And to estimate the errors of recovering the potential, one actually needs to estimate $\lim_{z\to\infty} z(E - I)$, where the limit can be chosen along any rays that are not parallel to $\mathbb{R}$. For simplicity, we will take the imaginary axis. The $\bar{\partial}-$problem is equivalent to the following Fredholm integral equation by a simple application of generalized Cauchy integral formula:

$$E(z) = I - \frac{1}{\pi}\int_{\mathbb{C}} \frac{E(s)W(s)}{s-z} dA(s). \tag{4.34}$$

In the following, we will show for each fixed $z \in \mathbb{C}$, $\mathcal{K}(W)(z) := \int_{\mathbb{C}} \frac{E(s)W(s)}{s-z} dA(s)$ is bounded and then by the dominated convergence theorem, we will show $\lim_{z\to\infty} z(E-I) = \mathcal{O}(t^{-3/4})$. First of all, since $m^{[3]}$ is uniformly bounded, setting $z = z_j + u + iv$, we have

$$\|W\| \lesssim \begin{cases} |\bar{\partial}E_{j,k}|e^{-2t\theta''(z_j)uv}, z \in \Omega_{j,k}, k = 1, 4, \\ |\bar{\partial}E_{j,k}|e^{2t\theta''(z_j)uv}, z \in \Omega_{j,k}, k = 3, 6, \end{cases} , \tag{4.35}$$

where $\|\cdot\|$ stands for matrix norm and $0 \le a \lesssim b$ means there exists $C > 0$ such that $a \le Cb$. Then we have

$$\mathcal{K}(W) \le \|E\| \int_{\mathbb{C}} \frac{\|W(s)\|}{|s-z|} dA(s). \tag{4.36}$$

We claim the following lemma:

Lemma 4.7. *Let* $\Omega = \{s : s = \rho e^{\phi}, \rho \ge 0, \phi \in [0, \pi/4]\}$*, and* $z \in \Omega$*, then*

$$\int_{\Omega} \frac{|u^2+v^2|^{-1/4}e^{-tuv}}{|u+iv-z|} dudv = \mathcal{O}(t^{-1/4}). \tag{4.37}$$

Proof. Since there are two singularities of the integrand at z and $(0,0)$. The first case, set $z \ne 0$, and let $d = dist(z, 0)$. We split Ω into three parts: $\Omega_1 \cup \Omega_2 \cup \Omega_3$, where $\Omega_1 = \{s : |s| < d/3\} \cap \Omega$,

$\Omega_2 = \{s : |s-z| < d/3\} \cap \Omega$ and $\Omega_3 = \Omega \backslash (\Omega_1 \cup \Omega_2)$. In the region Ω_1, $|s-z| \geq 2d/3$, and thus

$$
\begin{aligned}
|\int_{\Omega_1} \frac{|u^2+v^2|^{-1/4} e^{-tuv}}{|u+iv-z|} dudv| &\leq \frac{3}{2d} \int_0^\infty \int_0^u \frac{e^{-tuv}}{(u^2+v^2)^{1/4}} dvdu \\
& \text{substituted } v = wu \\
&\leq \frac{3}{2d} \int_0^\infty \int_0^1 \frac{e^{-tu^2 w}}{(1+w^2)^{1/4}} u^{1/2} dwdu \\
&\leq \frac{3}{2d} \int_0^\infty \int_0^1 e^{-tu^2 w} u^{1/2} dwdu \\
&= \frac{3}{2d} \int_0^\infty \frac{1-e^{-tu^2}}{tu^{3/2}} du \\
&= \frac{3}{2d} \frac{1}{2} t^{-3/4} \int_0^\infty \frac{1-e^{-u}}{u^{5/4}} du \\
&= \frac{3}{d} \Gamma(3/4) t^{-3/4}.
\end{aligned}
$$

In the region Ω_2, $|s|^{-1/2} \leq (2d/3)^{-1/2}$,

$$
\begin{aligned}
|\int_{\Omega_1} \frac{|u^2+v^2|^{-1/4} e^{-tuv}}{|u+iv-z|} dudv| &\leq \sqrt{\frac{3}{2d}} \int_{\Omega_2} \frac{e^{-tuv}}{((u-x)^2+(v-y)^2)^{1/2}} dvdu \\
&\leq \sqrt{\frac{3}{2d}} \int_0^{d/3} \int_0^{2\pi} e^{-t(x+\rho\cos(\theta))(y+\rho\sin(\theta))} d\theta d\rho \\
&\leq \frac{2\pi}{3} \sqrt{\frac{3d}{2}} e^{-txy}.
\end{aligned}
$$

While in the region Ω_3,

$$
|\int_{\Omega_3} \frac{|u^2+v^2|^{-1/4} e^{-tuv}}{|u+iv-z|} dudv| \leq \int_0^\infty \int_0^u e^{-tuv} dvdu = \mathcal{O}(t^{-1}).
$$

Now consider $z = 0$. We have

$$
\begin{aligned}
|\int_\Omega \frac{e^{-tuv}}{(u^2+v^2)^{3/4}} dA(u,v)| &= \int_0^\infty \int_0^u \frac{e^{-tuv}}{(u^2+v^2)^{3/4}} dv du \\
&= \int_0^\infty \int_0^1 \frac{e^{-tu^2 w}}{(1+w^2)^{3/4} u^{1/2}} dw du \\
&\leq \int_0^\infty \int_0^1 \frac{e^{-tu^2 w}}{u^{1/2}} dw du \\
&= \int_0^\infty \frac{1-e^{-tu^2}}{tu^{5/2}} du \\
&= \int_0^\infty \frac{1-e^{-u}}{tt^{-5/4} u^{5/4}} t^{-1/2} \frac{1}{2} u^{-\frac{1}{2}} du \\
&= \frac{1}{2} t^{-1/4} \int_0^\infty \frac{1-e^{-u}}{u^{7/4}} du \\
&= \frac{3}{8} t^{-1/4} \Gamma(1/4).
\end{aligned}
$$

Assembling all together, the proof is done. □

Remark 4.8. The essential fact that makes the above argument work is the rapid decaying of the exponential factor in the region. And this lemma also tells us that those mild singularities, which have a rational order grow, can be absorbed by the exponential factor. Back to our situation, after some elementary transformations (translation and rotation), the estimation of $\int_{\mathbb{C}} \frac{\|W(s)\|}{|s-z|} dA(s)$ will eventually reduce to similar situation discussed in the above lemma.

Based on the Lemma 4.7, we know when t is sufficiently large, $\|\mathcal{K}\| < 1$. Thus the resolvent is uniformly bounded and we obtain the following estimate by taking standard Neumann series, for some sufficiently large t_0,

$$
\|E - I\| = \|(1-\mathcal{K})^{-1}\mathcal{K}I\| \leq \frac{ct^{-1/4}}{1-ct^{-1/4}} \leq ct^{-1/4}, \quad t > t_0 \tag{4.38}
$$

Now since for each $z \in \Omega_{j,k}$, we have $|\bar{\partial} E_{j,k}(z)| \leq c(|z-z_j|^{-1/2} + |R'(u+z_j)|)$, and applying the dominated convergence theorem, we have

$$
\lim_{z\to\infty} |z(E-I)| \leq \frac{1}{\pi} \sum_{j=1}^{l} \sum_{k=1}^{4} \|E\|_{L^\infty} \int_{\Omega_{j,k}} \|W\| ds,
$$

and the estimate for the right hand side perfectly fits the situation $z \in \Omega_1$ of Lemma 4.7, and we eventually have:

$$
E_1 = \lim_{z\to\infty} |z(E-I)| = \mathcal{O}(t^{-3/4}). \tag{4.39}
$$

4.7 Asymptotics formulas

First, we summary all the steps as follows:

1. Initial RHP $m^{[0]}$.
2. Conjugate initial RHP to obtain $m^{[1]} = m^{[0]}\delta^{-\sigma_3}$.
3. Open lens to obtain $m^{[2]} = m^{[1]}O(z)$, where $O(z) = I + o(1), z \to \infty$ in all sectors.
4. Preparing for separating contributions and the phase reduction by removing some contours, which generate an error $\mathcal{O}(e^{-ct}), c > 0$.
5. Separating contributions and the phase reduction will generate an error $\mathcal{O}(t^{-1})$.
6. Connect each RHP($m^{[3,j]}$) near the stationary phase point to a Model RHP($m^{[4,j]}$).
7. Comparing $m^{[2]}$ and $m^{[3]}$ and computing the error by analysis a pure $\bar{\partial}-$problem. The error term is $\mathcal{O}(t^{-3/4})$.

Combining all previous results, we have

$$m^{[0]}(z) = E(z)m^{[3]}(z)O^{-1}(z)\delta^{\sigma_3}.$$

Since $O(z)$ uniformly converges to I as $z \to \infty$, and δ^{σ_3} is diagonal matrix, those two do not affect the recovering of the potential. Finally we obtain

$$\begin{aligned} q(x,t) &= -2i(m^{[3]}_{1,12} + E_{1,12}) \\ &= q_{as}(x,t) + \mathcal{O}(t^{-3/4}). \end{aligned}$$

Remark 4.9. q_{as} is $\mathcal{O}(t^{-1/2})$ as $t \to \infty$ and $x > 0$ and $|x/t|$ is bounded.

4.8 Fast decay region

Observe that the contour $\mathrm{Im}(z^{2k+1} + z) = 0$ has no intersection with real axis for $k \in \mathbb{N}$, which corresponding to the mKdV hierarchy, i.e., the odd parts of the AKNS hierarchy. For those phase functions $\theta(z) = z^n + z, n = 2k+1, k = 1, 2, ...$, we have the following properties:

1. There exits $\epsilon = \epsilon(n) > 0$, $\pm\mathrm{Im}(\theta) > 0$ in the strip $\{z : \pm\mathrm{Im}(z) \in (0, \epsilon)\}$.

2. There exits $M \in (0, 1/\epsilon)$ such that $\mathrm{Im}(\theta) \geq nvu^{n-1}$ for $|u| \geq M\epsilon$ and $\mathrm{Im}(\theta) \geq v(1-(M\epsilon)^2)$ for $|u| \leq M\epsilon$. Here $z = u + iv$.

Now, we will formulate a general RHP model as follows: Given $R(z) \in H^{1,1}(\mathbb{R})$, find a piecewise holomorphic matrix-value function m such that

1. $m_+ = m_- e^{-it\theta(z)\,\mathrm{ad}\,\sigma_3} v(z), z \in \mathbb{R}$, where the jump matrix is given by

$$v(z) = \begin{pmatrix} 1-|R|^2 & -\bar{R} \\ R & 1 \end{pmatrix} = \begin{pmatrix} 1 & -\bar{R} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ R & 1 \end{pmatrix} \tag{4.40}$$

2. $m = I + \mathcal{O}(z^{-1}), \quad z \to \infty$.

Theorem 4.10. *For the above RHP, the solution m enjoys the following asymptotics as $t \to \infty$:*

$$m_1(t) = \mathcal{O}(t^{-1}). \tag{4.41}$$

where $m = I + m_1(t)/z + \mathcal{O}(z^{-2}), z \to \infty$.

Now we will prove the theorem again using the idea of $\bar{\partial}$-steepest descent method.

Proof. We will only prove for the $z \in \{z : \mathrm{Im}\, z \in (0, \epsilon)\}$, for the other half, the same analysis works just by a slight modification. First we open the lens (i.e., the real line) by multiplying m by a smooth($\mathbb{R}^2$) matrix-valued function $O(z)$, where $O(z)$ is given by

$$e^{-it\theta(z)} \begin{pmatrix} 1 & 0 \\ -\frac{R(\mathrm{Re}\, z)}{1+(\mathrm{Im}\, z)^2} & 1 \end{pmatrix}. \tag{4.42}$$

Let us denote $\Sigma_1 = \{z : \mathrm{Im}\, z = \epsilon\}$ and

$$\tilde{m} = \begin{cases} m, & z \in \Omega_3 \\ mO, & z \in \Omega_1 \end{cases}$$

where $\Omega_1 = \{z : \mathrm{Im}\, z \in (0, \epsilon)\}$ and $\Omega_3 = \{z : \mathrm{Im}\, z > \epsilon\}$.

Now as usual, we obtain a $\bar{\partial}-$RHP, and based on a traditional small norm argument, the $\tilde{m} = I + o(1)$ [14]. Denote the solution to the pure RHP by $m^\sharp$, and consider

$$E = \tilde{m}(m^\sharp)^{-1}. \tag{4.43}$$

Then E doesn't have a jump on σ_1 and it satisfies a pure $\bar{\partial}-$problem:

$$\bar{\partial}E = EW, \tag{4.44}$$

where $W = -m^\sharp e^{-t\theta(z)}\bar{\partial}(\frac{R(\operatorname{Re} z)}{1+(\operatorname{Im} z)^2})(m^\sharp)^{-1}$, here $\bar{\partial} = \frac{1}{2}(\partial_{\operatorname{Re} z} + i\partial_{\operatorname{Im} z})$.

Since $R \in H^{1,1}$, $\bar{\partial}(\frac{R(\operatorname{Re} z)}{1+(\operatorname{Im} z)^2})$ is uniformly bounded by a nonnegative L^2 function $f(\operatorname{Re} z)$. Note that $m^\sharp$ is uniformly close to I, set $z = u + iv$. Then we have

$$\|W\| \leq f(u)e^{-t\operatorname{Im}\theta(u,v)}, u \in \mathbb{R}, v \in (0, \epsilon).$$

By the same procure as the previous sections, the error of approximating m by the identity matrix is given by the following integral:

$$\Delta := \int_0^\epsilon \int_{\mathbb{R}} f(u)e^{-t\operatorname{Im}\theta} du dv. \tag{4.45}$$

Split the u into two regions: (1) $|u| \leq M\epsilon$, (2) $|u| \geq M\epsilon$. And denote by Δ_1, Δ_2 respectively. Then $\Delta = \Delta_1 + \Delta_2$. On the one hand, we have

$$\begin{aligned}
\Delta_1 &\leq \int_0^\epsilon \int_{-M\epsilon}^{M\epsilon} f(u)e^{-tv(1-M^2\epsilon^2)} du dv \\
&\quad \text{by the Cauchy-Schwartz inequality} \\
&\leq \|f\|_{L^2(\mathbb{R})}(2M\epsilon)^{1/2}\frac{1-e^{-t\epsilon(1-M^2\epsilon^2)}}{t(1-M^2\epsilon^2)} \\
&= \mathcal{O}(t^{-1}).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\Delta_2 &\leq \int_0^\epsilon \int_{|u|\geq M\epsilon} f(u)e^{-ntvu^{n-1}} du dv \\
&= \int_{|u|\geq M\epsilon} f(u) \int_0^\epsilon e^{-ntvu^{n-1}} dv du \\
&\leq t^{-1}\|f\|_{L^2}(\int_{|u|\geq M\epsilon} (\frac{1-e^{-ntvu^{n-1}}}{nu^{n-1}})^2 du)^{1/2} \\
&\leq t^{-1}\|f\|_{L^2}\frac{n}{n-2}(M\epsilon)^{-(n-2)} \\
&= \mathcal{O}(t^{-1}).
\end{aligned}$$

Thus the error term is $\mathcal{O}(t^{-1})$ and this completes the proof. □

4.9 Asymptotics in Painlevé regions

4.9.1 Painlevé II hierarchy

It is well-known that one can generate the Painlevé II hierarchy from similarity reduction of the mKdV hierarchy [10]. In this section, we will provide an algorithm based on the Riemann-Hilbert problems to generate the Painlevé II hierarchy. Let's denote $\Theta(x,z) = xz + \frac{c}{n}z^n$. Suppose m solves the following RHP:

$$m_+ = m_- e^{i\Theta\sigma_3} v_0 e^{-i\Theta\sigma_3}, \quad z \in \Sigma_n,$$
$$m = I + \mathcal{O}(z^{-1}), \quad z \to \infty.$$

where the contour Σ_n consists of all stokes lines which are determined by Θ, and v_0 is a constant 2 by 2 matrix that is independent of x, z.

Now letting $\tilde{m} = me^{i\Theta\sigma_3}$, we arrive at a new RHP:

$$\tilde{m}_+ = \tilde{m}_- v_0, \quad z \in \Sigma_n,$$
$$\tilde{m} = (I + \mathcal{O}(z^{-1}))e^{i\Theta\sigma_3}, \quad z \to \infty.$$

Since v_0 is constant, it is easily to check, by Louisville's argument, that both $\partial_z \tilde{m}\tilde{m}^{-1}$ and $\partial_x \tilde{m}\tilde{m}^{-1}$ are polynomials of z. Hence we obtain the following two differential equations:

$$\partial_x \tilde{m}\tilde{m}^{-1} = A(x,z), \tag{4.46}$$

$$\partial_z \tilde{m}\tilde{m}^{-1} = B(x,z). \tag{4.47}$$

If we assume

$$m = I + \sum_{j=1}^{n-2} m_j(x) z^{-j} + \mathcal{O}(z^{-(n-1)}), \quad z \to \infty, \tag{4.48}$$

$$\underline{m} = m^{-1} = I + \sum_{j=1}^{n-2} \underline{m}_j(x) z^{-j} + \mathcal{O}(z^{-(n-1)}), \quad z \to \infty, \tag{4.49}$$

then direct computation shows

$$A = i[m_1, \sigma_3] + iz\sigma_3, \tag{4.50}$$

$$B = ix\sigma_3 + icz^{n-1}\sigma_3 + iz^{n-2}[m_1,\sigma_3] + \sum_{k=2}^{n-1} icz^{n-1-k}(m_k\sigma_3 + \sigma_3\underline{m}_k + \sum_{j=1}^{k-1} m_{k-j}\sigma_3\underline{m}_j). \tag{4.51}$$

Since $m_{x,z} = m_{z,x}$, we have

$$A_z - B_x + [A, B] = 0. \tag{4.52}$$

Letting the coefficients of z all vanish, and setting

$$m_j = \begin{pmatrix} 0 & u_j(x) \\ u_j(x) & 0 \end{pmatrix}, \tag{4.53}$$

we can solve the equations recursively from the high degree of z to low degree, and eventually, we will arrive at infinitely many nonlinear ODEs of u_1, which are in face a hierarchy of Painlevé II equations. We list the first few of them:

$$n = 3 : -8cu^3 + cu_{xx} - 4ux = 0, \tag{4.54}$$

$$n = 4 : 12icu^2 u_x - \frac{1}{2}icu_{xxx} - 4ux = 0, \tag{4.55}$$

$$n = 5 : -24cu^5 + 10cu^2 u_{xx} + 10cuu_x^2 - \frac{c}{4}u_{xxxx} - 4ux = 0. \tag{4.56}$$

In this dissertation, we are interested in the odd members. In particular, $n = 3$ corresponds to the mKdV equation, $n = 5$ corresponds to the 5th-order mKdV equation, and so on. In the following section, we will show how we can connect the long-time asymptotics of the mKdV hierarchy with solutions to the Painlevé II hierarchy.

4.9.2 Painlevé region

Recall the phase functions of the AKNS hierarchy of mKdV type equations are

$$\theta(z; x, t) = xz + ctz^n, \quad n \text{ is odd.} \tag{4.57}$$

The Painlevé region is the region of $|xt^{-1/n}|$ is bounded. By rescaling $z \to (nt)^{-\frac{1}{n}}\xi$, and letting $s = x(nt)^{-\frac{1}{n}}$, we have

$$\Theta(\xi) = s\xi + \frac{c}{n}\xi^n. \tag{4.58}$$

Now the modular of the stationary phase points of (4.57) is

$$|z_0| = \left|-\frac{x}{ct}\right|^{\frac{1}{n-1}} = \mathcal{O}(t^{-\frac{1}{n}}),$$

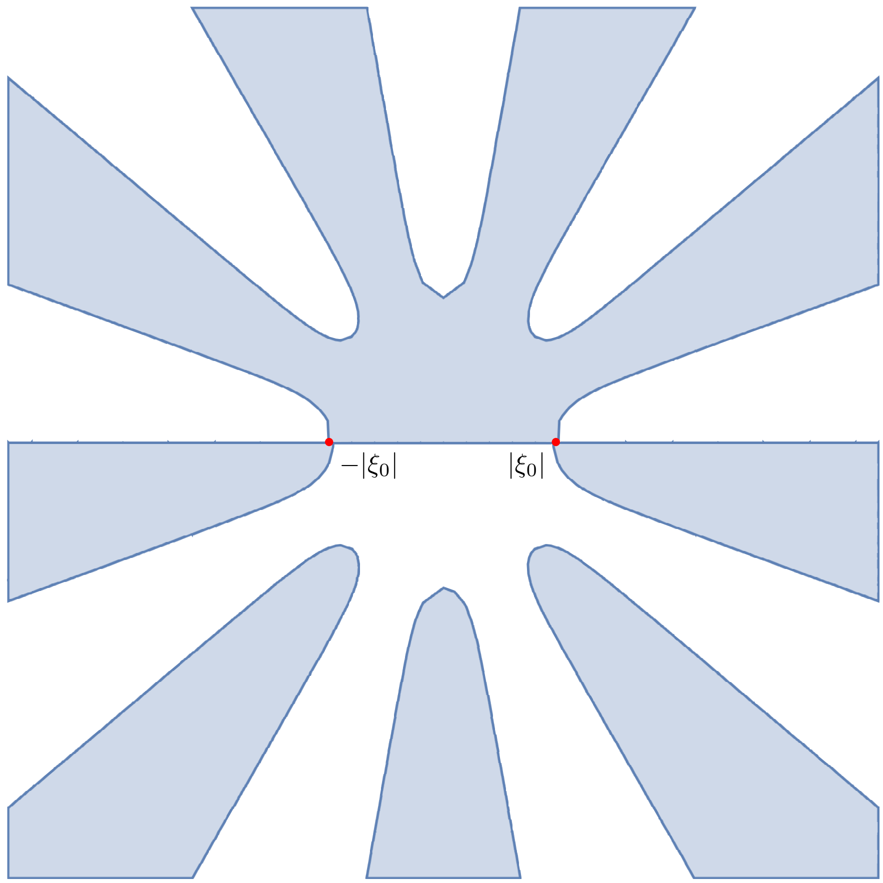

Figure 4. Signature of $\mathrm{Re}(i\theta)$. The gray region indicates $\mathrm{Re}(i\theta) > 0$.

however, after scaling, the modular of the stationary phase points of $\Theta(\xi)$ is

$$|\xi_0| = z_0 t^{\frac{1}{n}}, \tag{4.59}$$

which is fixed as $t \to \infty$. A direct computation shows for any odd n, the signature of $\mathrm{Re}\,(i\theta)$ is just similar to Fig 4. Since the original RHP only has a jump on the real line, all the stokes lines except those crossing the real line can be ignored.

Note that

$$
\begin{aligned}
e^{-i\theta(z)\,\mathrm{ad}\,\sigma_3}v(z) &= e^{-i\Theta(\xi)\,\mathrm{ad}\,\sigma_3}v(\xi) \\
&= \begin{pmatrix} 1-|R|^2 & -\bar{R}e^{-2i\Theta} \\ Re^{2i\Theta} & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & -\bar{R}e^{-2i\Theta} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ Re^{2i\Theta} & 1 \end{pmatrix} \\
&= e^{-i\Theta(\xi)\,\mathrm{ad}\,\sigma_3}v_-^{-1}v_+.
\end{aligned}
$$

We can deform the the contour $\{z \in \mathbb{R} : |z| > |\xi_0|\}$ as before and get the deformed contour as follows.

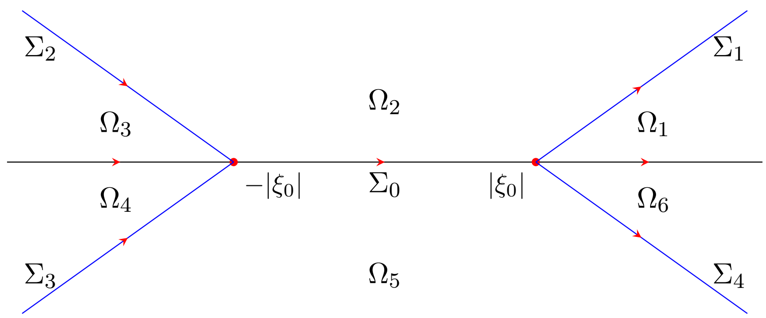

Figure 5. Contour for $\bar{\partial}-$RHP.

As before, we set the original RHP as $m^{[1]}$ with jump $e^{-i\theta(z)\,\mathrm{ad}\,\sigma_3}v(z)$. After rescaling and $\bar{\partial}-$ lens opening, we set $m^{[2]}(\xi) = m^{[1]}O(\gamma)$, where the lens-opening matrix reads

$$
O(\gamma) = \begin{cases} \begin{pmatrix} 1 & 0 \\ -E_+e^{2i\Theta(\gamma)} & 1 \end{pmatrix}, & \gamma \in \Omega_1 \cup \Omega_3, \\ \begin{pmatrix} 1 & -E_-e^{-2i\Theta(\gamma)} \\ 0 & 1 \end{pmatrix}, & \gamma \in \Omega_4 \cup \Omega_6, \\ I, \quad \gamma \in \Omega_2 \cup \Omega_5, \end{cases} \tag{4.60}
$$

where

$$
E_+(\gamma) = \mathcal{K}(\phi)R\left((nt)^{-\frac{1}{n}}\xi\right) + (1-\mathcal{K}(\phi))R(\xi_0(nt)^{-\frac{1}{n}}) \tag{4.61}
$$

$$
E_-(\gamma) = \overline{E_+(\gamma)}, \quad \gamma = \xi_0 + \rho e^{i\phi}, \xi = \mathrm{Re}(\gamma). \tag{4.62}
$$

Now we arrive at the following $\bar{\partial}-$RHP:

1. The RHP

   (1.a). $m^{[2]}(\gamma) \in C^1(\mathbb{R}^2\backslash\Sigma)$ and $m^{[2]}(z) = I + \mathcal{O}(\gamma^{-1}), \gamma \to \infty$.

   (1.b). The jumps on Σ_1 and Σ_2 are $e^{-i\Theta(\xi)\ \mathrm{ad}\ \sigma_3} v_+$, the jumps on Σ_3 and Σ_4 are $e^{-i\Theta(\xi)\ \mathrm{ad}\ \sigma_3} v_-$, and the jump on Σ_0 is $e^{-i\Theta\ \mathrm{ad}\ \sigma_3} v((nt)^{-\frac{1}{n}}\xi)$.

2. The $\bar{\partial}$ problem

   For $z \in \mathbb{C}$, we have

$$\bar{\partial} m^{[2]}(\xi) = m^{[2]}(\xi)\bar{\partial} O(\xi). \tag{4.63}$$

Again, we will need the following lemma in order to estimate the errors from the $\bar{\partial}-$problem.

Lemma 4.11. *For $\gamma \in \Omega_{1,3,4,6}$, $\xi = \operatorname{Re}\gamma$,*

$$|\bar{\partial} E_\pm(\gamma)| \leq (nt)^{-\frac{1}{n}} |(nt)^{-\frac{1}{n}}(\xi - \xi_0)|^{-\frac{1}{2}} \|R\|_{H^{1,0}} + (nt)^{-\frac{1}{n}} |R'((nt)^{-\frac{1}{n}}\xi)|. \tag{4.64}$$

Proof. For brevity, we only prove for the region Ω_1. Using polar coordinates, we have

$$|\bar{\partial} E_+(\gamma)| = \left| \frac{ie^{i\phi}}{2\rho} \mathcal{K}'(\phi) \left[R\left((nt)^{-\frac{1}{n}}\xi\right) - R(\xi_0 (nt)^{-\frac{1}{n}}) \right] + \mathcal{K}(\phi) R'\left((nt)^{-\frac{1}{n}}\xi\right)(nt)^{-\frac{1}{n}} \right|$$

by the Cauchy-Schwartz inequality

$$\leq \left| \frac{\|R\|_{H^{1,0}} |(nt)^{-\frac{1}{n}}\xi - \xi_0 (nt)^{-\frac{1}{n}}|^{1/2}}{\gamma - \xi_0} \right| + (nt)^{-\frac{1}{n}} \left| R'\left((nt)^{-\frac{1}{n}}\xi\right) \right|$$

$$\leq (nt)^{-\frac{1}{n}} |(nt)^{-\frac{1}{n}}(\xi - \xi_0)|^{-\frac{1}{2}} \|R\|_{H^{1,0}} + (nt)^{-\frac{1}{n}} |R'((nt)^{-\frac{1}{n}}\xi)|.$$

Similarly, we can prove for the other regions. $\square$

Next, consider a pure RHP $m^{[3]}$ which satisfies exactly the RHP part of $\bar{\partial}-$RHP($m^{[2]}$). Moreover, $m^{[3]}$ can be approximated by the RHP corresponding to a special solution of the Painlevé II hierarchy. Since for $\gamma \in \Omega_1$,

$$\begin{aligned} &\left| \left(R(\xi(nt)^{-\frac{1}{n}}) - R(0) \right) e^{2i\Theta(\gamma)} \right| \\ &\leq |\xi(nt)^{-\frac{2}{n}}|^{\frac{1}{2}} \|R\|_{H^{1,0}} e^{2\operatorname{Re} i\Theta(\gamma)} \\ &\leq (nt)^{-\frac{1}{n}} |\operatorname{Re}\gamma|^{\frac{1}{2}} \|R\|_{H^{1,0}} e^{2\operatorname{Re} i\Theta(\gamma)}, \end{aligned}$$

it is evident that

$$\|Re^{2i\Theta} - R(0)e^{2i\Theta}\|_{L^\infty \cap L^1 \cap L^2} \leq c(nt)^{-\frac{1}{n}}. \tag{4.65}$$

Let $m^{[4]}$ be the solution to the RHP by change the jumps of $m^{[3]}$ to $R(0)$ and $\bar{R}(0)$. Then, via the small norm technique, the errors between the corresponding potentials are given by

$$\begin{aligned} error_{3,4} &= \lim_{\gamma\to\infty} |\gamma(m_{12}^{[4]} - m_{12}^{[3]})| &(4.66)\\ &\leq c\int_\Sigma |(R(\mathrm{Re}(s)(nt)^{-\frac{1}{n}}) - R(0))e^{2i\Theta(s)}|ds &(4.67)\\ &\leq c(nt)^{-\frac{1}{n}}. &(4.68)\end{aligned}$$

Then since now the jumps are all analytic, we can perform the analytic deformation and arrive at the green contours as show in Fig6. Let's denote the new RHP by $m^{[5]}(\gamma)$, which is exactly equivalent to $m^{[4]}(\gamma)$.

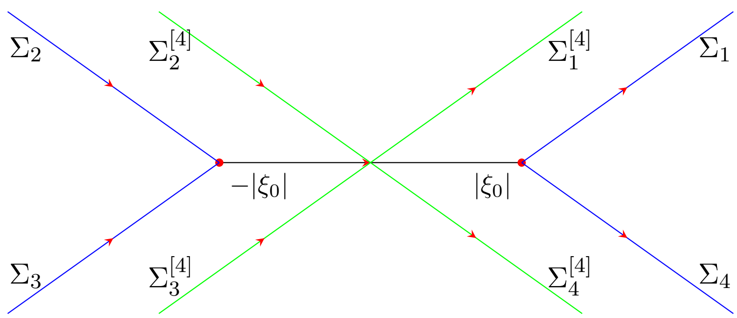

Figure 6. Contour for $m^{[4]}$(Green part).

The jumps of $m^{[5]}$ read:

$$e^{-i\Theta(\gamma)}v^{[4]}(0) = \begin{cases} \begin{pmatrix} 1 & 0 \\ R(0)e^{2i\Theta(\gamma)} & 1 \end{pmatrix}, & \gamma \in \Sigma_{1,2}^{[4]}, \\ \begin{pmatrix} 1 & \bar{R}(0)e^{-2i\Theta(\gamma)} \\ 0 & 1 \end{pmatrix}, & \gamma \in \Sigma_{3,4}^{[4]}. \end{cases} \tag{4.69}$$

Then according to the previous section, the $(1,2)$ entry of the solution $m^{[5]}$ (also of the solution $m^{[4]}$) is a solution to the Painlevé II hierarchy, i.e.,

$$m_{12}^{[4]}(\gamma) = m_{12}^{[5]}(\gamma). \tag{4.70}$$

Recall $P_n^{II}(s) = \lim_{\gamma\to\infty} \gamma m_{12}^{[5]}$ where P_n^{II} solves the n^{th} equation in the Painlevé II hierarchy.

Now let's consider the error generated from the $\bar\partial$-extension. Recall the error E satisfies the following pure $\bar\partial$ problem:

$$\begin{aligned}
&\bar\partial E = EW,\\
&W = m^{[3]}\bar\partial O(m^{[3]})^{-1}.
\end{aligned}$$

As before, the $\bar\partial$ equation is equivalent to an integral equation which reads

$$E(z) = I + \frac{1}{\pi}\int_{\mathbb{C}} \frac{E(s)W(s)}{z-s}ds = I + \mathcal{K}(E).$$

As before, we can show that the resolvent is always exist for large t. So we only need to estimate the true error which is:$\lim_{z\to\infty} z(E-I)$. In fact, we have

$$\begin{aligned}
\lim_{z\to\infty} |z(E-I)| &= |\int_{\mathbb{C}} EW ds|\\
&\leq c\|E\|_\infty \int_\Omega |\bar\partial O| ds.
\end{aligned}$$

For the sake of simplicity, we only estimate the integral on the right hand side in the region of the top right corner. Note there is only one entry is nonzero in $\bar\partial O$, which is one of the $E_\pm$ and we split the integral into two parts in the obvious way, i.e.,

$$\begin{aligned}
\int_\Omega |\bar\partial O| ds &\leq I_1 + I_2\\
&= \int_\Omega (nt)^{-\frac{1}{2n}} |\operatorname{Re} s - \xi_0| \|R\|_{H^{1,0}} e^{2\operatorname{Re} i\Theta(s)} ds\\
&+ \int_\Omega (nt)^{-\frac{1}{nt}} |R'((nt)^{-\frac{1}{n}} s)| e^{2\operatorname{Re} i\Theta(s)} ds.
\end{aligned}$$

As we know from the previous sections, $e^{\operatorname{Re} 2i\Theta(s)} \leq ce^{-2|\Theta''(\xi_0)|uv}$ in the region $\{z = u+iv : u > \xi_0, 0 < v < \alpha u\}$ for some small α , where $s = u + iv + \xi_0$. Then we have

$$\begin{aligned}
I_1 &\leq (nt)^{-\frac{1}{2n}} \int_\Omega |\operatorname{Re} s - \xi_0|^{-1/2} e^{-cuv} du dv\\
&\leq (nt)^{-\frac{1}{2n}} \int_0^\infty \int_0^{\alpha u} u^{-1/2} e^{-cuv} du dv\\
&\leq C(nt)^{-\frac{1}{2n}} \int_0^\infty \frac{1 - e^{-2\alpha|\Theta''(\xi_0)|}}{u^{3/2}} du\\
&= \mathcal{O}\left((nt)^{-\frac{1}{2n}}\right).
\end{aligned}$$

And

$$I_2 \le (nt)^{-\frac{1}{n}} \int |R'((nt)^{-\frac{1}{2n}} \operatorname{Re} s)| e^{-cuv} du dv$$

by the Cauchy-Schwartz inequality

$$\le (nt)^{-\frac{1}{n}} \|R\|_{H^{1,0}} \int_0^\infty \left(\int_{\alpha v}^\infty e^{-2cuv} du\right)^{1/2} dv$$

$$\le (nt)^{-\frac{1}{n}} \|R\|_{H^{1,0}} \int_0^\infty \frac{e^{-c\alpha v^2}}{\sqrt{2\alpha cv}} dc$$

$$= \mathcal{O}((nt)^{-\frac{1}{n}}).$$

Thus, we arrive at

$$\bar{\partial}\text{Error} = \mathcal{O}((nt)^{-\frac{1}{2n}}). \tag{4.71}$$

And we undo all the deformations, we obtain

$$\begin{aligned} m^{[1]}((nt)^{-\frac{1}{n}}\gamma) &= m^{[2]}(\gamma) O^{-1}(\gamma) \\ &= \left(1 + \frac{\mathcal{O}t^{\frac{1}{2n}}}{\gamma}\right) m^{[3]}(\gamma) O^{-1}(\gamma) \\ &= \left(1 + \frac{\mathcal{O}t^{\frac{1}{2n}}}{\gamma}\right)\left(1 + \frac{\mathcal{O}t^{\frac{1}{2n}}}{\gamma}\right) m^{[4]}(\gamma) O^{-1}(\gamma), \end{aligned}$$

and can be rewritten in terms variable z:

$$m^{[1]}(z) = \left(1 + \frac{\mathcal{O}(t^{-1/(2n)})}{z(nt)^{1/n}}\right) m^{[5]}((nt)^{1/n} z). \tag{4.72}$$

Since $m^[5]$ corresponds to the RHP for the Painlevé II hierarchy, we have

$$m^{[5]}(\gamma) = I + \frac{m_1^{[5]}(s)}{\gamma} + \mathcal{O}(\gamma^{-1}), \tag{4.73}$$

where $\gamma = z(nt)^{1/n}$. Thus,

$$m^{[1]}(z) = \left(1 + \frac{\mathcal{O}(t^{-\frac{1}{2n}})}{z(nt)^{1/n}}\right)\left(1 + \frac{m_1^{[5]}(s)}{z(nt)^{1/n}} + \mathcal{O}(z^{-2})\right) \tag{4.74}$$

$$= I + \frac{m_1^{[5]}(s)}{z(nt)^{1/n}} + \frac{\mathcal{O}(t^{-\frac{1}{2n}})}{z(nt)^{1/n}} + \mathcal{O}(z^{-2}). \tag{4.75}$$

Since $m_1^{[5]}(s)$ is connected to solutions of the Painlevé II hierarchy, we conclude that

$$q(x,t) = \lim_{z\to\infty} z(m^{[1]} - I) \tag{4.76}$$

$$= (nt)^{-\frac{1}{n}} u_n(x(nt)^{-\frac{1}{n}}) + \mathcal{O}(t^{-\frac{3}{2n}}), \tag{4.77}$$

where u_n solves the n^{th} member of the Painlevé II hierarchy. For the case of mKdV type defocusing reductions, we only take odd numbers for n.

Chapter 5

The AKNS system and the focusing/defocussing 5th-order mKdV equation

Note that the 5th-order mKdV equation generates so-called isospectral flow of the following AKNS spectral problem:

$$\psi_x(x,t;z) = \left(iz\sigma_3 + \begin{pmatrix} 0 & q(x,t) \\ \epsilon q(x,t) & 0 \end{pmatrix} \right) \psi(x,t;z), \tag{5.1}$$

$$\psi_t(x,t;z) = \left(16iz^5\sigma_3 + V_0(q,\epsilon,z)\right) \psi \tag{5.2}$$

where $\epsilon = 1$ and $\epsilon = -1$ correspond to the defocussing/focusing 5th-order mKdV equation. In the next section, we will discuss the time evolution part of this AKNS spectral problem in detail, i.e., how to construct V_0. For now, let us focus on the x-part. And refer to next section that the constant part of V_0 with respect to z is

$$V_5 := 6Q^5 - 10Q^2Q_{xx} - 10QQ_x^2 + Q_{xxxx}. \tag{5.3}$$

From which we obtain the unreduced 5th-order mKdV system

$$Q_t = V_{5x} := \frac{\partial V_x}{\partial x}. \tag{5.4}$$

Now let us consider the reductions, provided that q is a real potential:

- $\epsilon = 1$: the defocusing 5th-order mKdV equation

$$q_t = 30q^4q_x - 10q^2q_{xxx} - 40qq_{xx}q_x - 10q_x^3 + q_{xxxxx}. \tag{5.5}$$

- $\epsilon = -1$: the focusing 5th-order mKdV equation

$$q_t = 30q^4q_x + 10q^2q_{xxx} + 40qq_{xx}q_x + 10q_x^3 + q_{xxxxx}. \tag{5.6}$$

5.1 Time evolution of the focusing/defocussing 5th-order mKdV

Here following Ma's scheme [18] , the stationary zero curvature equation $W_x = [U, W]$,where

$$W = \sum_{i\geq 0} W_{0,i} z^{-i}, W_{0,i} = \begin{pmatrix} a_i & b_i \\ c_i & -a_i \end{pmatrix}$$

leads to the following recursion relation:

$$\begin{cases} b_{i+1} = \frac{1}{2I} b_{i,x} - Iqa_i, \\ c_{i+1} = -\frac{1}{2I} - I\bar{q}a_i, \\ a_{i+1,x} = qc_{i+1} - \bar{q}b_{i+1}, \end{cases} \tag{5.7}$$

upon taking the initial values

$$a_0 = 16I, b_0 = c_0 = 0, \tag{5.8}$$

also imposing the conditions of the integration for the third recursion relation:

$$a_i|_{q=0} = b_i|_{q=0} = c_i|_{q=0} = 0, \forall i \geq 1. \tag{5.9}$$

Now let

$$V^{[m]} = (z^m W)_+ \tag{5.10}$$

where $(\cdot)_+$ means the principle part of the Laurent expansion. Then the time-evolution problem is followed by

$$\Psi_t = V^{[m]} \Psi, \tag{5.11}$$

and the zero curvature equation

$$U_t - V_x^{[m]} + [U, V^{[m]}] = 0 \tag{5.12}$$

leads to the equivalent non-linear integrable PDEs. For $m = 2, 3$, we will obtain the NLS equation and the mKdV equation , respectively. In the current paper, letting $m = 5$, we obtain the time-evolution part for the 5th-order mKdV equation, which reads

$$\psi_t = (16iz^5\sigma_3 + V_0(q, \epsilon, z))\psi \equiv V\psi \tag{5.13}$$

where

$$V_0 = z^4(-16Q) - 8iz^3(Q^2 - Q_x)\sigma_3 + z^2(-8Q^3 + 4Q_{xx}) - iz(12Q_xQ^2 - 2Q_{xxx})\sigma_3 + V_5, \tag{5.14}$$

provided that $Q = \begin{pmatrix} 0 & q(x,t) \\ \epsilon q(x,t) & 0 \end{pmatrix}$.

Chapter 6

Exact solutions to the 5th-order focusing mKdV equation

6.1 Problem setup

From now on, we will study the exact solutions of the 5th-order mKdV equation in details. In the last chapter, we have already shown how to derive the 5th-order mKdV equation from reductions of the AKNS system. Let's recall the spectral problem:

$$\psi_x = (iz\sigma_3 + Q)\psi, \tag{6.1}$$

which satisfies the following boundary behaviors,

$$\psi^{(\pm)} = \mu^{(\pm)} e^{ixz\sigma_3}, \quad x \to \pm\infty,$$

where $\mu^{(\pm)} \to I, \quad x \to \pm\infty$. As studied in chapter 2, μ satisfies the following Volterra integral equation:

$$\mu^{\pm} = I + \int_{\pm\infty}^{x} e^{iz(x-y)\,\mathrm{ad}\,(\sigma_3)}(Q(y)\mu(y;z))dy. \tag{6.2}$$

Now, similarly analysis shows that the first col μ^+ can be analytically extended to upper half $z-$plane as well as the second row of μ^-. By introduce a new notations:

$$H_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, H_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

We construct $P_+ = \mu^+ H_1 + \mu^- H_2$ which is analytic in $\mathbb{C}^+$. Observe also that, denoting $\tilde{\mu} = \mu^{-1}$,

$$\tilde{\mu}^{\pm} = I - \int_{\pm\infty}^{x} e^{iz(x-y)\,\mathrm{ad}\,(\sigma_3)}(\tilde{\mu}(y;z)Q(y))dy, \tag{6.3}$$

We can construct $P_- = H_1\tilde{\mu}^+ + H_2\tilde{\mu}^-$, which is analytic in $\mathbb{C}^-$. Recall from chapter 2, we know

$$\psi^{(+)} = \psi^{(-)}S(z) = \psi^{(-)} \begin{pmatrix} a(z) & \breve{b}(z) \\ b(z) & \breve{a}(z) \end{pmatrix} \tag{6.4}$$

Also by setting $E = e^{ixz\sigma_3}$, $\mu^{(+)} = \mu^{(-)} ESE^{-1}$. then by multiplying P_- and P_+, we have

$$P_- P_+ = H_1^2 + H_2^2 + H_1 ES^{-1}E^{-1}H_1 + H_2 ESE^{-1}H_1 \tag{6.5}$$

$$= E \begin{pmatrix} 0 & -\breve{b} \\ b & 0 \end{pmatrix} E^{-1}. \tag{6.6}$$

From the construction, it is easy to check that

$$\begin{aligned} \det P_+ &= \det(\mu^{(-)} ESE^{-1}H_1 + \mu^{(-)}H_2) \\ &= \det \mu^{(-1)} \det(ESE^{-1}H_1 + H_2) \\ &= a(z). \end{aligned}$$

Similarly, $\det P_- = \breve{a}(z)$. Again, recall that $S(z)$ satisfies

$$\breve{a}(z) = a(-z), \tag{6.7}$$

$$\bar{a}(z) = a(-\bar{z}), \tag{6.8}$$

which in turn implies, that if z_k is a zero of $\det P_+$. Then $\bar{z}_k$ is a zero of $\det P_-$ with same multiplicity.

From the analytic Fredholm theory, the equation (6.2) has a solution which is entire in $z-$plane if the kernel is small. In fact, $\|Q\| \leq 1$ suffices. For a generic potential $q(x,0) \in L^1(dx)$, the number N of zeros of $a(z)$ is bounded by the following inequality [1]:

$$N \leq 1 + \int_{\mathbb{R}} |x||u(x,0)|dx. \tag{6.9}$$

Then due to the Theorem 1.6 and Proposition 1.7 in [26], the scattering data can be characterized as follows:

Theorem 6.1. *There is a bijection between the scattering data* $\{a(z), z \in \mathbb{R}\} \cup \{(z_k, c_k) : a(z_k) = 0, \operatorname{Im} z_k > 0\}$ *with the potential* $q(x,0)$.

Now we formulate the RHP as follows: Seeking a piecewise analytic matrix valued function P such that

1. For $z \in \mathbb{R}$, $P_- P_+ = G(z)$, with

$$G(z) = E \begin{pmatrix} 0 & -\breve{b} \\ b & 0 \end{pmatrix} E^{-1},$$

2. $P(z) = I + \mathcal{O}(1/z), \quad z \to \infty,$

3. $\det P_+(z_k) = 0, z_k \in \mathbb{C}^+$, and the $P_+(z_k)v_k = 0$, $k = 1, .., N$.

The solution (since to recovery the potential, we only need P_+) to the above RHP is given by the following formula:

$$\begin{aligned}P_+(z) &= \tilde{P}_+(z)(I + \Gamma(z)) \\ &= \tilde{P}_+(z)(I + \sum_{j,k=1}^{N} \frac{v_j (M^{-1})_{jk} v_k^*}{z - \bar{z}_k}),\end{aligned}$$

where

$$M_{jk} = \frac{v_j^* v_k}{\bar{z}_j - z_k}, \quad j, k = 1, \cdots, N, \tag{6.10}$$

and $\tilde{P}$ solves the RHP with jump:

$$\Gamma(z) G(z) \Gamma(z)^{-1}, z \in \mathbb{R}. \tag{6.11}$$

6.2 Time evolution

Note that the t-part in the Lax pair of the 5th-order mKdV equation, the coefficient matrix is of trace zero. Hence similarly, we only need consider the boundary at infinity, and we have

$$E_t = 16 i z^5 \sigma_3 E.$$

Then from the relation $\mu^{(+)} = \mu^{(-)} E S E^{-1}$, we finally grt

$$S_t = 16 i z^5 [S, \sigma_3]. \tag{6.12}$$

Writing out all the entries, we have

$$a(z; t) = a(z; 0) \tag{6.13}$$

$$b(z; t) = b(z; 0) e^{32 i z^5 t}. \tag{6.14}$$

Hence $\det(P_+)$ is independent of t variable. Now from the condition 3 in our RHP setup, i.e., $P_+(z_k)v_k(x, t) = 0$, differentiating with respect to x, t and using the Lax pair, we eventually get

$$P_+(z_k)(\partial_x v_k - i z \sigma_3 v_k) = 0, \tag{6.15}$$

$$P_+(z_k)(\partial_t v_k - 16 i z^5 \sigma_3 v_k) = 0. \tag{6.16}$$

Then the column vector $v_k(x,t) = e^{(iz_kx+16iz_k^5t)\sigma_3}v_{k0}$, and due to the formula of M_{jk}, the quantities $v_j(M^{-1})_{jk}v_k^*$ only depends on the ratios of $v_{k0,1}/v_{k0,2}$. So, without loss of generality, we can introduce $C_k = (1, c_k)^T$, and we have

$$v_k(x,t) = e^{(iz_kx+16iz_k^5t)\sigma_3}C_k. \tag{6.17}$$

6.3 Recover the potential

From the asymptotics conditions in our RHP setup, if we assume

$$P(z) = I + P_1/z + o(1/z), \quad z \to \infty. \tag{6.18}$$

and that P satisfies

$$\mu_x = iz[\sigma_3, \mu] + Q\mu. \tag{6.19}$$

Then the potential can be recovered by the following formula:

$$q(x,t) = 2i(P_1)_{12} = 2i\sum_{j,k=1}^{N} e^{\theta_j-\bar{\theta}_k}\bar{c}_k(M^{-1})_{jk}, \tag{6.20}$$

with

$$M_{jk} = \frac{e^{\theta_k+\bar{\theta}_j} + \bar{c}_jc_ke^{-\theta_k-\bar{\theta}_j}}{\bar{z}_j - z_k}, \tag{6.21}$$

where $\theta_k = iz_kx + 16iz_k^5t$.

The formula (6.20) represents the famous N-soliton solutions.

6.4 Darboux transformation and other interesting solutions

First we prove two fundamental theorems of the Darboux transformation for the AKNS hierarchy.

Theorem 6.2 (Classical Darboux Transformation)**.** *Consider a general spectral problem:*

$$\psi_x = M(z;x)\psi, \tag{6.22}$$

There exists an linear operator $T(z;x)$ such that $\tilde{\psi} = T\psi$ satisfying a new spectral problem:

$$\tilde{\psi}_x = \tilde{M}(z;x)\tilde{\psi}, \tag{6.23}$$

where $\tilde{M} - M$ does not depend on z.

Proof. Suppose such T exists and is differentiable with respect to x. Then we need to show such T has the property that $T_xT^{-1} + TMT^{-1} - M$ does not depend on z. This can be done by constructing T from an projection operator.

Let ψ_1 be the eigenfunction(vector) corresponding to eigenvalue z_1, i.e., $\psi_{1x} = M(z_1; x)\psi_1$, and define the projection operator

$$P = \frac{\psi_1\psi_1^\dagger}{\psi_1^\dagger\psi_1}.$$

Then direct computation shows

$$P_x = MP + PM^\dagger - P(M + M^\dagger)P. \tag{6.24}$$

For the sake of simplicity, denote $M_1 = M(z_1; x)$, $a = \frac{1}{z-\bar{z}_1}$,$b = \frac{1}{z-z_1}$ and $c = \bar{z}_1 - z_1$, and we claim that the following

$$T = \frac{1}{a} + (\frac{1}{b} - \frac{1}{a})P \tag{6.25}$$

fulfills our purpose. Indeed, due to the good property (i.e., $P(1-P) = 0$) of the projection operator P, it is easy to see

$$T^{-1} = a + (b - a)P. \tag{6.26}$$

Then we have

$$\begin{aligned} T_xT^{-1} &= [MP + PM^\dagger - P(M + M^\dagger)P](a(1-P) + bP)(z) \\ &= aPM_1^\dagger(1-P) + b(1-P)M_1P. \end{aligned}$$

While

$$\begin{aligned} TMT^{-1} &= (\frac{1}{a} + (1/b - 1/a)P)M(a + (b-a)P) \\ &= M + 1/aM(b-a)P + (1/b - 1/a)PMa + (1/b - 1/a)(b-a)PMP \\ &= M + bc(-1+P)MP + acPM(1-P). \end{aligned}$$

Adding them up, we obtain that

$$\begin{aligned} T_xT^{-1} + TMT^{-1} &= M + bc(1-P)(M_1 - M)P + acP(M + M_1^\dagger)(1-P) \\ &= M - ic(1-P)\sigma_3P + icP\sigma_3(1-P) \\ &= M + ic[P, \sigma_3]. \end{aligned}$$

This completes the proof. $\square$

Remark 6.3. From the proof we can see that the proof does not depend on the x variable, thus for if one consider the Lax pair t-part, it is easy to see that the new zero curvature equation $\tilde{M}_t - \tilde{N}_x + [\tilde{M}, \tilde{N}] = T(M_t - N_x + [M, N])T^{-1} = 0$, provided the initial potential is just constant (or the so-called 'seed' solution). This is guaranteed by the property of the AKNS hierarchy that for a constant potential, the t-part Lax pair shares the same structure as the x-part. Thus, the potential generated by the above procure will still satisfy the evolutionary PDEs, which correspond to the same zero curvature equation. This enables us to construct soliton solutions, rational solutions, breathers and other interesting solutions.

From this theorem, we have constructed a Darboux transformation. In fact, by choosing a "seed" potential $Q_0(x)$ and any initial eigenvalue $z_0, \mathrm{Im}(z_0) \neq 0$, we can get a new potential satisfying the AKNS spectral problem. One applies T to the eigenvector (ψ_0) generated by the seed potential and the initial eigenvalue, a new solution $\psi_1 = T\psi_0$ will be created satisfying the following new spectral problem:

$$\psi_{1x} = (iz\sigma_3 + Q_1(x))\psi_1, \tag{6.27}$$

with $Q_1 = Q_0 + i(\bar{z}_0 - z_0)\left[\frac{\psi_0\psi_0^\dagger}{\psi_0^\dagger\psi_0}, \sigma_3\right]$.

Theorem 6.4 (Generalized Darboux Transformation)**.** *Given a pair of eigenvalue and eigenfunction* $(z_1, \psi(z_1))$*, denote* $\psi[2](z_1) := \lim_{\delta\to 0} \frac{T[1](z_1+\delta)\psi(z_1+\delta)}{\delta}$*. Then*

$$T[2] = z - \bar{z}_1 + (\bar{z}_1 - z)P[2],$$

where $P[2] = \psi[2]\psi[2]^\dagger/(\psi[2]^\dagger\psi[2])$*, gives a so-called generalized Darboux Transformation (gDT).*

Proof. Due to the classical DT, $T[1](z)\psi(z_1)$ solves $\psi_x = (T[1](z)M)\psi$, by linearity, $T[1](z)/\delta$ is also a solution and hence $T[1](z_1+\delta)/\delta$ solves the the following linear DE:

$$\psi_x(z_1+\delta) = (T[1](z_1+\delta)M(z_1+\delta))\psi.$$

Assuming sufficient smoothness of ψ with respect to variable z_1, one can easily see that $\psi[2] = \psi[1] + T[1](z_1)\psi'[1](z_1)$ solves the limit DE, where we have used the property that $T[1](z_1)\psi(z_1) = 0$. □

Remark 6.5. From the generalized DT, the new potential is represented as

$$q[2] = q[1] + 2i(\bar{z}_1 - z_1)(P[2])_{12}. \tag{6.28}$$

And most importantly, the above construction can be iterated any times and any constant seed potentials to generate some new exact solutions such as rogue-wave solutions (nonzero background).

6.5 Exact Solutions of the 5th-order mKdV equation

6.5.1 Simple-pole-solitons with zero background: $q_0(x,t) = 0$

Starting with a trivial solution 0, we will apply the classical DT to get N-soliton solutions. Based on the seed solution, one can immediately solve the Lax equations, which gives

$$\psi_0(z) = e^{\omega(z;x,t)\sigma_3}(c_1(z), c_2(z))^T, \tag{6.29}$$

where $\omega(z;x,t) := izx + 16iz^5t + \gamma(z)$. Then applying the classical DT, we first construct a projection operator

$$P_1(z_1;x,t) = \frac{\psi_0(z_1)\psi_0^\dagger(z_1)}{\psi_0^\dagger(z_1)\psi_0(z_1)}, \tag{6.30}$$

then the DT operator is readily constructed as

$$T_1(z;x,t) = z - \bar{z}_1 + (\bar{z}_1 - z_1)P_1, \tag{6.31}$$

which in turn gives a new potential

$$Q_1(x,t) = 0 + i(\bar{z}_1 - z_1)(P_1\sigma_3 - \sigma_3 P). \tag{6.32}$$

This gives a one-soliton solution to the 5th-order mKdV equation:

$$q_{1ss}(x,t) = \frac{4\,\mathrm{Im}(z_1)c_1\bar{c}_2}{|c_1|^2e^{-2\,\mathrm{Im}\,A} + |c_2|^2e^{2\,\mathrm{Im}\,A}}, \tag{6.33}$$

where $A = 16z_1^5t + z_1x$, $z_1 \in i\mathbb{R}$ and c_1, c_2 are constants such that $c_1\bar{c}_2 \in \mathbb{R}$.

6.5.2 Double-pole-solitons with zero background: $q_0(x,t) = 0$

In this subsection, we will show how to apply the gDT to obtain so-called double-pole-solitons [25]. First, we need to choose a seed solution, say

$$\psi_1 := \psi_0(z_1) = \begin{pmatrix} c_1e^A \\ c_2e^{-A}, \end{pmatrix}$$

where

$$A = i(z_1x + 16z_1^5t)$$
$$z_1 = i\eta, \quad \eta \in \mathbb{R}.$$

Then we construct the projection operator $P[1]$ as well as the DT operator $T[1](z)$ as in the last section. Now in order to get the DPS, let new eigenvalue z_2 approach z_1 and calculating the limit will give us a new "seed" to construct a new projection operator $P[2]$ as well as a new DT operator $T[2]$. In fact, since

$$\begin{aligned}
&T[1](z_1+\delta)\psi(z_1+\delta)\\
&\quad=(T[1](z_1)+\delta)(\psi(z_1)+\psi'(z_1)\delta+\mathcal{O}(\delta^2))\\
&\quad=(\psi_1+T[1](z_1)\psi'(z_1))\delta+\mathcal{O}(\delta^2)\\
&\quad=:\psi_1[1](z_1)\delta+\mathcal{O}(\delta^2).
\end{aligned}$$

Then the projection operator is constructed as

$$P[2]=\frac{\psi_1[1]\psi_1^\dagger[1]}{\psi_1^\dagger[1]\psi_1[1]}.$$

And the new potential is

$$q_{1dps}(x,t)=q_{1ss}+2i(\bar{z}_1-z_1)(P[2])_{12} \tag{6.34}$$

$$=q_{1ss}+\frac{4\eta c_1\bar{c}_2(1-B^2)}{l(1+B^2)+2hB}, \tag{6.35}$$

where

$$h=|c_1|^2e^{2A}-|c_2|^2e^{-2A}, \tag{6.36}$$

$$l=|c_1|^2e^{2A}+|c_2|^2e^{-2A}, \tag{6.37}$$

$$B=-2\eta(x+80\eta^4t)(1-h). \tag{6.38}$$

Note that the denominator of the second term of p_{1dps} is $|c_1|^2e^{2A}|1+B|^2+|c_2|^2e^{-2A}|1-B|^2$, since $c_1c_2\neq 0$, that denominator is strictly greater than 0, hence the solution we obtained is real analytic in x,t.

6.5.3 Simple-pole-Solitons with nonzero background: $q_0(x,t)=b$

If we choose "seed" solution to be $q_0=b$. Then the Lax pair becomes:

$$\psi_x=(iz\sigma_3-b\sigma_1\sigma_3)\psi, \tag{6.39}$$

$$\psi_t=(izg\sigma_3-gb\sigma_1\sigma_3)\psi, \tag{6.40}$$

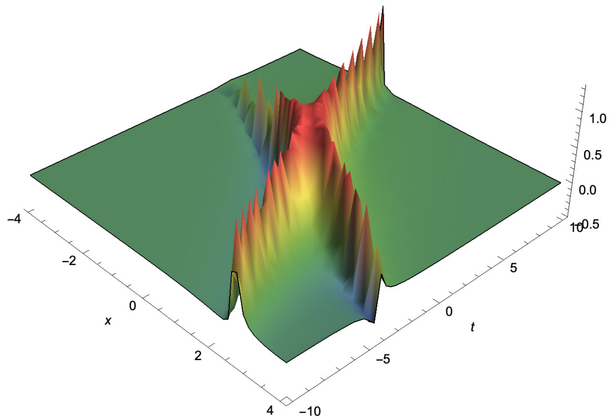

Figure 7. Double pole soliton solution

where

$$g = 6b^4 - 8b^2 z^2 + 16z^4.$$

Again, we need to construct a projection operator from a special solution of the Lax equations. In fact, since both equations only involve constant coefficient with respect to x, t, we can easily solve the system. And we then obtain a nontrivial special solution as follows:

$$\psi_1(z_1; x, t) = \begin{pmatrix} e^A + \frac{z_1 - \lambda}{b} e^{-A} \\ -i(\frac{z_1-\lambda}{b} e^A + e^{-A}) \end{pmatrix} := \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \tag{6.41}$$

where $\lambda = \sqrt{b^2 + z_1^2}$ with a properly-chosen branch cut, and $A = i\lambda x + ig\lambda t$. Then due to the classical DT, it is east to obtain the following one-soliton solution:

$$q_{n1ss}(x, t) = b + \frac{4\eta \phi_1 \bar{\phi}_2}{|\phi_1|^2 + |\phi_2|^2}, \tag{6.42}$$

where $z_1 = i\eta$.

6.6 N-fold Darboux transformation

In this section, we will formulate the N-fold DT in terms a quotient of two Vandermonde-like determinants follows the work of Steudel-Meinel-Neugebeauer's work and constraint on the focusing reduction only. As usually, given a Lax pair:

$$\psi_x = M(z; x, t)\psi \tag{6.43}$$

$$\psi_t = N(z; x, t)\psi. \tag{6.44}$$

Set DT matrix

$$T = \sum_{j=0}^{N} T_j z^j,$$

where

$$T_j = \begin{pmatrix} A_j(x,t) & B_j(x,t) \\ C_j(x,t) & -A_j(x,t) \end{pmatrix},$$

and $T_N = I$.

As noted in the last section, we want

$$T_x T^{-1} + T M_0 T^{-1} = iz\sigma_3 + Q, \tag{6.45}$$

where here we consider the AKNS hierarchy and M_0 is our initial spectral problem and α, which is independent of z, is to be determined. The reason why it works is based on the following relation:

$$M_t - N_x + [M, N] = T(M_{0t} - N_{0x} + [M_0, N_0])T^{-1}.$$

And since the nonlinear PDEs are uniquely determined by the zero curvature equation, the new potential which is contained in the pair (M, N) solves the same nonlinear PDEs as the one generated by the pair (M_0, N_0). The remaining job is to determine the DT matrix T.

From the ansatz, we have

$$\det T = \prod_{k=1}^{2N} (z - z_k),$$

with all the z_k's are distinguished.

At those zeros, $\det \psi = \det T \det \psi_0 = 0$, hence we can represent $\psi = (\psi_1, \psi_2)$ with $\psi_1 = b_k \psi_2$ at the zero z_k, where $b_k \neq 0$ is independent of x, t. In fact, since

$$\begin{aligned} \psi_{1x} &= M\psi_1 = M b_k \psi_2 \\ &= (b_k \psi_2)_x = b_{kx}\psi_2 + b_k \psi_{2x} \\ &= (b_{kx} + b_k M)\psi_2, \end{aligned}$$

we have,

$$b_{kx} = b_k M - M b_k = 0.$$

Similarly, one can show $b_{kt} = 0$; hence b_k is independent of x, t.

Then together with our DT matrix ansatz, we arrive at the following linear systems:

$$\sum_{j=0}^{N-1} z_k^j (A_j + \alpha_k B_j) = -z_k^N, \quad k = 1, \cdots, 2N, \tag{6.46}$$

where

$$\alpha_k = \frac{\psi_{021}(z_k) - b_k \psi_{022}(z_k)}{\psi_{011}(z_k) - b_k \psi_{012}(z_k)}.$$

Let us consider the focusing type reduction, i.e., let $M = iz\sigma_3 + Q$ with $\bar{Q} = -Q^T$. Under this reduction, it is evident that the zeros come in pairs: $(z_k, \bar{z}_k)$. Without loss of generality, we set $z_{N+k} = \bar{z}_k, k = 1, \cdots, N$. Note also that the reduction leads to the following property of the solutions to the spectral problem:

$$\psi(z)\psi^\dagger(\bar{z}) = I. \tag{6.47}$$

The same is true for ψ_0 if $\bar{Q}_0 = -Q_0^T$. Then it is easy to check that

$$b_{N+k} = -\bar{b}_k^{-1}.$$

Back to our linear systems, rewriting it in the matrix form, we have

$$\begin{pmatrix} \mathcal{V}_N & \alpha \mathcal{V}_N \\ \bar{\mathcal{V}}_N & \beta \bar{\mathcal{V}}_N \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = - \begin{pmatrix} Z_N \\ \bar{Z}_N, \end{pmatrix} \tag{6.48}$$

where

$$\mathcal{V}_N = \begin{pmatrix} 1 & z_1, \cdots, z_1^{N-1} \\ 1 & z_2, \cdots, z_2^{N-1} \\ \vdots & \vdots \\ 1 & z_N, \cdots, z_N^{N-1} \end{pmatrix},$$

and

$$A = \begin{pmatrix} A_0 \\ \vdots \\ A_{N-1} \end{pmatrix}, \quad B = \begin{pmatrix} B_0 \\ \vdots \\ B_{N-1} \end{pmatrix}, \quad Z_N = \begin{pmatrix} z_1^N \\ \vdots \\ z_N^N \end{pmatrix},$$

and

$$\alpha = \text{Diag}(\alpha_1, \cdots, \alpha_N),$$
$$\beta = -\bar{\alpha}^{-1}.$$

Due to the DT condition given in the equation (6.45), we have

$$Q = Q_0 + i[T_{N-1}, \sigma_3]. \tag{6.49}$$

To obtain a new potential, we only need to know B_{N-1}, which can be determined by applying Cramer's rule:

$$B_{N-1} = \frac{\tilde{\mathcal{V}}_{N,N}(\alpha,\beta)}{\mathcal{V}_{N,N}(\alpha,\beta)}, \tag{6.50}$$

where

$$\mathcal{V}_{N,N}(\alpha,\beta) = \begin{pmatrix} \mathcal{V}_N & \alpha\mathcal{V}_N \\ \bar{\mathcal{V}}_N & \beta\bar{\mathcal{V}}_N \end{pmatrix},$$

and

$$\tilde{\mathcal{V}}_{N,N}(\alpha,\beta) = \left([\mathcal{V}_{N,N}(\alpha,\beta)]_{2N\times(2N-1)}, -\begin{pmatrix} Z_N \\ \bar{Z}_N, \end{pmatrix} \right),$$

$[\cdot]_{i\times j}$ means choosing first i rows and first j columns. Moreover, applying the Laplace expansion theorem, one can reduce the Vandermonde-like matrix $\mathcal{V}_{N,N}$ into

$$\sum_{P_1,P_2} \sigma(P) \det \mathcal{V}(P_1) \det \mathcal{V}(P_2) \tag{6.51}$$

where $\sum_P$ means adding all possible permutations such that both P_1 and P_2 are increasing permutations with length N and $\sigma(P) = sign(P_1, P_2)$. For example, take $N = 2$, $P_1 \in \{(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\}$ while $P_2 \in \{(3,4),(2,4),(2,3),(1,4),(1,3),(1,2)\}$ in order. And for the notation $\mathcal{V}(P_1)$ is defined by choosing the rows and columns with respect to the permutation P_1. Taking $P_1 = (1,3)$ for example, by definition, we have

$$\mathcal{V}(P_1) = \begin{pmatrix} 1 & z_1 \\ 1 & \bar{z}_1 \end{pmatrix}, \quad \mathcal{V}(P_2) = \begin{pmatrix} \alpha_2 & \alpha_2 z_2 \\ \beta_2 & \beta_2 \bar{z}_2 \end{pmatrix}. \tag{6.52}$$

In fact, we have proved the following theorem.

Theorem 6.6 ($N-$fold classical DT for the AKNS hierarchy)**.** *Suppose $q_0(x,t) = Q_{1,2}$ solves the nonlinear PDE generated by the lax pair (M_0, N_0). Then the $N-$fold Darboux Transformation T constructed above generates a new solution $q(x,t)$ to the same nonlinear PDE as*

$$q(x,t) = q_0 - 2iB_{N-1}, \tag{6.53}$$

where B_{N-1} is defined in the equation (6.50).

Remark 6.7. If we set $Q_0 = 0$ and $z_k \in i\mathbb{R}$, it generates the famous $N-$soliton solutions. If $Q_0 = constant$, the formula gives the breather solutions to the mKdV hierarchy (odd parts of the AKNS hierarchy). In particular, for the focusing 5th-order mKdV, taking $Q_0 = 0$, we have $\psi_0 = \exp{(izx + 16iz^5 t)}\sigma_3$.

6.7 Generalized Darboux transformation in terms of generalized Vandermonde-like matrices

To construct the general DT matrix, it is sufficient to construct

$$T(z) = \sum_{j=0}^{N} T_j z^j, \tag{6.54}$$

such that

$$\det T(z) = \prod_{k=1}^{s} (z - z_k)^{n_k} (z - z_{k+N})^{n_k}, \tag{6.55}$$

where

$$\sum_{k=1}^{s} n_k = N.$$

Then if we also know the generalized co-linear coefficients for $\psi(z) = T(z)\psi_0(z)$ at each zeros of the determinant of $T(z)$, say ,

$$\begin{aligned} \psi_1(z_k) &= b_{k0}\psi_2(z_k) \\ \psi_1^{(1)}(z_k) &= b_{k1}\psi_2^{(1)}(z_k) \\ &\vdots \\ \psi_1^{(n_k-1)}(z_k) &= b_{k,n_k-1}\psi_2^{(n_k-1)}(z_k), \end{aligned}$$

with $k = 1, 2, \cdots, s$. For the sake of simplicity, we consider the case that

$$\det T(z) = (z - z_0)^N (z - z_N)^N. \tag{6.56}$$

Moreover, if the focusing reductions are performed. Then

$$\det T(z) = (z - z_0)^N (z - \bar{z}_0)^N. \tag{6.57}$$

Providing the co-linear coefficient data at each zero, and using the matrix polynomial ansatz for $T(z)$, we obtain, for each k,

$$\begin{aligned}&\sum_{l=0}^{k} \sum_{j=l}^{N-1} \frac{j!}{(j-l)!} z_0^{j-l} (\alpha_{kl} A_j + \beta_{kl} B_j) \\ &= -\sum_{l=0}^{k} \frac{N!}{(N-l)!} z_0^{N-l} \alpha_{kl}, \quad k = 0, 1, \cdots, N-1,\end{aligned}$$

where

$$\begin{aligned}\alpha_{kl} &= (\psi_{011}^{(k-l)}(z_0) - b_k \psi_{012}^{(k-l)}(z_0)) \binom{k}{l} \\ \beta_{kl} &= (\psi_{021}^{(k-l)}(z_0) - b_k \psi_{022}^{(k-l)}(z_0)) \binom{k}{l},\end{aligned}$$

To see the solvability, we rewrite the above equations in matrix form. First denote $\mathcal{V}^\sharp(\alpha)$ as follows:

$$\begin{pmatrix} \alpha_{00} & \alpha_{00} z_0 & \alpha_{00} z_0^2 & \cdots & \alpha_{00} z_0^{N-1} \\ \alpha_{10} & \alpha_{10} z_0 + \alpha_{11} & \alpha_{10} z_0^2 + \alpha_{11}(2z_0) & \cdots & \alpha_{10} z_0^{N-1} + \alpha_{11}(N-1) z_0^{N-2} \\ \alpha_{20} & \alpha_{20} z_0 + \alpha_{21} & \alpha_{20} z_0^2 + \alpha_{21}(2z_0) + \alpha_{22} 2 & \cdots & \sum_{l=0}^{k} \alpha_{k,l} \frac{k!}{(k-l)!} z_0^{k-l} \\ \vdots & \vdots & \vdots & & \vdots \\ \alpha_{N-1,0} & \alpha_{N-1,0} z_0 + \alpha_{N-1,1} & \alpha_{N-1,0} z_0^2 + \alpha_{N-1,1}(2z_0) + \alpha_{N-1,2} 2 & \cdots & \sum_{l=0}^{N-1} \alpha_{N-1,l} \frac{(N-1)!}{(N-1-l)!} z_0^{N-1-l} \end{pmatrix}. \tag{6.58}$$

Similarly, one can define $\mathcal{V}^\sharp(\beta)$. Then the system of linear equations can be represented as

$$\begin{pmatrix} \mathcal{V}^\sharp(\alpha) & \mathcal{V}^\sharp(\beta) \\ \bar{\mathcal{V}}^\sharp(-\bar{\alpha}^{-1}) & \bar{\mathcal{V}}^\sharp(\bar{\beta}^{-1}) \end{pmatrix} \begin{pmatrix} A_0 \\ \vdots \\ A_{N-1} \\ B_0 \\ \vdots \\ B_{N-1} \end{pmatrix} = -\begin{pmatrix} Z_N^\sharp \\ \bar{Z}_N^\sharp \end{pmatrix}, \tag{6.59}$$

where

$$Z_N^\sharp = \begin{pmatrix} \alpha_{00} z_0^N \\ \alpha_{10} z_0^N + \alpha_{11} N z_0^{N-1} \\ \vdots \\ \sum_{l=0}^{N-1} \frac{N!}{(N-l)!} \alpha_{N-1,l} z_0^{N-l} \end{pmatrix}. \tag{6.60}$$

Then by applying Cramer's rule, it is evident that

$$B_{N-1} = \frac{\det \tilde{\mathcal{V}}^\sharp(\alpha,\beta)}{\det \mathcal{V}^\sharp(\alpha,\beta)}, \tag{6.61}$$

where

$$\mathcal{V}^\sharp(\alpha,\beta) = \begin{pmatrix} \mathcal{V}^\sharp(\alpha) & \mathcal{V}^\sharp(\beta) \\ \bar{\mathcal{V}}^\sharp(-\bar{\alpha}^{-1}) & \bar{\mathcal{V}}^\sharp(\bar{\beta}^{-1}) \end{pmatrix}, \tag{6.62}$$

and by replacing the last column of $\mathcal{V}^\sharp(\alpha,\beta)$ by $-\begin{pmatrix} Z_N^\sharp \\ \bar{Z}_N^\sharp \end{pmatrix}$, we got $\tilde{\mathcal{V}}^\sharp(\alpha,\beta)$. By simple elementary column operations, we obtain

$$\det \mathcal{V}^\sharp(\alpha) = \prod_{j=0}^{N-1} \alpha_{jj} j!.$$

Again, by using the Laplace expansion theorem,

$$\mathcal{V}^\sharp(\alpha,\beta) = \sum_{P_1,P_2} \sigma(P) \mathcal{V}^\sharp(P_1) \mathcal{V}^\sharp(P_2) \tag{6.63}$$

Remark 6.8. The Laplace expansion theorem reduces the computation complexity from $2N!$ terms to $\binom{2N}{N}$ terms.

Chapter 7
Discussions

7.1 About the $\bar{\partial}-$steepest descent method

As we already saw, during the construction of the $\bar{\partial}-$extension function E, we only used the lowest regularity of the initial data. A nature question to ask here is how one can improve the estimate errors if more regularities of the initial data are provided? In Deift-Zhou 2003's work [12], they considered the long-time asymptotics for the defocusing NLS under H^1 norm. The method they used is a nonlinear version of the steepest descent method which can be extended to obtain smaller errors provided more regularities of the initial data. The main idea is to get a good rational approximation to the phase function so that the Cauchy transform of the approximation function as well as the large parameter (t) asymptotic of the oscillatory RHP can be estimated by the classical saddle point method. On the other hand, the $\bar{\partial}-$steepest descent simplified those complicated harmonic analysis. If the theorem of $\bar{\partial}-$steepest descent method for $H^{j,k}$ initial data can be established, the proceeding of long-time asymptotics for the integrable equations can be completely simplified to some fairly simple estimates of multiple integrals.

7.2 Higher dimensional generalization

The fundamental idea of the $\bar{\partial}-$steepest descent method is to apply the Cauchy-Green's theorem. While its higher dimensional version is Stokes' theorem, there is a possibility to extend the method to the higher dimensional case. So the problems such as long-time asymptotics of the KP equation and the DS equation, could be investigated in a similar way.

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