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Critical Axial Load

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Abstract

Our objective in this paper is to solve a second order differential equation for a long, simply supported column member subjected to a lateral axial load using Heun's numerical method. We will use the solution to find the critical load at which the column member will fail due to buckling. We will calculate this load using Euler's derived analytical approach for an exact solution, as well as Euler's Numerical Method. We will then compare the three calculated values to see how much they deviate from one another. During the critical load calculation, it will be necessary to calculate the moment of inertia for the column member.

Keywords

Axial load, Critical load calculation, Buckling

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Motivation

This problem is extremely important in the field of engineering, especially structural engineering. If a column member is subjected to an **axial load**—a load applied along (or parallel to) and concentric with the primary axis (ref. #1)—and the member fails, the result could be catastrophic. The failure in this situation is called **buckling** and is defined as the moment at which the axial load is greater than the column member is capable of supporting. The load at which buckling occurs is called the **critical load**. The critical load (F_{cr}) can be calculated by solving a differential equation for a simply supported column member,

$$\frac{d^2 w}{dx^2} + \frac{F}{EI} w = 0.$$

This equation can be solved by a variety of methods. We will use Euler's Numerical Method, Heun's Numerical Method, and Euler's analytical approach for an exact solution. We will then compare the calculations to determine how the three methods differ. Although three methods are being shown here, Heun's method is the main focus of this paper.

Before we begin, we will need a short review of bending and buckling Moments. A **moment** is the column's ability to resist the bending or buckling by applying a force, created by the member itself, with the same magnitude as the axial load multiplied by its deflection (δ) or (w), but in the opposite direction. Examine the free body diagram of the column (fig. 1): the red arrows represent the axial load applied to the column member, while the blue arrows represent the moment. Notice how the moment is resisting the deflection of the member by "pushing" the column back to its original position. As the axial load is increased, the deflection of the column is increased; this deflection is being overcome by the moment and the elasticity of the column member. It is this moment that determines the value of F_{cr} , because when the axial load exceeds the moment force, the column member is subjected to buckling.

Mathematical Description and Solution Approach

There are many computer programs available to solve problems of this nature. Here we use Microsoft Excel, but we could have easily used Maple, MatLab, or any of the number of computer software that have been designed for problems of this nature. These programs allow the engineer to calculate such values much faster than the traditional pen and paper. Even with powerful computer programs designed to solve equations, part of the procedure requires manual calculations. The following details the procedure in which these methods are performed, using Microsoft Excel.

The governing equation—known as Euler’s equation—for a column experiencing an axial load can be either found in a textbook, or determined from the **free body diagram** of the column (fig. 2), which depicts the forces acting on the member. For the column to remain in its vertical position, equilibrium of the moment and axial forces must be attained. The free body diagram shows that the downward force acting on the column member creates a deflection (w) and the moment (M) opposes the deflection by applying an internal force. Therefore, equilibrium is attained when

$$Fw - M = 0. \quad (1)$$

The moment is related to the transverse displacement (w) by

$$M = -EI \frac{d\theta}{dx} \quad \text{and} \quad \frac{d\theta}{dx} = \frac{d^2w}{dx^2}.$$

Therefore,

$$M = -EI \frac{d^2w}{dx^2}. \quad (2)$$

If M is removed from equations (1) and (2), the result is the governing equation for elastic buckling of a relatively long, slender column (ref. #4):

$$\frac{d^2w}{dx^2} + \frac{F}{EI}w = 0. \quad (3)$$

This second order homogeneous differential equation can be solved using numerous methods. Here, the equation will be solved using the **Heun’s Numerical Method** and then verified using Euler’s Numerical and Analytical methods.

First, let us review Heun’s Numerical Method. **Heun’s method** is referred to as the Improved Euler’s Method (ref. #5) because it is an improvement of Euler’s Numerical Method. Euler’s Numerical Method uses the tangent line endpoints as an approximation of the true solution. By examining the graph (ref. #5), it is obvious

that with an upward concavity the tangent line under-estimates the next point on the true solution curve, (x_{i+1}, y_{i+1}) . If the curve is concave down, then the tangent line will over-estimate

the next point on the actual solution curve, (x_{i+1}, y_{i+1}) . Therefore, Heun decided to adjust the tangent line. If the slope of the tangent line is increased, it will be a little closer to the next point on the actual solution curve. But, if the tangent line is too steep it will overestimate the next point on the actual solution curve. So, Heun decided to take the average of the overestimated/underestimated slopes to numerically determine a solution. (ref. #5).

The equation we are solving numerically is

$$\frac{d^2 w}{dx^2} + \frac{F}{EI} w = 0.$$

Now, we must use Euler's Numerical Method to begin the numerical approach. Remember that the Heun's method is an improvement of Euler's method. With this in mind, we let

$$P = \frac{F}{EI}.$$

Solving for $\frac{d^2 w}{dx^2}$ leaves the equation

$$\frac{d^2 w}{dx^2} = -Pw.$$

If $y = dw/dx$, then

$$\frac{dy}{dx} = -Pw.$$

Now, instead of having one second order differential equation, we have two first-order differential equations that need to be solved simultaneously. We need to start by estimating the slope:

$\frac{dw}{dx} = y$ $\Rightarrow \frac{\Delta w}{\Delta x} = y$ $\Rightarrow \frac{w_{i+1} - w_i}{\Delta x} = y_i$ $\Rightarrow w_{i+1} = y_i \cdot \Delta x + w_i$	$\frac{dy}{dx} = -Pw$ $\Rightarrow \frac{\Delta y}{\Delta x} = -Pw$ $\Rightarrow \frac{y_{i+1} - y_i}{\Delta x} = -Pw_i$ $\Rightarrow y_{i+1} = -Pw_i \cdot \Delta x + y_i$
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This is Euler's Numerical Method and is calculated in the Appendix. We must now modify, and

$$ka = P \cdot y_{i+1}$$

$$kb = P(y_{i+1} \cdot \Delta x + w_{i+1})$$

$$Ka = y_{i+1}$$

$$Kb = \Delta x \cdot P + y_{i+1}$$

$$w_{i+1} = w_i + \left(\frac{\Delta x}{2}\right)(Ka + Kb)$$

$$y_{i+1} = y_i - \left(\frac{\Delta x}{2}\right)(ka + kb)$$

take the average of the estimated slopes as formulated by Heun. If we continue to solve these equations simultaneously, we can perform the calculations on the left, where ka , kb , Ka , and Kb are the adjusted tangent line slopes and w_{i+1} and y_{i+1} are the averages of the two slopes for each of the two equations. Now we can place these six equations on a spreadsheet, and solve for P_{cr} (See the Appendix). Since $P_{cr} = F_{cr}/EI$, the critical axial load that a simply supported column member can support is $F_{cr} = P_{cr} \cdot EI$. When solving an equation numerically, the initial slope must be estimated by trial and error. Heun decided that it is necessary to assume a relatively small initial slope in order to

correctly determine a numerical solution. Heun also stated that it is important to choose a small Δx or "step size" to obtain the most accurate solution. For this problem, using Heun's method and Euler's numerical method, the initial slope of the solution is set at 0.1 inch. This small initial slope, with the aid of the adjusted tangent lines, allows us to obtain an accurate numerical solution to this problem. For a column member that is 120 inches long, the value calculated for P_{cr} is approximately 0.000686/in². Remember that this numerical solution is only an approximation to the actual solution. Since an actual solution to equation (3) exists, it makes sense to verify the numerical solutions by the actual solution. This will be shown after the Moment of Inertia is calculated.

The **Moment of Inertia** is a geometrical property of an area about a particular reference axis (ref. #6). The mathematical definition of the Moment of Inertia is

$$I_y = \int x^2 dA \quad \text{and} \quad I_x = \int y^2 dA.$$

where,

dA = the cross-sectional area;

x = the perpendicular distance to the element dA from the y -axis;

y = the perpendicular distance to the element dA from the x -axis (ref. #7).

These equations can only be used for a member that is symmetric to the coordinate plane.

In most cases—in the practice of engineering—the objects that are used have relatively basic geometric shapes (ref. #6); therefore, the moment of inertia formula has already been derived and is published in many textbooks. Below are the methods used for square and circular columns that are 120 inches long. This length was chosen, because it is the length that was considered on the Excel spreadsheet and P_{cr} has already been determined. The formula for

Inertia that was stated earlier will work for square and circular objects of any dimension. Here is the procedure for a rectangular member.

(Figure is from ref. # 10)

$$\begin{aligned}
 I &= \int_{-d/2}^{d/2} by^2 dy \\
 &= b \left[\frac{y^3}{3} \right]_{-d/2}^{d/2} \\
 &= b \left[\frac{d^3}{24} + \frac{d^3}{24} \right] \\
 &= b \left(\frac{d^3}{12} \right)
 \end{aligned}$$

Therefore, if we are considering a 4 in. by 120 in. steel column member, the moment of inertia is

$$b \cdot \frac{d^3}{12} = 120 \cdot \frac{4^3}{12} = 640 \text{ in}^4.$$

This is the inertia about the x -axis, the axis that experiences the buckling. If the inertia about the y -axis is calculated, we would cube 120 inches, so the moment of inertia in the y -direction is much stronger than that in the x -direction. We must note that in buckling problems, the coordinate plane is chosen so that if the column is rotated 90 degrees, the center of the member is on the positive axis. This means that the buckling will not occur along the y -axis (ref. #7).

For a cylindrical column member the inertia is calculated by using the polar coordinates r and θ , where, $y = r \sin \theta$.

(Figure is from ref. #7)

$$\begin{aligned}
 I_x &= \int y^2 dA \\
 &= \int_0^{2\pi} \int_0^r r^3 \sin^2 \theta dr d\theta \\
 &= \frac{r^4}{4} \int_0^{2\pi} \sin^2 \theta d\theta & dA = r dr d\theta \\
 &= \frac{r^4}{4} \int_0^{2\pi} \frac{1 - \cos 2\theta}{2} d\theta & y = r \sin \theta \\
 &= \frac{r^4}{4} \left[\frac{\theta}{2} - \frac{\sin 2\theta}{4} \right]_0^{2\pi} \\
 &= \frac{r^4}{4} [(\pi - 0) - (0 - 0)] \\
 &= \pi \frac{r^4}{4}.
 \end{aligned}$$

Above, we considered a column member with a round geometry, one inch in diameter and 120 in. long. The moment of inertia is

$$\pi \cdot \frac{r^4}{4} = \pi \cdot \frac{1}{4} = 0.7854 \text{ in}^4.$$

Notice that the moment of inertia for a circular object is independent of its length.

The elasticity of the material must now be determined. The elasticity is the Young's Modulus of Elasticity. The unit for elasticity is lb/in² and is tabulated in a variety of books. The Young's modulus (E) is a measure of the stiffness of an isotropic elastic material (ref. #7). When the axial load is increased, the elasticity of the material will allow the column to deflect slightly. If the axial load is then decreased, the elasticity, along with the moment of inertia, allows the column to return to its original vertical position. But when the critical load is reapplied, it will not be able to return to its original position when the load is removed, remaining in a permanent deformation. This is known as the *elastic limit* (ref. #3). At this point an axial load has the ability to create the buckling phenomena. Here, we will use the Elasticity (E) value of steel, which is 30,000ksi (kilo-pounds per square inch) (ref. #6).

Now we can calculate the maximum load that our round, steel column can support. Remember that $\frac{F_{cr}}{EI} = 0.000686 \text{ in}^{-2}$ and that $I = 0.7854 \text{ in}^4$ and $E = 300 \times 10^5 \text{ in}^2$. Therefore, the maximum force that the round column can support is

$$F_{cr} = \frac{686 \times 10^{-6}}{\text{in}^2} \cdot 300 \times 10^5 \frac{\text{lb}}{\text{in}^2} \cdot 785.4 \times 10^{-3} \text{ in}^4$$

$$\approx 16,163 \text{ lbs.}$$

For the 4 in. by 120 in. column member, $\frac{F_{cr}}{EI} = 686 \times 10^{-6} / \text{in}^2$, E is the same as before, and $I = 640 \text{ in}^4$. The maximum load that a column of this material and dimensions can support is

$$F_{cr} = \frac{686 \times 10^{-6}}{\text{in}^2} \cdot 300 \times 10^6 \frac{\text{lb}}{\text{in}^2} \cdot 640 \text{ in}^4$$

$$\approx 1.317 \times 10^7 \text{ lb.}$$

Euler's Numerical method gives approximately the same value of F_{cr}/EI , only slightly deviating from the Heun's Method—this is not the case for all differential equations. Therefore, Euler's numerical Method will have approximately the same F_{cr} for this equation.

Recall that there are a number of different methods available for solving this equation. So far, we have solved this equation numerically, using both the Euler's and Heun's Numerical Methods. Now, we will see how this equation is solved analytically (See the Appendix). It was stated that the numerical method is used when there is no analytical solution to the equation. This equation has an exact solution and it was Euler who derived this solution in the eighteenth century. Euler's solution is

$$F_{cr} = \frac{n^2 \pi^2}{L^2} \cdot EI,$$

where (n) is the mode in which the column will buckle and is an integer. The figure to the left shows two simply supported column members. The column on the left has $n = 1$, while the column on the right will have $n = 2$. This picture only shows the first and second modes of buckling, but there can be many modes. If the design called for another support between the top and bottom of the column member, meaning that the column has supports at four sections of its length—the two ends and two throughout its length—then the mode of buckling would be three. The other

variable, L , is the total length of the column member. For our simply supported column, $n = 1$ and $L = 120 \text{ in}$. Now, let us calculate the exact maximum load that a round, one inch diameter

column member subjected to an axial load F is capable of supporting. Remember that $E = 300 \cdot 10^5 \text{ lb/in}^2$ and $I = 0.785 \text{ in}^4$. Therefore,

$$F_{cr} = \frac{1^2 \cdot \pi^2}{120^2 \text{ in}^2} \cdot 300 \times 10^5 \frac{\text{lb}}{\text{in}^2} \cdot 0.785 \text{ in}^4 = 16,140 \text{ lb}.$$

Now for the square column, the Inertia $I = 640 \text{ in}^4$ and Elasticity $E = 300 \cdot 10^5 \text{ lb/in}^2$. With this in mind, the maximum load that a four inch-wide column member is able to support is

$$F_{cr} = \frac{1^2 \pi^2}{120^2 \text{ in}^2} \cdot 300 \times 10^5 \frac{\text{lb}}{\text{in}^2} \cdot 640 \text{ in}^4 = 1.316 \times 10^7 \text{ lb}.$$

Discussion

Now that we have calculated the critical load that a simply supported column can withstand, let us compare the values that we calculated. The Heun's Numerical Method gives approximately 16,163lb. But Euler's exact solution gives a critical load of 16,140lb for the same size column. This slight deviation is acceptable since the difference is only twenty-three lbs, and in practice the axial load would be considerably less than the critical load the member can support. What happens if the length of the column is changed? By changing the selected data in our spread sheet and allowing excel to recalculate F_{cr}/EI , we may easily calculate a new value F_{cr} using Heun's Method. For a column that is thirty inches long, having the same diameter as before, as well as the same material, the numerical analysis gives $F_{cr} = 264,702 \text{ lbs}$, and the exact solution is 258,255lbs. These values make sense, because a shorter column member should be able to support a greater load than a longer column. This deviation in calculation is a substantial amount, but the problem is not that the methods used to calculate the critical load of a column member are incorrect; it is that these methods are not used to calculate the critical load of a short column. Is there any deviation for a column member of the same material and diameter, but 170 inches in length? Again, using the spreadsheet to recalculate a new F_{cr} at this length, we obtain 8,054lbs and an exact solution of 8,042lbs. Do the values make sense? They do, because as the column increases in length, the critical load that the column is able to support decreases. Therefore, as the length of the column increases, Heun's Method deviates slightly from the exact calculation that was derived by Euler. This slight deviation allows one to assume that the Heun's Numerical Method is a good method to use, especially when there is no analytical solution to the differential equation being considered.

Conclusion

In this paper, we have seen three different approaches to finding the maximum axial load that a simply supported column member is able to support before it is subjected to buckling. We used Heun's and Euler's Numerical Methods to find this critical load by numerically solving the governing equation (3) for a simply supported column member subjected to an axial load. We also compared the values that we calculated numerically with Euler's derived analytical approach for an exact solution. In the comparison of the three methods, there is some deviations in the calculated critical loads. The deviation is small for relatively long, thin column members—only about ten pounds. These small amounts are overlooked in practice because they are only theoretical values and if a round column that is 120 inches long is subjected to an axial load of 6,000 pounds, the column's Moment of Inertia should be increased. Thus, the design will call for a larger column diameter.

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Appendix

$$\frac{d^2 w}{dx^2} + \frac{F}{EI} w = 0.$$

If $P^2 = \frac{F}{EI}$, then

$$\frac{d^2 w}{dx^2} + Pw = 0.$$

Solving this equation yields

$$w = A \cos Px + B \sin Px. \quad (a)$$

In equation (a), one must evaluate the two constants of integration A and B . Two boundary conditions must be considered for the column to be in equilibrium:

1. $w = 0$ at $x = 0$
2. $w = 0$ at $x = L$.

Using the conditions, (1) with equation (a),

$$0 = A \cos(0) + B \sin(0)$$

$$0 = A(1) + B(0)$$

$$A = 0.$$

Condition (2) applied to equation (a) gives

$$0 = B \sin PL. \quad (b)$$

To satisfy equation (b), either $B = 0$ or $\sin(PL) = 0$. If $B = 0$, the column is in equilibrium and the solution is trivial. Therefore B must have some non-zero value, even if it is indeterminate. Dividing both sides by B yields

$$\sin PL = 0.$$

This equation is an eigenvalue equation and the solutions are

$$PL = 0, \pi, 2\pi, 3\pi, \dots, n\pi.$$

From the general solution,

$$PL = n\pi; \quad \sqrt{\frac{F}{EI}} = \frac{n\pi}{L}; \quad F_{cr} = \frac{n^2\pi^2}{L^2} \cdot EI.$$

* This derivation was obtained from (ref. #6)

Wells: Critical Axial Load

Heun's Numerical and Euler's Exact

delta x= 1.0000 F/EI= 0.000686
 $\pi/120=$ 0.02618

x	y	w(1)	ka	kb	Ka	Kb	w(2)
0.000	0.1000	0	0	0.0001	0.1000	0.1007	0
1.000	0.1000	0.1003	0.0001	0.0001	0.1000	0.1007	0.1013
2.000	0.0999	0.2007	0.0001	0.0002	0.0999	0.1005	0.2026
3.000	0.0997	0.3009	0.0002	0.0003	0.0997	0.1004	0.3037
4.000	0.0995	0.4009	0.0003	0.0003	0.0995	0.1001	0.4046
5.000	0.0991	0.5007	0.0003	0.0004	0.0991	0.0998	0.5053
6.000	0.0988	0.6002	0.0004	0.0005	0.0988	0.0994	0.6056
7.000	0.0983	0.6993	0.0005	0.0005	0.0983	0.0990	0.7054
8.000	0.0978	0.7979	0.0005	0.0006	0.0978	0.0985	0.8048
9.000	0.0972	0.8961	0.0006	0.0007	0.0972	0.0979	0.9037
10.000	0.0966	0.9936	0.0007	0.0007	0.0966	0.0973	1.0019
11.000	0.0959	1.0906	0.0007	0.0008	0.0959	0.0965	1.0994
12.000	0.0951	1.1868	0.0008	0.0009	0.0951	0.0958	1.1962
13.000	0.0942	1.2822	0.0009	0.0009	0.0942	0.0949	1.2922
14.000	0.0933	1.3768	0.0009	0.0010	0.0933	0.0940	1.3872
15.000	0.0923	1.4704	0.0010	0.0011	0.0923	0.0930	1.4814
16.000	0.0913	1.5631	0.0011	0.0011	0.0913	0.0920	1.5745
17.000	0.0902	1.6548	0.0011	0.0012	0.0902	0.0909	1.6665
18.000	0.0890	1.7453	0.0012	0.0013	0.0890	0.0897	1.7574
19.000	0.0878	1.8347	0.0013	0.0013	0.0878	0.0885	1.8471
20.000	0.0865	1.9228	0.0013	0.0014	0.0865	0.0872	1.9355
21.000	0.0852	2.0097	0.0014	0.0014	0.0852	0.0859	2.0226
22.000	0.0838	2.0952	0.0014	0.0015	0.0838	0.0844	2.1083
23.000	0.0823	2.1793	0.0015	0.0016	0.0823	0.0830	2.1926
24.000	0.0808	2.2620	0.0016	0.0016	0.0808	0.0815	2.2753
25.000	0.0792	2.3431	0.0016	0.0017	0.0792	0.0799	2.3565
26.000	0.0776	2.4226	0.0017	0.0017	0.0776	0.0782	2.4361
27.000	0.0759	2.5005	0.0017	0.0018	0.0759	0.0766	2.5140
28.000	0.0741	2.5767	0.0018	0.0018	0.0741	0.0748	2.5902
29.000	0.0723	2.6512	0.0018	0.0019	0.0723	0.0730	2.6646
30.000	0.0705	2.7239	0.0019	0.0019	0.0705	0.0712	2.7372
31.000	0.0686	2.7947	0.0019	0.0020	0.0686	0.0693	2.8079
32.000	0.0667	2.8636	0.0020	0.0020	0.0667	0.0673	2.8767
33.000	0.0647	2.9306	0.0020	0.0021	0.0647	0.0654	2.9435
34.000	0.0626	2.9957	0.0021	0.0021	0.0626	0.0633	3.0083
35.000	0.0606	3.0586	0.0021	0.0021	0.0606	0.0612	3.0711
36.000	0.0584	3.1195	0.0021	0.0022	0.0584	0.0591	3.1317
37.000	0.0563	3.1783	0.0022	0.0022	0.0563	0.0570	3.1902
38.000	0.0541	3.2350	0.0022	0.0023	0.0541	0.0548	3.2465
39.000	0.0518	3.2894	0.0023	0.0023	0.0518	0.0525	3.3006
40.000	0.0496	3.3416	0.0023	0.0023	0.0496	0.0503	3.3524
41.000	0.0473	3.3915	0.0023	0.0024	0.0473	0.0479	3.4019
42.000	0.0449	3.4391	0.0024	0.0024	0.0449	0.0456	3.4491
43.000	0.0425	3.4844	0.0024	0.0024	0.0425	0.0432	3.4939
44.000	0.0401	3.5272	0.0024	0.0024	0.0401	0.0408	3.5363
45.000	0.0377	3.5677	0.0024	0.0025	0.0377	0.0384	3.5763
46.000	0.0352	3.6058	0.0025	0.0025	0.0352	0.0359	3.6139
47.000	0.0328	3.6414	0.0025	0.0025	0.0328	0.0334	3.6490
48.000	0.0303	3.6745	0.0025	0.0025	0.0303	0.0309	3.6815
49.000	0.0277	3.7051	0.0025	0.0026	0.0277	0.0284	3.7116
50.000	0.0252	3.7331	0.0026	0.0026	0.0252	0.0259	3.7391
51.000	0.0226	3.7586	0.0026	0.0026	0.0226	0.0233	3.7640
52.000	0.0200	3.7816	0.0026	0.0026	0.0200	0.0207	3.7864
53.000	0.0174	3.8019	0.0026	0.0026	0.0174	0.0181	3.8062
54.000	0.0148	3.8197	0.0026	0.0026	0.0148	0.0155	3.8233
55.000	0.0122	3.8348	0.0026	0.0026	0.0122	0.0129	3.8379
56.000	0.0095	3.8473	0.0026	0.0026	0.0095	0.0102	3.8498
57.000	0.0069	3.8572	0.0026	0.0027	0.0069	0.0076	3.8591
58.000	0.0042	3.8645	0.0027	0.0027	0.0042	0.0049	3.8657
59.000	0.0016	3.8690	0.0027	0.0027	0.0016	0.0023	3.8697
60.000	-0.0011	3.8710	0.0027	0.0027	-0.0011	-0.0004	3.8710

58.000	0.0042	3.8645	0.0027	0.0027	0.0042	0.0049	3.8657
59.000	0.0016	3.8690	0.0027	0.0027	0.0016	0.0023	3.8697
60.000	-0.0011	3.8710	0.0027	0.0027	-0.0011	-0.0004	3.8710
61.000	-0.0037	3.8703	0.0027	0.0027	-0.0037	-0.0030	3.8697
62.000	-0.0064	3.8669	0.0027	0.0026	-0.0064	-0.0057	3.8657
63.000	-0.0090	3.8609	0.0026	0.0026	-0.0090	-0.0083	3.8591
64.000	-0.0117	3.8522	0.0026	0.0026	-0.0117	-0.0110	3.8498
65.000	-0.0143	3.8409	0.0026	0.0026	-0.0143	-0.0136	3.8379
66.000	-0.0169	3.8269	0.0026	0.0026	-0.0169	-0.0162	3.8233
67.000	-0.0196	3.8103	0.0026	0.0026	-0.0196	-0.0189	3.8062
68.000	-0.0222	3.7911	0.0026	0.0026	-0.0222	-0.0215	3.7864
69.000	-0.0248	3.7693	0.0026	0.0026	-0.0248	-0.0241	3.7640
70.000	-0.0273	3.7449	0.0026	0.0026	-0.0273	-0.0266	3.7391
71.000	-0.0299	3.7179	0.0026	0.0025	-0.0299	-0.0292	3.7116
72.000	-0.0324	3.6883	0.0025	0.0025	-0.0324	-0.0317	3.6815
73.000	-0.0349	3.6563	0.0025	0.0025	-0.0349	-0.0343	3.6490
74.000	-0.0374	3.6216	0.0025	0.0025	-0.0374	-0.0368	3.6139
75.000	-0.0399	3.5845	0.0025	0.0024	-0.0399	-0.0392	3.5763
76.000	-0.0424	3.5450	0.0024	0.0024	-0.0424	-0.0417	3.5363
77.000	-0.0448	3.5029	0.0024	0.0024	-0.0448	-0.0441	3.4939
78.000	-0.0472	3.4585	0.0024	0.0023	-0.0472	-0.0465	3.4491
79.000	-0.0495	3.4117	0.0023	0.0023	-0.0495	-0.0488	3.4019
80.000	-0.0518	3.3625	0.0023	0.0023	-0.0518	-0.0512	3.3524
81.000	-0.0541	3.3110	0.0023	0.0022	-0.0541	-0.0535	3.3006
82.000	-0.0564	3.2572	0.0022	0.0022	-0.0564	-0.0557	3.2465
83.000	-0.0586	3.2012	0.0022	0.0022	-0.0586	-0.0579	3.1902
84.000	-0.0608	3.1429	0.0022	0.0021	-0.0608	-0.0601	3.1317
85.000	-0.0629	3.0825	0.0021	0.0021	-0.0629	-0.0622	3.0711
86.000	-0.0650	3.0199	0.0021	0.0020	-0.0650	-0.0643	3.0083
87.000	-0.0671	2.9552	0.0020	0.0020	-0.0671	-0.0664	2.9435
88.000	-0.0691	2.8885	0.0020	0.0019	-0.0691	-0.0684	2.8767
89.000	-0.0710	2.8198	0.0019	0.0019	-0.0710	-0.0703	2.8079
90.000	-0.0729	2.7491	0.0019	0.0018	-0.0729	-0.0722	2.7372
91.000	-0.0748	2.6765	0.0018	0.0018	-0.0748	-0.0741	2.6646
92.000	-0.0766	2.6021	0.0018	0.0017	-0.0766	-0.0759	2.5902
93.000	-0.0784	2.5258	0.0017	0.0017	-0.0784	-0.0777	2.5140
94.000	-0.0801	2.4478	0.0017	0.0016	-0.0801	-0.0794	2.4361
95.000	-0.0817	2.3681	0.0016	0.0016	-0.0817	-0.0810	2.3565
96.000	-0.0833	2.2867	0.0016	0.0015	-0.0833	-0.0826	2.2753
97.000	-0.0849	2.2037	0.0015	0.0015	-0.0849	-0.0842	2.1926
98.000	-0.0863	2.1192	0.0015	0.0014	-0.0863	-0.0857	2.1083
99.000	-0.0878	2.0332	0.0014	0.0013	-0.0878	-0.0871	2.0226
100.000	-0.0891	1.9458	0.0013	0.0013	-0.0891	-0.0884	1.9355
101.000	-0.0904	1.8570	0.0013	0.0012	-0.0904	-0.0897	1.8471
102.000	-0.0917	1.7669	0.0012	0.0011	-0.0917	-0.0910	1.7574
103.000	-0.0929	1.6756	0.0011	0.0011	-0.0929	-0.0922	1.6665
104.000	-0.0940	1.5831	0.0011	0.0010	-0.0940	-0.0933	1.5745
105.000	-0.0950	1.4895	0.0010	0.0010	-0.0950	-0.0943	1.4814
106.000	-0.0960	1.3948	0.0010	0.0009	-0.0960	-0.0953	1.3872
107.000	-0.0969	1.2991	0.0009	0.0008	-0.0969	-0.0963	1.2922
108.000	-0.0978	1.2025	0.0008	0.0008	-0.0978	-0.0971	1.1962
109.000	-0.0986	1.1051	0.0008	0.0007	-0.0986	-0.0979	1.0994
110.000	-0.0993	1.0068	0.0007	0.0006	-0.0993	-0.0986	1.0019
111.000	-0.1000	0.9079	0.0006	0.0006	-0.1000	-0.0993	0.9037
112.000	-0.1006	0.8082	0.0006	0.0005	-0.1006	-0.0999	0.8048
113.000	-0.1011	0.7080	0.0005	0.0004	-0.1011	-0.1004	0.7054
114.000	-0.1015	0.6073	0.0004	0.0003	-0.1015	-0.1008	0.6056
115.000	-0.1019	0.5061	0.0003	0.0003	-0.1019	-0.1012	0.5053
116.000	-0.1022	0.4045	0.0003	0.0002	-0.1022	-0.1015	0.4046
117.000	-0.1025	0.3027	0.0002	0.0001	-0.1025	-0.1018	0.3037
118.000	-0.1026	0.2005	0.0001	0.0001	-0.1026	-0.1020	0.2026
119.000	-0.1027	0.0982	0.0001	0.0000	-0.1027	-0.1021	0.1013
120.000	-0.1028	-0.0042	0.0000	-0.0001	-0.1028	-0.1021	0.0000

118.000	-0.1026	0.2005	0.0001	0.0001	-0.1026	-0.1020	0.2026
119.000	-0.1027	0.0982	0.0001	0.0000	-0.1027	-0.1021	0.1013
120.000	-0.1028	-0.0042	0.0000	-0.0001	-0.1028	-0.1021	0.0000

*where (y) is the slope,

(w1) is the distance of the deflection in inches using the Heun's Method

(w2) is the deflection in inches using the Euler's exact solution,

(x) is the length of the column in inches

ka, kb, Ka, and Kb are explained in text

Euler's Numerical and Exact Solutions

delta x	1	F/EI	-0.000686
		$\pi/120$	0.02618
x	y	w(1)	w(2)
0	0.1000	0	0
1	0.1000	0.1000	0.102043
2	0.0999	0.2000	0.204015
3	0.0998	0.2999	0.305848
4	0.0996	0.3997	0.407471
5	0.0993	0.4993	0.508815
6	0.0990	0.5986	0.60981
7	0.0986	0.6976	0.710388
8	0.0981	0.7962	0.810478
9	0.0975	0.8942	0.910013
10	0.0969	0.9918	1.008924
11	0.0962	1.0887	1.107144
12	0.0955	1.1850	1.204605
13	0.0947	1.2804	1.301241
14	0.0938	1.3751	1.396984
15	0.0929	1.4689	1.491771
16	0.0919	1.5618	1.585534
17	0.0908	1.6537	1.678212
18	0.0897	1.7444	1.769739
19	0.0885	1.8341	1.860053
20	0.0872	1.9226	1.949092
21	0.0859	2.0098	2.036796
22	0.0845	2.0956	2.123103
23	0.0831	2.1801	2.207956
24	0.0816	2.2632	2.291295
25	0.0800	2.3448	2.373064
26	0.0784	2.4248	2.453207
27	0.0767	2.5032	2.531668
28	0.0750	2.5799	2.608394
29	0.0733	2.6550	2.683333
30	0.0714	2.7282	2.756433
31	0.0696	2.7997	2.827643
32	0.0677	2.8693	2.896915
33	0.0657	2.9369	2.964203
34	0.0637	3.0026	3.029458
35	0.0616	3.0663	3.092638
36	0.0595	3.1279	3.153697
37	0.0574	3.1874	3.212596
38	0.0552	3.2447	3.269292
39	0.0530	3.2999	3.323749
40	0.0507	3.3529	3.375927
41	0.0484	3.4036	3.425791
42	0.0461	3.4519	3.473308
43	0.0437	3.4980	3.518444
44	0.0413	3.5417	3.561169
45	0.0389	3.5830	3.601453
46	0.0364	3.6219	3.639269
47	0.0339	3.6583	3.67459
48	0.0314	3.6922	3.707394
49	0.0289	3.7236	3.737656
50	0.0263	3.7525	3.765357
51	0.0238	3.7788	3.790477
52	0.0212	3.8026	3.813
53	0.0186	3.8237	3.832909
54	0.0159	3.8423	3.850191
55	0.0133	3.8582	3.864835
56	0.0107	3.8715	3.87683
57	0.0080	3.8822	3.886168
58	0.0053	3.8902	3.892842
59	0.0027	3.8955	3.896847
60	0.0000	3.8982	3.8982

Appe

ndix C:Appendix C:

58	0.0053	3.8902	3.892842
59	0.0027	3.8955	3.896848
60	0.0000	3.8982	3.8982
61	-0.0027	3.8982	3.896848
62	-0.0053	3.8955	3.892842
63	-0.0080	3.8902	3.886168
64	-0.0107	3.8821	3.87683
65	-0.0133	3.8715	3.864835
66	-0.0160	3.8581	3.850191
67	-0.0186	3.8421	3.832909
68	-0.0213	3.8235	3.813
69	-0.0239	3.8022	3.790477
70	-0.0265	3.7783	3.765357
71	-0.0291	3.7518	3.737656
72	-0.0317	3.7227	3.707394
73	-0.0342	3.6910	3.67459
74	-0.0368	3.6568	3.639269
75	-0.0393	3.6200	3.601453
76	-0.0417	3.5807	3.561169
77	-0.0442	3.5390	3.518444
78	-0.0466	3.4948	3.473308
79	-0.0490	3.4482	3.425791
80	-0.0514	3.3991	3.375927
81	-0.0537	3.3478	3.323749
82	-0.0560	3.2940	3.269292
83	-0.0583	3.2380	3.212596
84	-0.0605	3.1797	3.153697
85	-0.0627	3.1192	3.092638
86	-0.0648	3.0566	3.029458
87	-0.0669	2.9918	2.964203
88	-0.0690	2.9248	2.896915
89	-0.0710	2.8559	2.827643
90	-0.0729	2.7849	2.756433
91	-0.0748	2.7120	2.683333
92	-0.0767	2.6371	2.608394
93	-0.0785	2.5605	2.531668
94	-0.0803	2.4820	2.453207
95	-0.0820	2.4017	2.373064
96	-0.0836	2.3197	2.291295
97	-0.0852	2.2361	2.207956
98	-0.0867	2.1509	2.123103
99	-0.0882	2.0642	2.036796
100	-0.0896	1.9760	1.949092
101	-0.0910	1.8864	1.860053
102	-0.0923	1.7954	1.769739
103	-0.0935	1.7031	1.678212
104	-0.0947	1.6096	1.585534
105	-0.0958	1.5149	1.491771
106	-0.0968	1.4192	1.396984
107	-0.0978	1.3224	1.301241
108	-0.0987	1.2246	1.204605
109	-0.0995	1.1259	1.107144
110	-0.1003	1.0263	1.008924
111	-0.1010	0.9260	0.910013
112	-0.1016	0.8250	0.810478
113	-0.1022	0.7234	0.710388
114	-0.1027	0.6212	0.60981
115	-0.1031	0.5185	0.508815
116	-0.1035	0.4153	0.407471
117	-0.1038	0.3119	0.305848
118	-0.1040	0.2081	0.204015
119	-0.1041	0.1041	0.102043
120	-0.1042	0.0000	4.78E-16

118	-0.1040	0.2081	0.204015
119	-0.1041	0.1041	0.102043
120	-0.1042	0.0000	4.78E-16

(x) is the length of the column member in inches.

(w1) is the length of the deflection in inches using Euler's Numerical Method

(w2) is the deflection in inches using Euler's exact solution